Instituto de Matemática

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Dynamical Properties of N-distal homeomorphims

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Rio de Janeiro, Brasil

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Tese de doutorado apresentada ao Programa de Pós-graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Doutor em Matemática

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A mi familia.

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Resumo

Esta tese é dedicada ao estudo da propriedade N-distal para homeomorfismos em espaços métricos compactos. Definimos a N-equicontinuidade e provamos que cada sistema N-equicontínuo é N-distal. Introduzimos a noção de extensões N-distais e fatores N-distais. Também provamos que uma extensão M-distal de todo homeomorfismo N-distal é MN-distal e que se o semigrupo de Ellis de um homeomorfismo N-distal têm um único ideal mínimal então dito homeomosfismo têm um fator N-distal não trivial. Além disso, é mostrado que os homeomorfismos transitivos N-distais têm no máximo $N-1$ subsistemas minimais própios. Finalmente, mostramos que a entropia topológica de sistemas N-distais em espaços métricos compactos com certo comportamento no conjunto não-errante é zero. Estes resultados generalizam os já conhecidos para sistemas distais [24],[48].

Palavras-chave: Distalidade, N-distalidade, N-equicontinuidade, Extensão N-distal, Entropia Topológica.

Abstract

This thesis is dedicated to the study of N-distal property for homeomorphisms on compact metric spaces. For instance, we define N-equicontinuity and prove that every N-equicontinuous systems are N-distal. We introduce the notion of N-distal extensions and N-distal factors. We also prove that a M-distal extension of N-distal homeomorphisms is MN-distal and present a non-trivial N-distal factor for N-distal homeomorphisms having Ellis semigroup with a unique minimal ideal. It is also shown that transitive N-distal homeomorphisms have at most $N-1$ minimal proper subsystems. Finally, we prove that topological entropy vanishes for N-distal systems on compact metric spaces with some nice behavior on the non-wandering set. These results generalize previous ones for distal systems [24],[48].

Keywords: Distal, N-distal, N-equicontinuous, N-distal extensions, Topological Entropy.

List of Figures

List of abbreviations and acronyms

cf.	Compare, see
e.g.	For example
i.e.	That is

List of symbols

$\mathbb{N}$	Natural numbers with zero.
$\mathbb{N}^+$	Natural numbers without zero.
$\overline{A}$	The closure of set A.
$\#A$	Cardinality of the set A.
$n \mid m$	n divides m.
$\mathcal{O}_f(x)$ or $\mathcal{O}(x)$	The orbit of a point x under f.
$\inf$	Natural Infimum.
$\mathbb{T}^2$	Torus.
$\mathbb{S}^1$	The unit circle.
$\\| \cdot \\|$	Norm.

Contents

Introduction

The distal homeomorphisms were introduced by Hilbert in order to give a topological characterization for the concept of a rigid group of motions (see [61]). Such homeomorphisms have been widely studied in the literature. For instance, in [19] Ellis reduced them to the enveloping semigroups and the minimal distal systems, Fürstenberg proved in [24] a structure theorem and Parry without using Furstenberg's theorem on the structure of distal flows proved that they have zero entropy in [48].

Generalizations of distal systems include the *point distal flows* by Veech [56] who obtained a structure theorem from them and the more recent *mean distal systems* by Ornstein and Weiss [47]. From the measure-theoretic viewpoint we can mention Parry's systems with separating sieve also known as *measure distal systems*, see [48]. In [60] Zimmer proved a structure theorem for the measure distal systems. Lindenstrauss proved in [37] that any ergodic measure distal system can be realized as a minimal distal system with a fully supported invariant Borel measure. Fürstenberg introduced the notion of a *tight system* as one in which, after removing a negligible set, there are no distinct mean proximal points. Ornstein and Weiss also proved in [47] that tight systems have no finite positive entropy.

New classes of systems which generalize the notion of distality were recently introduced by Lee and Morales in [4] and [36]. They included the notion of *N-distal*, *countably distal*, *cw-distal* and *measurable distal*.

In this work we study N-distal self-homeomorphisms on compact metric spaces, discuss some of their basic dynamical properties and consider the extent to which certain classical results that are already available for distal systems are also valid for N-distal systems. Consequently we are interested in studying the relation between N-distality and other dynamical properties. For instance, in connection with the fact that equicontinuous systems are always distal, we introduce the notion of N-equicontinuity (see Definition 2.2.4) and prove a generalization of this result showing that every N-equicontinuous systems are N-distal in Theorem 2.2.6.

In addition we show how the N-distality behaves under homomorphisms. In this way we extend the notion of distal extensions and distal factors defining N-distal extensions and N-distal factors (see Definition 2.3.2) and prove in Theorem 2.3.3 that a M-distal extension of a N-distal homeomorphism is MN-distal. This result generalizes the previous one in [7]. Using the Ellis semigroups theory we give a criterion for the existence of a non trivial N-distal factor for N-distal homeomorphisms in Theorem 2.3.4.

We further investigate how N-distality interacts with topological transitivity and expansivity. Actually, in Theorem 3.1.5 we use the Ellis semigroups theory to obtain a restriction on the number of minimal subsystems for a transitive N-distal homeomorphism. This result is a N-distal version for the one given in [7]. On the other hand, after summarizing without proofs the relevant material on the relation between distality and expansivity, in Example 3.2.7 we present a *cw*-distal expansive homeomorphism which is not N-distal for every positive integer N.

Finally, we study the topological entropy of N-distal systems. In [48] W. Parry showed that the topological entropy of a distal system vanishes. One of the key facts for this result is that the phase space of a distal systems decomposes into a union of minimal subsets [19]. On the other hand, the same is not valid for N-distal systems as we show in Example 2.1.2. Thus, under a condition that guarantees this kind of decomposition on the non-wandering set, in Theorem 4.2.1, we prove that N-distal homeomorphims have zero topological entropy.

This work is divided as follows. In Chapter 1 we summarize the relevant preliminary material set up notation and terminology used through out this work. In Chapter 2 some basic properties, examples of N-distal systems and generalizations of classical results are established. In Chapter 3 we investigate how N-distality behaves together with topological transitivity and discuss the relation between N-distality and expansivity. The final Chapter 4 is devoted to the proof of the Theorem 4.2.1.

An appendix was added with the purpose of making the conceptual line proposed continuous, since although the topics they contain are a fundamental tool throughout the work, the insertion in a specific point where the theory established in them is used could divert the reader's attention subtracting intuitive properties in the configuration of the developed concepts. With the same purpose, the reference of the bibliographical source of each of the concepts that are presented without a previous development is given.

This thesis contains the work [49] joint work with Elias Rego.

Chapter 1

Preliminaries

In this chapter we define and give previous theory that we will use in this work. In sections 1.1 and 1.3 we summarize without proofs the relevant material on Topological Dynamics and Ergodic Theory, respectively. In section 1.2 we develop the Ellis semigroup theory that we will use throughout this work.

To start we set some basic notation. Throughout this work unless otherwise stated the pair (X, f) will be a *dynamical system* where X will denote a compact metric space with metric d and without loss of generality $f : X \to X$ will be a homeomorphism (or continuous). f^n is the n-fold self-composition of the map f if $n > 0$, on the other hand the n-fold composition of f^{-1} if $n < 0$ and f^0 is the identity map, denoted by Id. The orbit of a point x under f is the set $\{f^n(x); n \in \mathbb{Z}\}$ which we denote as $\mathcal{O}_f(x)$. The closure of $A \subseteq X$ is be denoted by $\overline{A}$ while the cardinality of A is be denoted by $\#A$. If $x \in X$ and $\delta > 0$ we denote the *open ball* around x by $B_\delta(x)$.

1.1 Topological Dynamics

In any mathematical system, one is interested in the maps which respect the structure of the system. The appropriate maps in topological dynamics are those which are continuous and equivariant. To be precise, this notion of equivalence is

Definition 1.1.1 (cf. [7, p. 21])**.** Let (X, f) and (Y, g) be dynamical systems. A *homomorphism* (or *semiconjugacy*) from (Y, g) to (X, f) is a continuous onto map $\pi : Y \to X$ satisfying $f \circ \pi = \pi \circ g$.

If there is a homomorphism π from (X, f) onto (Y, g), we say that (Y, g) is a *factor* of (X, f), and that (X, f) is an *extension* of (Y, g). The map $\pi : Y \to X$ is also called a *factor map* or *projection*. The simplest example of an extension is the *direct product*

$$f_1 \times f_2 : X_1 \times X_2 \to X_1 \times X_2$$

of the dynamical systems (X_1, f_1) and (X_2, f_2), where $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$. If f_1 and f_2 are homeomorphisms. The direct product turns out to be a homeomorphism on $X_1 \times X_2$ if we equip the space $X_1 \times X_2$ with the metric $d^2((x_1, z_1), (x_2, z_2)) = \max\{d_1(x_1, z_1), d_2(x_2, z_2)\}$ where d_1 and d_2 are the metrics on X_1 and X_2, respectively.

Note that (X_1, f_1) and (X_2, f_2) are factors of $(X_1 \times X_2, f_1 \times f_2)$, since the projections of $X_1 \times X_2$ onto X_1 and X_2 are homomorphisms. An extension (Y, g) of (X, f) with factor map $\pi : Y \to X$ is called a *skew product* over (X, f) if $Y = X \times F$, and π is the projection onto the first factor or, more generally, if Y is a fiber bundle over X with projection π.

Let (X, f) be a dynamical system and $x, y \in X$. The point x is said to be *proximal* to y if

$$\inf_{n \in \mathbb{Z}} d(f^n(x), f^n(y)) = 0.$$

Clearly, the proximal relation is reflexive, symmetric and invariant, but is in general neither transitive (see example 2.1.3) nor closed, example 2.1.3 also is not closed by [6, Corollary 1]. The pair (x, y) is a *proximal pair* if x is proximal to y and (x, y) is a *distal pair* if it is not a proximal pair. Let us denote by $P(x)$ the set of points $y \in X$ such that (x, y) are proximal pairs, i.e. the *proximal cell* (cf. [7, p. 66]) of x.

$$P(x) = \{y \in X \ : \ \inf_{n \in \mathbb{Z}} d(f^n(x), f^n(y)) = 0\}.$$

Notation $P_f(x)$ will be used to indicate dependence on f if necessary. Let us recall the definition of distality.

Definition 1.1.2 (cf. [14, p. 45])**.** Let (X, f) be a dynamical system. A point $x \in X$ is said to be *distal point* for f if $P(x)$ reduces to $\{x\}$. Let $Dist(f)$ denote the set of distal points of f. We say that (X, f) is *distal* if $Dist(f) = X$.

Remark 1.1.3. If we assume that f is just continuous in the definition above. The distality implies that the function f is bijective and so a homeomorphism [44, Theorem 26.6]. Indeed, the injectivity clearly follows from the distal definition. On the other hand, the innocent looking fact that distality also implies surjectivity is not transparent from the definition. In Proposition 1.2.10 we will present a proof for this fact using the Ellis enveloping semigroup of f, another proof via $\beta\mathbb{N}$ the Stone-Čech compactification of $\mathbb{N}$ can be found in [10, p. 33].

Basic examples of distal homeomorphisms are the identity map and isometries of a metric space. To give more examples let us recall the concept of equicontinuous homeomorphism.

Definition 1.1.4 (cf. [14, p. 45])**.** A homeomorphism f of a compact metric space (X, d) is said to be *equicontinuous* if the family of all iterates of f is an equicontinuous family, i.e., for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$d\left(f^n(x), f^n(y)\right) < \epsilon \text{ whenever } d(x,y) < \delta$$

for all $x, y \in X$ and $n \in \mathbb{Z}$

Clearly, equicontinuity implies continuity, but the converse is not true in general. For instance, consider $f : [0,1] \to [0,1]$ defined by $f(x) = x^2$. We claim that the family of all iterates $\{f^n(x) = x^{2^n}\}_{n\in\mathbb{Z}}$ is not equicontinuous. Indeed, we choose $\epsilon = \frac{1}{2}$ and $x = 1$. Let $1 > \delta > 0$ be given and fix $y \in (1-\delta, 1)$. Since $f^n(y) \to 0$ as $n \to \infty$, we can choose $n \in \mathbb{N}$ large enough that $f^n(y) < \frac{1}{2}$. Then $d\left(f^n(x), f^n(y)\right) > \frac{1}{2}$ and $d(x,y) < \delta$.

By the following Theorem other non-trivial examples of distal homeomorphisms are the equicontinuous homeomorphisms or equivalently uniform almost periodic homeomorphisms. Although its proof is very easy (cf., e.g., [24, 21, 7, 14]), yet it is a very useful important fact in topological dynamics that

Theorem 1.1.5 (cf. [14, Proposition 2.7.2.])**.** *Equicontinuous homeomorphisms are distal.*

An interesting proof of the above proposition that use the idea of Ellis semigroup (or Ellis enveloping semigroup) can be found in [21, p. 36]. In contrast, distal homeomorphisms are not necessary equicontinuous, the dynamical system given below is a counterexample.

Example 1.1.6 (cf. [33, p. 618])**.** *Let $f : D_1 \to D_1$ defined on the unit disc $D_1 = \{z \in \mathbb{C} : \|z\| \leqslant 1\}$ and given by the formula $f(z) = z\exp(2\pi i|z|)$.*

Now let us recall that a subset $A \subseteq X$ is said to be *minimal* if it is closed, non-empty, f-invariant (i.e. $f(A) \subset A$) and has no proper closed, non-empty, f-invariant subsets (cf. [14, p. 29]). Last condition is equivalent to the orbit of any point in A be dense in A. A homeomorphism f is *minimal* if X is a minimal set.

The example 1.1.6 is not minimal. However, there are examples of distal and minimal homeomorphism that are not equicontinuous.

The examples [7, Theorem 5.14] were constructed by L. Auslander, F. Hahn, and L. Markus, and were the first examples of distal minimal flows which are not equicontinuous. Their work in this general area is contained in the monograph [8]. An interesting historical sidelight is that at the time they were not aware of the notion of distal, but were trying to construct minimal actions of the real line on nilmanifolds. They were able to produce dense orbits, but could not prove minimality. W.H. Gottschalk suggested that these examples might be distal (and therefore necessarily minimal).

Independently and at about the same time, H. Furstenberg noted that the skew products over an equicontinuous basis with compact group translations as fiber maps are always distal, often minimal, but rarely equicontinuous. A typical examples are skew products on the torus (see Example 2.2.8), which are actually simpler than the flow on nilmanifolds.

Remark 1.1.7. The nature of these examples led Fürstenberg to his path-breaking structure theorem [24], describing the structure of a general minimal distal system as a (countable but maybe transfinite) inverse limit of isometric extensions starting with the one-point dynamical system. Modify the map in the Example 2.2.8 to $F(x, y) = (x + \alpha \pmod 1, y + 2x + \alpha \pmod 1)$. Again show that (X, F) is minimal and distal. Check that $F(0, 0) = (n\alpha, n^2\alpha)$ and deduce that the sequence $\{n^2\alpha\}_{n\in\mathbb{N}}$ is dense in $\mathbb{S}^1$; see Fürstenberg's book [25] for further development of these ideas.

There is a natural generalization of the distal notion to homomorphisms. This notion is defined in such a way that the trivial homomorphism $(X, f) \to (\{p\}, f)$ is distal if and only if the system (X, f) is distal.

Definition 1.1.8 (cf. [18, Definition 4.14])**.** Let (X, f) and (Y, g) be dynamical systems. We say that a map $\pi : Y \to X$ is *distal* if

$$\inf_{n\in\mathbb{Z}} d(g^n(y_1), g^n(y_2)) > 0$$

for every distinct $y_1, y_2 \in Y$ satisfying $\pi(y_1) = \pi(y_2)$. We say that (Y, g) is a *distal extension* of (X, f) (resp. (X, f) is a *distal factor* of (Y, g)) if there is a distal homomorphism from (Y, f) to (X, f).

The map $F : \mathbb{T}^2 \to \mathbb{T}^2$ in the Example 2.2.8 is a distal extension of a circle rotation $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ given by $R_\alpha(x) = x + \alpha \pmod 1$, with projection on the first coordinate as the distal homomorphism.

Theorem 1.1.9 (cf. [7, Proposition 5.8])**.** *A distal extension of a distal homeomorphism is distal.*

A more suitable problem is an analogous result for distal factors.

Let us recall that $A \subset \mathbb{Z}$ is said to be *syndetic* if there is $F \subset \mathbb{Z}$ finite such that $\mathbb{Z} = F + A$. We say that a point $x \in X$ is *almost periodic* with respect to a homeomorphism $f : X \to X$ if $\{n \in \mathbb{Z} : f^n(x) \in U\}$ is syndetic for every neighborhood U of x (cf. [3, p. 390]). A homeomorphism f is said to be *pointwise almost periodic* if every $x \in X$ is almost periodic w.r.t. f. Oftenly, almost periodic points are called *minimal points*. This is because a point x is almost periodic if and only if the closure of $\mathcal{O}(x)$ is a minimal set, for more details see [14, Section 2.1]. We note in this regard that

Proposition 1.1.10 (cf. [14, Proposition 2.7.5])**.** *A distal homeomorphism on a compact Hausdorff space X is pointwise almost periodic.*

That is a consequence of the following fact

Theorem 1.1.11 (cf. [14, Theorem 2.7.4])). *Let f be a homeomorphism of a compact Hausdorff space X. Then every point is proximal to an almost periodic point.*

From the Proposition 1.1.10 it follows that (X, f) is distal if and only if the product $(X \times X, f \times f)$ is pointwise almost periodic [14, Proposition 2.7.6]. This in turn implies that a factor of a distal homeomorphism is distal [14, Corollary 2.7.7].

The class of distal dynamical systems is of special interest because it is closed under factors and isometric extensions. The class of minimal distal systems is the smallest such class of minimal systems: According to Fürstenberg's structure theorem [24], every minimal distal homeomorphism (or flow) can be obtained by a (possibly transfinite) sequence of isometric extensions starting with the one-point dynamical system.

On the other hand, recall that a homeomorphism $f : X \to X$ is *transitive* if for any pair of non-empty open sets U and V there is an integer n such that $f^n(U) \cap V$ is nonempty (cf. [3, p. 36]). We say that a point x is a *transitive point* of f if its orbit is dense on X. Every point of minimal homeomorphism is a transitive point. We say that a f is *pointwise transitive* if there exists some transitive point for f.

We remark that for second countable spaces and in absence of isolated points, point transitivity is equivalent to topological transitivity; for more details we refer the reader to [3].

If the distal homeomorphism $f : X \to X$ is pointwise transitive with a transitive point x, i.e., $\mathcal{O}(x) = X$. Any point z in X is almost periodic by Proposition 1.1.10, and consequently $\mathcal{O}(z)$ is minimal. It follows that $\mathcal{O}(z) = X$. Therefore we have proved:

Proposition 1.1.12 (cf. [7, Corollary 5.7]). *A distal homeomorphism is minimal if and only if is pointwise transitive.*

The first researcher who considered the expansivity in dynamics was Utz in his seminal paper [54]. Indeed, he defined the notion of unstable homeomorphisms (today known as expansive homeomorphisms [30]) and studied their basic properties. Since then an extensive literature about these homeomorphisms has been developed.

Recall that a homeomorphism (resp. continuous map) $f : X \to X$ of a metric space X is *expansive* (res. positively expansive) if there is $\delta > 0$ such that for any two distinct points $x, y \in X$, there is some $n \in \mathbb{Z}$ (resp. $n \in \mathbb{N}$) such that $d(f^n(x), f^n(y)) \geqslant \delta$, or, equivalently, if $\{y \in X : d(f^n(x), f^n(y)) < \delta, \forall n \in \mathbb{Z}\} = \{x\}$ (resp. $\{y \in X : d(f^n(x), f^n(y)) < \delta, \forall n \in \mathbb{N}\} = \{x\}$) for all $x \in X$. Any number $\delta > 0$ with this property is called an *expansiveness constant* for f (cf. [14, p. 35]).

Proposition 1.1.13 ([14, Proposition 2.7.1])**.** *An expansive homeomorphism of an infinite compact metric space is not distal.*

Indeed, this result is a direct consequence of the following Theorem. Recall that two points $x, y \in X$ are *positively* (*negatively*) *asymptotic* if $x \neq y$ and $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$ ($n \to -\infty$). A space is *self-dense* if it has no isolated point.

Theorem 1.1.14 ([54, Theorem 2.1])**.** *For every expansive homeomorphism of a self-dense compact metric space there are two points asymptotic in at least one sense.*

Currently, some authors consider the Theorem 1.1.14 as a direct consequence of the Theorem bellow due to Schwartzman.

Theorem 1.1.15 ([52])**.** *If a compact metric space admits a positively expansive homeomorphism then it is finite.*

Utz [55, p. 222] recognized Theorem 1.1.15 as Schwartzman's improvement of his result Theorem 1.1.14. Also, Utz [55] explained that Schwartzman in his Thesis did not assume the self-dense hypothesis. In any case, in 1960 Bryant [5, Theorem 2] explained how to extend the result to spaces which are not self-dense.

The fact that the authors' result [1, Theorem p. 316] in 1965 implies Theorem 1.1.15 was noticed by Bryant and Walters [15] in 1969 and following the same techniques they essentially proved in [15, Theorem 6] that f being surjective is not a necessary hypothesis in Theorem 1.1.15. In 2010 Mai and Sun [38] gave another interesting proof where they just assume that f is continuous and injective. Furthermore, one cannot state a same theorem for noninvertible dynamical systems. For instance, the doubling map on $\mathbb{S}^1$ is a positively expansive continuous map, for more details see [5]

1.2 Ellis Semigroup Theory

An extremely useful tool to study the theory of topological dynamical systems is the enveloping semigroup of a dynamical system was introduced by R. Ellis in [23]. Henceforth, we refer to enveloping semigroup as Ellis semigroup. Let us briefly introduce this notion and some interesting facts about it that help us in our study of N-distality.

To start recall that a *semigroup* G is a nonempty set together with a closed binary associative operation. Any non-empty subset $H \subset G$ closed under the binary operation is a *subsemigroup* of G. A simple example of semigroup is the natural numbers set $\mathbb{N}$ with the sum operation and $2\mathbb{N}$ is a subsemigroup of $\mathbb{N}$.

Let G be a semigroup and $a \in G$. The *a-left multiplication* map $L_a : G \to G$ and the *a-right multiplication* map $R_a : G \to G$ are defined by $L_a(b) = ab$ and $R_a(b) = ba$.

A *right topological semigroup* consists of a semigroup G and a topology $\mathscr{T}$ on G such that for all $a \in G$, the right action L_a is a continuous mapping of the space G to itself.

Definition 1.2.1. [23, Definition 8] An *Ellis semigroup* is a compact hausdorff right topological semigroup.

Example 1.2.2. *Let X be a compact Hausdorff space. The set X^X of all (not necessarily continuous) functions from X to itself, provided with the topology of pointwise convergence, is a Ellis semigroup.*

Since the topology of pointwise convergence is the same as convergence in the product topology on the space $X^X = \prod_{x \in X} X_x$, where $X_x = X$ for all $x \in X$. Clearly, X^X is Hausdorff and by Tychonoff's Theorem is also compact. X^X has a natural semigroup structure defined by function composition. Evidently, if $f \in X^X$ then R_f is continuous.

Remark 1.2.3. In general the left multiplication is not continuous. For example, let $X = [0,1]$, $f(x) = 0$ for $x \in X$, $f_n(x) = \frac{x}{n}$ for $x \in X$, $g(x) = 1$ for $x \in (0,1]$ and $g(0) = 0$. Clearly, $f_n \to f$ but $L_g(f_n) \nrightarrow L_g(f)$ since $(g \circ f_n)(x) = 1$ and $(g \circ f)(x) = 0$ for every $x \in (0,1]$.

However, a direct computation shows that if $f \in X^X$ is continuous then L_f is continuous.

Note that, any closed subsemigroup subset H of a Ellis semigroup E, is also an Ellis semigroup.

Example 1.2.4. *Let (X,f) be a dynamical system. The closure $\overline{\{f^n : n \in \mathbb{Z}\}}$ in X^X is a closed semigroup, and so it is an Ellis group.*

First we denote $E(X,f) = \overline{\{f^n : n \in \mathbb{Z}\}}$. Trivially $E(X,f)$ is closed. Let $g, h \in E(X,f)$ then there is a sequence $\{f^{n_k}\}$ such that $f^{n_k} \to g$. Thus,

$$f^{n_k} \circ h = R_h(f^{n_k}) \to R_h(g) = g \circ h,$$

analogously $f^{n_k} \circ h$ is the limit of some sequence $\{f^{n_j}\}$. Since $E(X,f)$ is closed, $f^{n_k} \circ h \in E(X,f)$ and then we have $g \circ h \in E(X,f)$. Hence $E(X,f)^2 = E(X,f)E(X,f) \subset E(X,f)$, and $E(X,f)$ is a semigroup of X^X.

Definition 1.2.5. [23, Definition 8] Let (X,f) be a dynamical system. We define the Ellis semigroup of f by $E(f) = \overline{\{f^n : n \in \mathbb{Z}\}}$.

Explicit computations of Ellis semigroups are not very common. Some examples are to be found in [24, 29, 46, 40, 41, 22, 27, 9, 2, 16, 26]. One of the reasons for the scarcity of concrete examples of Ellis semigroups when X is a compact Hausdorff space is that these objects are usually non-metrizable, a notable exception is the case of weakly almost periodic metric systems (see [17] and [28, Theorem 1.48]).

Definition 1.2.6 (c.f. [20, Definition 3])**.** A nonempty closed subset I of an Ellis semigroup E is said to be a *left* (resp. *right*) ideal if $EI \subset I$ (resp. $IE \subset E$). A left (resp. right) ideal is said to be *minimal* if it does not properly contain a left (resp. right) ideal.

In particular, every left (resp. right) ideal is a closed subsemigroup, and so too an Ellis semigroup.

Let X be a compact metric space. As a first example of a minimal left (right) ideal of X^X, we have the collection of constant maps on X.

Example 1.2.7. *Let (X, f) be a dynamical system. A nonempty subset I of $E(f)$ such that $EI \subset I$ (resp. $IE \subset E$) is left (resp. right) ideal.*

Recall that an element a of a semigroup is called *idempotent* if $a^2 = a$. Let E be a semigroup and $a, b \in E$, we write $a \leqslant b$ if $a = ab = ba$. Clearly, "$\leqslant$" is a partial order on the collection of idempotents in E. An idempotent $a \in E$ is said to be *minimal* if for any idempotent $b \in E$, $b \leqslant a$ implies $b = a$.

Example 1.2.8. *Let X be a compact Hausdorff space. Then*

1. *$g \in X^X$ is an idempotent if and only if g restricted to $g(X)$ is the identity map.*
2. *Id_X and the constant maps of X are idempotents in X^X.*
3. *The minimal idempotents in X^X are the constant maps.*

In fact if $g \in X^X$ is an idempotent. Let $z \in g(X)$, then there is $x \in X$ such that $g(x) = z$ and then

$$g(z) = g(g(x)) = g(x) = z.$$

Conversely, if $g = Id_{g(X)}$ (in particular, if g is continuous, this means that g is a retraction onto $g(X)$), fix $x \in X$, we have

$$g^2(x) = g(g(x)) = Id(g(x)) = g(x).$$

It is clear that Id and constants maps are idempotent elements of X^X.

Let h be a minimal idempotent in X^X. Fix $z \in X$. Let g be the constant function define by $g(y) = h(z)$ for every $y \in X$. Let $x \in X$, we have

$$\begin{gathered} g(x) = h(z) = g(h(x)) \quad \text{and} \\ g(x) = h(z) = h^2(z) = h(h(z)) = h(g(x)), \end{gathered}$$

that is, $g = g \circ h = h \circ g$. Therefore $g \leqslant h$, and $h = g$ by minimality.

On the other hand, when (X, f) is a dynamical system the case for $E(f)$ is different, for instance $E(f)$ may not contain constant maps. However, some results are adapted from the Ellis semigroup of a dynamical systems [19, 20] to general case.

Proposition 1.2.9. *Let E be an Ellis semigroup. Then*

(i) There is an idempotent in E.

(ii) E has a minimal left ideal.

(iii) Every left ideal of E contains idempotents.

(iv) If I is a (minimal) left ideal of E, then so is Ia for every $a \in E$.

(v) If $u \in E$ is a minimal idempotent and I is a left ideal of E, then $u \in Iu$.

(vi) An idempotent $u \in E$ is minimal if and only if $u \in I$ for some minimal left ideal $I \subset E$.

Proof. Define
$$\mathcal{A} = \{A \subset E \,:\, A \text{ closed s.t. } A^2 \subset A\}.$$
$\mathcal{A} \neq \varnothing$ since $E \in \mathcal{A}$. The set inclusion is a partial order in $\mathcal{A}$, then applying Zorn's lemma we obtain a minimal element $A \in \mathcal{A}$. Let $a \in A$, we claim that $a^2 = a$. Indeed, set $B = R_a(A) = Aa \subset AA \subset A$. Since R_a is continuous, $B \in \mathcal{A}$. By minimality, $B = A$. Then, there is $b \in A$ such that $ba = a$. Consider

$$C = A \cap R_a^{-1}(A) = \{a \in A \,:\, ba = a\} \subset A,$$

hence $C \in \mathcal{A}$. Consequently $C = A$, the claim follows and (i) is proved.

To deduce (ii), note that E is a left ideal, applying Zorn's lemma to the collection of all left ideals of E.

As we have already note that a left ideal is in particular an Ellis semigroup. Then, (iii) follows from (i).

To prove (iv), let $a \in E$, if I is a left ideal then $Ia = R_a(I)$ is a left ideal by associativity. To prove it is minimal when I is minimal, consider a left ideal J of E with $J \subset Ia$, and define
$$K = I \cap R_a^{-1}(J) = \{i \in I \; ia \in J\}.$$
Since I and J are ideals, it is easy to check that K is a left ideal. Hence $K = I$ by the minimality of I. Consequently, $I \subset R_a^{-1}(J)$, and thus

$$Ia = R_a(I) \subset R_a\left(R_a^{-1}\right)(J) \subset J.$$

From (iv) and (iii), there is an idempotent $iu \in Iu$, where $i \in I$. Let $b = uiu \in Iu$. We have b is an idempotent because

$$bb = uiuuiu = uiuiu = u(iu)^2 = uiu = b.$$

Moreover, $b \leqslant u$ since

$$ub = uuiu = uiu = b \text{ and } bu = uiuu = uiu = b.$$

Since u is minimal, $u = b$, and so $u \in Iu$. This proves (v).

Let $u \in E$ be a minimal idempotent. E has a minimal left ideal I by (ii). It follows from (iv) and (v) that Iu is minimal and $u \in Iu$. Conversely, suppose that u is an idempotent belongs to a minimal left ideal $I \subset E$. Let $b \in E$ be an idempotent with $b \leqslant u$, then $b = bu \in I$ and $b = ub \in Ib$. By (iv), Ib is a minimal left ideal. Thus, $I \cap Ib \neq \varnothing$ and is a left ideal. As I is minimal we have $Ib = I$. So $u = cb$ for some $c \in I$, which implies

$$b = ub = cbb = cb = a.$$

Therefore u is minimal proving (vi). □

As an application we present the following result. It is proved as in [51, p. 959].

Proposition 1.2.10. *A distal continuous function on a compact metric space X is surjective.*

Proof. Let f be a distal continuous function on X. $f(X)$ is closed, since X is compact. If we prove that the image of f is dense, the assertion follows. For this purpose, we fix $x \in X$ and U be an open neighborhood of x. We proceed to show that U intersect $f(X)$. By the 1.2.9(i) there is an idempotent $u \in E(f)$. Let V be a neighborhood of $u(x) = u(u(x))$, it follows from the definition of $E(f)$ that there is $n \in \mathbb{N}$ such that $f^n(u(x)), f^n(x) \in V$. As f is distal we have $x = u(x)$. Taking $V = U$ we have $f^n(x) \in U$. That is, there is $z = f^{n-1}(x)$ such that $f(z) \in U$. □

The following Proposition is proved as in [23, Lemma 4] and [20, Remark 6].

Proposition 1.2.11. *Let (X, f) be a dynamical system and $x, y \in X$. The following properties are equivalent:*

(i) $y \in P(x)$.

(ii) $g(x) = g(y)$ *for some* $g \in E(f)$.

(iii) $g(x) = g(y)$ *for some minimal idempotent* $g \in E(f)$.

(iv) There is a minimal left ideal $I \subset E(f)$ *such that* $h(x) = h(y)$ *for every* $h \in I$.

Proof. Clearly (iv) implies (iii) and in turn (iii) implies (ii). It is sufficient to show that (i) is equivalent to (ii) and (ii) implies (iv).

To deduce (ii) from (i), assume that $y \in P(x)$. Let $\{n_k\}$ be the subsequence such that $d\left(f^{n_k}(x), f^{n_k}(y)\right) \to 0$. Since $E(f)$ is compact, there is $g \in E(f)$ and a subsequence $\{n_\alpha\}$ of $\{n_k\}$ such that $f^{n_\alpha} \to g$. Thus $g(x) = g(y)$. Conversely, suppose that $g(x) = g(y)$ for some $g \in E(f)$. If $g = f^n$ for some $n \in \mathbb{N}$, we are done. For the other case, if $g \neq f^n$ for every $n \in \mathbb{N}$, there is an eventually non-constant subsequence $\{n_\alpha\}$ such that $f^{n_\alpha} \to g$. Since X is compact, there are $a, b \in X$ such that $f^{n_\alpha}(x) \to a$ and $f^{n_\alpha}(y) \to b$. It follows that $a = g(x) = g(y) = b$. Therefore $y \in P(x)$.

It remains to prove that (ii) implies (iv). To this end, define

$$I = \{h \in E(f) \,:\, h(x) = h(y)\}$$

is non-empty by hypothesis. Moreover, I is closed and $EI \subset I$. That is, I is a left ideal. The result follows from applied Proposition 1.2.9(ii) to I. □

As a consequence of the above results the following fact due to Robert Ellis is obtained.

Theorem 1.2.12 (cf. [20, Theorem 2])**.** *Let (X, f) be a dynamical system. The following statements are equivalent.*

(i) The proximal relation is a equivalence relation in X.

(ii) $E(f)$ contains exactly one minimal right ideal.

1.3 Ergodic Theory

Let $(X, \mathcal{B}, \mu)$ be a probability space and $f : X \to X$ be a measurable transformation. We say that the measure is *f-invariant* (or *invariant under f*) if

$$\mu(B) = \mu(f^{-1}(B)) \quad \text{for every measurable set } B.$$

We also say that f *preserves* μ, to mean just the same. A probability space and a measure-preserving transformation on it $(X, \mathcal{B}, \mu, f)$ is called *measure-preserving dynamical system.*

Recall that a measurable set B is called f-invariant if $f^{-1}(B) = B$ modulo zero.

Definition 1.3.1. A measure μ is said to be *f-ergodic* (or simply *ergodic*) if every f-invariant measurable set has either total measure or null measure.

1.3.1 Metric and Topological Entropy

One way to determine the complexity of a dynamical is its topological entropy. Indeed, its positiveness is always related to some kind o chaoticity.

Let $(X, \mathcal{B}, \mu)$ be a probability space. By *partition* of $(X, \mathcal{B}, \mu)$ we mean a countable (finite or infinite) disjoint collection of elements of $\mathcal{B}$ whose union has full measure (see [58, Definition 4.1]).

Definition 1.3.2 (cf. [32, p. 74])**.** Let $\alpha = \{A_1, A_2, \ldots\}$ be a partition of a probability space $(X, \mathcal{B}, \mu)$. The *entropy of* α is defined to be

$$H_\mu(\alpha) = -\sum_{A \in \alpha} \mu(A) \log(\mu(A)).$$

where $0 \log(0) := 0$.

Suppose $\alpha = \{A_1, A_2, \ldots\}$ and $\beta = \{B_1, B_2, \ldots\}$ are two partitions. The *join* of α and β is the partition

$$\alpha \vee \beta = \{A_i \cap B_j \,:\, A_i \in \alpha, B_j \in \beta\}.$$

Let be a measure-preserving dynamical system.

Let $\xi = \{E_1, E_2, \ldots\}$ be a partition of a measure-preserving dynamical system $(X, \mathcal{B}, \mu, f)$. $f^{-j}(\xi)$ denotes the partition $\{f^{-j}(E_1), f^{-j}(E_2), \ldots\}$ (cf. [58, Definition 4.4]).

Definition 1.3.3 (cf. [53, Definition 14.5])**.** A finite measurable partition ξ of the space X is called *generating* for f if

$$\bigvee_{j=0}^{\infty} f^{-j}(\xi) = \epsilon,$$

Where ϵ is the partition of ξ into points and equality is understood modulo zero.

Let $\xi = \{E_1, E_2, \ldots\}$ be a partition of a measure-preserving dynamical system $(X, \mathcal{B}, \mu, f)$. We denote and define

$$\xi^n = \xi \vee f^{-1}(\xi) \vee \cdots \vee f^{-(n-1)}(\xi)$$

Since the sequence $H_\mu(\xi^n)$ is subadditive and $H_\mu(\xi) < \infty$, the following concept is well defined (see fro instance [58, Theorem 4.9]).

Definition 1.3.4 (cf. [32, Definition 3.7.2])**.** The *metric entropy* of the transformation f relative to the partition ξ.

$$h_\mu(f, \xi) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\xi^n).$$

Finally, we are in position to define

Definition 1.3.5 (cf. [32, p. 77])**.** The *metric entropy* of f with respect to μ is defined by

$$h_\mu(f) = \sup\{h_\mu(f, \xi)\},$$

where the supremum is taken over all finite or countable partitions ξ with $H_\mu(\xi) < \infty$.

In the above two definitions, finite or countable partitions with finite entropy are taken since the set of finite partitions is dense in the set of countable partitions with finite entropy (cf. [50, Section 6]).

On the other hand, topological entropy is defined in a similar way but in topological terms. The variational principle tells us that topological entropy is achieved by the supremum of the metric entropies (see [58, Theorem 8.6]). Moreover, this supremum can be taken on the metric entropies of ergodic measures (see for instance [58, Corollary 8.6.1]). Thus we can define

Definition 1.3.6. *topological entropy* of a continuous map $f : X \to X$ in a compact metric space as follows.

$$h(f) = \sup h_\mu(f),$$

where the supremum is taken over the set of ergodic f-invariant measures of f.

Remark 1.3.7. In particular, it follows that the topological entropy of f is zero if and only if $h_\mu(f) = 0$ for every ergodic f-invariant measures μ.

Kolmogorov-Sinai Theorem 1.3.8 (cf. [53, Theorem 14.3])**.** *Let ξ a measurable partition of the space X. If ξ is generating for f and $H_\mu(\xi) < \infty$, then $h(f) = h_\mu(f, \xi)$*

Chapter 2

N-distal homeomorphisms

In this chapter we introduce the class of N-distal systems, study some of its dynamical properties and their connection with other dynamical concepts. In section 2.1 we define the N-distality for homeomorphisms, prove its basic properties and give some examples. In section 2.2 we introduce the notion of N-equicontinuity for homeomorphism, prove that N-equicontinuos homeomosrphisms are N-distal and show that the Fürstenberg's example is a minimal distal homeomorphism that is not N-equicontunuos for every positive integer N. Setting the hierarchy between the two concepts. In section 2.3 we define the idea of proximal cell for a homomorpohism and used it to generalize the distal extension (distal factor) of an homeomorphism. Using this concepts to extend the Theorem 1.1.9 and give a criterion for existence of non-trivial N-distal factors for N-distal homeomorphisms.

2.1 N-distal homeomorphisms

Here we define the N-distal homeomorphims, state some of its properties and give a few examples. Recently in [4] the authors defined the following new classes of systems.

Definition 2.1.1. We say that f is a N-*distal* (for some $N \in \mathbb{N}^+$) map if $P(x)$ has at most N points and f is a *countably-distal* map if the set $P(x)$ is a countable subset of X, for all $x \in X$.

Our first remark is that distality clearly implies N-distality and N-distality clearly implies countably-distality, but the converse do not always hold. For instance, consider the following examples.

Example 2.1.2. *There is a compact metric space X and a 3-distal homeomorphism $f : X \to X$ which is not 2-distal.*

Proof. To see this, let $D = \{(\theta, r) \in \mathbb{R}^2 \;:\; 1 \leqslant |r| \leqslant 2\}$ be the annulus in polar coordinates. Define $F : D \to D$ through

$$F(\theta, r) = (\theta + k\,(\mathrm{mod}\,1), (r-1)^2 + 1) \tag{2.1}$$

with k an irrational number. Consider $p = (0, \frac{3}{2})$ and $X = S_1(\mathbf{0}) \cup \mathcal{O}_F(p) \cup S_2(\mathbf{0})$ where $\mathbf{0} = (0,0)$ and $S_r(x)$ denotes the circle of radius r centered at x. Set $f = F|_X$, the restriction of F to X. Thus the pairs $(p, (0,1))$ and $(p, (0,2))$ form proximal pairs for f, therefore f is 3-distal but it is not 2-distal. $\square$

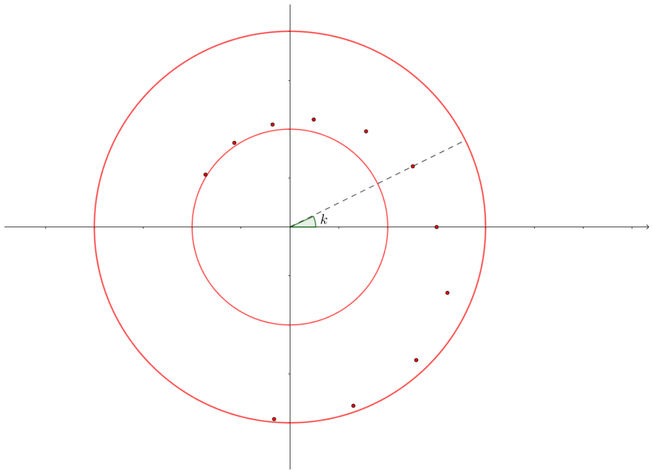

Figure 1 – 3-distal that is not 2-distal

If we slightly modify the previous example, we obtain

Example 2.1.3. *There are N-distal homeomorphisms which are not $N-1$-distal for $N \geqslant 4$ and there is a countably-distal homeomorphisms which is not N-distal for every positive integer N.*

Proof. Let $F : D \to D$ be the function defined in the previous example.

1. In D consider $p_n = (0, \frac{1}{n} + 1)$ with $2 \leqslant n \leqslant N-1$ for $N \geqslant 3$. Define $X = \partial D \cup (\cup_{N-1 \geqslant n \geqslant 2} \mathcal{O}(p_n))$, where ∂D denotes the boundary of D. Let f be given by $f = F|_X$. Then f is a N-distal homeomorphism which is not $N-1$-distal, since $P_f(0, \frac{3}{2}) = \{(0,1), (0,2)\} \cup \{p_n \;:\; 2 \leqslant n \leqslant N-1\}$.

2. Similarly, consider in D the points $p_n = (0, \frac{1}{n} + 1)$ with $n \geqslant 2$. Define $Y = \partial D \cup (\cup_{n \in \mathbb{N}} \mathcal{O}(p_n))$ and the homeomorphism $g : Y \to Y$ by $g = F|_Y$. Thereby $P_g(0, \frac{3}{2}) = \{(0,1), (0,2)\} \cup \{p_n \;:\; n \geqslant 2\}$, hence g countably-distal map but it is not N-distal for every positive integer N.

$\square$

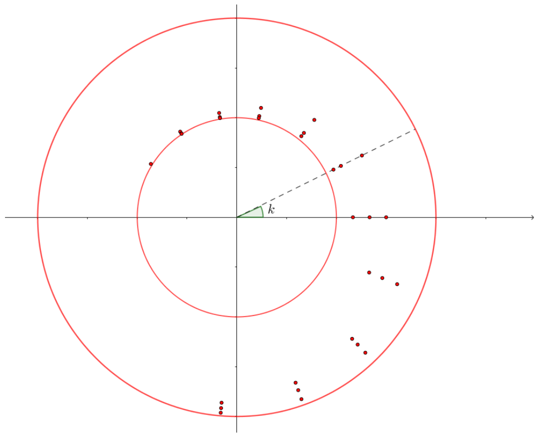

Figure 2 – 5-distal that is not 4-distal

Example 2.1.4. *There is a compact metric space X and a 2-distal homeomorphism $f : X \to X$ which is not distal.*

Proof. A modification of Example 2.1.2 can be made to obtain a 2-distal homeomorphism which is not distal. Indeed, the space in this example is the union of two concentric circles and the orbit of the point $p = (0, \frac{3}{2})$ between them. The application of the homeomorphism f makes the points in the circles stay in the circles, while the point p approach the inner circle in future and the outer circle in the past. If we consider only one circle $S_1(\mathbf{0})$ (or $S_2(\mathbf{0})$) a change in the dynamics of the point p to approach the circle $S_1(\mathbf{0})$ (or $S_2(\mathbf{0})$) for both future and past, we have the desired example. $\square$

The above examples show that these three levels of distality are different.

Remark 2.1.5. It is well known that every distal homeomorphism is pointwise almost periodic (see Proposition 1.1.10). Nevertheless, this is not true in general for N-distal homeomorphisms for the example 2.1.2.

Now we state our first result that deals with some elementary properties of N-distal homeomorphism.

Proposition 2.1.6. *Let $f : X \to X$ and $g : Y \to Y$ be homeomorphisms on X and Y compact metric spaces. The following properties hold:*

(i) Let k be a non-zero intenger. Then f is N-distal if and only if f^k is N-distal.

(ii) If f is N-distal and g is M-distal, then $f \times g$ is MN-distal.

(iii) If f and g are conjugated homeomorphisms, f is N-distal if and only if g is N-distal.

Proof. Observe that by the definition of proximal cell we have $P_{f^k}(x) \subseteq P_f(x)$ for every $x \in X$ and therefore the if part follows. Conversely, let $y \in P_f(x)$, there is a sequence $\{n_j\}$ goes to infinity such that

$$d(f^{n_j}(x), f^{n_j}(y)) \to 0.$$

If $k \mid n_j$ for infinite j. Set the subsequence $\{l_{j_m} : n_{j_m} = kl_{j_m}\}$. Then there is a sequence $\{l_{j_m}\}$ goes to infinity such that

$$d((f^k)^{l_{j_m}}(x), (f^k)^{l_{j_m}}(y)) = d(f^{n_{j_m}}(x), f^{n_{j_m}}(y)) \to 0.$$

Otherwise, there is $i_0 \in \{1, \ldots, k-1\}$ such that $n_j = kl_j - i_0$ for infinite j. Set the subsequence $\{l_{j_m} : n_{j_m} = kl_{j_m} - i_0\}$. Since f is uniformly continuous, for every $\epsilon > 0$ there is $\delta > 0$ such that

$$d((f^k)^{l_{j_m}}(x), (f^k)^{l_{j_m}}(y)) < \epsilon \text{ whenever } d(f^{kl_{j_m}-i_0}(x), f^{kl_{j_m}-i_0}(y)) < \delta,$$

which implies that for every $\epsilon > 0$ there is $M \in \mathbb{Z}$ such that

$$d((f^k)^{l_{j_m}}(x), (f^k)^{l_{j_m}}(y)) < \epsilon \text{ for every } l_{j_m} > M,$$

so $y \in P_{f^k}(x)$ proving (i).

Similarly, (ii) is a consequence of $P_{f\times g}(x, y) \subseteq P_f(x) \times P_g(y)$ for every $(x, y) \in X \times Y$, and this follows from the definitions of d^2 on $X \times Y$ and proximal cell.

Finally to prove (iii), suppose that h is the conjugacy homeomorphism between f and g. If g is not N-distal. Then, there exists $y \in Y$ such that $P_g(y)\backslash\{y\}$ has at least N points. Set $x = h^{-1}(y)$. We claim that $P_f(x)\backslash\{x\}$ has at least N points. Indeed, let $p_1, \ldots, p_N$ be distinct points in $P_g(y)\backslash\{y\}$. It follows that

$$d(g^{n_k^i}(y), g^{n_k^i}(p_i)) \to 0 \text{ as } k \to \infty$$

for $i = 1, \ldots, N$. Since h^{-1} is continuous, we have

$$d(h^{-1}(g^{n_k^i}(y)), h^{-1}(g^{n_k^i}(p_i))) = d(f^{n_k^i}(h^{-1}(y)), f^{n_k^i}(h^{-1}(p_i))) \to 0.$$

Thus $h^{-1}(p_i) \in P_f(x)\backslash\{x\}$ for every $i \in \{1, \ldots, N\}$. Therefore f is not N-distal. □

Remark 2.1.7. The above results are also valid for countably-distal homeomorphisms.

2.2 N-equicontinuity and N-distality

Now, we are going to investigate the relation between equicontinuity and N-distality. Since equicontinuous systems are examples of distal systems (see Theorem 1.1.5), every N-distal and equicontinuous homeomorphism must be distal.

In order to state a weaker form of equicontinuity we use the concept of N-diameter (for some $N \in \mathbb{N}^+$) defined by the authors in [39], which in turn were inspired by the definition of N-sensitivity given by Xiong in [59], namely,

Definition 2.2.1. Let X be a compact metric space and N be a positive integer. If A is a subset of X, define the N-*diameter* of A by

$$\operatorname{diam}_N(A) = \sup_{B \subseteq A} \{ \min_{x,y \in B} \{ d(x,y) \, : \, x \neq y \} \, : \, B \in \mathcal{C}_{N+1}(X) \}. \tag{2.2}$$

where $\mathcal{C}_N(X)$ denotes the set of subsets of X with N elements.

This concept satisfies the following properties

Lemma 2.2.2. *[39] Let X be a compact metric space and N a positive integer. If A,B are subsets of X. Then*

(i) $\operatorname{diam}_1(A) = \operatorname{diam}(A)$, *where* $\operatorname{diam}(A)$ *is the usual diameter of* A.

(ii) $\operatorname{diam}_N(A) \leqslant \operatorname{diam}_M(A)$, *whenever* $M \leqslant N$.

(iii) $\operatorname{diam}_N(A) \leqslant \operatorname{diam}_N(B)$, *whenever* $A \subseteq B$.

(iv) $\operatorname{diam}_N(A) = \operatorname{diam}_N(\bar{A})$.

(v) $\operatorname{diam}_N(A) = 0$ *if and only if* $\#A \leqslant N$.

(vi) $\frac{\operatorname{diam}(A)}{N} \leqslant \operatorname{diam}_N(A)$, *whenever* A *is connected.*

Proof. By direct computation we have

$$\operatorname{diam}_1(A) = \sup_{B \subseteq A} \{ \min_{x,y \in B} \{ d(x,y) \, : \, x \neq y \} \, : \, B \in \mathcal{C}_2(X) \} = \sup_{x,y \in A} \{ d(x,y) \} = \operatorname{diam}(A).$$

proving (i).

Observe that (ii) and (iii) follows from minimum and supremum definitions, respectively. Similarly, (iv) is a consequence of supremum properties.

Now to prove (v), first suppose that $\operatorname{diam}_N(A) = 0$. By Definition 2.2.1, we have

$$\min_{\substack{x_i, x_j \in B \\ \#B = N+1}} \{ d(x_i, x_j) \, : \, i \neq j \} = 0$$

for all $B \subseteq A$. It follows that $x_i = x_j$ for some $x_i, x_j \in B$, then $\#B \leqslant N$. Since B is arbitrary we conclude that $\#A \leqslant N$.

We will proof the contrapositive of the only if part. So, suppose that $\operatorname{diam}_N(A) > 0$. By Definition 2.2.1 there is $B \subseteq A$ with $\#B = N + 1$ such that

$$\min_{\substack{x_i, x_j \in B \\ \#B = N+1}} \{ d(x_i, x_j) \, : \, i \neq j \} > 0,$$

Thus $N + 1 = \#B \leqslant A$.

Finally we prove (vi), if $\mathrm{diam}(A) = 0$, there is nothing to prove. Otherwise, there is $x, x' \in A$ such that $d(x, x') > \mathrm{diam}(A) - \epsilon$ for every $\epsilon > 0$, by supremum's definition. Define $\varphi : A \to \mathbb{R}$ by $\varphi(y) = d(x, y)$. If follows that φ is continous, $\varphi(x) = 0$ and $\varphi(x') > \mathrm{diam}(A) - \epsilon$. Since A is connected, we have

$$[0, \mathrm{diam}(A) - \epsilon] \subseteq \varphi(A).$$

Hence there are $x_0 = x, x_2, \ldots, x_N$ such that

$$\varphi(x_i) = \frac{i}{N}(\mathrm{diam}(A) - \epsilon), \qquad \text{for all } i = 0, \ldots, N.$$

Thus, for all $i, j \in \{0, \ldots, N\}$ such that $i < j$, we have

$$\begin{aligned} d(x_i, x_j) &\geqslant d(x_j, x_0) - d(x_0, x_i) \\ &= \varphi(x_j) - \varphi(x_i) \\ &= \frac{j}{N}(\mathrm{diam}(A) - \epsilon) - \frac{i}{N}(\mathrm{diam}(A) - \epsilon) \\ &= \frac{j-i}{N}(\mathrm{diam}(A) - \epsilon) \\ &\geqslant \frac{\mathrm{diam}(A) - \epsilon}{N}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathrm{diam}_N(A) &\geqslant \min_{i,j \in \{0,\ldots,N\}} \{d(x_i, x_j) : i \neq j\} \\ &\geqslant \frac{\mathrm{diam}(A) - \epsilon}{N}. \end{aligned}$$

Since ϵ is arbitrary, we are done. $\square$

Example 2.2.3. *Consider $\mathbb{S}^1$ as a subset of $\mathbb{R}^2$ equipped with the induced metric. Then* $\mathrm{diam}_N(\mathbb{S}^1) = 2\sin(\frac{\pi}{N+1})$.

Proof. The problem to calculate 2.3 on $\mathbb{S}^1$, that is

$$\mathrm{diam}_N(\mathbb{S}^1) = \sup_{B \subseteq \mathbb{S}^1} \{ \min_{x,y \in B} \{d(x, y) \,:\, x \neq y\} \,:\, B \in \mathcal{C}_{N+1}(\mathbb{R}^2)\}. \tag{2.3}$$

Reduces to calculate the length of the side of a regular polygon of $N + 1$ points inscribed in $\mathbb{S}^1$. Indeed, the minimum of the distances between $N + 1$ points in $\mathbb{S}^1$ is the smallest length of the sides of the polygon connecting these points. The supremum over these minimums, is precisely the length of any regular polygon connecting $N + 1$ points in $\mathbb{S}^1$ that is equal to $2\sin(\frac{\pi}{N+1})$, as is illustrate in the Figure 2.2. $\square$

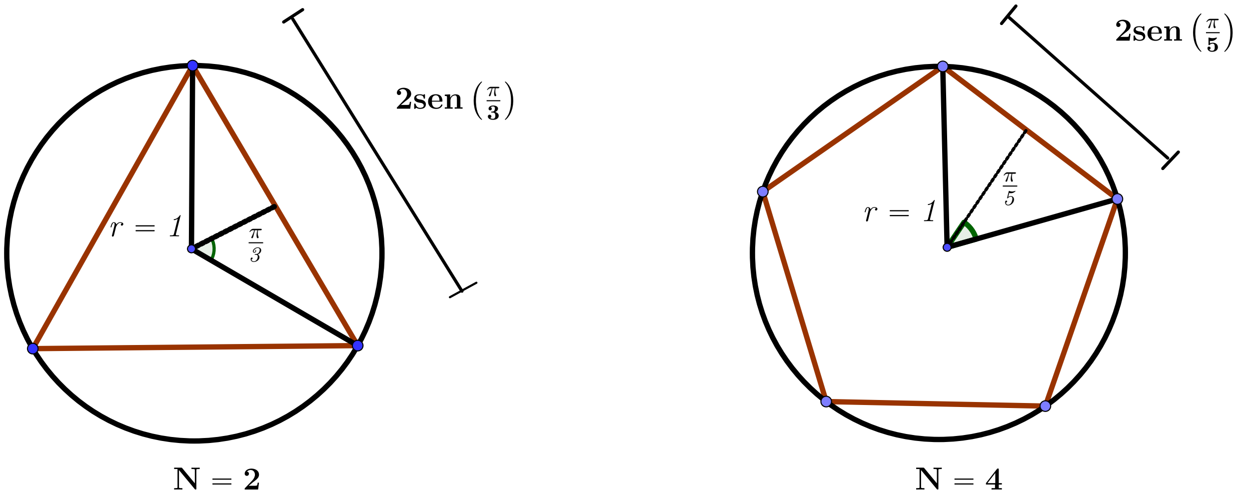

Figure 3 – $\operatorname{diam}_N(\mathbb{S}^1)$

Also we need the following notation for $\delta > 0$ and $x \in X$

$$R_\delta(x) = \{y \in X \ : \ d(f^m(y), f^m(x)) < \delta, \text{ for some } m \in \mathbb{Z}\}. \tag{2.4}$$

Notation $R_\delta^f(x)$ will be useful to indicate dependence of f. Note that

$$\begin{aligned} y \in R_\delta(x) \quad &\Leftrightarrow \quad d(f^m(y), f^m(x)) < \delta, \text{ for some } m \in \mathbb{Z} \\ &\Leftrightarrow \quad f^m(y) \in B_\delta(f^m(x)), \text{ for some } m \in \mathbb{Z} \\ &\Leftrightarrow \quad y \in f^{-m}\left(B_\delta(f^m(x))\right), \text{ for some } m \in \mathbb{Z} \\ &\Leftrightarrow \quad y \in \bigcup_{m \in \mathbb{Z}} f^{-m}\left(B_\delta(f^m(x))\right), \end{aligned}$$

then $R_\delta(x) = \bigcup_{m\in\mathbb{Z}} f^{-m}\left(B_\delta(f^m(x))\right)$. Clearly $R_\delta(x)$ is an open set and a superset of $B_\delta(x)$.

We can now define a generalization of equicontinuity. To motivate this definition we rewrite the classical definition of equicontinuity (see Definition 1.1.4). To be equicontinuous homeomorphism, it equivalent to say that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\operatorname{diam}(f^n(B_\delta(x))) < \varepsilon \text{ for all } x \in X \text{ and } n \in \mathbb{Z}.$$

This definition suggests the following one.

Definition 2.2.4. We say that f is N-*equicontinuous* (for some $N \in \mathbb{N}^+$) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\operatorname{diam}_N(f^n(R_\delta(x))) < \varepsilon \text{ for all } x \in X \text{ and } n \in \mathbb{Z}. \tag{2.5}$$

The 1-equicontinuous homeomorphisms are precisely the equicontinuous ones. In fact, by (i) and (iii) for $N = 1$ in Lemma 2.2.2, we have

$$\operatorname{diam}(f^n(B_\delta(x))) = \operatorname{diam}_1(f^n(B_\delta(x))) \leqslant \operatorname{diam}_1(f^n(R_\delta(x)))$$

for all $x \in X$ and $n \in \mathbb{Z}$. Thus, from N-equicontinuous definition we concluded that 1-equicontinuity implies equicontinuity.

Conversely, if $f : X \to X$ is equicontinuous. Let $\varepsilon > 0$ and $x \in X$, by equicontinuity, for $y, z \in X$ exists $\delta_1, \delta_2 > 0$ such that if

$$d\left(f^{k_1}(x), f^{k_1}(y)\right) < \delta_1 \text{ and } d\left(f^{k_2}(x), f^{k_2}(z)\right) < \delta_2,$$

for some $k_1, k_2 \in \mathbb{Z}$. Then

$$d\left(f^n(x), f^n(y)\right) < \frac{\epsilon}{2} \text{ and } d\left(f^n(x), f^n(z)\right) < \frac{\epsilon}{2},$$

for all $n \in \mathbb{Z}$. Set $\delta = \min\{\delta_1, \delta_2\}$. Hence it follows from the triangle inequality and supremum definition that

$$\operatorname{diam}_1(f^n(R_\delta(x))) = \sup_{y,z \in f^n(R_\delta(x))} \{d(y,z)\} = \sup_{y,z \in R_\delta(x)} \{d\left(f^n(y), f^n(z)\right)\} < \varepsilon,$$

for all $n \in \mathbb{Z}$. Therefore, f is 1-equicontinuous.

Clearly, every M-equicontinuous homeomorphism is N-equicontinuous homeomorphism for all $M \leqslant N$ by Lemma 2.2.2(ii). It follows that every equicontinuous homeomorphism is N-equicontinuous homeomorphism for every $N \geqslant 1$ and thus such dynamical systems exist. A more subtle problem is to find N-equicontinuous homeomorphisms which are not M-equicontinuous homeomorphisms for some $M \leqslant N$. Indeed, this question is answered positively in Example 2.2.7. Before we present this and a related example we first establish the following results.

Let $x \in X$, $n \in \mathbb{Z}$ and $\delta > 0$, we have

$$\begin{aligned} f^n(R_\delta(x)) &= f^n\left(\bigcup_{m\in\mathbb{Z}} f^{-m}\left(B_\delta(f^m(x))\right)\right) \\ &= \bigcup_{m\in\mathbb{Z}} f^{-(m-n)}\left(B_\delta(f^m(x))\right) \\ &= \bigcup_{m\in\mathbb{Z}} f^{-k}\left(B_\delta(f^{k+n}(x))\right), \text{ where } k = m-n \\ &= \bigcup_{m\in\mathbb{Z}} f^{-k}\left(B_\delta(f^k\left(f^n(x)\right))\right) \\ &= R_\delta(f^n(x)). \end{aligned} \tag{2.6}$$

Using it we obtain the lemma below.

Lemma 2.2.5. *Let $N \in \mathbb{N}^+$. The following properties are equivalent for every homeomorphism $f : X \to X$ on a compact metric spaces X:*

(i) f is N-equicontinuous.

(ii) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\operatorname{diam}_N(R_\delta(x)) < \varepsilon$ $\forall x \in X$.

Proof. Clearly (i) implies (ii) so it remains to prove that (ii) implies (i). To see this, let $\varepsilon > 0$ and $x \in X$, by (ii) for each $M \in \mathbb{Z}$ there is $\delta(M)$ such that

$$\operatorname{diam}_N(R_{\delta(M)}(f^M(x))) < \varepsilon.$$

Define

$$\delta = \inf_{M \in \mathbb{Z}} \{\delta(M) : \operatorname{diam}_N(R_{\delta(M)}(f^M(x))) < \varepsilon \wedge N < \# R_{\delta(M)}(f^M(x))\}.$$

Observe that $\delta > 0$. Indeed, if $\delta = 0$ then there exists a $M_0 \in \mathbb{Z}$ such that $\# R_{\delta(M_0)}(f^{M_0}(x)) < N$, a contradiction. Therefore, there is $\delta > 0$ such that

$$\operatorname{diam}_N(R_\delta(f^n(x))) < \varepsilon$$

for all $n \in \mathbb{Z}$. As x and ε are arbitrary we conclude from (2.6) that f is N-equicontinuous proving (i). $\square$

A direct application of the above Lemma is the following result that generalizes the Theorem 1.1.5.

Theorem 2.2.6. *N-equicontinuous homeomorphisms are N-distal.*

Proof. Let $f : X \to X$ be a N-equicontinuous homeomorphism of a compact metric space X. Suppose by contradiction that f is not N-distal. Then, there exists $x \in X$ such that $\# P(x) > N$. Thus let $A \in \mathcal{C}_{N+1}(X)$ be such that $x \in A$ and $A \subseteq P(x)$. Set $\varepsilon_n = \frac{1}{n}$ and let $0 < \delta_n \leqslant \varepsilon_n$ be given by the N-equicontinuity of f for each $n \in \mathbb{N}$. Notice that since any $y \in A$ is proximal to x, then

$$A \subseteq R_{\delta_n}(x)$$

for every $n \in \mathbb{N}^+$. Therefore by Lemma 2.2.5

$$\operatorname{diam}_N(R_{\delta_n}(x)) < \varepsilon_n$$

for every $n \in \mathbb{N}$. Since A is finite, then there exists $y \in A$ and a sequence $n_j \to \infty$ such that $d(x, y) < \frac{1}{n_j}$ for every $j \in \mathbb{N}$. It follows that $x = y$ and $\# A \leqslant N$ which is absurd. Therefore, f must be N-distal. $\square$

We now present some related examples. Combining the above Theorem with examples 2.1.4, 2.1.2 and 2.1.3.2.1 we obtain the following sentence.

Example 2.2.7. *There are N-equicontinuous homeomorphisms which are not $N-1$-equicontinuous for every $N \geqslant 2$.*

Proof. A simple computation shows that the N-distal homeomorphisms in Examples 2.1.4, 2.1.2 and 2.1.3.2.1 are also N-equicontinuous for $N = 2$, $N = 3$ and $N \geqslant 4$, respectively. As already verified these examples are not $N-1$-distal homeomorphisms. Consequently, are not $N-1$-equicontinuous homeomorphims by Theorem 2.2.6. □

The converse direction of the Theorem 2.2.6 motivate the question whether there are N-distal homeomorphims which are not N-equicontinuous. As is well known the answer for $N = 1$ is positive, as already mentioned an example can be found in [7, Theorem 5.14]. The answer also turns out to be positive for $N \geqslant 2$ by the following result. Even more, in the next remarkable Fürstenberg's example we deal with a more general problem on the existence of distal and minimal systems that are not necessarily N-equicontinuous.

Example 2.2.8. *There is a compact metric space X and a minimal distal homeomorphism $f : X \to X$ which is not N-equicontinuous for every positive integer N.*

Proof. In order to see this consider the homeomorphism defined by

$$\begin{array}{cccc} F: & \mathbb{T}^2 & \longrightarrow & \mathbb{T}^2 \\ & (x,y) & \mapsto & (x+\alpha \,(\mathrm{mod}\,1), y + x\,(\mathrm{mod}\,1)) \end{array}, \tag{2.7}$$

where $\alpha \in (0,1)\backslash\mathbb{Q}$. We view $\mathbb{T}^2$ represented as $[0,1) \times [0,1)$, the unit square with opposite sides identified and use the metric inherited from the Euclidean metric.

First, note that

$$\begin{array}{c} F^n(x,y) = (x + n\alpha(\mathrm{mod}\,1), y + nx + \sum_{j=0}^{n-1} j\alpha(\mathrm{mod}\,1)) \text{ and} \\ F^{-n}(x,y) = (x - n\alpha(\mathrm{mod}\,1), y - nx + \sum_{j=1}^{n} j\alpha(\mathrm{mod}\,1)) \end{array}$$

for all $n \geqslant 1$. Therefore, the distance between n-th iterate by F of two points (x,y) and (x',y') in $\mathbb{T}^2$ is

$$d(F^n(x,y), F^n(x',y')) = \begin{cases} \sqrt{(x-x')^2 + (y-y'+n(x-x'))^2} & \text{if } n \geqslant 0 \\ \sqrt{(x-x')^2 + (y-y'+n(x'-x))^2} & \text{if } n < 0 \end{cases}, \tag{2.8}$$

where the operations of the terms in parentheses are done in modulo 1.

We begin by verifying that F is distal. To see this, let (x_0, y_0) and (x_1, y_1) be distinct points in $\mathbb{T}^2$. If $x_0 \neq x_1$, then

$$d(F^n(x_0,y_0), F^n(x_1,y_1)) \geqslant d((x_0,y_0),(x_1,y_0)) \tag{2.9}$$

which is a positive constant. Analogously, if $x_0 = x_1$, then $y_0 \neq y_1$ and by (2.8) clearly

$$d(F^n(x_0,y_0), F^n(x_1,y_1)) = d((x_0,y_0),(x_1,y_1)). \tag{2.10}$$

In both cases 2.2 and 2.10 this hold for all $n \in \mathbb{Z}$. Thus, $((x_0, y_0), (x_1, y_1))$ is a distal pair. Since (x_0, y_0) and (x_1, y_1) are arbitrary we are done. Additionally, F is also minimal, see [25, Lemma 1.25] for details.

Finally, fix $N \in \mathbb{N}^+$, now we prove that F is not N-equicontinuous. For this purpose, let $\delta > 0$, in $\mathbb{T}^2$ consider $q = (0, 0)$ and $p_k = (\frac{\delta}{k}, 0)$ for $1 \leqslant k \leqslant N$. Define $B_0 = \{q\} \cup \{p_k \ : \ 1 \leqslant k \leqslant N\}$, then for each $n \in \mathbb{Z}$.

$$\begin{aligned}
\operatorname{diam}_N(F^n(R_\delta(q))) &= \sup_{\substack{B \in \mathcal{C}_{N+1}(X) \\ B \subseteq f^n(R_\delta(q))}} \{ \min_{\substack{z \neq w \\ z,w \in B}} \{d(z, w)\}\} \\
&\geqslant \min_{\substack{z \neq w \\ z,w \in B_0}} \{d(F^n(z), F^n(w))\} \\
&= \sqrt{\left(\frac{\delta}{M} (\operatorname{mod} 1)\right)^2 + \left(n\frac{\delta}{M} (\operatorname{mod} 1)\right)^2} \\
&\geqslant |n\frac{\delta}{M} (\operatorname{mod} 1)|,
\end{aligned}$$

for some $1 \leqslant M \leqslant N$. Since $\frac{\delta}{M} > 0$ we can find $n \in \mathbb{N}^+$ such that $|n\frac{\delta}{M} \operatorname{mod} 1| \geqslant \frac{1}{4}$. As N is arbitrary we are done. □

Thus, as a consequences of the Theorem 2.2.6, and Definitions 2.1.1 and 2.2.4 we obtain the following diagram which shows how the concepts of N-distality and N-equicontinuity interact.

$$\begin{array}{ccccc}
\text{Distality} & \Longrightarrow & N\text{-Distality} & \Longrightarrow & M\text{-Distality} \\
\Uparrow & & \Uparrow & & \Uparrow \\
\text{Equicontinuity} & \Longrightarrow & N\text{-Equicontinuity} & \Longrightarrow & M\text{-Equicontinuity}
\end{array}$$

for all $2 \leqslant N < M$. Moreover, the converse is not true in general for the Examples 2.1.2, 2.1.3.2.1, 2.1.4, 2.2.7 and 2.2.8.

2.3 Factors and extensions

Now we study how N-distality behaves under factors and extensions. In order to generalize the distal extensions (see Definition 1.1.8) to the setting of N-distal homeomorphisms, we introduce the following auxiliary definition.

Definition 2.3.1. Let $f : X \to X$ and $g : Y \to Y$ be homeomorphisms of compact metric spaces and $\pi : Y \to X$ a homomorphism from g to f. Let $y \in Y$ with $\pi(y) = x$ for some $x \in X$ we define and denote the *proximal cell* of y under π by

$$P^\pi(y) = \{z \in \pi^{-1}(x) \ : \ \inf_{n \in \mathbb{Z}} \{d(g^n(y), g^n(z))\} = 0\}.$$

Notation $P_g^\pi(y)$ will be used to indicate dependence on g if necessary.

We can now define the following notions. As in the case of distal homeomorphisms, we define a generalization of the N-distal notion to homomorphisms in such a way that the trivial homomorphism $(X, f) \to (\{p\}, f)$ is N-distal if and only if the system (X, f) is N-distal.

Definition 2.3.2. We say that homomorphism $\pi : Y \to X$ is N-*distal* if $P^\pi(y)$ has at most N points for every $y \in Y$. We say that g is a N-*distal extension* of f (or f is a N-*distal factor* of g) if there is a N-distal homomorphism π from g to f.

As we known (see Theorem 1.1.9) a distal extension of a distal homeomorphism is distal. Next we prove a generalization of this fact.

Theorem 2.3.3. *A M-distal extension of a N-distal homeomorphism is MN-distal.*

Proof. Let X and Y be compact metric spaces, $g : Y \to Y$ be a M-distal extension of a N-distal homeomorphism $f : X \to X$, with $\pi : Y \to X$ the N-distal homomorphism from g to f. Suppose by contradiction that g is not MN-distal. Then there is $y \in Y$ such that $P_g(y)$ has at lest $MN + 1$ elements. Let $p_1, \ldots, p_{MN}, p_{MN+1} = y$ be the different points in $P_g(y)$. As a consequence of the definition of proximal cell, we obtain

$$d(g^{n_k^i}(p_i), g^{n_k^i}(y)) \to 0 \text{ as } k \to \infty$$

for $i = 1, \ldots, NM$. Since π is continuous and $f \circ \pi = \pi \circ g$, we have

$$d(\pi(g^{n_k^i}(p_i))\pi(g^{n_k^i}(y))) = d(f^{n_k^i}(\pi(p_i)), f^{n_k^i}(\pi(y))) \to 0.$$

as $k \to \infty$ for $i = 1, \ldots, NM$. Since $\#P_f(\pi(y)) \leqslant N$, it follows that there are $p_{l_1}, \ldots, p_{l_{M+1}}$ different points such that $p_{l_1}, \ldots, p_{l_M} \in \pi^{-1}(p_{l_{M+1}})$. Thus $p_{l_1}, \ldots, p_{l_M} \in P_g^\pi(p_{l_{M+1}})$ and therefore there is a point $z = p_{l_{M+1}}$ in Y such that $\#P_g^\pi(z) > M$, a contradiction. □

We end this chapter by dealing with the problem of determining when a N-distal system has a non-trivial distal N-factor. Next, we use the Ellis semigroup (see Section 1.2) to obtain a criterion for existence of non-trivial N-distal factors for N-distal homeomorphisms.

Theorem 2.3.4. *Let f be a N-distal homeomorphism. If the Ellis semigroup $E(f)$ of f has a unique minimal ideal I, then f has a nontrivial N-distal factor. Moreover, if there is a continuous element in I, then this factor is distal.*

Proof. Suppose that $E(f)$ has a unique minimal ideal I. It is a classical fact that this condition is equivalent to the proximal relation "$\sim$" in X be an equivalence relation (see Theorem 1.2.12). Then define $Y = X/\sim$ to be the quotient space of X by proximality relation and let π denote the natural projection map. Let g be the homeomorphism induced on Y by f through the projection π. We notice that the conjugacy equation is trivially satisfied for g and f. It follows from the N-distality of f that $P_f^{\pi}(y)$ has at most N points for all $y \in Y$. Then g is a N-distal factor of f.

Since there is a continuous element in I and the proximal relation is an equivalence relation, we have that the proximal relation is closed (c.f. [35, Lemma 2.1]). It follows from [12, ch. I, Proposition 3.8] that Y is hausdorff. Therefore, Y is metrizable by [13, ch. IX, Proposition 10.17].

Next we prove that the homeomorphism g is distal. Indeed, suppose that $y, y' \in Y$ are distinct proximal points for g. Let us take $x \in \pi^{-1}(y)$ and $x' \in \pi^{-1}(y')$. By construction x and x' are distal. Compactness of Y implies that there are a sequence of $k \to \infty$ and a point $z \in Y$ such that $g^k(y), g^k(y') \to z$. We can assume by compactness of X that there exists $p, p' \in X$ such that $f^k(x) \to p$ a and $f^k(x') \to p'$.

We claim that p and p' are distal. Indeed, suppose that there is $i \to \infty$ and z' such that $f^i(p), f^i(p') \to z'$. Fix $\varepsilon > 0$ and i_0 such that $f^{i_0}(p), f^{i_0}(p') \in B\varepsilon(z')$. So there exists $\delta > 0$ such that

$$d(f^j(u), f^j(w)) < \epsilon \text{ whenever } d(u, w) \leqslant \delta$$

for $j = 0, ..., i_0$ and every $u, w \in X$. Take k big enough such that

$$f^k(x) \in B_\delta(p) \text{ and } f^k(x') \in B_\delta(p').$$

But this implies

$$d(f^{k+i_0}(x), f^{k+i_0}(x')) \leqslant 4\varepsilon.$$

Remember x and x' are distal and therefore it is impossible since $\varepsilon > 0$ was chosen arbitrarily. Thus p and p' must be distal.

Finally we must have $\pi(p) = \pi(p') = z$ by continuity of π, but this is impossible since p and p' cannot be in the same equivalent class. $\square$

Chapter 3

Transitive N-distal Systems and Expansiveness

In this chapter we study the relation betwenn N-distal property with transitivity and expansivity. Section 3.1 deals with the transitive N-distal systems, some results are proved in particular a generalization of the Proposition 1.1.12. Section 3.2 contains a brief summary of the close relation between distality and expansivity and present an example of a cw-distal expansive homeomorphism which is not N-distal for every positive integer N.

3.1 Transitive N-distal Systems

In this section we study some consequences of the topological transitivity for N-distal systems. In particular, we use the Ellis semigroup theory to prove the Theorem 3.1.5.

Next results deals with the existence of periodic orbits. But previously we need the following proposition.

Proposition 3.1.1. *Let $f : X \to X$ be a N-distal homeomorphism for $N \geqslant 2$. If $x \in X$ is periodic for f, then x is a distal point.*

Proof. Let x be a periodic point of f with period T. If $P(x) \neq \{x\}$, take $y \in P(x)\backslash\{x\}$. Then, there is a sequence $\{n_k\}$ goes to infinity such that

$$d(f^{n_k}(x), f^{n_k}(y)) \to 0.$$

Since x is periodic and $y \neq x$ then y cannot be periodic. Moreover, since the orbit of x is finite, we can assume that $f^{n_k}(x) = p$ for any $k \in \mathbb{N}$ and some $p \in \mathcal{O}(x)$. Last assumption implies that $n_k - n_{k'}$ is a multiple of T for any $k, k' \in \mathbb{Z}$. For any $k \in \mathbb{N}$ we set $y_k = f^{n_k}(y)$. We claim that y_k is proximal to p for every k. Indeed, fix k and define $m_j^k = n_j - n_k$ for $j \geqslant k$. Then we obtain that

$$d(f^{m_j^k}(y_k), f^{m_j^k}(p)) = d(f^{n_j}(y), f^{n_j}(p)) = d(f^{n_j}(y), p) \to 0$$

and this proves the claiming.

Finally, notice that y cannot be a periodic. Thus, we have infinitely many y_k's and therefore $\#P(p) = \infty$, a contradiction. □

As a consequence we show that the only way a transitive N-distal system can possess a periodic orbit is if the whole space is a periodic orbit.

Proposition 3.1.2. *Let $f : X \to X$ be a pointwise transitive N-distal homeomorphism which is not distal. Then either X is a periodic orbit, or f has not periodic points.*

Proof. Suppose that X is not a periodic orbit and let p be a periodic point. Suppose $x \in X$ is a transitive point. Since $\mathcal{O}(x)$ is dense, then for any point of $q \in \mathcal{O}(p)$ we can find a sequence $n_k \to \infty$ such that $f_{n_k}(x) \to q$. Let T denote the period of p. Since f is continuous, for every $k \in \mathbb{N}$ we can find $0 < \delta_k < \frac{1}{k}$ such that

$$d(f^i(x), f^i(y)) \leqslant \frac{1}{k} \text{ whenever } d(x, y) \leqslant \delta_k$$

for $|i| \leqslant T$. Up to take a subsequence of n_k, we can suppose that $f^{n_k}(x) \in B_{\delta_k}(p)$ for any $k \in \mathbb{N}$.

Now, by the choice of δ_k we have that

$$f^i(x_k) \in B_{\frac{1}{k}}(f^i(p)).$$

for $i = 0, 1, ..., T$. By the euclidean algorithm any n_k can be wrote as $n_k = q_k T + r_k$ with $q_k \in \mathbb{N}$ and $0 < r_k < T$. Since the orbit of p is finite, we can assume that $r_k = c$ for every k. Put $x_k = f^{n_k}(x)$ for every $k \in \mathbb{N}$.

We claim that the points x_k are proximal to p. Indeed, fix k and for any $j > k$ define $m_j^k = n_j - n_k + T = T(q_j - q_k + 1)$. Then we have that

$$\begin{aligned} d(f^{m_j^k}(x), f^{m_j^k}(p)) &= d(f^{n_j - n_k + T}(x_k), f^{n_j - n_k + T}(p')) \\ &= d(f^{n_j}(x), f^{T(q_j - q_k + 1)}(p)) \\ &= d(f^{n_j + T}(x), p) \\ &\leqslant \frac{1}{j}. \end{aligned}$$

Since this, in addiction to the fact that $\{x_k\}$ is an infinite set implies that $\#P(p) = \infty$, we obtain a contradiction by Proposition 3.1.1. This proves the result. $\square$

As it is well known, a distal homeomorphism is pointwise transitive if and only if is minimal (see Proposition 1.1.12). This is not valid for N-distal homeomorphism by the Example 2.1.2. Nevertheless, the transitive 3-distal homeomorphism given in this example has two minimal subsystems. It is then natural to ask if there is a relation between the number of minimal subsystems of a dynamical system and the transitive N-distal property. Indeed, the Theorem 3.1.5 gives an answer for this question.

An old result by Auslander [6, Lemma 2] that we will need is stated below. For completeness we include its proof here.

Lemma 3.1.3. *Let $f : X \to X$ be a homeomorphism on a compact metric space X and $x \in X$. If $A \subset \overline{\mathcal{O}(x)}$ be a minimal set, then there is $y \in A$ such that $y \in P(x)$.*

Proof. Let $E(f) = E$ be the Ellis semigroup of f. We claim that $H = \{h \in E \ : \ h(x) \in A\}$ is a minimal left ideal of E. Indeed, let $g \in E$ and $h \in H$. Then, there is a sequence $n_k \to \infty$ such that $f^{n_k} \to g$. It follows that

$$f^{n_k}(h(x)) \to g(h(x)) = (g \circ h)(x).$$

Since A is f-invariant, $f^{n_k}(h(x)) \in A$. As A is closed we have $(g \circ h)(x) \in A$. So $g \circ h \in H$. Hence, H is a left ideal. Moreover, the minimality condition for H follows from that of A. The claim is proved.

We have from Proposition 1.2.9(iii), 1.2.9(vi) and the claim that there is a minimal idempotent element k in H. Let $u(x) = y$, then $u(y) = u^2(x) = u(x)$. Therefore $y \in P(x)$ by 1.2.11(iii), which completes the proof. $\square$

Using this lemma we obtain the proposition below.

Proposition 3.1.4. *If f is a N-distal homeomorphism, then $f|_{\overline{\mathcal{O}(x)}}$ has at most $N-1$ proper minimal subsystems.*

Proof. First notice that two minimal subsets $A, B \subset X$ have non-empty intersection then they must be equal. Now, suppose f is N-distal and fix $x \in X$. Let us analyze the subsystem $f|_{\overline{\mathcal{O}(x)}}$. If f is distal then $\overline{\mathcal{O}(x)}$ is minimal [7, Corollary 5.7] or f is minimal, in both cases we are done. If it is not minimal, it is well known that there exists a non-trivial minimal subset $A \subset \overline{\mathcal{O}(x)}$. By Lemma 3.1.3 we have that there exists a $y \in A$ proximal to x. Clearly $x \neq y$. The latter fact is valid for any minimal subset of $\overline{\mathcal{O}(x)}$. Thus N-distality implies that there are at most $N-1$ minimal subsets on $\overline{\mathcal{O}(x)}$ and the proposition is proved. $\square$

A direct consequence of the preceding Proposition is the following N-distal version for the one given in [7]. Indeed, we just need to notice that if a system is transitive there exists some point $x \in X$ such that $\overline{\mathcal{O}(x)} = X$.

Theorem 3.1.5. *A transitive N-distal homeomorphism has at most $N-1$ minimal proper subsystems.*

3.2 Expansiveness vs distality

In this section we proceed to study the relation between N-distality and expansivity.

It is a classical result that a distal system cannot be expansive if the phase is sufficiently rich (see Proposition 1.1.13). We ask if the same is true for the weaker forms of distality and expansiveness. Actually, it is answered by the authors in [4] when the phase space has positive topological dimension. Before stating this result precisely, let us recall some weaker notions of distality and expansiveness.

The δ-*dynamical ball* centered at x (or δ-Bowen ball) is the set defined by

$$\Gamma_\delta(x) = \{y \in X \,:\, d(f^n(x), f^n(y)) < \delta, \forall n \in \mathbb{Z}\}.$$

Clearly,

$$\Gamma_\delta(x) \subseteq B_\delta(x) \subseteq R_\delta(x)$$

for all $x \in X$ and $\delta > 0$.

We can now define the notions of n-expansiveness and countably-expansiveness were defined in [42] and [43] respectively, and redefine expansiveness [54] in terms of the dynamical ball. Namely

Definition 3.2.1. Let $f : X \to X$ be a homeomorphism of a metric space. We say that

1. f is *expansive* if there exists $\delta > 0$ such that $\Gamma_\delta(x) = \{x\}$ for every $x \in X$.
2. f is N- *expansive* if there exists $\delta > 0$ such that $\#\Gamma_\delta(x) \leqslant N$ for every $x \in X$.
3. f is *countably-expansive* if there exists $\delta > 0$ such that $\Gamma_\delta(x)$ is countable for every $x \in X$.

In each case the number $\delta > 0$ is called an *expansiveness constant* for f.

In [34] incorporated the expansive systems into the Continuum Theory. In fact, Kato recognized that the equivalently a homeomorphism $f : X \to X$ of a metric space is expansive if there is $\delta > 0$ such that if $C \in 2^X$

$$\operatorname{diam}(f^n(C)) \leqslant \delta \text{ for every } n \in \mathbb{Z} \Rightarrow \operatorname{diam}(C) = 0. \tag{3.1}$$

where $\operatorname{diam}(C) = \sup\{d(x,y) : x, y \in C\}$ denotes the diameter of C and 2^X denotes the set of subsets of X.

Recall that a *continuum* is a nonempty, compact, connected metric space and *subcontinuum* is a subset which is itself a continuum under the induced topology. We say that a continuum is trivial if it is a singleton (cf. [45]). By restricting C to the class of nonempty continuum subsets of X Kato introduced the notion of continuum-wise expansive homeomorphisms and studied several properties [34].

Definition 3.2.2 (cf. [34])**.** A homeomorphism $f : X \to X$ of a metric space is *cw-expansive* if there exists $\delta > 0$ such that if a non-empty continuum $C \subseteq X$ satisfies $\operatorname{diam}(f^n(C)) < \delta$ for every $n \in \mathbb{Z}$, then C is a singleton.

Clearly, expansive implies N-expansive and N-expansivity implies countably-expansivity. The following proposition completes this hierarchy.

Proposition 3.2.3. *Any countably-expansive homeomorphism is cw-expansive.*

Proof. Let $\delta > 0$ be the countably-expansivity constant of f. If a continuum C satisfies $\operatorname{diam}(f^n(C)) < \delta$ for every $n \in \mathbb{Z}$, then $C \subset \Gamma_\delta(x)$ for any $x \in C$. Thus, C must to be countable, and this implies that C is a singleton. $\square$

Previous proposition allows us to classify the levels of expansiveness accordingly the following hierarchy.

$$Expansivity \Rightarrow N - Expansivity \Rightarrow Countably - Expansivity \Rightarrow cw - Expansivity.$$

In contrast, the converse is not true in general(see for instance [42] and [34] for examples).

On the other hand. In [4] the authors incorporated distal systems into the Continuum Theory in the same way that Kato. They noted that the definition 3.2.2 holds if we replace $\operatorname{diam}(f^n(C))$ by $\sup_{n\in\mathbb{Z}} \operatorname{diam}(f^n(C))$. Also they noticed that to be distal homeomorphism is then equivalent to say that if $C \in 2^X$

$$\inf_{n\in\mathbb{Z}} \operatorname{diam} f^n(C) = 0 \Rightarrow \operatorname{diam}(C) = 0. \tag{3.2}$$

If we incorporate $\sup_{n\in\mathbb{Z}}$ in 3.1, the only difference between 3.1 and 3.2 is that the supremum in the former was replaced by infimum in the latter.

By restricting C to the class of nonempty continuum subsets of X the authors [4] introduced the notion of continuum-wise distal homeomorphisms.

Definition 3.2.4 (cf. [4, Definition 1.1]). A homeomorphism of a compact metric space $f : X \to X$ is *cw-distal* if every subcontinuum $C \subset X$ satisfying $\inf_{n\in\mathbb{Z}} \operatorname{diam}(f^n(C)) = 0$ is degenerated, i.e. reduces to a singleton.

It is easy to see that for distality we have the following hierarchy.

$$Distality \Rightarrow N - Distality \Rightarrow Countably - Distality \Rightarrow cw - Distality.$$

For the examples 2.1.3.2.1 and 2.1.4 the converse of the first two implications is false in general. Also, the converse of the last implication do not always hold,

Example 3.2.5 (cf. [4, Example 1.7]). *There are cw-distal homeomorphisms which are not countably distal.*

Proof. Consider $X = [0,1] \times C$ where C is the ternary Cantor set of $[0,1]$. By [31] there is a homeomorphism $g : C \to C$ such that

$$g(0) = 0, \; g(1) = 1 \; \text{ and } \; \{g^n(y) : n \in \mathbb{Z}\}$$

is dense in C for every $y \in C\backslash\{0,1\}$. Define

$$\begin{array}{cccc} f: & X & \longrightarrow & X \\ & (x,y) & \mapsto & (x, g(y)) \end{array}.$$

Clearly, $P(x,y) \subseteq x \times C$ and so $P(x,y)$ is totally disconnected for every $(x,y) \in X$. It follows from [4, Theorem 1.3] that f is *cw*-distal.

On the other hand, since $g(0) = 0$ and $\{g^n(y) : n \in \mathbb{Z}\}$ is dense in C for every $y \in C\backslash\{0,1\}$, we get that $P(x,0) = x \times (C\backslash\{0,1\})$ is uncountable for every $x \in [0,1]$. Therefore, f is not countably distal. □

Next results gives us a distinction between all these levels of distality and expansiveness.

Theorem 3.2.6 (cf. [4, Theorem 1.2]). *A cw-expansive homeomorphism of a compact metric space of positive topological dimension cannot be cw-distal.*

Despite above result, we cannot distinguish between *cw*-expansivity and *cw*-distality in zero dimension. Indeed, by definition any system in a totally disconnected space is *cw*-distal and *cw* expansive. But as we see in the following example, as an application of the Theorem 3.1.5, we cannot say the same for N-distal and N-expansive.

Example 3.2.7. *There is cw-distal expansive homeomorphism which is not N-distal for every positive integer N.*

Proof. Let $\Sigma^2 = \{0,1\}^{\mathbb{Z}}$ with the metric $d(s,s') = \frac{1}{2^n}$ where $n = \min\{|n| : s_n \neq s'_n\}$ and $d(s,s') = 0$ if $s_i = s'_i\ \forall i \in \mathbb{Z}$. The shift map $\sigma : \Sigma^2 \to \Sigma^2$ is defined by $\sigma((s_i)) = (s_{i+1})$. It is well known that the shift is a transitive and expansive system. Since Σ^2 is totally disconnected it it also *cw*-distal. On the other hand, σ has infinitely many periodic orbits and therefore it cannot be N-distal by Theorem 3.1.5. $\square$

Chapter 4

Topological Entropy

In [48] W. Parry proved that distal homeomorphisms on a compact metric space have zero entropy. The purpose of this chapter is to extend this result to N-distal homeomorphisms. In section 4.1 some previous results are stated. In section 4.2 we prove the main theorem of this chapter.

4.1 Previous Lemmas

Before stating the main result to be proved in this chapter. In this section we give some necessary previous results and facts.

Lemma 4.1.1. *Let $f : X \to X$ be a minimal N-distal homeomorphism on a compact metric space X and μ a non-atomic f-invariant measure. There is a partition ξ such that*

$$\bigvee_{j=0}^{\infty} f^{-j}(\xi) = \epsilon_N,$$

where ϵ_N is the partition of X in sets with N or less elements. Furthermore, $H_\mu(\xi) < \infty$.

Proof. By Sierpiński's Theorem A.0.8 we can fix $0 < r < \frac{1}{e}$ and chose a sequence of open sets

$$X = S_0 \supseteq S_1 \supseteq \cdots \supseteq S_n \supseteq \cdots$$

Such that

$$\mu(S_n) \leqslant r^n \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad \bigcap_{n=0}^{\infty} S_n = \{z\} \quad \text{for some} \quad z \in X.$$

Define $\xi = \{E_0, E_1, \ldots\}$, where

$$E_0 = \{z\} \cup (S_0 \backslash S_1) \quad \text{and} \quad E_n = S_n \backslash S_{n+1} \quad \text{for all} \quad n \in \mathbb{N}^+. \tag{4.1}$$

Clearly ξ is a partition of X. Now we claim that

$$\bigvee_{j=0}^{\infty} f^{-j}(\xi) = \epsilon_N,$$

where ϵ_N is the partition of X in sets with N or less elements. Indeed, take $B \in \bigvee_{j=0}^{\infty} f^{-j}(\xi)$ and $x \in B$. If $y \in B$, then $f^n(x), f^n(y) \in E_{i_n}$ for the same sequence. In fact, since

$$\bigvee_{j=0}^{\infty} f^{-j}(\xi) = \{E_{i_0} \cap f^{-1}(E_{i_1}) \cap \cdots \cap f^{-n+1}(E_{i_{n-1}}) \cap \cdots \ : \ E_{i_j} \in \xi\},$$

it follows that $x, y \in B \subseteq f^{-n}(E_{i_n})$ for some sequence $(i_0, i_1, \ldots)$.

By the minimality of f, for each $m \in \mathbb{N}$ there is $k(m)$ such that $f^{k(m)}(x) \in S_m$. Then $f^{k(m)} \in E_{i_{k(m)}} \subseteq S_m$ for all $i_{k(m)} \geqslant m$, by (4.1). We have $\operatorname{diam}(S_n) \to 0$ when $n \to \infty$ because $S_n \to \{z\}$ when $n \to \infty$, therefore

$$\inf_n \{d\left(f^n(x), f^n(y)\right)\} = 0.$$

Hence $y \in P(x)$ and then $\#(B) \leqslant \#(P(x)) \leqslant N$, by N-distality of f. The claim is proved.

Moreover, since the function $-x\log(x)$ is increasing on $(0, \frac{1}{e})$. It follows that

$$H_\mu(\xi) \leqslant -\sum_n \mu\left(S_n\right) \log\left(\mu\left(S_n\right)\right) \leqslant -\sum_n e^{-n} \log\left(e^{-n}\right) = \sum_n n e^{-n} = \frac{e}{(e-1)^2} < \infty. \quad (4.2)$$

□

Remark 4.1.2. For non atomic measures $\epsilon_N = \epsilon \mod \mu$ (see Definition A.0.9), where ϵ and ϵ_N are the partition of X into singletons and a partition of X into sets with N or less elements, respectively. Then, the Lemma above can be rewrite saying that for f satisfying the conditions stated above there is a generator ξ (see Definition 1.3.3).

Finally consider a measure-preserving dynamical system $(X, \mathcal{B}, \mu, f)$. If μ is atomic and ergodic then there is $x \in X$ such that $\mu(x) > 0$ since $\mathcal{O}(x)$ is invariant and $\mu(\mathcal{O}(x)) \geqslant \mu(x) > 0$, then $\mu(\mathcal{O}(x)) = 1$. Therefore we have proved:

Lemma 4.1.3. *Let $(X, \mathcal{B}, \mu, f)$ be a measure-preserving dynamical system. If μ is atomic and ergodic then μ is concentrated on a single f orbit.*

4.2 Topological Entropy of N-distal homeomorphisms

Now we are in position to prove the main result of this chapter in which $AP(f)$ and $\Omega(f)$ denotes the set of almost periodic points and the set of non-wandering points of f, respectively.

Theorem 4.2.1. *Let $f : X \to X$ be a N-distal homeomorphism on a compact metric space X. If $\Omega(f) \subseteq AP(f)$, then f has zero entropy.*

Proof. We first prove the Theorem for minimal systems and later we will show how to remove this hypothesis.

By Remark 1.3.7 it is sufficient to prove that the metric entropy of f is zero for all ergodic f-invariant measure. Let μ be an ergodic f-invariant measure. By Lemma 4.1.3, if μ is atomic then it must be supported on a periodic orbit or a fixed point and therefore its entropy is null, thus we will assume μ is non-atomic.

By Lemma 4.1.1 there is a partition ξ such that $\bigvee_{j=0}^{\infty} f^{-j}(\xi) = \epsilon_N$, where ϵ_N is the partition of X in sets with N or less elements and $H_\mu(\xi) < \infty$.

Moreover, as previously remarked 4.1.2, since μ is non-atomic, $\epsilon_N = \epsilon \mod \mu$. Thus, ξ is a generating partition (see Definition 1.3.3).

Then, ξ satisfies the hypotheses of Kolmogorov-Sinai Theorem 1.3.8. We can use it and take r as in Lemma 4.1.1 to obtain

$$\begin{aligned}
h_\mu(f) = h_\mu(f, \xi) &\leqslant H_\mu(\xi) \\
&= -\sum_{n=0}^{\infty} \mu(E_n)\log(\mu(E_n)) \\
&= -\mu(E_0)\log(\mu(E_0)) - \sum_{n=1}^{\infty} \mu(E_n)\log(\mu(E_n)) \\
&\leqslant -\log(\mu(E_0)) - \sum_{n=1}^{\infty} r^n \log(r^n) \\
&= -\log(\mu(E_0)) - \sum_{n=1}^{\infty} n r^n \log(r) \\
&= -\log(\mu(E_0)) - \frac{r}{(1-r)^2}\log(r) \\
&\leqslant \log\left(\frac{1-r}{1-2r}\right) - \frac{1}{(1-r)^2} r \log(r).
\end{aligned}$$

As the last expression converges to zero when r goes to zero, we have $h(f) = 0$.

We now deal with the general case. If f is not minimal. Let ν be an f-invariant measure. It is well-known that $h_\nu(f) = h_\nu(f|_{\Omega}(f))$ (cf. [58, Theorem 6.15]). Since $\Omega(f) \subseteq AP(f)$, it follows from [7, Corollary 1.10] that $\Omega(f) = \bigcup M_\lambda$, where in the previous (necessarily disjoint) union each M_λ is a minimal subset. Moreover, $\eta = \{M_\lambda\}$ is a measurable partition of $\Omega(F)$ by minimality of each M_λ and second countable property of X. Then there exists a family of measures $\{\nu_\lambda\}$ decomposing ν. According to the above case, we have $h_{\nu_\lambda}(f|_{M_\lambda}) = 0$. Therefore,

$$h_\nu(f|_{\Omega(f)}) = \int_{\Omega(f)_\eta} h_{\nu_\lambda}(f|_{M_\lambda}) d\nu_\eta = 0,$$

where $\Omega(f)_\eta$ denotes the factor space of X with respect to η, and ν_η is the factor measure on $\Omega(f)_\eta$. This completes the proof. $\square$

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Appendix

APPENDIX A

Measure Theory Elements

In this chapter some definitions and results of the measure theory that will be used throughout the work are presented.

Definition A.0.1 (cf. [11, Vol I pp.4-5])**.** Let X be a set and $\mathcal{B} \subseteq \mathcal{P}(X)$, then

1. $\mathcal{B}$ is an *algebra* if $X \in \mathcal{B}$ and $A \backslash B \in \mathcal{B}$ whenever $A, B \in \mathcal{B}$.
2. An algebra is called *σ-algebra* if is closed under countable unions, i.e.
$$\bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$$
for any sequence of sets $\{B_n\}$ in $\mathcal{B}$.
3. A *measurable space* is a pair $(X, \mathcal{B})$ consisting of a set X and a σ-algebra $\mathcal{B}$ of subsets of X. The elements of $\mathcal{B}$ are called *measurable sets*.

Proposition A.0.2 (cf. [11, Proposition 1.2.6])**.** *Let X be a set. For any family $\mathcal{F}$ of subsets of X there is a unique σ-algebra generated by $\mathcal{F}$. Defined by*

$$\sigma(\mathcal{F}) = \bigcap_{\mathcal{F} \subset \mathcal{A}}^{\infty} B_n \in \mathcal{B},$$

the intersection of all σ-algebras of subsets of the space X containing $\mathcal{F}$.

Definition A.0.3 (cf. [11, Vol II p. 10])**.** The *Borel σ-algebra* on a topological space a X be a topological is the σ-algebra generated by all open sets. The elements of the Borel σ-algebra are called the *Borel sets* of X.

It is clear that the Borel σ-algebra is generated by all closed sets, too.

Definition A.0.4 (cf. [11, Vol I pp. 9-10])**.**

1. A real-valued set function μ on a class of sets $\mathcal{A}$ is called *countably additive* if
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu\left(A_n\right)$$
for all pairwise disjoint sets A_n in $\mathcal{A}$ such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. A countably additive set function defined on an algebra is called a *measure*.

2. A countably additive measure μ on a σ-algebra of subsets of a space X is called a *probability measure* if $\mu \geqslant 0$ and $\mu(X) = 1$.

Definition A.0.5 (cf. [11, Definition 1.3.5])**.** A triple $(X, \mathcal{B}, \mu)$ is called a *measure space* if μ is a nonnegative measure on a σ-algebra $\mathcal{B}$ of subsets of the set X. If μ is a probability measure, then $(X, \mathcal{B}, \mu)$ is called a *probability space*.

Definition A.0.6 (cf. [11, Definition 1.12.7])**.** A measurable set A is called an *atom* of the measure μ if $\mu(A) > 0$ and every measurable subset of A has measure either 0 or $\mu(A)$. A measure μ is called *non-atomic* if it has not atoms.

Clearly, a measure μ is non-atomic if for every measurable set with $\mu(A) > 0$ there is E a measurable subset of A such that $\mu(A) > \mu(E) > 0$.

Remark A.0.7 (cf. [11, Vol II p. 136])**.** A non-atomic measure has no points of positive mass, and conversely for regular Borel probability measures on compact Hausdorff spaces (cf. [11, Vol II, p. 136]). Further a non-atomic measure has no finite sets of positive measure.

Sierpiński's Theorem A.0.8 (cf. [11, Corollary 1.12.10])**.** *Let μ be a non-atomic measure. Then, μ is surjective onto* $[0, 1]$.

Definition A.0.9 (cf. [57, p. 435])**.** Given subsets ξ_1 and ξ_2 of the σ-algebra $\mathcal{B}$. We write $\xi_1 \subseteq \xi_2 \mod \mu$, if for every $E_1 \in \xi_1$ there exists $E_2 \in \xi_2$ such that $\mu(E_1 \triangle E_2) = 0$. ($A \triangle B$ denotes the symmetric difference of the sets A, B.) Thereby, we write $\xi_1 = \xi_2 \mod \mu$, if both inclusions hold $\mod \mu$.

Remark A.0.10 (cf. [11, Vol I, p. 53])**.** The expression $\mu(A \triangle B)$ used in the above definition, is also defined as
$$d(A, B) = \mu(A \triangle B),$$
where μ is a bounded nonnegative additive set function on an algebra $\mathcal{A}$ and $A, B \in \mathcal{A}$. The function d is called the Fréchet–Nikodym metric.

Definition A.0.11 (cf. [11, Definition 2.1.3])**.** Let $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ be two spaces with σ-algebras. A transformation $f : X_1 \to X_2$ is called *measurable with respect to the pair* $(\mathcal{A}_1, \mathcal{A}_2)$ (or $(\mathcal{A}_1, \mathcal{A}_2)$*-measurable*) if
$$f^{-1}(B) \in \mathcal{A}_1 \quad \text{for all} \quad B \in \mathcal{A}_2.$$

In the case where $(X_1, \mathcal{A}_1) = (X_2, \mathcal{A}_2)$, f is called *measurable*.