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Bounds for the degree and Betti sequences along a graded resolution.

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Resumo

O principal objetivo deste trabalho é dimensionar a resolução livre minimal graduada de um ideal homogêneo em termos dos graus de seus geradores. Em geral, isto é um um objetivo ambicioso. Conforme entendido, dimensionar significa olhar atentamente para os dois parâmetros disponíveis: os shifts e os números Betti. Como em geral as cotas para os shifts podem se comportar de forma bastante abrupta, filtramos esta dificuldade pela subaditividade das sizígias. Espera-se que o método que aplicamos seja novo e dê luz sobre a estrutura da resolução livre minimal. Para os números de Betti, aplicamos as técnicas de Boij-Söderberg para obter cotas superiores polinomiais para eles. Damos uma atenção especial para ideais que possuem resolução livre minimal graduada linear ou linear até uma certa etapa. A teoria de grafos se mostrou propícia para aplicarmos os resultados aqui estabelecidos, justamente por oferecer muitos exemplos de ideais com resoluções livres minimais graduadas lineares até uma certa etapa.

Palavras-chave: resoluções, números de Betti, regularidade, álgebra de Koszul, condição $N_{d,q}$.

Abstract

The main goal of this work is to size up the minimal graded free resolution of a homogeneous ideal in terms of its generating degrees. By and large, this is too ambitious an objective. As understood, sizing up means looking closely at the two available parameters: the shifts and the Betti numbers. Since, in general, bounds for the shifts can behave quite steeply, we filter the difficulty by the subadditivity of the syzygies. The method we applied is hopefully new and sheds additional light on the structure of the minimal free resolution. For the Betti numbers, we apply the Boij-Söderberg techniques in order to get polynomial upper bounds for them. We give a special attention to ideals which have linear or linear graded minimal free resolution up to a determined stage. The Graph theory proved to be suitable for us to apply the results here established, precisely because it offers several examples of ideals with linear graded minimal free resolution up to a certain stage.

Keywords: resolution, Betti numbers, regularity, Koszul algebra, $N_{d,q}$ conditions.

Contents

Introduction

Let $S = k[x_0, \ldots, x_n]$ be a standard graded polynomial ring over a field k, $I = (f_1, \cdots, f_r) \subset S$ be a homogeneous ideal, $p := \mathrm{pdim}_S(S/I)$ the projective dimension of S/I over S and $c := \mathrm{ht}(I)$ the height of I. Our main purpose is to understand the numerical details of the graded minimal free resolution of $R := S/I$ over S, by which one means the sequence of degrees (shifts) and the sequence of the Betti numbers. More specifically, in this work we are interested in obtaining bounds for the Betti numbers and shifts. By and large, the tese draws upon two major tools: spectral sequences and the Boij–Söderberg theory of Betti diagrams. Each of these is applied in a different direction to be detailed in a minute. It is importante to let clear that the choice of this research topic, as well as the main ideas of this work, came from the notable capacity and experience of Professor Hamid Hassanzadeh, especially after his works [19]. It is also worth to note that a large part of this thesis intersects with [6].

As for the Betti numbers, we focus on the d-equigenerated ideals. One of the questions that guided this research was:

There are polynomial upper bounds on d for Betti numbers?

We were able to answer this question positively in some cases as we can see in Propositions 3.1.6 and 3.2.1. In such cases, the ith Betti number of R has as its upper bound the polynomial function of d of degree $p-1$:

$$\binom{d+i-2}{i-1}\binom{d+p-1}{p-i}. \tag{1}$$

Furthermore, these bounds are attained.

Regarding the lower bounds for the Betti numbers, it is known [9, Proposition 1.9] that if an ideal $I \subset S$ has a d-linear free resolution, then the ith expression

$$\binom{d+i-2}{i-1}\binom{d+c-1}{c-i},$$

is a lower bound for the i-th Betti number.

In this work, in Corollary 3.1.8, we show that these lower bounds are still valid only assuming that I satisfies condition $N_{d,c}$, that is, when the minimal graded free resolution is linear up to step c.

The ith shifts of R is defined by

$$t_i^S(R) := \sup\{j \in \mathbb{N} \ ; \ (Tor_i^S(R,k))_j \neq 0\}.$$

It is known that, in general, the limits for the shifts in terms of $t_1^S(R)$ have doubly exponential behavior that cannot be avoided. However, when R is a Koszul algebra, that is , when $t_i^R(k) \leq i$ for all $i \geq 0$, Backelin [2] and Kempf [24] showed that the situation is totally different. They proved that

$$t_i^S(R) \leq 2i \ \text{ for } \ 1 \leq i \leq p. \tag{2}$$

In the paper [1] published in 2015, the authors relaxed the hypothesis. They only assumed

$$t_i^R(k) \leq i \ \text{ for } \ i \leq p+1. \tag{3}$$

In the Chapter 3, as a consequence of Proposition 2.1.1, we will relax even more the hypothesis above. In Corollary 2.2.5, we will prove (2) by requiring only that

$$t_i^R(k) \leq i+1 \ \text{ for } \ i \leq p+1. \tag{4}$$

We will give an example of a ring R which is not a koszul algebra and (3) is not satisfied, but corollary 2.2.5 can be applied.

We next briefly describe the contents of the thesis. Throughout, S denotes a standard graded k-algebra and $R = S/I$, where I is a homogeneous S-ideal.

Chapter 1 presents the preliminary concepts necessary to understand this work. The purpose of the Section 1.2 is to state a theorem of Boij-Söderberg Theory that is widely used in Chapter 3. In the Section 1.3 we briefly summarize the spectral sequence theory focused on spectral sequences that come from a bicomplex. This tool is only used in the demonstration of Proposition 2.1.1.

Chapter 2 is devoted to subadditivity estimates for the degrees of a resolution. The gist of the typical assertion, as compared, e.g., to the subadditivity results in [1], lies in two directions: first, we assume that the standard graded k-algebra S is a Koszul algebra (not just a polynomial ring); second, the subadditivity estimates involve the degrees of both the minimal free S-resolution of R and the minimal free R-resolution of k. Note that neither of the two free resolutions is finite in general. The basic subadditivity result is Proposition 2.1.1, where the results depend on a certain intertwining of the degrees from the two free resolutions. The main tool employed in the proof is the change of ring spectral sequence

$$\mathrm{Tor}_r^S(k, R) \otimes_k \mathrm{Tor}_s^R(k, k) \Rightarrow \mathrm{Tor}_{r+s}^S(k, k).$$

We draw some corollaries, first regarding estimates of the degrees of the S-resolution of R in a so-called 'linear slope" case; second, regularity intertwining estimates; third, estimates for the Green–Lazarsfeld invariant in certain condition.

In chapter 3 we assume again that S is a standard graded polynomial ring and deal with a more direct estimate of the degrees and Betti numbers of the minimal free S-resolution of $R = S/I$, where I is homogeneous and besides d-equigenerated. One main tool here is the Boij–Söderberg theory of Betti diagrams. Our first concern is to bound the first Betti number of I, which is its minimal number of generators under the present hypothesis. Though a well-know upper bound is known in terms of the generating degree d and the projective dimension of S/I over S, no efficient lower bound seems to be exhibited earlier. The first result of the section gives a lower bound for $\beta_1(S/I)$ in terms of the upper degree sequence of the minimal free S-resolution of S/I and $\operatorname{ht} I$ (Proposition 3.1.1). We believe that this lower bound in the non-pure case is new even in the case where S/I is Cohen–Macaulay. In addition, in the case the free resolution is d-linear it implies a binomial coefficient kind as lower bound. There is also a lower bound in terms of the Green–Lazarsfeld N_q condition.

In addition, in the case of projective dimension 4, assuming quadratic upper bounds for the upper degree terms of the resolution we deduce cubic upper bounds for the corresponding Betti numbers. The expressions involved are too technical to reproduce here, so we refer to the details in the appropriate proposition (Proposition 3.2.1).

In Chapter 4, we apply the results established in the Chapters 2 and 3 to the graph theory. Chordal, co-chordal, gap-free and Cameron-Walker graphs have been shown to be suitable to receiving such applications. The main results of this chapter are Propositions 4.2.5 and 4.2.6. The first one provides a linear lower bound for the number of edges of a q-co-chordal in terms of the vertex covering number, while the second provides a quadratic lower bound for the number of edges of a co-chordal graph in terms of the vertex covering number.

As a final note we mention a connection with the recent [7], where the authors pose questions on the Betti numbers of certain monomial ideals satisfying the $N_{d,q}$ property, Definition 1.1.8. Our results in Section 3 (Corollaries 3.1.2, 3.1.3, 3.1.5, 3.1.8 and 3.2.2) provide answers to some of these questions and not just for monomial ideals. For example, Corollary 3.1.2 explains why ideals with $N_{d,q}$ must have many generators.

Chapter 1

Preliminaries

In this thesis, by a ring we always understand as a commutative ring with unit.

The objective of this chapter is to establish notations and expose concepts necessary for the understanding of this work.

1.1 Resolutions and Betti diagrams

Let S denote a graded Noetherian ring over a field k. Let $d \in \mathbb{Z}$ and $S(-d)$ denote the rank one free S-module whose generator is in degree d. In other words, the ith graded part of $S(-d)$ is $S(-d)_i = S_{i-d}$. Given any finitely generated graded S-module M, we form the minimal graded free resolution.

$$\cdots \to \bigoplus_j S(-j)^{\beta_{l,j}(M)} \to \cdots \to \bigoplus_j S(-j)^{\beta_{1j}(M)} \to \bigoplus_j S(-j)^{\beta_{0,j}(M)} \to M \to 0. \tag{1.1}$$

Definition 1.1.1. *$\beta_{i,j}(M)$ are called the (i,j)th graded Betti numbers of M, $\beta_i(M) = \sum_j \beta_{i,j}(M)$ is the ith Betti number of M and $(\beta_0(M), \beta_1(M), \cdots)$ is called the Betti sequence of M.*

The integers $\beta_{i,j}(M)$ are commonly displayed in a matrix called the Betti diagram of M:

$$\begin{array}{c|cccccc}
\beta^S_{\bullet}(M) & 0 & 1 & \cdots & \text{i} & \cdots \\
\hline
0 & \beta_{0,0}(M) & \beta_{1,1}(M) & \cdots & \beta_{i,i}(M) & \cdots \\
1 & \beta_{0,1}(M) & \beta_{1,2}(M) & \cdots & \beta_{i,i+1}(M) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\text{j} & \beta_{0,j}(M) & \beta_{1,j+1}(M) & \cdots & \beta_{i,i+j}(M) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}.$$

When the module M is explicit, we will write $\beta_{i,j}$ instead of $\beta_{i,j}(M)$.

Definition 1.1.2. *The projective dimension of M is*

$$pdim_S(M) := max\{i \in \mathbb{N}_0 \mid \beta_{i,j} \neq 0 \text{ for some } j\}.$$

Definition 1.1.3. *Given an integer n, the nth Castelnuovo-Mumford regularity of M (or just nth regularity of M) over S is*

$$\operatorname{reg}^S_n(M) := max\{j \,|\, \beta_{i,i+j} \neq 0 \,|\, i \leq n\}$$

and the Castelnuovo-Mumford regularity of M, or simply regularity of M, over S is

$$\operatorname{reg}^S(M) := \max\{\operatorname{reg}^S_n(M) \,|\, n \in \mathbb{Z}\}.$$

We set

$$t^S_i(M) := \sup\{j \in \mathbb{N} \ ; \ (Tor^S_i(M,k))_j \neq 0\} = max\{j|\beta_{i,j} \neq 0\}.$$

Note that regularity also can be defined as

$$\operatorname{reg}(M) = max\{t_i(M) - i \mid 0 \leq i \leq pdim(M)\}.$$

Definition 1.1.4. *The graded ring S is said to be a Koszul algebra over a field k if $\operatorname{reg}^S(k) = 0$.*

Examples of Koszul algebras abound and include graded polynomial rings (see [5] for an account).

Now we will define two sequences of integers that are important for this work.

Definition 1.1.5. *Set* $\bar{d}_i := t_i^S(M) = \max\{j|\beta_{i,j}(M) \neq 0\}$ *and* $\underline{d}_i := \min\{j|\beta_{i,j}(M) \neq 0\}$*. The* upper degree sequence *of* M *is* $\bar{\mathbf{d}}(M) := (\bar{d}_0, \ldots, \bar{d}_p)$ *and the* lower degree sequence *of* M *is* $\underline{\mathbf{d}}(M) := (\underline{d}_0, \ldots, \underline{d}_p)$*, where* $p = pdim(M)$*.*

Example 1.1.6. *Let* $S = \mathbb{Q}[x, y, z, w]$ *be the polynomial ring,* $I = (x^2, y^2, z^2, xy, xz, xw)$ *an ideal of* S *and* $M := S/I$*. The graded minimal free resolution of* M *is given by*

$$0 \to S(-5) \to S(-5) \oplus S^4(-4) \to S(-4) \oplus S^8(-3) \to S^6(-2) \to S \to M \to 0.$$

On this example, we have that $pdim(M) = 4$*, that the Betti sequence of* M *is* $(1, 6, 9, 5, 1)$ *and that its Betti diagram is*

$\beta_\bullet^S(M)$	*0*	*1*	*2*	*3*	*4*
0	*1*	*0*	*0*	*0*	*0*
1	*0*	*6*	*8*	*4*	*1*
2	*0*	*0*	*1*	*1*	*0*

Furthermore, note that $\operatorname{reg}^S(M) = 2$*, the upper degree sequence is* $\bar{\mathbf{d}}(M) = (0, 2, 4, 5, 5)$ *and the lower degree sequence is* $\underline{\mathbf{d}}(M) = (0, 2, 3, 4, 5)$*.*

The following result is well-known. We give a proof for the reader's convenience.

Proposition 1.1.7. *With the above notation, let* $S = k[x_1, \ldots, x_n]$ *is a standard graded polynomial ring over a field k and* M *be a finitely generated graded* S*-module. Then:*

(i) $\underline{\mathbf{d}}(M)$ *is strictly increasing.*

(ii) *If* M *is Cohen-Macaulay then* $\bar{\mathbf{d}}(M)$ *is also strictly increasing.*

Proof. (i) Given $0 \leq v \leq p$, let

$$f_v : \bigoplus_j S(-j)^{\beta_{v,j}} \to \bigoplus_j S(-j)^{\beta_{(v-1),j}}$$

stand for the differential in the complex (1.1). Denote the basis of the free module $S(-j)^{\beta_{v,j}}$ by $\{e_i\}$ and that of $S(-j)^{\beta_{(v-1),j}}$ by $\{t_i\}$. Let $\deg(e_h) = \underline{d}_v$ and $\deg(t_r) = \underline{d}_{v-1}$. Set $f_v(e_h) = \sum_i a_i t_i$. Since the resolution is minimal, there are no null columns in the presentation matrix of f_v. Say, $a_i \neq 0$, for some i. Then

$$\underline{d}_v = \deg(e_h) = \deg(f_v(e_h)) \geq \deg(a_i) + \deg(t_i) \geq 1 + \deg(t_r) = 1 + \underline{d}_{v-1}.$$

(ii) Since M is Cohen-Macaulay, $\mathrm{Ext}^j_S(M,S) = 0$ if and only if $j \neq n-p$. Dualizing the minimal resolution 1.1 into S, we obtain a minimal free resolution of $\mathrm{Ext}^{n-p}_S(M,S)$. Moreover, $\bar{\mathbf{d}}(M) = \underline{\mathbf{d}}(\mathrm{Ext}^{n-p}_S(M,S))$. Thus, the result follows from part (i). □

The example (1.1.6) shows that the Cohen-Macaulayness property on the hypothesis is necessary in order to $\bar{\mathbf{d}}(M)$ be strictly increasing.

Definition 1.1.8. *Let S be a graded Noetherian ring over a field k and I be a homogeneous ideal in S generated by elements of degree d. Let $q \geq 1$.*

(i) We say that S/I satisfies the Green-Lazarsfeld condition N_q over S if $t^S_i(R) = i+1$ for all $1 \leq i \leq q$, or equivalently, if $\mathrm{reg}^S_q(R) = 1$.

(ii) We say that S/I satisfies the condition $N_{d,q}$ over S if $t^S_i(R) = d+i-1$ for $1 \leq i \leq q$, or equivalently, if $\mathrm{reg}^S_q(R) = d-1$.

(iii) We say that S/I has a d-linear free resolution over S if $t^S_i(R) = d+i-1$ for $1 \leq i \leq p$, or equivalently, if $\mathrm{reg}^S(R) = d-1$.

Although the last definition refers to the S-module S/I, in practice we will refer to the ideal I. For example, when we write that I has a d-linear free resolution, we mean that S/I has a d-linear free resolution over S.

Example 1.1.9. *Let* $S = \mathbb{Q}[x, y, z, w, t, s, l]$ *be the polynomial ring,* $I_1 = (xz, xw, xt, ys, zs, ws, yt, zt, yw)$*,* $I_2 = (xys, xyz, xtl, xyt, xws, xwl, xwt, xzl, ytl, yws, ywl, tsl)$ *and* $I_3 = (xy, yz, zw, wt, xt, xz)$ *ideals of* S*. The graded minimal free resolution of* S/I_1*,* S/I_2 *and* S/I_3 *are given respectively by*

$$0 \to S(-6) \to S^9(-4) \to S^{16}(-3) \to S^9(-2) \to S \to S/I_1 \to 0$$

$$0 \to S(-7) \to S(-6) \oplus S^9(-5) \to S^{20}(-4) \to S^{12}(-3) \to S \to S/I_2 \to 0$$

$$0 \to S^3(-4) \to S^8(-3) \to S^6(-2) \to S \to S/I_3 \to 0$$

Note that by definition, I_1 satisfies the Green-Lazarsfeld condition N_3, I_2 satisfies the condition $N_{3,2}$ and I_3 has a 2-linear free resolution.

Ideals with d-linear free resolution are not rare. For example, edge ideals of co-chordal graphs have 2-linear free resolution (see [16]). To know more examples of ideals with d-linear free resolution, see [28]. Below we present a simple example of a non-monomial ideal that has a 2-linear free resolution.

Example 1.1.10. *Let* $S = \mathbb{Q}[x, y, z, w, t, s, l]$ *be the polynomial ring and* $I = (x^2 - xz, x^2 + xz, xy - xz, x^2 - z^2, xy - xz - yz + z^2, xy - xz + yz - z^2)$ *ideal of* S*. The graded minimal free resolution of* S/I *is given by*

$$0 \to S^2(-4) \to S^6(-3) \to S^5(-2) \to S \to S/I \to 0.$$

Therefore, I *has a* d*-linear free resolution.*

In the case that the ideal I has 2-linear free resolution, we already know the lower bound for the Betti numbers.

Theorem 1.1.11. *(Herzog–Kühl, 1984 [21]) If M is a graded S-module of projective dimension p with a linear resolution, then $\beta_i(M) \geq \binom{p}{i}$.*

Definition 1.1.12. *By a diagram we shall mean a collection of rational numbers $(\beta_{i,j})$, $i = 1, \cdots, n$ and $j \in \mathbb{Z}$, with only a finite number of them being nonzero. By a pure diagram (of type $\mathbf{d} = (\mathbf{d_0}, \cdots \mathbf{d_t})$, we shall mean a diagram such that for each column i there is only one nonzero entry β_{i,d_i}, and the d_i form an increasing sequence.*

We finish this section with the construction of a pure diagram which is very important to our study.

Let $\mathbf{d} = (\mathbf{d_0}, ..., \mathbf{d_t})$ be a strictly increasing sequence of integers, the pure diagram having $\beta_{0,d_0} = 1$ and

$$\beta_{i,d_i} = \prod_{1 \leq j \leq t,\, j \neq i} \frac{|d_j - d_0|}{|d_j - d_i|}, \text{ for } 1 \leq i \leq t \tag{1.2}$$

is called the Herzog–Kühl diagram of $\mathbf{d}$ and will be denoted by $\beta(\pi(\mathbf{d}))$. We set $\beta_{i,d_i} = 0$ if $i > t$.

Example 1.1.13. *Let $\mathbf{d} = (0, 2, 5, 6)$, The pure diagram $\beta(\pi(\mathbf{d}))$ has $\beta_{0,0} = 1$, $\beta_{1,2} = \beta_{3,6} = 5/2$ and $\beta_{2,5} = 4$. In the same way as we represent the Betti diagrams, $\beta(\pi(\mathbf{d}))$ also can be displayed in a matrix*

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \frac{5}{2} & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 4 & \frac{5}{2} \end{array}.$$

1.2 A little about Boij-Söderberg theory

In Chapter 2 we deal with the question how the Betti sequence of the cyclic module S/I can be bounded by polynomial functions in terms of the degree of the generators of I. In order to do that, we need to know a fundamental result of the Boij-Söderberg theory. We start the explanation of this result through a simple example.

Throughout, we assume $S = k[x_1, \cdots, y_n]$ is a standard graded polynomial ring over a field k.

Example 1.2.1. *Let M be the S-module $S/(x^2, xy, y^3)$. Its minimal resolution is given by*

$$0 \to S(-4) \oplus S(-3) \to S(-3) \oplus S^2(-2) \to S \to M \to 0.$$

Hence its Betti diagram is

$\beta^S_\bullet(M)$	*0*	*1*	*2*
0	*1*	*0*	*0*
1	*0*	*2*	*1*
2	*0*	*1*	*1*

Now we will denote this Betti diagram just by

$$\beta(M) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Note that

$$\beta(M) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 3 \end{bmatrix}.$$

The interesting fact about the decomposition of $\beta(M)$ is that its coefficients are all positive rationals and their sum is 1. Furthermore, the diagrams that appear in this decomposition have only a single nonzero entry in each column, that is, they are pure diagrams. Then we could consider the following question: Will we always have a decomposition for the Betti diagram with such properties? A Theorem within the Boij-Söderberg theory answers this question in a positively and tells us how this decomposition is given. Eisenbud and Schreyer showed that the Betti diagram of any graded Cohen-Macaulay S-module M is a positive rational sum of pure diagrams. Boij and Söderberg extended this result to the non Cohen-Macaulay case.

Definition 1.2.2. *Fix an integer $t \leq n$. A sequence $\mathbf{d} = (d_0, ..., d_t) \in \mathbb{Z}^{t+1}$ is a* degree sequence *of length $t+1$ if $d_{i-1} < d_i$ for $i = 1, ..., t$.*

Let $\mathbb{Z}^{t+1}_{\text{deg}}$ denote the set of all degree sequences of length $t+1$. Given two degree sequences $\mathbf{d}$ and $\mathbf{d}'$ in $\mathbb{Z}^{t+1}_{\text{deg}}$, we say that $\mathbf{d} \preccurlyeq \mathbf{d}'$ if $d_i \leq d'_i$ for $i = 0, ..., t$.

For $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^{t+1}_{\text{deg}}$ with $\mathbf{a} \preccurlyeq \mathbf{b}$, we introduce the "window"

$$\mathcal{D}(\mathbf{a}, \mathbf{b}) := \{\mathbf{d} \in \mathbb{Z}^{t+1}_{\text{deg}} | \mathbf{a} \preccurlyeq \mathbf{d} \preccurlyeq \mathbf{b}\}.$$

If $\mathbf{d} = (d_0, \ldots, d_t) \in \mathbb{Z}^{t+1}_{\text{deg}}$ and $s \leq t$, then we set $\tau_s(\mathbf{d}) = (d_0, \ldots, d_s)$.

Theorem 1.2.3. *([12], [3]) Let M be a graded S-module of projective dimension p and codimension c. Then the Betti diagram $\beta(M)$ can be decomposed as a sum:*

$$\beta(M) = \sum_{c \leq s \leq p} \sum_{\mathbf{d} \in \mathcal{D}(\tau_{\mathbf{s}}(\underline{\mathbf{d}}(\mathbf{M})), \tau_{\mathbf{s}}(\overline{\mathbf{d}}(\mathbf{M})))} q_{\mathbf{d}} \beta(\pi(\mathbf{d})), \tag{1.3}$$

where $q_{\mathbf{d}}$'s are nonnegative rational numbers.

As an example, in the Example 1.2.1 the decomposition given by Theorem 1.2.3 is

$$\beta(M) = \frac{1}{2}\beta(\pi(0,2,3)) + \frac{1}{4}\beta(\pi(0,2,4)) + \frac{1}{4}\beta(\pi(0,3,4)).$$

In this section, we only establish the notations for understanding of Theorem 1.2.3. For more details about the Boij-Soderberg theory, the interested reader can consult [14]

1.3 Spectral sequences

The spectral sequence theory that was introduced by the French mathematician Jean Leray has a fundamental role in this work, more specifically in Chapter 2. Therefore, in order that the reader can remember this theory, we make a brief summary at this section. Case the reader does not know this tool, we recommend to consult [27] and [30].

We start by recalling the definitions of complex, bicomplex and total complex. In this section S is a ring.

Definition 1.3.1. *A* complex $C_\bullet$ *of S-modules is a family* $\{C_n\}_{n\in\mathbb{Z}}$ *of S-modules, together with S-module maps* $d_\bullet = \{d_n : C_n \to C_{n-1}\}$ *such that each composite* $d\, o\, d : C_n \to C_{n-2}$ *is zero. The maps d_n are called the differentials of $C_\bullet$. The kernel of d_n is the module of n-cycles of C, denoted $Z_n = Z_n(C_\bullet)$. The image of $d_{n+1} : C_{n+1} \to C_n$ is the module of n-boundaries of C, denoted $B_n = B_n(C_\bullet)$. Because $d\, o\, d = 0$, we have*

$$0 \subset B_n \subset Z_n \subset C_n$$

for all n. The n^{th} homology module of $C_\bullet$ is the subquotient $H_n(C_\bullet) = Z_n/B_n$ of C_n.

A chain complex C can also be represented by a diagram as below

$$C_\bullet : \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \dots$$

Definition 1.3.2. *A* bicomplex $C_{\bullet\bullet}$ *of S-modules is a family* $\{C_{p,q}\}_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$ *of S-modules, together with S-module maps*

$$d^h_{p,q} : C_{p,q} \to C_{p-1,q} \quad \textit{and} \quad d^v_{p,q} : C_{p,q} \to C_{p,q-1}$$

such that $d^h\, o\, d^h = d^v\, o\, d^v = d^v d^h + d^h d^v = 0$. *We will denote the bicomplex* $C_{\bullet\bullet}$

by $C_{\bullet\bullet} = (C_{p,q}, d^h, d^v)$. It is useful to picture the bicomplex $C_{\bullet\bullet}$ as a lattice

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
\cdots & C_{p-1,q+1} \arrow[l] \arrow[d, "d^v"'] & C_{p,q+1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & C_{p+1,q+1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
\cdots & C_{p-1,q} \arrow[l] \arrow[d, "d^v"'] & C_{p,q} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & C_{p+1,q} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
\cdots & C_{p-1,q-1} \arrow[l] \arrow[d] & C_{p,q-1} \arrow[l, "d^h"'] \arrow[d] & C_{p+1,q-1} \arrow[l, "d^h"'] \arrow[d] & \cdots \arrow[l] \\
 & \cdots & \cdots & \cdots &
\end{tikzcd} \tag{1.4}$$

in which the maps d^h go horizontally, the maps d^v go vertically, and each square anticommutes. Each row C_{*q} and each column C_{p*} is a complex. We say that a double complex C is bounded *if C has only finitely many nonzero terms along each diagonal line $p + q = n$, for example, if C is concentrated in the first quadrant of the plane (*a first quadrant bicomplex*).*

Definition 1.3.3. *Let $C_{\bullet\bullet} = (C_{p,q}, d^h, d^v)$ be a bicomplex of S-modules, its* total complex*, denoted by $Tot(C_{\bullet\bullet})$, is the complex with nth term*

$$Tot(C_{\bullet\bullet})_n = \bigoplus_{n=p+q} C_{p,q}$$

and with differentials $D_n : Tot(C_{\bullet\bullet})_n \to Tot(C_{\bullet\bullet})_{n-1}$ given by

$$D_n = \sum_{n=p+q} (d^h_{p,q} + d^v_{p,q})$$

.

Lemma 1.3.4. *If $C_{\bullet\bullet}$ is a bicomplex of S-modules, then $(Tot(C_{\bullet\bullet}), D)$ is a complex.*

Proof. See [27, Lemma 10.5]. □

Definition 1.3.5. *A* spectral sequence *of S-modules (starting with E^a) consists of the following data:*

(1) *for each $r \geq a$, a family $\{E^r_{p,q}\}_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$ of S-modules.*

(2) *Maps $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ that are differentials in the sense that $d^r d^r = 0$.*

(3) *$E^{r+1}_{p,q} \cong Ker(d^r_{p,q})/(d^r_{p+r,q-r+1})$.*

The total degree *of the term $E^r_{p,q}$ is $n = p + q$.*

Note that $E^{r+1}_{p,q}$ is a subquotient of $E^r_{p,q}$ and each differential $d^r_{p,q}$ decreases the total degree by one. We can think that for each r we have a page of the spectral sequence and for each (p, q) we have a point on this page.

Definition 1.3.6. *A spectral sequence of S-modules $\{E^r_{p,q}\}$ starting with E^a is said to be* bounded *if for all n there are $t(n)$ and $s(n)$ such that for p, q with $p+q = n$, $E^a_{p,q} = 0$ whenever $p < t(n)$ or $p > s(n)$. If so, then for each p and q there is an r_0 such that $E^r_{p,q} = E^{r+1}_{p,q}$ for all $r \geq r_0$. We write $E^\infty_{p,q}$ for this stable value of $E^r_{p,q}$.*

Definition 1.3.7. *A bounded spectral sequence of S-modules $\{E^r_{p,q}\}$ starting with E^a converges to a family of S-modules $\{H_n\}_{n\in\mathbb{Z}}$, denoted by*

$$E^a_{p,q} \Rightarrow H_{p+q},$$

if for all n and p, q with $p + q = n$, there is a filtration of H_n

$$0 = F_s H_n \subset \cdots \subset F_{p-1} H_n \subset F_p H_n \subset \cdots F_t H_n = H_n,$$

such that $E^\infty_{p,q} \cong F_p H_n / F_{p-1} H_n$

The next theorem is one of the main results of spectral sequence theory.

Theorem 1.3.8. *Let $C_{\bullet\bullet} = (C_{p,q}, d^h, d^v)$ be a bicomplex of S-modules in the first quadrant. Then there are two spectral sequences starting with E^1 denoted by $\{{}^{hor}E^r_{p,q}\}$ and $\{{}^{ver}E^r_{p,q}\}$ such that*

(1) ${}^{hor}E^1_{p,q} = H_p(C_{\bullet,q})$ *and* ${}^{hor}E^2_{p,q} = H^v_q H^h_p(C_{\bullet\bullet})$.

(2) ${}^{ver}E^1_{p,q} = H_q(C_{p,\bullet})$ *and* ${}^{ver}E^2_{p,q} = H^h_p H^v_q(C_{\bullet\bullet})$.

(3) ${}^{hor}E^1_{p,q} \Rightarrow H_{p+q}(Tot(C_{\bullet\bullet}))$ *and* ${}^{ver}E^1_{p,q} \Rightarrow H_{p+q}(Tot(C_{\bullet\bullet}))$

Proof. Consult [27, Proposition 10.17] and [27, Proposition 10.18]. □

Many spectral sequences that appear in applications are quite simple in the sense that: They start with E^1, $E^\infty_{p,q} = E^2_{p,q}$ and the terms on the second page are almost all null. The next result is focused on these cases.

Definition 1.3.9. *Let* $\{E^r_{p,q}\}$ *be a spectral sequence of* S*-modules. We say that* $\{E^r_{p,q}\}$ *collapses on the p-axis if* $E^2_{p,q} = 0$ *for all* $q \neq 0$ *and that* $\{E^r_{p,q}\}$ *collapses on the q-axis if* $E^2_{p,q} = 0$ *for all* $p \neq 0$.

Proposition 1.3.10. *Let* $\{E^r_{p,q}\}$ *be a spectral sequence of* S*-modules starting with* E^a *such that* $E^a_{p,q} \Rightarrow H_{p+q}$.

(1) *If* $\{E^r_{p,q}\}$ *collapses on either axis, then* $E^\infty_{p,q} = E^2_{p,q}$ *for all* p, q.

(2) *If* $\{E^r_{p,q}\}$ *collapses on the p-axis, then* $H_n \cong E^2_{n,0}$.

(3) *If* $\{E^r_{p,q}\}$ *collapses on the q-axis, then* $H_n \cong E^2_{0,n}$.

Proof. [27, Proposition 10.21] □

For the reader to adapt to the notation, we finish this section with a classic application of the spectral sequence theory. Let M, N be S - modules and $P_\bullet$ a deleted projective resolution of M. Remember that

$$\mathrm{Tor}^S_n(M, N) := H_n(P_\bullet \otimes_S N).$$

Example 1.3.11. *Let M, N be S - modules and $Q_\bullet$ a deleted projective resolution of N. Then $\operatorname{Tor}_n^S(M,N) = H_n(M \otimes_S Q_\bullet)$.*
Denote

$$Q_\bullet : \cdots \to Q_2 \xrightarrow{v_2} Q_1 \xrightarrow{v_1} Q_0 \to 0.$$

Let $P_\bullet$ be a deleted projective resolution of M, denoted by

$$P_\bullet : \cdots \to P_2 \xrightarrow{h_2} P_1 \xrightarrow{h_1} P_0 \to 0.$$

Consider the bicomplex $C_{\bullet\bullet}$ in the first quadrant

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & P_0 \otimes Q_2 \arrow[l] \arrow[d, "d^v_{02}"'] & P_1 \otimes Q_2 \arrow[l, "d^h_{12}"'] \arrow[d, "d^v_{12}"'] & P_2 \otimes Q_2 \arrow[l, "d^h_{22}"'] \arrow[d, "d^v_{22}"'] & \cdots \arrow[l] \\
0 & P_0 \otimes Q_1 \arrow[l] \arrow[d, "d^v_{01}"'] & P_1 \otimes Q_1 \arrow[l, "d^h_{11}"'] \arrow[d, "d^v_{11}"'] & P_2 \otimes Q_1 \arrow[l, "d^h_{21}"'] \arrow[d, "d^v_{21}"'] & \cdots \arrow[l] \\
0 & P_0 \otimes Q_0 \arrow[l] \arrow[d] & P_1 \otimes Q_0 \arrow[l, "d^h_{10}"'] \arrow[d] & P_2 \otimes Q_0 \arrow[l, "d^h_{20}"'] \arrow[d] & \cdots \arrow[l] \\
 & 0 & 0 & 0 &
\end{tikzcd} \quad , \tag{1.5}$$

where $d^v_{i,j} = I_{P_i} \otimes v_j$ with I_{P_i} the identity map in P_i and $d^h_{i,j} = h_i \otimes I_{Q_j}$ with I_{P_J} the identity map in Q_j.

Let $\{{}^{hor}E^r_{p,q}\}$ and $\{{}^{ver}E^r_{p,q}\}$ be the spectral sequences of Theorem 1.3.8. By this same theorem,we know what are the initial pages of these sequences and that ${}^{hor}E^1_{p,q} \Rightarrow H_{p+q}(Tot(C_{\bullet\bullet}))$ and ${}^{ver}E^1_{p,q} \Rightarrow H_{p+q}(Tot(C_{\bullet\bullet}))$.

The first page of $\{{}^{hor}E^r_{p,q}\}$ is constructed by calculating the homology of horizontal complexes of the bicomplex above. As each Q_i is a projective module, and in particular they are flat. We use the first isomorphism theorem to conclude that

$${}^{hor}E^1_{p,q} = \begin{cases} M \otimes Q_i & \textit{if } \; p = 0 \\ 0 & p \neq 0. \end{cases}$$

We can also represent this first page by the diagram below

$$\begin{array}{ccccc} \vdots & & \vdots & \vdots & \\ \downarrow & & \downarrow & \downarrow & \\ M \otimes Q_2 & & 0 & 0 & \cdots \\ {\scriptstyle d^v_{02}}\downarrow & & \downarrow & \downarrow & \\ M \otimes Q_1 & & 0 & 0 & \cdots \\ {\scriptstyle d^v_{01}}\downarrow & & \downarrow & \downarrow & \\ M \otimes Q_0 & & 0 & 0 & \cdots \\ \downarrow & & \downarrow & \downarrow & \\ 0 & & 0 & 0 & \cdots . \end{array} \tag{1.6}$$

By Theorem 1.3.8, the second page of the $\{{}^{hor}E^r_{p,q}\}$ *is constructed by calculating the homology of the above complexes. Therefore,*

$$ {}^{hor}E^1_{p,q} = \begin{cases} H_q(M \otimes Q_\bullet) & \text{if } p = 0 \\ 0 & p \neq 0. \end{cases}$$

This is, $\{{}^{hor}E^r_{p,q}\}$ *collapses on the q-axis. By the Theorem 1.3.10* $Tot(C_{\bullet\bullet})_n = E^2_{0,n} = H_n(M \otimes Q_\bullet)$.

For the spectral sequence $\{{}^{ver}E^r_{p,q}\}$, *we repeat an analogous argument as done above to show that* $Tot(C_{\bullet\bullet})_n = E^2_{n,0} = H_n(P_\bullet \otimes N)$. *Hence*

$$\mathrm{Tor}^S_n(M, N) = H_n(M, Q_\bullet) = H_n(P_\bullet \otimes N)$$

as we wanted to show.

Chapter 2

Subadditivity bounds via change of ring

Throughout this chapter we assume that S is a standard graded Koszul algebra over the field k and $R = S/I$ is a homogeneous residual algebra. We make use of the notations established in the first chapter

In the sequel we will consider the above invariants both over S and R, in particular the intertwining along the change of rings from S to R, both for the 'degrees' as for the regularity.

The main result of this chapter is Theorem 2.1.1 which was put separately in the Section 2.1

Section 4.2 is dedicated to obtain consequences of the Theorem 2.1.1

2.1 The key result

This section is devoted to prove the following proposition.

Proposition 2.1.1. *Let S be a standard graded Koszul algebra over k, and let $R = S/I$ denote a residual graded algebra with $p := \mathrm{pdim}_S(R)$, possibly infinite. Then, for any $i \geq 0$,*

(1) $t_i^S(R) \leq \max\{t_{i-j}^S(R) + t_{j+1}^R(k) \,|\, j = 1, \cdots, i\}$.

(2) $t^R_{i+1}(k) \le \max\{t^S_{i-j}(R) + t^R_j(k) \mid j = \max\{0, i-p\}, \cdots, i-1\}$, *with the convention that* $\max\{0, i-p\} = 0$ *if* p *is infinite.*

In particular, $t^R_2(k) = t^S_1(R)$.

Proof. The proof is based on the change of ring spectral sequence

$$\mathrm{Tor}^S_r(k, R) \otimes_k \mathrm{Tor}^R_s(k, k) \Rightarrow \mathrm{Tor}^S_{r+s}(k, k).$$

In order to study the maps in this spectral sequence, we introduce some basic intervening complexes. Thus, let $K^S_\bullet$ and $F_\bullet$ denote the minimal free resolution of k over S and the minimal free resolution of k over R, respectively. Since S is a Koszul algebra, we can write

$$K^S_\bullet(k) : \cdots \to S(-i)^{\beta^S_i(k)} \to S(-i+1)^{\beta^S_{i-1}(k)} \to \cdots \to S.$$

Set $K_i = S(-i)^{\beta^S_i(k)}$. Consider the bicomplex $(K^S_\bullet \otimes_S R) \otimes_R F_\bullet$ in the second quadrant

$$\begin{tikzcd}
 & & (K_2 \otimes_S R) \otimes_R F_0 \arrow[d] \\
\cdots & (K_1 \otimes_S R) \otimes_R F_1 \arrow[r] \arrow[d] & (K_1 \otimes_S R) \otimes_R F_0 \arrow[d] \\
(K_0 \otimes_S R) \otimes_R F_2 \arrow[r] & (K_0 \otimes_S R) \otimes_R F_1 \arrow[r] & (K_0 \otimes_S R) \otimes_R F_0
\end{tikzcd} \tag{2.1}$$

The horizontal spectral sequence collapses at the first step

$$^1\,\mathrm{E}^{-j,i}_{hor} = \begin{cases} 0 & \text{if } j \neq 0 \\ k^{\beta^S_i(k)}(-i) & j = 0. \end{cases} \tag{2.2}$$

Note that the shifts in this convergence are due to the fact that S is a Koszul algebra.

The vertical spectral sequence has first terms $^1\,\mathrm{E}^{-i,j}_{ver} = \mathrm{Tor}^S_j(k, R) \otimes_R F_i$ with connecting homorphisms $\mathrm{Tor}^S_j(k, R) \otimes F_i \to \mathrm{Tor}^S_j(k, R) \otimes F_{i-1}$. The map $\phi_i : F_i \to$

F_{i-1} comes from the minimal free resolution $F_\bullet$, hence $\phi_i(F_i) \subset \mathfrak{m}F_{i-1}$, where $\mathfrak{m}$ is the irrelevant maximal ideal of R. On the other hand $\operatorname{Tor}_j^S(k,R)$ is annihilated by $\mathfrak{m}$. Then, the connecting homomorphism $\operatorname{Tor}_j^S(k,R) \otimes F_i \to \operatorname{Tor}_j^S(k,R) \otimes F_{i-1}$ is the zero map. Thus ${}^2\,\mathrm{E}_{ver}^{-i,j} = \operatorname{Tor}_j^S(k,R) \otimes_R F_i$. By setting $T_i := \operatorname{Tor}_i^S(k,R)$, we draw the second vertical spectral as follows:

$$\begin{tikzcd}
 & & \cdots & T_3 \otimes F_1 & T_3 \otimes F_0 \\
\cdots & \cdots & T_2 \otimes F_2 \arrow[urr, dashed, "d^2"] & T_2 \otimes F_1 & T_2 \otimes F_0 \\
\cdots & T_1 \otimes F_3 \arrow[uurrr, dashed] & T_1 \otimes F_2 & T_1 \otimes F_1 & T_1 \otimes F_0 \\
T_0 \otimes F_4 \arrow[uuurrrr, dotted, no head] & T_0 \otimes F_3 & T_0 \otimes F_2 & T_0 \otimes F_1 & T_0 \otimes F_0
\end{tikzcd} \tag{2.3}$$

Let $i \geq 0$ be an integer. Consider the ith diagonal in the above picture. Since ${}^1\,\mathrm{E}_{hor}$ collapses, $k^{\beta_i^S(k)}(-i)$ is the ith homology of the total complex. The convergence of the vertical spectral sequence implies that the ${}^\infty\,\mathrm{E}_{ver}$ terms on the ith diagonal filter $k^{\beta_i^S(k)}(-i)$.

Now, let a be an integer such that $a > \max\{t_{i-j}^S(R) + t_{j+1}^R(k) | j = 1, \cdots, i\}$. Since $F_\bullet$ is minimal, if $t_{i-j}^S(R) \geq i-j$ and $t_{j+1}^R(k) \geq j+1$ then $a > i+1$. Therefore, considering these spectral sequences in degree a, one has $({}^1\mathrm{E}_{hor}^{0,i})_a = 0$, and hence, $({}^\infty\mathrm{E}_{ver}^{-j,i-j})_a = 0$ for all j.

We now show that $({}^\infty\mathrm{E}_{ver}^{0,i})_a = (T_i \otimes F_0)_a = \operatorname{Tor}_i^S(k,R)_a$. To see this, consider the map

$$(T_{i-1} \otimes F_2)_a \xrightarrow{d^2} (T_i \otimes F_0)_a.$$

Since $a > \operatorname{end}(T_{i-1}) + t_2^R(k) = \operatorname{end}(T_{i-1} \otimes F_2)$ then $(T_{i-1} \otimes F_2)_a = 0$. Thus, $({}^3\mathrm{E}_{ver}^{0,i})_a = ({}^2\mathrm{E}_{ver}^{0,i})_a = (T_i \otimes F_0)_a$. In the next page $({}^3\mathrm{E}_{ver}^{0,i})_a$ is target of a map from $({}^3\mathrm{E}_{ver}^{-3,i-2})_a$ which is a subquotient of $(T_{i-2} \otimes F_3)_a$. By a similar reasoning, the latter vanishes, hence eventually $({}^{\infty}\mathrm{E}_{ver}^{0,i})_a = \mathrm{Tor}_i^S(k,R)_a$ which must be zero since the abutment is zero. Therefore $t_i^S(R) \leq \max\{t_{i-j}^S(R) + t_{j+1}^R(k) : j = 1, \cdots, i\}$, thus proving item (1).

To prove item (2), we regard the above spectral sequences from a different angle. The vertical spectral in the second step is the following

$$\begin{tikzcd}
 & \cdots & \mathrm{Tor}_2^S(k,R)\otimes F_1 & \mathrm{Tor}_2^S(k,R)\otimes F_0 \\
 & \mathrm{Tor}_1^S(k,R)\otimes F_2 & \mathrm{Tor}_1^S(k,R)\otimes F_1 & \mathrm{Tor}_1^S(k,R)\otimes F_0 \\
\mathrm{Tor}_0^S(k,R)\otimes F_3 \arrow[urr, "d^2"] \arrow[uurrr, dashed, "d^3"] & \mathrm{Tor}_0^S(k,R)\otimes F_2 & \mathrm{Tor}_0^S(k,R)\otimes F_1 & \mathrm{Tor}_0^S(k,R)\otimes F_0
\end{tikzcd}$$

Set $t_i := t_i^S(R)$ and $\tau_i := t_i^R(k)$. We may assume that in the presentation $R = S/I$, the ideal I has no linear form; so that $t_i \geq i+1$ for any $i \geq 1$.

Notice that if p is finite it is the last index i for which $\mathrm{Tor}_i^S(k,R) \neq 0$. Let $i \geq 1$ and consider the $(i+1)$th diagonal in the above vertical spectral sequence ($i = 2$ is shown in the above picture). Let $a > \max\{t_{i-j}^S(R) + t_j^R(k) : j = \max\{0, i-p\} \cdots, i-1\}$. Then $k(-i-1)_a = 0$ since $a > i+1$. This implies that the infinity terms(in degree a) on the $(i+1)$th diagonal in the above vertical spectral sequence are all null. Next, since $a > t_{i-j} + \tau_j$, $(T_{i-j} \otimes_R F_j)_a = 0$. Therefore in any page, $n > 2$, $({}^{n}\mathrm{E}_{ver}^{-j,(i-j)})_a = 0$. Hence any map with source in $({}^{2}\mathrm{E}_{ver}^{i+1,0})_a$ maps to zero. This shows that $0 = ({}^{\infty}\mathrm{E}_{ver}^{i+1,0})_a = ({}^{2}\mathrm{E}_{ver}^{i+1,0})_a = (\mathrm{Tor}_0^S(k,R) \otimes F_{i+1})_a$. The latter shows that $a > \tau_{i+1}$, since $(\mathrm{Tor}_0^S(k,R) \otimes F_{i+1})_{\tau_{i+1}} \neq 0$. Hence, $\tau_{i+1} \leq \max\{t_{i-j} + \tau_j : j = \max\{0, i-p\} \cdots, i-1\}$, as was to be shown. $\square$

Next, we will see that the bounds given by Proposition 2.1.1 can be reached.

Example 2.1.2. *Let $S = \mathbb{Q}[x,y,z,w]$ be the polynomial ring, $I = (x^3, y^3, xz^2 - yw^2)$ be an ideal of S and $R = S/I$. Using Macaulay2, we have the following Betti tables:*

$\beta^S_\bullet(R)$	0	1	2	3	4
0	1	–	–	–	–
1	–	–	–	–	–
2	–	3	–	–	–
3	–	–	–	–	–
4	–	–	3	–	–
5	–	–	1	2	–
6	–	–	1	2	1
7	–	–	1	2	1

and

$\beta^R_\bullet(\mathbb{Q})$	0	1	2	3	4	5
0	1	4	6	4	1	–
1	–	–	3	12	18	12
2	–	–	–	–	6	24
3	–	–	–	–	–	–
4	–	–	–	1	6	15
5	–	–	–	1	6	21
6	–	–	–	1	6	21
7	–	–	–	–	–	6

Note that $9 = t_2^S(R) \leq \max\{t_1^S(R) + t_2^R(\mathbb{Q}), t_3^R(\mathbb{Q})\} = \max\{6, 9\}$ and $9 = t_3^R(\mathbb{Q}) \leq \max\{t_2^S(R), t_1^S(R) + t_1^R(\mathbb{Q})\} = \max\{9, 4\}$.

We finish this section by showing that hypothesis that S is a Koszul algebra in Proposition 2.1.1 can not be suppressed.

Example 2.1.3. *Let $H = \mathbb{Q}[x, y, z, w]$ be the polynomial ring, $I = (x^3, y^3, xz^2 - yw^2)$ an ideal of H and $S := H/I$. Since I is not generated in degree two, S is not a Koszul algebra. Let $I = (x^2, y^2, xz - yw)$ be an ideal of S and $R := S/I$. Using Macaulay2, we have the following Betti tables*

$\beta^S_\bullet(R)$	0	1	2	3	4	5
0	1	–	–	–	–	–
1	–	3	2	–	–	–
2	–	–	7	14	4	–
3	–	–	–	–	21	33
4	–	–	–	1	2	1
5	–	–	–	1	5	12
6	–	–	–	1	5	19
7	–	–	–	–	3	18
8	–	–	–	–	–	7

and

$\beta^R_\bullet(\mathbb{Q})$	0	1	2	3	4	5
0	1	4	9	16	25	36
1	–	–	1	8	33	98
2	–	–	–	–	1	12

Note that $9 = t_3^S(R) \not\leq \max\{t_2^S(R) + t_2^R(\mathbb{Q}), t_1^S(R) + t_3^R(\mathbb{Q}), t_4^R(\mathbb{Q})\} = \max\{7, 6\}$.

2.2 Its consequences

In this section we will develop consequences of Theorem 2.1.1. The first of these concerns the special situation informally known as 'linear slope'.

Corollary 2.2.1. *Let S and R be as in* Proposition 2.1.1. *Suppose that $2 \leq i \leq \mathrm{pdim}_S(R)$ and that for any $2 \leq j < i$, $t_j^S(R) \neq t_{j+1}^R(k)$. Then either $t_i^S(R) = t_{i+1}^R(k)$ or else the following inequalities hold:*

- $t_i^S(R) < t_{i+1}^R(k) \leq (i-1)t_1^S(R) + 1$, *or*
- $t_{i+1}^R(k) < t_i^S(R) \leq it_1^S(R)$.

Proof. The proof is by induction on i for which we consider the two sets of inequalities in Proposition 2.1.1. Setting $t_i := t_i^S(R)$ and $\tau_i := t_i^R(k)$, we have

$$\tau_{i+1} \leq \max\{t_i, t_{i-1} + \tau_1, \cdots, t_1 + \tau_{i-1}\} and \tag{2.4}$$

$$t_i \leq \max\{\tau_{i+1}, t_1 + \tau_i, \cdots, t_{i-1} + \tau_2\}. \tag{2.5}$$

The case $i = 2$ easily follows from 2.4 and 2.5. Assume that the result is valid for $2 \leq k \leq i-1$. Let $i \geq 3$

Consider the case that $t_i \leq \tau_{i+1}$, by 2.5 we have

$$\tau_{i+1} \leq \max\{t_{i-1} + \tau_1, \cdots, t_1 + \tau_{i-1}\} = \max\{t_{i-j} + \tau_j \,|\, j = 1, \cdots, i-1\}.$$

By the inductive hypothesis, for $j = 1, \ldots i-2$, we have

$$\begin{cases} t_{i-j} < \tau_{i-j+1} \leq (i-j-1)t_1 + 1 \\ \qquad \text{or} \\ \tau_{i-j+1} < t_{i-j} \leq (i-j)t_1. \end{cases} \tag{2.6}$$

and for $j = 3, \dots i-1$

$$\begin{cases} t_{j-1} < \tau_j \leq (j-2)t_1 + 1 \\ \quad\text{or} \\ \tau_j < t_{j-1} \leq (j-1)t_1. \end{cases} \tag{2.7}$$

Hence, for $j = 2, \dots i-2$, in any of the above alternatives, we conclude that $t_{i-j} + \tau_j \leq (i-1)t_1 + 1$. The verification that $t_{i-j} + \tau_j \leq (i-1)t_1 + 1$ for $j \in \{1, 2, i-1\}$ is done separately, using again 2.6 and 2.7.

The other case is shown in a similar way.

□

Corollary 2.2.2. *Let S and R be as in* Proposition 2.1.1. *Then, for any $i \geq 0$,*

(1) $t_i^S(R) \leq 2i + \sum_{l=2}^{i+1} \operatorname{reg}_l^R(k)$.

(2) $\operatorname{reg}_i^S(R) \leq \operatorname{reg}_{i-1}^S(R) + \operatorname{reg}_{i+1}^R(k) + 1$.

(3) $\operatorname{reg}_{i+1}^R(k) \leq \operatorname{reg}_i^S(R) + \operatorname{reg}_{i-1}^R(k) - 1$ *for* $i \leq p$; *and*
$\operatorname{reg}_{i+1}^R(k) \leq \operatorname{reg}_p^S(R) + \operatorname{reg}_{i-1}^R(k) - 1$ *for* $i \geq p$, *provided* $p < \infty$.

(4) ([20, Proposition 5.8]) If $\operatorname{reg}^S(R) = 1$, then R is a Koszul algebra.

Proof. (1) The proof is by induction on i. Setting $t_i := t_i^S(R)$ and $\tau_i := t_i^R(k)$. The case $i = 0$ is trivial. Assume that the result is valid for $1 \leq k \leq i-1$. By 2.1.1

$$t_i^S(R) \leq \max\{t_{i-j} + \tau_{j+1} \,|\, j = 1, \cdots, i\}.$$

Now, use the induction hypothesis to show that $t_{i-j} + \tau_{j+1} \leq 2i + \sum_{j=2}^{i+1} \operatorname{reg}_j^R(k)$, for $1 \leq j \leq i$.

Item (2) is merely the definition of the regularity as applied in Proposition 2.1.1(1), noting that $\operatorname{reg}_{j+1}^R(k) \geq \operatorname{reg}_j^R(k)$. Similarly, (3) follows from Proposition 2.1.1(2). To see (4), notice that (3) implies that $\operatorname{reg}^R(k) \leq \max\{\operatorname{reg}_1^R(k), \operatorname{reg}_0^R(k)\} = 0$. □

Recall that, given an integer $q \geq 0$, R satisfies the Green–Lazarsfeld condition N_q over S if $t_i^S(R) = i+1$ for $1 \leq i \leq q$; or equivalently, if $\mathrm{reg}_q^S(R) = 1$.

Corollary 2.2.3. *Let S and R be as in* Proposition 2.1.1. *Suppose that* $\mathrm{reg}_{n+1}^R(k) = 0$ *for some $n \geq 1$. Then, for every $i \leq n$,*

(1) $t_i^S(R) \leq 2i$.

(2) $\mathrm{reg}_i^S(R) \leq \mathrm{reg}_{i-1}^S(R) + 1$.

(3) $t_i^S(R) \leq t_{i-1}^S(R) + 2$ *for* $i \leq \min\{n, \mathrm{depth}(S) - \dim(R)\}$.

Proof. (1) It is an immediate consequence of Corollary 2.2.2(1).

(2) It follows from Corollary 2.2.2 (2).

(3) Set $m(R) \colon = \min\{i \geq 0 | t_i^S(R) \geq t_{i+1}^S(R)\}$. By the proof of [1, Lemma 6.1] we have $\mathrm{Ext}_S^{m(R)}(R, S) \neq 0$ hence $m(R) \geq \mathrm{depth}(S) - \dim(R)$. The result now follows from Proposition 2.1.1(1) as $t_{j+1}^R(k) = j+1$ for $j \leq n$.

□

Corollary 2.2.4. *Let S and R be as in* Proposition 2.1.1. *Suppose that* $\mathrm{reg}_{n+1}^R(k) \leq 1$ *for some $n \geq 1$ and that R satisfies the Green-Lazarsfeld condition N_q over S for some $q \geq 1$. Then*

$$t_i^S(R) \leq 2i - q + 1 \quad \textit{for} \quad q+1 \leq i \leq n.$$

Proof. Setting $t_i := t_i^S(R)$ and $\tau_i := t_i^R(k)$.

First we show that $\mathrm{reg}_q^S(R) \leq 1$ implies that $\mathrm{reg}_{q+1}^R(k) = 0$. This is done by induction on i, we will show that $\tau_i = i$ for $1 \leq i \leq q+1$.

For $i = 1$, $\tau_1 = 1$. Assume that $i \geq 2$ and that the result holds for $i < q+1$. By Proposition 2.1.1(2)

$$\tau_{i+1} \leq \max\{t_{i-j} + \tau_j \,|\, j = \max\{0, i-p\}, \cdots, i-1\}.$$

Note that $i-j \leq i \leq q$. Since R satisfies the Green-Lazarsfeld condition N_q, $t_{i-j} = i-j+1$. By the inductive hypothesis $\tau_j = j$, hence $\tau_{i+1} \leq i+1$.

Now we prove that $t_i^S(R) \leq 2i - q + 1$ for $q \leq i \leq n$. Proceed by induction on i. The result holds for $i = q$ by hypothesis. Assume that $i \geq q+1$ and that the result holds for i and below. By Proposition 2.1.1(1)

$$t_{i+1} \leq \max\{t_{i+1-j} + \tau_{j+1} \,|\, j = 1, \cdots, i+1\}.$$

First let's analyze the elements $t_{i+1-j} + \tau_{j+1}$ for $j = 1 \ldots q$. By the initial part of proof, $\tau_{j+1} = j+1$. On the other hand, using the hypothesis or induction hypothesis, $t_{i+1-j} \leq i+2-j$ or $t_{i+1-j} \leq 2(i+1-j) - q + 1$. In both cases, $t_{i+1-j} + \tau_{j+1} \leq 2(i+1) - q + 1$.

Now we analyze the terms $t_{i+1-j} + \tau_{j+1}$ for $j = q+1, \ldots i$. By hypothesis $\tau_{j+1} \leq j+2$. On the other hand, using the hypothesis or induction hypothesis, $t_{i+1-j} \leq i+2-j$ or $t_{i+1-j} \leq 2(i+1-j)-q+1$. In both cases, $t_{i+1-j}+\tau_{j+1} \leq 2(i+1)-q+1$. Finally, the case that $j = i+1$, $t_{i+1-j}+\tau_{j+1} = \tau_{j+2} \leq i+3 \leq 2(i+1)-q+1$. Thus concluding the proof. $\square$

Corollary 2.2.5. *Let S and $R = S/I$ be as in* Proposition 2.1.1. *Suppose that $\mathrm{reg}_{p+1}^R(k) \leq 1$ and that I is a 2-equigenerated ideal. Then*

$$t_i^S(R) \leq 2i \quad \textit{for} \quad i \leq p.$$

Proof. Follows from Corollary 2.2.4 $\square$

Next we will see an example of a ring that is not a koszul algebra and that does not satisfy the condition $\mathrm{reg}_{p+1}^R(k) \leq 0$, but satisfies the hypothesis of Corollary 2.2.5

Example 2.2.6. *Let $S = \mathbb{Q}[x,y,z,w]$ be the polynomial ring, $I = (x^2, y^2, z^2, w^2, xy+zw)$ be an ideal of S and $R = S/I$. Using Macaulay2, we have the following Betti tables.*

$\beta^R_\bullet(\mathbb{Q})$	*0*	*1*	*2*	*3*	*4*	*5*	*6*
0	*1*	*4*	*11*	*24*	*46*	*80*	*130*
1	–	–	–	*5*	*36*	*159*	*536*
2	–	–	–	–	–	–	*25*

and

$\beta^S_\bullet(R)$	*0*	*1*	*2*	*3*	*4*
0	*1*	–	–	–	–
1	–	*5*	–	–	–
2	–	–	*15*	*16*	*5*

Note that $\operatorname{reg}^R_5(k) \leq 1$.

Remark 2.2.7. *(a) Granted the assumption* $\operatorname{reg}^R_{n+1}(k) = 0$*, the inequality in Corollary 2.2.3(1) has been proved earlier in [1, Corollary 5.2] in the case where S is a polynomial ring. The argument there uses the structure of minimal model, a tool that may not be available for a Koszul algebra. Note that Corollary 2.2.5 is an extension of the Corollary 2.2.3(1).*

(b) Corollary 2.2.3(2) shows how the jumps happen along the way to compute the regularity of Koszul algebras. This inequality has also been shown in [1, Proposition 6.7] in the case where S is a polynomial ring and k has characteristic zero or prime characteristic p such that $p \nmid \binom{n}{j}$ for any $j \leq n$.

(c) Corollary 2.2.4 is a characteristic-free result, hence it improves the bound in [1, Theorem 7.1] in the case where k is a field, $q \geq 3$ and char(k) fails for the above restriction.

Chapter 3

Bounding the Betti sequence

Throughout this part, $S = k[x_1, \ldots, x_n]$ is a standard graded polynomial ring over a field k and $I \subset S$ is a homogeneous d-equigenerated ideal.

In this chapter we apply Theorem 1.2.3 to obtain bounds for the Betti sequence of modules of the form S/I. As before, let $(\beta_0, \beta_1, \cdots, \beta_p)$ denote the Betti sequence of S/I, where p is the projective dimension of S/I.

In the early of the Section 3.1 we will deal with the first and cth Betti numbers of S/I, where c is height of the ideal I. For $\beta_1(S/I)$, we give a lower bound in terms of height of I and the lower degree sequence of S/I. At the same time we produce an upper bound already known in terms of d and p. As for $\beta_c(S/I)$, we give a upper bound in terms of lower and upper degree sequence of ideal I and its height .

These lower bound for $\beta_1(S/I)$ provide us attractive consequences in the case that the ideal I satisfies the condition $N_{d,q}$. For example, Corollary 3.1.2 shows that such ideals have many generators, while Corollary 3.1.5 establishes a robust lower bound for the number of generators of monomial ideals. We finish this section showing bounds for all Betti numbers of ideals admiting d-linear free resolution and lower bounds for the Betti numbers of ideals satisfying the condition $N_{d,c}$.

In Section 2 we will establish upper bounds for the Betti numbers in terms of polynomial functions in d with degree $p-1$. In projective dimension 3 the bounds are best as possible in general. In projective dimension 4 we get cubic bounds whenever the highest degrees in a graded free resolution have certain quadratic upper bounds. We finish this section showing upper bounds for the Betti numbers of ideals satisfying the condition $N_{d,q}$.

3.1 Bounding β_1 and β_c

By assumption, $\beta_1(S/I) = \dim_k[I]_d = \mu(I)$, the minimal number of generators of I. The latter has an obvious upper bound in terms of d and $p = \mathrm{pdim}(S/I)$. To see this, we may assume that k is an infinite field. Since $\mathrm{pdim}(S/I) = p$, $\mathrm{depth}(S/I) = n-p$. Thus, one can specialize modulo a linear sequence of length $n-p$ which is regular both in S and on S/I. Letting $\bar{S}$ denote the residue of S modulo this regular sequence, one has $\mathrm{Tor}_i^{\bar{S}}(S/I,k) = \mathrm{Tor}_i^S(S/I,k)$ for all i ([25, p 140, Lemma 2], also [4, Proposition 1.1.5]). Thus, to compute the Betti numbers of S/I we may assume that $p = n$. Therefore,

$$\dim_k[I]_d \leq \dim_k(S_d) = \binom{d+p-1}{p-1}.$$

By drawing upon Theorem 1.2.3, in the next proposition we recover this bound, but also establish a non-trivial lower bound for β_1 and β_c, where $c = \mathrm{ht}(I)$.

Proposition 3.1.1. *Let $S = k[x_1, \ldots, x_n]$ be a standard graded polynomial ring over the field k and let $I \subset S$ be a homogeneous d-equigenerated ideal of height $c \geq 2$. With the notation of the previous chapters, one has*

(1)
$$\frac{d\underline{d}_2 \cdots \underline{d}_{c-1}}{(\overline{d}_c - d)(\overline{d}_c - \underline{d}_2) \ldots (\overline{d}_c - \underline{d}_{c-1})} \leq \beta_c(S/I).$$

(2)

$$\frac{\overline{d}_2 \dots \overline{d}_c}{(\overline{d}_2 - d) \dots (\overline{d}_c - d)} \leq \mu(I) \leq \binom{d+p-1}{p-1}.$$

Proof. Theorem 1.2.3 says that

$$\beta(S/I) = \sum_{c \leq s \leq p} \sum_{\mathbf{d} \in \mathcal{D}(\tau_\mathbf{s}(\underline{\mathbf{d}}(\mathbf{S/I})), \tau_\mathbf{s}(\overline{\mathbf{d}}(\mathbf{S/I})))} q_\mathbf{d} \beta(\pi(\mathbf{d})), \tag{3.1}$$

An element of $\mathcal{D}(\tau_s(\underline{\mathbf{d}}(S/I)), \tau_s(\overline{\mathbf{d}}(S/I)))$ is the form $\mathbf{d}_{i_2,\dots,i_s} = (0, d, \underline{d}_2 + i_2, \cdots, \underline{d}_s + i_s)$. Here for any $2 \leq j \leq s-1$,

$$\underline{d}_j + i_j + 1 \leq \underline{d}_{j+1} + i_{j+1}, \text{and} \tag{3.2}$$

$$0 \leq i_j \leq \overline{d}_j - \underline{d}_j. \tag{3.3}$$

Let $b^{\{j\}}_{i_2,\dots,i_s}$ denote the nonzero entry on the jth column of $\beta(\pi(\mathbf{d}_{i_2,\dots,i_s}))$. That is,

$$b^{\{j\}}_{i_2,\dots,i_s} = \frac{d(\underline{d}_2 + i_2) \dots (\widehat{\underline{d}_j + i_j}) \dots (\underline{d}_s + i_s)}{(\underline{d}_j + i_j - d) \dots (\underline{d}_j + i_j - \underline{d}_{j-1} - i_{j-1})(\underline{d}_{j+1} + i_{j+1} - \underline{d}_j - i_j)(\underline{d}_s + i_s - \underline{d}_j - i_j)}. \tag{3.4}$$

(1) According to the decomposition 3.1

$$\beta_c(S/I) = \sum_{c \leq s \leq p} \sum_{(i_2,\dots,i_s)} q_{i_2 \dots i_s} \cdot b^{\{c\}}_{i_2,\dots,i_s} \tag{3.5}$$

Note that by definition $b^{\{c\}}_{i_2,\dots,i_s} \geq b^{\{c\}}_{i_2,\dots,i_c}$ for all $s \geq c$. Consequently,

$$\beta_c(S/I) \geq \sum_{c \leq s \leq p} \sum_{(i_2,\dots,i_s)} q_{i_2 \dots i_s} \cdot b^{\{c\}}_{i_2,\dots,i_c}. \tag{3.6}$$

Theorem 1.2.3 also says that $\sum_{c \leq s \leq p} \sum_{(i_2,\dots,i_s)} q_{i_2 \dots i_s} = 1$. Therefore, an lower bound for $b^{\{c\}}_{i_2,\dots,i_c}$ which is independent of $(i_2, ..., i_c)$ provides an lower bound for $\beta_c(S/I)$.

Recall that

$$b^{\{c\}}_{i_2,\ldots,i_c} = \frac{d(\underline{d}_2+i_2)\ldots(\underline{d}_{c-1}+i_{c-1})}{(\underline{d}_c+i_t-d)(\underline{d}_c+i_c-\underline{d}_2-i_2)\ldots(\underline{d}_c+i_c-\underline{d}_{c-1}-i_{c-1})}.$$

Obviously,

$$\frac{\underline{d}_j+i_j}{\underline{d}_c+i_c-\underline{d}_j-i_j} \geq \frac{\underline{d}_j}{\underline{d}_c+i_c-\underline{d}_j} \geq \frac{\underline{d}_j}{\overline{d}_c-\underline{d}_j} \quad \text{for} \quad 2 \leq j \leq c-1.$$

Hence,

$$\beta_c(S/I) \geq \frac{d\underline{d}_2\cdots\underline{d}_{c-1}}{(\overline{d}_c-d)(\overline{d}_c-\underline{d}_2)\ldots(\overline{d}_c-\underline{d}_{c-1})}.$$

(2) We first consider the case where S/I is Cohen-Macaulay. Then $c=p$, hence the formula of Theorem 1.2.3 becomes

$$\beta(S/I) = \sum_{\mathbf{d}\in\mathcal{D}((\underline{\mathbf{d}}(\mathbf{S/I})),(\overline{\mathbf{d}}(\mathbf{S/I})))} q_{\mathbf{d}}(\beta(\pi(\mathbf{d}))). \tag{3.7}$$

$$b^{\{1\}}_{i_2,\ldots,i_p} = \frac{(\underline{d}_2+i_2)\ldots(\underline{d}_p+i_p)}{(\underline{d}_2+i_2-d)\ldots(\underline{d}_p+i_p-d)}, \tag{3.8}$$

According to the decomposition (3.7),

$$\beta_1(S/I) = \sum_{(i_2,\ldots,i_p)} q_{i_2\ldots i_p}\cdot b^{\{1\}}_{i_2,\ldots,i_p}.$$

Since $\sum_{(i_2,\ldots,i_p)} q_{i_2\ldots i_p}=1$, an upper bound (respectively, a lower bound) for $b^{\{1\}}_{i_2,\ldots,i_p}$ which is independent of $(i_2,\ldots,i_p)$ provides an upper bound (respectively, a lower bound) for $\beta_1(S/I)$.

Now, we can think of $b^{\{1\}}_{i_2,\ldots,i_p}$ as a positive real function. In terms of any of the variables $i_2,\cdots,i_p$, it is a hyperbolic function with negative vertical asymptotic; thus, the maximum value of $b^{\{1\}}_{i_2,\ldots,i_p}$ is attained at the minimum values of i_j's and the minimum values are attained at the maximum values of i_j's. We then have

$$\frac{\overline{d}_2\ldots\overline{d}_p}{(\overline{d}_2-d)\ldots(\overline{d}_p-d)} \leq \quad b^{\{1\}}_{i_2,\ldots,i_p} \leq \frac{\underline{d}_2\ldots\underline{d}_p}{(\underline{d}_2-d)\ldots(\underline{d}_p-d)}.$$

The function in the right hand side is hyperbolic in terms of any among $\underline{d}_2, \cdots, \underline{d}_p$ in the domain $[d+1, \infty)$. Thus, the maximum values are attained at the minimum values of each variable, so one gets

$$\frac{\overline{d}_2 \dots \overline{d}_p}{(\overline{d}_2 - d) \dots (\overline{d}_p - d)} \leq \quad b^{\{1\}}_{i_2,\dots,i_p} \leq \frac{(d+1)\dots(d+p-1)}{(p-1)!} = \binom{d+p-1}{p-1}.$$

Consequently, the decomposition (3.7) yields

$$\frac{\overline{d}_2 \dots \overline{d}_p}{(\overline{d}_2 - d) \dots (\overline{d}_p - d)} \leq \beta_1(S/I) \leq \binom{d+p-1}{p-1}.$$

Now, assume the general case, where $c \leq p$. Then, according to (3.7),

$$\beta_1(S/I) = \sum_{c \leq s \leq p} \sum_{(i_2,\dots,i_s)} q_{i_2\dots i_s} \cdot b^{\{1\}}_{i_2,\dots,i_s}.$$

According to (3.8), a similar argument as above shows that

$$\frac{\overline{d}_2 \dots \overline{d}_s}{(\overline{d}_2 - d) \dots (\overline{d}_t - d)} \leq \quad b^{\{1\}}_{i_2,\dots,i_s} \leq \frac{\underline{d}_2 \dots \underline{d}_s}{(\underline{d}_2 - d) \dots (\underline{d}_s - d)}.$$

Obviously,

$$\frac{\underline{d}_2 \dots \underline{d}_s}{(\underline{d}_2 - d) \dots (\underline{d}_s - d)} \leq \frac{\underline{d}_2 \dots \underline{d}_p}{(\underline{d}_2 - d) \dots (\underline{d}_p - d)} \quad \text{for } s \leq p$$

and

$$\frac{\overline{d}_2 \dots \overline{d}_s}{(\overline{d}_2 - d) \dots (\overline{d}_s - d)} \geq \frac{\overline{d}_2 \dots \overline{d}_c}{(\overline{d}_2 - d) \dots (\overline{d}_c - d)} \quad \text{for} \quad s \geq c.$$

Finally, since $\sum_{c \leq s \leq p} \sum_{(i_2,\dots,i_s)} q_{i_2\dots i_s} = 1$, we get

$$\frac{\overline{d}_2 \dots \overline{d}_c}{(\overline{d}_2 - d) \dots (\overline{d}_c - d)} \leq \sum_{c \leq s \leq p} \sum_{(i_2,\dots,i_s)} q_{i_2\dots i_s} \cdot b^{\{1\}}_{i_2,\dots,i_s} \leq \frac{\underline{d}_2 \dots \underline{d}_p}{(\underline{d}_2 - d) \dots (\underline{d}_p - d)}.$$

This yields the assertion.

$\square$

The first consequence of Proposition 3.1.1(2) shows that an ideal satisfying the condition $N_{d,q}$ has many generators.

Corollary 3.1.2. *Let S be a standard graded polynomial ring over a field k and let $I \subset S$ be an ideal of height $c \geq 2$ satisfying the condition $N_{d,q}$*

(1) *If $q \geq c$ then $\mu(I) \geq \binom{d+c-1}{c-1}$.*

(2) *If $q < c$ then $\mu(I) \geq \binom{d+q-1}{q-1} + 1$.*

Proof. (1) By hypothesis, we have $\overline{d}_j = d+j-1$ for $2 \leq j \leq c$. Now use Proposition 3.1.1(2).

(2) By hypothesis, we have $\overline{d}_j = d + j - 1$ for $2 \leq j \leq q < c$. Now use Proposition 3.1.1(2),

$$\mu(I) \geq \binom{d+q-1}{q-1} \frac{\overline{d}_{q+1} \ldots \overline{d}_c}{(\overline{d}_{q+1} - d) \ldots (\overline{d}_c - d)} > \binom{d+q-1}{q-1}$$

because $\frac{\overline{d}_{q+1} \ldots \overline{d}_c}{(\overline{d}_{q+1}-d) \ldots (\overline{d}_c-d)} > 1$. Therefore $\mu(I) \geq \binom{d+q-1}{q-1} + 1$. □

When I is a Cohen Macaulay ideal and satisfies the condition $N_{d,p-1}$, where $p := \operatorname{pdim}_S(S/I)$, Proposition 3.1.1(2) together with [26, Theorem 4.4] gives us an interesting lower bound for the minimum number of generators of I.

Corollary 3.1.3. *Let $S = k[x_1, \ldots, x_n]$ be a standard graded polynomial ring over a field k and let $I \subset S$ be a Cohen Macaulay ideal of height $c \geq 2$ and projective dimension p. If I satisfies condition $N_{d,p-1}$, then*

$$\mu(I) \geq \binom{d+p-2}{p-2} + \binom{d+p-3}{p-2}.$$

Proof. (1) By hypothesis, we have $\overline{d}_j = d + j - 1$ for $2 \leq j \leq p - 1$. Now use Proposition3.1.1(2),

$$\mu(I) \geq \binom{d+p-2}{p-2} \frac{\overline{d}_p}{(\overline{d}_p - d)}.$$

But, by the results [26, Theorem 4.4] or [22, Corollary 3], we have $\overline{d}_p \leq \overline{d}_{p-1}+\overline{d}_1 = 2d+p-2$. So, $\frac{\overline{d}_p}{(\overline{d}_p-d)} \geq \frac{2d+p-2}{d+p-2}$ and therefore

$$\begin{aligned}\mu(I) \geq \binom{d+p-2}{p-2}\frac{2d+p-2}{d+p-2} &= \binom{d+p-2}{p-2}\left(\frac{2d+p-2}{d+p-2}-1+1\right)\\ &= \binom{d+p-2}{p-2}+\binom{d+p-2}{p-2}\frac{d}{d+p-2}\\ &= \binom{d+p-2}{p-2}+\binom{d+p-3}{p-2}.\end{aligned}$$

□

Remark 3.1.4. *The results in Proposition 3.1.3 has been proved earlier in [11, Proposition 11.1] on the condition that I is an ideal $(x_1, \ldots, x_n)$-primary. Thus we got here an improvement of [11, Proposition 11.1].*

When I is a monomial ideal, we can generalize the inequality established in Corollary 3.1.3.

Corollary 3.1.5. *Let S be a standard graded polynomial ring over a field k and let $I \subset S$ be a monomial ideal of height $c \geq 2$ and projective dimension p. If I satisfies condition $N_{d,q}$ with $q < c$, then*

$$\mu(I) \geq \binom{d+q-1}{q-1}\frac{(c-q+1)d+q-1}{d+q-1}.$$

In particular, if $q = c-1$, then

$$\mu(I) \geq \binom{d+c-2}{c-2}+\binom{c+c-3}{c-2}.$$

Proof. (1) By hypothesis, we have $\overline{d}_j = d+j-1$ for $2 \leq j \leq q$. By Proposition 3.1.1(2),

$$\mu(I) \geq \frac{(d+1)(d+2)\ldots(d+q-1)}{1.2\ldots(q-1)}\frac{\overline{d}_{q+1}\ldots\overline{d}c}{(\overline{d}_{q+1}-d)\ldots(\overline{d}_c-d)}.$$

Since I is monomial, [22, Corollary 4] says that $\overline{d}_j \leq (j-q+1)d+q-1$ for $j \geq q+1$. So,

$$\begin{aligned}\mu(I) &\geq \binom{d+q-1}{q-1}\frac{(2d+q-1)(3d+q-1)\dots(c-q+1)d+q-1}{(d+q-1)(2d+q-1)\dots(c-q)d+q-1} \\ &\geq \binom{d+q-1}{q-1}\frac{(c-q+1)d+q-1}{d+q-1}.\end{aligned}$$

□

So far we have worked with bounds only for the first Betti number. Next, under the condition that the ideal I has d-linear free resolution, we will give lower bounds for all Betti numbers and not only for β_1 and β_c. Furthermore, we will show that Betti numbers have polynomial upper bounds, in terms of p and d, and that a Betti number reaches such a bound if and only if all Betti numbers reach their bounds.

Proposition 3.1.6. *Let S be a standard graded polynomial ring over a field k and let $I \subset S$ be an ideal of height $c \geq 2$, with projective dimension p and with d-linear free resolution.*

(1) $\max\left\{\binom{p}{t}, \binom{d+t-2}{t-1}\binom{d+c-1}{c-t}\right\} \leq \beta_t(S/I) \leq \binom{d+t-2}{t-1}\binom{d+p-1}{p-t} =: C_t$, *for* $1 \leq t \leq p$;

(2) $\beta_t(S/I) = C_t$ *for some t if and only if* $\beta_t(S/I) = C_t$ *for all t;*

(3) ([21]) *If S/I is a Cohen-Macaulay ring, then* $\beta_t(S/I) = C_t$ *for all t.*

Proof. (1). We keep the notation of Proposition 3.1.1. By Theorem 1.1.11 we already know that $\beta_i \geq \binom{p}{i}$. Keep the notation of Proposition 3.1.1. According to the decomposition 3.1

$$\beta_t(S/I) = \sum_{c\leq s\leq p}\sum_{(i_2,\dots,i_s)} q_{i_2\dots i_s}\cdot b^{\{t\}}_{i_2,\dots,i_s}.$$

Note that by definition $b^{\{t\}}_{i_2,\dots,i_c} \leq b^{\{t\}}_{i_2,\dots,i_s} \leq b^{\{t\}}_{i_2,\dots,i_p}$ for all $c \leq s \leq p$ and that $b^{\{t\}}_{i_2,\dots,i_c} = 0$ whenever $t > c$. Therefore, the next inequality from the left will be considered only for t¡c.

$$\sum_{t\leq s\leq p}\sum_{(i_2,\dots,i_s)} q_{i_2\dots i_s}\cdot b^{\{t\}}_{i_2,\dots,i_c} \leq \beta_t(S/I) \leq \sum_{t\leq s\leq p}\sum_{(i_2,\dots,i_s)} q_{i_2\dots i_s}\cdot b^{\{t\}}_{i_2,\dots,i_p}. \tag{3.9}$$

To shorten the notation, when $i_2 = \cdots = i_s = 0$, we denote $q_{i_2\dots i_s}$ by q_{s0}. Similarly, $b^{\{t\}}_{0,\dots,0}$ with s zeros in the index will be denoted $b^{\{t\}}_{s0}$. By hypothesis, $\underline{d}_j = \overline{d}_j = d + j - 1$, for $2 \leq j \leq p$ and $i_2 = i_3 = \cdots = i_p = 0$. Hence,

$$\sum_{s=c}^{p} q_{s0}\cdot b^{\{t\}}_{c0} \leq \beta_t(S/I) \leq \sum_{s=c}^{p} q_{s0}\cdot b^{\{t\}}_{p0}, \quad \text{and} \quad \sum_{s=c}^{p} q_{s0} = 1.$$

Since

$$b^{\{t\}}_{p0} = \frac{d(d+1)\dots\widehat{(d+t-1)}\dots(d+p-1)}{(t-1)!(p-t)!} = \binom{d+t-2}{t-1}\binom{d+p-1}{p-t}$$

and

$$b^{\{t\}}_{c0} = \frac{d(d+1)\dots\widehat{(d+t-1)}\dots(d+c-1)}{(t-1)!(c-t)!} = \binom{d+t-2}{t-1}\binom{d+c-1}{c-t}$$

we have the desired inequalities.

(2) Suppose that for some t $\beta_t(S/I) = C_t$, then

$$\beta_t(S/I) = C_t = \sum_{s=c}^{p} q_s b^{\{t\}}_s = q_c b^{\{t\}}_c + \cdots + q_{p-1} b^{\{t\}}_{p-1} + q_p C_t. \tag{3.10}$$

Adding the fact that $\sum_{s=c}^{p} q_s = 1$, we concluded that

$$q_c(C_t - b^{\{t\}}_c) + \cdots + q_{p-1}(C_t - b^{\{t\}}_{p-1}) = 0.$$

How $C_t - b^{\{t\}}_s > 0$ for $c \leq s \leq p-1$ and $q_s \geq 0$ for s, then $q_c = \cdots = q_{p-1} = 0$ and $q_p = 1$. Now, by 3.10 $\beta_l(S/I) = C_l$ for all l.

(3) Follows from item (2) with $c = p$

□

Remark 3.1.7. *The bounds deduced in Proposition 3.1.6*(1) *has been proved earlier in [9, Proposition 1.9(c) and Proposition 1.12]. However, it is easy to observe that the proof we presented for the lower bound is still valid if we only assume that the ideal I satisfies the condition $N_{d,c}$. So we have the following.*

Corollary 3.1.8. *Let S be a standard graded polynomial ring over a field k and let $I \subset S$ be a homogeneous ideal of height $c \geq 2$. If I satisfies condition $N_{d,c}$ (i.e., the minimal graded resolution is linear up to step c) then*

$$\binom{d+t-2}{t-1}\binom{d+c-1}{c-t} \leq \beta_t(S/I) \text{ for } 1 \leq t \leq c.$$

3.2 General polynomial bounds for Betti numbers

As mentioned earlier, in this section we will establish upper bounds for the Betti numbers in terms of polynomial functions in d with degree $p-1$.

Proposition 3.2.1. *Let S be a standard graded polynomial ring over the field k and let $I \subset S$ be a d-equigenerated ideal of height $c \geq 2$ and projective dimension p. Then:*

(1) *If $p = 3$ then $\beta_2 \leq d(d+2)$ and $\beta_3 \leq d(d+1)/2$.*

(2) *If $p = 4$, then*

 (a) *If $\overline{d}_2 \leq d^2+4d+2$ then $\beta_2 \leq d(d+2)(d+3)/2$, otherwise*

$$\beta_2 \leq \frac{d(\overline{d}_2+1)(\overline{d}_2+2)}{2(\overline{d}_2-d)} \leq \frac{((3d-2)d^2+3)((3d-2)d^2+4)}{2((3d-2)d^2-d+2)} < (d+\frac{1}{8})((3d-2)d^2+2).$$

 (b) *If $\overline{d}_3 \leq \max\{d+2, (1/2)(d^2+2d-1)\}$ then $\beta_3 \leq d(d+1)(d+3)/2$; else,*

$$\beta_3 \leq \frac{d(\overline{d}_3-1)(\overline{d}_3+1)}{(\overline{d}_3-d)} \leq \frac{d((3d-2)d^2+2)((3d-2)d^2+4)}{((3d-2)d^2-d+3)}.$$

(c) *If* $\overline{d}_4 \le \max\{d+3, (1/3)(d^2+2)\}$ *then* $\beta_4 \le d(d+1)(d+2)/6$; *else*

$$\beta_4 \le \frac{d(\overline{d}_4-2)(\overline{d}_4-1)}{2(\overline{d}_4-d)} \le \frac{d((3d-2)d^2+2)((3d-2)d^2+3)}{((3d-2)d^2-d+4)}.$$

(3) *If* $p \ge 5$ *then for any* $2 \le j \le p$,

$$\beta_j \le d \max\left\{ \frac{1}{j-1}\binom{d+j-2}{j-2}\binom{d+p-1}{p-j}, \quad \frac{1}{(\overline{d}_j - d)}\binom{\overline{d}_j+p-j}{p-j}\binom{\overline{d}_j-1}{j-2} \right\}.$$

Proof. We follow the same schedule of proof as in Proposition 3.1.1, where one could assume the Cohen–Macaulay case, as the general case will work quite the same way. Recall from this proof that $b^{\{j\}}_{i_2,\dots,i_p}$ denotes the nonzero entry on the jth column of the diagram $\beta(\pi(\mathbf{d}_{i_2,\dots,i_p}))$. One has:

$$\begin{aligned} b^{\{j\}}_{i_2,\dots,i_p} &= \frac{d(\underline{d}_2+i_2)\dots(\widehat{\underline{d}_j+i_j})\dots(\underline{d}_p+i_p)}{(\underline{d}_j+i_j-d)\dots(\underline{d}_j+i_j-\underline{d}_{j-1}-i_{j-1})(\underline{d}_{j+1}+i_{j+1}-\underline{d}_j-i_j)(\underline{d}_p+i_p-\underline{d}_j-i_j)} \\ &= \frac{d}{\underline{d}_j+i_j-d}\prod_{k=2}^{j-1}\frac{\underline{d}_k+i_k}{\underline{d}_j+i_j-\underline{d}_k-i_k}\prod_{l=j+1}^{p}\frac{\underline{d}_l+i_l}{\underline{d}_l+i_l-\underline{d}_j-i_j}. \end{aligned}$$

The relations (3.2) and (3.3) imply the inequalities

$$\frac{\underline{d}_k+i_k}{\underline{d}_j+i_j-\underline{d}_k-i_k} \le \frac{\underline{d}_j+i_j-j+k}{j-k} \quad \text{and} \quad \frac{\underline{d}_l+i_l}{\underline{d}_l+i_l-\underline{d}_j-i_j} \le \frac{\underline{d}_j+i_j+l-j}{l-j}.$$

One then gets

$$b^{\{j\}}_{i_2,\dots,i_p} \le \frac{d}{\underline{d}_j+i_j-d}\prod_{k=2}^{j-1}\frac{\underline{d}_j+i_j-j+k}{j-k}\prod_{l=j+1}^{p}\frac{\underline{d}_j+i_j+l-j}{l-j}.$$

We now inspect for which value of $\underline{d}_j+i_j$ the right hand side of the above inequality attains its maximum value. Setting $x=\underline{d}_j+i_j$, it becomes a hyperbolic function of $x=\underline{d}_j+i_j$

$$f(x) = \frac{d}{(x-d)}\frac{(x-j+2)\dots(x-1)}{(j-2)!}\frac{(x+1)\dots(x+p-j)}{(p-j)!}, \tag{3.11}$$

which we wish to analyze in the range $[d+j-1, \overline{d}_j]$.

The behavior of $f(x)$ for $p=3$ and that for $p>3$ will be quite different.

(1) ($p=3$) If $j=2$, $f(x)=\frac{d(x+1)}{(x-d)}$. The maximum value of this hyperbolic function occurs at the minimum value of x; so that $\beta_2 \leq f(d+1)=d(d+2)$. When $j=3$, $f(x)=\frac{d(x-1)}{(x-d}$, similarly, $\beta_3 \leq f(d+2)=d(d+1)/2$.

(2) ($p=4$) Say, $j=2$. $f(x)=\frac{d(x+1)(x+2)}{2(x-d)}$. We look for $x \in [d+1, \overline{d}_2]$ wherein $f(x)=f(d+1)=d(d+2)(d+3)/2$. This amounts to find the roots of $(x+1)(x+2)=(x-d)(d+2)(d+3)$. One of the roots of this equation is $d+1$ hence the other root is d^2+4d+2. Therefore, if $\overline{d}_2 \leq d^2+4d+2$ the maximum value in the range $[d+1, \overline{d}_2]$ is $d(d+2)(d+3)/2$. Otherwise, the maximum value is $f(\overline{d}_2)$. To settle the last inequality, we appeal to the bound for the Castelnuovo-Mumford regularity [4, Theorem 3.5(ii)]. Accordingly, $\operatorname{reg}(S/I) \leq (3d-2)d^2$ whenever $\dim(S/I) \leq 2$. Then the last inequality follows by the fact that $\overline{d}_2 - 2 \leq \operatorname{reg}(S/I)$.

The argument for $j=3,4$ is similar.

(3) ($p \geq 5$) Consider again the function $f(x)$ in (3.11). Since the numerator is a convex function in the range $x > j-2$, the intersection of $y=f(x)$ with any straight line in $\mathbb{A}^2$, in this range, consists of at most two points. Consequently, $f(x)$ has only one local minimum for $x > d+j-1$. Therefore

$$\max_{x \in [d+j-1, \overline{d}_j]} \{f(x)\} = \max\{f(d+j-1), f(\overline{d}_j)\}.$$

It is straightforward to see that

$$f(d+j-1) = \frac{d}{j-1}\binom{d+j-2}{j-2}\binom{d+p-1}{p-j}, \quad f(\overline{d}_j) = \frac{d}{(\overline{d}_j-d)}\binom{\overline{d}_j-1}{j-2}\binom{\overline{d}_j+p-j}{p-j}$$

□

Corollary 3.2.2. *Let S be a standard graded polynomial ring over the field k and let $I \subset S$ be an ideal of height $c \geq 2$, projective dimension p and satisfying the condition $N_{d,q}$. Then*

$$\beta_t \leq \binom{d+t-2}{t-1}\binom{d+p-1}{p-t} \text{ for } 2 \leq t \leq q.$$

Proof. Follows from Proposition 3.2.1 □

The results in Proposition 3.1.6, Proposition 3.2.1 and Corollary 3.2.2 encourage us to make the following Conjecture:

Conjecture 3.2.3. *Let S be a standard graded polynomial ring over the field k and let $I \subset S$ be a homogenous ideal generated in degree d and of projective dimension p. Then all of the Betti numbers of S/I are bounded by polynomials function of d of degree at most $p-1$.*

Chapter 4

Applications to graph theory

Let G be a graph with n vertices and $I = I(G) \subset S = K[x_1, \ldots, x_n]$ the edge ideal of the graph G, where k is a field. It is known that $R = S/I$ is a Koszul algebra and thus $\operatorname{reg}_R(k) = 0$, by Corollary 2.2.3(1), the minimal free resolution of I is of the form.

$$\cdots \to S(-6)^{\beta_{3,6}} \oplus S(-5)^{\beta_{3,5}} \oplus S(-4)^{\beta_{3,4}(R)} \to S(-4)^c \oplus S(-3)^b \to S(-2)^q \to S \to R \to 0.$$

In addition, we can determine $b = \beta_{2,3}(R)$ and $c = \beta_{2,4}(R)$ combinatorially (see Proposition 4.1.8).

The spacial form of this resolution motivated us to apply the results of the previous chapters in graph theory, mainly for gap-free and co-chordal graphs that have even more particular minimal free resolutions.

In the Section 1 we will establish the notations necessary to understand this chapter and remember some concepts of graph theory. For more coverage of this subject, see [29]. In the Section 2, we will apply the results obtained in this thesis in graph theory. The main results of this chapter are Propositions 4.2.1 and 4.2.6

4.1 Graph and edge ideals

Throughout this part, $G = (V(G), E(G))$ is a finite simple graph (i.e., a graph with no loops and no multiple edges) on the vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set E(G).

If $z = \{v_i, v_j\}$ is an edge of G one says that the vertices v_i and v_j are *adjacent or connected* by z, in this case it is also usual to say that the edge z is *incident* with the vertices v_i and v_j. The degree of a vertex v in $V(G)$, denoted by $deg(v)$, is the number of edges incident with v.

Definition 4.1.1. *Let G be a graph with vertex set V. A subset $A \subset V$ is a* minimal vertex cover *for G if (i) every edge of G is incident with one vertex in A, and (ii) there is no proper subset of A with the first property. The* vertex covering number *of G, $\alpha_0(G)$, is the smallest number of vertices in any minimal vertex cover. The* largest number of vertices in any minimal vertex cover *of G, is denoted by $\tau_{max}(G)$,*

Example 4.1.2. *The graph of the figure 4.1 has $\alpha_0(G) = 1$ and $\tau_{max}(G) = 6$.*

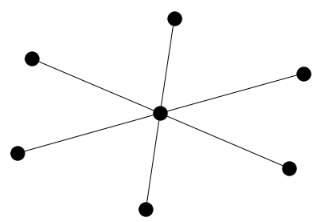

Figure 4.1: The star graph with seven vertices

Let G be a graph with vertices $v_1, \dots, v_n$ and $S = k[x_1, \dots, x_n]$ a polynomial ring over a field K, with one variable x_i for each vertex v_i.

Definition 4.1.3. *The* edge ideal $I(G)$ *associated to the graph G is the ideal of S generated by the set of square-free monomials $x_i x_j$ such that v_i is adjacent to v_j,* *that is,* $I(G) = (\{x_i x_j | \{v_i, v_j\} \in E(G)\}) \subset S$ *If all the vertices of* G *are isolated we set* $I(G) = (0)$.

Proposition 4.1.4. *(Corollary 6.1.18 [29]) If* G *is a graph and* $I(G)$ *its edge ideal, then the vertex covering number* $\alpha_0(G)$ *is equal to the height of the ideal* $I(G)$.

Definition 4.1.5. *The* edge graph *of* G*, denoted by* $L(G)$*, has vertex set equal to* $E = E(G)$ *with two vertices of* $L(G)$ *adjacent whenever the corresponding edges of* G *have exactly one common vertex.*

Example 4.1.6. *In the figure 4.2 we have the graphs* G *and its corresponding edge graph* $L(G)$.

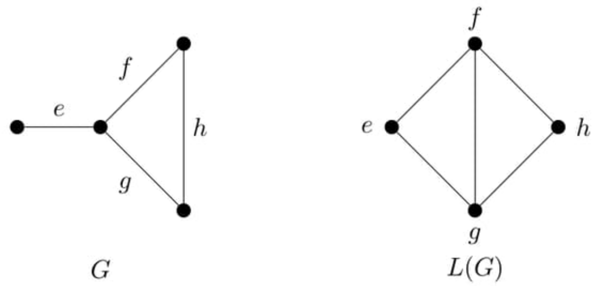

Figure 4.2: The graph G (left) and the graph $L(G)$ (right).

Proposition 4.1.7. *(Proposition 6.6.1 [29]) If* G *is a graph with vertices* $x_1, \ldots, x_n$ *and edge set* $E(G)$*, then the number of edges of the edge graph* $L(G)$ *is given by*

$$|E(L(G))| = \sum_{i=1}^{n} \binom{deg(x_i)}{2} = -|E(G)| + \sum_{i=1}^{n} \frac{deg^2 x_i}{2}.$$

Proposition 4.1.8. *[13] Let* $I \subset S$ *be the edge ideal of a graph* G*, let* V *be the vertex set of* G*, and let* $L(G)$ *be the edge graph of* G*. Let*

$$\cdots \to S(-4)^c \oplus S(-3)^b \to S^{|E|}(-2) \to S \to R \to 0. \tag{4.1}$$

be the minimal graded resolution of S/I. Then

$$b = |E(L(G))| - T(G),$$

where $T(G)$ is the number of triangles of G and $c := c(G)$ is the number of unordered pairs of edges $\{f, g\}$ such that $f \cap g = \emptyset$ and f and g cannot be joined by an edge.

Definition 4.1.9. *G is called* gap-free *if $c(G) = 0$. Equivalently, G is gap-free if for any two disjoint edges $e, f \in E(G)$, there exists an edge $g \in E(G)$ such that $e \cap g \neq \emptyset$ and $f \cap g \neq \emptyset$.*

Example 4.1.10. *In the figure 4.3 we have a gap-free graph and a non-gap-free graph*

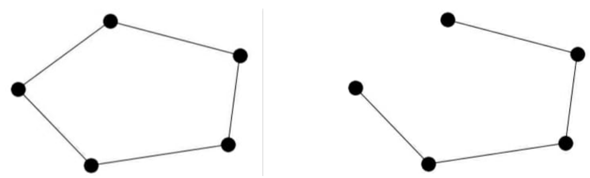

Figure 4.3: A gap-free graph (left) and a non-gap-free graph (right).

Definition 4.1.11. *Let G be a graph with vertex set V. The* complement $\overline{G}$ *of G is the graph whose vertex set is V and whose edges are the pairs of nonadjacent vertices of G.*

Definition 4.1.12. *We say that a graph G is* triangulated or chordal *if every cycle C_n in G of length $n \geq 4$ has a chord in G. A chord of C_n is an edge joining two non adjacent vertices of C_n. We say a graph G is* co-chordal *if its complement $\overline{G}$ is chordal.*

It is easy to see that a graph G is gap-free if and only if every cycle of length 4 in the complement $\overline{G}$ has a chord. In particular, all co-chordal graphs are gap-free.

The co-chordal graphs are interesting because their edge ideals are well behaved as it is expressed by the following theorem.

Theorem 4.1.13. *(Fröberg[16]) Let G be a graph and $I(G)$ its edge ideal. Then $I(G)$ has a 2-linear free resolution if and only if G is co-chordal.*

Example 4.1.14. *In figure 4.4 we present a graph G and its complement graph $\overline{G}$. Note that G is co-chordal.*

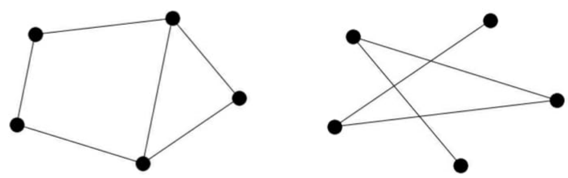

Figure 4.4: The graph G (left) and its complement graph $\overline{G}$ (right).

4.2 Applicatioins

In this Section we will apply the results of the previous chapters to graph theory. Keep the notations established so far.

Proposition 4.2.1. *Let G be a graph with edge set E and $\alpha_0(G)$ its vertex covering number. If G is a tree or is gap-free, then*

$$|E| \geq 2\alpha_0(G) - 1.$$

Furthermore, This bound is sharp whenever $|E|$ is odd

Proof. Firstly we will show the case where G is a gap-free graph.

Let G be a graph with vertices $v_1, \ldots, v_n$, $S = k[x_1, \ldots, x_n]$ a polynomial ring over a field k and $I(G) \subset S$ the edge ideal of G with $c = \mathrm{ht}(I(G))$.

Keeping the notation established in Chapter 1. By Proposition 3.1.1(2)

$$|E| \geq \frac{\overline{d}_2\overline{d}_3 \ldots \overline{d}_c}{(\overline{d}_2 - 2)\ldots(\overline{d}_c - 2)}.$$

Since G is gap-free, by Proposition 4.1.8 the edge ideal $I(G) \subset S$ satisfies the condition N_2, that is, $\overline{d}_2 = 3$. By Corollary 2.2.4, $\overline{d}_i \leq 2i - 1$ for $3 \leq i \leq p$, where $p = pdim(S/I(G))$. So

$$|E| \geq \frac{3.5.7\ldots(2c-1)}{1.3.5\ldots(2c-3)} = 2c - 1.$$

However, $\text{ht}(I(G)) = \alpha_0(G)$(Proposition 4.1.4) and so we get the desired inequality.

The proof for the case that G is a tree is done by induction on the number of vertices of G. Note that the proof is trivial in case G is a star. Suppose then that G is not a star.

If G has 4 vertices, it is easy to verify that the desired inequality is valid. Thus, let $n \geq 4$ and suppose the result is valid for trees with $n-1$ vertices or less.

Let G be a tree with n vertices. Denote by V the vertex set of G. Let $v_1, v_2 \in V$ such that $d(v', v'') \leq d(v_1, v_2) := r$ for all $v', v'' \in V$. Note that $deg(v_1) = 1$. Let u_1 be the unique vertice adjacent of v_1 and $\{v_1, u_1, u_2, \ldots, u_{r-2}, v_2\}$ the unique path from v_1, v_2.

Since G is not a star, $deg(u_2) \geq 2$. Furthermore, as G is a tree and v_1 and v_2 were taken with maximum distance, u_2 is the unique vertice adjacent to u_1 with degree greater than 1. Set $B = \{v \in V; d(v, u_1) = 1\} \cup \{u_1\}$ and $G' = G \setminus B$. We have that G' is a tree with less than n vertices. By the induction hypothesis, we have

$$|E| - deg(u_1) = |E(G')| \geq 2\alpha_0(G') - 1. \tag{4.2}$$

We claim that $\alpha_0(G') \geq \alpha_0(G) - 1$. In fact, suppose by contradiction that $\alpha_0(G') \leq \alpha_0(G) - 2 := \alpha_0 - 2$. So there is $V_1 = \{v_{i1}, \ldots v_{i(\alpha_0 - 2)}\} \subset V$ such that every edge of G' is incident with one vertex in V_1, em particular, every edge of G

is incident with one vertex in $V_1 \cup \{u_1\}$, which is a contradiction. Thus we prove the claim.

Hence, by 4.2,

$$|E| \geq 2\alpha_0(G) - 3 + deg(u_1) \geq 2\alpha_0(G) - 1,$$

because $deg(u_1) \geq 2$.

Now let's show that the bound in question is sharp. Suppose that $|E|$ is an odd number. Consider G the tree below

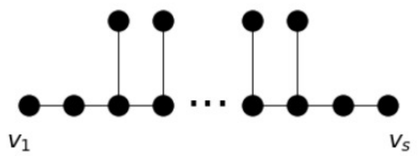

Figure 4.5

where $s = (E+5)/2$. It is possible to show that $\alpha_0(G) = (|E|+1)/2$. Therefore,

$$|E| = 2\alpha_0(G) - 1.$$

□

Corollary 4.2.2. *If I is an edge ideal of a gap-free graph or a tree, then*

$$\text{ht}(I) \leq \frac{\mu(I)+1}{2},$$

where $\mu(I)$ denotes the minimum number of generators of I.

Remark 4.2.3. *The inequality in Corollary 4.2.1 has been proved earlier in [17] in a more general context, just assuming that G is a connected graph. The argument used is purely combinatorial.*

Now we will see that for a gap-free graph G, the lower bounds for the number of edges in terms of $\alpha_0(G)$ grow as G 'approaches' of being a co-chordal graph.

Definition 4.2.4. *Let $q \geq 2$. We say that a graph G is* q-co-chordal *if every cycle C_i in the complement graph $\overline{G}$ in of length $i \leq q+2$ has a chord in $\overline{G}$. Note that a gap-free graph is a 2-co-chordal graph and that G is co-chordal if and only if G is q-co-chordal for all q.*

Proposition 4.2.5. *Let G be a graph with edge set E and $\alpha_0(G)$ its vertex covering number. If G is q-co-chordal, then*

$$|E| \geq \frac{q}{2}(2\alpha_0(G) - q + 1).$$

Proof. The proof is similar to that given in Proposition 4.2.1.

Let G be a graph with vertices $v_1, \ldots, v_n$, $S = k[x_1, \ldots, x_n]$ a polynomial ring over a field and $I(G) \subset S$ the edge ideal of G with $c = \mathrm{ht}(I(G)) = \alpha_0(G)$.

Keeping the notation established in Chapter 1. By Proposition 3.1.1(2)

$$|E| \geq \frac{\overline{d}_2\overline{d}_3 \ldots \overline{d}_c}{(\overline{d}_2 - 2) \ldots (\overline{d}_c - 2)}$$

Since G is q-co-chordal, by [10, Theorem 2.1] the edge ideal $I(G) \subset S$ satisfies the condition N_q, that is, $\overline{d}_i = i+1$ for $1 \leq i \leq q$. By Corollary 2.2.4 $\overline{d}_i \leq 2i-q+1$ for $q+1 \leq i \leq p$, where $p = pdim(S/I(G))$. If $q < c$, then

$$|E| \geq \frac{3.4.5 \ldots q(q+1)(q+3) \ldots (2c-q+1)}{1.2.3 \ldots (q-2)(q-1)(q+1) \ldots (2c-q-1)} = \frac{q}{2}(2c-q+1).$$

If $q \geq c$, then

$$|E| \geq \frac{3.4 \ldots c+1}{1.2 \ldots c-1} = \frac{c(c+1)}{2} \geq \frac{q}{2}(2c-q+1).$$

We have the desired inequalities. □

Proposition 4.2.6. *Let G be a graph with edge set E, n vertices and $\alpha_0(G)$ its vertex covering number. If G is $\alpha_0(G)$-co-chordal, then*

(1) $|E| \geq \frac{\alpha_0(G)(\alpha_0(G)+1)}{2}$.

(2) $|E(L(G))| - T(G) \geq \frac{\alpha_0(G)(\alpha_0(G)+1)(\alpha_0(G)-1)}{3}$.

Furthermore, these bounds are sharp for all n.

Proof. Follows from Proposition 4.2.5.

(2) By Proposition 4.1.8 $\beta_2(I(G)) = |E(L(G))| - T(G)$. We can assume that $\alpha_0(G) \geq 2$. Since G is $\alpha_0(G)$-co-chordal, by [10, Theorem 2.1] the edge ideal $I(G)$ satisfies the condition $N_{\alpha_0(G)}$. Now we use Corollary 3.1.8 to conclude that $|E(L(G))| - T(G) \geq \frac{\alpha_0(G)(\alpha_0(G)+1)(\alpha_0(G)-1)}{3}$.

To see that the bounds in (1) and (2) are sharp, consider complete graphs. $\square$

Definition 4.2.7. *Let G be a graph with edge set E. A set $M \subset E$ is said to be a* matching *if for all $e, e' \in M$ with $e \neq e'$ we have $e \cap e' = \emptyset$. The* matching number*, denoted $mat(G)$ is defined to be*

$$mat(G) := max\{|M| : M \text{ is a matching in } G\}.$$

Definition 4.2.8. *Let G be a graph with edge set E. A set $M \subset E$ is said to be an* induced matching *if for all $e, e' \in M$ with $e \neq e'$ there does not exist $f \in E$ such that $e \cap f \neq \emptyset$ and $e' \cap f \neq \emptyset$. The* induced matching number*, denoted $ind\,mat(G)$ is defined to be*

$$ind\,mat(G) := max\{|M| : M \text{ is an induced matching in } G\}$$

Because an induced matching is also a matching, we always have $ind\,mat(G) \leq mat(G)$. For example, if G is a cycle of length 6, then $ind\,mat(G) = 2$ and $mat(G) = 3$.

The numbers defined above are important. It is known that (see [18])

$$ind\,mat(G) \leq \operatorname{reg}(S(I(G))) \leq mat(G) \quad \text{and}$$

$$ind\,mat(G) = \text{reg}(S(I(G))) \text{ if G is chordal.}$$

As a consequence of these results and Proposition 3.1.1(2), we have.

Proposition 4.2.9. *Let G be a graph with edge set E and $\alpha_0(G)$ its vertex covering number. Then*

$$|E| \geq \frac{(mat(G) + \alpha_0(G) - 1)(mat(G) + \alpha_0(G))}{mat(G)(mat(G) + 1)} \quad \text{and}$$

$$|E| \geq \frac{(ind\,mat(G) + \alpha_0(G) - 1)(ind\,mat(G) + \alpha_0(G))}{(ind\,mat(G))(ind\,mat(G) + 1)} \quad \text{if } G \text{ is chordal.}$$

Remember that $c(G)$ is the number of unordered pairs of edges $\{f, g\}$ such that $f \cap g = \emptyset$ and f and g cannot be joined by an edge, $L(G)$ is the edge graph of G and $T(G)$ is the number of triangles de G.

Let's finish this chapter by giving an upper bounds for the number $c(G) + |E(L(G))| - T(G)$ in function of $\tau_{max}(G)$, when G is a graph such that $pdim(S/(I(G)))) = \tau_{max}(G)$. Dao–Schweig proved in 2013 [8] who in general, $pdim(S/(I(G)))) \geq \tau_{max}(G)$. However, the equality is valid when G is a chordal graph (see[15]) or a Cameron-Walker graphs(see[23]).

Definition 4.2.10. *A graph G is a* star graph *when G joining some paths of length 1 at one common vertex (see Figure 4.6). A graph G is a* star triangle *when G joining triangles at one common vertex (see Figure 4.6). A finite connected simple graph G is said to be a* Cameron–Walker graph *if $ind\,mat(G) = mat(G)$ and if G is neither a star graph nor a star triangle.*

Example 4.2.11. *The figure 4.7 represents a Cameron-Walker graph with $mat(G) = 4$*

Recall that for a graph G, $c(G)$ is is the number of unordered pairs of edges $\{f, g\}$ such that $f \cap g = \emptyset$ and f and g cannot be joined by an edge.

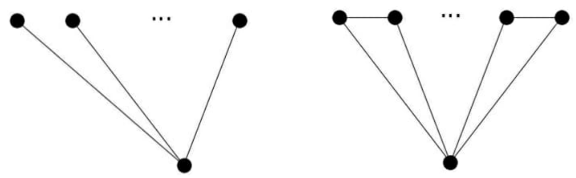

Figure 4.6: The star graph (left) and the star triangle (right).

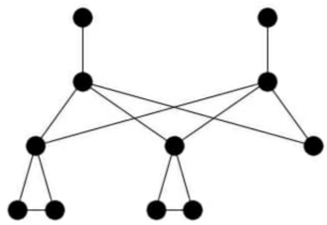

Figure 4.7: Example of Cameron-Walker graph.

Proposition 4.2.12. *Let G be a chordal graph or a Cameron-Walker and $\tau(G) := \tau_{max}(G)$ the* largest number of vertices in any minimal vertex cover of G. Then:

(1) If $2 \leq \tau(G) \leq 6$ then

$$c(G) + |E(L(G))| - T(G) \leq \frac{(\tau(G)+1)(\tau(G)\tau(G)-1)}{3}.$$

(2) If $\tau(G) \geq 7$ then

$$c(G) + |E(L(G))| - T(G) \leq \frac{(\tau(G)+2)(\tau(G)+1)\tau(G)(\tau(G)-1)}{4!}.$$

(3) If in addition, G is gap-free then

$$|E(L(G))| - T(G) \leq (\tau(G)+1)\tau(G)(\tau(G)-1)/3 \quad \text{for} \quad \tau(G) > 2.$$

Proof. We already know that $pdim(S/I(G)) = \tau_{max}(G)$ and that $\beta_2(S/I(G)) = c(G) + |E(L(G))| - T(G)$. Now just use Proposition 3.2.1 $\square$

Bibliography

[1] L. L. Avramov, A. Conca, and S. B. Iyengar. Subadditivity of syzygies of koszul algebras. *Mathematische Annalen*, 361(1):511–534, 2015.

[2] J. Backelin. *Relations between rates of growth of homologies*. Stockholms Universitet. Matematiska Institutionen, 1988.

[3] M. Boij and J. Söderberg. Betti numbers of graded modules and the multiplicity conjecture in the non-cohen–macaulay case. *Algebra & Number Theory*, 6(3):437–454, 2012.

[4] W. Bruns and H. J. Herzog. *Cohen-macaulay rings*. Number 39. Cambridge university press, 1998.

[5] A. Conca. Koszul algebras and their syzygies. In *Combinatorial algebraic geometry*, pages 1–31. Springer, 2014.

[6] W. da Silva, S. Hassanzadeh, and A. Simis. Bounds for the degree and betti sequences along a graded resolution. *arXiv preprint arXiv:2201.09994*, 2022.

[7] H. Dao and D. Eisenbud. Linearity of free resolutions of monomial ideals. *Research in the Mathematical Sciences*, 9(2):1–15, 2022.

[8] H. Dao and J. Schweig. Projective dimension, graph domination parameters, and independence complex homology. *Journal of Combinatorial Theory, Series A*, 120(2):453–469, 2013.

[9] D. Eisenbud and S. Goto. Linear free resolutions and minimal multiplicity. *Journal of Algebra*, 88(1):89–133, 1984.

[10] D. Eisenbud, M. Green, K. Hulek, and S. Popescu. Restricting linear syzygies: algebra and geometry. *Compositio Mathematica*, 141(6):1460–1478, 2005.

[11] D. Eisenbud, C. Huneke, and B. Ulrich. The regularity of tor and graded betti numbers. *American Journal of Mathematics*, 128(3):573–605, 2006.

[12] D. Eisenbud and F.-O. Schreyer. Betti numbers of graded modules and cohomology of vector bundles. *Journal of the American Mathematical Society*, 22(3):859–888, 2009.

[13] S. Eliahou and R. H. Villarreal. The second betti number of an edge ideal. In *XXXI National Congress of the Mexican Mathematical Society (Hermosillo, 1998)*, volume 25, pages 115–119. Soc. Mat. Mexicana México, 1999.

[14] G. Fløystad. Boij-söderberg theory: introduction and survey. *Progress in commutative algebra*, 1:1–54, 2012.

[15] C. Francisco and A. Van Tuyl. Sequentially cohen-macaulay edge ideals. *Proceedings of the American Mathematical Society*, 135(8):2327–2337, 2007.

[16] R. Fróberg. "on stanley-reisner rings". in. topics in algebra, 1990.

[17] I. Gitler and C. Valencia. On bounds for some graph invariants. *Boletín de la Sociedad Matemática Mexicana: Tercera Serie*, 16(2):73–94, 2010.

[18] H. T. Hà and A. Van Tuyl. Monomial ideals, edge ideals of hypergraphs, and their graded betti numbers. *Journal of Algebraic Combinatorics*, 27(2):215–245, 2008.

[19] S. H. Hassanzadeh and A. Simis. Plane cremona maps: saturation and regularity of the base ideal. *Journal of Algebra*, 371:620–652, 2012.

[20] J. Herzog and S. Iyengar. Koszul modules. *Journal of Pure and Applied Algebra*, 201(1-3):154–188, 2005.

[21] J. Herzog and M. Kühl. On the bettinumbers of finite pure and linear resolutions. *Communications in Algebra*, 12(13):1627–1646, 1984.

[22] J. Herzog and H. Srinivasan. On the subadditivity problem for maximal shifts in free resolutions. In *Commutative algebra and noncommutative algebraic geometry. Vol. II*, pages 245–249, 2015.

[23] T. Hibi, A. Higashitani, K. Kimura, and A. B. O'Keefe. Algebraic study on cameron–walker graphs. *Journal of Algebra*, 422:257–269, 2015.

[24] G. R. Kempf. Some wonderful rings in algebraic geometry. *Journal of Algebra*, 134(1):222–224, 1990.

[25] H. Matsumura. *Commutative ring theory.* Number 8. Cambridge university press, 1989.

[26] J. McCullough. A polynomial bound on the regularity of an ideal in terms of half of the syzygies. *Math. Res. Lett*, 19(03):555–565, 2012.

[27] J. J. Rotman and J. J. Rotman. *An introduction to homological algebra*, volume 2. Springer, 2009.

[28] S. O. Tohaneanu. On some ideals with linear free resolutions. *arXiv preprint arXiv:1906.02422*, 2019.

[29] R. H. Villarreal. *Monomial algebras*, volume 238. Marcel Dekker New York, 2001.

[30] C. A. Weibel. *An introduction to homological algebra*. Number 38. Cambridge university press, 1995.