INSTITUTO DE MATEMÁTICA
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On the Controllability and Unique Continuation Property for a Class of Dispersive Models

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Rio de Janeiro, Brasil

14 de setembro de 2022

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Tese de doutorado apresentada ao Programa de Pós-graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Doutor em Matemática

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Supervisor: Ademir Fernando Pazoto

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À minha mãe Anolfi Vieira Pineda, às minhas avós Elida Pineda e Rocio Perez (In memorian) e ao meu avô Dario Vieira (In memorian), que apoiaram todas as minhas iniciativas.

CIP - Catalogação na Publicação

S636c

Soto Vieira, Miguel Dario
Controllability and Unique Continuation Property for a Class of Dispersive Models / Miguel Dario Soto Vieira. -- Rio de Janeiro, 2022.
102 f.

Orientador: Ademir Fernando Pazoto.
Tese (doutorado) - Universidade Federal do Rio de Janeiro, Instituto de Matemática, Programa de Pós Graduação em Matemática, 2022.

1. Sistema de Boussinesq. 2. Controlabilidade. 3. Expansão de Fourier. 4. Problema de momentos. 5. Propriedade de continuação única.. I. Pazoto, Ademir Fernando, orient. II. Título.

Elaborado pelo Sistema de Geração Automática da UFRJ com os dados fornecidos pelo(a) autor(a), sob a responsabilidade de Miguel Romeu Amorim Neto - CRB-7/6283.

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Rio de Janeiro
Setembro/2022

Acknowledgements

À minha família, meu incondicional sustento e apoio. Este trabalho significa para eles o mesmo que para mim.

Sou eternamente grato pelo apoio e muita paciência que teve meu orientador Ademir Fernando Pazoto. Apesar da pandemia, esteve presente para me ajudar.

Aos meus amigos da sala de estudos da pós-graduação do IM.

À minha Namorada Graccyela, que nunca duvidou que eu poderia terminar este processo.

A todo o pessoal do IM-UFRJ (limpeza, secretaria, biblioteca, professores, etc.) que fazem um grande trabalho criando um ótimo ambiente de estudo, aconchegante e confortável.

Ao IM-UFRJ, pela oportunidade de fazer parte dele.

A Deus, por estar sempre comigo, guiando meus passos.

À Capes, ao CNPq e COLCIENCIAS (Colômbia) pelo suporte financeiro.

Abstract

O estudo dos fenômenos ondulatórios que surgem em meios dispersivos é de amplo interesse científico e pertence a uma moderna linha de pesquisa que é importante, tanto cientificamente, quanto para potenciais aplicações. O progresso no desenvolvimento de modelos matemáticos tornou possível compreender tais fenômenos em campos bastante distintos e resolver problemas que vem à tona nas discussões. Neste trabalho, o objetivo é avançar no estudo de problemas de valor inicial e de contorno explorando a dinâmica de modelos dispersivos usando análise matemática do ponto de vista da controlabilidade e continuação única. Considerações serão dadas para um sistema de Boussinesq, para a equação de Kawahara e para uma equação Korteweg-de Vries - Benjamin-Bona-Mahony (equação KdV-BBM) de ordem superior, definida em um domínio periódico. Primeiro provamos que o sistema de Boussinesq é exatamente controlável com controles atuando nas condições de fronteira. Em seguida, a propriedade de controlabilidade nula da equação de Kawahara é obtida por meio de um controle interno. Finalmente, provamos um resultado de continuação única para uma equação KdV-BBM de ordem superior.

Palavras-chave: Sistema de Boussinesq, controlabilidade, expansão de Fourier, análise não harmônica, Korteweg-de Vries, problema de momentos, propriedade de continuação única.

Abstract

The study of wave phenomena arising in dispersive media is of broad scientific interest and pertains to a modern line of research which is important both scientifically and for potential applications. Progress in the development of mathematical models has made it possible to understand such phenomena in quite distinct fields and to solve problems that come to the fore. In this work, the goal is to advance the study of the initial-boundary value problems exploring the dynamics of dispersive models by using mathematical analysis from both controllability and unique continuation point of view. Considerations will be given for a Boussinesq system, the Kawahara equation and a higher order Korteweg-de Vries-Benjamin-Bona-Mahony equation (KdV-BBM equation), posed on a periodic domain. We first prove that the Boussinesq system is exactly controllable with controls acting on the boundary conditions. Next, the null-controllability property of the Kawahara equation is derived by means of an internal control. Finally, we prove a unique continuation result for a higher order KdV-BBM equation.

Keywords: Boussinesq system, controllability, Fourier expansion, nonharmonic analysis, Korteweg-de Vries, moment problem, unique continuation property.

Contents

1 Introduction

This thesis is composed of three chapters devoted to the study of the controllability and unique continuation properties for a class of dispersive models, posed on a periodic domain. More precisely, the second chapter deals with the exact boundary controllability of a higher order Boussinesq system [8, 9], in the third chapter we study the internal null-controllability for the Kawahara equation and, finally, the fourth chapter is devoted to study of the unique continuation property for a higher order KdV-BBM equation (Kortewg-de Vries - Benjaming- Bona-Mahony equation).

1.1 Boundary controllability of a Boussinesq system

Water wave propagation phenomena is a classic research topic that has attracted researchers from many different areas with various objectives. Due to the complexity of the governing equations for water waves, physicists and mathematicians are led to derive simpler sets of equations which are likely to describe the dynamics of the water waves in some specific physical regimes. Because of their simplicity, Boussinesq systems have been used in the study of a variety of water wave phenomena in ports, channels, coastal areas, and in the open sea. They have also been used in studies of tsunami wave generation and propagation. In this context, Bona, Chen, and Saut [8, 9] derived a family of Boussinesq systems to describe the two-way propagation of small amplitude gravity waves on the surface of water in a canal. The systems were obtained from the classical Euler equations and have the following form:

$$\begin{cases} \eta_t + w_x + aw_{xxx} - b\eta_{txx} + a_1 w_{xxxxx} + b_1 \eta_{txxxx} \\ \qquad\qquad = -(\eta w)_x + b(\eta w)_{xxx} - \alpha(\eta w_{xx})_x \\ w_t + \eta_x + c\eta_{xxx} - dw_{txx} + c_1 \eta_{xxxxx} + d_1 w_{txxxx} \\ \qquad\qquad = -ww_x - c(ww_x)_{xx} - (\eta\eta_{xx})_x + \beta w_x w_{xx} + \rho\eta\eta_{xxx}. \end{cases} \tag{1.1}$$

In (1.1), η is the elevation of the fluid surface from the equilibrium position, $w = w_\theta$ is the horizontal velocity in the flow at height θh, where h is depth of the undisturbed liquid. The parameters $a, b, c, d, a_1, c_1, b_1, d_1$ are required to fulfill the relations

$$\begin{aligned} a+b &= \tfrac{1}{2}(\theta^2-\tfrac{1}{3}), \quad c+d=\tfrac{1}{2}(\theta^2-\tfrac{1}{2}),\\ a_1-b_1 &= -\tfrac{1}{2}(\theta^2-\tfrac{1}{3})b+\tfrac{5}{24}(\theta^2-\tfrac{1}{5})^2,\\ c_1-d_1 &= \tfrac{1}{2}(1-\theta^2)c+\tfrac{5}{24}(1-\theta^2)(\theta^2-\tfrac{1}{5}),\\ \alpha &= a+b-\tfrac{1}{3},\ \beta=c+d-1,\ \rho=c+d, \end{aligned} \tag{1.2}$$

where $\theta \in [0,1]$, specifies which horizontal velocity the variable w represents. Contrary to some classical wave models which assume that the waves travel only in one direction, system (1.1) is free of the presumption of unidirectionality and may have a wider range of applicability.

In mathematical studies, consideration has been given principally to pure initial-value problems where the wave profile is imagined to be determined everywhere at a given instant of time and the corresponding solution models the further wave motion. However, the practical use of the Boussinesq systems and their relatives does not always involve the pure initial-value problem. At this respect, a natural example arises when modeling the effect in a channel of a wave maker mounted at one end, or in modeling near-shore zone motions generated by waves propagating from deep water. The mathematical theory pertaining to the study of such boundary value problem is considerably less advanced, specially in what concerns the study of the controllability properties.

In this spirit, the present work is devoted to the study of initial-boundary-value problems associated to linearized Boussinesq system (1.1). We first consider the case in which the parameters given in (1.2) are such that $b=b_1=d=d_1=0$. The resulting system couples two Korteweg-de Vries type equations and it is called purely KdV-type Boussinesq system. Our attention, in particular, is given to the following system

$$\begin{cases} u_t+v_x+av_{xxx}+a_1v_{xxxxx}=0, & \text{in} \quad (0,T)\times(0,2\pi)\\ v_t+u_x+cu_{xxx}+c_1u_{xxxxx}=0, & \text{in} \quad (0,T)\times(0,2\pi), \end{cases} \tag{1.3}$$

with boundary conditions

$$\begin{cases} \partial_x^j u(t,2\pi)-\partial_x^j u(t,0)=f_j(t), & \text{in} \quad (0,T)\\ \partial_x^j v(t,2\pi)-\partial_x^j v(t,0)=g_j(t), & \text{in} \quad (0,T), \end{cases} \tag{1.4}$$

for $j=0,1,2,3,4$, and initial conditions

$$u(0,x)=u^0(x), \quad v(0,x)=v^0(x), \quad \text{in} \quad (0,2\pi). \tag{1.5}$$

In (1.4), the external forcing terms f_j and g_j, with $j=0,1,2,3,4$, are considered as control inputs. The purpose is to see whether one can force the solutions of the system to have certain desired properties by choosing appropriate control inputs acting at one end of the channel. More precisely, we are mainly concerned with the following problem which are fundamental in control theory:

Given $T>0$, initial states (u^0,v^0) and terminal states (u^1,v^1) in a certain space, can one find appropriate control inputs f_j and g_j, with $j=0,1,2,3,4$, so that the system (1.3)-(1.5) admits a solution (u,v) which satisfies $(u(0,x),v(0,x))=(u^0,v^0)$ and $(u(T,x),v(T,x))=(u^1,v^1)$?

If one can always find a control input to guide the system described by (1.3) from any given initial state to any given terminal state, then the system is said to be exactly controllable.

Our analysis does not depend on formulas (1.2) nor on other particular relations between the coefficients, but some sign conditions have to be imposed. More precisely, we first shall be mainly concerned with the case

$$a_1, c_1 \geq 0 \quad \text{and} \quad a \leq 0, c \leq 0. \tag{1.6}$$

As it will become clear during the proofs, assumptions (1.6) provide the tools needed to deal with the controllability, as well as, the well-posedness of the system.

The problem we address here was first addressed for the scalar KdV equation

$$y_t + y_{xxx} = 0, \ \text{ in } \ (0,T)\times(0,2\pi), \tag{1.7}$$

with the boundary conditions

$$\partial_x^j y(t,2\pi) - \partial_x^j y(t,0) = h_j(t), \ \text{ in } \ (0,T), \ \text{and } j=0,1,2, \tag{1.8}$$

and initial condition

$$y(0,x) = y_0(x), \ \text{ in } \ (0,2\pi). \tag{1.9}$$

By using the classical duality approach (see, for instance, [27]), the exact controllability of (1.7)-(1.9) was established by Rosier in [38]. More precisely, the following result was proved:

Theorem 1.1.1 (see, [38]). *Let $H_p^2(0,2\pi)=\{w\in H^2(0,2\pi): w(0)=w(2\pi)=w'(0)=w'(2\pi)\}$ and $T>0$. Then, for any $y_0,y_T\in(H_p^2(0,2\pi))'$, the dual space of $H_p^2(0,2\pi)$, there exist $h_0,h_1,h_2\in L^2(0,T)$, such that the solution $y\in\mathcal{C}([0,T];(H_p^2(0,2\pi))')$ of the initial-boundary-value KdV equation (1.7)-(1.7) satisfies $y(T,x)=y_T(x)$.*

Notice that explicit controls may be given, but since the state y is only known to belong to $\mathcal{C}([0,T];(H_p^2(0,2\pi))')$, the controllability results for nonlinear KdV equation was not studied.

Inspired by the work [38], we have proved that the higher-order linearized Boussinesq system (1.3)-(1.5) is exactly controllable as well. Following the classical duality approach, the exact controllability property is equivalent to an observability inequality for the adjoint system. Then, by means of a detailed spectral analysis developed in [3], the Fourier expansion of the solution and some results due to Komornik [25], condition (1.6) on the parameters of the system ensures that the observability property mentioned above holds. In what concerns the nonlinear model, the controllability properties are usually studied linearizing the problem at an equilibrium state, proving exact controllability results for this linear problem and applying next a fixed point argument (or the implicit function theorem). However, due to the structure of the nonlinear terms and the lack of a priori bound, including when higher order Sobolev norms are considered (e. g. H^s-norm), we only succeeded in deriving exact controllability results for the linear system. Indeed, the space of the controlled data for the associated linear system is a Hilbert space $V \sim [H_p^5(0,2\pi) \times H_p^5(0,2\pi)]'$, therefore it seems quite difficult to derive a controllability results for the nonlinear system.

As we remarked above, a similar approach was used in [38]. However, due to the complexity of the system, which couples two fifth-order KdV terms, the control problem presents new difficulties we have to deal with. Moreover, the techniques applicable to this more complicated situation can also be applied to other Boussinesq systems to derive positive controllability results:

- The linear KdV-BBM system ($b_1 = d_1 = 0$)

$$\begin{cases} u_t + v_x - bu_{txx} + av_{xxx} + a_1 v_{xxxxx} = 0, & \text{in} \quad (0,T) \times (0,2\pi) \\ v_t + u_x - du_{txx} + cu_{xxx} + c_1 u_{xxxxx} = 0, & \text{in} \quad (0,T) \times (0,2\pi), \end{cases}$$

with the following boundary conditions

$$\begin{cases} \partial_x^j u(t,2\pi) - \partial_x^j u(t,0) = 0, & \text{in} \quad (0,T) \\ \partial_x^j v(t,2\pi) - \partial_x^j v(t,0) = 0, & \text{in} \quad (0,T) \\ \partial_x^2 u(t,2\pi) - \partial_x^2 u(t,0) = f_2(t), & \text{in} \quad (0,T) \\ \partial_x^2 v(t,2\pi) - \partial_x^2 v(t,0) = g_2(t), & \text{in} \quad (0,T) \\ \partial_x^4 u(t,2\pi) - \partial_x^4 u(t,0) = f_4(t), & \text{in} \quad (0,T) \\ \partial_x^4 v(t,2\pi) - \partial_x^4 v(t,0) = g_4(t), & \text{in} \quad (0,T), \end{cases}$$

for $j = 0, 1, 3$.

- The full system (1.1) with the boundary conditions

$$\begin{cases} \partial_x^j u(t,2\pi) - \partial_x^j u(t,0) = 0 & \text{in} \quad (0,T) \\ \partial_x^j v(t,2\pi) - \partial_x^j v(t,0) = 0 & \text{in} \quad (0,T) \\ \partial_x^4 u(t,2\pi) - \partial_x^4 u(t,0) = f_4(t) & \text{in} \quad (0,T) \\ \partial_x^4 v(t,2\pi) - \partial_x^4 v(t,0) = g_4(t) & \text{in} \quad (0,T), \end{cases}$$

for $j = 0, 1, 2, 3$.

Observe that the presence of the BBM terms allow to control the system with less controls then boundary conditions. In addition, the presence of the higher-order terms in (1.3) ($b_1 = d_1 \neq 0$) provides a regularizing effect which allows to address the boundary controllability problem for the full system (1.1). By contrast, when two higher-order BBM type equations are coupled and both fifth order KdV term are not present ($b_1 = d_1 \neq 0$ and $a_1 = c_1 = 0$), the controllability property fails (see [5, 35]). This lack of exact controllability of the BBM-BBM system comes from the existence of a limit point in the spectrum of the operator associated with the state equations, a phenomenon already noticed in [29] for the single linear BBM equation.

It is also important to mention that the study of controllability and stabilization properties for Boussinesq systems was initiated in [30] by considering the following reduced form of the model (1.1), posed on a periodic domain:

$$\begin{cases} \eta_t + w_x + a w_{xxx} - b\eta_{txx} = -(\eta w)_x, & t > 0,\ x \in (0,2\pi) \\ w_t + \eta_x + c\eta_{xxx} - d w_{txx} = -w w_x, & t > 0,\ x \in (0,2\pi). \end{cases} \tag{1.10}$$

The parameters a, b, c and d are required to fulfill the relations

$$a + b = \frac{1}{2}(\theta^2 - \frac{1}{3}), \quad c + d = \frac{1}{2}(\theta^2 - \frac{1}{2}) \geq 0, \tag{1.11}$$

with $\theta \in [0,1]$. The work [30] deals with the internal controllability and stabilization of (1.10) on the torus. First, the space of the controllable data for the associated linear system is established for each possible value of the four parameters given in (1.11). Then, when $b, d > 0$ and $a, c < 0$ the local exact controllability of the nonlinear system is shown to hold. As an application of the established exact controllability results, some simple feedback controls are constructed for particular choices of the parameters a, b, c and d, such that the resulting closed-loop systems are exponentially stable. In [11], the exact boundary controllability of the linear Boussinesq system [11] of KdV-KdV type was studied. It was discovered that whether the associated linear system is exactly controllable or not depends on the length of the spatial domain. The extension of the exact controllability for the

Boussinesq system (1.10), when $b = d = 0$, is derived in the energy space by considering a control of Neumann type. It is obtained by incorporating a boundary feedback in the control in order to ensure the so-called Kato smoothing effect. In addition, proceeding as in [36], a local exponential stability result was also derived. As we mention above, the controllability problem was also addressed when (1.1) and (1.10) couples two BBB type equations. In [5, 35], the authors show that the model is approximately controllable but not spectrally controllable. This means that any state can be steered arbitrarily close to another state, but no finite linear combination of eigenfunctions, other than zero, can be steered to zero.

As far as we know, exact the controllability problem for the full system has been only addressed in when the model is posed on a periodic domain. General conditions are given to ensure both the well-posedness and the local exact controllability of the nonlinear problem by means of a control localized in the interior of the domain and acting on one equation only.

The chapter is organized as follows: In Section 1 we present some preliminary results used in our proofs, Section 2 are devoted to prove our main results and, finally, In Section 3 we describe some possible extensions of our results and also indicate open problems on the subject.

1.2 Null-controllability for the Kawahara equation

The study of wave phenomena arising in dispersive media is of broad scientific interest and pertains to a modern line of research which is important both scientifically and for potential applications. Progress in the development of mathematical models has made it possible to understand such phenomena in quite distinct fields and to solve problems that come to the fore. Within this context, the Korteweg-de Vries equation (KdV) has been derived as a model for the unidirectional propagation of nonlinear, dispersive waves in an impressive array of physical situations. In most cases when it is derived from more complex systems, the KdV equation appears in the form

$$u_t + u_x + \varepsilon u u_x + \delta u_{xxx} = 0,$$

where the small positive parameters ε and δ are related to a small-amplitude and a long-wavelength assumption, respectively. The unknown u is a real valued functions of the variables x and t and subscripts indicate partial differentiation.

Another relevant dispersive wave model is the Kawahara equation [23], also referred as fifth-order KdV equation. The Kawahara equation occurs in the theory of magneto-acoustic waves in a plasma and in the theory of shallow water waves with surface tension. In order to balance the nonlinear effect, Kawahara took into account the higher order effect of dispersion and established the following equation to describe solitary-wave propagation in media:

$$u_t + \gamma u_x + \alpha u_{xxx} + \beta u_{xxxxx} + \rho u u_x = 0. \tag{1.12}$$

The parameters $\gamma, \alpha, \beta, \rho \in \mathbb{R}$ with $\beta \neq 0$, and α and β represent the effect of dispersion.

There is a vast literature devoted to the study of water waves ranging from coastal engineering preoccupations to a very theoretical mathematical analysis of the equations. For instance, a large body of literature has been concerned with the questions of existence, uniqueness and continuous dependence of solutions corresponding to initial data. However, there are many issues still open that deserves further attention. In this work, the goal is to advance the study of the initial-boundary value problems exploring the dynamics of dispersive equations by using mathematical analysis from the controllability point of view. Due to the rapid development of new mathematical tools, since the late 1980s control theory of nonlinear dispersive wave equations have attracted a lot of attention. Particularly, control properties of the KdV equation has been intensively studied and significant progresses have been made. In contrast, there are relative few works on the Kawahara equation for its control theory.

Without loss of generality, we assume that the parameters given in (1.12) are such that $\gamma = \alpha = 1$ and $\beta = -1$. Thus, our attention is given to the following control system described by the linearized Kawahara equation posed on a periodic domain:

$$\begin{cases} u_t - u_{5x} + u_{3x} + u_x = f(x)v(t), & \text{in} \quad (0,T) \times (0,2\pi), \\ \partial_x^j u(t,0) = \partial_x^j u(t,2\pi), & \text{in} \quad (0,T), \\ u(0,x) = u_0(x), & \text{in} \quad (0,2\pi), \end{cases} \tag{1.13}$$

for $j = 0, 1, 2, 3, 4$. The goal is to drive the initial data u_0 to rest by using a control $v(t)$, depending only on time and acting on the system through a given function in space $f(x)$. This type of control is often used and sometimes called lumped or bilinear.

To be more precise, considerations will be given to the following null-controllability problem:

Given $T > 0$, an initial state u_0 and a profile f in a certain Hilbert space, find an appropriate control $v \in L^2(0,T)$, so that system (1.13) admits a solution u which satisfies $u(T,x) = 0$.

*If one can always find a control input to guide the system described by (1.13) from any given initial state to zero, then the system is said to be **null-controllable**.*

In order to make more precise the tools we employ to study this question, we introduce some notations: Given any $v \in L^2(0,2\pi)$ and $k \in \mathbb{Z}$, we denote by $\widehat{v}_k$ the $k-$Fourier coefficient of v,

$$\widehat{v}_k = \frac{1}{2\pi}\int_0^{2\pi} v(x)e^{-ikx}\ \mathrm{d}x.$$

Then, for any $s \in \mathbb{R}$, we define the Hilbert space

$$H_p^s(0,2\pi) = \left\{ v = \sum_{k\in\mathbb{Z}} \widehat{v}_k e^{ikx} \in L^2(0,2\pi) \;\middle|\; \sum_{k\in\mathbb{Z}} |\widehat{v}_k|^2(1+k^2)^s < \infty \right\} \tag{1.14}$$

endowed with the inner product

$$(v,w)_s = \sum_{k\in\mathbb{Z}} \widehat{v}_k \overline{\widehat{w}_k}(1+k^2)^s. \tag{1.15}$$

We denote by $\|\cdot\|_s$ the norm corresponding to the inner product given by (2.2). Then, we consider the following operator associated to the space variable:

$$\begin{cases} (D(A),A), \text{ where } D(A) = H_p^5(0,2\pi) \text{ and} \\ A : D(A) \subset L_p^2(0,2\pi) \to L_p^2(0,2\pi), \text{ such that } Au = \partial_x^5 u - \partial_x^3 u - \partial_x u. \end{cases} \tag{1.16}$$

Controllability properties of linear systems have been studied for a long time with the aid of Fourier techniques. In what concerns system (1.13), we employ Fourier series expansion to reduce the null control problem to a equivalent *moment problem*, whose solution is given in terms of an explicit biorthogonal sequence to a family of exponential $(e^{\lambda_m t})_{m\in\mathbb{Z}}$ in $L^2(0,T)$. Here, λ_m are the eigenvalues of the differential operator A defined in (1.16). We recall that a family of functions $(\phi_m)_{m\in\mathbb{Z}} \subset L^2(0,T)$ with the property that

$$\int_0^T \phi_m(t)e^{\overline{\lambda_m}t}dt = \delta_{mn}, \quad \forall\, m,n \in \mathbb{Z},$$

where δ_{mn} is the Kronecker symbol, is a biorthogonal sequence to $(e^{\lambda_m t})_{m\in\mathbb{Z}}$. In order to obtain this sequence, we introduce a family $\Psi_m(z)$ of entire functions of exponential type (see, for instance, [42]), such that $\Psi_m(i\lambda_n) = \delta_{mn}$. Then, by applying Paley–Wiener Theorem we obtain ϕ_m as the inverse Fourier transform of Ψ_m. Each Ψ_m is obtained from a Weierstrass product P_m multiplied by an appropriate function M_m with rapid decay on the real axis. Such a method was used for the first time by Paley and Wiener [34] and, in the context of control problems, by Fattorini and Russell [17, 18].

Once such family $(\phi_m)_{m\in\mathbb{Z}}$ is given, the control $v(t)$ for (1.13) is obtained by considering a linear combinations of functions ϕ_m. Indeed, if we consider $u_0(x) = \sum_{m\in\mathbf{Z}} \widehat{u}_m^0 e^{imx}$ and $f(x) = \sum_{m\in\mathbf{Z}} \widehat{f}_m e^{imx}$, $\widehat{f}_m \neq 0$, the Fourier expansions of u_0 and f, respectively, the function

$$v(t) = \sum_{m\in\mathbb{Z}} \frac{\hat{u}_m^0}{\widehat{f}_m} e^{T\lambda_m} \phi_m\left(T-t\right), \qquad t \in (0,T), \tag{1.17}$$

is a control for (1.13) in time T, if the series converges in $L^2(0,T)$. The convergence depends on some uniform boundedness, with respect to m, of the the family $(\phi_m)_{m\in\mathbb{Z}}$ in $L^2(0,T)$, which are obtained by applying Plancherel Theorem. In additon, some assumptions on f and u_0 are necessary. More precisely, let $f \in L^2(0,2\pi)$ be, such that

$$f(x) = \sum_{k\in\mathbf{Z}} \widehat{f}_k e^{ikx}, \text{ with } \widehat{f}_k \neq 0, \ \forall\, k \in \mathbf{Z}. \tag{1.18}$$

Assuming (1.18), for a given constant $\beta > 0$ define the space

$$\mathcal{H} = \left\{ h \in L_p^2(0,2\pi) : \sum_{k\in\mathbb{Z}} \left| \frac{\widehat{h}_k}{\widehat{f}_k} \right|^2 e^{\beta k^6} < \infty \right\}. \tag{1.19}$$

If $u_0 \in \mathcal{H}$ and $\widehat{f}_k$ satisfies (1.18), the convergence of (1.17) holds in $L_p^2(0,2\pi)$ and $v(t)$ is a control for (1.13). We remark that the choice of the space $\mathcal{H}$ defined in (1.19) is related to the form of the eigenvalues of the operator A defined in (1.16) and the growth of ϕ_m in $L^2(0,T)$. Indeed, the eigenvalues of the state operator corresponding to (1.13) are given by $\lambda_m = -im(m^4 + m^2 - 1)$ and $||\phi_m||_{L^2(0,T)}$ increases exponentially with m, i. e., $||\phi_m||_{L^2(0,T)} \leq ce^{\nu m^6 t}$, where c and ν are positive constants. The choice of the initial data in $\mathcal{H}$ compensates the growth of ϕ_m and ensure the converge of (1.17) in $L^2(0,T)$. When considering models in which the corresponding state operator has eigenvalues with negative real part, we can take $\beta = 0$ in (1.19).

The technique we describe above was used in the study of several control problems, being the pioneering articles of Fattorini and Russell [17, 18] one of the most relevant examples in the context of scalar parabolic equations. This method is very efficient in the one-dimensional space setting and has also been successfully applied in [10, 20, 31]. In particular, our analysis was inspired by the results obtained in [10, 28, 31] of the which we borrow some ideas.

Concerning the Kawahara equation posed on a periodic domain, the internal controllability and the stabilization problems were studied in [44, 45]. Particularly, in [45], the authors use the same approach as that developed in [26] to obtain the global exact control and global exponential stability for periodic solutions in H^s, for $s \geq 0$.

Bourgain spaces associated to the Kawahara equation, propagation of compactness and propagation of regularity for the linear Kawahara equation are three key ingredients in their proofs. More recently, in [19], the authors establish local exact control and local exponential stability of periodic solutions of fifth order Korteweg-de Vries type equations in H^s, for $s > 2$. A dissipative term is incorporated into the control which, along with a propagation of regularity property, yields a smoothing effect permitting the application of the contraction principle. It is important to emphasize that the results obtained in all papers mentioned above [19, 44, 45] do not give an answer to the null control problem addressed here. Moreover, they have been proved employing a different approach with a control input supported in a given open set $\omega \subset (0, 2\pi)$. To the best of our knowledge, the study we develop for the linear Kawahara equation has not been addressed in the literature yet. Moreover, the available results do not give an immediate answer to it.

This chapter is organized as follows: in Section 2, we present the well-posedness of the system and give an equivalent characterization of the controllability problem in terms of the moment problem. Section 3 is devoted to the construction of a biorthogonal sequence and in Section 4 we prove our main result. Finally, in Section 5, we present some open problems.

1.3 Unique continuation for a higher order KdV-BBM equation

In this section we investigate the Unique Continuation Property (UCP) of the following equation

$$u_t+u_x-b_1u_{txx}+a_1u_{xxx}+bu_{txxxx}+au_{xxxxx}+\frac{3}{2}uu_x+\gamma(u^2)_{xxx}-\frac{7}{48}(u_x^2)_x-\frac{1}{8}(u^3)_x=0, \quad (1.20)$$

where $(x,t) \in \mathbb{T} \times (0,T)$. The parameters $b_1, a_1, b, a, \gamma \in \mathbb{R}$ with $b_1, b > 0$. This higher order water wave model describing the unidirectional propagation of water waves was recently introduced by Bona et al. [7] by using the second order approximation in the two-way model, the so-called *abcd*-system introduced in [8, 9]. It is also known as the fifth order KdV-BBM type equation and, when posed on $\mathbb{R}$ and $\mathbb{T}$, it has been proved that the corresponding initial value problems are global well-posed [7, 13].

We say that the UCP holds in some class X of functions if, given any nonempty open set $\omega \subset \mathbb{T}$, the only solution $u \in X$ of (1.20) fulfilling

$$u(x,t) = 0 \text{ for } (x,t) \in \omega \times (0,T),$$

is the trivial one $u \equiv 0$. Such a property is very important in Control Theory, as it is equivalent to the approximate controllability for linear PDE, and it is involved in the classical uniqueness/compactness approach in the proof of the stability for a PDE with a localized damping. The UCP is usually proved with the aid of some Carleman estimate and, in the case of the KdV equations, it was established in [16, 40, 41]. For the BBM equation, the study of the UCP is only at its early age. However, it is important to note that, for the BBM equation, the underlying Cauchy problem is a characteristic one. Therefore, one cannot expect to apply Carleman-type estimates or the classical Holmgren uniqueness theorem to verify the UCP. This property has been proved in [46, Theorem 1.3] for the linearized BBM equation with a potential and in [39, Theorem 3.1] for the nonlinear BBM equation, under additional conditions concerning the initial data. We remark that the UCP for the nonlinear BBM equation does not follow from the UCP of the linearized BBM equation with only space dependent potential and, in the general case, it is still an open problem.

In what concerns model (1.20), the presence of the higher order KdV term au_{xxxxx} results in much better properties and allows to establish a unique continuation result. The equation is first split into a coupled system of an elliptic equation and a transport equation. Next, we prove some Carleman estimates with the same singular weights for both the elliptic and the hyperbolic equations, and we derive the UCP for (1.20) by combining these Carleman estimates with a regularization process. Our analysis was inspired by the results obtained in [39] from which we borrow some ideas. The authors prove the UCP for a KdV-BBM equation (by means of Carleman inequality) and apply the result to prove the exact controllability and the semiglobal exponential stability of the same equation with a localized damping term. We remark that the same arguments cannot be applied here, since the regularity of the solution required to apply the UCP is not fulfilled. Nonetheless, we address the issue in an appendix.

2 Boundary controllability of a Boussinesq system

In this chapter we consider a family of Boussinesq systems proposed by J. L. Bona, M. Chen and J.-C. Saut to describe the two-way propagation of small amplitude gravity waves on the surface of water in a canal. Our aim is to investigate the boundary controllability properties of this system posed on the interval $(0, 2\pi)$. Then, employing a classical duality approach, we prove that the linear system is exactly controllable by means of controls acting on the right endpoint of the interval. Moreover, we show that the spaces of the controllable data depend on the parameters involved in the system. When all the parameters are different from zero, the local exact controllability of the nonlinear system is also established.

2.1 Preliminaries

We first introduce a few notations. Given any $\phi \in L^2(0, 2\pi)$ and $k \in \mathbb{Z}$, we denote by $\widehat{\phi}_k$ the $k-$Fourier coefficient of ϕ,

$$\widehat{\phi}_k = \frac{1}{2\pi} \int_0^{2\pi} \phi(x) e^{-ikx} \, \mathrm{d}x.$$

Then, for any $s \geq 0$, we define the Hilbert space

$$H_p^s(0, 2\pi) = \left\{ \phi = \sum_{k \in \mathbb{Z}} \widehat{\phi}_k e^{ikx} \in L^2(0, 2\pi) \;\middle|\; \sum_{k \in \mathbb{Z}} |\widehat{\phi}_k|^2 (1 + k^2)^s < \infty \right\} \tag{2.1}$$

with respect to the inner product

$$(\phi, w)_s = \sum_{k \in \mathbb{Z}} \widehat{\phi}_k \overline{\widehat{w}_k} (1 + k^2)^s. \tag{2.2}$$

We denote by $\| \cdot \|_s$ the corresponding norm to the inner product given by (2.2).

For $s < 0$ we define the space $H_p^s(0, 2\pi)$ as the topological dual of $H_p^{-s}(0, 2\pi)$:

$$H_p^s(0, 2\pi) = \left(H_p^{-s}(0, 2\pi) \right)'.$$

Riesz representation theorem ensures that any $\phi \in H_p^0(0, 2\pi) = L^2(0, 2\pi)$ can be identified with an element $w_\phi \in \left(H_p^0(0, 2\pi) \right)'$ such that

$$w_\phi(z) = \int_0^{2\pi} z(x) \phi(x) \, \mathrm{d}x \qquad \left(z \in H_p^0(0, 2\pi) \right).$$

Traditionally, the same notation is used for ϕ and w_ϕ (the spaces $\left(H_p^0(0,2\pi)\right)'$ and $H_p^0(0,2\pi)$ are identified). Given $s<0$, any element $w \in H_p^s(0,2\pi)$ can be uniquely expanded as follows

$$w = \sum_{k\in\mathbb{Z}} \widehat{w}_k e^{ikx}, \tag{2.3}$$

where $\widehat{w}_k = \frac{1}{2\pi} w\left(e^{-ikx}\right)$ for each $k \in \mathbb{Z}$. The slight abuse of notation in (2.3) (the element w on the left hand side is not a function of x and the exponential function e^{ikx} on the right hand side is actually the representant of this L^2-function in the dual space) is compensated by the fact that expansion (2.3) looks exactly like one corresponding to an element in a space H^s with positive exponent s. On the other hand, the following map is a duality product between $H_p^s(0,2\pi)$ and $H_p^{-s}(0,2\pi)$, for any $s \geq 0$,

$$\langle \phi, w\rangle_s = \sum_{k\in\mathbb{Z}} \widehat{\phi}_k \widehat{w}_{-k} \qquad \left(\phi \in H_p^s(0,2\pi),\ w \in H_p^{-s}(0,2\pi)\right). \tag{2.4}$$

Consequently, if $s<0$, the space $H_p^s(0,2\pi)$ can be also defined by (2.1) and can be viewed as a Hilbert space with respect to the inner product (2.2).

Finally, for given δ, $\delta_1 \geq 0$, we introduce the operator $\mathcal{L}_{\delta,\delta_1}$ defined in the following way:

$$\begin{gathered} \mathcal{L}_{\delta,\delta_1} : H_p^s(0,2\pi) \to H_p^{s+s_0}(0,2\pi), \\ \mathcal{L}_{\delta,\delta_1}\left(\sum_{k\in\mathbb{Z}} \widehat{\psi}_k e^{ikx}\right) = \sum_{k\in\mathbb{Z}} \frac{\widehat{\psi}_k}{1+\delta k^2+\delta_1 k^4}, \end{gathered} \tag{2.5}$$

where $s_0 = \begin{cases} 4, & \text{if } \delta_1 \neq 0, \\ 2, & \text{if } \delta_1 = 0 \text{ and } \delta \neq 0, \\ 0, & \text{if } \delta_1 = \delta = 0. \end{cases}$ Let us remark that $\mathcal{L}_{\delta,\delta_1}$ represents the inverse of the elliptic operator $I - \delta\partial_x^2 + \delta_1\partial_x^4$ with periodic boundary conditions in $(0,2\pi)$.

Let us also introduce the numbers

$$w_1(k) = \frac{1-ak^2+a_1k^4}{1+bk^2+b_1k^4},\ w_2(k) = \frac{1-ck^2+c_1k^4}{1+dk^2+d_1k^4},\ \sigma(k) = \sqrt{w_1(k)w_2(k)}$$

and the number $l \in \mathbb{Z}$ with the property that

$$\sqrt{\frac{w_1(k)}{w_2(k)}} \sim C|k|^l, \quad \text{when} \quad |k| \to \infty,$$

where C is a positive constant that does not depend on k. Then, for each $s \in \mathbb{R}$ and l defined above, we define the space

$$V^s = H_p^s(0,2\pi) \times H_p^{s+l}(0,2\pi), \tag{2.6}$$

endowed with the inner product given by

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle_{V^s} = (f_1, g_1)_s + (\mathcal{H}f_2, \mathcal{H}g_2)_s,$$

where the operator $\mathcal{H} : H_p^{s+l}(0,2\pi) \to H_p^s(0,2\pi)$ is defined in the following way

$$\mathcal{H}\left(\sum_{k\in\mathbb{Z}} \widehat{\phi}_k e^{ikx}\right) = \sum_{k\in\mathbb{Z}} \sqrt{\frac{w_1(k)}{w_2(k)}}\widehat{\phi}_k e^{ikx}.$$

Let $\nu \in \mathbb{Z}$ be the number defined by the relation

$$\sigma(k) = \sqrt{w_1(k)w_2(k)} \sim C|k|^\nu, \quad \text{when} \quad |k| \to \infty, \tag{2.7}$$

where C is a positive constant not depending on k.

2.1.1 The linearized system: homogeneous boundary conditions

The aim of this section is to study the the following system

$$\begin{cases} u_t + v_x - bu_{txx} + b_1 u_{txxxx} + av_{xxx} + a_1 v_{xxxxx} = f(t,x), & \text{in} \quad (0,T)\times(0,2\pi) \\ v_t + u_x - dv_{txx} + d_1 v_{txxxx} + cu_{xxx} + c_1 u_{xxxxx} = g(t,x), & \text{in} \quad (0,T)\times(0,2\pi) \end{cases} \tag{2.8}$$

with periodic boundary conditions

$$\begin{cases} \partial_x^r u(t,2\pi) = \partial_x^r u(t,0), & \text{in} \quad (0,T), \quad 0 \le r \le r_0 \\ \partial_x^q v(t,2\pi) = \partial_x^q v(t,0), & \text{in} \quad (0,T), \quad 0 \le q \le q_0 \end{cases} \tag{2.9}$$

and initial condition

$$u(0,x) = u^0(x), \quad v(0,x) = v^0(x), \quad \text{in} \quad (0,2\pi). \tag{2.10}$$

The number of boundary conditions depend on the parameters of the system.

Let us first remark that (2.8)-(2.10) can be written as

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = A\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f^* \\ g^* \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix}(0,x) = \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix}, \tag{2.11}$$

where A is the unbounded linear operator defined by

$$A = -\begin{pmatrix} 0 & \mathcal{L}_{b,b_1}(\partial_x + a\partial_{xxx} + a_1\partial_{xxxxx}) \\ \mathcal{L}_{d,d_1}(\partial_x + c\partial_{xxx} + c_1\partial_{xxxxx}) & 0 \end{pmatrix} \tag{2.12}$$

and

$$f^* = \mathcal{L}_{b,b_1} f, \qquad g^* = \mathcal{L}_{d,d_1} g.$$

Then, using Fourier analysis, we obtain a group associated to the linear problem (2.11).

Theorem 2.1.1. *Let $D(A) = V^{s+1+\max\{-1,\nu\}}$ and A be given by (2.12). The operator $(D(A), A)$ is the infinitesimal generator of a group of isometries $(S(t))_{t\in\mathbb{R}}$ in V^s.*

As a consequence, the following well-posedness result holds:

Theorem 2.1.2. *Let $T>0$, $s\in\mathbb{R}$ and ν given by (2.7). If* $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \in V^s$ *and* $\begin{pmatrix} f^* \\ g^* \end{pmatrix} \in L^1(0,T;V^s)$*, then (2.8)-(2.10) admits a unique solution*

$$\begin{pmatrix} u \\ v \end{pmatrix} \in C^1\left([0,T];V^{s-1-\max\{-1,\nu\}}\right)\cap C\left([0,T];V^s\right).$$

Moreover, the following estimate holds

$$\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\|_{C([0,T];V^s)} \leq \left\|\begin{pmatrix} f^* \\ g^* \end{pmatrix}\right\|_{L^1(0,T;V^s)} + \left\|\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}\right\|_{V^s}. \tag{2.13}$$

Furthermore, the following remarks are in order:

Remark 2.1.1. *The eigenvalues of the operator A defined by (2.12) are given by*

$$\lambda_k = ik\sigma(k),\ \ k\in\mathbb{Z}, \tag{2.14}$$

where σ was defined in (2.7). Note that not all the eigenvalues in (2.14) are different. If we count only the distinct eigenvalues, we obtain the sequence $(\lambda_k)_{k\in\mathbb{I}}$, where $\mathbb{I}\subseteq\mathbb{Z}$ has the property that $\lambda_{k_1}\neq\lambda_{k_2}$ for any $k_1,k_2\in\mathbb{I}$. For each $k_1\in\mathbb{Z}$ set

$$I(k_1)=\{k\in\mathbb{Z}\,:\,k\sigma(k)=k_1\sigma(k_1)\}$$

and $|I(k_1)|=m(k_1)$. We have the following properties of $m(k_1)$:

- *$m(k_1)\leq 10$. This is a consequence of the fact that $m(k_1)$ is less than the number of entire roots of the equation $x\sigma(x)=\varrho$, where ϱ is an arbitrary real number. The roots of this equation are also roots of a polynomial of degree less or equal to 10.*

- *If the sequence of eigenvalues tends to infinity, there exists $k_1^*\in\mathbb{N}$ such that $m(k_1)=1$ for all $|k_1|>k_1^*$. This is a consequence of the fact that the function $x\sigma(x)$ is strictly increasing for $|x|$ large enough. Notice that, if $\nu\geq 0$, then*

  $$\lim_{|k|\to\infty}|\lambda_k|=\infty,$$

  and the above mentioned property holds.

- *The number of the eigenfunctions corresponding to an eigenvalue $\lambda_{k_1}\neq 0$ is $2m(k_1)$, for each $k_1\in\mathbb{I}$. These eigenfunctions read then*

  $$\begin{pmatrix} e^{ikx} \\ -\frac{\sigma(k)}{w_1}e^{ikx} \end{pmatrix},\quad \begin{pmatrix} e^{-ikx} \\ \frac{\sigma(k)}{w_1}e^{-ikx} \end{pmatrix},\qquad k\in I(k_1).$$

  On the other hand, under conditions (1.6) on the parameters of the system, the zero eigenvalue has multiplicity two, with associated eigenfunctions

  $$\begin{pmatrix} 1 \\ 0 \end{pmatrix},\quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then, taking into account the previous remark concerning the multiplicity of the eigenvalues of the operator A, we can give an equivalent expression for the solution of problem (2.11) when $f = g \equiv 0$. Indeed, if the initial data are given by $\begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix} = \sum_{k\in\mathbb{Z}} \begin{pmatrix} \hat{u}_k^0 \\ \hat{v}_k^0 \end{pmatrix} e^{ikx}$*, we have that*

$$\begin{pmatrix} u \\ v \end{pmatrix}(t,x) = \frac{1}{2}\sum_{k\in\mathbb{I}} e^{ik\sigma(k)t} \sum_{m\in I(k)} \left[\begin{pmatrix} 1 \\ -\sqrt{\frac{w_2(m)}{w_1(m)}} \end{pmatrix} e^{imx}\widehat{u}_m^0 + \begin{pmatrix} -\sqrt{\frac{w_1(m)}{w_2(m)}} \\ 1 \end{pmatrix} e^{imx}\widehat{v}_m^0 \right.$$
$$\left. + \begin{pmatrix} 1 \\ \sqrt{\frac{w_2(m)}{w_1(m)}} \end{pmatrix} e^{-imx}\widehat{u}_{-m}^0 + \begin{pmatrix} \sqrt{\frac{w_1(m)}{w_2(m)}} \\ 1 \end{pmatrix} e^{-imx}\widehat{v}_{-m}^0 \right]. \tag{2.15}$$

Remark 2.1.2. *Theorem 2.1.2 shows that, depending on the choice of the parameters, system (2.11) has an important regularizing property. For instance, if $b_1 \neq 0$ and $d_1 \neq 0$, then* $\begin{pmatrix} f \\ g \end{pmatrix} \in L^1(0,T;V^{s-4})$ *implies* $\begin{pmatrix} f^* \\ g^* \end{pmatrix} \in L^1(0,T;V^s)$*. Consequently, if* $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \in V^s$ *and* $\begin{pmatrix} f \\ g \end{pmatrix} \in L^1(0,T;V^{s-4})$*, then* $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T];V^s)$*. This regularizing effect is crucial in the study of the nonlinear system. Let us also mention that, under the above conditions on the parameters b_1 and d_1, there exists a constant $M > 0$, such that the following estimate for the solutions of (2.11) holds*

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{C([0,T];V^s)} \leq M \left(\left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{L^1(0,T;V^{s-4})} + \left\| \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \right\|_{V^s} \right). \tag{2.16}$$

2.1.2 The adjoint system

In order to characterize the controllability properties of our problem we need to introduce and study the following adjoint system:

$$\begin{cases} \varphi_t + \psi_x - b\varphi_{txx} + b_1\varphi_{txxxx} + c\psi_{xxx} + c_1\psi_{xxxxx} = 0, & t \in (0,T), x \in (0,2\pi) \\ \psi_t + \varphi_x - d\psi_{txx} + d_1\psi_{txxxx} + a\varphi_{xxx} + a_1\varphi_{xxxxx} = 0, & t \in (0,T), x \in (0,2\pi) \\ \partial_x^r\varphi(t,0) = \partial_x^r\varphi(t,2\pi), & t \in (0,T), \quad 0 \leq r \leq r_0 \\ \partial_x^q\psi(t,0) = \partial_x^q\psi(t,2\pi), & t \in (0,T), \quad 0 \leq q \leq q_0 \\ \varphi(T,x) = \varphi^T(x), \qquad \psi(T,x) = \psi^T(x), & x \in (0,2\pi). \end{cases} \tag{2.17}$$

Again, we remark that the number of boundary conditions depends on the values of the parameters of the system.

In order to show the well-posedness of (2.17), let us first define the spaces

$$\widetilde{V}^s = H_p^s(0,2\pi) \times H_p^{s+\tilde{l}}(0,2\pi), \tag{2.18}$$

where the number $\tilde{l} \in \mathbb{Z}$ has the property that

$$\sqrt{\frac{\widetilde{w}_1(k)}{\widetilde{w}_2(k)}} \sim C|k|^{\tilde{l}}, \quad \text{when} \quad |k| \to \infty,$$

being C a positive constant not depending on k. Furthermore, let

$$\widetilde{w}_1(k) = \frac{1 - ck^2 + c_1k^4}{1 + bk^2 + b_1k^4}, \text{ and } \widetilde{w}_2(k) = \frac{1 - ak^2 + a_1k^4}{1 + dk^2 + d_1k^4}.$$

Observe that

$$\sigma(k) = \sqrt{\widetilde{w}_1(k)\widetilde{w}_2(k)},$$

where $\sigma(k)$ is defined in (2.7). Moreover, the eigenvalues of the state operator associated to the adjoint system (2.17) coincide with λ_k, the eigenvalues of the operator A given by (2.14).

Following the notation introduced above, the properties of the solutions of the adjoint problem can be obtained proceeding as in Theorem 2.1.2. More precisely, we have the following result:

Theorem 2.1.3. *Let $T > 0$, $s \in \mathbb{R}$ and ν given by (2.7). If $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \in \widetilde{V}^s$, then (2.17) admits a unique solution*

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in C^1\left([0,T]; \widetilde{V}^{s-1-\max\{-1,\nu\}}\right) \cap C\left([0,T]; \widetilde{V}^s\right)$$

and the energy of solutions is conserved:

$$\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}(t) \right\|_{\widetilde{V}^s} = \left\| \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \right\|_{\widetilde{V}^s}, \quad \forall\, t \in [0,T], \tag{2.19}$$

Moreover, if the final data are given by

$$\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix}(x) = \sum_{k\in\mathbb{Z}} \begin{pmatrix} \widehat{\varphi}^T_k \\ \widehat{\psi}^T_k \end{pmatrix} e^{ikx} \tag{2.20}$$

the following representation formula holds:

$$\begin{aligned}\begin{pmatrix} \varphi \\ \psi \end{pmatrix}(t,x) = \frac{1}{2}\sum_{k\in\mathbb{I}} e^{ik\sigma(k)(T-t)} \sum_{m\in I(k)} \Bigg[& \begin{pmatrix} 1 \\ -\sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}} \end{pmatrix} e^{-imx}\widehat{\varphi}^T_m + \begin{pmatrix} -\sqrt{\frac{\widetilde{w}_1(m)}{\widetilde{w}_2(m)}} \\ 1 \end{pmatrix} e^{-imx}\widehat{\psi}^T_m \\ & + \begin{pmatrix} 1 \\ \sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}} \end{pmatrix} e^{imx}\widehat{\varphi}^T_{-m} + \begin{pmatrix} \sqrt{\frac{\widetilde{w}_1(m)}{\widetilde{w}_2(m)}} \\ 1 \end{pmatrix} e^{imx}\widehat{\psi}^T_{-m} \Bigg].\end{aligned} \tag{2.21}$$

2.2 Controllability

2.2.1 Higher order KdV-KdV system

This section is devoted to the study of the system

$$\begin{cases} u_t + v_x + av_{xxx} + a_1 v_{xxxxx} = 0, & \text{in} \quad (0,T)\times(0,2\pi) \\ v_t + u_x + cu_{xxx} + c_1 u_{xxxxx} = 0, & \text{in} \quad (0,T)\times(0,2\pi), \end{cases} \tag{2.22}$$

with boundary conditions

$$\begin{cases} \partial_x^j u(t,2\pi) - \partial_x^j u(t,0) = f_j(t), & \text{in} \quad (0,T) \\ \partial_x^j v(t,2\pi) - \partial_x^j v(t,0) = g_j(t), & \text{in} \quad (0,T), \end{cases} \tag{2.23}$$

for $j = 0,1,2,3,4$, and initial conditions

$$u(0,x) = u^0(x), \quad v(0,x) = v^0(x), \quad \text{in} \quad (0,2\pi). \tag{2.24}$$

The following result will be needed:

Proposition 2.2.1. *For any $s \in \mathbb{R}$, let V^s be the Hilbert space defined by (2.6) and $X(0,T) := (L^2(0,T))^2 \times (L^2(0,T))^2$, for $T > 0$. Then, the following well-posedness results hold:*

(i) Suppose that $f_j(t), g_j(t) \in C_0^2[0,T]$, for $j = 0,1,2,3,4$, and $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \in V^5$. Then, there exists a unique solution $\begin{pmatrix} u \\ v \end{pmatrix} \in C^1([0,T];V^0) \cap C([0,T];V^5)$ of (2.22)-(2.24). Moreover, for any $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \in V^5$ and $S \in [0,T]$, we have

$$\begin{aligned} \left\langle \begin{pmatrix} u(S,\cdot) \\ v(S,\cdot) \end{pmatrix}, \begin{pmatrix} \overline{\varphi(S,\cdot)} \\ \overline{\psi(S,\cdot)} \end{pmatrix} \right\rangle_{V^0\times V^0} &= \left\langle \begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \begin{pmatrix} \overline{\varphi(0,\cdot)} \\ \overline{\psi(0,\cdot)} \end{pmatrix} \right\rangle_{V^0\times V^0} \\ &- \left\langle \begin{pmatrix} f_0(t) \\ g_0(t) \end{pmatrix}, \begin{pmatrix} \overline{\psi(t,0)} + c\overline{\psi_{xx}(t,0)} + c_1\overline{\psi_{xxxx}(t,0)} \\ \overline{\varphi(t,0)} + a\overline{\varphi_{xx}(t,0)} + a_1\overline{\varphi_{xxxx}(t,0)} \end{pmatrix} \right\rangle_{X(0,S)} \\ &+ \left\langle \begin{pmatrix} f_1(t) \\ g_1(t) \end{pmatrix}, \begin{pmatrix} c\overline{\psi_x(t,0)} + c_1\overline{\psi_{xxx}(t,0)} \\ a\overline{\varphi_x(t,0)} + a_1\overline{\varphi_{xxx}(t,0)} \end{pmatrix} \right\rangle_{X(0,S)} \\ &- \left\langle \begin{pmatrix} f_2(t) \\ g_2(t) \end{pmatrix}, \begin{pmatrix} c\overline{\psi(t,0)} + c_1\overline{\psi_{xx}(t,0)} \\ a\overline{\varphi(t,0)} + a_1\overline{\varphi_{xx}(t,0)} \end{pmatrix} \right\rangle_{X(0,S)} \\ &+ \left\langle \begin{pmatrix} f_3(t) \\ g_3(t) \end{pmatrix}, \begin{pmatrix} c_1\overline{\psi_x(t,0)} \\ a_1\overline{\varphi_x(t,0)} \end{pmatrix} \right\rangle_{X(0,S)} - \left\langle \begin{pmatrix} f_4(t) \\ g_4(t) \end{pmatrix}, \begin{pmatrix} c_1\overline{\psi(t,0)} \\ a_1\overline{\varphi(t,0)} \end{pmatrix} \right\rangle_{X(0,S)}, \end{aligned} \tag{2.25}$$

where $\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in C^1([0,T];V^0)\cap C([0,T];V^5)$ *is the solution of the adjoint system* (2.17) *with initial data* $\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}$ *given by Theorem 2.1.3.*

(ii) If $\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix} \in V^4$, *there exist a unique solution* $\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in C([0,T];V^4)$ *of* (2.17) *and* $\begin{pmatrix}\varphi_{xxxx}(t,0)\\ \psi_{xxxx}(t,0)\end{pmatrix}$ *makes sense in* $(L^2(0,T))^2$.

(iii) Assume that $\begin{pmatrix}u^0\\ v^0\end{pmatrix} \in [V^4]'$ *and* $f_j, g_j \in L^2(0,T)$, *for* $j=0,1,2,3,4$. *Then, there exists a unique* $\begin{pmatrix}u\\ v\end{pmatrix} \in C([0,T];[V^4]')$, *such that, for any* $\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix} \in V^4$ *and any* $S\in[0,T]$, *we have*

$$\begin{aligned}
&\left\langle \begin{pmatrix}u(S,\cdot)\\ v(S,\cdot)\end{pmatrix}, \begin{pmatrix}\overline{\varphi(S,\cdot)}\\ \overline{\psi(S,\cdot)}\end{pmatrix}\right\rangle_{[V^4]'\times V^4} = \left\langle \begin{pmatrix}u^0\\ v^0\end{pmatrix}, \begin{pmatrix}\overline{\varphi(0,\cdot)}\\ \overline{\psi(0,\cdot)}\end{pmatrix}\right\rangle_{[V^4]'\times V^4}\\
&- \left\langle \begin{pmatrix}f_0(t)\\ g_0(t)\end{pmatrix}, \begin{pmatrix}\overline{\psi(t,0)} + c\overline{\psi_{xx}(t,0)} + c_1\overline{\psi_{xxxx}(t,0)}\\ \overline{\varphi(t,0)} + a\overline{\varphi_{xx}(t,0)} + a_1\overline{\varphi_{xxxx}(t,0)}\end{pmatrix}\right\rangle_{X(0,S)}\\
&+ \left\langle \begin{pmatrix}f_1(t)\\ g_1(t)\end{pmatrix}, \begin{pmatrix}c\overline{\psi_x(t,0)} + c_1\overline{\psi_{xxx}(t,0)}\\ a\overline{\varphi_x(t,0)} + a_1\overline{\varphi_{xxx}(t,0)}\end{pmatrix}\right\rangle_{X(0,S)}\\
&- \left\langle \begin{pmatrix}f_2(t)\\ g_2(t)\end{pmatrix}, \begin{pmatrix}c\overline{\psi(t,0)} + c_1\overline{\psi_{xx}(t,0)}\\ a\overline{\varphi(t,0)} + a_1\overline{\varphi_{xx}(t,0)}\end{pmatrix}\right\rangle_{X(0,S)}\\
&+ \left\langle \begin{pmatrix}f_3(t)\\ g_3(t)\end{pmatrix}, \begin{pmatrix}c_1\overline{\psi_x(t,0)}\\ a_1\overline{\varphi_x(t,0)}\end{pmatrix}\right\rangle_{X(0,S)} - \left\langle \begin{pmatrix}f_4(t)\\ g_4(t)\end{pmatrix}, \begin{pmatrix}c_1\overline{\psi(t,0)}\\ a_1\overline{\varphi(t,0)}\end{pmatrix}\right\rangle_{X(0,S)},
\end{aligned} \tag{2.26}$$

where $\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in C([0,T];V^4)$ *is the solution of the adjoint system* (2.17) *with initial data* $\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}$ *given by (ii).*

Proof. (i) Consider $\theta_j, \phi_j \in C^\infty(0,2\pi)$, with $\theta_j^{(k)}(0) = 0 = \phi_j^{(k)}(0)$ and $\theta_j^{(k)}(2\pi) = -\delta_j^k = \phi_j^{(k)}(2\pi)$, for $j,k = 0,1,2,3,4$. Denoting by $\begin{pmatrix}u_1\\ v_1\end{pmatrix}$ the solution of the corresponding homogeneous system given by Theorem 2.1.2, the change of functions

$$\begin{pmatrix}z\\ w\end{pmatrix} = \begin{pmatrix}u\\ v\end{pmatrix} - \begin{pmatrix}u_1\\ v_1\end{pmatrix} + \begin{pmatrix}\sum_{j=0}^4 \theta_j(x) f_j(t)\\ \sum_{j=0}^4 \phi_j(x) g_j(t)\end{pmatrix}$$

yields an equivalent problem: Find $\begin{pmatrix} z \\ w \end{pmatrix}$, such that

$$\begin{cases} z_t + w_x + aw_{xxx} + a_1 w_{xxxxx} = \displaystyle\sum_{j=0}^{4} [\theta_j(x) f_j'(t) + g_j(t)(\phi_j'(x) + a\phi_j^{(3)}(x) + a_1\phi_j^{(5)}(x))] := F \\ w_t + z_x + cz_{xxx} + c_1 zxxxxx = \displaystyle\sum_{j=0}^{4} [\phi_j(x) g_j'(t) + f_j(t)(\theta_j'(x) + c\theta_j^{(3)}(x) + c_1\theta_j^{(5)}(x))] := G \\ \partial_x^j z(t, 2\pi) = \partial_x^j z(t, 0), \quad \partial_x^j w(t, 2\pi) = \partial_x^j w(t, 0) \\ z(0, x) = 0, \quad w(0, x) = 0. \end{cases} \tag{2.27}$$

Since $F, G \in L^1(0, T; V^5)$, from Theorem 2.1.2 we deduce that (2.27) admits a unique solution $\begin{pmatrix} z \\ w \end{pmatrix} \in C([0, T]; V^5) \cap C^1([0, T]; V^0))$. Hence, we have a unique solution $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0, T]; V^5) \cap C^1([0, T]; V^0))$ of (2.22)-(2.24). To obtain identity (2.25), we multiply the first equation of (2.22) by $\overline{\varphi}$, the second one by $\overline{\psi}$, integrate in time and space over $(0, T) \times (0, 2\pi)$ and add the resulting identities.

(ii) If $t_1, t_2 \in [0, T]$, from (2.21) we obtain

$$\begin{aligned} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} (t_1, x) - \begin{pmatrix} \varphi \\ \psi \end{pmatrix} (t_2, x) = \frac{1}{2} \sum_{k \in \mathbb{I}} (e^{ik\sigma(k)(T-t_1)} - e^{ik\sigma(k)(T-t_2)}) \sum_{m \in I(k)} \Bigg[\begin{pmatrix} 1 \\ -\sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}} \end{pmatrix} e^{imx} \hat{\varphi}_m^T \\ + \begin{pmatrix} -\sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}} \\ 1 \end{pmatrix} e^{imx} \hat{\psi}_m^T + \begin{pmatrix} 1 \\ \sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}} \end{pmatrix} e^{-imx} \hat{\varphi}_{-m}^T + \begin{pmatrix} \sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}} \\ 1 \end{pmatrix} e^{-imx} \hat{\psi}_{-m}^T \Bigg]. \end{aligned}$$

Since $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \in V^4$, we have $\sum_{k \in \mathbb{Z}} (|\hat{\varphi}_k^T|^2 + |\hat{\psi}_k^T|^2)(1 + k^2)^4 < \infty$. Moreover, observe that, as $k \to \infty$, $\frac{\widetilde{w_2}(k)}{\widetilde{w_1}(k)} \to \frac{a_1}{c_1}$ and $\frac{\widetilde{w_1}(k)}{\widetilde{w_2}(k)} \to \frac{c_1}{a_1}$. Then, we obtain a constant $C > 0$ satisfying

$$\begin{aligned} \frac{1}{2} \sum_{k \in \mathbb{I}} \Bigg| (e^{ik\sigma(k)(T-t_1)} - e^{ik\sigma(k)(T-t_2)}) \sum_{m \in I(k)} \Bigg[\begin{pmatrix} 1 \\ -\sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}} \end{pmatrix} e^{imx} \hat{\varphi}_m^T + \begin{pmatrix} -\sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}} \\ 1 \end{pmatrix} e^{imx} \hat{\psi}_m^T \\ + \begin{pmatrix} 1 \\ \sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}} \end{pmatrix} e^{-imx} \hat{\varphi}_{-m}^T + \begin{pmatrix} \sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}} \\ 1 \end{pmatrix} e^{-imx} \hat{\psi}_{-m}^T \Bigg] \Bigg|^2 (1 + k^2)^4 \\ \leq C \sum_{k \in \mathbb{Z}} (|(e^{ik\sigma(k)(T-t_1)} - e^{ik\sigma(k)(T-t_2)})|^2 |\hat{\varphi}_k^T|^2 + |\hat{\psi}_k^T|^2)(1 + k^2)^4. \end{aligned}$$

Hence, by Lebesgue's theorem, it follows that $\left| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} (t_1, x) - \begin{pmatrix} \varphi \\ \psi \end{pmatrix} (t_2, x) \right| \to 0$, as $t_1 \to t_2$, which implies that $\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in C([0,T];V^4)$. If $\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix} \in V^4$, the same argument shows that

$$\begin{pmatrix}\varphi_{xxxx}(t,0)\\ \psi_{xxxx}(t,0)\end{pmatrix} = \frac{1}{2}\sum_{k\in\mathbb{I}} e^{ik\sigma(k)(T-t)} \sum_{m\in I(k)} m^4 \left[\begin{pmatrix}1\\ -\sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}}\end{pmatrix}\widehat{\varphi}^T_m + \begin{pmatrix}-\sqrt{\frac{\widetilde{w}_1(m)}{\widetilde{w}_2(m)}}\\ 1\end{pmatrix}\widehat{\psi}^T_m\right.$$
$$\left. + \begin{pmatrix}1\\ \sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}}\end{pmatrix}\widehat{\varphi}^T_{-m} + \begin{pmatrix}\sqrt{\frac{\widetilde{w}_1(m)}{\widetilde{w}_2(m)}}\\ 1\end{pmatrix}\widehat{\psi}^T_{-m}\right] \in (L^2(0,T))^2. \tag{2.28}$$

Remark that, if $\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix} \in V^4$, the sum above also makes sense in $(L^2(0,T))^2$, since

$$\sum_{k\in\mathbb{I}}\sum_{m\in I(k)} m^4 \left[\begin{pmatrix}1\\ -\sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}}\end{pmatrix}\widehat{\varphi}^T_m + \begin{pmatrix}-\sqrt{\frac{\widetilde{w}_1(m)}{\widetilde{w}_2(m)}}\\ 1\end{pmatrix}\widehat{\psi}^T_m\right.$$
$$\left. + \begin{pmatrix}1\\ \sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}}\end{pmatrix}\widehat{\varphi}^T_{-m} + \begin{pmatrix}\sqrt{\frac{\widetilde{w}_1(m)}{\widetilde{w}_2(m)}}\\ 1\end{pmatrix}\widehat{\psi}^T_{-m}\right] < \infty. \tag{2.29}$$

Moreover, the map $\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix} \mapsto \begin{pmatrix}\varphi_{xxxx}(t,0)\\ \psi_{xxxx}(t,0)\end{pmatrix}$ is continuous. Indeed, from (2.28) and (2.29) we deduce that

$$\left\|\begin{pmatrix}\varphi_{xxxx}(t,0)\\ \psi_{xxxx}(t,0)\end{pmatrix}\right\|_{(L^2(0,T))^2} \le C\left\|\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}\right\|_{V^4},$$

for some constant $C>0$. Thus, due to the considerations above, from now on $\begin{pmatrix}\varphi_{xxxx}(t,0)\\ \psi_{xxxx}(t,0)\end{pmatrix}$ denotes the sum (2.28), whenever $\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix} \in V^4$.

(iii) We proceed in several steps.

- (2.26) holds when $\begin{pmatrix}u^0\\ v^0\end{pmatrix} \in V^5$, $\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix} \in V^4$ and $f_i, g_i \in C_0^2([0,T])$, for $i,j = 0,1,2,3,4$.

First, suppose that $\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix} \in V^5$ and invoke (2.28). Since $V^4 \subset V^0 \subset [V^4]'$, where each embedding is dense, the result follows from (ii) and the density of V^5 in V^4.

- Let $S \in [0,T]$ be fixed. Then, (2.26) defines $\begin{pmatrix}u(S,\cdot)\\ v(S,\cdot)\end{pmatrix}$ in $[V^4]'$ in a unique manner.

Observe that, from the proof of (ii) we deduce that the map $\Gamma : V^4 \to \mathbb{C}$, given by

$$\Gamma\left(\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}\right) = -\left\langle \begin{pmatrix} f_0(t)\\ g_0(t)\end{pmatrix}, \begin{pmatrix} \overline{\psi(t,0)} + c\overline{\psi_{xx}(t,0)} + c_1\overline{\psi_{xxxx}(t,0)}\\ \overline{\varphi(t,0)} + a\overline{\varphi_{xx}(t,0)} + a_1\overline{\varphi_{xxxx}(t,0)}\end{pmatrix}\right\rangle_{(L^2(0,T))^2\times(L^2(0,T))^2}$$
$$+\left\langle \begin{pmatrix} f_1(t)\\ g_1(t)\end{pmatrix}, \begin{pmatrix} c\overline{\psi_x(t,0)} + c_1\overline{\psi_{xxx}(t,0)}\\ a\overline{\varphi_x(t,0)} + a_1\overline{\varphi_{xxx}(t,0)}\end{pmatrix}\right\rangle_{(L^2(0,T))^2\times(L^2(0,T))^2}$$
$$-\left\langle \begin{pmatrix} f_2(t)\\ g_2(t)\end{pmatrix}, \begin{pmatrix} c\overline{\psi(t,0)} + c_1\overline{\psi_{xx}(t,0)}\\ a\overline{\varphi(t,0)} + a_1\overline{\varphi_{xx}(t,0)}\end{pmatrix}\right\rangle_{(L^2(0,T))^2\times(L^2(0,T))^2}$$
$$\left\langle \begin{pmatrix} f_3(t)\\ g_3(t)\end{pmatrix}, \begin{pmatrix} c_1\overline{\psi_x(t,0)}\\ a_1\overline{\varphi_x(t,0)}\end{pmatrix}\right\rangle_{(L^2(0,T))^2\times(L^2(0,T))^2} - \left\langle \begin{pmatrix} f_4(t)\\ g_4(t)\end{pmatrix}, \begin{pmatrix} c_1\overline{\psi(t,0)}\\ a_1\overline{\varphi(t,0)}\end{pmatrix}\right\rangle_{(L^2(0,T))^2\times(L^2(0,T))^2},$$

is linear and continuous, where $\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$ is the solution of the adjoint system (2.17) with initial data $\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}$. On the other hand, the well-posedness of the adjoint system (see Theorem 2.1.3) allows us to conclude that the map $T_S : V^4 \to V^4$, given by

$$T_S\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix} = \begin{pmatrix}\varphi(S,\cdot)\\ \psi(S,\cdot)\end{pmatrix},$$

is an automorphism of Hilbert space. This implies that, for each $S \in [0,T]$, $\begin{pmatrix}u(S,\cdot)\\ v(S,\cdot)\end{pmatrix}$ is uniquely defined in $[V^4]'$. Moreover, for $S \in [0,T]$, we obtain the following estimate:

$$\begin{aligned}
\left\|\begin{pmatrix}u(S,\cdot)\\ v(S,\cdot)\end{pmatrix}\right\|_{[V^4]'} &= \sup_{\left\|\begin{pmatrix}\varphi(S,\cdot)\\ \psi(S,\cdot)\end{pmatrix}\right\|_{V^4}\le 1} \left|\left\langle \begin{pmatrix}u(S,\cdot)\\ v(S,\cdot)\end{pmatrix}, \begin{pmatrix}\varphi(S,\cdot)\\ \psi(S,\cdot)\end{pmatrix}\right\rangle\right| \\
&= \sup_{\left\|\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix}\right\|_{V^4}\le 1} \left|\left\langle \begin{pmatrix}u_0\\ v_0\end{pmatrix}, \begin{pmatrix}\varphi(0,\cdot)\\ \psi(0,\cdot)\end{pmatrix}\right\rangle_{[V^4]'\times V^4} - \Gamma\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix}\right| \\
&\le \sup_{\left\|\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix}\right\|_{V^4}\le 1} \left(\left\|\begin{pmatrix}u_0\\ v_0\end{pmatrix}\right\|_{[V^4]'}\left\|\begin{pmatrix}\varphi(0,\cdot)\\ \psi(0,\cdot)\end{pmatrix}\right\|_{V^4} + \left|\Gamma\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix}\right|\right) \qquad (2.30)\\
&\le C\left(\left\|\begin{pmatrix}u_0\\ v_0\end{pmatrix}\right\|_{[V^4]'} + \left\|\begin{pmatrix}f_0(t)\\ g_0(t)\end{pmatrix}\right\|_{(L^2(0,T))^2} + \left\|\begin{pmatrix}f_1(t)\\ g_1(t)\end{pmatrix}\right\|_{(L^2(0,T))^2}\right.\\
&\left.+\left\|\begin{pmatrix}f_2(t)\\ g_2(t)\end{pmatrix}\right\|_{(L^2(0,T))^2} + \left\|\begin{pmatrix}f_3(t)\\ g_3(t)\end{pmatrix}\right\|_{(L^2(0,T))^2} + \left\|\begin{pmatrix}f_4(t)\\ g_4(t)\end{pmatrix}\right\|_{(L^2(0,T))^2}\right),
\end{aligned}$$

where C is a positive constant which does not depend on S or on u_0, v_0, f_j, g_j, for $j = 0,1,2,3,4$.

- $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T];[V^4]')$.

First, observe that, from (i) we have that $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T];V^0) \subset C([0,T];[V^4]')$, whenever $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in V^5$ and $f_j, g_j \in C_0^2[(0,T)]$, for $j = 0,1,2,3,4$. Since V^5 is dense in V^0 and $C_0^2[(0,T)]$ is dense in $L^2(0,T)$, from (2.30) it follows that $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T];[V^4]')$. $\square$

Proposition 2.2.1 leads to the following definition:

Definition 2.2.1. *For* $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in [V^4]'$ *and* $f_j, g_j \in L^2(0,T)$, *with* $j = 0,1,2,3,4$, *a weak solution of* (2.22)-(2.24) *is a function* $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T];[V^4]')$, *such that* (2.26) *holds true for all* $\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \in V^4$ *and all* $S \in [0,T]$.

In order to prove the controllability result applying the Hilbert Uniqueness Method, we have to prove an *observability inequality* for the solutions of the corresponding adjoint system. Here, this is done using the so-called Ingham's inequality. For the sake of completeness and in order to facilitate the reading of the thesis, let us give below a generalization of Ingham's inequality (see, for instance, [2, 22]).

Theorem 2.2.1. *Let* $(\mu_k)_{k\in\mathbb{Z}}$ *be a sequence of distinct real numbers verifying*

$$\liminf_{|k|\to\infty} (\mu_{k+1} - \mu_k) \geq \gamma > 0. \tag{2.31}$$

For any $T > \dfrac{2\pi}{\gamma}$, *there exist constants* $C_i(T) > 0$, $i = 1,2$, *such that*

$$C_1 \sum_{k\in\mathbb{Z}} |a_k|^2 \leq \int_0^T \left| \sum_{k\in\mathbb{Z}} a_k e^{i\,\mu_k t} \right|^2 \, dt \leq C_2 \sum_{k\in\mathbb{Z}} |a_k|^2, \tag{2.32}$$

for any sequence of scalars $(a_k)_{k\in\mathbb{Z}} \in \ell^2$.

Then, we have the following result:

Proposition 2.2.2. *Let* $T > 0$. *Then, there exist positive constants* C *and* $\widetilde{C}$, *such that, for any* $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \in V^4$,

$$\begin{aligned} C \left\| \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \right\|_{V^4} &\leq \left\| \begin{pmatrix} \varphi + a\varphi_{xx} + a_1\varphi_{xxxx} \\ \psi + c\psi_{xx} + c_1\psi_{xxxx} \end{pmatrix} (t,0) \right\|^2_{(L^2(0,T))^2} \\ &+ \left\| \begin{pmatrix} a\varphi_x + a_1\varphi_{xxx} \\ c\psi_x + c_1\psi_{xxx} \end{pmatrix} (t,0) \right\|^2_{(L^2(0,T))^2} + \left\| \begin{pmatrix} a\varphi + a_1\varphi_{xx} \\ c\psi + c_1\psi_{xx} \end{pmatrix} (t,0) \right\|^2_{(L^2(0,T))^2} \\ &+ \left\| \begin{pmatrix} a_1\varphi_x \\ c_1\psi_x \end{pmatrix} (t,0) \right\|^2_{(L^2(0,T))^2} + \left\| \begin{pmatrix} a_1\varphi \\ c_1\psi \end{pmatrix} (t,0) \right\|^2_{(L^2(0,T))^2} \leq \widetilde{C} \left\| \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \right\|_{V^4}, \end{aligned} \tag{2.33}$$

where $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ *is a solution of the adjoint system* (2.17) *with initial data* $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix}$.

Proof. We first prove the left inequality assuming that the right one holds. Let us consider $\lambda_k, k \in \mathbb{Z}$, the eigenvalues of the operator A^*, the state operator associate to the adjoint system. Remark that they coincide with the eigenvalues of A defined by (2.14). Since $b = d = b_1 = d_1 = 0$,

$$\lim_{|k|\to\infty} |\lambda_k| = \infty.$$

Then, according to Remark 2.1.1, there exists $N_1 > 0$, such that, for $|k| > N_1$, the eigenvalues has multiplicity one. In particular, for $|k| > N_1$,

$$\begin{aligned}&\frac{1}{2}\sum_{k\in\mathbb{I}} e^{ik\sigma(k)(T-t)} \sum_{m\in I(k)} \left[\begin{pmatrix} 1 \\ -\sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}} \end{pmatrix} e^{imx}\hat{\varphi}^T_m + \begin{pmatrix} -\sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}} \\ 1 \end{pmatrix} e^{imx}\hat{\psi}^T_m \right. \\ &\left. + \begin{pmatrix} 1 \\ \sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}} \end{pmatrix} e^{-imx}\hat{\varphi}^T_{-m} + \begin{pmatrix} \sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}} \\ 1 \end{pmatrix} e^{-imx}\hat{\psi}^T_{-m} \right] = \sum_{|k|>N_1} e^{ik\sigma(k)(T-t)} \begin{pmatrix} \hat{\varphi}^T_k \\ \hat{\psi}^T_k \end{pmatrix} e^{ikx}.\end{aligned} \tag{2.34}$$

In addition, if we take $T_1 \in (0, T)$ and $\gamma > \frac{2\pi}{T_1}$, there exists $N_2 \in \mathbb{N}$, such that

$$k \in \mathbb{Z}, |k| \geq N_2 \Longrightarrow (k+1)\sigma(k+1) - k\sigma(k) \geq \gamma. \tag{2.35}$$

Also, taking Remark 2.1.1 into account, we introduce $W_n = \text{Span}\left\{ \begin{pmatrix} e^{ikx} \\ -\frac{\sigma(k)}{w_1} e^{ikx} \end{pmatrix}, \begin{pmatrix} e^{-ikx} \\ \frac{\sigma(k)}{w_1} e^{-ikx} \end{pmatrix} \right\}$, for $k \in I(n)$ and $n \in \mathbb{I}$, and consider $W = \bigoplus_{n\in\mathbb{I}} W_n \subset V^4$, whose embedding is dense. In W we define the following semi-norm:

$$\begin{aligned}\left[p\begin{pmatrix} u \\ v \end{pmatrix} \right]^2 = &\left| \begin{pmatrix} u(0) + au''(0) + a_1u''''(0) \\ v(0) + cv''(0) + c_1v''''(0) \end{pmatrix} \right|^2 + \left| \begin{pmatrix} au'(0) + a_1u'''(0) \\ cv'(0) + c_1v'''(0) \end{pmatrix} \right|^2 \\ &+ \left| \begin{pmatrix} au(0) + a_1u''(0) \\ cv(0) + c_1v''(0) \end{pmatrix} \right|^2 + \left| \begin{pmatrix} a_1u'(0) \\ c_1v'(0) \end{pmatrix} \right|^2 + \left| \begin{pmatrix} a_1u(0) \\ c_1v(0) \end{pmatrix} \right|^2, \ \forall \begin{pmatrix} u \\ v \end{pmatrix} \in W.\end{aligned}$$

Let $N = \max\{N_1, N_2\}$ and $\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \in W \cap (\bigoplus_{|n|<N} W_n)^\perp$, that is, $\begin{pmatrix} \hat{\varphi}^T_n \\ \hat{\psi}^T_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $|k| < N$ or for $|k|$ large enough. Then, by (2.34), (2.35) and Ingham's inequality, we obtain $C^{T_1} > 0$, such that

$$\begin{aligned}\left\| \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\|^2_{V^4} &\lesssim \sum_{|n|\geq N} \begin{pmatrix} (a_1 + a_1n + a_1n^2 + a_1n^3 + a_1n^4)^2|\hat{\varphi}^T_n|^2 \\ (c_1 + c_1n + c_1n^2 + c_1n^3 + c_1n^4)^2|\hat{\psi}^T_n|^2 \end{pmatrix}, \\ &\leq C^{T_1}\int_0^{T_1} \left| \sum_{|n|\geq N} \begin{pmatrix} (a_1 + a_1n + a_1n^2 + a_1n^3 + a_1n^4)\hat{\varphi}^T_n \\ (c_1 + c_1n + c_1n^2 + c_1n^3 + c_1n^4)\hat{\psi}^T_n \end{pmatrix} e^{in\sigma(n)(T-t)} \right|^2 dt\end{aligned}$$

thus

$$\left\|\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix}\right\|_{V^4}^2 \lesssim C^{T_1}\int_0^{T_1}\Big|\sum_{|n|\geq N}\Big[\begin{pmatrix}(1-an^2+a_1n^4)\hat{\varphi}_n^T\\ (1-cn^2+c_1n^4)\hat{\psi}_n^T\end{pmatrix}-\begin{pmatrix}(an-a_1n^3)\hat{\varphi}_n^T\\ (cn-c_1n^3)\hat{\psi}_n^T\end{pmatrix}$$

$$-\begin{pmatrix}\frac{a_1+a}{a_1}(a-a_1n^2)\hat{\varphi}_n^T\\ \frac{c_1+c}{c_1}(c-c_1n^2)\hat{\psi}_n^T\end{pmatrix}+\begin{pmatrix}\frac{a_1+a}{a_1}a_1n\hat{\varphi}_n^T\\ \frac{c_1+c}{c_1}c_1n\hat{\psi}_n^T\end{pmatrix}+\begin{pmatrix}\frac{a_1^2-a_1+aa_1+a^2}{a_1^2}a_1\hat{\varphi}_n^T\\ \frac{c_1^2-c_1+cc_1+c^2}{c_1^2}c_1\hat{\psi}_n^T\end{pmatrix}\Big]e^{in\sigma(n)(T-t)}\Big|^2dt$$

$$\leq C^{T_1}\int_0^{T_1}\Big[\Big|\sum_{|n|\geq N}\begin{pmatrix}(1-an^2+a_1n^4)\hat{\varphi}_n^T\\ (1-cn^2+c_1n^4)\hat{\psi}_n^T\end{pmatrix}\Big|^2+\Big|\sum_{|n|\geq N}\begin{pmatrix}(an-a_1n^3)\hat{\varphi}_n^T\\ (cn-c_1n^3)\hat{\psi}_n^T\end{pmatrix}\Big|^2$$

$$+\Big|\sum_{|n|\geq N}\begin{pmatrix}|\frac{a_1+a}{a_1}|(a-a_1n^2)\hat{\varphi}_n^T\\ |\frac{c_1+c}{c_1}|(c-c_1n^2)\hat{\psi}_n^T\end{pmatrix}\Big|^2+\Big|\sum_{|n|\geq N}\begin{pmatrix}|\frac{a_1+a}{a_1}|a_1n\hat{\varphi}_n^T\\ |\frac{c_1+c}{c_1}|c_1n\hat{\psi}_n^T\end{pmatrix}\Big|^2$$

$$+\Big|\sum_{|n|\geq N}\begin{pmatrix}\frac{a_1^2-a_1+aa_1+a^2}{a_1^2}a_1\hat{\varphi}_n^T\\ \frac{c_1^2-c_1+cc_1+c^2}{c_1^2}c_1\hat{\psi}_n^T\end{pmatrix}\Big|^2\Big]dt\leq C^{T_1}\int_0^{T_1}p\begin{pmatrix}\varphi(t,\cdot)\\ \psi(t,\cdot)\end{pmatrix}^2dt.$$

Since $T>T_1$, from the above estimate, [25, Theorem 5.3] and the right inequality in (2.33), we obtain $C^T>0$, such that

$$C^T\left\|\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix}\right\|_{V^4}\leq\int_0^T p\begin{pmatrix}\varphi(t,\cdot)\\ \psi(t,\cdot)\end{pmatrix}^2dt$$

$$\leq\left\|\begin{pmatrix}\varphi+a\varphi_{xx}+a_1\varphi_{xxxx}\\ \psi+c\psi_{xx}+c_1\psi_{xxxx}\end{pmatrix}(t,0)\right\|_{(L^2(0,T))^2}^2+\left\|\begin{pmatrix}a\varphi_x+a_1\varphi_{xxx}\\ c\psi_x+c_1\psi_{xxx}\end{pmatrix}(t,0)\right\|_{(L^2(0,T))^2}^2$$

$$+\left\|\begin{pmatrix}a\varphi+a_1\varphi_{xx}\\ c\psi+c_1\psi_{xx}\end{pmatrix}(t,0)\right\|_{(L^2(0,T))^2}^2+\left\|\begin{pmatrix}a_1\varphi_x\\ c_1\psi_x\end{pmatrix}(t,0)\right\|_{(L^2(0,T))^2}^2+\left\|\begin{pmatrix}a_1\varphi\\ c_1\psi\end{pmatrix}(t,0)\right\|_{(L^2(0,T))^2}^2,$$

$\forall\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}\in\mathbf{W}$. The general case follows from a density argument.

Now, we prove the right inequality in (2.33). Observe that, from the considerations above it follows that

$$\begin{pmatrix}\varphi_{xxxx}\\ \psi_{xxxx}\end{pmatrix}(t,0)=\frac{1}{2}\sum_{|k|<N_1}e^{ik\sigma(k)(T-t)}\sum_{m\in I(k)}m^4\Bigg[\begin{pmatrix}1\\ -\sqrt{\frac{w_2(m)}{w_1(m)}}\end{pmatrix}\widehat{\varphi}_m^T+\begin{pmatrix}-\sqrt{\frac{w_1(m)}{w_2(m)}}\\ 1\end{pmatrix}\widehat{\psi}_m^T$$

$$+\begin{pmatrix}1\\ \sqrt{\frac{w_2(m)}{w_1(m)}}\end{pmatrix}\widehat{\varphi}_{-m}^T+\begin{pmatrix}\sqrt{\frac{w_1(m)}{w_2(m)}}\\ 1\end{pmatrix}\widehat{\psi}_{-m}^T\Bigg]$$

$$+\frac{1}{2}\sum_{|k|\geq N_1}e^{ik\sigma(k)(T-t)}k^4\Bigg[\begin{pmatrix}1\\ -\sqrt{\frac{w_2(k)}{w_1(k)}}\end{pmatrix}\widehat{\varphi}_k^T+\begin{pmatrix}-\sqrt{\frac{w_1(k)}{w_2(k)}}\\ 1\end{pmatrix}\widehat{\psi}_k^T$$

$$+\begin{pmatrix}1\\ \sqrt{\frac{w_2(k)}{w_1(k)}}\end{pmatrix}\widehat{\varphi}_{-k}^+\begin{pmatrix}\sqrt{\frac{w_1(k)}{w_2(k)}}\\ 1\end{pmatrix}\widehat{\psi}_{-k}^T\Bigg].$$

Then,

$$\left\|\begin{pmatrix}\varphi_{xxxx}\\ \psi_{xxxx}\end{pmatrix}(t,0)\right\|^2_{(L^2(0,T))^2} \leq C_1\left(\left\|\sum_{|k|<N_1} k^4 e^{ik\sigma(k)(T-t)}\begin{pmatrix}\widehat{\varphi}_k^T\\ \widehat{\psi}_k^T\end{pmatrix}\right\|^2_{(L^2(0,T))^2}\right.$$
$$\left.+\left\|\sum_{|k|\geq N_1} k^4 e^{ik\sigma(k)(T-t)}\begin{pmatrix}\widehat{\varphi}_k^T\\ \widehat{\psi}_k^T\end{pmatrix}\right\|^2_{(L^2(0,T))^2}\right) \leq C_2\left\|\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}\right\|^2_{V^4},$$

where C_1 and C_2 are positive constants. Analogously, we obtain $C_3>0$, such that

$$\left\|\begin{pmatrix}\varphi_{xx}\\ \psi_{xx}\end{pmatrix}(t,0)\right\|^2_{(L^2(0,T))^2} \leq C_3\left\|\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}\right\|^2_{V^4}.$$

Thus,

$$\left\|\begin{pmatrix}\varphi+a\varphi_{xx}+a_1\varphi_{xxxx}\\ \psi+c\psi_{xx}+c_1\psi_{xxxx}\end{pmatrix}(t,0)\right\|^2_{(L^2(0,T))^2} \leq 3\left(\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}(t,0)\right\|^2_{(L^2(0,T))^2}+\left\|\begin{pmatrix}a\varphi_{xx}\\ c\psi_{xx}\end{pmatrix}(t,0)\right\|^2_{(L^2(0,T))^2}\right.$$
$$\left.+\left\|\begin{pmatrix}a_1\varphi_{xxxx}\\ c_1\psi_{xxxx}\end{pmatrix}(t,0)\right\|^2_{(L^2(0,T))^2}\right) \leq C_4\left\|\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}\right\|^2_{V^4},$$

for some constant $C_4>0$. The remaining terms in (2.33) are estimated in a similar way. □

Using Proposition 2.2.2 we prove our main result:

Theorem 2.2.2. *Let $T>0$. Then, for any $\begin{pmatrix}u^0\\ v^0\end{pmatrix}, \begin{pmatrix}u^T\\ v^T\end{pmatrix} \in [V^4]'$, there exist $f_j, g_j \in L^2(0,T)$, with $j=0,1,2,3,4$, such that the solution $\begin{pmatrix}u\\ v\end{pmatrix} \in C([0,T];[V^4]')$ of problem (2.22)-(2.24) satisfies $\begin{pmatrix}u(T,\cdot)\\ v(T,\cdot)\end{pmatrix} = \begin{pmatrix}u^T\\ v^T\end{pmatrix}$.*

Proof. We can assume that $\begin{pmatrix}u^0\\ v^0\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$. Let Λ denote the map

$$\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix} \in V^4 \mapsto \begin{pmatrix}u(T,\cdot)\\ v(T,\cdot)\end{pmatrix} \in [V^4]',$$

where $\begin{pmatrix}u\\ v\end{pmatrix}$ is the solution (weak) of (2.22)-(2.24) with f_j, g_j given by

$$\begin{aligned}
&f_0 = -(\psi(t,0)+c\psi_{xx}(t,0)+c_1\psi_{xxxx}(t,0)), && g_2 = -(a\varphi(t,0)+a_1\varphi_{xx}(t,0))\\
&g_0 = -(\varphi(t,0)+a\varphi_{xx}(t,0)+a_1\varphi_{xxxx}(t,0)), && f_3 = c_1\psi_x(t,0)\\
&f_1 = c\psi_x(t,0)+c_1\psi_{xxx}(t,0), && g_3 = a_1\varphi_x(t,0)\\
&g_1 = a\varphi_x(t,0)a_1+\varphi_{xxx}(t,0), && f_4 = -c_1\psi(t,0)\\
&f_2 = -(c\psi(t,0)+c_x\psi_{xx}(t,0)), && g_4 = -a_1\varphi(t,0),
\end{aligned} \tag{2.36}$$

where $\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$ is the solution of the adjoint system associated with $\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}$. Λ is linear and continuous (see (2.30) and (2.33)). Moreover, using Propositions 2.2.1 and 2.2.2 it follows that Λ is coercive, since

$$
\begin{aligned}
&\left\langle \Lambda \begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}, \begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}\right\rangle \\
&\quad = \left\| \begin{pmatrix}\varphi + a\varphi_{xx} + a_1\varphi_{xxxx}\\ \psi + c\psi_{xx} + c_1\psi_{xxxx}\end{pmatrix}(t,0)\right\|^2_{(L^2(0,T))^2} + \left\| \begin{pmatrix}a\varphi_x + a_1\varphi_{xxx}\\ c\psi_x + c_1\psi_{xxx}\end{pmatrix}(t,0)\right\|^2_{(L^2(0,T))^2} \\
&\quad + \left\| \begin{pmatrix}a\varphi + a_1\varphi_{xx}\\ c\psi + c_1\psi_{xx}\end{pmatrix}(t,0)\right\|^2_{(L^2(0,T))^2} + \left\| \begin{pmatrix}a_1\varphi_x\\ c_1\psi_x\end{pmatrix}(t,0)\right\|^2_{(L^2(0,T))^2} + \left\| \begin{pmatrix}a_1\varphi\\ c_1\psi\end{pmatrix}(t,0)\right\|^2_{(L^2(0,T))^2} \\
&\quad \geq C \left\| \begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}\right\|_{V^4}.
\end{aligned}
$$

Thus, by Lax-Milgram theorem it follows that Λ is invertible. Consequently, given $\begin{pmatrix}u^T\\ v^T\end{pmatrix} \in [V^4]'$, we can define $\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix} := \Lambda^{-1}\begin{pmatrix}u^T\\ v^T\end{pmatrix}$ to solve the adjoint system and get $\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in V^4$. Then, if the boundary functions f_j, g_j, with $j = 0,1,2,3,4$, are given by (2.36), the corresponding solution $\begin{pmatrix}u\\ v\end{pmatrix}$ of the system (2.22)-(2.24) satisfies

$$
\begin{pmatrix}u(0,\cdot)\\ v(0,\cdot)\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}u(T,\cdot)\\ v(T,\cdot)\end{pmatrix} = \begin{pmatrix}u^T\\ v^T\end{pmatrix}.
$$

□

If we assume that $\begin{pmatrix}f_0\\ g_0\end{pmatrix} = \begin{pmatrix}f_2\\ g_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$ in (2.22)-(2.24), the same arguments as above yield the following result in smaller spaces of initial data.

Theorem 2.2.3. *For every $T > 0$ and $\begin{pmatrix}u^0\\ v^0\end{pmatrix}, \begin{pmatrix}u^T\\ v^T\end{pmatrix} \in [V^3]'$, there exist $\begin{pmatrix}f_1\\ g_1\end{pmatrix}, \begin{pmatrix}f_3\\ g_3\end{pmatrix}, \begin{pmatrix}f_4\\ g_4\end{pmatrix} \in (L^2(0,T))^2$, such that the solution $\begin{pmatrix}u\\ v\end{pmatrix} \in C([0,T];[V^3]')$ of (2.22)-(2.24), with $\begin{pmatrix}f_0\\ g_0\end{pmatrix} = \begin{pmatrix}f_2\\ g_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$, satisfies $\begin{pmatrix}u(T,\cdot)\\ v(T,\cdot)\end{pmatrix} = \begin{pmatrix}u^T\\ v^T\end{pmatrix}$.*

Proof. Let $\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$ be a solution of the adjoint system with final data $\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix} \in V^3$. Proceedings as in (2.28)-(2.29) it can be shown that $\begin{pmatrix}\varphi_{xxx}(t,0)\\ \psi_{xxx}(t,0)\end{pmatrix}$ makes sense in $(L^2(0,T))^2$.

Moreover, the following observability inequality holds

$$C^{T_1}\left\|\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix}\right\|^2_{V^3}$$
$$\leq \left\|\begin{pmatrix}a\varphi_x(t,0)+a_1\varphi_{xxx}(t,0)\\ c\psi_x(t,0)+c_1\psi_{xxx}(t,0)\end{pmatrix}\right\|^2_{(L^2(0,T))^2}+\left\|\begin{pmatrix}a_1\varphi_x\\ c_1\psi_x\end{pmatrix}(t,0)\right\|^2_{(L^2(0,T))^2}+\left\|\begin{pmatrix}a_1\varphi\\ c_1\psi\end{pmatrix}(t,0)\right\|^2_{(L^2(0,T))^2},$$

for some $C^{T_1}>0$. Indeed,

$$\left\|\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix}\right\|^2_{V^3}\lesssim\sum_{|n|\geq N}\begin{pmatrix}(1+n^3)^2|\widehat{\varphi}^T_n|^2\\ (1+n^3)^2|\widehat{\psi}^T_n|^2\end{pmatrix}$$
$$\leq C^{T_1}\int_0^{T_1}\left|\sum_{|n|\geq N}\begin{pmatrix}(1+n^3)\widehat{\varphi}^T_n\\ (1+n^3)\widehat{\psi}^T_n\end{pmatrix}e^{in\sigma(n)t}\right|^2dt$$
$$=C^{T_1}\int_0^{T_1}\left|-\sum_{|n|\geq N}\begin{pmatrix}\frac{1}{a_1}(an-a_1n^3)\widehat{\varphi}^T_n\\ \frac{1}{c_1}(cn-c_1n^3)\widehat{\psi}^T_n\end{pmatrix}e^{in\sigma(n)t}+\begin{pmatrix}\frac{a}{a_1}n\widehat{\varphi}^T_n\\ \frac{c}{c_1}n\widehat{\psi}^T_n\end{pmatrix}e^{in\sigma(n)t}+\begin{pmatrix}\widehat{\varphi}^T_n\\ \widehat{\psi}^T_n\end{pmatrix}e^{in\sigma(n)t}\right|^2dt$$
$$\leq C^{T_1}\int_0^{T_1}\left(\left|\sum_{|n|\geq N}\begin{pmatrix}\frac{1}{a_1}(an-a_1n^3)\widehat{\varphi}^T_n\\ \frac{1}{c_1}(cn-c_1n^3)\widehat{\psi}^T_n\end{pmatrix}\right|^2+\left|\sum_{|n|\geq N}\begin{pmatrix}\frac{a}{a_1}n\widehat{\varphi}^T_n\\ \frac{c}{c_1}n\widehat{\psi}^T_n\end{pmatrix}\right|^2+\left|\sum_{|n|\geq N}\begin{pmatrix}\widehat{\varphi}^T_n\\ \widehat{\psi}^T_n\end{pmatrix}\right|^2\right)dt$$
$$\leq C^{T_1}\int_0^{T_1}\widetilde{p}\begin{pmatrix}\varphi(t,\cdot)\\ \psi(t,\cdot)\end{pmatrix}^2dt,$$

where

$$\left[\widetilde{p}\begin{pmatrix}u\\ v\end{pmatrix}\right]^2=\left|\begin{pmatrix}au'(0)+a_1u'''(0)\\ cv'(0)+c_1v'''(0)\end{pmatrix}\right|^2+\left|\begin{pmatrix}a_1u'(0)\\ c_1v'(0)\end{pmatrix}\right|^2+\left|\begin{pmatrix}a_1u(0)\\ c_1v(0)\end{pmatrix}\right|^2.$$

Then, the result is obtained following the arguments developed in the proofs of Proposition 2.2.2 and Theorem 2.2.2, respectively. □

The following remarks are in order:

Remark 2.2.1.

1. *If $T=2\pi$ the left inequality in Proposition 2.2.2 is easy to prove. Indeed, for any $\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix}\in V^4$ it follows that*

$$\left\|\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix}\right\|^2_{V^4}=\sum_{k\in\mathbb{Z}}\left(1+n^2+n^4+n^6+n^8\right)\left|\begin{pmatrix}\varphi^T_k\\ \psi^T_k\end{pmatrix}\right|^2$$
$$\leq C\sum_{k\in\mathbb{Z}}\left(\left|\begin{pmatrix}a_1\varphi^T_k\\ c_1\psi^T_k\end{pmatrix}\right|^2+\left|\begin{pmatrix}a_1n\varphi^T_k\\ c_1n\psi^T_k\end{pmatrix}\right|^2+\left|\begin{pmatrix}(a-a_1n^2)\varphi^T_k\\ (c-c_1n^2)\psi^T_k\end{pmatrix}\right|^2+\left|\begin{pmatrix}(an-a_1n^3)\varphi^T_k\\ (cn-c_1n^3)\psi^T_k\end{pmatrix}\right|^2\right.$$
$$\left.+\left|\begin{pmatrix}(1-an^2+a_1n^4)\varphi^T_k\\ (1-cn^2+c_1n^4)\psi^T_k\end{pmatrix}\right|^2\right),$$

thus

$$\left\|\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix}\right\|_{V^4}^2 \leq C\left(\left\|\begin{pmatrix}\varphi+a\varphi_{xx}+a_1\varphi_{xxxx}\\ \psi+c\psi_{xx}+c_1\psi_{xxxx}\end{pmatrix}(t,0)\right\|_{(L^2(0,2\pi)^2}^2+\left\|\begin{pmatrix}a\varphi_x+a_1\varphi_{xxx}\\ c\psi_x+c_1\psi_{xxx}\end{pmatrix}(t,0)\right\|_{(L^2(0,2\pi)^2}^2\right.$$
$$\left.+\left\|\begin{pmatrix}a\varphi+a_1\varphi_{xx}\\ c\psi+c_1\psi_{xx}\end{pmatrix}(t,0)\right\|_{(L^2(0,2\pi)^2}^2+\left\|\begin{pmatrix}a_1\varphi_x\\ c_1\psi_x\end{pmatrix}(t,0)\right\|_{(L^2(0,2\pi)^2}^2+\left\|\begin{pmatrix}a_1\varphi\\ c_1\psi\end{pmatrix}(t,0)\right\|_{(L^2(0,2\pi)^2}^2\right),$$

where C only depends on the parameters a, a_1, c, c_1.

2. *A similar approach as the one given in the proofs of Theorems 2.2.2 and 2.2.3 allows to show that the lower order Boussinesq system of KdV-KdV type (system (1.10) with $b = d = 0$) is controllable.*

2.2.2 KdV-BBM

This section is devoted to the study of the system

$$\begin{cases} u_t+v_x-bu_{txx}+av_{xxx}+a_1v_{xxxxx}=0, & \text{in} \quad (0,T)\times(0,2\pi)\\ v_t+u_x-du_{txx}+cu_{xxx}+c_1u_{xxxxx}=0, & \text{in} \quad (0,T)\times(0,2\pi), \end{cases} \tag{2.37}$$

with boundary conditions

$$\begin{cases} \partial_x^j u(t,2\pi)-\partial_x^j u(t,0)=0, & \text{in} \quad (0,T)\\ \partial_x^j v(t,2\pi)-\partial_x^j v(t,0)=0, & \text{in} \quad (0,T)\\ \partial_x^2 u(t,2\pi)-\partial_x^2 u(t,0)=f_2(t), & \text{in} \quad (0,T)\\ \partial_x^2 v(t,2\pi)-\partial_x^2 v(t,0)=g_2(t), & \text{in} \quad (0,T)\\ \partial_x^4 u(t,2\pi)-\partial_x^4 u(t,0)=f_4(t), & \text{in} \quad (0,T)\\ \partial_x^4 v(t,2\pi)-\partial_x^4 v(t,0)=g_4(t), & \text{in} \quad (0,T), \end{cases} \tag{2.38}$$

for $j = 0, 1, 3$, and initial conditions

$$u(0,x)=u^0(x), \quad v(0,x)=v^0(x), \text{ in } \quad (0,2\pi). \tag{2.39}$$

We first prove that system (2.37)-(2.39) is well-posed.

Proposition 2.2.3. *For any $s \in \mathbb{R}$, let V^s be the Hilbert space defined by (2.6). Then, the following well-posedness results hold:*

(i) Suppose that $f_2(t), g_2(t), f_4(t), g_4(t) \in C_0^2[0,T]$ and $\begin{pmatrix}u^0\\ v^0\end{pmatrix} \in V^5$. Then, there exists a unique solution $\begin{pmatrix}u\\ v\end{pmatrix} \in C^1([0,T];V^2)\cap C([0,T];V^5)$ of (2.37)-(2.39). Moreover, for

any $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \in V^5$ *and* $S \in [0,T]$, *we have*

$$\begin{aligned}
&\left\langle \begin{pmatrix} u(S,\cdot) - bu_{xx}(S,\cdot) \\ v(S,\cdot) - dv_{xx}(S,\cdot) \end{pmatrix}, \begin{pmatrix} \overline{\varphi(S,\cdot)} \\ \overline{\psi(S,\cdot)} \end{pmatrix} \right\rangle_{V^0 \times V^0} \\
&= \left\langle \begin{pmatrix} u^0 - bu^0_{xx} \\ v^0 - dv^0_{xx} \end{pmatrix}, \begin{pmatrix} \overline{\varphi(0,\cdot)} \\ \overline{\psi(0,\cdot)} \end{pmatrix} \right\rangle_{V^0 \times V^0} \\
&- \left\langle \begin{pmatrix} f_2(t) \\ g_2(t) \end{pmatrix}, \begin{pmatrix} \overline{c\psi(t,0) + c_1\psi_{xx}(t,0)} \\ \overline{a\varphi(t,0) + a_1\varphi_{xx}(t,0)} \end{pmatrix} \right\rangle_{(L^2(0,S))^2 \times (L^2(0,2\pi))^2} \\
&- \left\langle \begin{pmatrix} f_4(t) \\ g_4(t) \end{pmatrix}, \begin{pmatrix} \overline{c_1\psi(t,0)} \\ \overline{a_1\varphi(t,0)} \end{pmatrix} \right\rangle_{(L^2(0,S))^2 \times (L^2(0,2\pi))^2},
\end{aligned} \tag{2.40}$$

where $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in C^1([0,T];V^2) \cap C([0,T];V^5)$ *is the solution of the adjoint system* (2.17) *with initial data* $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix}$ *given by Theorem 2.1.3.*

(ii) If $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \in V^2$, *there exists a unique solution* $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in C([0,T];V^2)$ *of* (2.17) *and* $\begin{pmatrix} \varphi_{xx}(t,0) \\ \psi_{xx}(t,0) \end{pmatrix}$ *makes sense in* $(L^2(0,T))^2$.

(iii) Assume that $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \in V^0$ *and* $f_j, g_j \in L^2(0,T)$, *for* $j = 2,4$. *Then, there exists a unique* $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T];[V^0]^2)$, *such that, for any* $\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \in V^2$ *and any* $S \in [0,T]$, *we have*

$$\begin{aligned}
&\left\langle \begin{pmatrix} u(S,\cdot) - bu_{xx}(S,\cdot) \\ v(S,\cdot) - dv_{xx}(S,\cdot) \end{pmatrix}, \begin{pmatrix} \overline{\varphi(S,\cdot)} \\ \overline{\psi(S,\cdot)} \end{pmatrix} \right\rangle_{[V^2]' \times V^2} = \left\langle \begin{pmatrix} u^0 - bu^0_{xx} \\ v^0 - dv^0_{xx} \end{pmatrix}, \begin{pmatrix} \overline{\varphi(0,\cdot)} \\ \overline{\psi(0,\cdot)} \end{pmatrix} \right\rangle_{[V^2]' \times V^2} \\
&- \left\langle \begin{pmatrix} f_2(t) \\ g_2(t) \end{pmatrix}, \begin{pmatrix} \overline{c\psi(t,0) + c_1\psi_{xx}(t,0)} \\ \overline{a\varphi(t,0) + a_1\varphi_{xx}(t,0)} \end{pmatrix} \right\rangle_{(L^2(0,S))^2 \times (L^2(0,2\pi))^2} \\
&- \left\langle \begin{pmatrix} f_4(t) \\ g_4(t) \end{pmatrix}, \begin{pmatrix} \overline{c_1\psi(t,0)} \\ \overline{a_1\varphi(t,0)} \end{pmatrix} \right\rangle_{(L^2(0,S))^2 \times (L^2(0,2\pi))^2},
\end{aligned} \tag{2.41}$$

where $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in C([0,T];V^2)$ *is the solution of the adjoint system* (2.17) *with initial data* $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix}$ *given by (ii).*

Proof. (i) Consider $\theta_j, \phi_j \in C^\infty(0, 2\pi)$, with $\theta_j^{(k)}(0) = 0 = \phi_j^{(k)}(0)$ and $\theta_j^{(k)}(2\pi) = -\delta_j^k = \phi_j^{(k)}(2\pi)$, for $j = 2, 4,\ k = 0, 1, 2, 3, 4$. Denoting by $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ the solution of the corresponding homogeneous system given by Theorem 2.1.2, the change of functions

$$\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} \sum_{j=2,4} \theta_j(x) f_j(t) \\ \sum_{j=2,4} \phi_j(x) g_j(t) \end{pmatrix}$$

yields an equivalent problem: Find $\begin{pmatrix} z \\ w \end{pmatrix}$, such that

$$\begin{cases} z_t + w_x - bz_{txx} + aw_{xxx} + a_1 w_{xxxxx} = \\ \qquad \displaystyle\sum_{j=2,4} [(\theta_j(x) - b\theta_j''(x)) f_j'(t) + g_j(t)(\phi_j'(x) + a\phi_j^{(3)}(x) + a_1 \phi_j^{(5)}(x))] := F \\ w_t + z_x - dw_{txx} + cz_{xxx} + c_1 z_{xxxxx} = \\ \qquad \displaystyle\sum_{j=2,4} [(\phi_j(x) - d\phi_j''(x)) g_j'(t) + f_j(t)(\theta_j'(x) + c\theta_j^{(3)}(x) + c_1 \theta_j^{(5)}(x))] := G \\ \partial_x^j z(t, 2\pi) = \partial_x^j z(t, 0), \quad \partial_x^j w(t, 2\pi) = \partial_x^j w(t, 0), \quad j = 0, 1, 2, 3, 4, \\ z(0, x) = 0, \quad w(0, x) = 0. \end{cases} \tag{2.42}$$

Since $F, G \in C^1([0, T]; C^\infty(0, 2\pi)) \subset L^1((0, T); V^5)$, from Theorem 2.1.2 we deduce that (2.42) admits a unique solution $\begin{pmatrix} z \\ w \end{pmatrix} \in C([0, T]; V^5) \cap C^1([0, T]; V^2)$. Hence, we have a unique solution $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0, T]; V^5) \cap C^1([0, T]; V^2)$ of (2.42). To obtain identity (2.40), we multiply the first equation of (2.37) by $\overline{\varphi}$, the second one by $\overline{\psi}$, integrate in time and space over $(0, T) \times (0, 2\pi)$ and add the resulting identities.

(ii) If $t_1, t_2 \in [0, T]$, from (2.21) we obtain

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix}(t_1, x) - \begin{pmatrix} \varphi \\ \psi \end{pmatrix}(t_2, x) = \frac{1}{2} \sum_{k \in \mathbb{I}} (e^{ik\sigma(k)(T - t_1)} - e^{ik\sigma(k)(T - t_2)}) \sum_{m \in I(k)} \left[\begin{pmatrix} 1 \\ -\sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}} \end{pmatrix} e^{imx} \hat{\varphi}_m^T \right.$$
$$\left. + \begin{pmatrix} -\sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}} \\ 1 \end{pmatrix} e^{imx} \hat{\psi}_m^T + \begin{pmatrix} 1 \\ \sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}} \end{pmatrix} e^{-imx} \hat{\varphi}_{-m}^T + \begin{pmatrix} \sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}} \\ 1 \end{pmatrix} e^{-imx} \hat{\psi}_{-m}^T \right].$$

Since $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \in V^2$, we have $\sum_{k \in \mathbb{Z}} (|\hat{\varphi}_k^T|^2 + |\hat{\psi}_k^T|^2)(1 + k^2)^2 < \infty$. Moreover, observe that, as $k \to \infty$, $\dfrac{\widetilde{w_2}(k)}{\widetilde{w_1}(k)} \to \dfrac{a_1}{c_1}$ and $\dfrac{\widetilde{w_1}(k)}{\widetilde{w_2}(k)} \to \dfrac{c_1}{a_1}$.

Then, we obtain a constant $C > 0$ satisfying

$$\frac{1}{2}\sum_{k\in\mathbb{I}}\left|(e^{ik\sigma(k)(T-t_1)}-e^{ik\sigma(k)(T-t_2)})\sum_{m\in I(k)}\left[\begin{pmatrix}1\\ -\sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}}\end{pmatrix}e^{imx}\hat{\varphi}^T_m+\begin{pmatrix}-\sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}}\\ 1\end{pmatrix}e^{imx}\hat{\psi}^T_m\right.\right.$$
$$\left.\left.+\begin{pmatrix}1\\ \sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}}\end{pmatrix}e^{-imx}\hat{\varphi}^T_{-m}+\begin{pmatrix}\sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}}\\ 1\end{pmatrix}e^{-imx}\hat{\psi}^T_{-m}\right]\right|^2(1+k^2)^2$$
$$\leq C\sum_{k\in\mathbb{Z}}(|(e^{ik\sigma(k)(T-t_1)}-e^{ik\sigma(k)(T-t_2)})|^2|\hat{\varphi}^T_k|^2+|\hat{\psi}^T_k|^2)(1+k^2)^2.$$

Hence, by Lebesgue's theorem, it follows that $\left|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}(t_1,x)-\begin{pmatrix}\varphi\\ \psi\end{pmatrix}(t_2,x)\right|\to 0$, as $t_1\to t_2$, which implies that $\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\in C([0,T];V^2)$. If $\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}\in V^5$, the same argument shows that

$$\begin{pmatrix}\varphi_{xx}(t,0)\\ \psi_{xx}(t,0)\end{pmatrix}=\frac{1}{2}\sum_{k\in\mathbb{I}}e^{ik\sigma(k)(T-t)}\sum_{m\in I(k)}m^2\left[\begin{pmatrix}1\\ -\sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}}\end{pmatrix}\widehat{\varphi}^T_m+\begin{pmatrix}-\sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}}\\ 1\end{pmatrix}\widehat{\psi}^T_m\right.$$
$$\left.+\begin{pmatrix}1\\ \sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}}\end{pmatrix}\widehat{\varphi}^T_{-m}+\begin{pmatrix}\sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}}\\ 1\end{pmatrix}\widehat{\psi}^T_{-m}\right]\in(L^2(0,T))^2. \tag{2.43}$$

Remark that, if $\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}\in V^2$, the sum above also makes sense in $(L^2(0,T))^2$, since

$$\sum_{k\in\mathbb{I}}\sum_{m\in I(k)}m^2\left[\begin{pmatrix}1\\ -\sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}}\end{pmatrix}\widehat{\varphi}^T_m+\begin{pmatrix}-\sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}}\\ 1\end{pmatrix}\widehat{\psi}^T_m\right.$$
$$\left.+\begin{pmatrix}1\\ \sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}}\end{pmatrix}\widehat{\varphi}^T_{-m}+\begin{pmatrix}\sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}}\\ 1\end{pmatrix}\widehat{\psi}^T_{-m}\right]<\infty. \tag{2.44}$$

Moreover, the map $\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}\mapsto\begin{pmatrix}\varphi_{xx}(t,0)\\ \psi_{xx}(t,0)\end{pmatrix}$ is continuous. Indeed, from (2.43) and (2.44) we deduce that

$$\left\|\begin{pmatrix}\varphi_{xx}(t,0)\\ \psi_{xx}(t,0)\end{pmatrix}\right\|_{(L^2(0,T))^2}\leq C\left\|\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}\right\|_{V^2},$$

for some constant $C > 0$. Thus, due to the considerations above, from now on $\begin{pmatrix}\varphi_{xx}(t,0)\\ \psi_{xx}(t,0)\end{pmatrix}$ denotes the sum (2.43), whenever $\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}\in V^2$.

(iii) We proceed in several steps.

• (2.41) holds when $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \in V^5$, $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \in V^2$ and $f_i, g_i \in C_0^2([0,T])$, for $i,j = 2,4$.

First, suppose that $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \in V^5$ and invoke (2.43). Since $V^2 \subset V^0 \subset [V^2]'$, where each embedding is dense, the result follows from (ii) and the density of V^5 in V^2.

• Let $S \in [0,T]$ be fixed. Then, (2.41) defines $\begin{pmatrix} u(S,\cdot) - u_{xx}(S,\cdot) \\ v(S,\cdot) - v_{xx}(S,\cdot) \end{pmatrix}$ in $[V^2]'$ in a unique manner.

Observe that, from the proof of (ii) we deduce that the map $\Gamma : V^2 \to \mathbb{C}$, given by

$$\begin{aligned}\Gamma\left(\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix}\right) = &-\left\langle \begin{pmatrix} f_2(t) \\ g_2(t) \end{pmatrix}, \begin{pmatrix} c\psi(t,0) + c_1\psi_{xx}(t,0) \\ a\varphi(t,0) + a_1\varphi_{xx}(t,0) \end{pmatrix} \right\rangle_{(L^2(0,S))^2 \times (L^2(0,2\pi))^2} \\ &- \left\langle \begin{pmatrix} f_4(t) \\ g_4(t) \end{pmatrix}, \begin{pmatrix} c_1\psi(t,0) \\ a_1\varphi(t,0) \end{pmatrix} \right\rangle_{(L^2(0,S))^2 \times (L^2(0,2\pi))^2},\end{aligned}$$

is linear and continuous, where $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ is the solution of the adjoint system (2.17) with initial data $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix}$. On the other hand, the well-posedness of the adjoint system (see Theorem 2.1.3) allows us to conclude that the map $T_S : V^2 \to V^2$, given by

$$T_S \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} = \begin{pmatrix} \varphi(S,\cdot) \\ \psi(S,\cdot) \end{pmatrix},$$

is an automorphism of Hilbert space. This implies that, for each $S \in [0,T]$, $\begin{pmatrix} u(S,\cdot) - bu_{xx}(S,\cdot) \\ v(S,\cdot) - dv_{xx}(S,\cdot) \end{pmatrix}$ is uniquely defined in $[V^2]'$. Moreover, for $S \in [0,T]$, we obtain the following estimate:

$$\begin{aligned}\left\| \begin{pmatrix} u(S,\cdot) - u_{xx}(S,\cdot) \\ v(S,\cdot) - v_{xx}(S,\cdot) \end{pmatrix} \right\|_{[V^2]'} &= \sup_{\left\| \begin{pmatrix} \varphi(S,\cdot) \\ \psi(S,\cdot) \end{pmatrix} \right\|_{V^2} \leq 1} \left| \left\langle \begin{pmatrix} u(S,\cdot) - u_{xx}(S,\cdot) \\ v(S,\cdot) - v_{xx}(S,\cdot) \end{pmatrix}, \begin{pmatrix} \varphi(S,\cdot) \\ \psi(S,\cdot) \end{pmatrix} \right\rangle \right| \\ &= \sup_{\left\| \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\|_{V^2} \leq 1} \left| \left\langle \begin{pmatrix} u_0 - u_{0,xx} \\ v_0 - v_{0,xx} \end{pmatrix}, \begin{pmatrix} \varphi(0,\cdot) \\ \psi(0,\cdot) \end{pmatrix} \right\rangle_{[V^2]' \times V^2} - \Gamma \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right| \\ &\leq \sup_{\left\| \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\|_{V^2} \leq 1} \left(\left\| \begin{pmatrix} u_0 - u_{0,xx} \\ v_0 - v_{0,xx} \end{pmatrix} \right\|_{[V^2]'} \left\| \begin{pmatrix} \varphi(0,\cdot) \\ \psi(0,\cdot) \end{pmatrix} \right\|_{V^2} + \left| \Gamma \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right| \right) \\ &\leq C \left(\left\| \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right\|_{V^0} + \left\| \begin{pmatrix} f_2(t) \\ g_2(t) \end{pmatrix} \right\|_{(L^2(0,T))^2} + \left\| \begin{pmatrix} f_4(t) \\ g_4(t) \end{pmatrix} \right\|_{(L^2(0,T))^2} \right),\end{aligned} \tag{2.45}$$

where C is a positive constant which does not depend on S or on u_0, v_0, f_j, g_j, for $j = 2, 4$.

- $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T]; V^0)$.

First, observe that, from (i) we have that $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T]; V^5) \subset C([0,T]; V^0)$, whenever $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in V^5$ and $f_j, g_j \in C_0^2[(0,T)]$, for $j = 2, 4$. Since V^5 is dense in V^0 and $C_0^2(0,T)$ is dense in $L^2(0,T)$, it is follow from (2.45) that $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T]; V^0)$. □

Proposition 2.2.3 leads to the following definition:

Definition 2.2.2. *For $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in V^0$ and $f_j, g_j \in L^2(0,T)$, with $j = 2, 4$, a weak solution of (2.37)-(2.39) is a function $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T]; V^0)$, such that (2.41) holds true for all $\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \in V^2$ and all $S \in [0,T]$.*

In order to prove the controllability result applying the Hilbert Uniqueness Method, we have to prove an *observability inequality* for the solutions of the corresponding adjoint system. Here, this is done using the so-called Ingham's inequality (see, for instance, [2, 22]) . For the sake of completeness and in order to facilitate the reading of the tese, we give a generalization of Ingham's inequality in Theorem 2.2.1.

Then, we have the following result:

Proposition 2.2.4. *Let $T > 0$. Then, there exist positive constants C and $\widetilde{C}$, such that, for any $\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \in V^2$,*

$$\begin{aligned} C \left\| \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\|_{V^2}^2 &\leq \left\| \begin{pmatrix} a\varphi + a_1\varphi_{xx} \\ c\psi + c_1\psi_{xx} \end{pmatrix} (t,0) \right\|_{(L^2(0,T))^2}^2 + \left\| \begin{pmatrix} a_1\varphi \\ c_1\psi \end{pmatrix} (t,0) \right\|_{(L^2(0,T))^2}^2 \\ &\leq \tilde{C} \left\| \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\|_{V^2}^2, \end{aligned} \tag{2.46}$$

where $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ is the solution of the adjoint system with data $\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix}$.

Proof. We first prove the left inequality assuming that the right one holds.

Let us consider λ_k, $k \in \mathbb{Z}$, the eigenvalues of the operator A^*, the state operator associate to the adjoint system. Remark that they coincide with the eigenvalues of A, defined by (2.14). Since $b_1 = d_1 = 0$,

$$\lim_{|k|\to\infty} |\lambda_k| = \infty.$$

Then, according to Remark 2.1.1, there exists $N_1 > 0$, such that, for $|k| > N_1$, the eigenvalues has multiplicity one. In particular, for $|k| > N_1$,

$$\begin{aligned}&\frac{1}{2}\sum_{k\in\mathbb{I}} e^{ik\sigma(k)(T-t)} \sum_{m\in I(k)} \left[\begin{pmatrix} 1 \\ -\sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}} \end{pmatrix} e^{imx}\hat{\varphi}^T_m + \begin{pmatrix} -\sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}} \\ 1 \end{pmatrix} e^{imx}\hat{\psi}^T_m \right. \\ &\left. + \begin{pmatrix} 1 \\ \sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}} \end{pmatrix} e^{-imx}\hat{\varphi}^T_{-m} + \begin{pmatrix} \sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}} \\ 1 \end{pmatrix} e^{-imx}\hat{\psi}^T_{-m} \right] = \sum_{|k|>N_1} e^{ik\sigma(k)(T-t)} \begin{pmatrix} \hat{\varphi}^T_k \\ \hat{\psi}^T_k \end{pmatrix} e^{ikx}.\end{aligned} \tag{2.47}$$

In addition, if we take $T_1 \in (0, T)$ and $\gamma > \frac{2\pi}{T_1}$, there exists $N_2 \in \mathbb{N}$, such that

$$k \in \mathbb{Z}, |k| \geq N_2 \Longrightarrow (k+1)\sigma(k+1) - k\sigma(k) \geq \gamma. \tag{2.48}$$

Also, taking Remark 2.1.1 into account, we introduce $W_n = \operatorname{Span}\left\{ \begin{pmatrix} e^{ikx} \\ -\frac{\sigma(k)}{w_1}e^{ikx} \end{pmatrix}, \begin{pmatrix} e^{-ikx} \\ \frac{\sigma(k)}{w_1}e^{-ikx} \end{pmatrix} \right\}$, for $k \in I(n)$ and $n \in \mathbb{I}$, and consider $W = \bigoplus_{n\in\mathbb{I}} W_n \subset V^2$, whose embedding is dense. In W we define the following semi-norm:

$$\left[p\begin{pmatrix} u \\ v \end{pmatrix} \right]^2 = \left| \begin{pmatrix} au(0) + a_1u''(0) \\ cv(0) + c_1v''(0) \end{pmatrix} \right|^2 + \left| \begin{pmatrix} a_1u(0) \\ c_1v(0) \end{pmatrix} \right|^2, \ \forall \begin{pmatrix} u \\ v \end{pmatrix} \in W.$$

Let $N = \max\{N_1, N_2\}$ and $\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \in W \cap (\bigoplus_{|n|<N} W_n)^\perp$, that is, $\begin{pmatrix} \hat{\varphi}^T_n \\ \hat{\psi}^T_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $|k| < N$ or for $|k|$ large enough. Then, by (2.59), (2.60) and Ingham's inequality, we obtain $C^{T_1} > 0$, such that

$$\begin{aligned}\left\| \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\|^2_{V^2} &\gtrsim \sum_{|n|\geq N} \begin{pmatrix} (1+a_1n^2)^2|\hat{\varphi}^T_n|^2 \\ (1+c_1n^2)^2|\hat{\psi}^T_n|^2 \end{pmatrix} \leq C^{T_1}\int_0^{T_1} \left| \sum_{|n|\geq N} \begin{pmatrix} (1+a_1n^2)\hat{\varphi}^T_n \\ (1+c_1n^2)\hat{\psi}^T_n \end{pmatrix} e^{in\sigma(n)t} \right|^2 dt \\ &= C^{T_1}\int_0^{T_1} \left| \sum_{|n|\geq N} \begin{pmatrix} -(a-an^2)\hat{\varphi}^T_n \\ -(c-c_1n^2)\hat{\psi}^T_n \end{pmatrix} e^{in\sigma(n)t} + \begin{pmatrix} \frac{1+a}{a_1}a_1\hat{\varphi}^T_n \\ \frac{1+c}{c_1}c_1\hat{\psi}^T_n \end{pmatrix} e^{in\sigma(n)t} \right|^2 dt \\ &\leq C^{T_1}\int_0^{T_1} \left[\left| \sum_{|n|\geq N} \begin{pmatrix} (a-a_1n^2)\hat{\varphi}^T_n \\ (c-c_1n^2)\hat{\psi}^T_n \end{pmatrix} e^{in\sigma(n)t} \right|^2 + \left| \sum_{|n|\geq N} \begin{pmatrix} |\frac{1+a}{a_1}|a_1\hat{\varphi}^T_n \\ |\frac{1+c}{c_1}|c_1\hat{\psi}^T_n \end{pmatrix} e^{in\sigma(n)t} \right|^2 \right] dt \\ &\leq C^{T_1}\int_0^{T_1} \left[\left| \sum_{|n|\geq N} \begin{pmatrix} (a-a_1n^2)\hat{\varphi}^T_n \\ (c-c_1n^2)\hat{\psi}^T_n \end{pmatrix} e^{in\sigma(n)t} \right|^2 + \left| \sum_{|n|\geq N} \begin{pmatrix} a_1\hat{\varphi}^T_n \\ c_1\hat{\psi}^T_n \end{pmatrix} e^{in\sigma(n)t} \right|^2 \right] dt \\ &\leq C^{T_1}\int_0^{T_1} p\begin{pmatrix} \varphi(t,\cdot) \\ \psi(t,\cdot) \end{pmatrix}^2 dt.\end{aligned}$$

Since $T > T_1$, from the above estimate, [25, Theorem 5.3] and the right inequality in (2.46), we obtain $C^T > 0$, such that

$$C^T \left\| \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\|^2_{V^2} \leq \int_0^T p\begin{pmatrix} \varphi(t,\cdot) \\ \psi(t,\cdot) \end{pmatrix}^2 \leq \left\| \begin{pmatrix} a\varphi + a_1\varphi_{xx} \\ c\psi + c_1\psi_{xx} \end{pmatrix}(t,0) \right\|^2_{(L^2(0,T))^2} + \left\| \begin{pmatrix} a_1\varphi \\ c_1\psi \end{pmatrix}(t,0) \right\|^2_{(L^2(0,T))^2},$$

$\forall \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \in W$. The general case follows from a density argument.

Now, we prove the right inequality in (2.33). Observe that, from the considerations above it follows that

$$
\begin{aligned}
\begin{pmatrix} \varphi_{xx} \\ \psi_{xx} \end{pmatrix}(t,0) = \; & \frac{1}{2} \sum_{|k|<N_1} e^{ik\sigma(k)(T-t)} \sum_{m\in I(k)} m^2 \left[\begin{pmatrix} 1 \\ -\sqrt{\frac{w_2(m)}{w_1(m)}} \end{pmatrix} \widehat{\varphi}^T_m + \begin{pmatrix} -\sqrt{\frac{w_1(m)}{w_2(m)}} \\ 1 \end{pmatrix} \widehat{\psi}^T_m \right. \\
& \left. + \begin{pmatrix} 1 \\ \sqrt{\frac{w_2(m)}{w_1(m)}} \end{pmatrix} \widehat{\varphi}^T_{-m} + \begin{pmatrix} \sqrt{\frac{w_1(m)}{w_2(m)}} \\ 1 \end{pmatrix} \widehat{\psi}^T_{-m} \right] \\
& + \frac{1}{2} \sum_{|k|\geq N_1} e^{ik\sigma(k)(T-t)} k^2 \left[\begin{pmatrix} 1 \\ -\sqrt{\frac{w_2(k)}{w_1(k)}} \end{pmatrix} \widehat{\varphi}^T_k + \begin{pmatrix} -\sqrt{\frac{w_1(k)}{w_2(k)}} \\ 1 \end{pmatrix} \widehat{\psi}^T_k \right. \\
& \left. + \begin{pmatrix} 1 \\ \sqrt{\frac{w_2(k)}{w_1(k)}} \end{pmatrix} \widehat{\varphi}^{+}_{-k} \begin{pmatrix} \sqrt{\frac{w_1(k)}{w_2(k)}} \\ 1 \end{pmatrix} \widehat{\psi}^T_{-k} \right].
\end{aligned}
$$

Then,

$$
\begin{aligned}
\left\| \begin{pmatrix} \varphi_{xx} \\ \psi_{xx} \end{pmatrix}(t,0) \right\|^2_{(L^2(0,T))^2} & \leq C_1 \left(\left\| \sum_{|k|<N_1} k^2 e^{ik\sigma(k)(T-t)} \begin{pmatrix} \widehat{\varphi}^T_k \\ \widehat{\psi}^T_k \end{pmatrix} \right\|^2_{(L^2(0,T))^2} \right. \\
& \left. + \left\| \sum_{|k|\geq N_1} k^2 e^{ik\sigma(k)(T-t)} \begin{pmatrix} \widehat{\varphi}^T_k \\ \widehat{\psi}^T_k \end{pmatrix} \right\|^2_{(L^2(0,T))^2} \right) \leq C_2 \left\| \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \right\|^2_{V^2},
\end{aligned}
$$

where C_1 and C_2 are positive constants. Analogously, we obtain $C_3 > 0$, such that

$$
\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}(t,0) \right\|^2_{(L^2(0,T))^2} \leq C_3 \left\| \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \right\|^2_{V^4}. \tag{2.49}
$$

Thus,

$$
\begin{aligned}
\left\| \begin{pmatrix} a\varphi + a_1\varphi_{xx} \\ c\psi + c_1\psi_{xx} \end{pmatrix}(t,0) \right\|^2_{(L^2(0,T))^2} & \leq 2 \left(\left\| \begin{pmatrix} a\varphi \\ c\psi \end{pmatrix}(t,0) \right\|^2_{(L^2(0,T))^2} \right. \\
& \left. + \left\| \begin{pmatrix} a_1\varphi_{xx} \\ c_1\psi_{xx} \end{pmatrix}(t,0) \right\|^2_{(L^2(0,T))^2} \right) \leq C_4 \left\| \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \right\|^2_{V^2},
\end{aligned}
$$

for some constant $C_4 > 0$. Then, the right inequality in (2.46) follows from (2.49) and the estimate above. □

Using Proposition 2.2.4 we prove our main result:

Theorem 2.2.4. *Let $T > 0$. Then, for any* $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \begin{pmatrix} u^T \\ v^T \end{pmatrix} \in V^0$, *there exist* $f_j, g_j \in L^2(0,T)$, *with* $j = 2, 4$, *such that the solution* $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T]; V^0)$ *of problem* (2.37)-(2.39) *satisfies* $\begin{pmatrix} u(T,\cdot) \\ v(T,\cdot) \end{pmatrix} = \begin{pmatrix} u^T \\ v^T \end{pmatrix}$.

Proof. We can assume that $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Let Λ denote the map

$$\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \in V^2 \mapsto \begin{pmatrix} u(T,\cdot) - bu_{xx}(T,\cdot) \\ v(T,\cdot) - dv_{xx}(T,\cdot) \end{pmatrix} \in [V^2]',$$

where $\begin{pmatrix} u \\ v \end{pmatrix}$ is the solution (weak) of (2.37)-(2.39) with f_2, g_2, f_4 and g_4 given by

$$\begin{aligned} f_2 &= -(c\psi(t,0) + c_1\psi_{xx}(t,0)) \\ g_2 &= -(a\varphi(t,0) + a_1\varphi_{xx}(t,0)) \\ f_4 &= -c_1\psi(t,0) \\ g_4 &= -a_1\varphi(t,0), \end{aligned} \tag{2.50}$$

where $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ is the solution of the adjoint system associated with $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix}$. Λ is linear and continuous (see (2.45) and (2.46)). Moreover, using Propositions 2.2.3 and 2.2.4 it follows that Λ is coercive, since

$$\begin{aligned} \left\langle \Lambda \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix}, \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\rangle &= \left\| \begin{pmatrix} a\varphi + a_1\varphi_{xx} \\ c\psi + c_1\psi_{xx} \end{pmatrix}(t,0) \right\|^2_{(L^2(0,T))^2} + \left\| \begin{pmatrix} a_1\varphi \\ c_1\psi \end{pmatrix}(t,0) \right\|^2_{(L^2(0,T))^2} \\ &\geq C^T \left\| \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\|_{V^2}. \end{aligned}$$

Thus, by Lax-Milgram theorem it follows that Λ is invertible. Consequently, given $\begin{pmatrix} u^T \\ v^T \end{pmatrix} \in V^0$, we can define $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} := \Lambda^{-1} \begin{pmatrix} u^T \\ v^T \end{pmatrix}$ to solve the adjoint system and get $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in V^2$. Then, if the boundary functions f_j, g_j, with $j = 2, 4$, are given by (2.50), the corresponding solution $\begin{pmatrix} u \\ v \end{pmatrix}$ of the system (2.37)-(2.39) satisfies

$$\begin{pmatrix} u(0,\cdot) \\ v(0,\cdot) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u(T,\cdot) \\ v(T,\cdot) \end{pmatrix} = \begin{pmatrix} u^T \\ v^T \end{pmatrix}.$$

□

If we assume that $\begin{pmatrix} f_2 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in (2.37)-(2.39), the same arguments as above yields the following result in smaller spaces of initial data.

Theorem 2.2.5. *For every $T > 0$ and $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \begin{pmatrix} u^T \\ v^T \end{pmatrix} \in V^2$, there exists $\begin{pmatrix} f_4 \\ g_4 \end{pmatrix} \in (L^2(0,T))^2$, such that the solution $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T]; V^2)$ of (2.37)-(2.39), with $\begin{pmatrix} f_2 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, satisfies $\begin{pmatrix} u(T,\cdot) \\ v(T,\cdot) \end{pmatrix} = \begin{pmatrix} u^T \\ v^T \end{pmatrix}$.*

Proof. Let $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ be a solution of the adjoint system with final data $\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \in V^0$. Proceedings as in (2.43)-(2.44) it can be shown that $\begin{pmatrix} \varphi(t,0) \\ \psi(t,0) \end{pmatrix}$ makes sense in $(L^2(0,T))^2$. Moreover, the following observability inequality holds

$$C^{T_1} \left\| \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\|_{V^0}^2 \leq \left\| \begin{pmatrix} a_1\varphi(t,0) \\ c_1\psi(t,0) \end{pmatrix} \right\|_{(L^2(0,T))^2}^2, \tag{2.51}$$

for some $C^{T_1} > 0$. Indeed,

$$\left\| \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\|_{V^0}^2 = \sum_{|n|\geq N} \begin{pmatrix} |\widehat{\varphi}_n^T|^2 \\ |\widehat{\psi}_n^T|^2 \end{pmatrix} \leq C^{T_1} \int_0^{T_1} \left| \sum_{|n|\geq N} \begin{pmatrix} \widehat{\varphi}_n^T \\ \widehat{\psi}_n^T \end{pmatrix} e^{in\sigma(n)t} \right|^2 dt \leq C^{T_1} \int_0^{T_1} \widetilde{p} \begin{pmatrix} \varphi(t,\cdot) \\ \psi(t,\cdot) \end{pmatrix}^2 dt,$$

where

$$\left[\widetilde{p} \begin{pmatrix} u \\ v \end{pmatrix} \right]^2 = \left| \begin{pmatrix} a_1 u(0) \\ c_1 v(0) \end{pmatrix} \right|^2.$$

Then, the result is obtained following the arguments developed in the proofs of Proposition 2.2.4 and Theorem 2.2.4, respectively. □

The following remarks are in order:

Remark 2.2.2.

1. *If $T = 2\pi$ the left inequality in Proposition 2.2.4 is easy to prove. Indeed, for any $\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \in V^2$ it follows that*

$$\left\| \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\|_{V^2}^2 = \sum_{k\in\mathbb{Z}} \left(1+n^2\right)^2 \left| \begin{pmatrix} \varphi_k^T \\ \psi_k^T \end{pmatrix} \right|^2 \leq C \sum_{k\in\mathbb{Z}} \left(\left| \begin{pmatrix} a_1\varphi_k^T \\ c_1\psi_k^T \end{pmatrix} \right|^2 + \left| \begin{pmatrix} (a - a_1 n^2)\varphi_k^T \\ (c - c_1 n^2)\psi_k^T \end{pmatrix} \right|^2 \right)$$

$$= C \left(\left\| \begin{pmatrix} a\varphi + a_1\varphi_{xx} \\ c\psi + c_1\psi_{xx} \end{pmatrix} (t,0) \right\|_{(L^2(0,2\pi))^2}^2 + \left\| \begin{pmatrix} a_1\varphi \\ c_1\psi \end{pmatrix} (t,0) \right\|_{(L^2(0,2\pi))^2}^2 \right),$$

where C only depends on the parameters a, a_1, c, c_1.

2. *A similar approach as the one given in the proofs of Theorems 2.2.4 and 2.2.5 allows to show that the lower order Boussinesq system (see (1.10)) is controllable.*

2.2.3 Higher order KdV-BBM system

This section is devoted to study the system

$$\begin{cases} u_t + v_x - bu_{txx} + b_1 u_{txxxx} + av_{xxx} + a_1 v_{xxxxx} = 0 & \text{in} \quad (0,T)\times(0,2\pi) \\ v_t + u_x - dv_{txx} + d_1 v_{txxxx} + cu_{xxx} + c_1 u_{xxxxx} = 0 & \text{in} \quad (0,T)\times(0,2\pi), \end{cases} \tag{2.52}$$

with boundary conditions

$$\begin{cases} \partial_x^j u(t,2\pi) - \partial_x^j u(t,0) = 0 & \text{in} \quad (0,T) \\ \partial_x^j v(t,2\pi) - \partial_x^j v(t,0) = 0 & \text{in} \quad (0,T) \\ \partial_x^4 u(t,2\pi) - \partial_x^4 u(t,0) = f_4(t) & \text{in} \quad (0,T) \\ \partial_x^4 v(t,2\pi) - \partial_x^4 v(t,0) = g_4(t) & \text{in} \quad (0,T), \end{cases} \tag{2.53}$$

for $j = 0,1,2,3$, and initial data

$$u(0,x) = u^0(x), \quad v(0,x) = v^0(x), \text{ in} \quad (0,2\pi). \tag{2.54}$$

We first prove that system (2.52)-(2.54) is well-posed.

Proposition 2.2.5. *For any $s \in \mathbb{R}$, let V^s be the Hilbert space defined by (2.6) and $X(0,T) := (L^2(0,T))^2 \times (L^2(0,T))^2$, $T > 0$. Then, the following well-posedness results hold:*

(i) Suppose that $f_4(t), g_4(t) \in C_0^2[0,T]$ and $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \in V^5$. Then, there exists a unique solution $\begin{pmatrix} u \\ v \end{pmatrix} \in C^1([0,T];V^4) \cap C([0,T];V^5)$ of (2.52)-(2.54). Moreover, for any $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \in V^5$ and $S \in [0,T]$, we have

$$\begin{aligned} &\left\langle \begin{pmatrix} u(S,x) - bu_{xx}(S,x) + b_1 u_{xxxx}(S,x) \\ v(S,x) - dv_{xx}(S,x) + d_1 v_{xxxx}(S,x) \end{pmatrix}, \begin{pmatrix} \varphi(S,x) \\ \psi(S,x) \end{pmatrix} \right\rangle_{V^0\times V^0} = \\ &\left\langle \begin{pmatrix} u^0 - bu^0_{xx} + b_1 u^0_{xxxx} \\ v^0 - dv^0_{xx} + d_1 v^0_{xxxx} \end{pmatrix}, \begin{pmatrix} \varphi(0,x) \\ \psi(0,x) \end{pmatrix} \right\rangle_{V^0\times V^0} - \left\langle \begin{pmatrix} f_4(t) \\ g_4(t) \end{pmatrix}, \begin{pmatrix} c_1\psi(t,0) \\ a_1\varphi(t,0) \end{pmatrix} \right\rangle_{X(0,S)}, \end{aligned} \tag{2.55}$$

where $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in C^1([0,T];V^4) \cap C([0,T];V^5)$ is the solution of the adjoint system (2.17) with initial data $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix}$ given by Theorem 2.1.3.

(ii) If $\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix} \in V^0$*, then* $\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in C([0,T];V^0)$ *and* $\begin{pmatrix}\varphi(t,0)\\ \psi(t,0)\end{pmatrix}$ *makes sense in* $(L^2(0,T))^2$.

(iii) Assume that $\begin{pmatrix}u^0\\ v^0\end{pmatrix} \in V^4$ *and* $f_4, g_4 \in L^2(0,T)$*. Then, there exists a unique* $\begin{pmatrix}u\\ v\end{pmatrix} \in C([0,T];V^4)$*, such that, for any* $\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix} \in V^0$ *and any* $S \in [0,T]$*, we have*

$$\left\langle \begin{pmatrix}u(S,x) - bu_{xx}(S,x) + b_1 u_{xxxx}(S,x)\\ v(S,x) - dv_{xx}(S,x) + d_1 v_{xxxx}(S,x)\end{pmatrix}, \begin{pmatrix}\overline{\varphi(S,x)}\\ \overline{\psi(S,x)}\end{pmatrix}\right\rangle_{V^0\times V^0} = \\ \left\langle \begin{pmatrix}u^0 - bu^0_{xx} + b_1 u^0_{xxxx}\\ v^0 - dv^0_{xx} + d_1 v^0_{xxxx}\end{pmatrix}, \begin{pmatrix}\overline{\varphi(0,x)}\\ \overline{\psi(0,x)}\end{pmatrix}\right\rangle_{V^0\times V^0} - \left\langle \begin{pmatrix}f_4(t)\\ g_4(t)\end{pmatrix}, \begin{pmatrix}c_1\overline{\psi(t,0)}\\ a_1\overline{\varphi(t,0)}\end{pmatrix}\right\rangle_{X(0,S)}, \tag{2.56}$$

where $\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in C([0,T];V^0)$ *is the solution of the adjoint system* (2.17) *with initial data* $\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}$ *given by Theorem 2.1.3.*

Proof. (i) The proof of (2.55) is similar to the proof of (2.40), therefore we omit it.

(ii) Arguing as in (ii) of Proposition 2.2.3, we deduce that, if $\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix} \in V^0$, the sum

$$\begin{pmatrix}\varphi\\ \psi\end{pmatrix}(t,0) = \frac{1}{2}\sum_{k\in\mathbb{I}} e^{ik\sigma(k)t} \sum_{m\in I(k)} \left[\begin{pmatrix}1\\ -\sqrt{\frac{w_2(m)}{w_1(m)}}\end{pmatrix}\hat{\varphi}^T_m + \begin{pmatrix}-\sqrt{\frac{w_1(m)}{w_2(m)}}\\ 1\end{pmatrix}\hat{\psi}^T_m \right. \\ \left. + \begin{pmatrix}1\\ -\sqrt{\frac{w_2(m)}{w_1(m)}}\end{pmatrix}\hat{\varphi}^T_{-m} - \begin{pmatrix}\sqrt{\frac{w_1(m)}{w_2(m)}}\\ 1\end{pmatrix}\hat{\psi}^T_{-m}\right]$$

makes sense in $L^2(0,T)$. Moreover, we have the following estimate

$$\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}(t,0)\right\|_{(L^2(0,T))^2} \le C\left\|\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix}\right\|_{V^0},$$

where $C > 0$ is a positive constant.

(iii) We proceed in several steps.

- (2.56) holds when $\begin{pmatrix}u_0\\ v_0\end{pmatrix} \in V^4$, $\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix} \in V^0$ and $f_4,\ g_4 \in C^2_0([0,T])$.

First, suppose that $\begin{pmatrix}\varphi_T\\ \psi_T\end{pmatrix} \in V^5$. Since V^5 is dense in V^0, using (ii) we obtain the result.

- Let $S \in [0,T]$ be fixed. Then, (2.56) defines $\begin{pmatrix} u(S) - bu_{xx}(S) + b_1 u_{xxxx}(S) \\ v(S) - dv_{xx}(S) + d_1 v_{xxxx}(S) \end{pmatrix}$ in V^0 in a unique manner.

Observe that, from the proof of (ii) we deduce that the map $\Gamma : V^0 \to \mathbb{C}$ given by

$$\Gamma\left(\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix}\right) = -\left\langle \begin{pmatrix} f_4(t) \\ g_4(t) \end{pmatrix}, \begin{pmatrix} c_1\psi(t,0) \\ a_1\varphi(t,0) \end{pmatrix} \right\rangle_{(L^2(0,T))^2}$$

is linear and continuous. On the other hand, the well-posedness of the adjoint system (see Theorem 2.1.3) allows us to conclude that the map $T_S : V^0 \to V^0$, given by

$$T_S \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} = \begin{pmatrix} \varphi(S,\cdot) \\ \psi(S,\cdot) \end{pmatrix},$$

is an automorphism of Hilbert space. This implies that $\begin{pmatrix} u(S,\cdot) - bu_{xx}(S,\cdot) + b_1 u_{xxxx}(S,\cdot) \\ v(S,\cdot) - dv_{xx}(S,\cdot) + d_1 u_{xxxx}(S,\cdot) \end{pmatrix}$ is uniquely defined in V^0, for each $S \in [0,T]$. Moreover, for $S \in [0,T]$, we obtain the following estimate:

$$\begin{aligned}
&\left\| \begin{pmatrix} u(S) - bu_{xx}(S) + b_1 u_{xxxx}(S) \\ v(S) - dv_{xx}(S) + d_1 v_{xxxx}(S) \end{pmatrix} \right\|_{V^0} = \\
&\sup_{\left\| \begin{pmatrix} \varphi(S,\cdot) \\ \psi(S,\cdot) \end{pmatrix} \right\|_{V^0} \le 1} \left| \left\langle \begin{pmatrix} u(S) - bu_{xx}(S) + b_1 u_{xxxx}(S) \\ v(S) - dv_{xx}(S) + d_1 v_{xxxx}(S) \end{pmatrix}, \begin{pmatrix} \varphi(S,\cdot) \\ \psi(S,\cdot) \end{pmatrix} \right\rangle \right| \\
&= \sup_{\left\| \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\|_{V^0} \le 1} \left| \left\langle \begin{pmatrix} u^0 - bu^0_{xx} + b_1 u^0_{xxxx} \\ v_0 - dv^0_{xx} + d_1 v^0_{xxxx} \end{pmatrix}, \begin{pmatrix} \varphi(0,\cdot) \\ \psi(0,\cdot) \end{pmatrix} \right\rangle_{V^0 \times V^0} - \Gamma \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right| \qquad (2.57) \\
&\le \sup_{\left\| \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\|_{V^0} \le 1} \left(\left\| \begin{pmatrix} u^0 - bu^0_{xx} + b_1 u^0_{xxxx} \\ v_0 - dv^0_{xx} + d_1 v^0_{xxxx} \end{pmatrix} \right\|_{V^0} \left\| \begin{pmatrix} \varphi(0,\cdot) \\ \psi(0,\cdot) \end{pmatrix} \right\|_{V_0} + \left| \Gamma \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right| \right) \\
&\le C \left(\left\| \begin{pmatrix} u^0 - bu^0_{xx} + b_1 u^0_{xxxx} \\ v_0 - dv^0_{xx} + d_1 v^0_{xxxx} \end{pmatrix} \right\|_{V^0} + \left\| \begin{pmatrix} f_4(t) \\ g_4(t) \end{pmatrix} \right\|_{[L^2(0,T)]^2} \right),
\end{aligned}$$

where C is a positive constant which does not depend on S or on u_0, v_0, f_4, g_4.

- $u \in C([0,T];V^4)$

First, observe that, from (i) we have that $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T];V^5) \subset C([0,T];V^4)$, whenever $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in V^5$ and $f_j, g_j \in C^2_0[(0,T)]$, for $j = 4$. Since V^5 is dense in V^4 and $C^2_0(0,T)$ is dense in $L^2(0,T)$, it is follow from (2.57) that $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T];V^4)$. □

Proposition 2.2.5 leads to the following definition:

Definition 2.2.3. *For* $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in V^4$ *and* $f_j, g_j \in L^2(0,T)$*, with* $j = 4$*, a weak solution of* (2.52)-(2.54) *is a function* $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T]; V^4)$*, such that* (2.56) *holds true for all* $\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \in V^0$ *and all* $S \in [0,T]$*.*

In order to prove the controllability result applying the Hilbert Uniqueness Method, we have to prove an *observability inequality* for the solutions of the corresponding adjoint system. Here, this is done using the so-called Ingham's inequality (see, for instance, [2, 22]) . For the sake of completeness and in order to facilitate the reading of the tese, we give a generalization of Ingham's inequality in Theorem 2.2.1.

Then, we have the following result:

Proposition 2.2.6. *Let* $T > 0$*. Then, there exist positive constants* C *and* $\widetilde{C}$*, such that, for any* $\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \in V^0$*,*

$$C \left\| \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\|_{V^0}^2 \leq \left\| \begin{pmatrix} a_1 \varphi \\ c_1 \psi \end{pmatrix} (t,0) \right\|_{(L^2(0,T))^2}^2 \leq \widetilde{C} \left\| \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\|_{V^0}^2, \tag{2.58}$$

where (φ, ψ) *is solution of the adjoint system with data* $(\varphi_T, \psi_T) \in V^0$*.*

Proof. We first prove the left inequality assuming that the right one holds.

Let us consider λ_k, $k \in \mathbb{Z}$, the eigenvalues of the operator A^*, the state operator associate to the adjoint system. Remark that they coincide with the eigenvalues of A, defined by (2.14), and

$$\lim_{|k| \to \infty} |\lambda_k| = \infty.$$

Then, according to Remark 2.1.1, there exists $N_1 > 0$, such that, for $|k| > N_1$, the eigenvalues has multiplicity one. In particular, for $|k| > N_1$,

$$\begin{aligned} &\frac{1}{2} \sum_{k \in \mathbb{I}} e^{ik\sigma(k)(T-t)} \sum_{m \in I(k)} \left[\begin{pmatrix} 1 \\ -\sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}} \end{pmatrix} e^{imx} \hat{\varphi}_m^T + \begin{pmatrix} -\sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}} \\ 1 \end{pmatrix} e^{imx} \hat{\psi}_m^T \right. \\ &\left. + \begin{pmatrix} 1 \\ \sqrt{\frac{\widetilde{w_2}(m)}{\widetilde{w_1}(m)}} \end{pmatrix} e^{-imx} \hat{\varphi}_{-m}^T + \begin{pmatrix} \sqrt{\frac{\widetilde{w_1}(m)}{\widetilde{w_2}(m)}} \\ 1 \end{pmatrix} e^{-imx} \hat{\psi}_{-m}^T \right] = \sum_{|k| > N_1} e^{ik\sigma(k)(T-t)} \begin{pmatrix} \hat{\varphi}_k^T \\ \hat{\psi}_k^T \end{pmatrix} e^{ikx}. \end{aligned} \tag{2.59}$$

In addition, if we take $T_1 \in (0,T)$ and $\gamma > \frac{2\pi}{T_1}$, there exists $N_2 \in \mathbb{N}$, such that

$$k \in \mathbb{Z}, |k| \geq N_2 \Longrightarrow (k+1)\sigma(k+1) - k\sigma(k) \geq \gamma. \tag{2.60}$$

Also, taking Remark 2.1.1 into account, we introduce $W_n = \text{Span}\left\{\begin{pmatrix} e^{ikx} \\ -\frac{\sigma(k)}{w_1}e^{ikx}\end{pmatrix}, \begin{pmatrix} e^{-ikx} \\ \frac{\sigma(k)}{w_1}e^{-ikx}\end{pmatrix}\right\}$, for $k \in I(n)$ and $n \in \mathbb{I}$, and consider $W = \bigoplus_{n\in\mathbb{I}} W_n \subset V^0$, whose embedding is dense. In W we define the following semi-norm:

$$\left[p\begin{pmatrix} u \\ v\end{pmatrix}\right]^2 = \left|\begin{pmatrix} a_1 u(0) \\ c_1 v(0)\end{pmatrix}\right|^2, \quad \forall \begin{pmatrix} u \\ v\end{pmatrix} \in W.$$

Let $N = \max\{N_1, N_2\}$ and $\begin{pmatrix} \varphi_T \\ \psi_T\end{pmatrix} \in W \cap (\bigoplus_{|n|<N} W_n)^{\perp}$, that is, $\begin{pmatrix} \hat{\varphi}_n^T \\ \hat{\psi}_n^T\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix}$ for $|k| < N$ or for $|k|$ large enough. Then, by (2.59), (2.60) and Ingham's inequality, we obtain $C^{T_1} > 0$, such that

$$\begin{aligned}\left\|\begin{pmatrix} \varphi_T \\ \psi_T\end{pmatrix}\right\|_{V^0}^2 &= \sum_{|n|\geq N}\begin{pmatrix} |\hat{\varphi}_n^T|^2 \\ |\hat{\psi}_n^T|^2\end{pmatrix} \leq C^{T_1}\int_0^{T_1}\left|\sum_{|n|\geq N}\begin{pmatrix} \hat{\varphi}_n^T \\ \hat{\psi}_n^T\end{pmatrix} e^{n\sigma(n)t}\right|^2 dt \\ &= C^{T_1}\int_0^{T_1}\left|\sum_{|n|\geq N}\begin{pmatrix} \frac{1}{a_1}a_1\hat{\varphi}_n^T \\ \frac{1}{c_1}c_1\hat{\psi}_n^T\end{pmatrix} e^{n\sigma(n)t}\right|^2 dt \leq C^{T_1}\int_0^{T_1} p\begin{pmatrix} \varphi(t,\cdot) \\ \psi(t,\cdot)\end{pmatrix}^2 dt.\end{aligned}$$

Since $T > T_1$, from the above estimate, [25, Theorem 5.3] and the right inequality in (2.58), we obtain $C^T > 0$, such that

$$C^T\left\|\begin{pmatrix} \varphi_T \\ \psi_T\end{pmatrix}\right\|_{V^0}^2 \leq \int_0^T p\begin{pmatrix} \varphi(t,\cdot) \\ \psi(t,\cdot)\end{pmatrix}^2 \leq \left\|\begin{pmatrix} a_1\varphi \\ c_1\psi\end{pmatrix}(t,0)\right\|_{(L^2(0,T))^2}^2,$$

for all $\begin{pmatrix} \varphi_T \\ \psi_T\end{pmatrix} \in W$. The general case follows from a density argument.

Now, we prove the right inequality in (2.58). Observe that, from the considerations above it follows that

$$\begin{aligned}\begin{pmatrix} \varphi \\ \psi\end{pmatrix}(t,0) = &\ \frac{1}{2}\sum_{|k|<N_1} e^{ik\sigma(k)(T-t)}\sum_{m\in I(k)}\left[\begin{pmatrix} 1 \\ -\sqrt{\frac{w_2(m)}{w_1(m)}}\end{pmatrix}\widehat{\varphi}_m^T + \begin{pmatrix} -\sqrt{\frac{w_1(m)}{w_2(m)}} \\ 1\end{pmatrix}\widehat{\psi}_m^T\right. \\ &\left. + \begin{pmatrix} 1 \\ \sqrt{\frac{w_2(m)}{w_1(m)}}\end{pmatrix}\widehat{\varphi}_{-m}^T + \begin{pmatrix} \sqrt{\frac{w_1(m)}{w_2(m)}} \\ 1\end{pmatrix}\widehat{\psi}_{-m}^T\right] \\ &+\frac{1}{2}\sum_{|k|\geq N_1} e^{ik\sigma(k)(T-t)}\left[\begin{pmatrix} 1 \\ -\sqrt{\frac{w_2(k)}{w_1(k)}}\end{pmatrix}\widehat{\varphi}_k^T + \begin{pmatrix} -\sqrt{\frac{w_1(k)}{w_2(k)}} \\ 1\end{pmatrix}\widehat{\psi}_k^T\right. \\ &\left. + \begin{pmatrix} 1 \\ \sqrt{\frac{w_2(k)}{w_1(k)}}\end{pmatrix}\widehat{\varphi}_{-k}^+ \begin{pmatrix} \sqrt{\frac{w_1(k)}{w_2(k)}} \\ 1\end{pmatrix}\widehat{\psi}_{-k}^T\right].\end{aligned}$$

Then,

$$\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} (t,0) \right\|^2_{(L^2(0,T))^2} \leq C_1 \left(\left\| \sum_{|k|<N_1} e^{ik\sigma(k)(T-t)} \begin{pmatrix} \widehat{\varphi}_k^T \\ \widehat{\psi}_k^T \end{pmatrix} \right\|^2_{(L^2(0,T))^2} + \left\| \sum_{|k|\geq N_1} e^{ik\sigma(k)(T-t)} \begin{pmatrix} \widehat{\varphi}_k^T \\ \widehat{\psi}_k^T \end{pmatrix} \right\|^2_{(L^2(0,T))^2} \right) \leq C_2 \left\| \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \right\|^2_{V^4}.$$

where C_1 and C_2 are positive constants. □

Using Proposition 2.2.6 we prove our main result:

Theorem 2.2.6. *Let $T > 0$. Then, for any $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \begin{pmatrix} u^T \\ v^T \end{pmatrix} \in V^4$, there exist $f_4, g_4 \in L^2(0,T)$, such that the solution $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T]; V^4)$ of (2.52)-(2.54) satisfies $\begin{pmatrix} u(T,\cdot) \\ v(T,\cdot) \end{pmatrix} = \begin{pmatrix} u_T \\ v_T \end{pmatrix}$.*

Proof. We can assume that $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Let Λ denote the map

$$\begin{pmatrix} \varphi_T \\ \varphi_T \end{pmatrix} \in V^0 \mapsto \begin{pmatrix} u^T - bu^T_{xx} + b_1 u^T_{xxxx} \\ v^T - dv^T_{xx} + d_1 v^T_{xxxx} \end{pmatrix} \in V^0,$$

where $\begin{pmatrix} u \\ v \end{pmatrix}$ is the solution (weak) of (2.52)-(2.54) with f_4, g_4 given by

$$f_4 = -c_1\psi(t,0) \quad \text{and} \quad g_4 = -a_1\varphi(t,0), \tag{2.61}$$

where $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ is a solution of adjoint system with initial condition $\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix}$. Λ is linear and continuous (see (2.57) and (2.58)). Moreover, using Propositions 2.2.5 and 2.2.6 it follows that Λ is coercive, since

$$\left\langle \Lambda \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix}, \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\rangle = \left\| \begin{pmatrix} a_1\varphi \\ c_1\psi \end{pmatrix} (t,0) \right\|^2_{(L^2(0,T))^2} \geq C^T \left\| \begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \right\|_{V^0}.$$

Thus, by Lax-Milgran theorem it follows that Λ is invertible. Consequently, given $\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \in V^0$, we can define $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} := \Lambda^{-1} \begin{pmatrix} u^T \\ v^T \end{pmatrix}$ to solve the adjoint system and get $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in V^0$. Then, if the boundary functions f_4, g_4 are given by (2.61), the corresponding solution $\begin{pmatrix} u \\ v \end{pmatrix}$ of the system (2.52)-(2.54) satisfies

$$\begin{pmatrix} u(0,\cdot) \\ v(0,\cdot) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u(T,\cdot) \\ v(T,\cdot) \end{pmatrix} = \begin{pmatrix} u^T \\ v^T \end{pmatrix}.$$

□

2.2.3.1 The nonlinear system

In this section we are concerned with the analyze of the controllability properties of the full system (1.1)-(1.2):

$$\begin{cases} u_t + v_x - bu_{txx} + b_1 u_{txxxx} + av_{xxx} + a_1 v_{xxxxx} = \\ \qquad -(uv)_x + b(uv)_{xxx} - \alpha(uv_{xx})_x, & \text{in} \quad (0,T)\times(0,2\pi) \\ v_t + u_x - dv_{txx} + d_1 v_{txxxx} + cu_{xxx} + c_1 u_{xxxxx} = \\ \qquad -vv_x - c(vv_x)_{xx} - (uu_{xx})_x + \beta v_x v_{xx} + \rho u u_{xxx}, & \text{in} \quad (0,T)\times(0,2\pi) \\ \partial_x^j u(t,2\pi) - \partial_x^j u(t,0) = 0, & \text{in} \quad (0,T) \\ \partial_x^j v(t,2\pi) - \partial_x^j v(t,0) = 0, & \text{in} \quad (0,T) \\ \partial_x^4 u(t,2\pi) - \partial_x^4 u(t,0) = f_4(t), & \text{in} \quad (0,T) \\ \partial_x^4 v(t,2\pi) - \partial_x^4 v(t,0) = g_4(t), & \text{in} \quad (0,T) \\ u(0,x) = u^0(x), \quad v(0,x) = v^0(x), & \text{in} \quad (0,2\pi), \end{cases} \tag{2.62}$$

for $j = 0,1,2,3$.

Let us begin introducing the nonlinear operator $\mathcal{N} : V^4 \to V^4$ defined by

$$\mathcal{N}\begin{pmatrix}\eta\\ \zeta\end{pmatrix} = \begin{pmatrix} \mathcal{L}_{b,b_1}[-(\eta\zeta)_x + b(\eta\zeta)_{xxx} - \alpha(\eta\zeta_{xx})_x] \\ \mathcal{L}_{d,d_1}[-\zeta\zeta_x - c(\zeta\zeta_x)_{xx} - (\eta\eta_{xx})_x + \beta\zeta_x\zeta_{xx} + \rho\eta\eta_{xxx}] \end{pmatrix}. \tag{2.63}$$

Some of its most important properties are given in the following result. The proof can be found in [3]:

Theorem 2.2.7. *Suppose that $b_1, d_1 > 0$. Then, the operator $\mathcal{N} : V^4 \to V^4$ given in (2.63) is well-defined and there is $\tilde{K} > 0$, such that the following estimates are verified:*

$$\left\| \mathcal{N}\begin{pmatrix}\eta\\ \zeta\end{pmatrix} \right\|_{V^4} \le \tilde{K} \left\| \begin{pmatrix}\eta\\ \zeta\end{pmatrix} \right\|_{V^4}^2, \tag{2.64}$$

$$\left\| \mathcal{N}\begin{pmatrix}\eta\\ \zeta\end{pmatrix} - \mathcal{N}\begin{pmatrix}\theta\\ \gamma\end{pmatrix} \right\|_{V^4} \le \tilde{K} \left(\left\| \begin{pmatrix}\eta\\ \zeta\end{pmatrix} \right\|_{V^4} + \left\| \begin{pmatrix}\theta\\ \gamma\end{pmatrix} \right\|_{V^4} \right) \left\| \begin{pmatrix}\eta\\ \zeta\end{pmatrix} - \begin{pmatrix}\theta\\ \gamma\end{pmatrix} \right\|_{V^4}, \tag{2.65}$$

for any $\begin{pmatrix}\eta\\ \zeta\end{pmatrix}, \begin{pmatrix}\theta\\ \gamma\end{pmatrix} \in V^4$.

To study the nonlinear system, we observe that the solutions of (2.62) can be written as

$$\begin{pmatrix}u\\ v\end{pmatrix} = S(t)\begin{pmatrix}u_0\\ v_0\end{pmatrix} + \begin{pmatrix}\varphi\\ \psi\end{pmatrix} + \begin{pmatrix}\eta\\ \zeta\end{pmatrix},$$

where $(S(t))_{t\in\mathbb{R}}$ is the group of isometries associated to the linear homogeneous system (see Theorem 2.1.1) and $\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$ and $\begin{pmatrix}\eta\\ \zeta\end{pmatrix}$ satisfy, respectively,

$$\begin{cases}\varphi_t+\psi_x-b\varphi_{txx}+b_1\varphi_{txxxx}+a\psi_{xxx}+a_1\psi_{xxxxx}=0, & \text{in} \quad (0,T)\times(0,2\pi)\\ \psi_t+\varphi_x-d\psi_{txx}+d_1\psi_{txxxx}+c\varphi_{xxx}+c_1\varphi_{xxxxx}=0, & \text{in} \quad (0,T)\times(0,2\pi).\\ \partial_x^j\varphi(t,2\pi)-\partial_x^j\varphi(t,0)=0, & \text{in} \quad (0,T)\\ \partial_x^j\psi(t,2\pi)-\partial_x^j\psi(t,0)=0, & \text{in} \quad (0,T)\\ \partial_x^4\varphi(t,2\pi)-\partial_x^4\varphi(t,0)=f_4(t), & \text{in} \quad (0,T)\\ \partial_x^4\psi(t,2\pi)-\partial_x^4\psi(t,0)=g_4(t), & \text{in} \quad (0,T)\\ \varphi(0,x)=0, \quad \psi(0,x)=0, & \text{in} \quad (0,2\pi)\end{cases}. \tag{2.66}$$

and

$$\begin{cases}\eta_t+\zeta_x-b\eta_{txx}+b_1\eta_{txxxx}+a\zeta_{xxx}+a_1\zeta_{xxxxx}=h_1, & \text{in} \quad (0,T)\times(0,2\pi)\\ \zeta_t+\eta_x-d\zeta_{txx}+d_1\zeta_{txxxx}+c\eta_{xxx}+c_1\eta_{xxxxx}=h_2, & \text{in} \quad (0,T)\times(0,2\pi).\\ \partial_x^j\eta(t,2\pi)-\partial_x^j\eta(t,0)=0, & \text{in} \quad (0,T)\\ \partial_x^j\zeta(t,2\pi)-\partial_x^j\zeta(t,0)=0, & \text{in} \quad (0,T)\\ \partial_x^4\eta(t,2\pi)-\partial_x^4\eta(t,0)=f_4(t), & \text{in} \quad (0,T)\\ \partial_x^4\zeta(t,2\pi)-\partial_x^4\zeta(t,0)=g_4(t), & \text{in} \quad (0,T)\\ \eta(0,x)=0, \quad \zeta(0,x)=0, & \text{in} \quad (0,2\pi),\end{cases}. \tag{2.67}$$

with $h_1=-(uv)_x+b(uv)_{xxx}-\alpha(uv_{xx})_x$ and $h_2=-vv_x-c(vv_x)_{xx}-(uu_{xx})_x+\beta v_xv_{xx}+\rho uu_{xxx}$.

The existence and uniqueness of solutions of the nonlinear system (2.62) can be proved if the initial data and the boundary conditions are small enough. More precisely, we have the following result:

Theorem 2.2.8. *Assume that $b_1,d_1\neq 0$ and let $T>0$. Then, there exists a constant $\delta>0$, such that, for any $\begin{pmatrix}u_0\\ v_0\end{pmatrix}\in V^4$ and any $\begin{pmatrix}f_4\\ g_4\end{pmatrix}\in (L^2(0,T))^2$ satisfying*

$$\left\|\begin{pmatrix}u_0\\ v_0\end{pmatrix}\right\|_{V^4}<\delta \qquad \text{and} \qquad \left\|\begin{pmatrix}f_4\\ g_4\end{pmatrix}\right\|_{(L^2(0,T))^2}<\delta, \tag{2.68}$$

system (2.62) has an unique weak solution $\begin{pmatrix}u\\ v\end{pmatrix}\in C([0,T];V^4)$.

Proof. The arguments used to prove this result are similar to those used to prove the main result of this section. So, we have omitted the details.

Let us define the maps

$$
\begin{aligned}
\Theta &: (L^2(0,T))^2 \to C([0,T];V^4) \\
\Theta &\begin{pmatrix} f_4 \\ g_4 \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \end{pmatrix},
\end{aligned}
\tag{2.69}
$$

where $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ is the solution of (2.66), and

$$
\begin{aligned}
\widetilde{\Theta} &: L^1(0,T;(L^2(0,2\pi))^2) \to C([0,T];V^4) \\
\widetilde{\Theta} &\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \eta \\ \zeta \end{pmatrix},
\end{aligned}
\tag{2.70}
$$

where $\begin{pmatrix} \eta \\ \zeta \end{pmatrix}$ solves (2.67). Remark that $\begin{pmatrix} \eta \\ \zeta \end{pmatrix}$ is given by $\begin{pmatrix} \eta \\ \zeta \end{pmatrix}(t) := \int_0^t S(t-\tau)\mathcal{N}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} d\tau.$

With the notation introduced above, we define the map $G : C([0,T];V^4) \to C([0,T];V^4)$ by

$$
G\begin{pmatrix} u \\ v \end{pmatrix} = S(\cdot)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \Theta\begin{pmatrix} f_4 \\ g_4 \end{pmatrix} + \widetilde{\Theta}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix},
$$

with

$$
h_1 = -(uv)_x + b(uv)_{xxx} - \alpha(uv_{xx})_x \quad \text{and} \quad h_2 = -vv_x - c(vv_x)_{xx} - (uu_{xx})_x + \beta v_x v_{xx} + \rho u u_{xxx}.
\tag{2.71}
$$

From Proposition 2.2.5 it follows that Θ is linear, continuous and well defined. Theorem 2.2.7 and [3, Theorem 5.1] ensure that the same properties remains valid for $\widetilde{\Theta}$. Consequently, we deduce that G has a unique fixed point $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T];V^4)$, which is the unique solution of the system (2.62). □

Theorem 2.2.9. *Assume that $b_1, d_1, a_1, c_1 \neq 0$ and let $T > 0$. Then, there exists a constant $\delta > 0$, such that, for any $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \in V^4$ satisfying*

$$
\left\| \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right\|_{V^4} < \delta \quad \text{and} \quad \left\| \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \right\|_{V^4} < \delta,
\tag{2.72}
$$

there exist controls $f_4, g_4 \in L^2(0,T)$, such that the solution of system (2.62) satisfies

$$
\begin{pmatrix} u \\ v \end{pmatrix}(T) = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}.
$$

Proof. Let us first define the following map

$$
\begin{aligned}
\Theta_1 &: V^4 \to (L^2(0,T))^2 \\
\Theta_1 &\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} f_4 \\ g_4 \end{pmatrix},
\end{aligned}
$$

where $f_4, g_4 \in L^2(0,T)$ are the controls given by the Theorem 2.2.6, which take the solution of (2.52)-(2.54) from the initial data $(0,0)$ to the final data (u_1, v_1). Remark that the observability inequality (2.58) ensures that Θ_1 is continuous, this is, there exists $K_1 > 0$, such that

$$\left\| \Theta_1 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \right\|_{(L^2(0,T))^2} \leq K_1 \left\| \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \right\|_{V^4}.$$

Next, we define the nonlinear map $F : C([0,T]; V^4) \to C([0,T]; V^4)$ as follows:

$$F\begin{pmatrix} u \\ v \end{pmatrix} = S(\cdot)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \Theta \circ \Theta_1 \left(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - S(T)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \widetilde{\Theta}\begin{pmatrix} -h_1 \\ -h_2 \end{pmatrix}(T,\cdot) \right) + \widetilde{\Theta}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix},$$

where $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$, Θ and $\widetilde{\Theta}$ are given in (2.71), (2.69) and (2.70), respectively. Remark that, if $\begin{pmatrix} u \\ v \end{pmatrix}$ is a fixed point of F, then $\begin{pmatrix} u \\ v \end{pmatrix}$ is a solution of (2.62) and satisfies $\begin{pmatrix} u(T,x) \\ v(T,x) \end{pmatrix} = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$. Therefore, we show that there exists $R > 0$ with the following properties:

(i) F maps the ball $\overline{B_R(0)} \subset C([0,T]; V^4)$ into itself.

From Proposition 2.2.5 it follows that Θ is linear, continuous and well defined. Therefore, we obtain $K > 0$, such that

$$\left\| \Theta \begin{pmatrix} f_4 \\ g_4 \end{pmatrix} \right\|_{C([0,T];V^4)} \leq K \left\| \begin{pmatrix} f_4 \\ g_4 \end{pmatrix} \right\|_{(L^2(0,T))^2}.$$

Theorem 2.2.7 and [3, Theorem 5.1] ensure that the same properties remains valid for $\widetilde{\Theta}$. Thus,

$$\left\| \widetilde{\Theta} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|_{C([0,T];V^4)} \leq \tilde{K} \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2_{C([0,T];V^4)}.$$

Let $R > 0$, to be chosen latter on, and $\begin{pmatrix} u \\ v \end{pmatrix} \in \overline{B(0,R)}$. Then, we have that

$$\begin{aligned}
&\left\| F\begin{pmatrix} u \\ v \end{pmatrix} \right\|_{C([0,T];V^4)} \\
&\leq \left\| \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right\|_{V^4)} + KK_1 \left\| \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - S(T)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \widetilde{\Theta}\begin{pmatrix} -h_1 \\ -h_2 \end{pmatrix}(T,\cdot) \right\|_{V^4} + \tilde{K} \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2_{C([0,T];V^4)} \\
&\leq \delta + 2KK_1\delta + KK_1\tilde{K} \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2_{C([0,T];V^4)} + \tilde{K} \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2_{C([0,T];V^4)} \\
&\leq \delta + 2KK_1\delta + (KK_1 + 1)\tilde{K}R^2.
\end{aligned}$$

Therefore, $F(\overline{B_R(0)}) \subset \overline{B_R(0)}$ for any $R > 0$ satisfying

$$(1 + 2KK_1)\delta + (KK_1 + 1)\tilde{K}R^2 \leq R. \tag{2.73}$$

(ii) F is a contraction

If $\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix} \in \overline{B(0,R)}$,

$$F\begin{pmatrix} u \\ v \end{pmatrix} - F\begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix} = \Theta \circ \Theta_1 \left(\widetilde{\Theta} \left(\begin{pmatrix} -h_1 \\ -h_2 \end{pmatrix} - \begin{pmatrix} -\widehat{h_1} \\ -\widehat{h_2} \end{pmatrix} \right)(T, \cdot) \right) + \widetilde{\Theta} \left(\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - \begin{pmatrix} \widehat{h_1} \\ \widehat{h_2} \end{pmatrix} \right),$$

where $h_1 = -(\widehat{u}\widehat{v})_x + b(\widehat{u}\widehat{v})_{xxx} - \alpha(\widehat{u}\widehat{v}_{xx})_x$ and $h_2 = -\widehat{v}\widehat{v}_x - c(\widehat{v}\widehat{v}_x)_{xx} - (\widehat{u}\widehat{u}_{xx})_x + \beta\widehat{v}_x\widehat{v}_{xx} + \rho\widehat{u}\widehat{u}_{xxx}$.

Then, we obtain

$$\begin{aligned} \left\| F\begin{pmatrix} u \\ v \end{pmatrix} - F\begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix} \right\| &\leq KK_1\tilde{K} \left\| \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix} \right\|^2_{C([0,T];V^4)} + \tilde{K} \left\| \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix} \right\|^2_{C([0,T];V^4)} \\ &\leq 2R\widetilde{K}(KK_1 + 1) \left\| \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix} \right\|_{C([0,T];V^4)}. \end{aligned}$$

Consequently, F is a contraction if R verifies

$$2R\widetilde{K}(KK_1 + 1) \leq 1. \tag{2.74}$$

Hence, if R satisfies (2.74), by choosing $\delta = \dfrac{R}{2(1 + 2KK_1)}$, it follows that (2.73) also holds and the proof ends. □

2.3 Further Coments

In this section, we present an extension of our results. More precisely, we have that the absence of the BBM terms, as well as, the higher order KdV terms in the first equation of the linear system also provides positive controllability result.

We consider the following system

$$\begin{cases} u_t + v_x + av_{xxx} = 0, & \text{in} \quad (0,T) \times (0,2\pi) \\ v_t + u_x - dv_{txx} + d_1 v_{txxxx} + cu_{xxx} = 0, & \text{in} \quad (0,T) \times (0,2\pi), \end{cases} \tag{2.75}$$

with periodic boundary conditions

$$\begin{cases} \partial_x^j u(t,2\pi) - \partial_x^j u(t,0) = f_j(t), & \text{in} \quad (0,T) \\ \partial_x^j v(t,2\pi) - \partial_x^j v(t,0) = g_j(t), & \text{in} \quad (0,T) \\ \partial_x^3 v(t,2\pi) - \partial_x^3 v(t,0) = g_3(t), & \text{in} \quad (0,T) \end{cases} \tag{2.76}$$

with $j = 0, 1, 2$, and initial data

$$u(0, x) = u^0(x), \quad v(0, x) = v^0(x), \quad \text{in} \quad (0, 2\pi). \tag{2.77}$$

Again, we remark that the number of boundary conditions depends on the values of the parameters of the system.

Proposition 2.3.1. *For any $s \in \mathbb{R}$, let V^s be the Hilbert space defined by (2.6) and $X(0,T) = (L^2(0,T))^2 \times (L^2(0,T))^2$. Then, the following well-posedness results hold:*

(i) Suppose that $f_j(t), g_j(t), g_3(t) \in C_0^2[0,T]$, for $j = 0, 1, 2$ and $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \in V^3$. Then, there exists a unique solution $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T]; V^3)$ of (2.75)-(2.76). Moreover, for any $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \in V^3$ and $S \in [0,T]$, we have

$$\begin{aligned}
&\left\langle \begin{pmatrix} u(S,\cdot) \\ v(S,\cdot) - dv_{xx}(S,\cdot) + d_1 v_{xxxx}(S,\cdot) \end{pmatrix}, \begin{pmatrix} \overline{\varphi(S,\cdot)} \\ \overline{\psi(S,\cdot)} \end{pmatrix} \right\rangle_{[L^2(0,2\pi)\times L^2(0,2\pi))]^2} = \\
&\left\langle \begin{pmatrix} u^0 \\ v^0 - dv^0_{xx} + d_1 v^0_{xxxx} \end{pmatrix}, \begin{pmatrix} \overline{\varphi(0,\cdot)} \\ \overline{\psi(0,\cdot)} \end{pmatrix} \right\rangle_{[L^2(0,2\pi)\times L^2(0,2\pi))]^2} \\
&- \left\langle \begin{pmatrix} f_0(t) \\ g_0(t) \end{pmatrix}, \begin{pmatrix} \overline{\psi(t,0)} + c\overline{\psi_{xx}(t,0)} \\ \overline{\varphi(t,0)} - d\overline{\psi_{tx}(t,0)} + d_1 \overline{\psi_{txxx}(t,0)} + a\overline{\varphi_{xx}(t,0)} \end{pmatrix} \right\rangle_{X(0,S)} \\
&+ \left\langle \begin{pmatrix} f_1(t) \\ g_1(t) \end{pmatrix}, \begin{pmatrix} c\overline{\psi_x(t,0)} \\ a\overline{\varphi_x(t,0)} - d\overline{\psi_t(t,0)} + d_1 \overline{\psi_{txx}(t,0)} \end{pmatrix} \right\rangle_{X(0,S)} \\
&- \left\langle \begin{pmatrix} f_2(t) \\ g_2(t) \end{pmatrix}, \begin{pmatrix} c\overline{\psi(t,0)} \\ a\overline{\varphi(t,0)} + d_1 \overline{\psi_{tx}(t,0)} \end{pmatrix} \right\rangle_{X(0,S)} \\
&+ \left\langle \begin{pmatrix} 0 \\ g_3(t) \end{pmatrix}, \begin{pmatrix} 0 \\ d_1 \overline{\psi_t(t,0)} \end{pmatrix} \right\rangle_{X(0,S)},
\end{aligned} \tag{2.78}$$

where $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in C([0,T]; V^3)$ is the solution of the adjoint system (2.17) with initial data $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix}$ given by Theorem 2.1.3.

(ii) If $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \in V^0$, there exist a unique solution $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in C([0,T]; V^0)$ of (2.17) and $\psi_{txxx}(t,0)$, $\varphi_{xx}(t,0)$ makes sense in $H^{-2}(0,T)$, $\psi_{txx}(t,0)$, $\varphi_x(t,0)$ make sense in $H^{-1}(0,T)$ and $\psi_{tx}(t,0)$ make sense in $L^2(0,T)$.

(iii) Assume that $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \in V^0$, $f_2 \in H^2(0,T)$, $f_1, g_3 \in H^1(0,T)$, $f_0, g_2 \in L^2(0,T)$, $g_1 \in H^{-1}(0,T)$ *and* $g_0 \in H^{-2}(0,T)$. *Then, there exists a unique* $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T];V^0)$, *such that, for any* $\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \in V^0$ *and any* $S \in [0,T]$, *we have*

$$
\begin{aligned}
&\left\langle \begin{pmatrix} u(S,\cdot) \\ v(S,\cdot) - dv_{xx}(S,\cdot) + d_1 v_{xxxx}(S,\cdot) \end{pmatrix}, \begin{pmatrix} \overline{\varphi(S,\cdot)} \\ \overline{\psi(S,\cdot)} \end{pmatrix} \right\rangle_{[V^0]'\times V^0} = \\
&\left\langle \begin{pmatrix} u^0 \\ v^0 - dv^0_{xx} + d_1 v^0_{xxxx} \end{pmatrix}, \begin{pmatrix} \overline{\varphi(0,\cdot)} \\ \overline{\psi(0,\cdot)} \end{pmatrix} \right\rangle_{[V^0]'\times V^0} \\
&- \left\langle \begin{pmatrix} f_0(t) \\ g_0(t) \end{pmatrix}, \begin{pmatrix} \overline{\psi(t,0)} + c\overline{\psi_{xx}(t,0)} \\ \overline{\varphi(t,0)} - d\overline{\psi_{tx}(t,0)} + d_1 \overline{\psi_{txxx}(t,0)} + a\overline{\varphi_{xx}(t,0)} \end{pmatrix} \right\rangle_{[L^2(0,S)\times H^{-2}(0,S)]^2} \\
&+ \left\langle \begin{pmatrix} f_1(t) \\ g_1(t) \end{pmatrix}, \begin{pmatrix} c\overline{\psi_x(t,0)} \\ a\overline{\varphi_x(t,0)} - d\overline{\psi_t(t,0)} + d_1\overline{\psi_{txx}(t,0)} \end{pmatrix} \right\rangle_{[H^1(0,S)\times H^{-1}(0,S)]^2} \\
&- \left\langle \begin{pmatrix} f_2(t) \\ g_2(t) \end{pmatrix}, \begin{pmatrix} c\overline{\psi(t,0)} \\ a\overline{\varphi(t,0)} + d_1\overline{\psi_{tx}(t,0)} \end{pmatrix} \right\rangle_{[H^2(0,S)\times L^2(0,S)]^2} \\
&+ \left\langle \begin{pmatrix} 0 \\ g_3(t) \end{pmatrix}, \begin{pmatrix} 0 \\ d_1\overline{\psi_t(t,0)} \end{pmatrix} \right\rangle_{[H^3(0,S)\times H^1(0,S)]^2},
\end{aligned}
\tag{2.79}
$$

where $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in C([0,T];V^0)$ *is the solution of the adjoint system* (2.17) *with initial data* $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix}$ *given by (ii).*

Proof. (i) To obtain (2.78) we proceed as in the proof of (2.25) in Proposition 2.2.1.

(ii) Since $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \in V^0$, we have $\sum_{k\in\mathbb{Z}} (|\hat{\varphi}^T_k|^2 + |\hat{\psi}^T_k|^2(1+k^2)^2) < \infty$. Moreover, observe that, $\frac{\widetilde{w}_2(k)}{\widetilde{w}_1(k)} \sim \frac{c}{(1+k^2)^2}$, $\frac{\widetilde{w}_1(k)}{\widetilde{w}_2(k)} \sim C(1+k^2)^2$ and $\sigma(k) \sim \widetilde{C}$, for some positive constants c, C and $\widetilde{C}$. Since

$$
\begin{aligned}
\psi_{txxx}(t,x) = \frac{1}{2}\sum_{k\in\mathbb{I}} k\sigma(k) e^{ik\sigma(k)(T-t)} \sum_{m\in I(k)} \Bigg[&-\sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}} e^{-imx}\widehat{\varphi}^T_m + e^{-imx}\widehat{\psi}^T_m \\
&+ \sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}} e^{imx}\widehat{\varphi}^T_{-m} + e^{imx}\widehat{\psi}^T_{-m} \Bigg]
\end{aligned}
$$

we have that

$$
\begin{aligned}
\psi_{txxx}(t,0) =& \frac{1}{2}\sum_{k\in\mathbb{I}} k\sigma(k)e^{ik\sigma(k)(T-t)} \sum_{m\in I(k)} m^3 \left[-\sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}}\widehat{\varphi}^T_m + \widehat{\psi}^T_m\right.\\
&\left. + \sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}}\widehat{\varphi}^T_{-m} + \widehat{\psi}^T_{-m}\right]\\
\sim& \frac{1}{2}\sum_{k\in\mathbb{I}} ke^{ik\sigma(k)(T-t)} \sum_{m\in I(k)} m^3\left[-\frac{c}{1+m^2}\widehat{\varphi}^T_m + \widehat{\psi}^T_m + \frac{1}{1+m^2}\widehat{\varphi}^T_{-m} + \widehat{\psi}^T_{-m}\right]\\
\sim& \frac{1}{2}\sum_{k\in\mathbb{I}}\sum_{m\in I(k)} \left[m^2\widehat{\varphi}^T_m + m^4\widehat{\psi}^T_m + m^2\widehat{\varphi}^T_{-m} + m^4\widehat{\psi}^T_{-m}\right].
\end{aligned}
$$

Then,

$$
\begin{aligned}
\|\psi_{txxx}\|^2_{H^{-2}(0,T)} \sim& \frac{1}{2}\sum_{k\in\mathbb{I}}\sum_{m\in I(k)} \left[m^4|\widehat{\varphi}^T_m|^2 + m^8|\widehat{\psi}^T_m|^2\right.\\
&\left. + m^4|\widehat{\varphi}^T_{-m}|^2 + m^8|\widehat{\psi}^T_{-m}|^2\right](1+k^2)^{-2}\\
\sim& (\|\varphi^T\|^2_{L^2(0,2\pi)} + \|\psi^T\|^2_{H^2(0,2\pi)}).
\end{aligned}
$$

The remaining cases are proved in a similar way.

(iii) We proceed in several steps.

• (2.79) holds when $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in V^3$, $\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \in V^0$ and $f_0, f_1, f_2, g_0, g_1, g_2, g_3 \in C^2_0([0,T])$.

First, suppose that $\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix} \in V^3$. Since V^3 is dense in V^0, using (ii) we obtain the result.

• Let $S\in[0,T]$ be fixed. Then, (2.79) defines $\begin{pmatrix} u(S) \\ v(S) - dv_{xx}(S) + d_1 v_{xxxx}(S) \end{pmatrix}$ in V^0 in a unique manner.

Observe that, from the proof of (ii) we deduce that the map $\Gamma: V^0 \to \mathbb{C}$ given by

$$
\begin{aligned}
&\Gamma\left(\begin{pmatrix} \varphi_T \\ \psi_T \end{pmatrix}\right) =\\
&-\left\langle \begin{pmatrix} f_0(t) \\ g_0(t) \end{pmatrix}, \begin{pmatrix} \overline{\psi(t,0)} + c\overline{\psi_{xx}(t,0)} \\ \overline{\varphi(t,0)} - d\overline{\psi_{tx}(t,0)} + d_1\overline{\psi_{txxx}(t,0)} + a\overline{\varphi_{xx}(t,0)} \end{pmatrix}\right\rangle_{[L^2(0,S)\times H^{-2}(0,S)]^2}\\
&+\left\langle \begin{pmatrix} f_1(t) \\ g_1(t) \end{pmatrix}, \begin{pmatrix} c\overline{\psi_x(t,0)} \\ a\overline{\varphi_x(t,0)} - d\overline{\psi_t(t,0)} + d_1\overline{\psi_{txx}(t,0)} \end{pmatrix}\right\rangle_{[H^1(0,S)\times H^{-1}(0,S)]^2}\\
&-\left\langle \begin{pmatrix} f_2(t) \\ g_2(t) \end{pmatrix}, \begin{pmatrix} c\overline{\psi(t,0)} \\ a\overline{\varphi(t,0)} + d_1\overline{\psi_{tx}(t,0)} \end{pmatrix}\right\rangle_{[H^2(0,S)\times L^2(0,S)]^2}\\
&+\left\langle \begin{pmatrix} 0 \\ g_3(t) \end{pmatrix}, \begin{pmatrix} 0 \\ d_1\overline{\psi_t(t,0)} \end{pmatrix}\right\rangle_{[H^3(0,S)\times H^1(0,S)]^2},
\end{aligned}
$$

is linear and continuous. On the other hand, the well-posedness of the adjoint system (see Theorem 2.1.3) allows us to conclude that the map $T_S : V^0 \to V^0$, given by

$$T_S \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} = \begin{pmatrix} \varphi(S,\cdot) \\ \psi(S,\cdot) \end{pmatrix},$$

is an automorphism of Hilbert space. This implies that $\begin{pmatrix} u(S,\cdot) \\ v(S,\cdot) - dv_{xx}(S,\cdot) + d_1 u_{xxxx}(S,\cdot) \end{pmatrix}$ is uniquely defined in V^0, for each $S \in [0,T]$.

□

Proposition 2.3.2. *Let $T > 0$. Then, there is $C_1 > 0$, such that*

$$\begin{aligned} C(\|\varphi_T\|^2_{L^2(0,2\pi)} + \|\psi_T\|^2_{H^2(0,2\pi)}) &\le \|\psi(t,0) + c\psi_{xx}(t,0)\|^2_{L^2(0,T)} \\ &+ \|\varphi(t,0) - d\psi_{tx}(t,0) + a\varphi_{xx}(t,0) + d_1\psi_{txxx}(t,0)\|^2_{H^{-2}(0,T)} + \|c\psi_x(t,0)\|^2_{H^1(0,T)} \\ &+ \|a\varphi_x(0,T) - d\psi_t(0,T) + d_1\psi_{txx}(0,T)\|^2_{H^{-1}(0,T)} + \|c\psi(0,T)\|^2_{H^2(0,T)} \\ &+ \|a\varphi(t,0) + d_1\psi_{tx}(t,0)\|^2_{L^2(0,T)} + \|d\psi_t(t,0)\|^2_{H^1(0,T)}, \end{aligned}$$

where (φ,ψ) is solution of the adjoint system with data $(\varphi_T,\psi_T) \in V^0$.

Proof. Recall that, $\sigma(k) \sim \widetilde{C}$. Then, for

$$\psi(t,x) = \frac{1}{2}\sum_{k\in\mathbb{I}} e^{ik\sigma(k)(T-t)} \sum_{m\in I(k)} \left[-\sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}} e^{-imx}\widehat{\varphi}^T_m + e^{-imx}\widehat{\psi}^T_m + \sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}} e^{imx}\widehat{\varphi}^T_{-m} + e^{imx}\widehat{\psi}^T_{-m} \right],$$

we have that

$$\begin{aligned} \psi_{txxx}(t,0) &= \frac{1}{2}\sum_{k\in\mathbb{I}} k\sigma(k) e^{ik\sigma(k)(T-t)} \sum_{m\in I(k)} m^3 \left[-\sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}}\widehat{\varphi}^T_m + \widehat{\psi}^T_m + \sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}}\widehat{\varphi}^T_{-m} + \widehat{\psi}^T_{-m} \right] \\ &\sim \frac{1}{2}\sum_{k\in\mathbb{I}} k e^{ik\sigma(k)(T-t)} \sum_{m\in I(k)} m^3 \left[-\sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}}\widehat{\varphi}^T_m + \widehat{\psi}^T_m + \sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}}\widehat{\varphi}^T_{-m} + \widehat{\psi}^T_{-m} \right] \\ &\sim \frac{1}{2}\sum_{k\in\mathbb{I}} e^{ik\sigma(k)(T-t)} \sum_{m\in I(k)} m^4 \left[-\sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}}\widehat{\varphi}^T_m + \widehat{\psi}^T_m + \sqrt{\frac{\widetilde{w}_2(m)}{\widetilde{w}_1(m)}}\widehat{\varphi}^T_{-m} + \widehat{\psi}^T_{-m} \right] = \psi_{xxxx}(t,0). \end{aligned}$$

Hence

$$\psi_{txxx}(t,0) \sim \psi_{xxxx}(t,0).$$

Similarly, we get $\psi_{tx}(t,0) \sim \psi_{xx}(t,0)$ and $\psi_t(t,0) \sim \psi_x(t,0)$. Now, applying the same previous technique, we obtain the desired result for our new norm. □

Theorem 2.3.1. *Let $T > 0$. Then, for any $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}$, $\begin{pmatrix} u^T \\ v^T \end{pmatrix} \in V^0$, there exist $f_0, g_0 \in L^2(0,T)$, $f_2 \in H^2(0,T)$, $f_1, g_3 \in H^1(0,T)$, $g_1 \in H^{-1}(0,T)$ and $g_0 \in H^{-2}(0,T)$, such that the solution $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T];V^0)$ of (2.75)-(2.76) satisfies $\begin{pmatrix} u(T,\cdot) \\ v(T,\cdot) \end{pmatrix} = \begin{pmatrix} u_T \\ v_T \end{pmatrix}$.*

Proof. We can assume that $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Let Λ denote the map

$$\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \in V^0 \mapsto \begin{pmatrix} u(T,\cdot) \\ v(T,\cdot) - dv_{xx}(T,\cdot) + d_1 v_{xxxx}(T,\cdot) \end{pmatrix} \in [V^0]',$$

where $\begin{pmatrix} u \\ v \end{pmatrix}$ is the solution (weak) of (2.75)-(2.76) with $f_0, g_0, f_1, g_1, f_2, g_2$ and g_3 given by

$$\begin{aligned}
f_0 &= -(\psi(t,0) + c\psi_{xx}(t,0)) \\
g_0 &= -(\varphi(t,0) - d\psi_{tx}(t,0) + d_1\psi_{txxx}(t,0) + a\varphi_{xx}(t,0)) \\
f_1 &= c\psi_x(t,0) \\
g_1 &= a\varphi_x(t,0) - d\psi_t(t,0) + d_1\psi_{txx}(t,0) \\
f_2 &= -c\psi(t,0) \\
g_2 &= -(a\varphi(t,0) + d_1\psi tx(t,0)) \\
g_3 &= d_1\psi_t(t,0).
\end{aligned}$$

Then, proceding as in the previos cases, we obtain the result. □

3 Null-controllability for the Kawahara equation.

This chapter deals with the controllability properties of the linear Kawahara equation posed on a periodic domain. We show that the equation is null-controllable by means of a control depending only on time and acting on the system through a given shape function in space. The method we apply is based on Fourier expansion of solutions and the analysis of a biorthogonal sequence to a family of complex exponential functions.

3.1 The Moment Problem

Let us first present a well-posedness result for system (1.13).

Theorem 3.1.1. *Given any $T > 0$, $F \in L^1(0,T;L^2(0,2\pi))$ and $u^0 \in L^2(0,2\pi)$, there exists a unique weak solution $u \in C([0,T];L^2(0,2\pi))$ of the problem*

$$\begin{cases} u_t - u_{5x} + u_{3x} + u_x = F(t,x), & in \quad (0,T)\times(0,2\pi), \\ \partial_x^j u(t,0) = \partial_x^j u(t,2\pi), & in \quad (0,T), \\ u(0,x) = u_0(x), & in \quad (0,2\pi), \end{cases} \tag{3.1}$$

for $j = 0,1,2,3,4$.

Proof. According to [44], the operator A defined in (1.16) generates a group of isometries in $L^2_p(0,2\pi)$. Hence, the result follows from the semigroup theory. □

Having the well-posedness of (1.13) in hand, we can give now the characterization of the controllability property in terms of a moment problem. We refer to [1, 25, 43] for a detailed discussion of the subject.

Theorem 3.1.2. *Let $T > 0$, $f \in L^2(0,2\pi)$ and $u_0 \in L^2_p(0,2\pi)$, such that*

$$u_0(x) = \sum_{n\in\mathbb{Z}} \widehat{u}_n^0 e^{inx} \quad \text{and} \quad f(x) = \sum_{n\in\mathbb{Z}} \widehat{f}_n e^{inx}.$$

Then, there exists a control $v \in L^2(0,T)$ such that the solution u of (1.13) verifies $u(T,x) = 0$ if, and only if, $v \in L^2(0,T)$ satisfies

$$\widehat{f}_n \int_0^T v(T-s)e^{\lambda_n s}ds = -\widehat{u}_n^0 e^{T\lambda_n}, \tag{3.2}$$

where $\lambda_n = -in(n^4 + n^2 - 1)$ are the eigenvalues of the operator A defined in (1.16).

Proof. We consider the "adjoint" system

$$\begin{cases} \varphi_t - \varphi_{5x} + \varphi_{3x} + \varphi_x = 0, & \text{in} \quad (0,T) \times (0,2\pi), \\ \partial_x^j \varphi(t,0) = \partial_x^j \varphi(t,2\pi), & \text{in} \quad (0,T), \\ \varphi(T,x) = \varphi_T(x), & \text{in} \quad (0,2\pi), \end{cases} \tag{3.3}$$

for $j = 0,1,2,3,4$. If we multiply the equation in (1.13) by $\overline{\varphi}$ and integrate for parts in $(0,T) \times (0,2\pi)$, we deduce that $v \in L^2(0,T)$ is a control for (1.13) if, and only if, it verifies

$$\int_0^T v(t) \int_0^{2\pi} f(x)\overline{\varphi}(t,x)dxdt = -\int_0^{2\pi} u_0(x)\overline{\varphi}(0,x)dx, \tag{3.4}$$

for any solution φ of (3.3). Since $(e^{-inx})_{n\in\mathbb{Z}}$ is a basis for $L_p^2(0,2\pi)$, it is sufficient to check (3.4) for solutions of (3.3) of the form $\varphi(t,x) = e^{(t-T)\lambda_n} e^{-inx}$, $n \in \mathbb{Z}$. Thus, it is straightforward to deduce that (3.2) holds. □

3.2 A Biorthogonal Sequence

This section is devoted to construct a biorthogonal sequence $(\Phi_m)_{m\in\mathbb{Z}}$ mentioned in the previous sections. By using Paley-Wiener Theorem, it is obtained as the inverse Fourier transform of a family Ψ_m of entire functions of exponential type, such that $\Psi_m(i\lambda_n) = \delta_{mn}$, where δ_{mn} is the Kronecker symbol. Each Ψ_m is obtained from a Weierstrass product P_m multiplied by an appropriate function M_m with rapid decay on the real axis. Therefore, for any $m \in \mathbb{Z}^*$, we first introduce the function

$$P_m(z) = \prod_{n\in\mathbb{Z}^*, n\neq m} \left(1 + \frac{iz}{\lambda_n}\right)\left(\frac{\lambda_n}{\lambda_n - \lambda_m}\right), \tag{3.5}$$

where λ_m are the eigenvalues of the operator A defined in (1.16). Since $\lambda_{-m} = \overline{\lambda_m}$, we prove the following result:

Lemma 3.2.1. *P_m is an entire function of the exponential type, such that*

$$P_m(i\lambda_n) = \delta_{mn}, \quad m \in \mathbb{Z}^*,$$

where δ_{mn} is the Kronecker symbol.

Proof. We obtain the result by analyzing the following products:

$$E_m(z) = \prod_{n\in\mathbb{Z}^*, n\neq m} \left|1 + \frac{iz}{\lambda_n}\right| \quad \text{and} \quad Q_m = \prod_{n\in\mathbb{Z}^*, n\neq m} \left|\frac{\lambda_n}{\lambda_n - \lambda_m}\right|. \tag{3.6}$$

First, observe that, for any $z \in \mathbb{C}$,

$$\begin{aligned} E_m(z) &= \prod_{n\in\mathbb{Z}^+, n\neq m} \left|1 + \frac{iz}{\lambda_n}\right| \prod_{n\in\mathbb{Z}^-, n\neq m} \left|1 + \frac{iz}{\lambda_n}\right| = \prod_{n\in\mathbb{N}^*, n\neq m} \left|1 + \frac{iz}{\lambda_n}\right| \left|1 + \frac{iz}{\overline{\lambda_n}}\right| \\ &= \exp\left(\sum_{n=1}^{\infty} \ln\left|1 - \frac{z^2}{|\lambda_n|^2} + 2iz\mathcal{R}\left(\frac{1}{\lambda_n}\right)\right|\right) = \exp\left(\sum_{n=1}^{\infty} \ln\left|1 - \frac{z^2}{|\lambda_n|^2}\right|\right). \end{aligned}$$

Since

$$\sum_{n=1}^{\infty} \ln\left|1-\frac{z^2}{|\lambda_n|^2}\right| \le \sum_{n=1}^{\infty} \ln\left(1+2\frac{|z|^2}{|\lambda_n|^2}\right) \le \sum_{n=1}^{\infty} \ln\left(1+2\frac{|z|^2}{n^2}\right) \le \int_0^{\infty} \ln\left(1+2\frac{|z|^2}{x^2}\right) dx,$$
$$= \sqrt{2}\pi|z|,$$

we get

$$E_m(z) \le \exp(\sqrt{2}\pi|z|). \tag{3.7}$$

For Q_m have that:

$$\begin{aligned} Q_m &= \prod_{n\in\mathbb{Z}^*, n\neq m} \left|\frac{\lambda_n}{\lambda_n-\lambda_m}\right| = \frac{1}{2}\prod_{n\in\mathbb{N}^*, n\neq m} \frac{|\lambda_n|^2}{|\lambda_n-\lambda_m||\lambda_n+\lambda_m|} \\ &= \underbrace{\frac{1}{2}\prod_{n\in\mathbb{N}^*, n\neq m} \frac{|\lambda_n|^2}{|\lambda_{n-m}||\lambda_{n+m}|}}_{Q_m^1} \underbrace{\prod_{n\in\mathbb{N}^*, n\neq m} \frac{|\lambda_{n-m}||\lambda_{n+m}|}{|\lambda_n-\lambda_m||\lambda_n+\lambda_m|}}_{Q_m^2}. \end{aligned}$$

Then, the next steps are devoted to estimate Q_m^1 and Q_m^2.

$$Q_m^1 = \prod_{n\in\mathbb{N}^*, n\neq m} \frac{|\lambda_n|^2}{|\lambda_{n-m}||\lambda_{n+m}|} \le \frac{|\lambda_1|^2|\lambda_2|^2\cdots|\lambda_{m-1}|^2|\lambda_{m+1}|^2\cdots|\lambda_{2m-1}|^2|\lambda_{2m}|^2|\lambda_{2m+1}|^2\cdots}{|\lambda_{m-1}|\cdots|\lambda_1||\lambda_{m+1}|\cdots|\lambda_{2m-1}|\prod_{n=1}^{\infty}|\lambda_n|\prod_{n=2m+1}^{\infty}|\lambda_n|}$$
$$\le \frac{|\lambda_{2m}|}{|\lambda_m|} = \frac{|32m^5+8m^3-2m|}{|m^5+m^3-m|} \le C,$$

where C is a positive constant.

To evaluate Q_m^2, we proceed as follows:

$$\begin{aligned} Q_m^2 &= \prod_{n\in\mathbb{N}^*, n\neq m} \frac{|\lambda_{n-m}||\lambda_{n+m}|}{|\lambda_n-\lambda_m||\lambda_n+\lambda_m|} = \prod_{n\in\mathbb{N}^*, n\neq m}\left(1+\frac{|\lambda_{n-m}||\lambda_{n+m}|-|\lambda_n-\lambda_m||\lambda_n+\lambda_m|}{|\lambda_n-\lambda_m||\lambda_n+\lambda_m|}\right) \\ &= \prod_{n\in\mathbb{N}^*, n\neq m}\left(1+\frac{|\lambda_{n-m}\lambda_{n+m}|-|(\lambda_n-\lambda_m)(\lambda_n+\lambda_m)|}{|\lambda_n-\lambda_m||\lambda_n+\lambda_m|}\right) \\ &\le \prod_{n\in\mathbb{N}^*, n\neq m}\left(1+\frac{|\lambda_{n-m}\lambda_{n+m}-(\lambda_n-\lambda_m)(\lambda_n+\lambda_m)|}{|\lambda_n-\lambda_m||\lambda_n+\lambda_m|}\right) \\ &= \exp\left(\sum_{n=1, n\neq m}^{\infty} \ln\left(1+\frac{|\lambda_{n-m}\lambda_{n+m}-(\lambda_n-\lambda_m)(\lambda_n+\lambda_m)|}{|\lambda_n-\lambda_m||\lambda_n+\lambda_m|}\right)\right) \\ &\le \exp\left(\sum_{n=1, n\neq m}^{\infty}\left(\frac{|\lambda_{n-m}\lambda_{n+m}-(\lambda_n-\lambda_m)(\lambda_n+\lambda_m)|}{|\lambda_n-\lambda_m||\lambda_n+\lambda_m|}\right)\right) \\ &\le \exp\left(\sum_{n=1, n\neq m}^{\infty} \frac{5m^8 f(\frac{n}{m})+4m^6 g(\frac{n}{m})+13m^4 h(\frac{n}{m})}{\alpha(m,n)}\right), \end{aligned}$$

where

$$f(t) = t^6-t^4+t^2, \quad g(t) = t^4+t^2, \quad h(t) = t^2,$$
$$\alpha(m,n) = (n^4+n^3m+n^2m^2+nm^3+m^4+n^2+nm+m^2-1)\times$$
$$|n^4-n^3m+n^2m^2-nm^3+m^4+n^2-nm+m^2-1|.$$

In the remaining part of the proof C will denote a positive constant that may change from one estimate to another, but it is independent of m.

Observe that the function $f(t)$ satisfy

$$f(t) \leq \begin{cases} t^2, & \text{if} \quad 0 \leq t \leq 1 \\ t^6, & \text{if} \quad t \geq 1. \end{cases}$$

Then, if $n \leq m$,

$$\sum_{n=1}^{m-1} \frac{5m^8 f(\frac{n}{m})}{\alpha(m,n)} \leq 5m^8 \sum_{n=1}^{m-1} \frac{\frac{n^2}{m^2}}{n^4} \leq 5m^6 \sum_{n=1}^{m-1} \frac{1}{n^2} \leq 5m^6 \int_1^{m-1} \frac{1}{t^2} dt = 5m^6 \frac{m-2}{m-1} \leq 5m^6.$$

If $n \geq m$,

$$\sum_{n=m+1}^{\infty} \frac{5m^8 f(\frac{n}{m})}{\alpha(m,n)} \leq 5m^8 \sum_{n=m+1}^{\infty} \frac{\frac{n^6}{m^6}}{n^4(n-m)^4} \leq 5m^2 \sum_{n=m+1}^{\infty} \frac{n^2}{(n-m)^4} \leq 5m^2 \sum_{k=1}^{\infty} \frac{(k+m)^2}{k^4} \leq Cm^4. \tag{3.8}$$

In what concerns the function $g(t)$, have that

$$g(t) \leq \begin{cases} (t+1)^2, & \text{if} \quad 0 \leq t \leq 1, \\ 2t^6, & \text{if} \quad t \geq 1. \end{cases}$$

When $n \leq m$,

$$\begin{aligned} \sum_{n=1}^{m-1} \frac{2m^6 g(\frac{n}{m})}{\alpha(m,n)} &\leq 2m^6 \sum_{n=1}^{m-1} \frac{(\frac{n}{m}+1)^2}{n^4} \leq 2m^4 \sum_{n=1}^{m-1} \frac{(n+m)^2}{n^4} \leq 2m^4 \sum_{n=1}^{m-1} \frac{n^2+2nm+m^2}{n^4} \\ &\leq 2m^4 \int_1^{m-1} \left(\frac{1}{t^2} + \frac{2m}{t^3} + \frac{m^2}{t^4} \right) dt \leq Cm^6. \end{aligned}$$

If $n \geq m$, we proceed as in (3.8). In this case, we use the fact that $g(t) \leq 4t^6$, for $t \geq 1$. Finally, to estimate the term involving the function h, we also proceed as before using the following estimate:

$$h(t) \leq \begin{cases} t^2, & \text{if} \quad 0 \leq t \leq 1, \\ t^6, & \text{if} \quad t \geq 1. \end{cases}$$

Combining the estimates above, we deduce that

$$Q_m = Q_m^1 Q_m^2 \leq \exp(Cm^6).$$

From (3.6), (3.7) and the above estimate we conclude the proof. □

Remark 3.2.1. *Lemma 3.2.1 remains valid if we consider the following linear equation associated to (1.12): $u_t + \gamma u_x + \alpha u_{xxx} - \beta u_{xxxxx} = 0$. In fact, the differential operator associated to the space variable is given by $A_1 := \beta \partial_x^5 u - \alpha \partial_x^3 u - \gamma \partial_x u : H_p^5(0, 2\pi) \to L^2(0, 2\pi)$, whose eigenvalues are*

$$\lambda_k = -ik(\beta k^4 + \alpha k^2 - \gamma), \quad k \in \mathbb{Z}.$$

Hence, it may occur that not all eigenvalues are different. If we count only the distinct eigenvalues, we get a sequence $\{\lambda_k\}_{k\in\mathbb{I}}$*, where* $\mathbb{I}\subset\mathbb{Z}$ *have a property of* $\lambda_{k_1}\neq\lambda_{k_2}$ *for any* $k_1,k_2\in\mathbb{I}$*. Then, for all* $k_1\in\mathbb{Z}$*, we define*

$$I(k_1)=\{k\in\mathbb{Z}:k(\beta k^4+\alpha k^2-\gamma)=k_1(\beta k_1^4+\alpha k_1^2-\gamma)\}$$

and $|I(k_1)|=m(k_1)$*, which has the following properties:*

- $m(k_1)\leq 5$*. This is a consequence of the fact that the polynomial* $p(x)=x(\beta x^4+\alpha x^2-\gamma)$ *has a maximum of* 5 *distinct roots.*
- $\lambda_k\to\pm\infty$*, as* $k\to\pm\infty$*. Then, there exists* $k^*\in\mathbb{N}$*, such that* $m(k)=1$*, for all* $|k|\geq k^*$*.*

To prove Lemma 3.2.1, we have assumed that $I(k_1)$ *is a unitary set. This is due to the fact that, in the original model, we have assumed that* $\alpha=\beta=\gamma=1$*. If this is not the case, we can also prove the result by using the same approach. Indeed, following the notation introduced in the proof of the lemma, we have that*

$$\begin{aligned}Q_m^1&=\prod_{n\in\mathbb{N}^*,n\notin I(m)}\frac{|\lambda_n|^2}{|\lambda_{n-m}||\lambda_{n+m}|}\\&=\prod_{n=1}^{m_1-1}\frac{|\lambda_n|^2}{|\lambda_{n-m}||\lambda_{n+m}|}\prod_{m_1+1}^{m_2-1}\frac{|\lambda_n|^2}{|\lambda_{n-m}||\lambda_{n+m}|}\cdots\prod_{m_5+1}^{\infty}\frac{|\lambda_n|^2}{|\lambda_{n-m}||\lambda_{n+m}|}.\end{aligned}$$

Then, proceeding in a similar way, we can estimate each term of the product above. For Q_m^2*, we use a similar argument.*

From Lemma 3.2.1 we obtain the following estimate for P_m, defined in (3.5):

$$|P_m(z)|\leq\exp(C\pi(|z|+m^6),$$

where C is a positive constant. Consequently, on the real axis, it follows that

$$|P_m(x)|\leq\exp(C_1(|x|+m^6), \tag{3.9}$$

for some $C_1>0$.

The next proposition guarantees the existence of a entire function (of exponential type) which plays an important role in the construction of the biorthogonal sequence. It is an appropriate multiplier that compensates the growth of P_m on the real axis. In order to prove the proposition, the following technical lemma is needed.

Lemma 3.2.2. *If* $x\geq m^6$*, then*

$$\sum_{j=m^6}^{[x]}\ln\left|\frac{j}{x}\right|=-\int_{m^6}^{x}\frac{B(u)-m^6+1}{u}du, \tag{3.10}$$

where $B(u)=\#\{n:n\leq u\}$*.*

Proof. Firstly, we remark that the function B has the following properties:

- If $j \le u < j+1$, we have $B(u) = j$.
- If $[x] \le u \le x$, then $B(u) = [x]$ and $B(u) \ge x - 1$.

Hence, we have that

$$\begin{aligned}
-\int_{m^6}^{x} \frac{B(u)}{u} du &= -\sum_{j=m^6}^{[x]-1} \int_j^{j+1} \frac{B(u)}{u} du - \int_{[x]}^{x} \frac{B(u)}{u} du \\
&= -\sum_{j=m^6}^{[x]-1} \int_j^{j+1} \frac{j}{u} du - \int_{[x]}^{x} \frac{[x]}{u} du \\
&= \sum_{j=m^6}^{[x]-1} j \ln \left| \frac{j}{j+1} \right| + [x] \ln \left| \frac{[x]}{x} \right| = \ln \left| \prod_{j=m^6}^{[x]-1} \frac{(j)^j}{(j+1)^j} \frac{([x])^{[x]}}{(x)^{[x]}} \right| \\
&= \ln \left| \frac{(m^6)^{m^6}}{(m^6+1)^{m^6}} \frac{(m^6+1)^{m^6+1}}{(m^6+2)^{m^6+1}} \cdots \frac{([x]-1)^{[x]-1}}{([x])^{[x]-1}} \frac{([x])^{[x]}}{(x)^{[x]}} \right| \\
&= \ln \left| \frac{(m^6)^{m^6-1}}{([x])^{m^6-1}} \prod_{j=m^6}^{[x]-1} \frac{j}{x} \right| = -\int_{m^6}^{x} \frac{m^6-1}{u} du + \sum_{j=m^6}^{[x]} \ln \left| \frac{j}{x} \right|.
\end{aligned}$$

□

As remarked above, Lemma 3.2.2 allows us to prove the following result, inspired in [21]:

Proposition 3.2.1. *For each $m \ge 1$, there exists a function $M_m : \mathbb{C} \to \mathbb{C}$ and positive constants $K_1, K_2 > 0$, such that:*

- *M_m is a function of the exponential type,*
- $|M_m(x)| \le \exp(K_1(m^6 - |x|)), \forall\, x \in \mathbb{R}$,
- $|M_m(i\lambda_m)| \ge \exp(-K_2 m^6)$,

where $\lambda_m = -im(m^4 + m^2 - 1)$ are the eigenvalues of the operator A defined in (1.16).

Proof. We follow the ideas introduced in [21] and define a function $M_m : \mathbb{C} \to \mathbb{C}$ as follows:

$$M_m(z) = \prod_{n=m^3}^{\infty} \frac{\sin(\frac{z}{n^2})}{\frac{z}{n^2}}. \tag{3.11}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, the first property is a consequence of the following estimate:

$$\prod_{n=m^3}^{N} \left| \frac{\sin(\frac{z}{n^2})}{\frac{z}{n^2}} \right| \le \prod_{n=m^3}^{N} \exp\left(\left| \frac{z}{n^2} \right| \right) = \exp(|z| \sum_{n=m^3}^{N} \frac{1}{n^2}) \le \exp(C|z|),$$

for some $C > 0$.

To prove the second property, we proceed in two steps, as follows:

- If $|x| \le m^6$, then

$$|M_m(x)| = \prod_{n=m^3}^{\infty} \left| \frac{\sin(\frac{x}{n^2})}{\frac{x}{n^2}} \right| \le 1 \le \exp(m^6 - |x|).$$

- If $|x| > m^6$, we apply Lemma 3.2.2 to deduce that

$$\begin{aligned} |M_m(x)| &= \prod_{n=m^3}^{\infty} \left| \frac{\sin(\frac{x}{n^2})}{\frac{x}{n^2}} \right| \le \prod_{n=m^3}^{[|x|^{\frac{1}{2}}]} \frac{n^2}{|x|} = \exp\left(\sum_{n=m^3}^{[|x|^{\frac{1}{2}}]} \ln \frac{n^2}{|x|} \right) \le \exp\left(\sum_{n=m^6}^{[|x|]} \ln \frac{n^2}{|x|} \right) \\ &= \exp\left(-\int_{m^6}^{|x|} \frac{B(u) - m^6 + 1}{u} du \right). \end{aligned}$$

Since $m^6 \le [|x|]$, from the estimate above, we obtain a positive constant satisfying

$$\begin{aligned} |M_m(x)| &\le \exp\left(-\int_{[|x|]}^{|x|} \frac{B(u) - m^6 + 1}{u} du \right) \le \exp\left(-\int_{[|x|]}^{|x|} \frac{|x| - 1 - m^6 + 1}{u} du \right) \\ &= \exp\left((m^6 - |x|) \ln \frac{|x|}{[|x|]} \right) \le C \exp(m^6 - |x|), \end{aligned}$$

where C is a positive constant.

In what concerns the third property, we observe that $m^6 \ge |\lambda_m|$, i. e., $\left| \frac{\lambda_m}{n^2} \right| \le 1$. Then,

$$\begin{aligned} |M_m(i\lambda_m)| &= \prod_{n=m^3}^{\infty} \left| \frac{\sin\left(\frac{i\lambda_m}{n^2}\right)}{\frac{i\lambda_m}{n^2}} \right| = \prod_{n=m^3}^{\infty} \frac{\sin(\frac{|\lambda_m|}{n^2})}{\frac{|\lambda_m|}{n^2}} \ge \prod_{n=m^3}^{\infty} \left| 1 - \frac{1}{6} \frac{|\lambda_m|^2}{n^4} \right| \\ &= \exp\left(\sum_{n=m^3}^{\infty} \ln\left(1 - \frac{1}{6} \frac{|\lambda_m|^2}{n^4} \right) \right) \ge \exp\left(-\frac{|\lambda_m|^2}{30} \sum_{n=m^3}^{\infty} \frac{1}{n^4} \right) \\ &\ge \exp\left(-\frac{m^6}{30} \sum_{n=m^3}^{\infty} \frac{1}{n^2} \right) \ge \exp\left(-\frac{m^6}{30} C \right), \end{aligned}$$

for some $C > 0$. □

Now we have the tools we need to construct a biorthogonal sequence to the family $(e^{\lambda_n t})_{n \in \mathbb{Z}^*}$ in $L^2(-\frac{T}{2}, \frac{T}{2})$, $T > 0$.

Theorem 3.2.1. *There exists a constant $T_1 > 0$ and a biorthogonal sequence $(\Theta_m)_{m \in \mathbb{Z}^*}$ to the family $(e^{-\lambda_n t})_{n \in \mathbb{Z}^*}$ in $L^2(-\frac{T_1}{2}, \frac{T_1}{2})$. Moreover,*

$$\|\Theta_m\|_{L^2(-\frac{T_1}{2}, \frac{T_1}{2})} \le C \exp(bm^6), \tag{3.12}$$

where C and b are positive constants.

Proof. For all $m \in \mathbf{Z}^*$, let P_m and M_m be the functions defined in (3.5) and (3.11), respectively. We also define the function

$$\Psi_m(z) = P_m(z)\left(\frac{M_{|m|}(z)}{M_{|m|}(i\lambda_m)}\right)^{\frac{C_1}{K_1}} \frac{\sin(\delta(z - i\lambda_m))}{\delta(z - i\lambda_m)},$$

where $\delta > 0$ is an arbitrary constant, C_1 is given in (3.9) and K_1 in Proposition 3.2.1. Let

$$\Theta_m(t) = \frac{1}{2\pi}\int_{\mathbf{R}} \Psi_m(x)e^{itx}dx. \tag{3.13}$$

From Lemma 3.2.1 and Proposition 3.2.1, we deduce that there exists $\widetilde{T} > 0$, such that Ψ_m is an entire function of the exponential type $\frac{\widetilde{T}}{2}$. Moreover, from the estimates for P_m and M_m on the real axis (see (3.9) and Proposition 3.2.1) we obtain

$$\begin{aligned}\int_{\mathbb{R}} |\Psi_m(x)|^2 dx &\leq Ce^{2(2C_1+\frac{C_1K_2}{K_1})m^6}\int_{\mathbf{R}}\left|\frac{\sin(\delta(x-i\lambda_m))}{\delta(x-i\lambda_m)}\right|^2 dx \\ &\leq \frac{C}{\delta}e^{2(2C_1+\frac{C_1K_2}{K_1})m^6}\int_{\mathbf{R}}\left|\frac{\sin t}{t}\right|^2 dt \leq C_1 e^{bm^6},\end{aligned} \tag{3.14}$$

where $b = 2\left(2C_1 + \frac{C_1K_2}{K_1}\right)$. Taking into account the properties of Ψ_m and applying Paley-Wiener Theorem, we deduce that $\widehat{\Theta}_m$ has support included in $\left(-\frac{\widetilde{T}}{2}, \frac{\widetilde{T}}{2}\right)$ and $\Theta_m \in L^2(-\frac{\widetilde{T}}{2}, \frac{\widetilde{T}}{2})$. Moreover, from the properties of the inverse Fourier transform we have that the sequence Θ_m is biorthogonal to $(e^{-\lambda_m t})_{m\in\mathbf{Z}}$ in $L^2(-\widetilde{T}, \widetilde{T})$. In fact,

$$\int_{-\frac{\widetilde{T}}{2}}^{\frac{\widetilde{T}}{2}} \Theta_m(t)e^{\lambda_n t}dt = \int_{-\frac{\widetilde{T}}{2}}^{\frac{\widetilde{T}}{2}} \Theta_m(t)e^{-i(i\lambda_n)t}dt = \Psi_m(i\lambda_n) = P_m(i\lambda_n)\frac{\sin(\delta i(\lambda_n - \lambda_m))}{\delta i(\lambda_n - \lambda_m)} = \delta_{nm}.$$

Finally, the estimative (3.12) follows from (3.14) by using Plancherel Theorem. □

Remark 3.2.2. *Let Θ_m be given by* (3.13). *From the proof of Theorem 3.2.1, it follows that $\widehat{\Theta}_m$ has support included in $(-\frac{\widetilde{T}}{2}, \frac{\widetilde{T}}{2})$ and*

$$\|\widehat{\Theta}_m\|_{L^\infty(\mathbb{R})} \leq C\exp(bm^6).$$

The following result gives the existence of a new biorthogonal sequence with better norm properties than the one from Theorem 3.2.1. In order to prove it, for $a > 0$, we define the following auxiliary functions:

$$\kappa_a = \frac{\sqrt{2\pi}}{a^2}(\chi_a * \chi_a) \quad \text{and} \quad \rho_m(x) = e^{x\lambda_m}\kappa_a(x), \tag{3.15}$$

where χ_a is the characteristic function of the interval $[-\frac{a}{2}, \frac{a}{2}]$. Observe that κ_a and ρ_m satisfy the following properties:

- *supp*$(\kappa_a) \subset [-a, a]$,
- $\widehat{\kappa_a}(\xi) = \frac{4}{a^2}\frac{\sin^2((\frac{a}{2})\xi)}{\xi^2}$,

- $\widehat{\kappa_a}(0) = 1$,

- $supp(\rho_m) \subset [-a, a]$,

- $\widehat{\rho}_m(x) = \widehat{\kappa}_a(x - \lambda_m)$.

Then, we have the following result:

Theorem 3.2.2. *There exist positive constants $T > 2\pi$, b and C and a biorthogonal sequence $(\zeta_m)_{m\in\mathbb{Z}}$ to the family $(e^{-\lambda_m t})_{m\in\mathbb{Z}}$ in $L^2(-\frac{T}{2}, \frac{T}{2})$, with the property*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{n\in\mathbb{Z}^*} c_m \zeta_m(t) \right|^2 dt \leq C \sum_{n\in\mathbb{Z}^*} |c_n|^2 e^{2bm^6},$$

for any sequence $(c_n)_{n\in\mathbb{N}}$.

Proof. Let $(\Theta_m)_{m\in\mathbf{Z}^*} \subset L^2(-\widetilde{T}, \widetilde{T})$ be the biorthogonal sequence given by Theorem 3.2.1. Define

$$\zeta_m(t) = \frac{1}{2\pi\widehat{\rho}_m(i\lambda_m)} (\Theta_m * \rho_m)(t), \qquad m \in \mathbb{Z}^*,$$

where $\widehat{\rho}_m$ is the Fourier transform of ρ_m defined in (3.15). Since $\zeta_m \in L^2(-\widetilde{T} - a, \widetilde{T} + a)$, take $\frac{T}{2} = \widetilde{T} + a$. Then, applying the properties of convolution, it follows that $(\zeta_m)_{m\in\mathbb{Z}}$ is a biorthogonal sequence to $(e^{-\lambda_m t})_{m\in\mathbb{Z}}$. In fact,

$$\begin{aligned}
\int_{-\frac{T}{2}}^{\frac{T}{2}} \zeta_m(t) e^{\lambda_n t} dt &= \int_{-\frac{T}{2}}^{\frac{T}{2}} \zeta_m(t) e^{-i(i\lambda_n)t} dt = 2\pi \widehat{\zeta}_m(i\lambda_n) = \frac{2\pi}{2\pi\widehat{\rho}_m(i\lambda_m)} \widehat{\Theta}_m(i\lambda_n)\widehat{\rho}_m(i\lambda_n) \\
&= \frac{1}{\widehat{\rho}_m(i\lambda_m)} \Psi_m(i\lambda_n)\widehat{\rho}_m(i\lambda_n) = \delta_{nm}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{m\in\mathbb{Z}^*} c_m \zeta_m(t) \right|^2 dt &= \int_{-\infty}^{\infty} \left| \sum_{m\in\mathbb{Z}^*} c_m \widehat{\Theta}_m(x)\widehat{\rho}_m(x) \right|^2 dx \\
&\leq \int_{-\infty}^{\infty} \left(\sum_{m\in\mathbb{Z}^*} |c_m| \|\widehat{\Theta}_m\|_{L^\infty(\mathbb{R})} |\widehat{\kappa}_a(x - \lambda_m)| \right)^2 dx \\
&= \int_{-\infty}^{\infty} \left| \sum_{m\in\mathbb{Z}^*} |c_m| \|\widehat{\Theta}_m\|_{L^\infty(\mathbb{R})} \kappa_a(t) e^{i\lambda_m t} \right|^2 dt \\
&\leq \int_{-a}^{a} \left| \sum_{m\in\mathbb{Z}^*} |c_m| \|\widehat{\Theta}_m\|_{L^\infty(\mathbb{R})} e^{i\lambda_m t} \right|^2 dt.
\end{aligned}$$

Remark that $|\lambda_{m+1}-\lambda_m|>1$, for all $m\in\mathbb{Z}^*$. Hence, from Ingham inequality and Remark 3.2.2, we get

$$\int_{-a}^{a}\left|\sum_{m\in\mathbb{Z}^*}|c_m|\|\widehat{\Theta}_m\|_{L^\infty(\mathbb{R})}e^{i\lambda_m t}\right|^2 dt\leq\sum_{m\in\mathbb{Z}^*}|c_m|^2\|\widehat{\Theta}_m\|^2_{L^\infty(\mathbb{R})}\leq\sum_{m\in\mathbb{Z}^*}|c_m|^2e^{bm^6}. \tag{3.16}$$

□

3.3 Controllability

This section is devoted to prove the main result of the chapter. In order to do that, for any $\beta\geq b$, where b is given by Theorem 3.2.2, and f as in (1.18), we define the space

$$\widetilde{\mathcal{H}}=\left\{h\in L^2(0,2\pi):\sum_{k\in\mathbb{Z}}\left|\frac{\hat{h}_k}{\hat{f}_k}\right|^2e^{\beta k^6}<\infty\right\}. \tag{3.17}$$

Then, our main result reads as follows:

Theorem 3.3.1. *Let $f\in L^2(0,2\pi)$ a function verifying (1.18) and $\widetilde{\mathcal{H}}$ defined by (3.17). There exists $T>0$, such that, for any initial data $u_0\in\widetilde{\mathcal{H}}$, there exist a control $v\in L^2(0,T)$ for which the solution of (1.13) satisfies $u(T,x)=0$.*

Proof. Let $T>2\pi$ and $(\zeta_m)_{m\in\mathbb{Z}^*}$ given by Theorem 3.2.2. For $u_0\in\widetilde{\mathcal{H}}$ given by $u_0(x)=\sum_{k\in\mathbf{Z}}\widehat{u}_k^0e^{ikx}$, define v as follows:

$$v(t)=-\sum_{m\in\mathbb{Z}}\frac{\hat{u}_m^0}{\hat{f}_m}e^{\frac{T}{2}\lambda_m}\zeta_m\left(t-\frac{T}{2}\right),\qquad t\in(0,T). \tag{3.18}$$

From the properties of the biorthogonal sequence $(\zeta_m)_{m\in\mathbb{Z}}$, we deduce that v is a control that satisfies (3.2), i. e., leads the solution to zero. Moreover, $v\in L^2(0,T)$. In fact,

$$\int_0^T|v(t)|^2dt=\int_0^T\left|-\sum_{m\in\mathbb{Z}}\frac{\hat{u}_m^0}{\hat{f}_m}e^{-\frac{T}{2}\lambda_m}\zeta_m\left(t-\frac{T}{2}\right)\right|^2dt\leq C\sum_{m\in\mathbb{Z}}\frac{|\hat{u}_m^0|^2}{|\hat{f}_m|^2}e^{bm^6}\leq C, \tag{3.19}$$

for some $C>0$. □

3.4 Comments and Open Problems

We close this chapter with some comments and open problems:

- Following the approach employed in this work, Theorem 3.3.1 can be proved for the KdV equation with similar statements. In this case, our analysis can be simplified due to the absence of the fifth order dispersive term.

- In [31], the authors consider the following parabolic type control system

$$\begin{cases} u_t + i(-\partial_{xx}^2)^{\frac{1}{2}} u - \varepsilon \partial_{xx}^2 u = f(x) v_\varepsilon(t), & \text{in} \quad (0,T) \times (0,\pi), \\ u(t,0) = u(t,\pi) = 0 & \text{in} \quad (0,T), \\ u(0,x) = u_0(x), & \text{in} \quad (0,\pi), \end{cases}$$

  where v_ε is a control and f is a given profile. For $\varepsilon = 0$ the system is of hyperbolic type and the authors show that the control steering the hyperbolic system to rest can be approximated by a sequence $(v_\varepsilon)_{\varepsilon>0}$ of controls of the parabolic system when $\varepsilon \to 0$. The proof is based on the moment problem with respect to the nonharmonic Fourier family $(e^{\lambda_n})_{n\in\mathbb{N}}$, where $\lambda_n = in - \varepsilon n^2$, $n \geq 1$, are the eigenvalues of the corresponding differential state operator. More recently, in [10], the same problem was studied for the linear wave equation by introducing a viscous term which contains a fractional power of the Dirichlet Laplace operator. It is a difficult problem that remains unanswered for the Kawahara equation.

4 Unique continuation for a higher order KdV-BBM equation

In this chapter we are interested in the unique continuation issue for the initial value problem associated with a higher order water wave model on a one dimensional torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. In the literature, the model under consideration is also known as the higher order KdV-BBM type equation.

4.1 Carleman estimates

In this section we prove some UCP for the following KdV-BBM equation

$$u_t - b_1 u_{txx} + b u_{txxxx} + a u_{xxxxx} + q(u)u_x + p(u)u_{xxx} + r(u)u_{xx} = 0, \quad (x,t) \in \mathbb{T}\times(0,T), \tag{4.1}$$

where $q(u) = 1 + \frac{3}{2}u - \frac{3}{8}u^2$, $p(u) = a_1 + 2\gamma u$ and $r(u) = (6\gamma - \frac{7}{24})u_x$.

Theorem 4.1.1. *Let $a, b \neq 0$, $T > \frac{2\pi b}{|a|}$ and $q, p, r \in L^\infty(0,T; L^\infty(\mathbb{T}))$. Let $\omega \subset \mathbb{T}$ be a nonempty open set. Let $u \in L^2(0,T; H^4(\mathbb{T})) \cup L^\infty(0,T; H^3(\mathbb{T}))$ satisying (4.1) and*

$$u(x,t) = 0 \quad \text{for a.e } \ (x,t) \in \omega \times (0,T). \tag{4.2}$$

Then $u = 0$ in $\mathbb{T} \times (0,T)$

Proof. Assume that

$$u \in L^2(0,T; H^4(\mathbb{T})). \tag{4.3}$$

Let $w = u - b_1 u_{xx} + b u_{xxxx} \in L^2(0,T; L^2(\mathbb{T}))$. Then (u, w) solves the following system

$$u - b_1 u_{xx} + b u_{xxxx} = w \tag{4.4}$$

$$w_t + \frac{a}{b} w_x = (\frac{a}{b} - q)u_x - (\frac{ab_1}{b} + p)u_{xxx} - r u_{xx}. \tag{4.5}$$

We shall establish some Carleman estimates for the elliptic equation (4.4) and the transport equation (4.5) with the "same weights", and combine both Carleman estimates into the single one for (4.1).

Remark 4.1.1. *There is a finite speed propagation for KdV-BBM: assuming for simplicity that $q(x) = \frac{a}{b}$, $p(x) = \frac{-ab_1}{b}$ and $r(x) = 0$ for all $x \in \mathbb{T}$, where $a > 0$ is given, and that $\omega = (2\pi - \epsilon, 2\pi)$ for a small $\epsilon > 0$, then the UCP fails in time $T \le \frac{b(2\pi - 2\epsilon)}{a}$. Indeed, picking any nontrivial initial state $u_0 \in C_0^\infty(0,\epsilon)$, we easily see that the solution (u,w) of (4.4)-(4.5) is $u(x,t) = u_0(x - \frac{a}{b}t)$, $w(x,t) = w_0(x - \frac{a}{b}t)$ where $w_0 = (I - \partial_x^2)u_0$. Then $u(x,t) = 0$ for $(x,t) \in \omega \times (0, \frac{b(2\pi-2\epsilon)}{a})$ although $u \neq 0$. Hence, the condition $T > \frac{2b\pi}{|a|}$ in the Theorem 4.1.1 is sharp.*

Introduce a few notations. We identify $\mathbb{T}$ with $[0,2\pi)$ by chossing a system of coordinates. Without loss of generality, we can assume that $c>0$, and that $\omega=(2\pi-\eta,2\pi+\eta)\sim[0,\eta)\cup(2\pi-\eta,2\pi)$ for some $\eta\in(0,\pi)$(by choosing the origin of the coordinates inside ω). Assume given a time T fulfilling

$$T>\frac{2b\pi}{a}. \tag{4.6}$$

Pick some numbers $\delta>0$ and $\rho\in(0,1)$, such that

$$\rho cT>2\pi+\delta \tag{4.7}$$

and a function $\psi\in C^\infty([0,2\pi]\times\mathbb{R})$ satisfying

$$\psi(x)=(x+\delta)^2 \qquad \text{for } x\in[\frac{\eta}{2},2\pi-\frac{\eta}{2}], \tag{4.8}$$

$$\frac{d^k\psi}{dx^k}(0)=\frac{d^k\psi}{dx^k}(2\pi) \qquad \text{for } k=1,2,3,4,5,6,7, \tag{4.9}$$

$$2\delta\le\frac{d\psi}{dx}(x)\le 2(2\pi+\delta) \quad \text{for } x\in[0,2\pi]. \tag{4.10}$$

Introduce the function $\varphi\in C^\infty([0,2\pi]\times\mathbb{R})$ defined by

$$\varphi(x,t)=\psi(x)-\rho c^2t^2. \tag{4.11}$$

Then, the following Carleman estimate for (4.1) will be derived.

Proposition 4.1.1. *Let ω, c and T be as above. Then, there exists some positive numbers s_2 and C_2, such that, for all $s\ge s_2$ and all $u\in L^2(0,T;H^4(\mathbb{T}))$ satisfying (4.1), we have*

$$\begin{aligned}\int_0^T\int_{\mathbb{T}}[s|u_{xxxx}|^2+s|u_{xxx}|^2+s^3|u_{xx}|^2+s^5|u_x|^2+s^7|u|^2]e^{2s\varphi}dxdt \\ +s\int_{\mathbb{T}}[|u-b_1u_{xx}+bu_{xxxx}|^2e^{2s\varphi}]_{t=0}dx \\ \le C_2\int_0^T\int_\omega[s|u_{xxxx}|^2+s^3|u_{xx}|^2+s^7|u|^2]e^{2s\varphi}dxdt.\end{aligned} \tag{4.12}$$

Note that the Carleman estimate (4.12) yields at once the observability inequality

$$\|u(\cdot,0)\|_{H^4(\mathbb{T})}\le C\int_0^T\|u(\cdot,t)\|^2_{H^4(\omega)}dt. \tag{4.13}$$

Proof. The proof of Proposition 4.1.1 is outlined as follows. In the first step, we prove a Carleman estimate for the elliptic equation (4.4) with the weight $e^{s\psi}$. In the second step, we prove a Carleman estimate for the transport equation (4.5) with the weight $e^{s\varphi}$. In the last step, we combine the two above Carleman estimates into a single one to obtain (4.12). Step 1. Carleman estimate for the elliptic equation

Lemma 4.1.1. *There exist $s_0\ge1$ and $C_0>0$ such that for all $s\ge s_0$ and all $u\in H^4(\mathbb{T})$ the following holds*

$$\begin{aligned}\int_{\mathbb{T}}[s|u_{xxx}|^2+s^3|u_{xx}|^2+s^5|u_x|^2+s^7|u|^2]e^{2s\psi}dx \\ \le C_0\left(\int_{\mathbb{T}}|u_{xxxx}|^2e^{2s\psi}dx+\int_\omega(s^7|u|^2+s^3|u_{xx}|^2)e^{2s\psi}dx\right).\end{aligned} \tag{4.14}$$

Proof. Let $v = e^{s\psi}u$ and $P = \partial_x^4$. Then,

$$e^{s\psi}Pu = e^{s\psi}P(e^{-s\psi}v) = P_p v + P_n v$$

where

$$P_p v = (s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)v + 12s^2\psi_x\psi_{xx}v_x + 6s^2\psi_x^2 v_{xx} + v_{xxxx}, \tag{4.15}$$

$$P_n v = -(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})v - (4s^3\psi_x^3 + 4s\psi_{xxx})v_x - 6s\psi_{xx}v_{xx} - 4s\psi_x v_{xxx}. \tag{4.16}$$

It follows that

$$\|e^{s\psi}Pu\|^2 = \|P_p v\|^2 + \|P_n v\|^2 + 2(P_p v, P_n v),$$

where $(f,g) = \int_{\mathbb{T}} fg dx$, and $\|f\| = (f,f)$. Then,

$$\begin{aligned}
&(P_p v, P_n v) = ((s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)v, -(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})v) \\
&+ ((s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)v, -(4s^3\psi_x^3 + 4s\psi_{xxx})v_x) \\
&+ ((s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)v, -6s\psi_{xx}v_{xx}) \\
&+ ((s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)v, -4s\psi_x v_{xxx}) + (12s^2\psi_x\psi_{xx}v_x, -(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})v) \\
&+ (12s^2\psi_x\psi_{xx}v_x, -(4s^3\psi_x^3 + 4s\psi_{xxx})v_x) + (12s^2\psi_x\psi_{xx}v_x, -6s\psi_{xx}v_{xx}) \\
&+ (12s^2\psi_x\psi_{xx}v_x, -4s\psi_x v_{xxx}) + (6s^2\psi_x^2 v_{xx}, -(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})v) \\
&+ (6s^2\psi_x^2 v_{xx}, -(4s^3\psi_x^3 + 4s\psi_{xxx})v_x) + (6s^2\psi_x^2 v_{xx}, -6s\psi_{xx}v_{xx}) + (6s^2\psi_x^2 v_{xx}, -4s\psi_x v_{xxx}) \\
&+ (v_{xxxx}, -(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})v) + (v_{xxxx}, -(4s^3\psi_x^3 + 4s\psi_{xxx})v_x) + (v_{xxxx}, -6s\psi_{xx}v_{xx}) \\
&+ (v_{xxxx}, -4s\psi_x v_{xxx}) \\
&= \sum_{n=1}^{16} I_n.
\end{aligned}$$

After some integrations by parts in x, from (4.9) we obtain that

$$\begin{aligned}
I_1 &= -\int_{\mathbb{T}} (s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})v^2 dx \\
I_2 &= -\int_{\mathbb{T}} (s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)(4s^3\psi_x^3 + 4s\psi_{xxx})v_x v dx \\
&= \int_{\mathbb{T}} [(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)(2s^3\psi_x^3 + 2s\psi_{xxx})]_x v^2 dx \\
I_3 &= -\int_{\mathbb{T}} (s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)(6s\psi_{xx})v v_{xx} dx \\
&= \int_{\mathbb{T}} [(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)(6s\psi_{xx})]_x v v_x dx \\
&+ \int_{\mathbb{T}} [(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)(6s\psi_{xx})] v_x^2 dx \\
&= -\int_{\mathbb{T}} [(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)(3s\psi_{xx})]_{xx} v^2 dx \\
&+ \int_{\mathbb{T}} [(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)(6s\psi_{xx})] v_x^2 dx
\end{aligned}$$

$$
\begin{aligned}
I_4 &= -\int_{\mathbb{T}}(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)4s\psi_x v v_{xxx}dx\\
&= \int_{\mathbb{T}}[(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)4s\psi_x]_x v v_{xx}dx\\
&+ \int_{\mathbb{T}}(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)4s\psi_x v_x v_{xx}dx\\
&= -\int_{\mathbb{T}}[(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)4s\psi_x]_{xx} v v_x dx\\
&- \int_{\mathbb{T}}[(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)4s\psi_x]_x v_x^2 dx\\
&- \int_{\mathbb{T}}[(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)2s\psi_x]_x v_x^2 dx\\
&= \int_{\mathbb{T}}[(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)2s\psi_x]_{xxx} v^2 dx\\
&- \int_{\mathbb{T}}[(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)6s\psi_x]_x v_x^2 dx\\
I_5 &= -\int_{\mathbb{T}} 12s^2\psi_x\psi_{xx}(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})v v_x dx\\
&= \int_{\mathbb{T}}[6s^2\psi_x\psi_{xx}(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_x v^2 dx\\
I_6 &= -\int_{\mathbb{T}} 12s^2\psi_x\psi_{xx}(4s^3\psi_x^3 + 4s\psi_{xxx})v_x^2 dx\\
I_7 &= -\int_{\mathbb{T}} 12s^2\psi_x\psi_{xx}6s\psi_{xx}v_x v_{xx}dx = \int_{\mathbb{T}}[36s^3\psi_x\psi_{xx}^2]_x v_x^2 dx\\
I_8 &= -\int_{\mathbb{T}} 12s^2\psi_x\psi_{xx}4s\psi_x v_x v_{xxx}dx = -\int_{\mathbb{T}} 48s^3\psi_x^2\psi_{xx}v_x v_{xxx}dx\\
&= \int_{\mathbb{T}}[48s^3\psi_x^2\psi_{xx}]_x v_x v_{xx}dx + \int_{\mathbb{T}} 48s^3\psi_x^2\psi_{xx}v_{xx}^2 dx\\
&= -\int_{\mathbb{T}}[24s^3\psi_x^2\psi_{xx}]_{xx} v_x^2 dx + \int_{\mathbb{T}} 48s^3\psi_x^2\psi_{xx}v_{xx}^2 dx\\
I_9 &= -\int_{\mathbb{T}} 6s^2\psi_x^2(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})v v_{xx}dx\\
&= \int_{\mathbb{T}}[6s^2\psi_x^2(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_x v v_x dx + \int_{\mathbb{T}} 6s^2\psi_x^2(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})v_x^2 dx\\
&= -\int_{\mathbb{T}}[3s^2\psi_x^2(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_{xx} v^2 dx + \int_{\mathbb{T}} 6s^2\psi_x^2(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})v_x^2 dx\\
I_{10} &= -\int_{\mathbb{T}} 6s^2\psi_x^2(4s^3\psi_x^3 + 4s\psi_{xxx})v_x v_{xx}dx = \int_{\mathbb{T}}[3s^2\psi_x^2(4s^3\psi_x^3 + 4s\psi_{xxx})]_x v_x^2 dx\\
I_{11} &= -\int_{\mathbb{T}} 6s^2\psi_x^2 6s\psi_{xx}v_{xx}^2 dx = -\int_{\mathbb{T}} 36s^3\psi_x^2\psi_{xx}v_{xx}^2 dx\\
I_{12} &= -\int_{\mathbb{T}} 6s^2\psi_x^2 4s\psi_x v_{xx}v_{xxx}dx = \int_{\mathbb{T}}[12s^3\psi_x^3]_x v_{xx}^2 dx\\
I_{13} &= -\int_{\mathbb{T}}(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})v v_{xxxx}dx = \int_{\mathbb{T}}[6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_x v v_{xxx}dx\\
&+ \int_{\mathbb{T}}(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})v_x v_{xxx}dx\\
&= -\int_{\mathbb{T}}[6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_{xx} v v_{xx}dx - \int_{\mathbb{T}}[6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_x v_x v_{xx}dx\\
&- \int_{\mathbb{T}}[6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_x v_x v_{xx}dx - \int_{\mathbb{T}}(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})v_{xx}^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{T}}[6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_{xxx} v v_x dx + \int_{\mathbb{T}}[6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_{xx} v_x^2 dx \\
&+ \int_{\mathbb{T}}[6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_{xx} v_x^2 dx - \int_{\mathbb{T}}(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}) v_{xx}^2 dx \\
&= -\int_{\mathbb{T}} \frac{1}{2}[6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_{xxx} v^2 dx + \int_{\mathbb{T}} 2[6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_{xx} v_x^2 dx \\
&- \int_{\mathbb{T}}(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}) v_{xx}^2 dx \\
I_{14} &= -\int_{\mathbb{T}}(4s^3\psi_x^3 + 4s\psi_{xxx}) v_x v_{xxxx} dx = \int_{\mathbb{T}}[(4s^3\psi_x^3 + 4s\psi_{xxx})]_x v_x v_{xxx} dx \\
&+ \int_{\mathbb{T}}(4s^3\psi_x^3 + 4s\psi_{xxx}) v_{xx} v_{xxx} dx \\
&= -\int_{\mathbb{T}}[(4s^3\psi_x^3 + 4s\psi_{xxx})]_{xx} v_x v_{xx} dx - \int_{\mathbb{T}}[(4s^3\psi_x^3 + 4s\psi_{xxx})]_x v_{xx}^2 dx \\
&- \int_{\mathbb{T}}[(2s^3\psi_x^3 + 2s\psi_{xxx})]_x v_{xx}^2 dx \\
&= \int_{\mathbb{T}}[(2s^3\psi_x^3 + 2s\psi_{xxx})]_{xxx} v_x^2 dx - \int_{\mathbb{T}} 3[(2s^3\psi_x^3 + 2s\psi_{xxx})]_x v_{xx}^2 dx \\
I_{15} &= -\int_{\mathbb{T}} 6s\psi_{xx} v_{xx} v_{xxxx} dx = \int_{\mathbb{T}} 6s\psi_{xxx} v_{xx} v_{xxx} dx + \int_{\mathbb{T}} 6s\psi_{xx} v_{xxx}^2 dx \\
&= -\int_{\mathbb{T}} 3s\psi_{xxxx} v_{xx}^2 dx + \int_{\mathbb{T}} 6s\psi_{xx} v_{xxx}^2 dx \\
I_{16} &= -\int_{\mathbb{T}} 4s\psi_x v_{xxx} v_{xxxx} dx = \int_{\mathbb{T}} 2s\psi_{xx} v_{xxx}^2 dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|e^{s\psi}Pu\|^2 = \|P_p v\|^2 + \|P_n v\|^2 + 2\int_{\mathbb{T}} h_1(\psi) v^2 dx + 2\int_{\mathbb{T}} h_2(\psi) v_x^2 dx + \\
2\int_{\mathbb{T}} h_3(\psi) v_{xx}^2 dx + 2\int_{\mathbb{T}} h_4(\psi) v_{xxx}^2 dx
\end{aligned}
\tag{4.17}
$$

where

$$
\begin{aligned}
h_1(\psi) &= [(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)(2s^3\psi_x^3 + 2s\psi_{xxx})]_x \\
&- (s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}) \\
&- [(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)(3s\psi_{xx})]_{xx} \\
&+ [(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)2s\psi_x]_{xxx} \\
&+ [12s^2\psi_x\psi_{xx}(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_x - [3s^2\psi_x^2(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_{xx} \\
&- \frac{1}{2}[6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_{xxx} \\
h_2(\psi) &= [(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)(6s\psi_{xx})] - [(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)6s\psi_x]_x \\
&- 12s^2\psi_x\psi_{xx}(4s^3\psi_x^3 + 4s\psi_{xxx}) + [36s^3\psi_x\psi_{xx}^2]_x - [24s^3\psi_x^2\psi_{xx}]_{xx} \\
&+ [3s^2\psi_x^2(4s^3\psi_x^3 + 4s\psi_{xxx})]_x \\
&+ 2[6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_{xx} + [(2s^3\psi_x^3 + 2s\psi_{xxx})]_{xxx} + 6s^2\psi_x^2(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}) \\
h_3(\psi) &= 48s^3\psi_x^2\psi_{xx} - 36s^3\psi_x^2\psi_{xx} + [12s^3\psi_x^3]_x - (6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}) \\
&- 3[(2s^3\psi_x^3 + 2s\psi_{xxx})]_x - 3s\psi_{xxxx} \\
h_4(\psi) &= 6s\psi_{xx} + 2s\psi_{xx} = 8s\psi_{xx}.
\end{aligned}
$$

From (4.8), we infer that there exist some numbers $s_0 \geq 1$, $K > 0$ and $K_1 > 0$ such that for all $s \geq s_0$,

$$\begin{aligned}
2h_1(\psi) \geq Ks^7 \quad &\text{for} \quad (x,t) \in (\frac{\eta}{2}, 2\pi - \frac{\eta}{2}) \times (0,T), \\
2h_2(\psi) \geq Ks^5 \quad &\text{for} \quad (x,t) \in (\frac{\eta}{2}, 2\pi - \frac{\eta}{2}) \times (0,T), \\
2h_3(\psi) \geq Ks^3 \quad &\text{for} \quad (x,t) \in (\frac{\eta}{2}, 2\pi - \frac{\eta}{2}) \times (0,T), \\
2h_4(\psi) \geq Ks \quad &\text{for} \quad (x,t) \in (\frac{\eta}{2}, 2\pi - \frac{\eta}{2}) \times (0,T),
\end{aligned}$$

while, setting $\omega_0 = [0, \frac{\eta}{2}) \cup (2\pi - \frac{\eta}{2}, 2\pi)$,

$$\begin{aligned}
|2h_1(\psi)| \leq K_1 s^7 \quad &\text{for} \quad (x,t) \in \omega_0 \times (0,T), \\
|2h_2(\psi)| \leq K_1 s^5 \quad &\text{for} \quad (x,t) \in \omega_0 \times (0,T), \\
|2h_3(\psi)| \leq K_1 s^3 \quad &\text{for} \quad (x,t) \in \omega_0 \times (0,T), \\
|2h_4(\psi)| \leq K_1 s \quad &\text{for} \quad (x,t) \in \omega_0 \times (0,T).
\end{aligned}$$

Then, from (4.17)

$$\begin{aligned}
&\|P_p v\|^2 + \int_{\mathbb{T}} [s^7|v|^2 + s^5|v_x|^2 + s^3|v_{xx}|^2 + s|v_{xxx}|^2] dx \\
&= \|P_p v\|^2 + \int_{\omega_0} [s^7|v|^2 + s^5|v_x|^2 + s^3|v_{xx}|^2 + s|v_{xxx}|^2] dx \\
&+ \int_{\mathbb{T}\setminus\omega_0} [s^7|v|^2 + s^5|v_x|^2 + s^3|v_{xx}|^2 + s|v_{xxx}|^2] dx \\
&\leq \|P_p v\|^2 + \int_{\omega_0} [s^7|v|^2 + s^5|v_x|^2 + s^3|v_{xx}|^2 + s|v_{xxx}|^2] dx \\
&+ C \int_{\mathbb{T}\setminus\omega_0} [2h_1(\psi)|v|^2 + 2h_2(\psi)|v_x|^2 + 2h_3(\psi)|v_{xx}|^2 + 2h_4(\psi)|v_{xxx}|^2] dx \\
&\leq \|P_p v\|^2 + \int_{\omega_0} [s^7|v|^2 + s^5|v_x|^2 + s^3|v_{xx}|^2 + s|v_{xxx}|^2] dx \\
&+ C \int_{\mathbb{T}} [2h_1(\psi)|v|^2 + 2h_2(\psi)|v_x|^2 + 2h_3(\psi)|v_{xx}|^2 + 2h_4(\psi)|v_{xxx}|^2] dx \\
&\leq C \left(\|e^{s\psi} P u\|^2 + \int_{\omega_0} [s^7|v|^2 + s^5|v_x|^2 + s^3|v_{xx}|^2 + s|v_{xxx}|^2] dx \right).
\end{aligned}$$

We conclude that for $s \geq s_0$ and $C > 0$

$$\begin{aligned}
&\|P_p v\|^2 + \int_{\mathbb{T}} [s^7|v|^2 + s^5|v_x|^2 + s^3|v_{xx}|^2 + s|v_{xxx}|^2] dx \\
&\leq C \left(\|e^{s\psi} P u\|^2 + \int_{\omega_0} [s^7|v|^2 + s^5|v_x|^2 + s^3|v_{xx}|^2 + s|v_{xxx}|^2] dx \right).
\end{aligned} \tag{4.18}$$

Next we show that $\int_{\mathbb{T}} s^{-1}|v_{xxxx}|^2 dx$ is also less than the rigth hand side of (4.18). We have

$$\begin{aligned}
&\int_{\mathbb{T}} s^{-1}|v_{xxxx}|^2 dx = \int_{\mathbb{T}} s^{-1}|P_p v - (s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)v - 12s^2\psi_x\psi_{xx}v_x - 6s^2\psi_x^2 v_{xx}|^2 dx \\
&\leq C \int_{\mathbb{T}} s^{-1}(|P_p v|^2 + |s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x|^2|v|^2 + |12s^2\psi_x\psi_{xx}|^2|v_x|^2 + |6s^2\psi_x^2|^2|v_{xx}|^2) dx \\
&\leq C s^{-1} \left(\|P_p v\|^2 + \int_{\mathbb{T}} (s^8|v|^2 + s^4|v_x|^2 + s^4|v_{xx}|^2) \right) \\
&= C \left(s^{-1}\|P_p v\|^2 + \int_{\mathbb{T}} (s^7|v|^2 + s^3|v_x|^2 + s^3|v_{xx}|^2) \right).
\end{aligned}$$

Combined with (4.18), this gives

$$\begin{aligned}&\int_{\mathbb{T}}\{s^{-1}|v_{xxxx}|^2+s|v_{xxx}|^2+s^3|v_{xx}|^2+s^5|v_x|^2+s^7|v|^2\}dx\\
&\le C\left(s^{-1}\|P_p v\|^2+\int_{\mathbb{T}}(s^7|v|^2+s^3|v_x|^2+s^3|v_{xx}|^2)+\int_{\mathbb{T}}(s|v_{xxx}|^2+s^3|v_{xx}|^2+s^5|v_x|^2+s^7|v|^2)\right)\\
&\le C\left(s^{-1}\|P_p v\|^2+\int_{\mathbb{T}}(s|v_{xxx}|^2+s^3|v_{xx}|^2+s^5|v_x|^2+s^7|v|^2)\right),\end{aligned}$$

then we have

$$\begin{aligned}&\int_{\mathbb{T}}\{s^{-1}|v_{xxxx}|^2+s|v_{xxx}|^2+s^3|v_{xx}|^2+s^5|v_x|^2+s^7|v|^2\}dx\\
&\le C\left(s^{-1}\|P_p v\|^2+\int_{\mathbb{T}}(s|v_{xxx}|^2+s^3|v_{xx}|^2+s^5|v_x|^2+s^7|v|^2)\right)\end{aligned}\tag{4.19}$$

where C does not depend on s and v. Finally, we show that we can drop the terms v_x, v_{xxx} on the right hand side of (4.19). Let $\xi \in C_0^\infty(\omega)$ with $0 \le \xi \le 1$ for $x \in \omega_0$. Then,

$$\begin{aligned}\int_{\omega_0}|v_x|^2dx \le \int_\omega \xi|v_x|^2dx &= \int_\omega \xi v_x v_x dx\\
&= -\int_\omega(\xi_x v_x+\xi v_{xx})v dx\\
&= \frac{1}{2}\int_\omega \xi_{xx}v^2dx-\int_\omega \xi v_{xx}v dx\end{aligned}$$

so that

$$2\int_{\omega_0}s|v_x|^2dx \le \|\xi_{xx}\|_{L^\infty(\mathbb{T})}\int_\omega s|v|^2dx+\kappa\int_\omega s^{-1}|v_{xx}|^2dx+\int_\omega s^3|v|^2dx \tag{4.20}$$

and

$$\begin{aligned}\int_{\omega_0}|v_{xxx}|^2dx \le \int_\omega \xi|v_{xxx}|^2dx &= \int_\omega \xi v_{xxx}v_{xxx}dx\\
&= -\int_\omega(\xi_x v_{xxx}+\xi v_{xxxx})v_{xx}dx\\
&= \frac{1}{2}\int_\omega \xi_{xx}v_{xx}^2dx-\int_\omega \xi v_{xxxx}v_{xx}dx.\end{aligned}$$

Then,

$$2\int_{\omega_0}s|v_{xxx}|^2dx \le \|\xi_{xx}\|_{L^\infty(\mathbb{T})}\int_\omega s|v_{xx}|^2dx+\kappa\int_\omega s^{-1}|v_{xxxx}|^2dx+\kappa^{-1}\int_\omega s^3|v_{xx}|^2dx, \tag{4.21}$$

where $\kappa > 0$ is a constant that can be chosen as small as desired. Combining (4.20) and (4.21) with κ small enough gives for $s \ge s_0$(with a possibly incresed value of s_0) and some constant C that does not depend on s and v

$$\begin{aligned}&\int_\omega\{s^{-1}|v_{xxxx}|^2+s|v_{xxx}|^2+s^3|v_{xx}|^2+s^5|v_x|^2+s^7|v|^2\}dx \le\\
&\qquad C\left(\|e^{s\psi}Pu\|^2+\int_\omega(s^7|v|^2+s^3|v_{xx}|^2)dx\right).\end{aligned}\tag{4.22}$$

Replacing v by $e^{s\psi}u$ in (4.22) gives at once (4.14). The proof of Lemma 4.1.1 is complete. □

Step 2. Carleman estimate for the transport equation.

Lemma 4.1.2. *There exist $s_1 \geq s_0$ and $C_1 > 0$ such that for all $s \geq s_1$ and all $w \in L^2(\mathbb{T} \times (0,T))$ with $w_t + \frac{a}{b} w_x \in L^2(\mathbb{T} \times (0,T))$, the following holds*

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}} s|w|^2 e^{2s\varphi} dxdt + \int_{\mathbb{T}} s[|w|^2 e^{2s\varphi}]_{t=0} dx + \int_{\mathbb{T}} s[|w|^2 e^{2s\varphi}]_{t=T} dx \\ &\quad \leq C_1 \left(\int_0^T \int_{\mathbb{T}} |w_t + \frac{a}{b} w_x|^2 e^{2s\varphi} dxdt + \int_0^T \int_\omega s|w|^2 e^{2s\varphi} dxdt \right). \end{aligned} \tag{4.23}$$

Proof. The result was proved in [39, Lemma 5.5]. For the sake of completeness we have included the proof in Appendix 4.3. □

Let us complete the proof of Proposition 4.1.1. Let $u \in L^2(0,T;H^4(\mathbb{T}))$ satisfying (4.1) and let $w = u - b_1 u_{xx} + b u_{xxxx} \in L^2(0,T;L^2(\mathbb{T}))$.
Then $w_t + \frac{a}{b} w_x = (\frac{a}{b} - q)u_x - (\frac{ab_1}{b} + p)u_{xxx} - r u_{xx} \in L^2(0,T;L^2(\mathbb{T}))$. Combining (4.4),(4.5),(4.14) (multiplied by $e^{-2s\rho c^2 t^2}$ and next integrated over $(0,T)$), and (4.23), we obtain for $s \geq s_1$ that

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}} [s|u_{xxx}|^2 + s^3|u_{xx}|^2 + s^5|u_x|^2 + s^7|u|^2 + s|u - b_1 u_{xx} + b u_{xxxx}|^2] e^{2s\varphi} dxdt \\ &+ \int_{\mathbb{T}} [s|u - b_1 u_{xx} + b u_{xxxx}|^2 e^{2s\varphi}]_{t=0} dx \\ &\leq C \int_0^T \int_{\mathbb{T}} [|u_{xxxx}|^2 + |(\frac{a}{b} - q)u_x - (\frac{ab_1}{b} + p)u_{xxx} - r u_{xx}|^2] e^{2s\varphi} dxdt \\ &+ C \int_0^T \int_\omega [s|u - b_1 u_{xx} + b u_{xxxx}|^2 + s^7|u|^2 + s^3|u_{xx}|^2]] e^{2s\varphi} dxdt. \end{aligned} \tag{4.24}$$

Then choosing $s_2 \geq s_1$ and $C_2 > C$ large enough, we obtain (4.12) for any $s \geq s_2$ and any $u \in L^2(0,T;H^4(\mathbb{T}))$ satisfying (4.1). □

We are now in a position to prove Theorem 4.1.1. Pick any function fulfilling (4.1) and (4.2). If $u \in L^2(0,T;H^4(\mathbb{T}))$, then it follows from (4.12) that $u = 0$ in $\mathbb{T} \times (0,T)$. Assume now that $u \in L^\infty(0,T;H^3(\mathbb{T}))$. We proceed as in [39]. Since u and $w = u - b_1 u_{xx} + b u_{xxxx}$ are not regulat enough to apply Lemmas 4.1.1 and 4.1.2, we smooth them by some convolution in time. For any function $v = v(x,t)$ and any number $h > 0$, we set

$$v^{[h]}(x,t) = \frac{1}{h} \int_t^{t+h} v(x,s) ds.$$

Recall that if $v \in L^p(0,T,X)$, where $1 \leq p \leq \infty$ and X denote any Banach space, then $v^{[h]} \in W^{1,p}(0,T-h;X)$, $\|v^{[h]}\|_{L^p(0,T-h;X)} \leq \|v\|_{L^p(0,T;X)}$ and, for $p < \infty$ and $T' < T$,

$$v^{[h]} \to v \quad \text{in } L^p(0,T';X) \quad \text{as} \quad h \to 0.$$

In the sequel, $v_t^{[h]}$ denote $(v^{[h]})_t$, $v_x^{[h]}$ denote $(v^{[h]})_x$, etc. Assume again that $a > 0$. Pick any $T' \in (\frac{2b\pi}{a}, T)$, any pair (ρ,δ) such that (4.7) still holds with T', and define the functions ψ

and φ as above. Then, for any positive number $h < h_0 = T - T'$, $u^{[h]} \in W^{1,\infty}(0, T', H^3(\mathbb{T}))$ and it solves

$$u_t^{[h]} - b_1 u_{txx}^{[h]} + b u_{txxxx}^{[h]} + a u_{xxxxx}^{[h]} + (q(u)u_x)^{[h]} + (p(u)u_{xxx})^{[h]} + (r(u)u_{xx})^{[h]} \tag{4.25}$$
$$= 0 \text{ in } L^\infty(0, T'; H^{-2}(\mathbb{T}))$$
$$u^{[h]}(x,t) = 0 \quad \text{in} \quad (x,t) \in \omega \times (0, T'). \tag{4.26}$$

From (4.25), we infer that

$$u_{xxxxx}^{[h]}$$
$$= a^{-1}(-u_t^{[h]} + b_1 u_{txx}^{[h]} - b u_{txxxx}^{[h]} - (q(u)u_x)^{[h]} - (p(u)u_{xxx})^{[h]} - (r(u)u_{xx})^{[h]}) \in L^\infty(0, T'; H^{-1}(\mathbb{T})),$$

hence

$$u^{[h]} \in L^\infty(0, T'; H^4(\mathbb{T})). \tag{4.27}$$

This yields, with (4.4)-(4.5),

$$w^{[h]} = u^{[h]} - b_1 u_{xx}^{[h]} + b u_{xxxx}^{[h]} \in L^\infty(0, T'; L^2(\mathbb{T})) \tag{4.28}$$
$$w_t^{[h]} + \frac{a}{b} w_x^{[h]} = [(\frac{a}{b} - q)u_x]^{[h]} - [(\frac{ab_1}{b} + p)u_{xxx}]^{[h]} - [ru_{xx}]^{[h]} \in L^\infty(0, T'; L^2(\mathbb{T})). \tag{4.29}$$

From (4.25)-(4.29) and Lemma 4.1.1 and 4.1.2, we infer that exist some constants $s_1 > 0$ and $C_1 > 0$, such that, for all $s \geq s_1$ and all $h \in (0, h_0)$, we have

$$\int_0^{T'} \int_{\mathbb{T}} [s|u_{xxx}^{[h]}|^2 + s^3|u_{xx}^{[h]}|^2 + s^5|u_x^{[h]}|^2 + s^7|u^{[h]}|^2 + s|u_{xxxx}^{[h]}|^2] e^{2s\varphi} dx dt \tag{4.30}$$
$$\leq C \int_0^{T'} \int_{\mathbb{T}} [|u_{xxxx}^{[h]}|^2 + \left| [(\frac{a}{b} - q)u_x]^{[h]} - [(\frac{ab_1}{b} + p)u_{xxx}]^{[h]} - [ru_{xx}]^{[h]} \right|^2] e^{2s\varphi} dx dt$$
$$\leq C \int_0^{T'} \int_{\mathbb{T}} [|u_{xxxx}^{[h]}|^2 + \left| [(\frac{a}{b} - q)u_x]^{[h]} \right|^2 + \left| [(\frac{ab_1}{b} + p)u_{xxx}]^{[h]} \right|^2 + \left| [ru_{xx}]^{[h]} \right|^2] e^{2s\varphi} dx dt$$
$$\leq C \int_0^{T'} \int_{\mathbb{T}} [|u_{xxxx}^{[h]}|^2 + \left| [(\frac{a}{b} - q)u_x^{[h]} \right|^2 + \left| [(\frac{ab_1}{b} + p)u_{xxx}^{[h]} \right|^2 + \left| [ru_{xx}^{[h]} \right|^2] e^{2s\varphi} dx dt$$
$$+ \int_0^{T'} \int_{\mathbb{T}} \left| [(\frac{a}{b} - q)u_x]^{[h]} - (\frac{a}{b} - q)u_x^{[h]} \right|^2 e^{2s\varphi} dx dt$$
$$+ \int_0^{T'} \int_{\mathbb{T}} \left| [(\frac{ab_1}{b} + p)u_{xxx}]^{[h]} - (\frac{ab_1}{b} + p)u_{xxx}^{[h]} \right|^2 e^{2s\varphi} dx dt \tag{4.31}$$
$$+ \int_0^{T'} \int_{\mathbb{T}} \left| [ru_{xx}]^{[h]} - ru_{xx}^{[h]} \right|^2 e^{2s\varphi} dx dt.$$

Comparing the powers of s in (4.30), we obtain that for $s \geq s_3 > s_1, h \in (0, h_0)$ and some constant $C_3 > C_1$(That does depend of s, h)

$$\begin{aligned}
&\int_0^{T'}\int_{\mathbb{T}}[s|u^{[h]}_{xxx}|^2 + s^3|u^{[h]}_{xx}|^2 + s^5|u^{[h]}_{x}|^2 + s^7|u^{[h]}|^2 + s|u^{[h]}_{xxxx}|^2]e^{2s\varphi}dxdt\\
&\le C\int_0^{T'}\int_{\mathbb{T}}|[(\frac{a}{b}-q)u_x]^{[h]} - (\frac{a}{b}-q)u^{[h]}_x|^2 e^{2s\varphi}dxdt\\
&+C\int_0^{T'}\int_{\mathbb{T}}|[(\frac{ab_1}{b}+p)u_{xxx}]^{[h]} - (\frac{ab_1}{b}+p)u^{[h]}_{xxx}|^2 e^{2s\varphi}dxdt\\
&+C\int_0^{T'}\int_{\mathbb{T}}|[ru_{xx}]^{[h]} - ru^{[h]}_{xx}|^2 e^{2s\varphi}dxdt.
\end{aligned}$$

Fix s to value s_3 and let $h \to 0$, we claim that

$$\int_0^{T'}\int_{\mathbb{T}}|[(\frac{a}{b}-q)u_x]^{[h]} - (\frac{a}{b}-q)u^{[h]}_x|^2 e^{2s_3\varphi}dxdt \to 0, \text{ as } h \to 0.$$

$$\int_0^{T'}\int_{\mathbb{T}}|[(\frac{ab_1}{b}+p)u_{xxx}]^{[h]} - (\frac{ab_1}{b}+p)u^{[h]}_{xxx}|^2 e^{2s_3\varphi}dxdt \to 0, \text{ as } h \to 0.$$

and

$$\int_0^{T'}\int_{\mathbb{T}}|[ru_{xx}]^{[h]} - ru^{[h]}_{xx}|^2 e^{2s_3\varphi}dxdt \to 0, \text{ as } h \to 0.$$

Indeed, if $h \to 0$,

$$\begin{aligned}
&[(\frac{a}{b}-q)u_x]^{[h]} \to (\frac{a}{b}-q)u_x \quad \text{in} \quad L^2(0,T';L^2(\mathbb{T}))\\
&(\frac{a}{b}-q)u^{[h]}_x \to (\frac{a}{b}-q)u_x \quad \text{in} \quad L^2(0,T';L^2(\mathbb{T}))\\
&[(\frac{ab_1}{b}+p)u_{xxx}]^{[h]} \to (\frac{ab_1}{b}+p)u_{xxx} \quad \text{in} \quad L^2(0,T';L^2(\mathbb{T}))\\
&(\frac{ab_1}{b}+p)u^{[h]}_{xxx} \to (\frac{ab_1}{b}+p)u_{xxx} \quad \text{in} \quad L^2(0,T';L^2(\mathbb{T}))\\
&[ru_{xx}]^{[h]} \to ru_{xx} \quad \text{in} \quad L^2(0,T';L^2(\mathbb{T}))\\
&ru^{[h]}_{xx} \to ru_{xx} \quad \text{in} \quad L^2(0,T';L^2(\mathbb{T}))
\end{aligned}$$

while $e^{2s_3\varphi} \in L^\infty(\mathbb{T}\times(0,T'))$. Therefore,

$$\int_0^{T'}\int_{\mathbb{T}}|u^{[h]}|^2 e^{2s_3\varphi}dxdt \to 0, \quad \text{as} \quad h \to 0.$$

On the other hand, $u^{[h]} \to u$ in $L^2(0,T';L^2(\mathbb{T}))$, hence

$$\int_0^{T'}\int_{\mathbb{T}}|u^{[h]}|^2 e^{2s_3\varphi}dxdt \to \int_0^{T'}\int_{\mathbb{T}}|u|^2 e^{2s_3\varphi}dxdt,$$

as $h \to 0$. We conclude that $u = 0$ in $\mathbb{T}\times(0,T')$. As T' may be taken arbitrarily close to T, we infer that $u = 0$ in $\mathbb{T}\times(0,T)$ as desired. The proof of Theorem 4.1.1 is complete. □

4.2 Appendix-Unique Continuation Property Conjecture

Having the UCP in hands, it is natural to expect that some stability property could be derived by incorporating some dissipation in a fixed subset of the domain. The conclusion is that the solutions indeed decay to zero in the energy space $H^2(\mathbb{T})$, as $t \to \infty$, provided that the following conjecture is true:

Unique Continuation Property Conjecture: For any $u_0 \in H^2(\mathbb{T})$, if the solution $u = u(x,t)$ of

$$\begin{cases} u_t + u_x - b_1 u_{txx} + a_1 u_{xxx} + b u_{txxxx} + a u_{xxxxx} + \frac{3}{2} u u_x + \gamma (u^2)_{xxx} - \frac{7}{48}(u_x^2)_x - \frac{1}{8}(u^3)_x = 0, \\ u(x,0) = u_0(x), \end{cases}$$

with $x \in \mathbb{T}$, satisfies

$$u(x,t) = 0, \ \forall\, (x,t) \in \omega \times (0,T),$$

for some nonempty open set $\omega \subset \mathbb{T}$ and some $T > 0$, then $u_0 = 0$ (and hence $u \equiv 0$).

We are concerned with the stabilization of

$$\begin{cases} u_t + u_x - b_1 u_{txx} + a_1 u_{xxx} + b u_{txxxx} + a u_{xxxxx} \\ \qquad = -\dfrac{3}{2} u u_x - \gamma (u^2)_{xxx} + \dfrac{7}{48}(u_x^2)_x + \dfrac{1}{8}(u^3)_x + a(x) h, \\ u(x,0) = u_0(x), \end{cases} \tag{4.32}$$

where $(x,t) \in \mathbb{T} \times (0,T)$ and $a \in C^\infty(\mathbb{T})$ is a given nonzero function. Let

$$\omega = \{x \in \mathbb{T} : a(x) \neq 0\} \neq \emptyset.$$

To guess the expression of h, it is convenient to write the linearized system of (4.32) as

$$u_t = Au + Bk, \tag{4.33}$$

$$u(0) = u_0, \tag{4.34}$$

where $A = -(I - b_1\partial_x^2 + b\partial_x^4)^{-1}(\partial_x + a_1\partial_x^3 + a\partial_x^5)$, $k(t) = (I - b_1\partial_x^2 + b\partial_x^4)^{-1} h(t) \in L^2(0,T; H^s(\mathbb{T}))$ is a control input and

$$B = (I - b_1\partial_x^2 + b\partial_x^4)^{-1} a (I - b_1\partial_x^2 + b\partial_x^4).$$

We already noticed that A is skew adjoint in $H^s(\mathbb{T})$, and that (4.33)-(4.34) is exactly controllable in $H^s(\mathbb{T})$ (see [6]). If we choose the simple feedback law

$$k = -B^{*,s} u \tag{4.35}$$

the resulting closed-loop system

$$u_t = Au - BB^{*,s} u, \tag{4.36}$$

$$u(0) = u_0 \tag{4.37}$$

is exponentially stable in $H^s(\mathbb{T})$, where $B^{*,s}$ denotes the adjoint of B in $\mathcal{L}(H^s(\mathbb{T}))$. $B^{*,s}$ is given by

$$B^{*,s} = (1 - b\partial_x^2 + b_1\partial_x^4)^{1-\frac{s}{2}} a(1 - b\partial_x^2 + b_1\partial_x^4)^{\frac{s}{2}-1}. \tag{4.38}$$

Indeed, observe that

$$c_{b,b_1}(1 + bx^2 + b_1x^4)^{\frac{s}{2}} \leq [(1 + x^2)^2]^{\frac{s}{2}} \leq C_{b,b_1}(1 + bx^2 + b_1x^4)^{\frac{s}{2}},$$

for $s \geq 2$ and some positive constants c_{b,b_1}, C_{b,b_1}. Then, we can define the following equivalent inner product in $H^s(\mathbb{T})$ as

$$(u, v)_s = \int_{\mathbb{T}} (1 + bx^2 + b_1x^4)^{\frac{s}{2}} \mathcal{F}u(x)\overline{\mathcal{F}v}(x)d,$$

where $\mathcal{F}\varphi$ denote the Fourier transform of φ. Hence, employing Plancherel Theorem, we get

$$\begin{aligned}
(B\varphi, \psi)_s &= \int_{\mathbb{T}} (1 + bx^2 + b_1x^4)^{\frac{s}{2}} \mathcal{F}[(1 - b\partial_x^2 + b_1\partial_x^4)^{-1} a(x)(1 - b\partial_x^2 + b_1\partial_x^4)\varphi(x)]\ \overline{\mathcal{F}\psi}(x)dx \\
&= \int_{\mathbb{T}} (1 + bx^2 + b_1x^4)^{\frac{s}{2}-1} \mathcal{F}[a(x)(1 - b\partial_x^2 + b_1\partial_x^4)\varphi(x)]\overline{\mathcal{F}\psi}(x)dx \\
&= \int_{\mathbb{T}} \mathcal{F}[a(x)(1 - b\partial_x^2 + b_1\partial_x^4)\varphi(x)]\overline{\mathcal{F}[(1 - b\partial_x^2 + b_1\partial_x^4)^{\frac{s}{2}-1}\psi(x)]}dx \\
&= (a(1 - b\partial_x^2 + b_1\partial_x^4)\varphi, (1 - b\partial_x^2 + b_1\partial_x^4)^{\frac{s}{2}-1}\psi)_{L^2(\mathbb{T})} \\
&= ((1 - b\partial_x^2 + b_1\partial_x^4)\varphi, a(1 - b\partial_x^2 + b_1\partial_x^4)^{\frac{s}{2}-1}\psi)_{L^2(\mathbb{T})} \\
&= \int_{\mathbb{T}} \mathcal{F}[(1 - b\partial_x^2 + b_1\partial_x^4)\varphi(x)]\overline{\mathcal{F}[a(x)(1 - b\partial_x^2 + b_1\partial_x^4)^{\frac{s}{2}-1}\psi(x)]}dx \\
&= \int_{\mathbb{T}} (1 + bx^2 + b_1x^4)^{\frac{s}{2}} \mathcal{F}\varphi(x)(1 + bx^2 + b_1x^4)^{1-\frac{s}{2}}\overline{\mathcal{F}[a(x)(1 - b\partial_x^2 + b_1\partial_x^4)^{\frac{s}{2}-1}\psi(x)]}dx \\
&= \int_{\mathbb{T}} (1 + bx^2 + b_1x^4)^{\frac{s}{2}} \mathcal{F}\varphi(x)\overline{\mathcal{F}[(1 - b\partial_x^2 + b_1\partial_x^4)^{1-\frac{s}{2}} a(x)(1 - b\partial_x^2 + b_1\partial_x^4)^{\frac{s}{2}-1}\psi(x)]}dx \\
&= (\varphi, (1 - b\partial_x^2 + b_1\partial_x^4)^{1-\frac{s}{2}} a(x)(1 - b\partial_x^2 + b_1\partial_x^4)^{\frac{s}{2}-1}\psi)_s,
\end{aligned}$$

for all $\varphi, \psi \in H^s(\mathbb{T})$. From the computation above we deduce that

$$B^{*,2}u = au.$$

Let $\tilde{A} = A - BB^{*,2}$, where $(BB^{*,2})u = (I - b_1\partial_x^2 + b\partial_x^4)[a(I - b_1\partial_x^2 + b\partial_x^4)(au)]$. Since $BB^{*,2} \in \mathcal{L}(H^s(\mathbb{T}))$ and A is skew adjoint in $H^s(\mathbb{T})$, $\tilde{A}$ is the infinitesimal generator of a group $\{W_a(t)\}_{t\in\mathbb{R}}$ on $H^s(\mathbb{T})$ (See [37, Theorem 3.4]).

We have the following exponentially stabilization result for (4.33)-(4.32) in $H^s(\mathbb{T})$ for $s \geq 2$ proved in [6].

Lemma 4.2.1. *Let $a \in C^\infty(\mathbb{T})$ with $a \neq 0$. Then, there exist a constant $\beta > 0$, such that, for $s \geq 2$, one can find constant $C_s > 0$ for which the following holds for all $u_0 \in H^s(\mathbb{T})$:*

$$\|W_a(t)u_0\|_{H^s} \leq C_s e^{-\beta t}\|u_0\|_{H^s} \quad \textit{for all} \quad t \geq 0. \tag{4.39}$$

Plugging the feedback law $k = -B^{*,2}u = -au$ in the nonlinear equation gives the following closed-loop system

$$\begin{cases} u_t + u_x - b_1 u_{txx} + a_1 u_{xxx} + b u_{txxxx} + a u_{xxxxx} = \\ -\dfrac{3}{2} u u_x - \gamma (u^2)_{xxx} + \dfrac{7}{48}(u_x^2)_x + \dfrac{1}{8}(u^3)_x - a(I - b_1\partial_x^2 + b\partial_x^4)[au] \\ u(x,0) = u_0(x), \end{cases} \tag{4.40}$$

where $(x,t) \in \mathbb{T} \times (0,T)$. Proceding as in [13], we can show that the system (4.40) is globally well-posed in the space $H^s(\mathbb{T})$ for $s \geq 1$.

Theorem 4.2.1. *Let $s \geq 2$ and $T > 0$ be given. For any $u_0 \in H^s(\mathbb{T})$, the system (4.40) admits a unique solution $u \in C([0,T]; H^s(\mathbb{T}))$.*

Now we show that the system (4.40) is globally exponentially stable in the space $H^2(\mathbb{T})$, but first we show a observability inequality.

Proposition 4.2.1. *Let $R_0 > 0$ be given. Then, there exist two positive number T and θ, such that for any $u_0 \in H^2(\mathbb{T})$ satisfying*

$$\|u_0\|_{H^2(\mathbb{T})} \leq R_0, \tag{4.41}$$

the corresponding solution u of (4.40) satisfies

$$\|u_0\|^2_{H^2(\mathbb{T})} \leq \theta \int_0^T \|au(t)\|^2_{H^2(\mathbb{T})} dt. \tag{4.42}$$

Proof. Let $T > \frac{2b\pi}{a}$. We prove the estimate (4.42) by contradiction. If (4.42) is not true, for any $n \geq 1$, (4.40) admits a solution $u_n \in C([0,T]; H^2(\mathbb{T}))$ satisfying

$$\|u_{0,n}\|_{H^2(\mathbb{T})} \leq R_0, \tag{4.43}$$

and

$$\int_0^T \|au_n(t)\|^2_{H^2(\mathbb{T})} dt \leq \frac{1}{n}\|u_{0,n}\|^2_{H^2(\mathbb{T})}, \tag{4.44}$$

where $u_{0,n} = u_n(0)$. Since $\alpha_n = \|u_{0,n}\|_{H^2(\mathbb{T})} \leq R_0$ we can choose a subsequence of (α_n), still denoted by $(\alpha_n)_{n\in\mathbb{N}}$, such that $\lim_{n\to\infty} \alpha_n = \alpha$. Note that $\alpha_n > 0$ by (4.44). Set $v_n = \frac{u_n}{\alpha_n}$ for all $n \geq 1$. Then,

$$\begin{aligned} v_{n,t} + v_{n,x} - b_1 v_{n,txx} + a_1 v_{n,xxx} + b v_{n,txxxx} + a v_{n,xxxxx} + \frac{3}{2}\alpha_n v_n v_{n,x} - \gamma \alpha_n (v_n^2)_{xxx} \\ - \frac{7}{48}\alpha_n (v_{n,x}^2)_x - \frac{1}{8}\alpha_n^2 (v_n^3)_x = -a(I - b_1\partial_x^2 + b\partial_x^4) a v_n \end{aligned}$$

and

$$\int_0^T \|av_n(t)\|^2_{H^2(\mathbb{T})} dt \leq \frac{1}{n}. \tag{4.45}$$

Moreover

$$\|v_n(0)\|_{H^2(\mathbb{T})} = 1. \tag{4.46}$$

Since $\|v_n(0)\|_{H^2(\mathbb{T})} = 1$, for all $n \in \mathbb{N}$, the sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H^2(\mathbb{T}))$ and $(v_{n,t})_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H^1(\mathbb{T}))$. From Aubin-Lions Lemma $(v_n)_{n\in\mathbb{N}}$ is bounded in $C(0,T;H^s(\mathbb{T}))$ with $1<s<2$, then we infer that we can extract a subsequence of $(v_n)_{n\in\mathbb{N}}$, still denoted $(v_n)_{n\in\mathbb{N}}$, such that

$$v_n \to v \quad \text{in} \quad C(0,T;H^s(\mathbb{T})), \tag{4.47}$$

$$v_n \to v \quad \text{in} \quad L^\infty(0,T;H^2(\mathbb{T})) \ \text{ weak-*}, \tag{4.48}$$

for some $v \in L^\infty(0,T;H^2(\mathbb{T})) \cup C([0,T];H^s(\mathbb{T}))$, for all $1<s<2$. Note that, from (4.47)-(4.48), we have that

$$\alpha_n v_n v_{n,x} \to \alpha v v_x \ \text{ in } \ L^\infty(0,T,L^2(\mathbb{T})) \ \text{ weak-*}, \tag{4.49}$$

$$\alpha_n (v_n^2)_{xxx} \to \alpha (v^2)_{xxx} \ \text{ in } \ L^\infty(0,T,H^{-1}(\mathbb{T})) \ \text{ weak-*}, \tag{4.50}$$

$$\alpha_n (v_{n,x}^2)_x \to \alpha (v_x^2)_x \ \text{ in } \ L^\infty(0,T,L^2(\mathbb{T})) \ \text{ weak-*}, \tag{4.51}$$

$$\alpha_n^2 (v_n^3)_x \to \alpha^2 (v^3)_x \ \text{ in } \ L^\infty(0,T,L^2(\mathbb{T})) \ \text{ weak-*}. \tag{4.52}$$

Furthermore, by (4.45),

$$\int_0^T \|av\|_{H^2(\mathbb{T})}^2 dt \le \liminf_{n\to\infty} \int_0^T \|av_n\|_{H^2(\mathbb{T})}^2 dt = 0. \tag{4.53}$$

Thus, v solves

$$\begin{aligned}
&v_t + v_x - b_1 v_{txx} + a_1 v_{xxx} + b v_{txxxx} + a v_{xxxxx} \\
&+ \frac{3}{2}\alpha v v_x - \gamma\alpha (v^2)_{xxx} - \frac{7}{48}(v_x^2)_x - \frac{1}{8}\alpha^2 (v^3)_x = -a(I - b_1\partial_x^2 + b\partial_x^4) av, \quad (x,t) \in \mathbb{T}\times(0,T), \\
&v = 0, \quad \text{in} \quad \omega\times(0,T).
\end{aligned}$$

If the UCP holds, $v=0$ in $\mathbb{T}\times(0,T)$. We claim that $(v_n)_{n\in\mathbb{N}}$ is linearizable in the sense of [15, Proposition 9]; that is, if $(w_n)_{n\in\mathbb{N}}$ denotes the sequence of solution of the linear higher order KdV-BBM equation with the same initial data

$$\begin{cases} w_{n,t} + w_{n,x} - b_1 w_{n.txx} + a_1 w_{xxx} + b w_{txxxx} + a w_{xxxxx} = -a(I - b_1\partial_x^2 + b\partial_x^4)[a w_n], \\ w_n(x,0) = v_n(x,0), \end{cases}$$

then

$$\sup_{0\le t\le T} \|v_n(t) - w_n(t)\|_{H^2(\mathbb{T})} \to 0, \qquad \text{as} \quad n\to\infty. \tag{4.54}$$

Indeed, if $d_n = v_n - w_n$, then d_n solves

$$\begin{aligned}
&d_{n,t} + d_{n,x} - b_1 d_{n,txx} + a_1 d_{n,xxx} + b d_{txxxx} + a d_{xxxxx} = \\
&-\frac{3}{2}\alpha_n v_n v_{n,x} - \gamma\alpha_n (v_n^2)_{xxx} + \frac{7}{48}\alpha_n (v_{n,x}^2)_x + \frac{1}{8}\alpha_n^2 (v_n^3)_x - a(I - b_1\partial_x^2 + b\partial_x^4)[a d_n], \quad (4.55) \\
&d_n(0) = 0.
\end{aligned}$$

Since $\|W_a(t)\|_{\mathcal{L}(H^2(\mathbb{T}))} \leq Me^{\omega t} \leq Me^{\omega T}$ with $\omega, M > 0$, we have from Duhamel formula that, for $t \in [0, T]$,

$$\begin{aligned}\|d_n(t)\|_{H^2(\mathbb{T})} \leq Me^{\omega T} \Bigg(& \int_0^T \|(I - b_1\partial_x^2 + b\partial_x^4)^{-1}\frac{3}{2}\alpha_n v_n v_{n,x}\|_{H^2(\mathbb{T})}dt \\ & + \int_0^T \|(I - b_1\partial_x^2 + b\partial_x^4)^{-1}\gamma\alpha_n(v_n^2)_{xxx}\|_{H^2(\mathbb{T})}dt \\ & + \int_0^T \|(I - b_1\partial_x^2 + b\partial_x^4)^{-1}\frac{7}{48}\alpha_n(v_{n,x}^2)_x\|_{H^2(\mathbb{T})}dt \\ & + \int_0^T \|(I - b_1\partial_x^2 + b\partial_x^4)^{-1}\frac{1}{8}\alpha_n(v_n^3)_x\|_{H^2(\mathbb{T})}dt \Bigg).\end{aligned}$$

The above estimate combined with (4.47)-(4.48) and the fact of $v = 0$ give us (4.54). By Lemma 4.2.1, we have that

$$\|w_n(t)\|_{H^2(\mathbb{T})} \leq c_1 e^{-\beta t}\|w_n(0)\|_{H^2(\mathbb{T})} \quad \text{for all} \quad t \geq 0. \tag{4.56}$$

From the energy identity for (4.55), we get

$$\|w_n(t)\|_{H^2(\mathbb{T})}^2 - \|w_n(0)\|_{H^2(\mathbb{T})}^2 = -2\int_0^T \|aw_n(t)\|_{H^2(\mathbb{T})}^2 dt$$

or

$$\|w_n(0)\|_{H^2(\mathbb{T})}^2 - \|w_n(t)\|_{H^2(\mathbb{T})}^2 = 2\int_0^T \|aw_n(t)\|_{H^2(\mathbb{T})}^2 dt.$$

Therefore, from (4.56) it follows that

$$\|w_n(0)\|_{H^2(\mathbb{T})}^2 \leq 2(1 - c_1^2 e^{-2\beta T})^{-1}\left[\int_0^T \|aw_n(t) - av_n(t)\|_{H^2(\mathbb{T})}^2 dt + \int_0^T \|av_n(t)\|_{H^2(\mathbb{T})}^2 dt\right].$$

Combining (4.54) and (4.45), this yields $\|v_n(0)\|_{H^2(\mathbb{T})}^2 = \|w_n(0)\|_{H^2(\mathbb{T})}^2 \to 0$, which contradicts (4.46). □

Theorem 4.2.2. *Let $a \in C^\infty(\mathbb{T})$ with $a \neq 0$, and $\beta > 0$ be as given in Lemma 4.2.1. Then, for any $R_0 > 0$, there exists a constant $C > 0$, such that, for any $u_0 \in H^3(\mathbb{T})$ with $\|u_0\|_{H^3(\mathbb{T})} \leq R_0$, the corresponding solution u of (4.40) satisfies*

$$\|u(\cdot, t)\|_{H^2(\mathbb{T})} \leq Ce^{-\beta t}\|u_0\|_{H^2(\mathbb{T})}. \tag{4.57}$$

Proof. From Proposition 4.2.1 and the energy identity

$$\|u(t)\|_{H^2(\mathbb{T})}^2 = \|u(0)\|_{H^2(\mathbb{T})}^2 - 2\int_0^t \|au(\tau)\|_{H^2(\mathbb{T})}^2 d\tau, \qquad t \geq 0,$$

we have

$$\|u(T)\|_{H^2(\mathbb{T})}^2 \leq (1 - 2\theta^{-1})\|u(0)\|_{H^2(\mathbb{T})}^2.$$

Thus,

$$\|u(kT)\|_{H^2(\mathbb{T})}^2 \leq (1 - 2\theta^{-1})^k\|u(0)\|_{H^2(\mathbb{T})}^2, \quad k \in \mathbb{N},$$

which gives by the semigroup property

$$\|u(t)\|_{H^2(\mathbb{T})}^2 \leq Ce^{-\kappa t}\|u(0)\|_{H^2(\mathbb{T})}^2, \quad \text{for all} \quad t \geq 0.$$

□

4.3 Appendix-A

Proof of Lemma 4.1.2: (See [39, Lemma 5.5])

Proof. We first assume that $w \in H^1(\mathbb{T}\times(0,T))$. Let $v = e^{s\varphi}$ and $P = \partial_t + \frac{a}{b}\partial_x$. Then

$$\begin{aligned} e^{s\varphi}Pw &= e^{s\varphi}P(e^{-s\varphi}v) \\ &= (-s\varphi_t v - \frac{a}{b}s\varphi_x v) + (v_t + \frac{a}{b}v_x) \\ &= P_n v + P_p v. \end{aligned}$$

It follows that

$$\|e^{s\varphi}Pw\|^2_{L^2(\mathbb{T}\times(0,T))} = \|P_p v\|^2_{L^2(\mathbb{T}\times(0,T))} + \|P_n v\|^2_{L^2(\mathbb{T}\times(0,T))} + 2(P_p v, P_n v)^2_{L^2(\mathbb{T}\times(0,T))} \tag{4.58}$$

After some integrations by parts in t and x in the last term in (4.58), we obtain

$$\begin{aligned} 2(P_p v, P_n v)^2_{L^2(\mathbb{T}\times(0,T))} = &\int_0^T\int_{\mathbb{T}} s(\varphi_{tt} + 2\frac{a}{b}\varphi_{xt} + \frac{a^2}{b^2}\varphi_{xx})v^2 dxdt \\ &- \int_{\mathbb{T}} s(\varphi_t + \frac{a}{b}\varphi_x)v^2|_0^T dx - - \int_0^T \frac{a}{b}s(\varphi_t + \frac{a}{b}\varphi_x)v^2|_0^{2\pi} dt \end{aligned} \tag{4.59}$$

Using (4.9)-(4.11) and the fact that $v(0,t) = v(2\pi,t)$, we notice that the last term in (4.59) is null. From (4.8)-(4.11), we infer that

$$\begin{aligned} &\varphi_{tt} + 2\frac{a}{b}\varphi tx + \frac{a^2}{b^2}\varphi_{xx} = 2(1-\rho)\frac{a^2}{b^2} > 0 \quad \text{for} \quad (x,t) \in (\frac{\eta}{2}, 2\pi - \frac{\eta}{2}) \times (0,T), \\ &-(\varphi_t + \frac{a}{b}\varphi_x) \ge 2\frac{a}{b}(\frac{aT\rho}{b} - 2\pi - \delta) > 0 \quad \text{for} \quad x \in (0,2\pi), t = T \\ &\varphi_t + \frac{a}{b}\varphi_x \ge 2\frac{a}{b}\delta > 0 \quad \text{for} \quad x \in (0,2\pi), t = 0. \end{aligned}$$

Thus

$$\int_0^T\int_{\mathbb{T}} s|v|^2 dxdt + \int_{\mathbb{T}} s(|v|^2_{t=0} + |v^2_{t=T})dx \le C\left(\int_0^T\int_{\mathbb{T}} |e^{s\varphi}Pw|^2 dxdt + \int_0^T\int_{\omega} s|v|^2 dxdt\right).$$

Which gives at once (4.23) by replacing v by $e^{s\varphi}w$. The proof of Lemma 4.1.2 is achieved when $w \in H^1(\mathbb{T}\times(0,T))$. We now claim that Lemma 4.1.2 is still true when w and f are in $L^2(0,T;L^2(\mathbb{T}))$. Indeed, in the case $w \in C([0,T];L^2(\mathbb{T}))$, and if (w_n^0) and f_n are two sequences in $H^1(\mathbb{T})$ and $L^2(0,T;H^1(\mathbb{T}))$, respectively, such that

$$\begin{aligned} &w_n^0 \to w(0) \quad \text{in} \quad L^2(\mathbb{T}) \\ &f^n \to f \quad \text{in} \quad L^2(0,T;L^2(\mathbb{T}), \end{aligned}$$

then the solution $w^n \in C([0,T];H^1(\mathbb{T}))$ of

$$\begin{aligned} w_t^n + \frac{a}{b}w_x^n + f^n, \\ w^n(0) = w_0^n \end{aligned}$$

satisfies $w^n \in H^1(\mathbb{T}\times(0,T))$ and $w^n \to w$ in $C([0,T];L^2(\mathbb{T}))$, so that we can apply (4.23) to w^n and next pass to the limit $n \to \infty$ in (4.23). The proof of Lemma 4.1.2 is complete.

□

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