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PÓS-GRADUCÃO EM MATEMÁTICA

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On Anosov Geodesic Flows

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Dissertation submitted to the Faculty of the Institute Mathematics at the Universidade Federal do Rio de Janeiro, as part of the necessary requirements for the obtaining a Doctor's degree.

Advisor: Sergio A. Romaña Ibarra

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Dissertação apresentada ao Instituto de Matemática da Universidade Federal do Rio de Janeiro, como parte do requisito necessário para a obtenção do grau de Doutor.

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Summary

We study the relationship between the dynamic properties of a geodesic flow $\varphi^t : SM \to SM$ and the rigidity of the geometry of the manifold M.

We study some conditions for two geodesic flows defined in compact Riemannian manifolds of the same dimension, admit a certain type of conjugacy. Such conditions are imposed on the sectional curvatures of the manifolds. For this purpose, we extend a result on conjugacy and rigidity of Romaña - Melo, they show that under some condition in the sectional curvatures of two compact Riemannian manifolds of the same dimension, if there is a certain special type of conjugacy (1-conjugacy) between the corresponding geodesic flows then the sectional curvatures are constants.

On the other hand, a result that relates Lyapunov exponents and rigidity is due to Clark Butler, he shows that if all Lyapunov exponents of a geodesic flow $\varphi^t : SM \to SM$ defined in a compact Riemannian manifold of negative curvature are constants along periodic orbits then the sectional curvature of M is a negative constant. We extend that result in the following two context. First, for non-compact manifold of finite volume with pinched negative curvature and some restriction on the values of Lyapunov exponents. Second, for compact surfaces, changing the negative curvature condition for the geodesic flow to be Anosov.

Keywords: Conjugacy, Lyapunov exponents, Rigidity and Anosov geodesic flow.

Resumo

Nós estudamos a relação entre as propriedades dinâmicas de um fluxo geodésico $\varphi^t : SM \to SM$ e a rigidez da geometria da variedade M.

Nós estudaremos algumas condições para dois fluxos geodésicos definidos em variedades Riemannianas compactas da mesma dimensão, admitam certo tipo de conjugação. Tais condições são impostas sobre as curvaturas seccionais das variedades. Especificamente, na interseção dos intervalos de curvatura seccional. Para isso, nós estendemos um resultado sobre conjugação e rigidez de Romaña-Melo, eles mostram que baixo alguma condição nas curvaturas seccionais de duas variedades compactas da mesma dimensão, se existe um certo tipo especial de conjugação (1-conjugação) entre os correspondentes fluxos geodésicos então as curvaturas seccionais das variedades são constantes.

Por outro lado, um resultado que relaciona exponentes de Lyapunov e rigidez é devido a Clark Butler, quem mostra que se todos os exponentes de Lyapunov de um fluxo geodésico $\varphi^t : SM \to SM$ definido em uma variedade Riemanniana compacta de curvatura negativa são constantes ao longo de orbitas periódicas, então a curvatura seccional de M é constante negativa. Nós estendemos esse resultado nos seguintes dois contextos. Primeiro, para variedades não compactas de volume finito com curvatura pinched negativa e alguma restrição nos valores dos exponentes de Lyapunov. E segundo, para superfícies compactas, trocando a condição de curvatura negativa pela condição do fluxo geodésico ser Anosov.

Palavras-Chaves: Conjugação, exponente de Lyapunov, Rigidez e Fluxo Geodésico Anosov.

Abstract

The density property of periodic orbits for an Anosov flow in a compact manifold is well known. We prove that this property is still valid for an Anosov geodesic flow on manifolds of finite volume (Theorem 1.1). This property will be used to know when two geodesic flows defined in manifolds of the same dimension admit certain types of conjugacy (See Chapter 4, Section 4.3). Finally, we prove, for the Anosov geodesic flow, a rigidity result on the sectional curvature under the condition that the Lyapunov exponents are constant along periodic orbits, which is an extension of the Butler result [Bu] in the following two contexts: in any n-dimensional manifold of finite volume (Theorem 1.3) and in the compact case in dimension 2 (Theorem 1.5).

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Chapter 1

Introduction

The geodesic flow plays a significant role in modern theories of both differential geometry and dynamical systems and, primarily on Riemannian manifolds, has been extensively studied (cf. [P] and [Kn] for a comprehensive introduction).

In several works some properties of an Anosov geodesic flow are highlighted. For example in [AK], for a compact Riemannian manifold negatively curved (M, g), we have that the geodesic flow $\varphi^t : SM \to SM$ is Anosov, which implies that the geodesic flows has special submanifolds W^{ss} and W^{uu} called stable and unstable manifolds, respectively. Using these submanifolds we have special properties of the geodesic flow as Shadowing, expansiveness, local structure product, and ergodicity.

In the non-compact case, this mean when the manifold M is a complete Riemannian manifold; the hyperbolicity of the geodesic flow is proved under restriction over negative pinched curvature K_M (this mean, $-c^2 \leq K_M \leq -\alpha^2 < 0$, for some $0 < \alpha < c$) (cf. [Kn]). In [E], Eberlein gives equivalent conditions for the geodesic flow to be Anosov. In this case, there is no reference where the basic properties of the stable and unstable are studied to obtain good properties of the periodic orbits. In the chapter 1, we study the existence of local stable and unstable manifolds for non-compact case and local product structure.

On the other hand, with the property of the hyperbolicity of a flow $\Psi^t : N \to N$, it is of interest to know the density of periodic orbits, the shadowing property and transitivity. In the compact case, when the flow is Anosov, we have that density of periodic orbits of Ψ^t on N is equivalent to transitivity of Ψ^t (See [FH]). In the non-compact case, if $\varphi^t : SM \to SM$ is an Anosov geodesic flow, it is not know the density of the periodic orbits in SM, although is results seen easy. In the chapter 3, we deal with the density of the periodic orbits for an Anosov geodesic flow on a manifold of $\mathrm{vol}(M) < \infty$. Thus, we prove

Theorem 1.1. *If $\varphi : SM \to SM$ is an Anosov geodesic flow and M has finite volume, then then $\overline{Per(\varphi)} = SM$.*

We hope that this kind of result can be used to understand the topological entropy of Anosov geodesic flow on manifold of finite volume.

In the chapter 4, we deal with a problem of rigidity and conjugacy between the geodesic flow of two manifolds. An interesting related problem in this context is knowing that types of manifolds admit a certain type of conjugacy between their geodesic flows under some conditions of their curvatures. In this chapter, we extended a result of Romaña and Melo [MR] and we use it to find results of rigidity of conjugacy (see [CLUV], for more details).

Moreover, we have the result of Feldman and Ornstein [FO] on the regularity of conjugacy of horocycle flows defined on surfaces of negative curvature. In the case of Anosov flows in a compact 3 dimensional manifold, we have the result due to De la LLave and Moriyón [DM]. They show that if two C^∞ transitive Anosov Flows in a 3-dimensional manifold are topologically conjugate and the Lyapunov exponents on corresponding periodic orbits agree, then the conjugacy homeomorphism is C^∞. Recently Gogolev and Rodriguez Hertz in [GF] introduce the matching functions technique in the setting of Anosov flows, that can be used to improve regularity of the conjugacy between conservative codimension one Anosov flows defined in manifolds with dimension ≥ 4. Specifically they prove that a continuous conjugacy must, in fact, be a C^1 diffeomorphism for an open and dense set of codimension one conservative Anosov flows. In this context, we prove the following result, which improved the result of [MR]

Theorem 1.2. *Let M and N be two compact Riemannian manifolds with the same dimension. Assume that the sectional curvature satisfies $\inf K_M \geq a^2 \sup K_N$ for some $a > 0$ and M has no conjugate points. If h is a 1-conjugacy in orbits between φ_M^t and φ_N^t with odd reparametrization $f(t)$ satisfying $f(t) \geq at$ for all $t \geq 0$, then $K_M \equiv a^2 \sup K_N \equiv a^2 K_N$.*

Other interesting problems related to the rigidity of Lyapunov exponents of a geodesic flow appear in [Bu], where Butler show that if all Lyapunov exponents of a geodesic flow $\varphi^t : SM \to SM$ defined on a compact negatively curved Riemannian manifold M are constant on periodic orbits, then the sectional curvature of M is negative constant. In chapter 5, we extended the result of Butler in two context, first for non-compact manifolds of finite volume, with pinched negative curvature and some restriction on the value of the Lyapunov exponents (see Theorem 1.3), and second for compact surfaces, change the condition of negative curvature by the condition of Anosov geodesic flow (see Theorem 1.5). More specifically, we prove the following two theorem:

Theorem 1.3. *Let (M, g) be a complete Riemannian manifold of finite volume and such that $-c^2 \leq K_M \leq -\alpha^2 < 0$. Let $\varphi^t : SM \to SM$ be the geodesic flow. Consider $b \in \{c, \alpha\}$ and assume that for all $\theta \in Per(\varphi^t)$ we have*

$$\chi^+(\theta, \xi) = \lim_{t \to +\infty} \frac{1}{t} \log \|d_\theta \varphi^t(\xi)\| = b,$$

for all $\xi \in E_\theta^u \backslash \{0\}$. Then $K = -b^2$.

Note that the condition $-c^2 \leq K_M \leq -\alpha^2$ implies that $\alpha \leq \chi^+(\theta, \xi) \leq c$ for all $\theta \in Per(\varphi^t)$ and $\xi \in E_\theta^u \backslash \{0\}$. So, the Theorem 1.3 claims that if $\chi^+(\theta, \xi)$ (unstable Lyapunov exponent) is equal to α or c(the endpoints of closed interval $[\alpha, c]$) then we have rigidity in the sectional curvature of M.

The case case $b = c$ has been prove by Melo - Romaña (cf. [MR, Corollary 3.6]). Our proof, to the case $b = \alpha$, used the Theorem 1.1 and one Mañé-Freire result (cf. [MF]).

In [MR], support by the Butler's result and [MR, Corollary 3.6] was conjectured that

Conjecture 1.4. *Let M be a complete Riemannian manifold with finite volume, whose geodesic flow is Anosov. If the unstable Lyapunov exponents are constants over all periodic orbits, then M has constant negative sectional curvature.*

The Theorem 1.3 shows the conjecture in the case of pinched negative curvature, when the unstable Lyapunov exponent is the minimal or maximal possible. The conjecture remains open for general case, even in compact manifolds. Thus, our last result in this work is to give a positive answer of this conjecture in the 2-dimensional compact case.

Theorem 1.5. [**Main Theorem**] *Let $\varphi^t : SM \to SM$ be the C^k-geodesic flow with $k \geq 5$ and M a compact surface. Suppose that φ^t is Anosov and for all $\theta \in Per(\varphi)$ holds*

$$\chi^+(\theta, \xi) = \lim_{t \to +\infty} \frac{1}{t} \log \|d_\theta \varphi^t(\xi)\| = \alpha,$$

for all $\xi \in E_\theta^u \backslash \{0\}$. Then $K = -\alpha^2$.

The idea behind of the proof of this theorem is to use a Kalinin's result (cf. [Kal]) to show that the Liouville measure is a measure of maximum entropy (MME), so find some rigidity on the curvature. In fact, we prove that the Theorem 1.5 is equivalent to having the Liouville measure as a MME in the 2-dimensional case (see Corollary 5.14).

As the Liouville measure is ergodic for Anosov geodesic flow defined on the unitary tangent bundle of a compact Riemannian surface (cf. [VO]), then using the Kalinin result (cf. [Kal]) and the hypothesis of the Theorem 1.5 (in any dimension) implies that the Liouville measure is a measure of maximal entropy (see Theorem 5.13). This property is linked with the *Katok Entroy Conjecture* (see Conjecture 5.9 and Section 5.2).

Chapter 2

Preliminaries

2.1 Anosov Geodesic Flow

In this section we will give some definitions of Anosov flows and results concerning the Anosov geodesic flow.

Definition 2.1. *Let N be a complete Riemannian manifold and $\psi^t : N \to N$ a flow of class C^r ($r \geq 1$). We say that the flow ψ^t is Anosov if there is a ψ^t-invariant continuous splitting of the tangent bundle of TN, given by $TN = E^s \oplus E^0 \oplus E^u$, where E^0 is the line bundle tangent to the flow ψ^t and E^s, E^u satisfy the following conditions: There are $C > 0$ and $0 < \lambda < 1$ such that for all $\theta \in N$:*

$$\|d_\theta \psi^t|_{E^s(\theta)}\| \leq C\lambda^t, \ \ \forall\, t \geq 0$$
$$\|d_\theta \psi^{-t}|_{E^u(\theta)}\| \leq C\lambda^t, \ \ \forall\, t \geq 0$$

E^s and E^u are called the stable and unstable subbundles of TN, respectively.

Let (M, g) be a complete Riemannian manifold. Denote by $SM = \{\theta = (p, v) : p \in M, v \in T_pM\}$ unit tangent bundle of M. For $\theta = (p, v) \in SM$. Let $\gamma_\theta(t)$ denote the unique geodesic with initial conditions $\gamma_\theta(0) = p$ and $\gamma'_\theta(0) = v$. For $t \in \mathbb{R}$, let $\varphi^t : SM \to SM$ be the diffeomorphism given by $\varphi^t(\theta) = (\gamma_\theta(t), \gamma'_\theta(t))$. Recall that this family is a flow, it is called the **Geodesic flow** in the sense that $\varphi^{t+s} = \varphi^t \circ \varphi^s$ for all $s, t \in \mathbb{R}$.

Let $V := \ker D\pi$ (where $\pi : TM \to M$ is the canonical projection) denote the **vertical subbundle** of TTM (tangent bundle of TM). Let $K : TTM \to TM$ be the Levi-Civita connection map of M. Recall K is definded as follow: let $\xi \in T_\theta TM$ and $z : (-\varepsilon, \varepsilon) \to TM$ be a curve adapted to ξ, that is, $z(0) = \theta$ and $z'(0) = \xi$, where $z(t) = (\alpha(t), Z(t))$, $\alpha(-\varepsilon, \varepsilon) \to M$, $\alpha = \pi \circ z$, and Z is a vector field along of α.

Define

$$K_\theta(\xi) = \nabla_{\alpha'} Z(t)|_{t=0}$$

Let $H := \ker K$ be the **horizontal sub-bundle**. For each θ, the maps $d_\theta\pi|_{H(\theta)} : H(\theta) \to T_pM$ and $K_\theta|_{V(\theta)} : V(\theta) \to T_pM$ are linear isomorphisms. Furthermore, $T_\theta TM = H(\theta) \oplus V(\theta)$ and the map $j_\theta : T_\theta TM \to T_pM \times T_pM$ given by

$$j_\theta(\xi) = (d_\theta\pi(\theta), K_\theta(\xi))$$

is a linear isomorphism.

From now on, whenever we write $\xi = (\xi_h, \xi_v)$ we mean that we identify ξ with $j_\theta(\xi)$, where $\xi_h = d_\theta\pi(\xi)$ and $\xi_v = K_\theta(\xi)$.

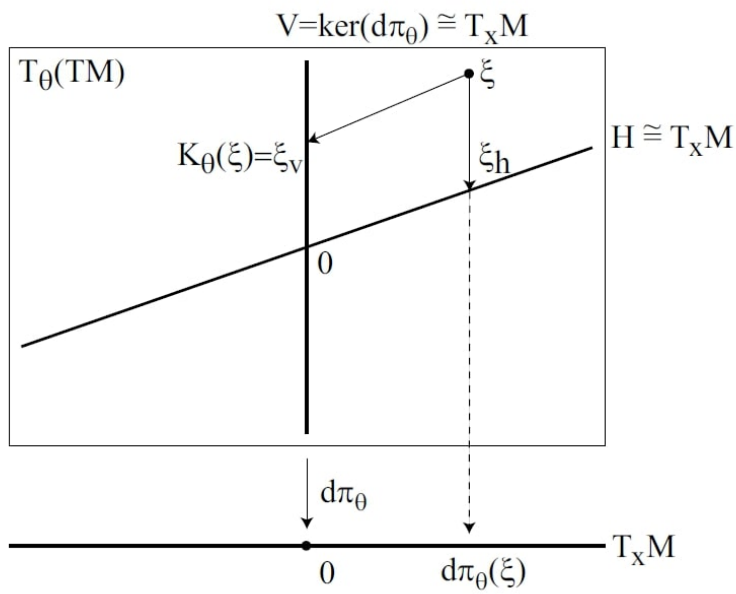

Figure 2.1: Decomposition on horizontal and vertical bundle.

Using the decomposition $T_\theta TM = H(\theta) \oplus V(\theta)$, we can define in a natural way a Riemannian metric on TM that makes $H(\theta)$ and $V(\theta)$ orthogonal subspaces. This metric is called the **Sasaki's metric** and is given by

$$g^S_\theta(\xi, \eta) = \langle d_\theta\pi(\xi), d_\theta\pi(\eta)\rangle + \langle K_\theta(\xi), K_\theta(\eta)\rangle$$

for all ξ and $\eta \in TTM$.

Using the identification j_θ, the geodesic vector field has a very simple expression. The geodesic vector field $G : TM \to TTM$ is given by:

$$G(\theta) := \frac{\partial}{\partial t}\bigg|_{t=0} \varphi^t(\theta) = \frac{\partial}{\partial t}\bigg|_{t=0} (\gamma_\theta(t), \gamma'_\theta(t))$$

where $\theta = (p, v)$ and γ_θ is, as usual, the unique geodesic with initial conditions $\gamma_\theta(0) = p$ and $\gamma'_\theta(0) = v$. Note that $t \to \gamma'_\theta(t)$ is the parallel transport of v along γ_θ. So, using the identification j_θ, we have $G(\theta) = (v, 0)$.

From now on, we consider the Sasaki's metric restricted to the unit tangent bundle SM. It is easy to proof that the geodesic flow preserves the volume measure generated by this Riemannian metric on SM. However, this volume measure on SM coincides with the Liouville measure $\mathcal{L}$ up to a constant. When M has finite volume, the Liouville measure is finite. (for more details see [P])

Consider the one-form β in TM define for $\theta = (p, v)$ by

$$\beta_\theta(\xi) = g^S_\theta(\xi, G(\theta)) = \langle d_\theta\pi(\xi), v\rangle_p.$$

Observed that $\ker \beta_\theta \supset V(\theta)$. It is possible prove that a vector $\xi \in T_\theta TM$ lies in $T_\theta SM$ with $\theta = (p, v)$ if and only if $\langle K_\theta(\xi), v\rangle = 0$. Furthermore, when restricted to SM the one-form β becomes a contact form invariant by the geodesic flow whose Reeb vector field is the geodesic vector field G. However, the sub-bundle $S = \ker \beta$ is the orthogonal complement of the subspace spanned by G. Since β is invariant by the geodesic flow, then the sub-bundle S is invariant by φ^t, that is, $\varphi^t(S(\theta)) = S(\varphi^t(\theta))$ for all $\theta \in SM$ and for all $t \in \mathbb{R}$.

To understand the behavior of $d_\theta\varphi^t$ let us introduce the definition of Jacobi field. A vector field J along of a geodesic γ_θ is called a **Jacobi field** if it satisfies the following equation:

$$J'' + R(\gamma'_\theta, J)\gamma'_\theta = 0$$

where R is the Riemannian curvature tensor of M and "$'$" denotes the covariant derivative along γ_θ. Note that, for $\xi = (w_1, w_2) \in T_\theta SM$ (in the horizontal and vertical decomposition), with $w_1, w_2 \in T_pM$ and $\langle v, w_2\rangle = 0$, it is known that $d_\theta\varphi^t(\xi) = (J_\xi(t), J'_\xi(t))$, where J_ξ denotes the unique Jacobi field along γ_θ such that $J_\xi(0) = w_1$ and $J'_\xi(0) = w_2$. (for more details see [P])

An important example historically and mathematically of a hyperbolic flow is the geodesic flow of a negatively curved manifold. Indeed, the concept of an Anosov flow arose as Anosov axiomatized the arguments used in working with geodesic flow on manifolds of negative sectional curvature. The motivation was that these are mechanical (in particular, physical) systems because this represents the motion of a free particle on the manifold. From that point of view, it is natural to think of geodesic flow as Hamiltonian flows for the Hamiltonian $H(x, v) = \frac{1}{2}g(v, v)$, which is the kinetic energy.

One result that relationed the hyperbolicity of the geodesic flow and the sectional curvature of the manifold is the following result. (See [Kn] and [E])

Theorem 2.2. [**Anosov**] *The geodesic flow on the unit tangent bundle SM of a complete Riemannian manifold M with negative pinched sectional curvature is an Anosov flow with respect to the Sasaki metric on SM.*

The **negative pinched** condition of the sectional curvature K_M means: $-c^2 \leq K_M \leq -\alpha^2 < 0$, for some constants $c \geq \alpha > 0$.

Example 2.3. *The pseudosphere is a surface of revolution generated by a curve called tractrix about its asypmtote. It has constant negative curvature and finite volume. By* Theorem 2.2 *we have that the geodesic flow on the unit tangent bundle is Anosov.*

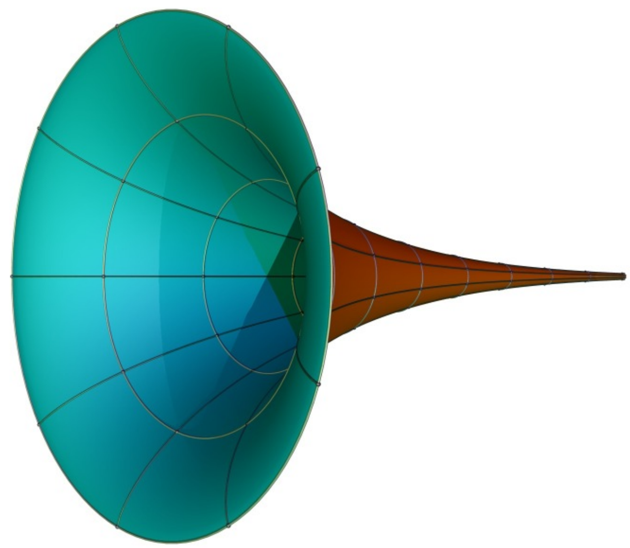

Figure 2.2: Pseudosphere.

2.2 No Conjugate Points and Riccati Equation

Suppose p and q are two points on a Riemannian manifold, we say that p and q are *conjugates* if there is a geodesic γ that connects p and q and a non-zero Jacobi field along γ that vanishes at p and q. When neither two points in M are conjugated, we say the manifold M *has no conjugate points*. Another important kind of manifolds are the manifold without focal points, we say that a manifold M *has no focal points*, if for any unit speed geodesic γ in M and for any Jacobi field J on γ such that $J(0) = 0$ and $J'(0) \neq 0$ we have $(\|J\|^2)'(t) > 0$, for any $t > 0$. It is clear that if a manifold has no focal points, then it has no conjugate points.

The more classical example of manifolds without focal points and therefore without conjugate points, are the manifolds of non-positive sectional curvature. It is possible to construct a manifold having positive curvature in somewhere, and without conjugate points. There are many examples of manifold without conjugate points. We emphasize here, for example, in [M] Mañé proved that, when the volume is finite and the geodesic flow is Anosov, then the manifold has no conjugate points. This latter had been proved by Klingenberg in the compact case (see [K]). In the case of infinite volume the result of Mañé was solved by Melo - Romaña in one pre-print paper [MR2] over the assumption of below bounded of sectional curvature.

Now suppose that M has no conjugate points and its sectional curvatures are bounded below by $-c^2$. In this case, if the geodesic flow $\varphi^t : SM \to SM$ is Anosov, then in [B] Bolton showed that there exists a positive constant δ such that for all $\theta \in SM$ the angle between E^s_θ and E^u_θ is greather than δ. Furthermore, if J is a perpendicular Jacobi vector field along γ_θ such that $J(0) = 0$ then exist $A > 0$ and $s_0 \in \mathbb{R}$ such that $\|J(t)\| \geq A\|J(s)\|$ for $t \geq s \geq s_0$. Therefore, for $\xi \in E^s_\theta$ and $\eta \in E^u_\theta$ since $\|J_\xi(t)\| \to 0$ as $t \to +\infty$ and $\|J_\eta(t)\| \to 0$ as $t \to -\infty$ follows that $J_\xi(0) \neq 0$ and $J_\eta(0) \neq 0$. In particular, $E^s_\theta \cap V(\theta) = \{0\}$ and $E^u_\theta \cap V(\theta) = \{0\}$ for all $\theta \in SM$.

For $\theta = (p, v) \in SM$, we denote by $N(\theta) := \{w \in T_pM : \langle w, v\rangle = 0\}$. Moreover, by the identification with the horizontal and vertical space, the horizontal space can be write as $H(\theta) = \{0\} \times N(\theta)$ and the vertical space as $V(\theta) = N(\theta) \times \{0\}$. Thus, if $E \subset S(\theta) := \ker\beta = N(\theta) \times N(\theta)$ is a subspace, $\dim E = n - 1$, and $E \cap (V(\theta) \cap S(\theta)) = \{0\}$ then $E \cap (H(\theta) \cap S(\theta))^\perp = \{0\}$. Hence, there exist a unique linear map $T : H(\theta) \cap S(\theta) \to V(\theta) \cap S(\theta)$ such that E is the graph of T. In other words, there exists a unique linear map $T : N(\theta) \to N(\theta)$ such that $E = \{(v, Tv) : v \in N(\theta)\}$. Furthermore, the linear map T is symmetric if and only if E is Lagrangian (see [P]).

It is known that if the geodesic flow is Anosov, then for each $\theta \in SM$, the subbundles E^u_θ and E^s_θ are Lagrangian (see [P] for the definition of Lagrangian subspace). Therefore, for each $t \in \mathbb{R}$, we can write $d_\theta\varphi^t(E^s_\theta) = E^s_{\varphi^t(\theta)} = Graph(U_s(t))$ and $d_\theta\varphi^t(E^u_\theta) = E^u_{\varphi^t(\theta)} = Graph(U_u(t))$, where $U_s(t) : N(\varphi^t(\theta)) \to N(\varphi^t(\theta))$ and $U_u(t) : N(\varphi^t(\theta)) \to N(\varphi^t(\theta))$ are symmetric maps.

Now we describe a usefull method of L. Green (see [L]) to see what properties the maps $U_s(t)$ and $U_u(t)$ satisfies.

Let γ_θ be a geodesic and consider $V_1, V_2, \ldots, V_n$ a system of parallel orthonormal vector fields along γ_θ with $V_n(t) = \gamma'_\theta(t)$. If $Z(t)$ is a perpendicular vector field along $\gamma_\theta(t)$, we can write

$$Z(t) = \sum_{i=1}^{n-1} y_i(t)V_i(t).$$

Note that the covariant derivative $Z'(s)$ is identified with the curve $\alpha'(s) = (y'_1(t), y'_2(t), \ldots, y'_{n-1}(t))$. Conversely, any curve in $\mathbb{R}^{n-1}$ (where $n = \dim(M)$) can be identified with a perpendicular vector field on $\gamma_\theta(t)$.

For each $t \in \mathbb{R}$, consider the symmetric matrix $R(t) = (R_{i,j}(t))$, where $1 \leq i, j \leq n-1$, $R_{i,j}(t) = \langle R(\gamma'_\theta(t), V_i(t))\gamma'_\theta(t), V_j(t))\rangle$ and R is the curvature tensor of M. Consider $(n-1) \times (n-1)$ matrix Jacobi equation

$$Y''(t) + R(t)Y(t) = 0 \tag{2.1}$$

If $Y(t)$ is a solution of (2.1) then for each $x \in \mathbb{R}^{n-1}$, the curve $B(t) = Y(t)x$ corresponds to a Jacobi perpendicular vector field on $\gamma_\theta(t)$. For $\theta \in SM$, $r \in \mathbb{R}$, we consider $Y_{\theta,r}(t)$ be the unique solution of (2.1) satisfying $Y_{\theta,r}(0) = I$ and $Y_{\theta,r}(r) = 0$. In [L], Green proved that $\lim_{r\to-\infty} Y_{\theta,r}(t)$ exists for all $\theta \in SM$ (see also [E], Sect. 2). Moreover, if we define:

$$Y_{\theta,u}(t) := \lim_{r\to-\infty} Y_{\theta,r}(t), \tag{2.2}$$

we obtain a solution of Jacobi equation (2.1) such that $\det Y_{\theta,u}(t) \neq 0$. Furthermore, it is proved in [L] (see also [MF] and [E]) that

$$\frac{DY_{\theta,u}}{dt}(t) = \lim_{r\to-\infty} \frac{DY_{\theta,r}}{dt}(t).$$

However, if

$$U_r(\theta) = \frac{DY_{\theta,r}}{dt}(0); \;\; U_u(\theta) = \frac{DY_{\theta,u}}{dt}(0)$$

then

$$U_u(\theta) = \lim_{r\to-\infty} U_r(\theta),$$

and follows that (see [MF])

$$U_u(\varphi^t(\theta)) = \frac{DY_{\theta,u}}{dt}(t)Y_{\theta,u}^{-1}(t)$$

for all $t \in \mathbb{R}$. Therefore, U_u is a symmetric solution of the matrix Riccati equation:

$$U'(t) + U^2(t) + R(t) = 0. \tag{2.3}$$

Analogously, taking the limit when $r \to +\infty$, we have defined $U_s(\theta)$, that also satisfies the Riccati equation (2.3). Furthermore, in [L], Green also showed that, in the case of curvature bounded below by $-c^2$, symmetric solutions of the Riccati equation which are definded for all $t \in \mathbb{R}$ are bounded by c, that means

$$\sup_t \|U_s(t)\| \leq c \;\; \textit{and} \;\; \sup_t \|U_u(t)\| \leq c. \tag{2.4}$$

2.3 Stable and Unstable Foliations

For compact manifold, Anosov flows have an important result called the Stable - Unstable Manifold Theorem, which guarantees that the condition of hyperbolicity implies the existence of certain submanifolds W^{ss} and W^{uu}, called Stable and Unstable Manifolds, whose tangent spaces are the subspaces E^s and E^u, respectively. They are smoothly immersed manifolds and these manifolds form foliations tangent to the stable and unstable subbundles.

2.3.1 Compact Case

Theorem 2.4. [**Stable and Unstable Manifold Theorem**] *Let $\varphi^t : M \to M$ be a Anosov C^r - flow (with $r \geq 1$), M compact and $t_0 > 0$. Then for each $x \in M$ there is a pair of embedded C^r disks W^s_{loc}, W^u_{loc}, depending continuosly on x in the C^1- topology and called the local strong-stable and the local strong-unstable manifold of x, respectively, such that*

(1) $T_xW^s_{loc}(x) = E^s_x$, $T_xW^u_{loc} = E^u_x$,

(2) $\varphi^t(W^s_{loc}(x)) \subset W^s_{loc}(\varphi^t(x))$ *and* $\varphi^{-t}(W^u_{loc}(x)) \subset W^u_{loc}(\varphi^{-t}(x))$ *for* $t \geq t_0$,

(3) for every $\delta > 0$ there exist $C(\delta)$ such that

$$
\begin{aligned}
d(\varphi^t(x), \varphi^t(y)) &< C(\delta)(\lambda+\delta)^t d(x.y) \;\; for \;\; y \in W^s_{loc}(x), \;\; t > 0 \\
d(\varphi^{-t}(x), \varphi^{-t}(y)) &< C(\delta)(\lambda^{-1}-\delta)^{-t} d(x.y) \;\; for \;\; y \in W^u_{loc}(x), \;\; t > 0,
\end{aligned}
$$

(4) there exists a continuous family U_x of neighborhoods of x such that

$$
\begin{aligned}
W^s_{loc}(x) &= \{y | \varphi^t(y) \in U_{\varphi^t(x)} \;\; for \;\; t > 0, \; \lim_{t\to+\infty} d(\varphi^t(x), \varphi^t(y)) = 0\}, \\
W^u_{loc}(x) &= \{y | \varphi^{-t}(y) \in U_{\varphi^{-t}(x)} \;\; for \;\; t > 0, \; \lim_{t\to+\infty} d(\varphi^{-t}(x), \varphi^{-t}(y)) = 0\}.
\end{aligned}
$$

Observation 2.5. *A proof of Theorem 2.4 is using the Hadamard-Perron Theorem (See [AK]) applied to the time t_0 map φ^{t_0} with $T_xM = E^s_x \oplus (E^0_x \oplus E^u_x)$ to obtains the existence of $W^s_{loc}(x)$ satisfying (1)-(4) for $t \in \mathbb{N}t_0 = \{nt_0 : n \in \mathbb{N}\}$. The same with $T_xM = (E^s_x \oplus E^0_x) \oplus E^u_x$ yields $W^u_{loc}(x)$ satisfying (1)-(4) with $t \in \mathbb{N}t_0$. Once (3) holds for $t \in \mathbb{N}t_0$ then holds for $t > 0$ by adjusting the constant $C(\delta)$ since $\{\varphi^t\}_{t\in[0,t_0]}$ is equicontinuous and M is compact (see* [AK]*).*

Remark 2.6. *With a little care one can replace the condition $t \geq t_0$ in (2) by $t > 0$. The sets*

$$
\begin{aligned}
W^{ss}(x) := \bigcup_{t>0} \varphi^{-t}(W^s_{loc}(\varphi^t(x))) &= \{y \in M : \lim_{t\to+\infty} d(\varphi^t(x), \varphi^t(y)) = 0\}, \\
W^{uu}(x) := \bigcup_{t>0} \varphi^{t}(W^u_{loc}(\varphi^{-t}(x))) &= \{y \in M : \lim_{t\to+\infty} d(\varphi^{-t}(x), \varphi^{-t}(y)) = 0\}
\end{aligned}
$$

are defined independently of a particular choice of local stable and unstable manifolds, and are smooth injectively immersed manifolds called the global ***strong-stable*** *and* ***strong-unstable*** *manifolds. The manifolds*

$$
W^{cs}(x) := \bigcup_{t\in\mathbb{R}} \varphi^t(W^{ss}(x)) \;\; and \;\; W^{cu}(x) := \bigcup_{t\in\mathbb{R}} \varphi^t(W^{uu}(x))
$$

are called ***center-stable*** *and* ***cente-unstable*** *manifolds (or* ***weak-stable*** *and* ***weak-unstable*** *manifolds) of x. Note that $T_xW^{cs} = E^s_x \oplus E^0_x$ and $T_xW^{cu} = E^0_x \oplus E^u_x$. (See Figure 2.3)*

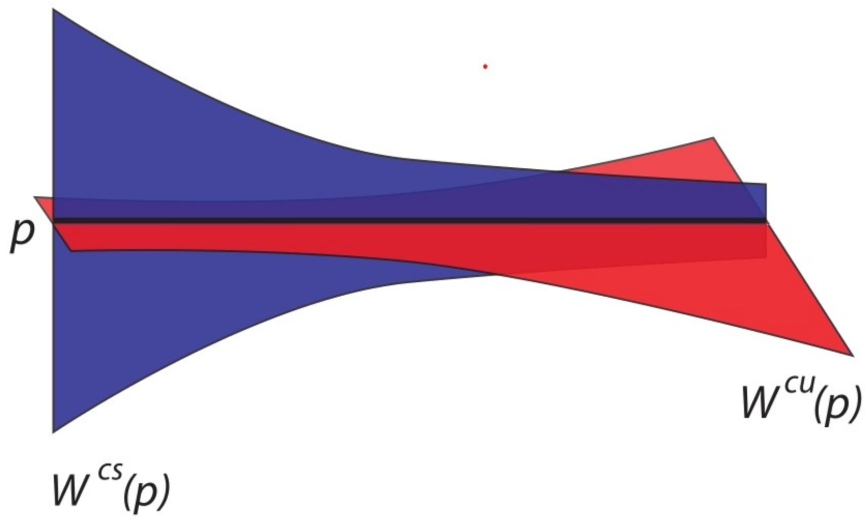

Figure 2.3: Local center-stable and center-unstable leaves.

2.3.2 Finite Volume Case

An important observation in the Theorem 2.4 on the proof of existence of the local stable-unstable manifolds is the construction of special charts depending of the injectivity radius of exponential map for its construction. (see [AK])

Now in the non compact case, with the only condition of hyperbolicity of our manifold M, we can imitated the proof of the Stable-Unstable Manifold Theorem (Theorem 2.4) for the construction of local stable-unstable manifold on each point $x \in M$, one crucial diference here in the non compact case is that the sizes of local stable-unstable manifolds vary with the point $x \in M$. Since in the non compact case, for every point $x \in M$ there exists $\varepsilon(x) > 0$ (injectivity radius of exponential map) such that the exponential map $exp_x : B(0_x, \varepsilon(x)) \subset T_xM \to M$ is an diffeomorphism over its image and that charts are used in the construction. Now in the compact case, we can choose $0 < \varepsilon_M := \inf\{\varepsilon(x) > 0 : x \in M\}$. In the non compact case, we can not guarantee that $\varepsilon_M > 0$. Even in the Finite Volume case, it could happen that the injectivity radius of exponential map tends to 0 in the ends.

However, by the observation above, in the finite volume case we can proof the existence of local stable-unstable manifolds, but in this case we have the size of local stable-unstable manifold vary with the point $x \in M$. This mean, the size of local stable-unstable manifold is not uniform.

Other important observation on the proof of the existence of local stable-unstable manifold (Theorem 2.4), as already mentioned, is that as the size of local stable-unstable manifolds depends of injectivity radius of exponential map over each point $x \in M$, we can restrict ourselves to any compact set K of M for choose $\varepsilon_{_K} := \inf\{\varepsilon(x) > 0 : x \in K\} > 0$ (injectivity radius of exponential map restricted to the compact set $K \subset M$) and thus we can choose one fix size of the local stable-unstable manifolds for every point $x \in K$.

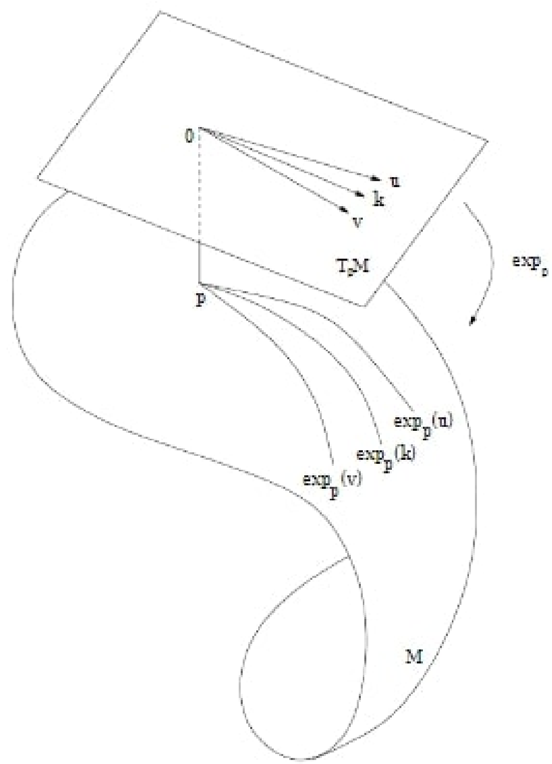

Figure 2.4: Action of the exponential map of three different initial velocities $u, k, v \in T_pM$.

We summarize the observations above over existence of local stable-unstable manifold in the following theorem:

Theorem 2.7. [**Finite Volume**] *Let $\varphi^t : M \to M$ be a Anosov C^r - flow (with $r \geq 1$) and $Vol(M) < +\infty$. Then for each $x \in M$ there is a pair of embedded C^1 disks $W^s_{\varepsilon(x)}$, $W^u_{\varepsilon(x)} \subset M$ called the* ***local strong-stable*** *and* ***local strong-unstable*** *manifolds of $x \in M$ of size $\varepsilon(x) > 0$ such that:*

(1) $T_x W^s_{\varepsilon(x)}(x) = E^s_x$, $T_x W^u_{\varepsilon(x)}(x) = E^u_x$.

(2) *For every compact subset $K \subset M$ there exists $\varepsilon_K := \inf\{\varepsilon(x) > 0 : x \in K\} > 0$ such that for every $x \in K$ we have that $W^s_{\varepsilon_K}(x)$ and $W^u_{\varepsilon_K}(x)$ with uniform size.*

(3) $W^s_{\varepsilon(x)}(x) \cap W^u_{\varepsilon(x)}(x) = \{x\}$.

Proof. The proof follows the same arguments that ([AK]), □

2.3.3 Local Product Structure

In this subsection we comment about the Local Product Structure on the compact case and the non compact case. The following result appears in the compact case and characterizes the local maximility through local stable and stable manifolds. (See [FH])

Theorem 2.8. [**Bowen Bracket**] *For a hyperbolic flow $\varphi^t : M \to M$ and $\varepsilon > 0$ sufficiently small there exists a $\delta > 0$ such that if $x, y \in M$ such that $d(x, y) < \varepsilon$, then there exists some $t = t(x, y) \in (-\varepsilon, \varepsilon)$ such that*

$$W^s_\varepsilon(\varphi^t(x)) \cap W^u_\varepsilon(y)) = \{[x, y]\}$$

consists of a single point. This intersection point $[x, y]$ of $W^{cs}_\varepsilon(x)$ and $W^u_\varepsilon(y)$ is called the ***Bowen Bracket*** *of x and y, and there exists $C_0 = C_0(\delta) > 0$ such that if x, $y \in M$ and $d(x, y) < \delta$, then $d_s(\varphi^{t(x,y)}(x), [x, y]) < C_0 d(x, y)$ and $d_u(y, [x, y]) < C_0 d(x, y)$, where d_s and d_u denote the distances along the stable and unstable manifolds.*

Remark 2.9. *In the Theorem above a complementary choice would be $W^s_\varepsilon(x) \cap W^{cu}_\varepsilon(y) = \{[x, y]\}$.*

Now, when we have a hiperbolic flow $\varphi^t : M \to M$ and M is not necessarily compact, the Theorem 2.8 above is still true for compact balls of M, more specifically:

Theorem 2.10. *Given $x_0 \in M$ and $\varepsilon > 0$. Consider the compact ball $B[x_0, \varepsilon] \subset M$. Then there exist $\delta > 0$ and $\eta > 0$ such that for all $x, y \in B(x_0, \delta)$ we have $W^{cu}_\eta(x) \cap W^{ss}_\eta(y) = \{w\} \in B(x_0, \varepsilon)$.*

Proof. The same proof on [FH] holds in this case because we restrict ourselves to one compact ball and the compactness of ball allows us to obtain uniform constants $\delta > 0$ and $\eta > 0$. □

An analogous result holds for $W^{cs}_\eta(x)$ and $W^{uu}_\eta(y)$.

Chapter 3

Anosov geodesic flow and periodic orbits

3.1 Shadowing Lemma

The orbit structure of hyperbolic dynamical system has a distinctive and iconic richness and complexity, and these features can be derived from what thereby appears as a core feature of hyperbolic dynamics: the shadowing of orbits. In this section, we remember the Shadowing Lemma for flows and the expansivity in the compact case (See [FH] and [AK]).

Definition 3.1. *Let $\varphi^t : M \to M$ be a flow on a complete Riemannian manifold.*

1 [**Pseudo Orbit**] *An ε-pseudo orbit or ε-chain for φ^t on M is a map $h : I \to M$ on a non trivial interval $I \subset \mathbb{R}$ such that*

$$d(h(t+\tau), \varphi^{\tau}(h(t))) < \varepsilon, \quad for \quad t, t+\tau \in I \quad and \quad |\tau| < 1$$

2 [**Shadowing**] *Let h be an ε-pseudo orbit for φ^t. Then h is said to be δ-shadowed if there exist a point $p \in M$ and a homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(t) - t$ has Lipschitz constant δ and $d(h(t), \varphi^{\alpha(t)}(p)) \leq \delta$ for all $t \in \mathbb{R}$. A set $K \subset M$ has the* ***Shadowing property*** *if for any $\delta > 0$ there is an $\varepsilon > 0$ such that any ε-pseudo orbit in K is δ-shadowed by a point $p \in M$. We say that φ^t has the shadowing property if this holds for $K = M$. A set $K \subset M$ has L-**Lipschitz shadowing** for $\varepsilon_0 > 0$ if any ε-pseudo orbit in K with $\varepsilon \leq \varepsilon_0$ is $L\varepsilon$-shadowed by a point $p \in M$.*

Definition 3.2. [**Expansiveness**] *Let $\varphi^t : M \to M$ be a flow on a compact Riemannian manifold. We said that φ^t is* ***expansive*** *if for all $\varepsilon > 0$ there exists a $\delta > 0$, called an* ***expansivity constant*** *(for ε), such that if $p, q \in M$, $h : \mathbb{R} \to \mathbb{R}$ continuous, $h(0) = 0$, and $d(\varphi^t(p), \varphi^{h(t)}(q)) < \delta$, for all $t \in \mathbb{R}$, then $q = \varphi^{t_0}(p)$ for some $t_0 \in (-\varepsilon, \varepsilon)$.*

Theorem 3.3. [**Shadowing Lemma**] *A hyperbolic flow has a property of L-Lipschitz shadowing for some $\epsilon_0 > 0$ and for some $L > 0$. The shadowing point need not be unique because neither is the choice of the parametrization. But the shadowing orbit is unique and the shadowing point is determined up to a small shift within that orbit.*

The uniqueness assertion of Theorem 3.3 implies that no two orbits can shadow each other:

Corollary 3.4. *A hyperbolic flow $\varphi^t : M \to M$ in a compact Riemannian manifold is expansive.*

The uniqueness assertion of Theorem 3.3 implies not only expansivity but also that the shadowing orbit is periodic when one starts with a periodic pseudo orbit.

Theorem 3.5. [**Anosov Closing Lemma**] *For a Anosov flow $\varphi^t : M \to M$ in a compact Riemannian manifold there exist ε_0, $L > 0$ such that for $\varepsilon \leq \varepsilon_0$ any periodic ε-pseudo orbit is $L\varepsilon$-shadowed by a unique periodic orbit for φ^t.*

3.2 Density of Periodic Orbits in the Compact Case

The knowledge of the distribution and density of periodic orbits of Anosov flows and more generally hyperbolic flows is a very interesting problem because it gives us more information about the dynamics of the system. One classical example is the Anosov geodesic flow over the unit tangent bundle SM of a compact manifold M with negative curvature (see [A]). A proof for the density of the periodic orbits of a hyperbolic flow over one compact manifold is using the Spectral Decomposition Theorem for flows (see [AK] and [FH]). Moreover, the information about the density of periodic orbits of a hyperbolic flow over a compact manifold also guarantees the topological transitivity of the flow. We can summarize all remarks above in the case of the hyperbolic flow on a compact Riemannian manifold in the following result: (see [AK] and [FH])

Theorem 3.6. *Let $\varphi^t : M \to M$ be an Anosov flow defined over a compact Riemannian manifold M. The following are equivalent:*

(1) *The spectral decomposition has one piece (the whole manifold),*

(2) *The flow is topologically transitive,*

(3) *Periodic points are dense,*

(4) *All center-unstable leaves are dense,*

(5) *All center-stable leaves are dense.*

3.3 Density of Periodic Orbits in the Finite Volume Case

For $\eta > 0$, we will consider the following sets:

$$\begin{aligned}
W^{ss}(x) &= \{z \in SM : \lim_{t\to+\infty} d(\varphi^t(z), \varphi^t(x)) = 0\},\\
W^{uu}(x) &= \{z \in SM : \lim_{t\to+\infty} d(\varphi^{-t}(z), \varphi^{-t}(x)) = 0\},\\
W^{cs}(x) &= \bigcup_{t\in\mathbb{R}} W^{ss}(\varphi^t(x)),\\
W^{cu}(x) &= \bigcup_{t\in\mathbb{R}} W^{uu}(\varphi^t(x)),\\
W^{ss}_\eta(x) &= \{z \in SM : d(\varphi^t(x), \varphi^t(z)) \le \lambda^t\eta,\ \forall\, t \ge 0\},\\
W^{uu}_\eta(x) &= \{z \in SM : d(\varphi^{-t}(x), \varphi^{-t}(z)) \le \lambda^t\eta,\ \forall\, t \ge 0\},\\
W^{cu}_\eta(x) &= \{z \in SM : \exists\, \theta_1 \in W^{uu}_\eta(x)\ \ and\ \ \varphi_{s_1}(\theta_1) = z\ \ for\ \ some\ \ |s_1| < \eta\},\\
W^{cs}_\eta(x) &= \{z \in SM : \exists\, \theta_2 \in W^{ss}_\eta(x)\ \ and\ \ \varphi_{s_2}(\theta_2) = z\ \ for\ \ some\ \ |s_2| < \eta\}.
\end{aligned}$$

Lemma 3.7. *Let $x \in SM$. Then*

(a) For all $t \in \mathbb{R}$ holds $\varphi^t(W^{uu}(x)) = W^{uu}(\varphi^t(x))$.

(b) For all $t \in \mathbb{R}$ holds $\varphi^t(W^{cu}(x)) = W^{cu}(x)$.

(c) If $y \in W^{cu}(x)$ then $W^{cu}(y) \subset W^{cu}(x)$.

Proof. **(a)** Apply the definition of $W^{uu}(x)$.
(b) Let $t \in \mathbb{R}$ fixed, using (a) we have that:

$$\begin{aligned}
\varphi^t(W^{cu}(x)) &= \varphi^t\left(\bigcup_{s\in\mathbb{R}} W^{uu}(\varphi^s(x)))\right)\\
&= \bigcup_{s\in\mathbb{R}} \varphi^t(W^{uu}(\varphi^s(x)))\\
&= \bigcup_{s\in\mathbb{R}} W^{uu}(\varphi^{t+s}(x))\\
&= \bigcup_{s\in\mathbb{R}} W^{uu}(\varphi^s(x))\\
&= W^{cu}(x).
\end{aligned}$$

(c) Let $y \in W^{cu}(x) = \cup_{t\in\mathbb{R}} W^{uu}(\varphi^t(x))$ then by (b) for all $t \in \mathbb{R}$ holds $\varphi^t(W^{cu}(x)) = W^{cu}(x)$. Now, let $z \in W^{cu}(y)$ then there exists $t_1 \in \mathbb{R}$ sucha that $z \in W^{uu}(\varphi^{t_1}(y))$, so

$$\lim_{t\to+\infty} d(\varphi^{-t}(z), \varphi^{-t}(\varphi^{t_1}(y))) = 0. \tag{3.1}$$

As $y \in W^{cu}(x)$ then $\varphi^{t_1}(y) \in W^{cu}(x)$, thus there exists $t_2 \in \mathbb{R}$ such that $\varphi^{t_1}(y) \in W^{uu}(\varphi^{t_2}(x))$, so

$$\lim_{t\to+\infty} d(\varphi^{-t}(\varphi^{t_1}(y)), \varphi^{-t}(\varphi^{t_2}(x))) = 0, \tag{3.2}$$

thus of (3.1) and (3.2) we have that:

$$\begin{aligned} 0 &\leq \lim_{t\to+\infty} d(\varphi^{-t}(z), \varphi^{-t}(\varphi^{t_2}(x))) \\ &\leq \lim_{t\to+\infty} d(\varphi^{-t}(z), \varphi^{-t}(\varphi^{t_1}(y))) + \lim_{t\to+\infty} d(\varphi^{-t}(\varphi^{t_1}(y)), \varphi^{-t}(\varphi^{t_2}(x))) \\ &= 0, \end{aligned}$$

this implies that

$$\lim_{t\to+\infty} d(\varphi^{-t}(z), \varphi^{-t}(\varphi^{t_2}(x))) = 0.$$

Hence $z \in W^{uu}(\varphi^{t_2}(x)) \subset W^{cu}(x)$. □

Lemma 3.8. *Let $A = W^{cu}(x)$ and $B = \bigcup_{y\in A} W^{cu}(y)$. Then $A = B$. Moreover $\bigcup_{y\in\overline{A}} W^{cu}(y) = \overline{A} = \overline{W^{cu}(x)}$.*

Proof. Let $z \in A$, then there exist $t_1 \in \mathbb{R}$ such that $z \in W^{uu}(\varphi^{t_1}(x)) \subset W^{cu}(\varphi^{t_1}(x))$. Note that $x \in W^{uu}(x) \subset W^{cu}(x)$ and by Lemma 3.7 we have that $\varphi^{t_1}(x) \in \varphi^{t_1}(W^{cu}(x)) = W^{cu}(x) = A$, hence $z \in \bigcup_{y\in A} W^{cu}(y) = B$.

On the other hand, if $z \in B$ there exists $y \in A$ such that $z \in W^{cu}(y)$. As $y \in A$ by Lemma 3.7, (c), we have that $W^{cu}(y) \subset W^{cu}(x) = A$. Hence $A = B$. The subset $A \subset SM$ is called saturated in W^{cu}, and by property of foliations we have that $\overline{A} = \overline{W^{cu}(x)}$ is saturated, and this means

$$\bigcup_{y\in\overline{A}} W^{cu}(y) = \overline{A} = \overline{W^{cu}(x)}.$$

□

Proposition 3.9. *Let $\varphi : SM \to SM$ be a Anosov geodesic flow with M connected and $vol(M) < \infty$. If $\overline{Per(\varphi)}$=SM then for all $x \in SM$ we have $W^{cu}(x)$ and $W^{cs}(x)$ are denses on SM.*

Proof. Consider $x \in SM$ and $W^{cu}(x)$ (the proof for W^{cs} is analogous). We know that $\overline{W^{cu}(x)}$ is a closed subset of SM. We will show that $\overline{W^{cu}(x)}$ is an open subset, thus by connectedness of SM we have $\overline{W^{cu}(x)} = SM$.

Let $z \in \overline{W^{cu}(x)}$, $U \subset SM$ a neighborhood of z and $p \in U \cap Per(\varphi)$ (this last is possible because $\overline{Per(\varphi)} = SM$). Taking U sufficiently small such that local structure product hold‘s, then $\phi \neq W^{cs}(p) \cap W^{cu}(z)$. Thus take $y \in W^{cs}(p) \cap W^{cu}(z)$. Denote $A = W^{cu}(x)$ and consider

$B = \cup_{v\in A} W^{cu}(v)$ as on Lema 3.8 we have $\overline{W^{cu}(x)} = \bigcup_{v\in\overline{W^{cu}(x)}} W^{cu}(v)$. As $z \in \overline{W^{cu}(x)}$ then there is $y_0 \in \overline{W^{cu}(x)}$ such that $z \in W^{cu}(y_0)$. By Lemma 3.7, (c) we obtain:

$$W^{cu}(z) \subset W^{cu}(y_0) \subset \bigcup_{v\in\overline{W^{cu}(x)}} W^{cu}(v) = \overline{W^{cu}(x)},$$

thus $W^{cu}(z) \subset \overline{W^{cu}(x)}$. As $y \in W^{cu}(z)$ then by Lemma 3.7, (b), $O(y) \subset W^{cu}(z) \subset \overline{W^{cu}(x)}$. On the other hand, $y \in W^{cs}(p)$, then for all $t \in \mathbb{R}$, $\varphi^t(y) \in \varphi^t(W^{cs}(p)) = W^{cs}(p)$, this implies $O(y) \subset W^{cs}(p)$.

Now fixed $r \in \mathbb{R}$, we have $\varphi^r(y) \in W^{cs}(p) = \cup_{t\in\mathbb{R}} W^{ss}(\varphi^t(p))$, then there is $t_r \in \mathbb{R}$ such that $\varphi^r(y) \in W^{ss}(\varphi^{t_r}(p))$, this implies $\lim_{t\to+\infty} d(\varphi^t(\varphi^r(y)), \varphi^t(\varphi^{t_r}(p))) = 0$, thus $O(y)$ accumulates in the orbit $O(p)$ and as $O(y) \subset \overline{W^{cu}(x)}$, we have $O(p) \subset \overline{W^{cu}(x)}$. Hence $p \in \overline{W^{cu}(x)}$ and as $Per(\varphi)$ is dense on SM, then $Per(\varphi)$ is dense in U, this implies $U \subset \overline{W^{cu}(x)}$. □

Theorem 3.10. *Let $\varphi^t : SM \to SM$ be a Anosov geodesic flow with $vol(M) < \infty$. If $W^{cs}(x)$ and $W^{cu}(x)$ are dense on SM for all $x \in SM$ then φ is transitive.*

Proof. Let U and V be two open sets of SM. Fix $x \in V$ such that $\alpha(x) \neq \phi$. Take $y \in \alpha(x)$, then there is $n_k \to +\infty$ such that $\lim\limits_{n_k\to+\infty} \varphi^{-n_k}(x) = y$. Now by hypothesis $\overline{W^{cs}(y)} = SM$ then $W^{cs}(y) \cap U \neq \phi$, thus there exists $z \in W^{cs}(y) \cap U$.

As $z \in W^{cs}(y) = \cup_{t\in\mathbb{R}} W^{ss}(\varphi^t(y))$, then there is $t_1 \in \mathbb{R}$ such that $z \in W^{ss}(\varphi^{t_1}(y) \cap U$. By continuity of flow φ^{t_1} we obtain

$$\lim_{n_k\to+\infty} \varphi^{-n_k}(\varphi^{t_1}(x)) = \lim_{n_k\to+\infty} \varphi^{t_1}(\varphi^{-n_k}(x)) = \varphi^{t_1}(y)$$

As $W^{ss}(\varphi^{t_1}(y)) \cap U$ and U is a open set, we can fix a real number $b > 0$ such that for all k large there is a disk $D_k \subset W^{ss}(\varphi^{-n_k}(\varphi^{t_1}(x)))$ centered at $\varphi^{-n_k}(\varphi^{t_1}(x))$ of radius at most $b > 0$ such that $D_k \cap U \neq \phi$.

Since φ^{-t_1} is continuous and $x \in V$, we can fix a neighbourhood Q of $\varphi^{t_1}(x)$ such that $\varphi^{-t_1}(Q) \subset V$. So for all k large we have $\varphi^{n_k}(D_k) \subset Q$ (here by Lambda Lemma, we choose $D_k \subset W^{ss}(\varphi^{-n_k}(\varphi^{t_1}(x)))$ such that $\varphi^{n_k}(D_k) \subset Q$). Thus as $\varphi^{n_k}(D_k) \subset Q$ then $\varphi^{-t_1}(\varphi^{n_k}(D_k)) \subset \varphi^{-t_1}(Q) \subset V$. Now taking k large we have

$$\begin{aligned} D_k &\cap U \subset U \Longrightarrow \varphi^{-t_1}(\varphi^{n_k}(D_k \cap U)) \subset \varphi^{-t_1}(\varphi^{n_k}(U)) \\ D_k &\cap U \subset D_k \Longrightarrow \varphi^{-t_1}(\varphi^{n_k}(D_k \cap U)) \subset \varphi^{-t_1}(\varphi^{n_k}(D_k)) \subset V \end{aligned}$$

by the last two inequalities above we get $\varphi^{-t_1}(\varphi^{n_k}(D_k \cap U)) \subset \varphi^{-t_1}(\varphi^{n_k}(U)) \cap V$ and as $D_k \cap U \neq \phi$ then $\phi \neq \varphi^{-t_1}(\varphi^{n_k}(U)) \cap V$. □

3.3.1 Shadowing and Expansiveness in Finite Volume Case

In this section we will give some definitions about the shadowing and expansiveness that we need for proof the density of periodic orbits when we have a geodesic flow on a finite volume manifold.

Definition 3.11. [**Shadowing forward and backward**]

Let $x_0, y \in SM$, $t_0 > 0$ and $\varepsilon > 0$. Consider the orbit segment $\varphi^{[0,t_0]}(x_0) = \{\varphi^t(x_0) : t \in [0, t_0]\}$ and the future orbit and past orbit of y, $O^+(y) = \{\varphi^t(y) : t \geq 0\}$ and $O^-(y) = \{\varphi^t(y) : t \leq 0\}$, respectively.

1 *We said that $O^+(y)$ ε-**shadowed forward by piecewise** the orbit segment $\varphi^{[0,t_0]}(x_0)$ if*

$$d(\varphi^t(y), \varphi^t(x_0)) \leq \varepsilon, \quad \forall\ t \in [0, t_0],$$

*and there exists a sequence $\{s_j\}_{j\geq 0}$ with $s_j \geq 0$ (called **transitions times**), such that for all $k \geq 1$ holds*

$$d(\varphi^t(\varphi^{kt_0+\sum_{i=0}^{k-1} s_i}(y)), \varphi^t(x_0)) \leq \varepsilon, \quad \forall\ t \in [0, t_0].$$

2 *Analogously we said that $O^-(y)$ ε-**shadowed backward by piecewise** the orbit segment $\varphi^{[0,t_0]}(x_0)$ if there exists a sequence $\{r_j\}_{j\geq 0}$ with $r_j \geq 0$ (called **transitions times**) such that for all $k \geq 0$ holds*

$$d(\varphi^{-t}(\varphi^{-kt_0-\sum_{j=0}^{k} r_j}(y)), \varphi^{-t}(\varphi^{t_0}(x_0))) \leq \varepsilon, \quad \forall\ t \in [0, t_0],$$

*and we said that $O(y) = O^+(y) \cup O^-(y)$ ε-**shadowed by piecewise** the orbit segment $\varphi^{[0,t_0]}(x_0)$ if $O^+(y)$ ε-shadowed forward and $O^-(y)$ ε-shadowed backward by piecewise respectively the orbit segment $\varphi^{[0,t_0]}(x_0)$.*

The definition above of the shadowing forward and backward by piecewise is similar to the specification property for flows (see [FH]), the diference between specification property and the shadowing by piecewise above, is that the specification property its appling for orbit segment arising from a recurrent point. (see Proposition 3.12 below)

Proposition 3.12. *Let $\varphi : SM \to SM$ be a Anosov geodesic flow with $Vol(M) < +\infty$. Fix $x_0 \in Rec(\varphi)$. For all $\varepsilon > 0$ there are $t_0 > 0$ and $\tilde{y} \in SM$ such that $O(\tilde{y}) = \{\varphi^t(\tilde{y}) :\ t \in \mathbb{R}\}$ ε-shadowed by piecewise the segment orbit $\varphi^{[0,t_0]}(x_0)$.*

Proof. As $x_0 \in Rec(\varphi)$ then given $\varepsilon > 0$, there is $t_0 \in \mathbb{N}$ such that $\varphi^{t_0}(x_0) \in B(x_0, \varepsilon)$.

By the hypothesis $Vol(M) < \infty$, we still have a local product structure (see Theorem 2.10), this mean for all $\varepsilon > 0$ there exists $\delta > 0$ and $\eta > 0$ such that for all $x, y \in B(x_0, \delta)$ we have $W^{cu}_\eta(x) \pitchfork W^{ss}_\eta(y) = \{w\} \in B(x_0, \varepsilon)$.

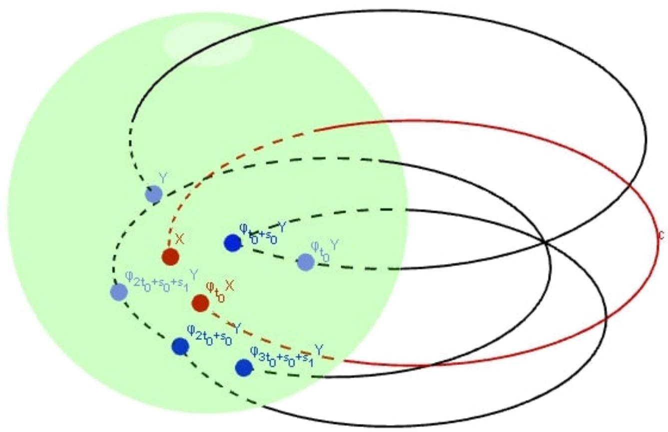

Figure 3.1: Shadowing forward by piecewise.

On the other hand, we denote by $\lambda \in (0,1)$ the hyperbolicity constant. We define the following sequence: for all $m \in \mathbb{N}$

$$\begin{aligned} S_m(t) &= \sum_{j=1}^{m} \lambda^{t+jt_0+\sum_{i=1}^{j} c_i} \\ &= \lambda^t \sum_{j=1}^{m} \left(\lambda^{t_0+\frac{1}{j}\sum_{i=1}^{j} c_i}\right)^j, \end{aligned} \tag{3.3}$$

where for all $i \in \mathbb{N}$, $c_i \in \mathbb{R}$ and $|c_i| < \eta$.

As for all $i \in \mathbb{N}$, $|c_i| < \eta$ then for all $j \in \mathbb{N}$ we have

$$\begin{array}{rcl} -\eta < & c_i & < \eta \\ j(-\eta) < & \sum_{i=1}^{j} c_i & < j\eta \\ -\eta < & \dfrac{1}{j}\sum_{i=1}^{j} c_i & < \eta \\ t_0 - \eta < & t_0 + \dfrac{1}{j}\sum_{i=1}^{j} c_i & < t_0 + \eta \end{array}$$

Thus if we choose $t_0 > 0$ such that $t_0 - \eta < \frac{t_0}{2}$ then for all $j \in \mathbb{N}$ we have

$$0 < \frac{t_0}{2} - \eta \leq t_0 + \frac{1}{j}\sum_{i=1}^{j} c_i$$
$$\lambda^{(t_0 + \frac{1}{j}\sum_{i=1}^{j} c_i)} \leq \lambda^{\frac{t_0}{2}}$$
$$\left(\lambda^{(t_0 + \frac{1}{j}\sum_{i=1}^{j} c_i)}\right)^j \leq (\lambda^{\frac{t_0}{2}})^j$$

summing over $j \in \mathbb{N}$

$$\begin{aligned}
\sum_{j=1}^{m}\left(\lambda^{t_0 + \frac{1}{j}\sum_{i=1}^{j} c_i}\right)^j &\leq \sum_{j=1}^{m}(\lambda^{\frac{t_0}{2}})^j \\
&\leq \sum_{j=1}^{\infty}(\lambda^{\frac{t_0}{2}})^j \\
&= \frac{\lambda^{\frac{t_0}{2}}}{1 - \lambda^{\frac{t_0}{2}}} = L
\end{aligned}$$

Now as $0 < \lambda < 1$ then for all $t \in [0, t_0]$ we have $0 < \lambda^t \leq 1$ and $0 < \lambda^{t_0 - t} \leq 1$, thus for all $t \in [0, t_0]$ holds

$$\lambda^t + \lambda^{t_0 - t} + \sum_{j=1}^{m} \lambda^{t + jt_0 + \sum_{i=1}^{j} c_i} \leq 2 + L = K, \ \forall \ m \in \mathbb{N} \tag{3.4}$$

We denote $f(t) = \dfrac{\varepsilon(1 - \lambda^{\frac{t}{2}})}{2 - \lambda^{\frac{t}{2}}}$. Since, $\lim_{t \to +\infty} f(t) = \dfrac{\varepsilon}{2}$, then given $\varepsilon > 0$ we can choose $\delta > 0$, $\eta > 0$ and $t_0 > 0$ such that

$$\eta < \frac{\varepsilon}{4} < f(t_0) = \frac{\varepsilon}{K} \ \textit{and} \ \lambda^{t_0}\eta < \delta. \tag{3.5}$$

Moreover we can choose $t_0 > 0$ such that $x_1 = \varphi^{t_0}(x_0) \in B(x_0, \delta)$, thus by local product structure (see Theorem 2.10) there is $z_1 \in W^{cu}_{\eta}(x_1) \pitchfork W^{ss}_{\eta}(x_0)$, this means

$$\begin{aligned}
z_1 \in W^{cu}_{\eta}(x_1) &\implies \exists\, \theta_1 \in W^{uu}_{\eta}(x_1) : \varphi^{s_1}(\theta_1) = z_1, \ \ for \ \ some \ |s_1| < \eta, \\
\theta_1 \in W^{uu}_{\eta}(x_1) &\implies d(\varphi^{-t}(\theta_1), \varphi^{-t}(x_1)) \leq \lambda^t \eta, \ \ \forall\, t \geq 0,
\end{aligned}$$

taking $y_1 = \varphi^{-t_0}(\theta_1)$, for all $0 \leq t \leq t_0$ we have

$$\begin{aligned}
d(\varphi^t(y_1), \varphi^t(x_0)) &= d(\varphi^{t - t_0}(\theta_1), \varphi^{t - t_0}(\varphi^{t_0}(x_0))) \\
&= d(\varphi^{t - t_0}(\theta_1), \varphi^{t - t_0}(x_1)) \\
&\leq \lambda^{t_0 - t}\eta, \qquad (3.6)
\end{aligned}$$

thus by (3.5) for $t = 0$ on (3.6) we get that $d(y_1, x_0) \leq \lambda^{t_0}\eta < \delta$, this implies $y_1 \in B(x_0, \delta)$ and again by local product structure (see Theorem 2.10) there is $z_2 \in W^{cu}_\eta(x_1) \pitchfork W^{ss}_\eta(y_1)$, this means

$$\begin{aligned}
z_2 \in W^{cu}_\eta(x_1) &\implies \exists\, \theta_2 \in W^{uu}_\eta(x_1) : \varphi^{s_2}(\theta_2) = z_2, \;\; for \;\; some \;\; |s_2| < \eta, \\
\theta_2 \in W^{uu}_\eta(x_1) &\implies d(\varphi^{-t}(\theta_2), \varphi^{-t}(x_1)) \leq \lambda^t\eta, \;\; \forall\, t \geq 0, \\
z_2 \in W^{ss}_\eta(y_1) &\implies d(\varphi^{t}(z_2), \varphi^{t}(y_1)) \leq \lambda^t\eta, \;\; \forall\, t \geq 0,
\end{aligned} \tag{3.7}$$

taking $y_2 = \varphi^{-t_0}(\theta_2)$ (where $\varphi^{t_0+s_2}(y_2) = \varphi^{s_2}(\theta_2) = z_2$), for all $0 \leq t \leq t_0$ we have

$$\begin{aligned}
d(\varphi^t(y_2), \varphi^t(x_0)) &= d(\varphi^{t-t_0}(\theta_2), \varphi^{t-t_0}(x_1)) \\
&\leq \lambda^{t_0-t}\eta,
\end{aligned} \tag{3.8}$$

also by (3.6) and (3.7) we get

$$\begin{aligned}
d(\varphi^t(z_2), \varphi^t(x_0)) &\leq d(\varphi^t(z_2), \varphi^t(y_1)) + d(\varphi^t(y_1), \varphi^t(x_0)) \\
&\leq \lambda^t\eta + \lambda^{t_0-t}\eta = (\lambda^t + \lambda^{t_0-t})\eta,
\end{aligned} \tag{3.9}$$

thus by (3.5) for $t = 0$ on (3.8) we get that $d(y_2, x_0) \leq \lambda^{t_0}\eta < \delta$, this implies $y_2 \in B(x_0, \delta)$ and again by local product structure (see Theorem 2.10) there is $z_3 \in W^{cu}_\eta(x_1) \pitchfork W^{ss}_\eta(y_2)$, this means

$$\begin{aligned}
z_3 \in W^{cu}_\eta(x_1) &\implies \exists\, \theta_3 \in W^{uu}_\eta(x_1) : \varphi^{s_3}(\theta_3) = z_3, \;\; for \;\; some \;\; |s_3| < \eta, \\
\theta_3 \in W^{uu}_\eta(x_1) &\implies d(\varphi^{-t}(\theta_3), \varphi^{-t}(x_1)) \leq \lambda^t\eta, \;\; \forall\, t \geq 0, \\
z_3 \in W^{ss}_\eta(y_2) &\implies d(\varphi^{t}(z_3), \varphi^{t}(y_2)) \leq \lambda^t\eta, \;\; \forall\, t \geq 0,
\end{aligned} \tag{3.10}$$

taking $y_3 = \varphi^{-t_0}(\theta_3)$, for all $0 \leq t \leq t_0$ we have

$$\begin{aligned}
d(\varphi^t(y_3), \varphi^t(x_0)) &= d(\varphi^{t-t_0}(\theta_3), \varphi^{t-t_0}(x_1)) \\
&\leq \lambda^{t_0-t}\eta,
\end{aligned} \tag{3.11}$$

also by (3.10) and (3.8) we get

$$\begin{aligned}
d(\varphi^t(z_3), \varphi^t(x_0)) &\leq d(\varphi^t(z_3), \varphi^t(y_2)) + d(\varphi^t(y_2), \varphi^t(x_0)) \\
&\leq \lambda^t\eta + \lambda^{t_0-t}\eta = (\lambda^t + \lambda^{t_0-t})\eta
\end{aligned} \tag{3.12}$$

remember that $\varphi^{t_0+s_2}(y_2) = z_2$, by (3.10) and (3.9) we get

$$\begin{aligned}
d(\varphi^t(\varphi^{t_0+s_2}(z_3)), \varphi^t(x_0)) &\leq d(\varphi^{t_0+s_2}(z_3)), \varphi^t(\varphi^{t_0+s_2}(y_2))) + d(\varphi^t(z_2), \varphi^t(x_0)) \\
&\leq \lambda^{t+t_0+s_2}\eta + (\lambda^t + \lambda^{t_0-t})\eta \\
&= (\lambda^{t+t_0+s_2} + \lambda^t + \lambda^{t_0-t})\eta
\end{aligned} \tag{3.13}$$

thus by (3.5) for $t=0$ on (3.11) we get that $d(y_3,x_0)\le\lambda^{t_0}\eta<\delta$, this implies $y_3\in B(x_0,\delta)$ and again by local product structure (see Theorem 2.10) there is $z_4\in W^{cu}_\eta(x_1)\pitchfork W^{ss}_\eta(y_3)$

$$
\begin{aligned}
z_4\in W^{cu}_\eta(x_1) &\implies \exists\,\theta_4\in W^{uu}_\eta(x_1):\varphi^{s_4}(\theta_4)=z_4,\ \ for\ \ some\ \ |s_4|<\eta,\\
\theta_4\in W^{uu}_\eta(x_1) &\implies d(\varphi^{-t}(\theta_4),\varphi^{-t}(x_1))\le\lambda^t\eta,\ \ \forall\, t\ge 0,\\
z_4\in W^{ss}_\eta(y_3) &\implies d(\varphi^{t}(z_4),\varphi^{t}(y_3))\le\lambda^t\eta,\ \ \forall\, t\ge 0,
\end{aligned}
\tag{3.14}
$$

taking $y_4=\varphi^{-t_0}(\theta_4)$ and proceeding as we did in the case of y_1, y_2 and y_3, we get that for all $0\le t\le t_0$ holds

$$
\begin{aligned}
d(\varphi^t(y_4),\varphi^t(x_0)) &\le \lambda^{t_0-t}\eta\\
d(\varphi^t(z_4),\varphi^t(x_0)) &\le (\lambda^t+\lambda^{t_0-t})\eta\\
d(\varphi^t(\varphi^{t_0+s_3}(z_4)),\varphi^t(x_0)) &\le (\lambda^{t+t_0+s_3}+\lambda^t+\lambda^{t_0-t})\eta\\
d(\varphi^t(\varphi^{2t_0+s_2+s_3}(z_4)),\varphi^t(x_0)) &\le (\lambda^{t+2t_0+s_2+s_3}+\lambda^{t+t_0+s_2}+\lambda^{t_0-t}+\lambda^t)\eta.
\end{aligned}
$$

Thus we get four sequence $\{y_n\}_{n\ge0}$, $\{\theta_n\}_{n\ge0}$, $\{z_n\}_{n\ge0}$ on SM and $\{s_n\}$ in $\mathbb{R}$ satisfying $y_n=\varphi^{-t_0}(\theta_n)$, $\varphi^{s_n}(\theta_n)=z_n$ with $|s_n|<\eta$,and for $0\le t\le t_0$ we have:

$$
\begin{aligned}
d(\varphi^t(y_n),\varphi^t(x_0)) &\le \lambda^{t_0-t}\eta\\
d(\varphi^t(\varphi^{t_0+\sum_{i=0}^{0}s_{n-i}}(y_n)),\varphi^t(x_0)) &\le (\lambda^{t_0-t}+\lambda^t)\eta\\
d(\varphi^t(\varphi^{2t_0+\sum_{i=0}^{1}s_{n-i}}(y_n)),\varphi^t(x_0)) &\le (\lambda^{t_0-t}+\lambda^t+\sum_{j=1}^{1}\lambda^{t+jt_0+\sum_{i=0}^{j-1}s_{(n-1)+i}})\eta\\
d(\varphi^t(\varphi^{3t_0+\sum_{i=0}^{2}s_{n-i}}(y_n)),\varphi^t(x_0)) &\le (\lambda^{t_0-t}+\lambda^t+\sum_{j=1}^{2}\lambda^{t+jt_0+\sum_{i=0}^{j-1}s_{(n-2)+i}})\eta\\
&\ \ \vdots\\
d(\varphi^t(\varphi^{(n-1)t_0+\sum_{i=0}^{n-2}s_{n-i}}(y_n)),\varphi^t(x_0)) &\le (\lambda^{t_0-t}+\lambda^t+\sum_{j=1}^{n-2}\lambda^{t+jt_0+\sum_{i=0}^{j-1}s_{2+i}})\eta.
\end{aligned}
\tag{3.15}
$$

Note that on the right side of (3.15) appear sums of the following type:

$$
\sum_{j=1}^{m}\lambda^{t+jt_0+\sum_{i=0}^{j-1}s_{(n-m)+i}},
$$

for $0<m<n-1$. This sum is same as in (3.3), except for an index change and where c_i are defined in function of s_i and by (3.4) we have

$$
\lambda^{t_0-t}+\lambda^t+\sum_{j=1}^{m}\lambda^{t+jt_0+\sum_{i=0}^{j-1}s_{(n-m)+i}}\le K.
$$

Hence all the summations that appear on the right side of (3.15) are less than or equal to $K\eta$ and this last by (3.5) is less than ε, this means

$$d(\varphi^t(y_n), \varphi^t(x_0)) \leq K\eta < \varepsilon, \tag{3.16}$$

and for all $j = 1, 2, ..., n-1$

$$d(\varphi^t(\varphi^{jt_0+\sum_{i=0}^{j-1} s_{n-i}}(y_n)), \varphi^t(x_0)) \leq K\eta < \varepsilon. \tag{3.17}$$

Now taking $t = 0$ in the first inequality in (3.15) we have $d(y_n, x_0) \leq \lambda^{t_0}\eta < \delta$ this implies $y_n \in B(x_0, \delta) \subset B[x_0, \delta] \subset B(x_0, \varepsilon)$, then passing to subsequence of $\{y_n\}_{n\geq 0}$ if necessary, there is $y \in B(x_0, \varepsilon)$ such that $\lim_{n\to+\infty} y_n = y$. Moreover, by (3.16) for all $n \in \mathbb{N}$, $d(\varphi^t(y_n), \varphi^t(x_0)) \leq \varepsilon$, for $0 \leq t \leq t_0$, by continuity of flow we have

$$d(\varphi^t(y), \varphi^t(x_0)) = \lim_{n\to+\infty} d(\varphi^t(y_n), \varphi^t(x_0)) \leq \varepsilon, \ \forall\, 0 \leq t \leq t_0.$$

As for all $n \in \mathbb{N}$, $|s_n| < \eta$, again passing to subsequence if necessary, there is $s \in \mathbb{R}$ such that $\lim_{n\to+\infty} s_n = s$ and $|s| \leq \eta$. Also as for all $n \in \mathbb{N}$, $\varphi^{t_0+s_n}(y_n) = z_n$, then by continuity of flow, $\lim_{n\to+\infty} \varphi^{t_0+s_n}(y_n) = \varphi^{t_0+s}(y)$. We denote by $z = \varphi^{t_0+s}(y)$; by (3.17) for all $n \in \mathbb{N}$, $d(\varphi^t(z_n), \varphi^t(x_0)) \leq \varepsilon$ then

$$d(\varphi^t(\varphi^{t_0+s}(y)), \varphi^t(x_0)) = d(\varphi^t(z), \varphi^t(x_0)) = \lim_{n\to+\infty} d(\varphi^t(z_n), \varphi^t(x_0)) \leq \varepsilon, \ \forall\, 0 \leq t \leq t_0;$$

continuing like this way, by (3.17), for all $k \in \mathbb{N}$ we get

$$d(\varphi^t(\varphi^{k(t_0+s)}(y)), \varphi^t(x_0)) \leq \varepsilon, \ \forall\, 0 \leq t \leq t_0. \tag{3.18}$$

In summary, by (3.18) we have $y \in B(x_0, \varepsilon)$ and such that $O^+(y)$ ε-shadowed forward by piecewise the orbit segment

$$\varphi^{[0,t_0]}(x_0) = \{\varphi^t(x_0) : \ \ 0 \leq t \leq t_0\},$$

with transitions times of size $|s| \leq \eta$. Moreover, analogously to the construction of $y \in SM$, as $O^+(y) = \{\varphi^t(y) : \ \ t \geq 0\}$ ε-shadowed forward by piecewise the orbit segment $\varphi^{[0,t_0]}(x_0)$, we can find $\hat{y} \in B(x_0, \varepsilon)$ such that $O^-(\hat{y})$ ε-shadowed backward by piecewise the orbit segment $\varphi^{[0,t_0]}(x_0)$, and hence we can find $\tilde{y} \in B(x_0, \varepsilon)$ such that $O(\tilde{y})$ ε-shadowed by piecewise the orbit segment $\varphi^{[0,t_0]}(x_0)$.

$\square$

The Proposition 3.12 and the local product structure still hold in the finite volume case (Theorem 2.10), allows us to conclude that for Anosov geodesic flow of a complete Riemannian manifold of finite volume, the periodic orbits are dense over the unit tangent bundle.

Proof of the Theorem 1.1. As $Vol(M) < \infty$ then $\Omega(\varphi) = SM$ (see [P]). Now we consider the recurrent points set

$$Rec(\varphi) = \{x \in SM : x \in \omega(x)\}.$$

By hypothesis we also know that the Liouville measure is φ-invariant, thus Liouville almost every point on SM is recurrent and $\overline{Rec(\varphi)} = SM$.

Now let $x_0 \in SM$, if $x_0 \in SM \backslash Rec(\varphi)$ then we can take $x_1 \in Rec(\varphi)$ sufficiently close to x_0. Thus we can suppose that $x_0 \in Rec(\varphi)$. Given $\varepsilon > 0$, on the proof of Proposition 3.12 we obtain $\delta > 0$ and $\eta > 0$ satisfying

$$\forall\, x, y \in B(x_0, \delta) : \ W^{cu}_{\eta}(x) \pitchfork W^{ss}_{\eta}(y) = \{w\}. \tag{3.19}$$

Moreover $\delta > 0$ and $\eta > 0$ can be choose satisfying (3.5). Now taking $l \in \mathbb{N}$ such that $\frac{\eta}{l} < \delta$ then for $\frac{\eta}{2l} > 0$ by Proposition 3.12 there is $y \in B(x_0, \frac{\eta}{2l}) \subset B(x_0, \varepsilon)$ such that $O(y)$ $\frac{\eta}{2l}$-shadowed piecewise the orbit arc $\varphi_{[0,t_0]}(x_0)$ for some $t_0 > 0$, this mean that for some $s \in \mathbb{R}$ we have:

$$\forall\, j \in \mathbb{Z} : \ d(\varphi^t(\varphi^{j(t_0+s)}(y)), \varphi^t(x_0)) \leq \frac{\eta}{2l}, \ \forall\, 0 \leq t \leq t_0. \tag{3.20}$$

On the other hand, if there is $\tilde{y}$ such that $O(\tilde{y})$ $\frac{\eta}{2l}$-shadowed piecewise the arc orbit $\varphi^{[0,t_0]}(x_0)$ (this is, $O(\tilde{y})$ sitisfy (3.20)) then

$$\forall\, j \in \mathbb{Z} : \ d(\varphi^t(\varphi^{j(t_0+s)}(\tilde{y})), \varphi^t(x_0)) \leq \frac{\eta}{2l}, \ \forall\, 0 \leq t \leq t_0; \tag{3.21}$$

this implies that for all $j \in \mathbb{Z}$ and $0 \leq t \leq t_0$ holds

$$\begin{aligned} d(\varphi^t(\varphi^{j(t_0+s)}(y)), \varphi^t(\varphi^{j(t_0+s)}(\tilde{y}))) &\leq d(\varphi^t(\varphi^{j(t_0+s)}(y)), \varphi^t(x_0)) + d(\varphi^t(\varphi^{j(t_0+s)}(\tilde{y})), \varphi^t(x_0)) \\ &\leq \frac{\eta}{2l} + \frac{\eta}{2l} = \frac{\eta}{l}; \end{aligned} \tag{3.22}$$

if in addition holds (suppose that $s > 0$)

$$\forall\, j \in \mathbb{Z}, \ d(\varphi^{j(t_0+r)}(\tilde{y}), \varphi^{j(t_0+r)}(y)) \leq \frac{\eta}{l}, \ \forall\, r \in (0, s) \tag{3.23}$$

then by (3.22) and (3.23) we obtain

$$d(\varphi^t(\tilde{y}), \varphi^t(y)) \leq \frac{\eta}{l} \leq \eta, \ \forall\, t \in \mathbb{R}, \tag{3.24}$$

and this last equation implies

$$\begin{aligned} \forall\, t \geq 0, \ d(\varphi^t(\tilde{y}), \varphi^t(y)) &\leq \eta \implies \tilde{y} \in W^{ss}_{\eta}(y), \\ \forall\, t \geq 0, \ d(\varphi^{-t}(\tilde{y}), \varphi^{-t}(y)) &\leq \eta \implies \tilde{y} \in W^{uu}_{\eta}(y) \subset W^{cu}_{\eta}(y), \end{aligned}$$

then $\tilde{y} \in W^{ss}_{\eta}(y) \pitchfork W^{cu}_{\eta}(y)$, but as $y \in B(x_0, \frac{\eta}{l}) \subset B(x_0, \delta)$ then by (3.19) $W^{ss}_{\eta}(y) \pitchfork W^{cu}_{\eta}(y) = \{y\}$, thus $\tilde{y} = y$.

Now consider the point $\tilde{y} = \varphi^{t_0+s}(y)$, then this point satisfies (3.23), hence for the above we obtain $y = \tilde{y} = \varphi^{t_0+s}(y)$, this is $O(y)$ is a periodic orbit such that $y \in B(x_0, \varepsilon)$.

$\square$

Chapter 4

Conjugacy and Rigidity

In this section we will see some aspects related between the conjugacy and rigidity of the two Geodesic flows defined over two complete Riemannian manifolds. Specifically, we want to answer the following question:

Question 4.1. *When two geodesic flows defined in manifolds of the same dimension admit a certain type o conjugacy?. (*1*-conjugacy)*

4.1 Conjugacy Rigid, Conjugacy and Equivalence

In this section, we will see the definitions of conjugacy and equivalence between two flows. We will also see the definition of conjugacy rigid.

Definition 4.2. *If M_1 and M_2 are two Riemannian manifold without boundary, a map $F : SM_1 \to SM_2$ between the unit tangent bundles is called a C^k conjugacy between the geodesic flows if it is a C^k diffeomorphism and $\varphi_2^t \circ F = F \circ \varphi_1^t$ where φ_1^t and φ_2^t are the geodesic flows on SM_1 and SM_2 respectively. $\mathcal{F}^k U(M)$ will refer to the C^k conjugacy class of the geodesic flow on M. That is, $M_1 \in \mathcal{F}^k U(M)$ if there is a C^k conjugacy between the geodesic flows of M_1 and M. A manifold M is called C^k **conjugacy rigid** if it is isometric to all spaces $M_1 \in \mathcal{F}^k U(M)$.*

Some examples of C^∞ conjugacy rigid manifolds are given by the real projectives spaces $\mathbb{RP}^n$ with the standard metric, compact surfaces of nonpositive curvature, compact flat manifolds and compact locally symmetric spaces of negative curvature. (see [CLUV])

Some properties are preserved under C^k conjugacies. For example, C^1 conjugacies always preserve the volume (see [C-K]). It is still not known if C^0 conjugacies do, even in the negative curvature setting. It is also known that the property of **no conjugate points** is preserved under C^0 conjugacies. (see [CK2])

Now in the nonpositive curvature setting, we have that conjugacy rigid in two-dimensional manifolds. Some works was done by Otal [Ot1], [Ot2] and [Cr] under a negative curvature assumption and was extended to the nonpositive case (using the method of Otal) in [CFF].

The best statement in the 2 dimensional case can be found in [CLUV] and the statement is follow:

Theorem 4.3. [**Croke**] *Every 2-dimensional compact Riemannian manifold of nonpositively curvature is C^0 conjugacy rigid.*

One important observation in all results above is that the Gauss Bonnet Theorem is used in some stages of the proofs, which seems to be why the arguments have not extended to higher dimensions.

Now let $(M, g_{_M})$ and $(N, g_{_N})$ be two complete Riemannian manifolds and $\varphi^t_M : SM \to SM$ and $\varphi^t_N : SN \to SN$ the geodesic flows of M and N, respectively.

Definition 4.4. *The flows φ^t_M and φ^t_N are said **orbit equivalents** if there exists a continuous map $h : SM \to SN$ and a real function $f : \mathbb{R} \to \mathbb{R}$ such that $h \circ \varphi^t_M = \varphi^{f(t)}_N \circ h$ for all $t \in \mathbb{R}$. The function h is called an "**equivalence in orbits**" between φ^t_M and φ^t_N and f a reparametrization. When h is a homeomorphism it is called a "**conjugacy in orbits**". We say that h is a $\alpha-$**equivalence** ($\alpha-$**conjugacy**) in orbits between φ^t_M and φ^t_N if h is a equivalence (conjugacy) in orbits and there are two constants $C_1 > 0$ and $C_2 > 0$ such that for all θ_1, $\theta_2 \in SM$*

$$C_1 d_M(\theta_1, \theta_2)^\alpha \leq d_N(h(\theta_1), h(\theta_2)) \leq C_2 d_M(\theta_1, \theta_2)^\alpha. \tag{4.1}$$

Note that 1-equivalences are actually bi-Lipschitz equivalences.

4.2 Conjugacy and Rigidity

In [MR, Corollary 1.4] was proved the following result:

Theorem 4.5. [**Melo-Romaña**] *Let M and N be two compact Riemannian manifolds of the same dimension. Assume that $\inf K_M \geq \sup K_N \geq -b^2$ and that geodesic flow φ^t_M is Anosov. If h is a 1-conjugacy in orbits between φ^t_M and φ^t_N with reparametrization $f(t) \geq t$ for $t \geq 0$ and odd, then $K_M = \sup K_N = K_N$.*

Next we prove the Theorem 1.2, which improve the above theorem.

Proof of the Theorem 1.2. Since that M has no conjugate points and the hypothesis on the sectional curvatures, we obtain that $\sup K_N \leq 0$. Thus we consider two cases:

Case 1. $\sup K_N < 0$.

Case 2. $\sup K_N = 0$.

Case 1. Put $-b^2 = \sup K_N < 0$, then φ_N^t is Anosov and we denote by E^s and E^u the stable and unstable bundle on SN. By hypothesis $h \circ \varphi_M^t = \varphi_N^{f(t)} \circ h$ where h is $1-$conjugacy then by Definition 4.4, there exist two constants $C_1 > 0$ and $C_2 > 0$ such that:

$$C_1 d_M(\varphi_M^t(\theta_1), \varphi_M^t(\theta_2)) \leq d_N(h(\varphi_M^t(\theta_1)), h(\varphi_M^t(\theta_2))) \leq C_2 d_M(\varphi_M^t(\theta_1), \varphi_M^t(\theta_2)),$$

hence

$$C_1 d_M(\varphi_M^t(\theta_1), \varphi_M^t(\theta_2)) \leq d_N(\varphi_N^{f(t)}(h(\theta_1)), \varphi_N^{f(t)}(h(\theta_2))) \leq C_2 d_M(\varphi_M^t(\theta_1), \varphi_M^t(\theta_2)). \tag{4.2}$$

Let Γ the set of points of SM where there exists $d_\theta h$, then as h is a Lipschitz map then Γ has full Liouville measure. Let $\theta \in \Gamma$, $\xi \in T_\theta SM$ and $\beta(r) \subset SM$ a curve differentiable such that $\beta(0) = \theta$ and $\beta'(0) = \xi$ by Lemma 4.1 in [MR] for all $t \in \mathbb{R}$, we have

$$\lim_{s \to 0} \frac{d_M(\varphi_M^t(\beta(s)), \varphi_M^t(\beta(0)))}{s} = \|d_\theta \varphi^t(\xi)\|$$

and

$$\lim_{s \to 0} \frac{d_N(\varphi_N^{f(t)}(h(\beta(s))), \varphi_N^{f(t)}(h(\beta(0))))}{s} = \|d_{h(\theta)} \varphi_N^{f(t)}(d_\theta h(\xi))\|.$$

The last two equalities above and the equation (4.2) imply that for all $t \in \mathbb{R}$,

$$C_1 \|d_\theta \varphi_M^t(\xi)\| \leq \|d_{h(\theta)} \varphi_N^{f(t)}(d_\theta h(\xi))\| \leq C_2 \|d_\theta \varphi_M^t(\xi)\|. \tag{4.3}$$

In particular for $t = 0$ above we obtain that

$$C_1 \|\xi\| \leq \|d_\theta h(\xi)\| \leq C_2 \|\xi\|$$

and this last inequality implies that $d_\theta h$ is an isomorphism. For $0 \neq \xi \in T_\theta SM$ the equation (4.3) implies

$$C_1 \frac{\|d_\theta \varphi_M^t(\xi)\|}{\|\xi\|} \leq \frac{\|d_{h(\theta)} \varphi_N^{f(t)}(d_\theta h(\xi))\|}{\|d_\theta h(\xi)\|} . \frac{\|d_\theta h(\xi)\|}{\|\xi\|} \leq C_2 \frac{\|d_\theta \varphi_M^t(\xi)\|}{\|\xi\|}. \tag{4.4}$$

Now as $d_\theta h$ is an isomorphism we can define the subspaces $F_\theta^{s(u)}$ of $T_\theta SM$ satisfying the equation

$$d_\theta h(F_\theta^{s(u)}) = E_{h(\theta)}^{s(u)}.$$

Note that Γ is φ_M^t-invariant and

$$d_{\varphi_M^t(\theta)} h = d_{h(\theta)} \varphi_N^{f(t)} \circ d_\theta h \circ (d_\theta \varphi_M^t)^{-1},$$

which implies that the subspaces $F_\theta^{s(u)}$ are $d\varphi_M^t-$invariants. Thus, for all $t \in \mathbb{R}$

$$T_{\varphi_M^t(\theta)}(SM) = F^s_{\varphi_M^t(\theta)} \oplus \langle \varphi_M \rangle \oplus F^u_{\varphi_M^t(\theta)}. \tag{4.5}$$

Now for $\xi \in F_\theta^u$ by (4.4) for all $t \in \mathbb{R}$ we have

$$\frac{C_1}{C_2}.\|d_\theta \varphi_M^t \mid_{F_\theta^u} \| \leq \|d_{h(\theta)} \varphi_N^{f(t)} \mid_{E^u_{h(\theta)}} \| \leq \frac{C_2}{C_1}.\|d_\theta \varphi_M^t \mid_{F_\theta^u} \|. \tag{4.6}$$

Analogously, for $\xi \in F_\theta^s$ by (4.4) we obtain

$$\frac{C_1}{C_2}.\|d_\theta \varphi_M^t \mid_{F_\theta^s} \| \leq \|d_{h(\theta)} \varphi_N^{f(t)} \mid_{E^s_{h(\theta)}} \| \leq \frac{C_2}{C_1}.\|d_\theta \varphi_M^t \mid_{F_\theta^s} \|. \tag{4.7}$$

By classical result (see [Kn]) $e^{-\sqrt{-\sup K_N}} = e^{-b}$ is a constant of contraction for φ_N^t, then the equations (4.6) and (4.7) and the hypothesis on the reparametrization $f(t)$ be odd, provide that for all $t \geq 0$

$$\|d_\theta \varphi_M^t \mid_{F_\theta^s} \| \leq C\frac{C_2}{C_1}e^{-abt} \ \textit{and} \ \|d_\theta \varphi_M^{-t} \mid_{F_\theta^u} \| \leq C\frac{C_2}{C_1}e^{-abt}. \tag{4.8}$$

The last inequalities and (4.5) provide a hyperbolic behavior of φ_M^t along of the orbit of θ , therefore as M has no conjugate points and $K_M \geq a^2 \sup K_N = -a^2b^2 = -(ab)^2$ then the by similar arguments used in the proof of the Theorem 1.1 in [MR], provides that for all $\theta \in \Gamma$

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t Ric(\varphi_M^s(\theta))ds = -(ab)^2. \tag{4.9}$$

Since Γ has full Liouville measure then the Birkhoff ergodic theorem and (4.9) give us

$$\int_{SM} Ric(\theta) d\mathcal{L}_M = -(ab)^2,$$

where $\mathcal{L}_M$ denotes the Liouville measure on SM.

Since $K_M \geq -(ab)^2$ then the last equality implies that $Ric(\theta) = -(ab)^2$ for $\mathcal{L}_M - a.e.$ point $\theta \in SM$. Thus we conclude that $K_M \equiv -(ab)^2$ and therefore the splitting given by (4.5) coincide with its hyperbolic splitting.

To conclude the proof, we will show that $K_N \equiv -b^2$. For this sake, since $K_M \equiv -(ab)^2$ for $\xi \in F_\theta^u$ (cf. [Kn]) we have that

$$\|d_\theta \varphi_M^t(\xi)\| = \sqrt{1+(ab)^2}e^{abt}\|\pi_1(\xi)\|,$$

where $\pi_1(\cdot)$ is the projection on the first coordinate in the horizontal and vertical decomposition of TSM. Thus

$$\lim_{t\to+\infty} \frac{1}{t}\log \|d_\theta \varphi_M^t(\xi)\| = ab,$$

uniformly in bounded regions of pair (θ, ξ) with $\theta \in SM$ and $\xi \in F^u_\theta$. Put $\Lambda = h(\Gamma)$, then as h is Lipschitz and Γ has full Liouville measure on SM, then Λ has full Liouville measure on SN.

So, by (4.3) we obtain

$$\lim_{t\to+\infty} \frac{1}{t} \log \|d_{h(\theta)}\varphi_N^{f(t)}(d_\theta h(\xi))\| = ab. \tag{4.10}$$

Claim 4.6. $\lim_{t\to+\infty} \frac{1}{t} \log \|d_w\varphi_N^t(\eta)\| = b$, *where* $w = h(\theta)$ *and* $\eta = d_\theta h(\xi)$.

Proof of the Claim 4.6. As $\sup K_N = -b^2 < 0$ then we know that the function $t \longmapsto \|J_u(t)\|^2$ is strictly increasing (where $J_u(t)$ is the unstable Jacobi field associated to η), this implies that the function $t \longmapsto \|J_u(t)\|$ is also strictly incrasing. Since the reparametrization $f(t)$ satisfies $f(t) \geq at$ for all $t \geq 0$ then

$$\|J_u(f(t))\| \geq \|J_u(at)\|,$$

for all $t \geq 0$, thus

$$\begin{aligned} ab &= \lim_{t\to+\infty} \frac{1}{t} \log \|d_w\varphi_N^{f(t)}(\eta)\| \\ &= \lim_{t\to+\infty} \frac{1}{t} \log \|J_u(f(t))\| \\ &\geq \lim_{t\to+\infty} \frac{1}{t} \log \|J_u(at)\| \\ &= \lim_{t\to+\infty} \frac{1}{t} \log \|d_w\varphi_N^{at}(\eta)\|. \end{aligned} \tag{4.11}$$

On the other hand, as $\eta \in E^u_w$ then for all $t \geq 0$

$$\|d_w\varphi_N^t(\eta)\| \geq C_3 e^{bt}\|\eta\|.$$

Thus,

$$\lim_{t\to+\infty} \frac{1}{t} \log \|d_w\varphi_N^{at}(\eta)\| \geq ab,$$

this last inequality and (4.11) imply that

$$\lim_{t\to+\infty} \frac{1}{t} \log \|d_w\varphi_N^{at}(\eta)\| = ab.$$

Hence

$$\lim_{t\to+\infty} \frac{1}{at} \log \|d_w\varphi_N^{at}(\eta)\| = b.$$

$\square$

Then by Claim.4.6 we have

$$\lim_{t\to+\infty}\frac{1}{t}\log\|d_w\varphi_N^t(\eta)\| = b$$

uniformly in bounded regions of pair (w,η) with $w\in\Lambda$ and $\eta\in E_w^u$.

Now, to conclude, we follow similar arguments of the proof of [MR, Theorem 1.1]

Claim 4.7. *For $w\in SN$ and $\eta\in E_w^u$ hold that*

$$\lim_{t\to+\infty}\frac{1}{t}\log\|d_w\varphi_N^t(\eta)\| = b.$$

Proof of the Claim 4.7. Let $w\in SN$ be and $\eta\in E_w^u$. By density of Λ and continuity of E^u, we have that there are $w_m\in\Lambda$ and $\eta_m\in E_{w_m}^u$ such that (w_m,η_m) converge to (w,η). On the other hand, for $t\in\mathbb{R}$ holds:

$$\left|b-\frac{1}{t}\log\|d_w\varphi_N^t(\eta)\|\right| \le \left|b-\frac{1}{t}\log\|d_{w_m}\varphi_t(\eta_m)\|\right| + \left|\frac{1}{t}\log\frac{\|d_{w_m}(\varphi^t(\eta_m))\|}{\|d_w\varphi^t(\eta)\|}\right|$$

By the uniformity in the convergence in the Claim 4.6, given $\varepsilon>0$ there is t_0 such that:

$$\left|b-\frac{1}{t}\log\|d_{w_m}\varphi_t(\eta_m)\|\right| < \frac{\varepsilon}{2},$$

for each $t\ge t_0$ and all $m\in\mathbb{N}$. Also by continuity of $d\varphi_N^t$, for each $t\ge t_0$ there is $m(t)$ such that

$$\left|\frac{1}{t}\log\frac{\|d_{w_m}(\varphi^t(\eta_m))\|}{\|d_w\varphi^t(\eta)\|}\right| < \frac{\varepsilon}{2},$$

for each $m\ge m(t)$. Hence for each $t\ge t_0$ holds:

$$\left|b-\frac{1}{t}\log\|d_w\varphi_N^t(\eta)\|\right| < \varepsilon.$$

□

Finally by the Claim.4.7 above and Theorem 1.1 at [Bu] imply that $K_N\equiv -b^2$, which concluded the proof of Case 1.

Case 2. Assume that $\sup K_N\equiv 0$, then as $\inf K_M\ge a^2\sup K_N = 0$ and by hipothesis M has no conjugate points then $K_M\equiv 0$ (cf. [G]). We will show that $K_N\equiv 0$. For this sake, note that the condition on the curvature on M implies that

$$\lim_{t\to+\infty}\frac{1}{t}\log\|d_\theta\varphi_M^t(\xi)\| = 0$$

uniformly in bounded regions of pair (θ, ξ) with $\theta \in SM$ and $\xi \in T_\theta SM$. We can conclude by similar arguments as **Case 1**, that

$$\lim_{t\to+\infty} \frac{1}{t} \log \|d_w \varphi_N^{f(t)}(\eta)\| = 0$$

for each pair (w, η) with $w \in SN$ and $\eta \in T_w SN$.

As $\sup K_N \equiv 0$ then N has no conjugate points and as N is compact, there is a constant $c \geq 0$ such that $-c^2 \leq K_N \leq 0$.

Now for each $w = (y, v) \in SN$ we denote $N(w)$ the subspace of $T_y N$ orthogonal to v. Then by construction, for all $x \in N(w)$ the Jacobi field $Y_w^u(t)x$ is a unstable Jacobi field (where $Y_w^u(t)$ is a unstable Jacobi tensor). We know that

$$E^u_{\varphi_N^t(w)} = graph(U_w^u(t)) = \{(x, U_w^u(t)x) : x \in N(w)\},$$

where $U_w^u(t) = (Y_w^u(t))'(Y_w^u(t))^{-1}$ is the unstable Riccati tensor.

Let $\pi_w : E_w^u \to N(w)$ the projection in the first coordinate. Then

$$\pi_w^{-1}(v) = (v, U_w^u(0)v).$$

Note that for all $w \in SN$ hold

$$\|\pi_w\| \leq 1 \ \ and \ \ 1 \leq \|\pi_w^{-1}\| \leq \sqrt{1 + c^2} \tag{4.12}$$

and

$$d_w \varphi_N^t \mid_{E_w^u} = \pi^{-1}_{\varphi_N^t(w)} \circ Y_w^u(t) \circ \pi_w. \tag{4.13}$$

Thus (4.12) and (4.13) imply that for all $\eta \in E_w^u$ hold

$$0 = \lim_{t\to+\infty} \frac{1}{t} \log \|d_w \varphi_N^{f(t)}(\eta)\| \leq \limsup_{t\to+\infty} \frac{1}{t} \|Y_w^u(f(t))(\pi_w(\eta))\|. \tag{4.14}$$

On the other hand, by (4.13) we also have

$$\|Y_w^u(f(t))(\pi_w(\eta))\| = \|\pi_{\varphi_N^{f(t)}(w)}(d_w \varphi_N^{f(t)}(\eta))\| \leq \|\pi_{\varphi_N^{f(t)}(w)}\| \cdot \|d_w \varphi_N^{f(t)}(\eta)\|.$$

Again by (4.12) we obtain

$$\limsup_{t\to+\infty} \frac{1}{t} \|Y_w^u(f(t))(\pi_w(\eta))\| \leq \lim_{t\to+\infty} \frac{1}{t} \log \|d_w \varphi_N^{f(t)}(\eta)\| = 0 \tag{4.15}$$

Thus of (4.14) and (4.15) we conclude

$$\limsup_{t\to+\infty} \frac{1}{t}\|Y_w^u(f(t))(\pi_w(\eta))\| = 0 \tag{4.16}$$

On the other hand, as $K_N \leq 0$ then the function $t \longmapsto \|Y_w^u(t)(\pi_w(\eta))\|$ is increasing. As by hipothesis $f(t) \geq at$ for all $t \geq 0$, then

$$\|Y_w^u(f(t))(\pi_w(\eta))\| \geq \|Y_w^u(at)(\pi_w(\eta))\|.$$

Thus, the last inequality with (4.12), (4.13) and (4.16) give us

$$\limsup_{t\to+\infty} \frac{1}{t}\log\|d_w\varphi_N^{at}(\eta)\| \leq \limsup_{t\to+\infty} \frac{1}{t}\|Y_w^u(f(t))(\pi_w(\eta))\| = 0. \tag{4.17}$$

Claim 4.8. *For each $w \in SN$ hold that*

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t Ric(\varphi_N^{as}(w))ds = 0.$$

Proof of the Claim 4.8. By contradiction, assume that exist $w \in SN$ such that:

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t Ric(\varphi_N^{as}(w))ds = B < 0. \tag{4.18}$$

As the function $t \longmapsto \|Y_w^u(t)x\|^2$ is increasing, for $x \in \mathbb{R}^{n-1}$, then

$$\langle y, U_w^u(t)y\rangle \geq 0, \quad for\ \ y \in \mathbb{R}^{n-1}.$$

Since $U_w^u(t)$ is symmetric, the last equation implies that all eigenvalues of $U_w^u(t)$ are non-negative. Let $\lambda_1(t) \geq \ldots \geq \lambda_{n-1}(t) \geq 0$ the eigenvalues of $U_w^u(t)$. We have that $|U_w^u(t)| \leq c$, then

$$\begin{aligned} tr(U_w^u(t))^2 &= \lambda_1^2(t) + \lambda_2^2(t) + \ldots + \lambda_{n-1}^2(t) \\ &\leq c \cdot trU_w^u(t) \end{aligned} \tag{4.19}$$

Taking trace in the equation (2.3) and integrating we obtain:

$$\begin{aligned} 0 &= \frac{1}{t}\int_0^t tr(U_w^u)'(r)dr + \frac{1}{t}\int_0^t tr(U_w^u(r))^2dr + \frac{1}{t}\int_0^t trR(r)dr \\ &= \frac{trU_w^u(t) - trU_w^u(0)}{t} + \frac{1}{t}\int_0^t tr(U_w^u(r))^2dr + \frac{1}{t}\int_0^t trR(r)dr. \end{aligned} \tag{4.20}$$

Remember that $R(t) = (R_{ij}(t))$ is a matrix, where $R_{ij}(t) = \langle R(\gamma_\theta'(t), V_i(t))\gamma_\theta'(t), V_j(t)\rangle$ and R is the curvature tensor.

By our hypothesis (equation 4.18) we have that there exists $t_0 > 0$ such that for $t \geq t_0$ holds

$$\frac{1}{t}\int_0^t trR(r)dr < (n-1)B, \quad t \geq t_0. \tag{4.21}$$

From (4.19), (4.20) and (4.21) there is $t_1 > 0$ such that for $t \geq t_1$:

$$\begin{aligned} B(n-1) &> \frac{1}{t}\int_0^t trR(r)dr \\ &\geq -\frac{1}{t}\int_0^t tr(U_w^u(r))^2 dr \\ &\geq -\frac{c}{t}\int_0^t trU_w^u(r)dr. \end{aligned} \tag{4.22}$$

Remember the Liouville's formula (see [MF]), which state that:

$$\frac{d}{dt}\log|\det Y_w^u(r)| = trU_w^u(r), \quad Y_w^u(0) = Id. \tag{4.23}$$

Integrating (4.23) and using (4.22) we have that:

$$\log|\det Y_w^u(t)| \geq -\frac{1}{c}(n-1)Bt, \quad t \geq t_1. \tag{4.24}$$

Finally, the Proposition 4.9 below implies that

$$\limsup_{t\to+\infty}\frac{1}{t}\log|\det d\varphi_N^{at}\big|_{E_w^u}| \geq -\frac{B(n-1)}{2c} > 0.$$

As $|\det d\varphi_N^t|_{E_w^u}| \leq \| d\varphi_N^t|_{E_w^u}\|^{\dim(E_w^u)}$ and $\dim(E_w^u) = \dim(N) - 1$, then the last inequality provides that

$$\limsup_{t\to+\infty}\frac{1}{t}\log\| d\varphi_N^{at}\big|_{E_w^u}\| \geq -\frac{B}{2c} > 0,$$

which is a contradiction to (4.17). $\square$

Thus by using a change of variable on Claim. 4.8 we obtain

$$\limsup_{t\to+\infty}\frac{1}{at}\int_0^{at} Ric(\varphi_N^s(w))ds = 0.$$

Hence

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^{t} Ric(\varphi_N^s(w))ds = 0. \tag{4.25}$$

Now the Birkhoff ergodic theorem implies that for Liouville almost every point $w \in SN$

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t Ric(\varphi_N^s(w))ds = \int_{SN} Ric(w)d\mathcal{L}_N.$$

Thus (4.25) implies that

$$\int_{SN} Ric(w)d\mathcal{L}_N = 0.$$

Since $Ric(\cdot)$ is a continuous function and $-c^2 \leq Ric(w) \leq 0$ then should be $Ric(\cdot) \equiv 0$. Also, since $-c^2 \leq K_N \leq 0$ and the definition of $Ric(\cdot)$ we have $K_N \equiv 0$. □

Proposition 4.9. *In the same conditions of the* **Case 2**, *there exists* $t_2 > 0$ *such that for each* $w \in SN$ *we have that:*

$$\frac{1}{t}\log|\det d\varphi_N^t\big|_{E_w^u}| \geq -\frac{B(n-1)}{2c}, \quad t \geq t_2.$$

Proof. The projection $\pi_w : E_w^u \to N(w)$ satisfies:

$$1 \leq |det\pi^{-1}_{\varphi_N^t(w)}| \leq (1+c^2)^{\frac{n-2}{2}}. \tag{4.26}$$

By hypothesis there is $t_2 > 0$ such that for $t \geq t_2$, using the equation (4.13) and remember that $|\det \pi_w| = |\det \pi_w^{-1}|^{-1} \geq (1+c^2)^{-\frac{n-2}{2}}$, we have that:

$$\begin{aligned}\frac{1}{t}\log|\det d\varphi_N^t\big|_{E_w^u}| &= \frac{1}{t}\log|\det\pi^{-1}_{\varphi_N^t(w)}| + \frac{1}{t}\log|\det Y_w^u(t)| + \frac{1}{t}\log|\det\pi_w| \\ &\geq -\frac{(n-1)B}{c} + \frac{(1+c^2)^{-\frac{n-2}{2}}}{t} \\ &\geq -\frac{(n-1)B}{2c}\end{aligned}$$

□

The Theorem 1.2 above gives us a rigidity between the sectional curvatures of the two compact Riemannian manifolds when we have some relation over the sectional curvatures and the existence of a 1-conjugacy between the geodesic flows of them. In fact, on the conditions above, if there exist a 1-conjugacy between the geodesic flows then we have that the sectional curvatures are constants, in general, could be different up to a positive constant $a > 0$ that is related with the parametrization of the conjugacy h.

Another observation on the Theorem 1.2 is that in the relationship over the sectional curvatures $\inf K_M \geq a^2 \sup K_N$, the sectional curvature of M could be zero in some regions.

Now we could ask what happens if we change the condition of the sectional curvatures by $\inf K_M \leq a^2 \sup K_N$, the Theorem 4.12 in the next section below gives us an answer in the case that $\sup K_N < 0$ and some condition on the reparametrization of the conjugacy.

4.3 Conjugacy between certain types of manifolds

In this section, first, we will study some relation between the geodesic flows defined on the same manifold but with different metrics. Specifically, we will be interested in conformal metrics with a positive constant. After we will see how the existence of a certain type between geodesic flows of two manifolds of the same dimension implies rigidity in the sectional curvatures.

Let (M, g) be a Riemannian manifold and consider a family of conformal metrics $g_r = e^{2r}g$, with $r \in \mathbb{R}$, on M. It is easy to see that $\nabla_r = \nabla$ where ∇_r and ∇ are the conections compatible with g_r and g, respectively. This implies $R_r = R$, where R_r and R are the curvature tensors of M_r and M respectively. Hence $g_r(R_r(X_i, X_j)X_k, X_s) = e^{2r}g(R(X_i, X_j)X_k, X_s)$, where X_i, X_j, X_k, and X_s are the coordinates fields around a neighborhood, thus the sectional curvatures K_r and K of M_r and M, respectively are equal up to a constant. More specifically, for all $x \in M$ and $v, w \in T_xM$ linearly independent we have:

$$K_r(v, w) = e^{-2r}K(v, w). \tag{4.27}$$

Since $\nabla_r = \nabla$, we can see that the geodesics of (M, g_r) and (M, g) are the same but they have different velocities, this means, if $\gamma(t)$ is a geodesic of (M, g_r), then $\|\gamma'(t)\|_r = e^r\|\gamma'(t)\|$, where $\|\cdot\|_r$ and $\|\cdot\|$ are the norms induced by the metrics g_r and g on M, respectively.

Now if $\gamma(t)$ is a geodesic on (M, g_r) and $V(t)$ is a vector field along $\gamma(t)$, as $\nabla_r = \nabla$ then $\dfrac{D_rV}{dt} = \dfrac{DV}{dt}$ where $\dfrac{D_r}{dt}$ and $\dfrac{D}{dt}$ are the covariant derivatives of (M, g_r) and (M, g), respectively. This last with $R_r = R$ implies that (M, g_r) and (M, g) has the same Jacobi fields but with differents norms, this means, that if $J(t)$ is a Jacobi field along of $\gamma(t)$ then $\|J(t)\|_r = e^r\|J(t)\|$. With this we can conclude the following:

Lemma 4.10. *(M, g_r) has no conjugate points if and only if (M, g) has no conjugate points.*

Proof. It is enough to note that $\|J(t)\|_r = e^r\|J(t)\|$. $\square$

Let (M, g) be a complete Riemannian manifold and for $r \in \mathbb{R}$ fixed we consider (M_r, g_r) where $g_r = e^{2r}g$. Let $\varphi^t_M : SM \to SM$ and $\varphi^t_{M_r} : SM_r \to SM_r$ be the geodesic flows of M and M_r respectively.

First, we want to see if there exists some conjugacy h as in Definition 4.4 above, between the geodesic flows induced by the metrics $\|\cdot\|_r$ and $\|\cdot\|$ on M.

Lemma 4.11. *Let (M, g) be a complete Riemannian manifold and consider $(M_r = M, g_r)$. Then there exists $h : SM \to SM_r$ 1-conjugacy in orbits between φ^t_M and $\varphi^t_{M_r}$.*

Proof. We define the function $h : SM \to SM_r$ by $h(x, v) = (x, e^{-r}v)$ and $f : \mathbb{R} \to \mathbb{R}$ define by $f(t) = e^rt$. Note that h is a C^1-diffeomorphism. Let $\theta = (x, v) \in SM$ and $\widetilde{\theta} = (x, e^{-r}v) =$

$h(\theta) \in SM_r$ and $\gamma_\theta(t)$ and $\gamma_{\widetilde{\theta}}(t)$ the geodesic satisfying $\gamma_\theta(0) = x$, $\gamma_\theta'(0) = v$, $\gamma_{\widetilde{\theta}}(0) = x$ and $\gamma_{\widetilde{\theta}}'(0) = e^{-r}v$. By definition of h and the geodesic flows φ_M^t and $\varphi_{M_r}^t$ we have that for all $t \in \mathbb{R}$

$$\begin{aligned} h \circ \varphi_M^{f^{-1}(t)}(\theta) &= h \circ \varphi_M^{e^{-r}t}(\theta) \\ &= h(\gamma_\theta(e^{-r}t), \gamma_\theta'(e^{-r}t)) \\ &= (\gamma_\theta(e^{-r}t), e^{-r}\gamma_\theta'(e^{-r}t)) \\ &= (\gamma_{\widetilde{\theta}}(t), \gamma_{\widetilde{\theta}}'(t)) \\ &= \varphi_{M_r}^t(x, e^{-r}t) \\ &= \varphi_{M_r}^t(\widetilde{\theta}) \\ &= \varphi_{M_r}^t \circ h(\theta). \end{aligned}$$

Hence for all $t \in \mathbb{R}$ holds $h \circ \varphi_M^t = \varphi_{M_r}^{f(t)} \circ h$.

□

Now we will see how the existence of a certain type of conjugacy between the geodesic flows of certain kinds of manifolds of the same dimension implies rigidity in the sectional curvatures.

Theorem 4.12. *Let M and N be two compact Riemannian manifolds with the same dimension. Fixed $a > 0$ and assume that the sectional curvatures satisfy $\inf K_M \leq a^2 \sup K_N < 0$ and M has no conjugate points. If h is a $1-$conjugacy in orbits between φ_M^t and φ_N^t with odd reparametrization $l(t)$ satisfying $l(t) \geq \sqrt{\frac{\inf K_M}{\sup K_N}}t$ for all $t \geq 0$ then K_M and K_N are constant.*

Proof. We consider the deformation $(M_s = M, g_s = e^{2s}g)$ of (M, g). As $\inf K_M$ and $\sup K_N$ has the same sign, then there exist $s_0 \geq 0$ such that $\inf K_{M_{s_0}} = a^2 \sup K_N$. As $K_{M_{s_0}} = e^{-2s_0}K_M$ then

$$e^{-2s_0} \inf K_M = \inf K_{M_{s_0}} = a^2 \sup K_N$$

and this implies

$$e^{-s_0} = a\sqrt{\frac{\sup K_N}{\inf K_M}}. \tag{4.28}$$

Now, by Lemma 4.11, we know that there is $h_1 : SM_{s_0} \to SM$ a $1-$conjugacy in orbits between $\varphi_{M_{s_0}}^t$ and φ_M^t, this means $h_1 \circ \varphi_{M_{s_0}}^t = \varphi_M^{f^{-1}(t)} \circ h_1$, where $f^{-1}(t) = e^{-s_0}t$.

On the other hand, by hipothesis, we have that $h \circ \varphi_M^t = \varphi_N^{l(t)} \circ h$. Thus we consider the composition $h \circ h_1 : SM_{s_0} \to SN$. Nothe that

$$\begin{aligned} h \circ h_1 \circ \varphi_{M_{s_0}}^t &= h \circ \varphi_M^{f^{-1}(t)} \circ h_1 \\ &= \varphi_N^{l(f^{-1}(t))} \circ h \circ h_1. \end{aligned}$$

Since h and h_1 are $1-$conjugacy in orbits, then the last equality above gives us that $h \circ h_1$ is a $1-$conjugacy in orbits between $\varphi^t_{M_{s_0}}$ and φ^t_N with the reparametrization $l(f^{-1}(t))$.

Now for all $t \geq 0$ we have $e^{-s_0}t \geq 0$ then (4.28) and the hypothesis in $l(t)$ implies

$$l(f^{-1}(t)) = l(e^{-s_0}t) \geq \sqrt{\frac{\inf K_M}{\sup K_N}} \cdot e^{-s_0}t = at.$$

As M has no conjugate points and $h \circ h_1$ satisfies the conditions of Theorem 1.2 then $K_{M_{s_0}} \equiv a^2 \sup K_N \equiv a^2 K_N$ and this implies that $K_M \equiv e^{2s_0}a^2 \sup K_N \equiv e^{2s_0}a^2 K_N$. $\square$

The Theorem 1.2 and 4.12 help us to know when two compact manifolds of the same dimension have a certain type of conjugacy in orbits and therefore answers the Question 4.1. For this sake, for two compacts Riemannian manifolds M and N we denote by $I_M = [\inf K_M, \sup K_M]$ and $I_N = [\inf K_N, \sup K_N]$ the intervals of sectional curvature of M and N, respectively. Then we have two cases or $I_M \cap I_N = \emptyset$ or $I_M \cap I_N \neq \emptyset$.

The following two results below (Corollary 4.13 and Corollary 4.14) give us some information about when a conjugacy does not exist between two compact manifolds of the same dimension under the hypothesis that one of them has no conjugate points and the other one has negative curvature and the intersection of the intervals of curvature are empty or not empty.

Corollary 4.13. *Let M and N be two compact Riemannian manifolds with the same dimension and I_M and I_N the intervals of curvature of M and N, respectively. Suppose that M has no conjugate points,* $\sup K_N < 0$ *and $I_M \cap I_N = \emptyset$. Then:*

(a) *If* $\inf K_M > \sup K_N$ *then there is no $h : SM \to SN$ a $1-$conjugacy in orbits between φ^t_M and φ^t_N (i.e. $h \circ \varphi^t_M = \varphi^{f(t)}_N \circ h$) with reparametrization odd $f(t)$ satisfying $f(t) \geq t$ for all $t \geq 0$.*

(b) *If* $\inf K_N > \sup K_M$ *then there is no $h : SN \to SM$ a $1-$conjugacy in orbits between φ^t_N and φ^t_M (i.e. $h \circ \varphi^t_N = \varphi^{f(t)}_M \circ h$) with reparametrization odd $f(t)$ satisfying $f(t) \geq t$ for all $t \geq 0$.*

Proof. **a)** If there is $h : SM \to SN$ a $1-$conjugacy in orbits between φ^t_M and φ^t_N (i.e. $h \circ \varphi^t_M = \varphi^{f(t)}_N \circ h$) with reparametrization odd $f(t)$ satisfying $f(t) \geq t$ for all $t \geq 0$, then by Theorem 1.2 with $a = 1$ implies that $K_M \equiv \sup K_N \equiv K_N$, hence $I_M \cap I_N \neq \emptyset$ contradiction!.

(b) Analogous to item **(a)**. $\square$

Now when $I_M \cap I_N \neq \emptyset$ we have two cases, or $I_M \cap I_N = \{k_0\}$(in this case we have I_M and I_N intersect at the endpoints or one of the curvature intervals is unitary) or $I_M \cap I_N = I$ where $I \subset \mathbb{R}$ is a non-degenerate interval.

Corollary 4.14. *Let M and N be two compact Riemannian manifolds with the same dimension and I_M and I_N the intervals of curvature of M and N respectively. Suppose that M has no conjugate points, $\sup K_N < 0$ and $I_M \cap I_N = \{k_0\}$. Then:*

(1) *I_M and I_N intersect at endpoints*

- **(a)** *If $\sup K_N = \inf K_M$ and there is $h : SM \to SN$ a $1-$conjugacy in orbits between φ_M^t and φ_N^t (i.e. $h \circ \varphi_M^t = \varphi_N^{f(t)} \circ h$) with reparametrization odd $f(t)$ satisfying $f(t) \geq t$ for all $t \geq 0$ then $K_M \equiv \sup K_N \equiv K_N = \{k_0\}$.*
- **(b)** *If $\sup K_M = \inf K_N$ and there is $h : SN \to SM$ a $1-$conjugacy in orbits between φ_N^t and φ_M^t (i.e. $h \circ \varphi_N^t = \varphi_M^{f(t)} \circ h$) with reparametrization odd $f(t)$ satisfying $f(t) \geq t$ for all $t \geq 0$ then $K_N \equiv \sup K_M \equiv K_M = \{k_0\}$.*

(2) *$I_M = \{k_0\}$ or $I_N = \{k_0\}$*

- **(a)** *If I_N is a non-degenerate interval and $I_M = \{k_0\} \subset int(I_N)$ then there is no $h : SM \to SN$ a $1-$conjugacy in orbits between φ_M^t and φ_N^t (i.e. $h \circ \varphi_M^t = \varphi_N^{f(t)} \circ h$) with reparametrization odd $f(t)$ satisfying $f(t) \geq \sqrt{\dfrac{\inf K_N}{\sup K_M}} \cdot t$, for all $t \geq 0$.*
- **(b)** *If I_M is a non-degenerate interval and $I_N = \{k_0\} \subset int(I_M)$ then there is no $h : SN \to SM$ a $1-$conjugacy in orbits between φ_N^t and φ_M^t (i.e. $h \circ \varphi_N^t = \varphi_M^{f(t)} \circ h$) with reparametrization odd $f(t)$ satisfying $f(t) \geq \sqrt{\dfrac{\inf K_M}{\sup K_N}} \cdot t$, for all $t \geq 0$.*

Proof. **(1), (a)** (the proof of item **(b)** is similar) Apply the Theorem 1.2 with $a = 1$.

(2), (a) (the proof of item **(b)** is similar) As $\{k_0\} = I_M \subset int(I_N)$ then $\inf K_M = \{k_0\} < \sup K_N < 0$. Now if there is $h : SM \to SN$ satisfying such conditions then by Theorem 4.12 with $a = 1$ implies that K_N is constant, thus I_N is unitary, contradiction! □

Corollary 4.15. *Let M and N be two compact Riemannian manifolds with the same dimension and I_M and I_N the intervals of curvature of M and N, respectively. Suppose tha M has no conjugate points, $\sup K_N < 0$ and $I_M \cap I_N = I$ where $I \subset \mathbb{R}$ is a non degenerate interval. If* $\inf K_M < \sup K_N$ *then there is no $h : SM \to SN$ a $1-$conjugacy in orbits between φ_M^t and φ_N^t (i.e. $h \circ \varphi_M^t = \varphi_N^{f(t)} \circ h$) with reparametrization odd $f(t)$ satisfying $f(t) \geq \sqrt{\dfrac{\inf K_M}{\sup K_N}} \cdot t$, for all $t \geq 0$.*

Proof. By Theorem 4.12 with $a = 1$ we can conclude that K_M and K_N are constants and this implies that I_M and I_N are unitaries, contradiction!. □

Looking at the proof of Theorem 1.2, **Case 1** more carefully, one can find that the hyperbolic behavior of φ_N^t is passed to that of φ_M^t by the $1-$conjugacy in orbits $h : SM \to SN$ at each point, where h is diferentiable and the set of such points has full Liouville measure. Specifically if we assume that h is C^1 and K_M has some relation with λ_N, the constant contraction of φ_N^t, then we can conclude the following:

Theorem 4.16. *Let M and N be two compact Riemannian manifolds with the same dimension and fixed $a > 0$. Assume that $\varphi_N^{f(t)}$ is Anosov with contraction constant $\lambda_N = e^{-\alpha}$ for some $\alpha > 0$. If h is C^1 and a $1-$conjugacy in orbits between φ_M^t and φ_N^t with odd reparametrization $f(t)$ satisfying $f(t) \geq at$ for all $t \geq 0$ then φ_M^t is also Anosov and $\lambda_M = \lambda_N^a$ is a contraction constant. Moreover if $K_M \geq -(\alpha a)^2$ then $K_M = -(\alpha a)^2$.*

To prove the Theorem 4.16 we need the following theorem:

Theorem 4.17. [Theorem 1.1 in [MR]] *Let M be a complete Riemannian manifold with finite volume and sectional curvature bounded below by $-c^2$ If the geodesic flow is Anosov with constant of contraction λ, then $\lambda \geq e^{-c}$. Moreover, the equality hold if and only if the sectional curvature of M is constant equal to $-c^2$.*

Proof of the Theorem 4.16. Assume that φ_N^t is Anosov and we denote by E^s and E^u the stable and unstable bundle. By hypohese $h \circ \varphi_M^t = \varphi_N^{f(t)} \circ h$ where h is a $1-$conjugacy then by definition 4.4, there exists two constants $C_1 > 0$ and $C_2 > 0$ such that:

$$C_1 d_M(\varphi_M^t(\theta_1), \varphi_M^t(\theta_2)) \leq d_N(h(\varphi_M^t(\theta_1)), h(\varphi_M^t(\theta_2))) \leq C_2 d_M(\varphi_M^t(\theta_1), \varphi_M^t(\theta_2)).$$

Hence

$$C_1 d_M(\varphi_M^t(\theta_1), \varphi_M^t(\theta_2)) \leq d_N(\varphi_N^{f(t)}(h(\theta_1)), \varphi_N^{f(t)}(h(\theta_2))) \leq C_2 d_M(\varphi_M^t(\theta_1), \varphi_M^t(\theta_2)). \quad (4.29)$$

Let $\theta \in SM$, $\xi \in T_\theta SM$ and $\beta(r) \subset SM$ a curve differentiable such that $\beta(0) = \theta$ and $\beta'(0) = \xi$ by Lemma 4.1 in [MR], for all $t \in \mathbb{R}$, we have

$$\lim_{s \to 0} \frac{d_M(\varphi_M^t(\beta(s)), \varphi_M^t(\beta(0)))}{s} = \|d_\theta \varphi^t(\xi)\|$$

and

$$\lim_{s \to 0} \frac{d_N(\varphi_N^{f(t)}(h(\beta(s))), \varphi_N^{f(t)}(h(\beta(0))))}{s} = \|d_{h(\theta)} \varphi_N^{f(t)}(d_\theta h(\xi))\|.$$

The last two equalities above and the equation (4.29) imply that for all $t \in \mathbb{R}$

$$C_1 \|d_\theta \varphi^t(\xi)\| \leq \|d_{h(\theta)} \varphi_N^{f(t)}(d_\theta h(\xi))\| \leq C_2 \|d_\theta \varphi^t(\xi)\|, \quad (4.30)$$

in particular for $t = 0$ above we obtain that

$$C_1 \|\xi\| \leq \|d_\theta h(\xi)\| \leq C_2 \|\xi\|.$$

This last inequality implies that $d_\theta h$ is an isomorphism and as M and N are compacts then h is a C^1- diffeomorphism. For $0 \neq \xi \in T_\theta SM$ the equation (4.30) implies

$$C_1 \frac{\|d_\theta \varphi^t(\xi)\|}{\|\xi\|} \leq \frac{\|d_{h(\theta)} \varphi_N^{f(t)}(d_\theta h(\xi))\|}{\|d_\theta h(\xi)\|} . \frac{\|d_\theta h(\xi)\|}{\|\xi\|} \leq \frac{\|d_\theta \varphi^t(\xi)\|}{\|\xi\|}. \tag{4.31}$$

Now, as $d_\theta h$ is an isomorphism we can define the subspaces $F_\theta^{s(u)}$ of $T_\theta SM$ satisfying the equation

$$d_\theta h(F_\theta^{s(u)}) = E_{h(\theta)}^{s(u)}$$

and

$$d_{\varphi_M^t(\theta)} h = d_{h(\theta)} \varphi_N^{f(t)} \circ d_\theta h \circ (d_\theta \varphi_M^t)^{-1},$$

which implies that the subspaces $F_\theta^{s(u)}$ are $d\varphi_M^t-$invariants. Thus for all $t \in \mathbb{R}$

$$T_{\varphi_M^t(\theta)}(SM) = F^s_{\varphi_M^t(\theta)} \oplus \langle \varphi_M \rangle \oplus F^u_{\varphi_M^t(\theta)}. \tag{4.32}$$

Now for $\xi \in F_\theta^u$ by (4.31) for all $t \in \mathbb{R}$ we have

$$\frac{C_1}{C_2} . \|d_\theta \varphi_M^t \mid_{F_\theta^u} \| \leq \|d_{h(\theta)} \varphi_N^{f(t)} \mid_{E^u_{h(\theta)}} \| \leq \frac{C_2}{C_1} . \|d_\theta \varphi_M^t \mid_{F_\theta^u} \|. \tag{4.33}$$

Analogously for $\xi \in F_\theta^s$ by (4.31) we obtain

$$\frac{C_1}{C_2} . \|d_\theta \varphi_M^t \mid_{F_\theta^s} \| \leq \|d_{h(\theta)} \varphi_N^{f(t)} \mid_{E^s_{h(\theta)}} \| \leq \frac{C_2}{C_1} . \|d_\theta \varphi_M^t \mid_{F_\theta^s} \|. \tag{4.34}$$

Since $\lambda_N = e^{-\alpha}$ is a contraction constant for φ_N^t, then (4.33) and (4.34), and the hypothesis over the reparametrization $f(t)$ provide that for all $t \geq 0$

$$\|d_\theta \varphi_M^t \mid_{F_\theta^s} \| \leq C \frac{C_2}{C_1} e^{-a\alpha t} \ and \ \|d_\theta \varphi_M^{-t} \mid_{F_\theta^u} \| \leq C \frac{C_2}{C_1} e^{-a\alpha t}. \tag{4.35}$$

Thus φ_M^t is Anosov and $\lambda_M = e^{-a\alpha}$ is a contraction constant. Now if $K_M \geq -(a\alpha)^2$ then by Theorem 4.17 conclude that $K_M \equiv -(a\alpha)^2$. $\square$

Chapter 5

Lyapunov Exponents and Rigidity

In this chapter, we will see that under certain conditions on the Lyapunov exponents of an Anosov geodesic flow that is define on the unitary tangent bundle SM with M a compact Riemannian manifold, we can obtain rigidity on the sectional curvature of M.

The goal of this chapter is to extend the result over the rigidity of equality of Lyapunov exponents for geodesic flows of Butler in [Bu].

For a periodic point θ of the geodesic flow $\varphi^t : SM \to SM$ defined on a compact Riemannian manifold M, let $l(\theta)$ be the period of θ. Let $\chi_1^{(\theta)}, \ldots, \chi_{m-1}^{(\theta)}$ are the complex eigenvalues of $d_\theta \varphi^{l(\theta)} : E_\theta^u \to E_\theta^u$, counted with the multiplicity of their generalized eigenspaces.

Theorem 5.1. [**Butler**] *Let M be an m-dimensional closed negatively curved Riemannian manifold. Suppose that*

$$\left|\chi_i^{(\theta)}\right| = \left|\chi_j^{(\theta)}\right|; \;\; 1 \leq i, j \leq m-1,$$

for every periodic point θ of the geodesic flow $\varphi : SM \to SM$. Then M is homothetic to a compact quotient of $\mathbb{H}_{\mathbb{R}}^m$.

In particular, the Butler's result above claims that if each periodic orbit of the geodesic flow defined on a manifold of negative curvature, has exactly one Lyapunov exponent on the unstable (or stable) bundle then the manifold has constant negative curvature.

One first extension of the result of Butler is in the finite volume case and can be done using the techniques of Romaña and Melo in [MR]. More specifically, if $-c^2 \leq K_M$, $Vol(M) < +\infty$, Anosov geodesic flow, and such that every periodic point has exactly the Lyapunov exponent on the unstable bundle equal to c then $K_M = -c^2$.

If the curvature satisfies $-c^2 \leq K_M \leq -\alpha^2$, we expect that if only every periodic point has exactly the Lyapunov exponent on the unstable bundle equal to α then $K_M = -\alpha^2$. In fact, in the next section we prove it, which is the prove of the Conjecture 1.4 in this case.

5.1 Pinched Negative Curvature, Lyapunov Eponents and Rigidity in the Finite Volume Case

We consider (M, g) a Complete Riemannian manifold with pinched negative curvature $-c^2 \leq K_M \leq -\alpha^2$ for some $c \geq \alpha > 0$. Let $\theta = (x, v) \in SM$, $\gamma_\theta(t)$ the geodesic throught $x \in M$ and initial velocity $v \in S_xM$. Let $Y_{\theta,s}(t)$ and $Y_{\theta,u}(t)$ the stable and unstable Jacobi tensor along $\gamma_\theta(t)$, respectively and $U_{\theta,s}(t) = Y'_{\theta,s}(t)Y^{-1}_{\theta,s}(t)$ and $U_{\theta,u}(t) = Y'_{\theta,u}(t)Y^{-1}_{\theta,u}(t)$ the stable and unstable Riccati solution, respectively.

Now for each $x \in \mathbb{R}^{n-1}\backslash\{0\}$ consider the following functions $f_u : \mathbb{R} \to \mathbb{R}$ and $f_s : \mathbb{R} \to \mathbb{R}$ define by $f_u(t) = |Y_{\theta,u}(t)x|^2$ and $f_s(t) = |Y_{\theta,s}(t)x|^2$, respectively.

Lemma 5.2. [[Kn]], [[E]] *Let (M, g) be a complete Riemannian manifold with $-c^2 \leq K_M \leq -\alpha^2$. Let $\theta = (x, v) \in SM$ and $\gamma_\theta(t)$ the geodesic such that $\gamma_\theta(0) = x$ and $\gamma'_\theta(0) = v$. Then for all $y \in \mathbb{R}^{n-1}\backslash\{0\}$ the functions $f_s(t) = |Y_{\theta,s}(t)y|^2$ and $f_u(t) = |Y_{\theta,u}(t)y|^2$ are strictly decreasing and increasing, respectively. Therefore, $f'_u(t)$ and $f'_s(t)$ are strictly increasing and for all $t \in \mathbb{R}$ we have $f'_u(t) > 0$ and $f'_s(t) < 0$.*

Proof. Since $K_M \leq -\alpha^2$ then have that the functions $f_u(t)$ and $f_s(t)$ are increasing and decreasing respectively, this means: (see [E])

$$\frac{d}{dt}(f_u(t)) = 2\langle Y'_{\theta,u}(t)x, Y_{\theta,u}(t)x\rangle \geq 0 \tag{5.1}$$

and

$$\frac{d}{dt}(f_s(t)) = 2\langle Y'_{\theta,u}(t)x, Y_{\theta,u}(t)x\rangle \leq 0. \tag{5.2}$$

Therefore, by the Jacobi equation and the condition on the curvature we obtain that for all $t \in \mathbb{R}$

$$\begin{aligned}
\frac{d^2}{dt^2}(f_u(t)) &= 2(\langle Y''_{\theta,u}(t)x, Y_{\theta,u}(t)x\rangle + |Y'_{\theta,u}(t)x|^2) \\
&= 2(\langle -R(t)Y_{\theta,u}(t)x, Y_{\theta,u}(t)x\rangle + |Y'_{\theta,u}(t)x|^2) \\
&\geq 2(\alpha^2|Y_{\theta,u}(t)x|^2 + |Y'_{\theta,u}(t)x|^2) \\
&> 0.
\end{aligned} \tag{5.3}$$

Analogously for all $t \in \mathbb{R}$

$$\frac{d^2}{dt^2}(f_s(t)) > 0. \tag{5.4}$$

Thus of (5.3) and (5.4), $f_u(t)$ and $f_s(t)$ are convex functions (strictly) in $\mathbb{R}$.

Of (5.1) and (5.3)(resp. (5.2) and (5.4)), we obtain that $f_u(t)$ ($f_s(t)$) is strictly incrasing (strictly decreasing).

Also (5.3) and (5.4) implies that $f'_u(t)$ and $f'_s(t)$ are strictly increasing. Thus, the convexity of $f_\star(t)$ implies that $f_\star(t) \neq 0$, for all $t \in \mathbb{R}$, $\star = u, s$. $\square$

Now, for $\star = u, s$ we know that $U_{\theta,\star}(t) = Y'_{\theta,\star}(t)Y^{-1}_{\theta,\star}(t)$ are symmetric solutions of Riccati equation. Since for $\star = u, s$, $Y_{\theta,\star}(t)$ are invertible, by (5.1) and (5.2) we obtain that

$$\langle y, U_{\theta,s}(t)y\rangle = \langle y, Y'_{\theta,s}(t)Y^{-1}_{\theta,s}(t)y\rangle \leq 0, \;\; y \in \mathbb{R}^{n-1} \tag{5.5}$$

and

$$\langle y, U_{\theta,u}(t)y\rangle = \langle y, Y'_{\theta,u}(t)Y^{-1}_{\theta,u}(t)y\rangle \geq 0, \;\; y \in \mathbb{R}^{n-1}. \tag{5.6}$$

Fix $y \in \mathbb{R}^{n-1}\backslash\{0\}$, if there is $t_0 \in \mathbb{R}$ such that $0 = \langle y, U_{\theta,u}(t_0)y\rangle = \langle y, Y'_{\theta,u}(t_0)Y^{-1}_{\theta,u}(t)y\rangle$ then put $x = Y^{-1}_{\theta,u}(t_0)y \in \mathbb{R}^{n-1}\backslash\{0\}$ we obtain

$$\begin{aligned} 0 &= \langle y, U_{\theta,u}(t_0)y\rangle \\ &= \langle y, Y'_{\theta,u}(t_0)Y^{-1}_{\theta,u}(t_0)y\rangle \\ &= \langle Y'_{\theta,u}(t_0)x, Y_{\theta,u}(t_0)x\rangle, \end{aligned}$$

thus $f'_u(t_0) = 2\langle Y'_{\theta,u}(t_0)x, Y_{\theta,u}(t_0)x\rangle = 0$ which contradicts the Lemma 5.2. Hence of (5.6) we conclude that $\langle y, U_{\theta,u}(t)y\rangle > 0$ for all $t \in \mathbb{R}$. Analogously using (5.5), $\langle y, U_{\theta,s}(t)y\rangle < 0$, for all $t \in \mathbb{R}$.

Now, we will show that for all $t \in \mathbb{R}$ and for all $x \in \mathbb{R}^{n-1}$ holds $\langle U_{\theta,u}(t)x, x\rangle \geq \alpha\langle x, x\rangle$. (see [Kn]).

Proposition 5.3. *Suppose that $R(t) \leq -\alpha^2$ for all $t \in \mathbb{R}$ and M has no conjugate points. Let $A(t)$ be a solution of the Jacobi equation*

$$A''(t) + R(t)A(t) = 0$$

and satisfying the initial conditions $A(0) = 0$ and $A'(0) = Id$. Then for all $y \in \mathbb{R}^{n-1}$

$$\langle A'(t)A(t)^{-1}y, y\rangle \geq \alpha\langle y, y\rangle \coth(\alpha t),$$

for all $t > 0$.

We denote by $s_\alpha(t) \in C^\infty(\mathbb{R})$ the solution of the following Jacobi equation:

$$s''(t) - \alpha^2 s(t) = 0,$$

with initial conditions $s(0) = 0$ and $s'(0) = 1$, this mean $s_\alpha(t) = \dfrac{\sinh(\alpha t)}{\alpha}$.

Lemma 5.4. *Suppose that $R(t) \leq -\alpha^2$ for all $t \in \mathbb{R}$ and M without conjugate points. Let $J(t)$ be a solution of Jacobi equation:*

$$J''(t) + R(t)J(t) = 0,$$

with the initial conditions $J(0) = 0$ and $\|J'(0)\| = 1$. Then for all $t > 0$

$$\|J(t)\| \geq s_\alpha(t).$$

Proof. Fix $t_0 > 0$ and consider the following two functions $X(t) = s_\alpha(t)J(t_0)$ and $Y(t) = s_\alpha(t_0)J(t)$. We denote $\overline{R}(t) = -\alpha^2 Id$, then $X(t)$ satisfies that

$$X''(t) + \overline{R}(t)X(t) = X''(t) - \alpha^2 X(t) = 0.$$

Also $Y(t)$ satisfies

$$Y''(t) + R(t)Y(t) = 0.$$

Note that $X(0) = Y(0)$ and $X(t_0) = Y(t_0)$, this implies that

$$\begin{aligned} I^{\overline{R}}_{[0,t_0]}(X,X) &\leq I^{\overline{R}}_{[0,t_0]}(Y,Y) \\ &= \int_0^{t_0} (\langle Y'(t), Y'(t)\rangle - \langle \overline{R}(t)Y(t), Y(t)\rangle dt \\ &\leq \langle Y'(t_0), Y(t_0)\rangle - \int_0^{t_0} \langle Y''(t) + R(t)Y(t), Y(t)\rangle dt \\ &= s_\alpha(t_0)^2 \langle J'(t_0), J(t_0)\rangle. \end{aligned}$$

Where $I^{\overline{R}}$ is the index form with relation to the curvature operator $\overline{R}$ (see Appendix).

On the other hand:

$$I^{\overline{R}}_{[0,t_0]}(X,X) = \int_0^{t_0} \Big(\langle X'(t), X'(t)\rangle - \langle \overline{R}(t)X(t), X(t)\rangle\Big)\, dt = \|J(t_0)\|^2 s'_\alpha(t_0)s_\alpha(t_0).$$

Hence by inequality above, we can conclude that

$$(s_\alpha(t_0))^2\langle J'(t_0), J(t_0)\rangle - \|J(t_0)\|^2 s'_\alpha(t_0)s_\alpha(t_0) \geq 0.$$

This last implies

$$\frac{d}{dt}\left(\frac{\|J(t)\|^2}{(s_\alpha(t))^2}\right)|_{t=t_0} \geq 0.$$

As $t_0 > 0$ was fixed arbitrarily, then the last inequality holds for all $t > 0$. Hence the function

$$\frac{\|J(t)\|^2}{(s_\alpha(t))^2}$$

is increasing on $(0, +\infty)$. Finally, note that

$$\lim_{t \to 0^+} \frac{\|J(t)\|^2}{s_\alpha(t)^2} = 1$$

and the Lemma 5.4 follows. $\square$

Proof of the Proposition 5.3. Let $A(t)$ be a Jacobi tensor with $A(0) = 0$ and $A'(0) = Id$. Let $x \in \mathbb{R}^{n-1}$ with $\|x\| = 1$ and consider the Jacobi field $J(t) = A(t)x$ which satisfies $J(0) = 0$ and $\|J'(0)\| = 1$. By the Lemma 5.4 we have that the function

$$\frac{\langle A(t)x, A(t)x \rangle}{\alpha(s_\alpha(t))^2}$$

is increasing on $[0, +\infty)$. Thus for all $t > 0$ we have

$$\frac{d}{dt}\left(\frac{\langle A(tx), A(t)x \rangle}{\alpha(s_\alpha(t))^2}\right) \geq 0$$

and this implies

$$\langle A'(t)x, A(t)x \rangle \geq \alpha \|A(t)x\|^2 \coth(\alpha t).$$

The last inequality above holds for all $x \in \mathbb{R}^{n-1}$ with $\|x\| = 1$ and for all $t > 0$. Now given $y \in \mathbb{R}^{n-1}$ and fixed $t_0 > 0$, define $x_0 = A(t_0)^{-1}y$. By above we obtain

$$\langle A'(t_0)A^{-1}(t_0)y, y \rangle \geq \alpha \|y\|^2 \coth(\alpha t_0).$$

Thus as $y \in \mathbb{R}^{n-1}$ and $t_0 > 0$ are arbitrary, the Proposition follows. $\square$

Proposition 5.5. *Suppose that M has no conjugate points and $R(t) \leq -\alpha^2$, for all $t \in \mathbb{R}$. Let $Y_u(t)$ be the unstable Jacobi tensor of Jacobi equation $Y''(t) + R(t)Y(t) = 0$. Then for all $y \in \mathbb{R}^{n-1}$ holds $\langle Y'(0)y, y \rangle \geq \alpha \langle y, y \rangle$.*

Proof. Let $r > 0$ and consider $A(t)$ the solution of Jacobi equation

$$A''(s) + R(-r + s)A(s) = 0,$$

with $A(0) = 0$ and $A'(0) = Id$. Also consider the Jacobi equation:

$$B''(t) + R(t)B(t) = 0. \tag{5.7}$$

Let $S_{-r}(t)$ be the solution of (5.7) with $S_{-r}(0) = Id$ and $S_{-r}(-r) = 0$. Now define $G(t) = A(t+r)A(r)^{-1}$ and note that it satisfies the equation: $G''(t) + R(t)G(t) = 0$ with $G(0) = S_{-r}(0)$ and $G(-r) = S_{-r}(-r)$. This means that $G(t)$ is the solution of (5.7) with $G(0) = S_{-r}(0)$ and $G(-r) = S_{-r}(-r)$, then $S_{-r}(t) = G(t) = A(t + r)A(r)^{-1}$.

By Proposition 5.3 for all $y \in \mathbb{R}^{n-1}$ we obtain that

$$\langle S'_{-r}(0)y, y\rangle = \langle A'(r)A(r)^{-1}y, y\rangle \geq \alpha\langle y, y\rangle \coth(\alpha r).$$

Taking of limit when r goes to $+\infty$ on the last inequality above:

$$\begin{aligned}\langle Y'_u(0)y, y\rangle &= \lim_{r\to+\infty}\langle S'_{-r}(0)y, y\rangle \\ &\geq \alpha\langle y, y\rangle \lim_{r\to+\infty}\coth(\alpha r) \\ &= \alpha\langle y, y\rangle.\end{aligned}$$

□

Finally, suppose that $R(t) \leq -\alpha^2$, for all $t \in \mathbb{R}$ and $Y_u(t)$ the unstable tensor of Jacobi equation $B''(t) + R(t)B(t) = 0$. For $t_0 \in \mathbb{R}$, the Jacobi tensor $\widetilde{Y}_u(t) = Y_u(t + t_0)Y_u(t_0)^{-1}$ is the unstable solution of the Jacobi equation $B''(t) + R(t + t_0)B(t) = 0$. As $R(t + t_0) \leq -\alpha^2$ for all $t \in \mathbb{R}$, then by Proposition 5.5 for all $x \in \mathbb{R}^{n-1}$ holds

$$\langle Y'_u(t_0)Y_u(t_0)^{-1}x, x\rangle = \langle \widetilde{Y}'_u(0)x, x\rangle \geq \alpha\langle x, x\rangle.$$

As $t_0 \in \mathbb{R}$ was arbitrary, then for all $t \in \mathbb{R}$ and for all $x \in \mathbb{R}^{n-1}$ holds

$$\langle U_{\theta,u}(t)x, x\rangle = \langle Y'_u(t)Y_u(t)^{-1}x, x\rangle \geq \alpha\langle x, x\rangle. \tag{5.8}$$

For analogous arguments, for all $t \in \mathbb{R}$, and for all $x \in \mathbb{R}^{n-1}$ we can obtain:

$$\langle U_{\theta,s}(t)x, x\rangle \leq -\alpha\langle x, x\rangle. \tag{5.9}$$

Proof of the Theorem 1.3.

(1) If $b = \alpha$. Note that the hypothesis $\chi^+(\theta, \xi) = \alpha$ for all $\xi \in E^u_\theta$ implies that

$$\lim_{t\to+\infty}\frac{1}{t}\log|\det d_\theta\varphi^t|_{E^u_\theta}| = \alpha \cdot \dim E^u_\theta = \alpha(n-1).$$

Now by [MF], we obtain

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t tr(U^u(\varphi^s(\theta)))ds = \lim_{t\to+\infty}\frac{1}{t}\log|\det d_\theta\varphi^t|_{E^u_\theta}| = \alpha(n-1), \tag{5.10}$$

for all $\theta \in Per(\varphi)$.

On the other hand, as $K \leq -\alpha^2$ and $U^u(\varphi^t(\theta)) := U_{\theta,u}(t)$ is symmetric. By (5.8) we can conclude that all eigenvalues of $U^u(\varphi^t(\theta))$ are non-negative and they are great than or equal to α. Let $\lambda_{n-1}(t) \geq \lambda_{n-2}(t) \geq \cdots \geq \lambda_1(t) \geq \alpha$ the eigenvalues of $U^u(\varphi^t(\theta))$. Then

$$tr(U^u(\varphi^t(\theta))) = \sum_{i=1}^{n-1}\lambda_i(t) \geq \alpha(n-1).$$

As $\theta \in Per(\varphi)$, we have that the function $t \to tr(U^u(\varphi^t(\theta)))$ is periodic. Thus the last inequality and (5.10) imply that for all $t \in \mathbb{R}$:

$$tr(U^u(\varphi^t(\theta))) = \alpha(n-1).$$

Hence, for all $j = 1, 2, ..., n-1$ holds $\lambda_j(t) = \alpha$ for all $t \in \mathbb{R}$. Finally this implies for all $t \in \mathbb{R}$

$$tr((U^u(\varphi^t(\theta)))^2) = \sum_{j=1}^{n-1} (\lambda_j(t))^2 = \alpha^2(n-1).$$

Then, since $\|U^u(\varphi^s(\theta))\| \le c$, integrating the Riccati equation and taking of limit when t goes $+\infty$, we obtain:

$$\begin{aligned}
\alpha^2(n-1) &= \lim_{t\to+\infty} \frac{1}{t}\int_0^t tr((U^u(\varphi^s(\theta)))^2)ds \\
&= -\lim_{t\to+\infty} \frac{1}{t}\int_0^t tr(R(\varphi^s(\theta)))ds \\
&= -\lim_{t\to+\infty} \frac{(n-1)}{t}\int_0^t Ric(\varphi^s(\theta))ds.
\end{aligned}$$

Hence

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t Ric(\varphi^s(\theta))ds = -\alpha^2.$$

As $\theta \in Per(\varphi)$ and $K \le -\alpha^2$ this implies that for all $t \in \mathbb{R}$ we have $Ric(\varphi^t(\theta)) = -\alpha^2$. Hence we show that for all $\theta \in Per(\varphi)$ holds $Ric(\theta) = -\alpha^2$. As $Vol(M) < +\infty$ then by Theorem 1.1 we have that $\overline{Per(\varphi)} = SM$ and by continuity of function $Ric(\cdot)$ we get for all $\theta \in SM$ holds $Ric(\theta) = -\alpha^2$ and hence $K = -\alpha^2$.

(2) If $b = c$. Here we use the same arguments of [MR]. By the hypothesis $\chi^+(\theta, \xi) = c$ for all $\xi \in E^u_\theta$ we have that

$$\lim_{t\to+\infty} \frac{1}{t} \log|\det d_\theta\varphi^t|_{E^u_\theta}| = c \cdot dim E^u_\theta = c(n-1).$$

Now by [MF], we obtain

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t tr(U^u(\varphi^s(\theta)))ds = \lim_{t\to+\infty} \frac{1}{t} \log\left|\det d_\theta\varphi^t|_{E^u_\theta}\right| = c(n-1). \tag{5.11}$$

The last inequality above holds for all $\theta \in Per(\varphi)$.

Since $U^u(\varphi^s(\theta))$ is symmetric then easy to see that

$$(tr(U^u(\varphi^s(\theta))))^2 \le (n-1)tr((U^u(\varphi^s(\theta)))^2).$$

As the sectional curvature satisfies $K \geq -c^2$, then taking trace and integrating of 0 to t the Riccati equation, we have that by Cauchy-Schwarz inequality that

$$\begin{aligned}
\frac{1}{t}\int_0^t tr(U^u(\varphi^s(\theta)))ds &\leq \sqrt{\frac{1}{t}\int_0^t (tr(U^u(\varphi^s(\theta))))^2 ds} \\
&\leq \sqrt{\frac{(n-1)}{t}\int_0^t tr((U^u(\varphi^s(\theta)))^2)ds} \\
&= \sqrt{-\frac{(n-1)}{t}\Big(\int_0^t tr((U^u)'(\varphi^s(\theta)))ds + \int_0^t tr(R(s))ds\Big)} \\
&= \sqrt{-\frac{n-1}{t}\left(tr(U^u(\varphi^t(\theta))) - tr(U^u(\theta))\right) - \frac{(n-1)^2}{t}\int_0^t Ric(\varphi^s(\theta))ds} \\
&\leq \sqrt{-\frac{n-1}{t}\left(tr(U^u(\varphi^t(\theta))) - tr(U^u(\theta))\right) + (n-1)^2 c^2}.
\end{aligned}$$

Hence, since $\|U^u(\varphi^t(\theta))\| \leq c$, we get

$$\lim_{t\to+\infty} -\frac{(n-1)}{t}\Big(tr(U^u(\varphi^t(\theta))) - tr(U^u(\theta))\Big) = 0.$$

Thus, taking the limit as $t \to +\infty$ in the last inequality above, we have that

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t Ric(\varphi^s(\theta))ds = -c^2.$$

As $\theta \in Per(\varphi)$ and $K \geq -c^2$, this implies that for all $t \in \mathbb{R}$ we have $Ric(\varphi^t(\theta)) = -c^2$. Hence we show that for all $\theta \in Per(\varphi)$ holds $Ric(\theta) = -c^2$. As $Vol(M) < +\infty$ then by Theorem 1.1 we have that $\overline{Per(\varphi)} = SM$ and by continuity of function $Ric(\cdot)$ we get for all $\theta \in SM$ holds $Ric(\theta) = -c^2$ and hence $K = -c^2$.

□

5.2 Rigidity in Dimension 2

5.2.1 Setting

In this subsection, we will extend the Butler's result [Bu] for dimension 2.

Now we assume that (M, g) is a compact Riemannian manifold of $dim(M) = 2$. this means M is a compact surface. Also we assume that the metric g is Hölder C^3.

The main result of this section is a generalization in dimension 2 of a result of Butler in ([Bu]).

The Theorem 1.5 extend the Butler‘s result since we only assume that the geodesic flow is Anosov without any restrictions on the sectional curvature of the manifold.

To prove the Theorem 1.5, we will use some results about the aproximation of Lyapunov exponents of an invariant measure of a flow by Lyapunov exponents of measures concentrated on periodic orbits.

The result about values of Lyapunov exponents along periodic orbits due to Kalinin [Kal], which enable us to aproximate the Lyapunov exponents of any ergodic φ^t- invariant measure by Lyapunov exponents of measures concentrated on periodic orbits.

For each periodic point θ, we let μ_θ denote the unique φ^t-invariant probability measure supported on the orbit of θ, which may be obtained as the normalized push-forward of Lebesgue measure on $\mathbb{R}$ by the map $t \to \varphi^t(\theta)$.

Theorem 5.6. [**Kalinin**] *Let $\mathcal{E}$ be an n-dimensional Hölder continuous vector bundle over a manifold N and Λ a Hölder continuous cocycle on $\mathcal{E}$ over φ^t. Let μ be an ergodic φ^t-invariant measure and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the Lyapunov exponents of Λ with respect to μ, counted with the multiplicity. Then for every $\varepsilon > 0$, there is a periodic point $\theta \in N$ of φ^t such that the Lyapunov exponents $\lambda_1^\theta \leq \lambda_2^\theta \leq \cdots \leq \lambda_n^\theta$ of Λ with respect to μ_θ satisfy*

$$|\lambda_i - \lambda_i^\theta| < \varepsilon,$$

for each $1 \leq i \leq n$.

In our case, $N = SM$, the vector bundle $\mathcal{E} = TN$, and the cocycle Λ is the cocycle derivative of the geodesic flow φ^t, this means:

$$\begin{array}{rcl} \Lambda : N \times \mathbb{R} & \to & GL(n, \mathbb{R}) \\ (x,t) & \mapsto & \Lambda(x,t) = d_x\varphi^t, \end{array}$$

and $n = 2\dim(M) - 1$.

Now for the following step, we need the following definition of **Algebraic flows**, which can be found in [SLVY] and [T].

Definition 5.7. *An Anosov flow $\Phi : N \to N$ on a 3-dimensional compact manifold N is* ***algebraic*** *if it is finitely covered by*

(1) *a suspension of a hyperbolic automorphism of the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$;*

(2) *or the geodesic flow on some closed Riemannian surface of constant negative curvature, i.e., a flow on a homogeneous space $\Gamma\backslash\widetilde{SL(2,\mathbb{R})}$ corresponding to the right translations by diagonal matrices $diag(e^t, e^{-t})$, $t \in \mathbb{R}$, where $\widetilde{SL(2,\mathbb{R})}$ denotes the universal cover of $SL(2,\mathbb{R})$ and Γ is a uniform subgroup.*

The following result can be found in [SLVY], it gives us a characterization of the conjugacy of a flow to an algebraic flow when we have that the Liouville measure is a measure of maximal entropy.

Theorem 5.8. [**De Simoi, Leguil, Vinhage and Yang**] *Let $k \geq 5$ be some integer and let Φ be a C^k anosov flow on a compact connected 3-manifold N such that $\Phi_* \mu = \mu$ for some smooth volume μ. Then $h_{top}(\Phi) = h_\mu(\Phi)$ if and only if Φ is $C^{k-\epsilon}$-conjugate to an algebraic flow, for $\epsilon > 0$ arbitrarily small.*

The Theorem 5.8 is related with the following conjecture:

Conjecture 5.9. [**Katok Entropy Conjecture**] *Let (M, g) be a connected Riemannian manifold of negative curvature and ψ be the corresponding geodesic flow. Then $h_{top}(\psi) = h_{\mathcal{L}}(\psi)$ if and only if (M, g) is a locally symmetric space. Here $\mathcal{L}$ is the Liouville measure in SM.*

A weak version of this was obtained by [BCG], which still highly depends on the structures coming from the geometry of the flow. Other generalizations work with broader classes of Anosov flows. Foulon in [F], showed that in the case of a contact Anosov flow Φ on a closed three-manifold, Φ is, up to finite cover, smoothly conjugate to the geodesic flow of a metric of constant negative curvature on a closed surface if and only if the measure of maximal entropy is the contact volume. There, he asks the following question generalizing Conjecture 5.9:

Question 5.10. *Let Φ be a smooth Anosov flow on a 3-manifold which preserves a smooth volume μ. If $h_{top}(\Phi) = h_\mu(\Phi)$, smoothly conjugate to an algebraic flow?*

Recall that the geodesic flow on M occurs on the unit tangent bundle SM, which has dimension $2\dim(M) - 1$. Therefore, the Question 5.10 corresponds to the case of the geodesic flow on surfaces, which was proved by Katok in [Ka] and [Ka2]. The low-dimensionality assumption of Question 5.10 is required for a theorem in this generality, It is not difficult to construct non-algebraic systems whose maximal entropy measure is a volume when the stable an unstable distributions are multidimensional. The Theorem 5.8 provides a positive answer to Question 5.10.

We need the following result, which is due to Plante (see [Pl]). Basically, the result gives us two alternatives for the strong stable and strong unstable manifold are or not dense on the manifold when the Anosov flow satisfy $\Omega(\psi) = M$.

Theorem 5.11. [**Plante**] *Let $\psi^t : M \to M$ be an Anosov flow such that $\Omega(\psi) = M$. Then there are two possibilities:*

(a) *Each strong stable and each strong unstable manifold is dense in M, or*

(b) *ψ^t is a suspension (modulo time scale change by a constant factor) of an Anosov diffeomorphism of a compact C^1 submanifold of codimension one in M.*

We use the Theorem 5.8, Theorem 5.11, Ruelle's inequality and Pesin's formula to prove the Theorem 1.5 discarding some cases that may appear for our geodesic flow.

Proof of the Theorem 1.5.

We denote by $\mathcal{M}_e(\varphi)$ the set of all ergodic φ^t-invariant measures.

Let $\mu \in \mathcal{M}_e(\varphi)$ and $\lambda_1 \leq \lambda_2 \leq \lambda_3$ be the Lyapunov exponents concerning for to μ. Let $\theta \in Per(\varphi)$ and $\mu_\theta \in \mathcal{M}_e(\varphi)$ the unique φ^t-invariant probability measure supported on the orbit of θ. Let $\lambda_1^\theta \leq \lambda_2^\theta \leq \lambda_3^\theta$ the Lyapunov exponents concerning for to μ_θ. In our case by hypothesis $\lambda_1^\theta = -\alpha$, $\lambda_2^\theta = 0$ and $\lambda_3^\theta = \alpha$. Then by Theorem 5.6 we can aproximate the Lyapunov exponents of μ by Lyapunov exponents of μ_θ and we can conclude that for $\mu \in \mathcal{M}_e(\varphi)$ holds $\lambda_1 = -\alpha$, $\lambda_2 = 0$ and $\lambda_3 = \alpha$.

We denote by $\mathcal{L}$ the Liouville measure on SM. Note that in this case, $\mathcal{L}$ is ergodic. (see [VO])

Now we will show that the Liouville measure on SM is a maximal measure of entropy. Indeed, by Ruelle's inequality, for $\mu \in \mathcal{M}_e(\varphi)$ we have that

$$\begin{aligned} h_\mu(\varphi) &\leq \int_{SM} \chi^+(\theta) d\mu(\theta) \\ &= \int_{SM} \lambda_3 d\mu(\theta) \\ &= \int_{SM} \alpha d\mu(\theta) \\ &= \alpha. \end{aligned} \tag{5.12}$$

Since φ^t is C^2, by Pesin's formula we obtain that

$$\begin{aligned} h_{\mathcal{L}}(\varphi) &= \int_{SM} \chi^+(\theta) d\mathcal{L}(\theta) \\ &= \int_{SM} \alpha d\mathcal{L}(\theta) \\ &= \alpha. \end{aligned} \tag{5.13}$$

Thus by (5.12) and (5.13), we conclude that for all $\mu \in \mathcal{M}_e(\varphi)$ holds

$$h_\mu(\varphi) \leq \alpha = h_{\mathcal{L}}(\varphi).$$

Thus

$$h_{top}(\varphi) = \sup_{\mu \in \mathcal{M}_e(\varphi)} h_\mu(\varphi) \leq \alpha = h_{\mathcal{L}}(\varphi).$$

Hence

$$h_{top}(\varphi) = \alpha = h_{\mathcal{L}}(\varphi). \tag{5.14}$$

In our case taking the smooth measure $\mu = \mathcal{L}$ as the Liouville measure on SM and the flow $\Phi = \varphi$ as an Anosov geodesic flow on SM, then by (5.14) we have that the topological entropy is equal to metric entropy of Liouville measure, then the Theorem 5.8 above implies that our Anosov geodesic is $C^{k-\epsilon}$-conjugate to an algebraic flow, for $\epsilon > 0$ arbitrarily small.

Now by Definition 5.7 of Algebraic flow, we have that our Anosov geodesic flow is $C^{k-\epsilon}$-conjugate up to finite covers to a suspension of hyperbolic automorphism of the 2-torus or to the geodesic flow on some Riemannian surface of negative curvature.

Since our flow φ is an Anosov geodesic flow, then it can not be conjugate to a suspension of an automorphism of the 2-torus $\mathbb{T}^2$ (that by Theorem 5.11), thus it should be conjugate to the geodesic flow of some closed Riemannian surface of constant negative curvature.

To conclude the proof of Theorem 1.5, we use the Theorem 4.3, the result about C^k conjugacy rigidity in dimension 2 due to Croke (see [CLUV]).

$\square$

The Theorem 1.5 provides the following Corollary, following the techniques of Chapter 4, which has some relation to the Theorem 4.3 by Croke.

Corollary 5.12. *If $\varphi^t : SM \to SM$ is Anosov geodesic flow, then φ^t can not be 1-conjugacy to the geodesic flow of a manifold of constant negative curvature, unless M has constant negative curvature.*

On the other hand, note that the Theorem 5.6 due to Kalinin is true in any dimension and any C^2 Anosov geodesic flow where the Liouville measure is ergodic (see [VO]) then we obtain the following result:

Theorem 5.13. *If (M, g) is a compact Riemannian manifold with the same hypothesis of Theorem 1.5 and assuming that the Liouville measures $\mathcal{L}$ is ergodic, we have that:*

$$h_{top}(\varphi) = h_{\mathcal{L}}(\varphi).$$

In other words, the Liouville measure is a measure of maximal entropy (MME).

Now, in dimension 2 joing the Theorem 1.5 and the Theorem 5.13, we have the following Corollary:

Corollary 5.14. *Let (M, g) be a compact Riemannian surface. If $\varphi^t : SM \to SM$ is C^k, $k \geq 5$, an Anosov geodesic flow. Then the following are equivalents:*

(1) *For all $\theta \in Per(\varphi)$ hold:*

$$\chi^+(\theta, \xi) = \lim_{t \to +\infty} \frac{1}{t} \log \|d_\theta \varphi^t(\xi)\| = \alpha,$$

for all $\xi \in E^u_\theta \backslash \{0\}$.

(2) *The Liouville measure $\mathcal{L}$ is a measure of maximal entropy.*

(3) *The surface M has constant negative curvature $K_M = -\alpha^2$.*

On the hypothesis of Theorem 1.5 over the geodesic flow be C^k, $k \geq 5$, we believe the theorem is true for $k \geq 2$, beacuse observation of the Theorem 5.8 on [SLVY] says that this theorem can be true for $k \geq 2$, but technical obstructions prevent finding the precise boundary of required regularity. Also it is emphasized that regularity is extremely important for the rigidity phenomenon.

Appendix

Appendix A

Index Form

Let $(E, \langle , \rangle)$ be a vector space with scalar product $\langle , \rangle$. Let $End(E)$ denote the set of endomorphisms and let $Sym(E)$ denote the set of symmetric endomorphisms on E. If $R : \mathbb{R} \to Sym(E)$ is a C^∞-map (curvature operator), the linear differential equation:

$$\ddot{J}(t) + R(t)J(t) = 0, \tag{A.1}$$

is called the **Jacobi equation** and its solutions $J : \mathbb{R} \to E$ are called **Jacobi fields**. If $B : \mathbb{R} \to End(E)$ is a solution of the matrix equation

$$\ddot{B}(t) + R(t)B(t) = 0, \tag{A.2}$$

we call B a **Jacobi tensor**. B is a Jacobi tensor if and only if $B(t)x$ is a Jacobi field for all $x \in E$. The solutions of the Jacobi equations are generated by a time-dependent Hamiltonian function. Define for each $t \in \mathbb{R}$ the Hamiltonian

$$H_t(x, y) = \frac{1}{2}(\langle y, y\rangle + \langle R(t)x, x\rangle), \tag{A.3}$$

then the Jacobi equation is equivalent to the Hamiltonian equation:

$$\begin{aligned} \dot{J}_1(t) &= \frac{\partial H_t}{\partial y}(J_1(t), J_2(t)) = J_2(t) \\ \dot{J}_2(t) &= -\frac{\partial H_t}{\partial x}(J_1(t), J_2(t)) = -R(t)J_1(t)) \end{aligned}$$

Definition A.1. *Let $R : \mathbb{R} \to Sym(E)$ be a C^∞-map. We say that Jacobi equation*

$$\ddot{J}(t) + R(t)J(t) = 0$$

has no conjugate points on $[a, b]$ if for all Jacobi fields J with $J(a) = 0$ and $\dot{J}(a) \neq 0$, we have that $J(t) \neq 0$ for all $t \in (a, b]$.

The variational spects of the Jacobi equation are revealed by the index form.

Definition A.2. *Let $R : \mathbb{R} \to Sym(E)$ be a curvature operator and $V = \{X : [a,b] \to E | X \text{ is piecewise differentiable}\}$ be the vector space continuous and piecewise differentiable curves in E. The symmetric bilinear form*

$$I := I_{[a,b]} : E \times E \to \mathbb{R},$$

with

$$I_{[a,b]}(X,Y) = \int_a^b \Big(\langle \dot{X}(t), \dot{Y}(t) \rangle - \langle R(t)X(t), Y(t) \rangle \Big)\, dt \tag{A.4}$$

*is called the **index form**. If we want to specify the curvature tensor operator, we write I^R or $I^R_{[a,b]}$.*

Let $X, Y \in V$ be differentiable on each interval of the subdivision $a = t_0 < t_1 < ... < t_k = b$. Then, using integration by parts, we obtain:

$$I(X,Y) = \sum_{i=1}^{k} \langle \dot{X}, Y \rangle \Big|_{t_{i-1}}^{t_i} - \int_a^b \langle \ddot{X}(t) + R(t)X(t), Y(t) \rangle dt. \tag{A.5}$$

This implies that X is a Jacobi field if and only if $I(X,X) = 0$ for all $Y : [a,b] \to E$ with $Y(a) = Y(b) = 0$. Since

$$\frac{\partial}{\partial s}\Big|_{s=0} I(X + sY, X + sY) = 2I(X,Y), \tag{A.6}$$

X is a Jacobi field if and only if it is a critical point of the action $I(X) := I(X,X)$ on the space vector fields with fixed endpoints.

Lemma A.3. (**Index Lemma**) *Assume that the Jacobi equation has no conjugate points on $[a,b]$. Then $I_{[a,b]}$ is positive definite on the subspace*

$$V^0 := \{X \in V | X(a) = X(b) = 0\},$$

this mean,

$$I(X,X) = 0, \;\; for \;\; all \;\; X \in V^0, \;\; X \neq 0.$$

Proof. Let us first assume that $X \in V^0$ is smooth. Consider the Jacobi tensor A with initial conditions $A(a) = 0$ and $\dot{A}(a) = Id$. Since the Jacobi equation has no conjugate points, there is a smooth curve $Y \in V$ such that $X(t) = A(t)Y(t)$. In particular, $Y(b) = 0$. Then we obtain

$$\begin{aligned} I(X,X) &= \langle \dot{X}, X\rangle\Big|_a^b - \int_a^b \langle \ddot{X} + R(t)X, X\rangle dt \\ &= -\int_a^b \langle \ddot{A}Y + R(t)AY, AY\rangle dt - \int_a^b \langle 2\dot{A}\dot{Y} + a\ddot{Y}, AY\rangle dt \\ &= -\int_a^b 2\langle \dot{A}\dot{Y}, AY\rangle + \langle \ddot{Y}, A^T AY\rangle dt \\ &= 2\int_a^b \langle \dot{Y}, \dot{A}^T\rangle dt + \int_a^b \langle \dot{Y}, \dot{\widehat{A^T AY}}\rangle dt. \end{aligned}$$

Since

$$0 = A^T \dot{A} - \dot{A}^T A,$$

we obtain

$$(\dot{\widehat{A^T AY}}) = \dot{A}^T AY + A^T \dot{A}Y + A^T A\dot{Y} = 2\dot{A}^T AY + A^T A\dot{Y}.$$

Hence

$$I(X,X) = \int_a^b \langle \dot{Y}, A^T A\dot{Y}\rangle dt.$$

In the general case, we deduce the same formula by integrating piecewise. This shows $I(X,X) \geq 0$ for $X \in V^0$. Furthermore, $I(X,X) = 0$ implies that Y is constant. Since $Y(b) = 0$, we deduce that $0 = Y(t) = X(t)$ for all $t \in [a,b]$.

□

This lemma shows for Jacobi equations without conjugate points on $[a,b]$ that there are no conjugate points inside the interval $[a,b]$ as well. To see this, one considers for $a \leq t_1 < t_2 \leq b$ be piecewise differentiable curve X which is 0 on $[a,t_1]$ and $[t_2,b]$ and coincides with a Jacobi field J on $[t_1,t_2]$ with $J(t_1) = J(t_2) = 0$. Then it follows from (A.5) that $I(X,X) = 0$ and the index Lemma A.3 implies that X vanishis identically.

The next corollary shows that for Jacobi equations without conjugate points, Jacobi fields are not only critical points but minimize of the action $I(X,X)$ on the space of piecewise differentiable curves with fixed endpoints.

Corollary A.4. (**Minimizing property of Jacobi fields**) *Assume that the Jacobi equation* $\ddot{J}(t) + R(t)J(t) = 0$ *has no conjugate points on* $[a, b]$*. Let* J *be a Jacobi field and* $X \in V$ *be a piecewise differentiable field with* $X(a) = J(a)$ *and* $X(b) = J(b)$*. Then*

$$I(J, J) \leq I(X, X), \tag{A.7}$$

where the inequality is strict unless $J = X$*.*

Proof. From the Index Lemma A.3 and (A.5), we deduce

$$0 \leq I(J - X, J - X) = I(J, J) - 2I(J, X) + I(X, X) = -I(J, J) + I(X, X). \tag{A.8}$$

□

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