Wrapped Floer Cohomology and the Symmetric Symplectic Capacity

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The purpose of this dissertation is to use wrapped Floer cohomology, which is basically an homology theory similar to standard (periodic) Floer homology but intended for open strings with Lagrangian boundary conditions instead of closed orbits, to conclude some results which have been discovered in the periodic case. Particularly, we focus on the symmetric symplectic capacity, the analogue of the symplectic capacity but accounting for an anti-symplectic involution on the manifold.

We arrive on a result of finiteness of the symmetric symplectic capacity under the hypothesis of vanishing wrapped Floer cohomology which can then be applied to prove the finiteness of such capacity under some conditions. This can be pushed further by introducing local coefficients suitable for vanishing the wrapped Floer cohomology while still restricting to the trivial local system when restricted to the Lagrangian L.

Contents

Chapter 1

Introduction

Poincaré's contributions and interest on the dynamical aspects of the solar system and, more generally, astronomy mechanics is often regarded as the origin of the modern study of dynamical systems as a Mathematical area. A major point of curiosity was whether planetary movement was bound to repeat itself. This curiosity has evolved through time into the study of periodic orbits, which receives attention from a lot of professionals from different areas and from different points of view.

A similar question can be asked, although it is a more specific one. To keep the same context as above, we phrase it as follows: given an ensemble of planets, starting from rest, is this ensemble bound to return to rest at some other point in time? This type of orbit is called a *brake orbit*, and is the motivation for the definitions that will come in place later.

The main objective of this dissertation is to present adaptations to the aforementioned context of brake orbits of some tools and invariants that are commonly used in symplectic topology in the case of periodic orbits of Hamiltonian flows. Most notably, we will deal with an adaptation of the Hofer-Zehnder capacity which we will refer to as *symmetric Hofer-Zehnder capacity*. The reason for the use of the term "symmetric" will soon become clear. This invariant was introduced in [15], although we make an adaptation to its definition in order to encompass the case of compact manifolds more appropriately.

Let us briefly recall the definition of the Hofer-Zehnder capacity: Given a symplectic manifold (M, ω), possibly with boundary, the *Hofer-Zehnder capacity* is defined as

$$c(M, \omega) := \sup\{\max(H) \mid H \in \mathcal{H}\},$$

where $\mathcal{H}$ is the set of smooth functions H such that:

- There is a compact set K, depending on H, such that $K \subset \operatorname{int}(M)$ and H is constant and equal to its maximum outside K.
- There is an open set, also depending on H, where H is constant and equal to zero, zero being the minimum value of H.
- Every periodic orbit is either constant or has minimal period greater than 1.

Even though the definition mentions periodic orbits, it is not clear why this value can help in finding them. Formally, this happens due to results such as the classical statement of existence of periodic orbits in level-sets corresponding to a set of real values of full Lebesgue measure if the capacity of the thickening is finite. (More information will be given in Chapter 4.) More intuitively, we can see that the capacity is actually measuring the supremum of the oscillations (in the sense of $\max H - \min H$) of Hamiltonians which respect some "boundary" conditions and that do not have periodic orbits which are "fast". We could hope that if we take Hamiltonians of sufficiently high oscillation, that the gradient of H should be large, and thus also the speed of the orbits, and we could expect some periodic one to be fast enough. When the capacity is finite, this expectation is somewhat fulfilled. But, of course, it should be said that we have plenty of cases where the capacity is not finite.

Let us now briefly introduce the framework that we are mainly studying in this dissertation. We have a symplectic manifold (X, ω), together with an involution $\varphi : X \to X$ that is anti-symplectic, i.e. $\varphi^*\omega = -\omega$. We suppose that the fixed-point set of this involution is not empty. In this case, it is a Lagrangian submanifold of X, which we will call L. This data will be referred to as (X, ω, φ, L). Since L is determined by φ, we may suppress it in the notation. We usually also suppress ω since it is the symplectic form, resulting in the notation being often simply (X, φ). We will call such manifold a *symmetric symplectic manifold*. A Hamiltonian satisfying $H \circ \varphi = H$ is called a *symmetric Hamiltonian*, and an orbit x of the Hamiltonian flow for which there is a $T \in \mathbb{R}$ such that

$$x(-t) = \varphi(x(t)), \quad x(t+T) = x(t)$$

is called a *brake orbit*.

The connection between the above and the situation described as the motivation for brake orbits is seen by looking at the case of X being the cotangent bundle of some manifold M, i.e. $X = T^*M$. In this case, if we take $\varphi = (x,p) \mapsto (x,-p)$ and $H(x,p) = \|p\|^2 + V(x)$ where $V(x)$ is some potential, then brake orbits in this last sense correspond precisely to brake orbits in the mechanical sense of beginning and ending with zero momentum. Indeed, note that the condition of being a brake orbit says that x is a periodic orbit and $x(0)$ is a fixed point of φ, and thus is a point of zero momentum. Reciprocally, if x is an orbit of the Hamiltonian flow that begins and ends at the zero section, then by continuing x we get a periodic orbit which is a brake orbit.

With the framework of symmetric symplectic manifolds in mind, we then investigate a notion of capacity that takes φ into consideration by considering only symmetric Hamiltonians. In order to have some grounded and easily visualized examples, we investigate how this notion of capacity behaves in surfaces. (It is known that the Hofer-Zehnder capacity is equal to the area in closed surfaces, c.f. [29]. We reach similar results for the symmetric Hofer-Zehnder capacity with mild assumptions.)

Some main results of this dissertation illustrate nicely that the way in which L is positioned topologically inside M may be of great relevance for the symmetric capacity. More specifically, we have results for the finiteness of the symmetric capacity which include hypotheses such as the inclusion $L \hookrightarrow M$ being nullhomotopic, or $H^2(M, L; \mathbb{R})$ being nonzero.

The dissertation is structured as follows, where we do not mention this chapter as it is an introduction.

Chapter 2 gives a quick description of Floer homology in aspherical manifolds with vanishing first Chern class. This is arguably the most traditional, well-known and simple example of Floer homology. The analytical details of the construction are not the focus of this dissertation so we will not provide much, but we will give references and brief discussions. This observation also holds for the other chapters which deal with different kinds of Floer homology.

Chapter 3 has as its core objective to describe wrapped Floer homology. In order to do so, we start by dealing with symplectic homology, as it is a midpoint between the Floer homology case of chapter 2 and the wrapped Floer homology case.

Chapter 4 contains a large bulk of the content of the dissertation and discusses the symmetric capacity, which is the symmetric analogue of the Hofer-Zehnder capacity. We also introduce the symmetric Gromov capacity. We give definitions and prove basic theorems similar to the standard case. We also explore the symmetric capacity in the realm of surfaces and prove an equivariant version of the Moser theorem on volume-preserving diffeomorphisms on closed surfaces being symplectomorphisms.

Chapter 5 involves the application of the wrapped Floer homology to conclude that, if the wrapped Floer homology vanishes, then the symmetric capacity is finite. The proof of this follows very closely the strategy of the standard case via spectral invariants, although some significant differences arise considering the restriction to symmetric Hamiltonians in the interior of the symmetric manifold. We will point the differences out explicitly so that the reader can be aware.

Chapter 6 makes use of the results of Chapter 5 by searching for suitable local systems that make the wrapped Floer homology vanish.

The appendix is intended as a brief look in how the isomorphism between the symplectic homology of the cotangent bundle and the homology of the free loop space of the base manifold can fail when we don't take into account a twist in the orientations. This is relevant since we use results in this direction in Chapter 6.

Chapter 2

Floer Homology

In this chapter, we will briefly recall the main facts regarding Floer homology on the simplified compact aspherical case with vanishing first Chern class.

2.1 Introduction and motivation from Morse homology

The definition of Floer homology is inspired by the case of Morse homology. Let us briefly describe the idea of Morse homology.

Let M be a compact manifold together with a nondegenerate smooth function $f : M \to \mathbb{R}$. Here, nondegenerate means that all critical points have an invertible Hessian. For each natural number i, we define the abelian group C_i to be the free abelian group generated by the critical points with index i, where the index of a critical point is given by the maximum number of different linearly independent eigenvectors with negative eigenvalues. We can then define a map

$$\partial_i C_i \to C_{i-1}$$

which takes a critical point q_i to $\sum n_{c_i} c_i$, where n_{c_i} is the number of minus-gradient-flow paths that tend to q_i as $t \to -\infty$ and to c_i as $t \to +\infty$, counted with orientations in mind, and considering two such paths equivalent if they are just shifts of one another.

If the unstable manifolds intersect the stable manifolds transversely, we then have a chain complex, and thus an homology. This homology is the so-called *Morse homology*, usually denoted by $HM(M;\mathbb{Z})$. For quick explicit examples and details about the orientation, c.f. [18]. It can be proved that $HM(M;\mathbb{Z}) \simeq H(M;\mathbb{Z})$, where the latter is singular homology. This can be found in [25], for example. The technical details about Morse homology can also be seen there.

Leaving aside these details about the construction, note that it seems natural from the definition that we can infer facts about critical points when we know the Morse homology. Indeed, a trivial observation is that given any compact manifold M and non-degenerate smooth function $f : M \to \mathbb{R}$, then there are at least as many critical points of f as the total rank of the homology. For example, every non-degenerate smooth function on $S^1 \times S^1$ must have at least 4 critical points. [1]

[1] Non-degeneracy is important here. There are smooth functions on the torus with 3 critical points.

This ability of Morse homology to find critical points can be seen as a hook for an idea to investigate periodic orbits of Hamiltonian flows via a similar strategy: that of considering those orbits as critical points of a certain action functional on the loop space.

2.2 Floer homology of compact symplectic aspherical manifolds with vanishing first Chern class

For the purposes of this section, we assume that $\omega|_{\pi_2(M)} = 0$, which is to say that $\int_s \omega = 0$ for any $s : S^2 \to M$, and that the first Chern class of M is zero, which will be relevant for grading the homology.

The situation can be seen to arise from a variational principle: we consider $\Lambda_0(M)$ (the connected component of the contractible orbits of the free loop space[2]) and, given a Hamiltonian H on M, we define the action functional

$$\begin{aligned} A_H : \Lambda_0(M) &\to \mathbb{R} \\ x &\to -\int u^*\omega + \int H \circ x, \end{aligned}$$

where u is a capping disk for x. (i.e., a map from the disk which coincides with x in the boundary.) This is possible due to the fact that x is contractible, and well-defined since M is assumed to satisfy the requirement of $\omega|_{\pi_2(M)} = 0$.

The critical points of such a functional are the periodic orbits of the Hamiltonian flow, as can be seen by the following calculation.

Proposition 2.2.1. The critical points of A_H are precisely the periodic orbits of the Hamiltonian flow.

Proof. Let x_s be a variation of x which gives rise to a variational vector field Y and a variation v of u. Then

$$\begin{aligned} dA_H(Y) = \frac{d}{ds} A_h(x_s) &= -\int_{D^2} j_0^* \mathcal{L}_{\partial_s} v^*\theta + \int_0^1 dH(Y)dt \\ &= -\int_{D^2} j_0^*(d\iota_{\partial_s} v^*\omega) + \int_0^1 dH(Y)dt \\ &= -\int_{D^2} d(j_0^*\iota_{\partial_s} v^*\omega) + \int_0^1 dH(Y)dt \\ &= -\int_{S^1} j_0^*\iota_{\partial_s} v^*\omega + \int_0^1 dH(Y)dt \\ &= -\int_{S^1} j_0^*\omega(\partial_s v_*, v_*) + \int_0^1 dH(Y)dt \end{aligned}$$

[2]Of course, more care should be taken when considering the smoothness of paths, and the proper course of action is easily seen *a posteriori* to be to consider a Sobolev space of curves. But this will not be relevant here. We assume $\Lambda_0(M)$ to consist of smooth paths.

$$= \int_0^1 -\omega(Y, \dot{x}) + dH(Y)dt.$$

Since x_s is arbitrary and therefore also Y, we have that $dA_H = 0$ if and only if $dH = \omega(\cdot, \dot{x})$, from which the result follows. □

Following now the analogy with Morse theory, we would like to know what is the gradient of A_H so that we can study what should be the gradient flow. This is almost contained in the previous proposition.

Proposition 2.2.2. Let g, J be a compatible Riemann metric and J-complex structure, respectively, with the symplectic form ω. Then the gradient of A_H with respect to the inner product

$$\langle X, Y \rangle = \int_{S^1} g(X, Y)$$

is given by

$$(\nabla A_H)_x = J\dot{x} + \nabla H.$$

Proof. From Proposition 2.2.1, we have that

$$dA_H(Y) = \int_0^1 -\omega(Y, \dot{x}) + dH(Y)dt$$

Therefore,

$$\begin{aligned} dA_H(Y) &= \int_0^1 g(Y, J\dot{x}) + g(\nabla H, Y)dt \\ &= \int_0^1 g(J\dot{x} + \nabla H, Y)dt \end{aligned}$$

□

Therefore, if we are looking for the flow of $-\nabla$, we are searching for cylinders $u : S^1 \times \mathbb{R} \to M$ that satisfy

$$\partial_s u = -J\partial_t u - \nabla H,$$

i.e. solutions u for the equation

$$\partial_s u + J\partial_t u + \nabla H \circ u = 0. \tag{2.1}$$

Equation 2.1 is called *Floer's equation.* We will denote the operator $\partial_s + J\partial_t + \nabla H$ by $\mathcal{F}$.

It would be convenient if we could proceed *a la* Morse Theory, but this is not the case: the flow of $-\nabla A_H$ is not well-behaved enough for that. (e.g., the indices of the critical points can very well be infinite. Furthermore, the flow is also not globally defined.)

The workaround is to directly consider a moduli space of solutions connecting two critical points and build the theory from there, establishing the boundary map and the fact that

$\partial^2 = 0$, for instance. In a sense, we avoid considering the unstable manifold of a critical point x and instead consider just the paths that converge to some other critical point.

Recall that in Morse homology we can assign indices to the critical points by taking the largest possible number of linearly independent negative eigenvectors. Here this assignment would not be possible, since that number is infinite in general. Instead, we assign to each periodic orbit x an index in the following way:

- We pick a trivialization of x^*TM.
- We look at the symplectic path $\widetilde{x} : S^1 \to \mathrm{Sp}(2n)$ that the linearized flow of x gives with respect to such trivialization.
- We take the Conley-Zehnder index of such path, and denote this by $i(x)$. (c.f. [6].)

The fact that this does not depend on the chosen trivialization of x^*TM is due to the vanishing of the first Chern class. This discussion of the assignment of indices exists since we are attempting to keep a close parallel with the Morse case in this first moment. However, we will not need the grading of the homology for our future purposes. This will become clear later.

In any case, we now have an index assignment for the periodic orbits, and thus we can construct a chain complex like in the Morse case: we consider C_i to be the free $\mathbb{Z}_2$-module generated by the periodic orbits. We are now using $\mathbb{Z}_2$ coefficients to avoid further discussions about orientation.

Now, given two periodic orbits x_1, x_2, define

$$\widetilde{\mathcal{M}}(x_1, x_2) := \{u : S^1 \times \mathbb{R} \to M \mid \mathcal{F}u = 0; \ \lim_{s \to -\infty} u(\cdot, s) = x_1; \ \lim_{s \to +\infty} u(\cdot, s) = x_2\}.$$

In other words, $\widetilde{\mathcal{M}}(x_1, x_2)$ consists of the solutions of Floer's equation which converge uniformly to x_1 as $s \to -\infty$ and to x_2 as $s \to +\infty$. In a sense, we are looking at the negative gradient flow, but we are restricting our attention to the ones that converge to some other periodic orbit. It can be shown that, if H is generic in a certain sense, then $\widetilde{\mathcal{M}}(x_1, x_2)$ is a manifold with dimension being $i(x_1) - i(x_2)$.

Note that if u satisfies the Floer equation, then so does $u(\cdot, c + \cdot)$ where c is constant. Therefore, there is an $\mathbb{R}$-action on $\widetilde{\mathcal{M}}(x_1, x_2)$ given by such translation. This is a free (if $x_1 \neq x_2$) and proper action, and thus the quotient is a manifold, which we denote by $\mathcal{M}(x_1, x_2)$.

It can be shown that $\mathcal{M}(x_1, x_2)$ can be compactified by adding broken orbits, which are essentially concatenations of intermediary solutions to the Floer equation. If $i(x_1) = i(x_2) + 1$, it follows that $\mathcal{M}(x_1, x_2)$ is already compact. Being also zero-dimensional, we have a finite number of elements in $\mathcal{M}(x_1, x_2)$. This is what we need to define the boundary map, which is

$$\begin{aligned} \partial_i : C_i &\to C_{i-1} \\ x &\mapsto \sum_y (\#\mathcal{M}(x, y) \mod 2) y. \end{aligned}$$

The fact that $\mathcal{M}(x_1, x_2)$ can be compactified by adding broken orbits also proves that $\partial_{i+1}\partial_i = 0$, establishing that this is indeed a chain complex and therefore it generates an homology

$$HF(M, (H, \omega, J); \mathbb{Z}_2).$$

We are intentionally overloading the notation in order to point out that, a priori, the definition of the Floer homology depends on that entire data.

These aspects of compactness (i.e. compactification by broken trajectories, the compactness of the set of solutions of the Floer equation with finite energy etc) come up in the development of other Floer-like homology theories and are important for the same reasons that they are here. It is often a subtle issue; indeed, if we do not assume asphericity (i.e., $\omega|_{\pi_2(M)} = 0$), then we do not have compactness of the set of Floer solutions with finite energy. This phenomenom is usually referred to as "bubbling", due to it being associated with the creation of extraneous spheres under a limit process. (For more information, c.f. [6].)

The fact that $\widetilde{M}(x_1, x_2)$ is a manifold for a generic H is related to the issue of *transversality*. Just as *compactness* appears many times as an important point in construction of Floer-like homologies, transversality also does. The situation is this:

We have a map

$$\begin{aligned} \mathcal{F} : C^\infty(S^1 \times \mathbb{R}; M) &\to C^\infty(S^1 \times \mathbb{R}; TM) \\ u &\mapsto \mathcal{F}(u). \end{aligned}$$

The space $C^\infty(S^1 \times \mathbb{R}; TM)$ projects in $C^\infty(S^1 \times \mathbb{R}; M)$ by $\Pi : U \mapsto \pi \circ U$, where π is the projection $\pi : TM \to M$. As such, we have that $\Pi^{-1}(u)$ consists of vector fields along u. We then have that $C^\infty(S^1 \times \mathbb{R}; TM)$ can be understood as a bundle over $C^\infty(S^1 \times \mathbb{R}; TM)$. Let us denote this bundle by E.

Under this point of view, $\mathcal{F}$ is a section of E. It follows that if $\mathcal{F}$ is *transverse* to the zero-section, then the space of solutions to the Floer equation is a manifold[3], and so the issue of transversality manifests itself.

It will also be relevant to make a quick comment regarding why

$$H(M, (H_1, J_1); \mathbb{Z}_2) \cong H(M(H_2, J_2); \mathbb{Z}_2).$$

We can consider an homotopy H_s between H_1 and H_2, an homotopy J_s between J_1 and J_2 and look at the solutions of the modified Floer equation

$$\partial_s u + J_s \partial_t u + \nabla H_s \circ u = 0.$$

We can, just as before, consider moduli spaces for this equation. This allows us to define chain maps between the two chain complexes, and those end up inducing isomorphisms on the homology level. These types of maps are called *continuation maps* and are important due to several reasons, the above being only one. They also are used in the construction of the symplectic homology which we will see in the next section.

One thing which is also worthy of notice is that we can also do a *filtration* in the chain complex by considering only orbits in a given interval of action, i.e. given an interval $I \subset \mathbb{R}$, define

$$C_i^I := \{c = \sum n_\alpha x_\alpha \in C_i \mid \forall x_\alpha, A_H(x_\alpha) \in I\}.$$

[3]Here it is important to consider Sobolev maps instead of smooth ones in order to have a Banach manifold, but as previously mentioned we do not enter in technical details.

Since the action is decreasing from one orbit to its boundaries, it follows that C_*^I together with ∂ is still a chain complex. Indeed, it suffices to show that $\delta^2 x = 0$ for x a closed orbit, where we are denoting the restriction of ∂ to the subcomplex by δ for clarity.. We have that

$$\begin{aligned}\delta\delta x &= \delta(\sum n_\alpha y_\alpha + \widehat{\sum n_\beta y_\beta}) \\ &\sum n_\alpha \delta y_\alpha + \widehat{\sum n_\beta \delta y_\beta} \\ &= \sum n_\alpha n_{\alpha'} y_{\alpha\alpha'} + \sum \widehat{n_\alpha n_{\alpha''} y_{\alpha\alpha''}} + \sum \widehat{n_\beta n_{\beta'} y_{\beta\beta'}},\end{aligned}$$

where a hat indicates that the term is not present in the filtration due to the action of its elements not being in I. Since $\partial^2 = 0$, this last term without the hats is equal to 0. But since each term under the hats must necessarily have lower action than what is not in the hat, it follows that in order for things to cancel out, $\sum n_\alpha n_{\alpha'} y_{\alpha\alpha'}$ must itself be zero, and thus it follows that the filtered complex is indeed a chain complex. Its homology will be denoted by

$$HF^I(M, (H_1, J_1); \mathbb{Z}_2).$$

2.3 Grading

Although for our purposes we do not need to develop the index theory to make (wrapped) Floer homology graded, it is nonetheless worth it to address it.

In the periodic case, the procedure is summarized as follows:

(i) Given a periodic orbit x and a capping disk u, we trivialize u^*TM, inducing a trivialization on the boundary and thus a trivialization of x^*TM. Therefore, we have a path $\widetilde{x} : S^1 \to Sp(2n)$ given by the linearized flow. By the assumption of nondegeneracy, this path ends at a matrix which does not have 1 as an eigenvalue.

(ii) Using an appropriate map $\rho : Sp(2n) \to S^1$ that induces as isomorphism in homology groups and has good properties, we can compose ρ with the inclusion $\iota : Sp(2n)^* \to Sp(2n)$, where $Sp(2n)^*$ consists of symplectic matrices which do not have 1 as eigenvalue.

(iii) Given a path $\gamma : [0,1] \to Sp(2n)$, $\rho \circ \gamma$ lifts to a map to the universal cover of S^1, which is $\mathbb{R}$. Call this lift L_γ, and define

$$\Delta(\gamma) := \frac{L_\gamma(0) - L_\gamma(1)}{\pi}.$$

(iv) Having fixed matrices A^+, A^- on the two connected components of $Sp(2n)^*$, we define the *Maslov index* of an orbit x to be

$$\mu(x) := \Delta(\widetilde{x}) + \Delta(\gamma_{\widetilde{x}(1),A^\pm}),$$

where $\gamma_{\widetilde{x}(1),A^\pm}$ is a path on $Sp(2n)^*$ connecting $\widetilde{x}(1)$ and $A^\pm$, the later depending on which connected component of $Sp(2n)^*$ is $\widetilde{x}(1)$.

Remark 2.3.1. There are some choices done in the previous construction. The choice of the path connecting $\gamma_{\widetilde{x}(1),A^{\pm}}$ is a nonissue because the inclusion of $Sp(2n)^* \to Sp(2n)$ induces the zero map on the fundamental group. This is a nontrivial topological fact of the symplectic group which can be found in [6], along with a full description of this entire approach to grading. Furthermore, the choice of a capping disk made in item (i) is a nonissue since we assume that the first Chern class vanishes.

As we can see, the fundamental part is essentially the isomorphism $\pi_1(Sp(2n)) \cong S^1$. We will return to this when we discuss Floer homology from the Lagrangian point of view in the next chapter. However, we seize the opportunity and mention that the set of Lagrangian subspaces of $\mathbb{R}^{2n}$ is diffeomorphic to $U(n)/O(n)$, and thus also has fundamental group isomorphic to $\mathbb{Z}$.

The relationship between the (difference of) the indices and the dimension of the moduli space is not clear at all from the above definition, but can be seen to follow from Riemann-Roch-type procedures.

2.4 Symplectic homology of Liouville domains

We now turn to describe symplectic homology of Liouville domains, which is close to the wrapped Floer homology context which we are ultimately aiming at. Indeed, the domain of discourse will be the same (i.e., Liouville domains); what will change are the orbits and the action functional.

A *Liouville domain* is a symplectic $2n$-manifold X with boundary where the symplectic form is given by $d\lambda$, with λ being a contact form positively-oriented on the boundary, in the sense that $\lambda \wedge d\lambda^{n-1} > 0$. Liouville domains come automatically equipped with a special vector field Z, the *Liouville vector field*, determined by the equation $\iota_Z d\lambda = \lambda$.

This vector field allows us to *complete* X with rays in order to embed it into a symplectic manifold without boundary. We do this as follows:

- Consider the embedding $\mathrm{Tub} : \partial X \times (0, 1] \to X$ defined by

$$\mathrm{Tub}(z, 1) = z, \quad \partial_r \mathrm{Tub}(z, r) = \frac{1}{r} Z(\mathrm{Tub}(z, r)).$$

  This is essentially just integrating radially the Liouville vector field to parametrize a tubular neighbourhood of the boundary of X.

- Then, glue $\partial X \times [1, +\infty)$ along ∂X via Tub. More precisely, let

$$\widehat{X} := X \cup_{\mathrm{Tub}} \partial X \times (0, +\infty)$$

  and define $\widehat{\lambda}$ as

$$\widehat{\lambda}(x) := \begin{cases} \lambda(x), & x \in X \\ r\lambda(z), & x \in \partial X \times (0, +\infty). \end{cases}$$

The manifold $\widehat{X}$ obtained above is called the *completion* of X. We will often refer to the second coordinate of $\partial X \times (0, +\infty)$ as the radial coordinate.

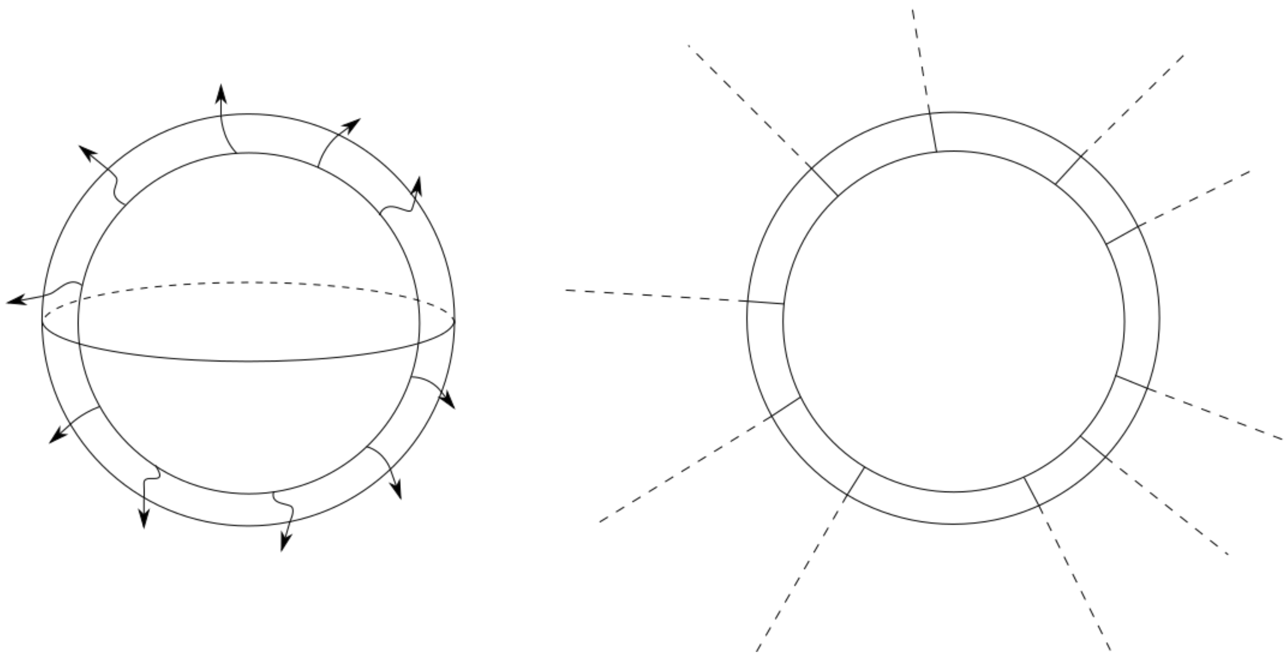


Figure 2.1: On the left, we have the image of Tub. On the right, we have $\partial X \times [0, +\infty)$. The $[0, 1)$ part is shown with the corresponding segments.

A Hamiltonian is said to be *homology-admissible* [4] if all its periodic orbits are nondegenerate and there exists $R > 0$ such that H is linear with respect to the radial coordinate when $r > R$, i.e. there exists a_H, b_H such that $H(z, r) = a_H r + b_H$ for $r > R$.

Before defining the symplectic homology for those Hamiltonians, we need one further restriction: we will require that almost-complex structures J are of *contact type* on $\partial X \times I$, meaning that $dr \circ J = -\lambda$.

Now, given an homology-admissible H, we consider the action functional

$$\begin{aligned} A_H : \Lambda(\widehat{X}) &\to \mathbb{R} \\ x &\mapsto -\int x^*\lambda + \int H \circ x. \end{aligned}$$

A similar computation to that of Proposition 2.2.1 shows that the critical points are the periodic orbits. Note that the actional functional is almost equal to the previous case, but this time the fact that the symplectic form is exact simplifies it. For example, we do not need to choose capping disks for x, nor do we need to restrict to contractible orbits. Also just as before, given x, y periodic orbits, we have the moduli space $\mathcal{M}(x, y)$, which is the space of solutions of Floer equations which converge to x, y as $s \to -\infty$ and $+\infty$ respectively.

As before, some restriction on the first Chern class must be imposed if a $\mathbb{Z}$-grading is desired. In any case, though, we can define the total complex

$$C(\widehat{X}, H) := \text{FreeAb}(\{\text{Periodic orbits of } X_H\}),$$

[4] This nomenclature is nonstandard. We introduce it due to the fact that the term *admissible* is overused in the literature among different non-equivalent meanings.

where, given a set S, FreeAb(S) stands for the free abelian group generated by S, and the boundary map

$$\partial x = \sum_{\alpha} n_\alpha y_\alpha,$$

where y_α runs over all periodic orbits such that $\mathcal{M}(x, y_\alpha)$ is zero-dimensional, and n_α is the cardinality of $\mathcal{M}(x, y_\alpha)$ mod 2.

The homology of this complex will be denoted by

$$SH(\widehat{X}, H).$$

Note that the construction above does *not* give a grading. As mentioned, in order for a grading to exist, we need to have some topological hypothesis related to how trivializations behave, which is codified by the first Chern class, so that we can consistently assign indices to the orbits.

Differently from the case of the previous section, changing H here can yield different homologies. This is inherently related to the behaviour of the slope at infinity of H. (Recall that our Hamiltonians must be linear at infinity.) It is now crucial then to understand what happens when we change H, and the continuation maps alluded to in the previous section are fundamental for that.

Let H_-, H_+ be two homology-admissible Hamiltonians, J_-, J_+ be two contact-type almost-complex structures and $(H_s, J_s)_{s\in\mathbb{R}}$ be an interpolation between (H_-, J_-) and (H_+, J_+) in the sense that for large $|s|$, we have $H_s, J_s = H_\pm, J_\pm$. Some care should be taken: we need this deformation to be monotone for large enough r in order for the maximum principle to apply and deal with compactness, and we need it to be generic in order to solve the issues of transversality. This means that for large enough r, the Hamiltonians are a radial function h with respect to the second variable of the parametrization (z, r) and $\partial_s h'_z \leq 0$ there.

We then consider the *parametrized Floer equation*

$$\partial_s u + J_s \partial_t u + \nabla H_s \circ u = 0.$$

This allows us to define a *continuation map* between the homologies with respect to the data (H_+, J_+) and (H_-, J_-), by defining on the chain level

$$\Psi_{(H_+,J_+),(H_-,J_-)}(x) = \sum n_\alpha y_\alpha,$$

where the y_α run over all periodic orbits for which the space of solutions of the Floer equation is a 0-dimensional manifold, and n_α is the cardinality of such space mod 2. Note that in this case there is no quotient by a $\mathbb{R}$-action, indeed the solutions do not respect translation on the s-coordinate this time.

Moreover, we have the following result which can be found in [23]:

Lemma 2.4.1. With the previous context in mind, the following are true.

(i) Different interpolations induce chain homotopic maps on the chain level, and thus equal maps in homology. More precisely, if (H_s^1, J_s^1) and (H_s^2, J_s^2) are interpolations between (H_-, J_-) and (H_+, J_+) respecting the aforementioned asymptotic requirement, then $\Psi^1_{(H_+,J_+),(H_-,J_-)}$ is chain homotopic to $\Psi^2_{(H_+,J_+),(H_-,J_-)}$, and thus these maps coincide on the homology level.

(ii) The constant interpolation induces the identity on homology.

(iii) If we have an interpolation between H_- and K, and then K and H_+, those interpolations provide a composition of continuation maps

$$SH(\widehat{X}, H_-) \to SH(\widehat{X}, K) \to SH(\widehat{X}, H_+)$$

that coincides with Ψ_{H_-,H_+}.

(iv) If H_- and H_+ have the same slope at infinity, then Ψ_{H_-,H_+} is an isomorphism.

Note that *(iv)* is a consequence of the others. Indeed, since they have the same slope at infinity, we can interpolate them in a way that is constant for large enough r, and thus the interpolation and its reverse are both monotone. Therefore, we have

$$SH(\widehat{X}, H_-) \to SH(\widehat{X}, H_+) \to SH(\widehat{X}, H_-).$$

By *(iii)*, this coincides with the map originated from the constant interpolation. By *(ii)*, this is the identity, so *(iv)* follows.

If H_-, H_+ are two different Hamiltonians, then if their linear coefficients at infinity a_-, a_+, respectively, are such that $a_- \geq a_+$, then we have a continuation map

$$\Psi_{H_-,H_+} : SH(\widehat{X}, H_-) \to SH(\widehat{X}, H_+)$$

by taking a monotone homotopy, which is possible due to the fact that $a_- \geq a_+$. This tells us that the homologies $SH(\widehat{X}, H)$ together with the continuation maps form an inverse system. The inverse limit of such system is defined as the *symplectic homology* of $\widetilde{X}$, i.e.

$$SH(\widehat{X}) := \lim_{\leftarrow} SH(\widehat{X}, H).$$

We should note that there are various ways of constructing symplectic homology. This one is due to Viterbo, but one can also find others such as the one by Cieliebak-Floer-Hofer.

Chapter 3

Wrapped Floer (co)homology

3.1 Introduction

In this chapter, we will define the wrapped Floer (co)homology. As it will become clear, it can be seen as an adaptation of symplectic (co)homology that switches the attention towards orbits that connect a Lagrangian to itself instead of periodic orbits.

3.2 Action functional

Let (X^{2n}, λ) be a Liouville domain. Let $L^n \subset X$ be an exact Lagrangian submanifold, i.e., the pull-back $\lambda|_L$ is exact, i.e., $\lambda|_L = df$ for some smooth function f, and that $\lambda|_L$ vanishes near ∂L. Suppose also that $\partial L = L \cap \partial X$ and that ∂L is Legendrian. Similarly to the extension of X to the completion $\widehat{X}$ that was done in the last chapter, we can extend L to

$$\widehat{L} = L \cup ([1, \infty) \times \partial L),$$

and f will be locally constant in $\widehat{L} \backslash L$ since $\lambda|_L$ vanishes near ∂L and we naturally extend it to 0.

Given a Hamiltonian $H : \widehat{X} \to \mathbb{R}$, a *Hamiltonian chord* is a path $x : [0, 1] \to \widehat{X}$ satisfying Hamilton's equation $\dot{x} = X_H$ and such that $x(0), x(1) \in \widehat{L}$. The space of smooth paths connecting $\widehat{L}$ to $\widehat{L}$, i.e.

$$\Omega(\widehat{X}, \widehat{L}) = \{x \in C^\infty([0, 1], M) : x(0), x(1) \in \widehat{L}\}$$

is where we define our action functional

$$\begin{aligned} A_H : \Omega(\widehat{X}, \widehat{L}) &\to \mathbb{R} \\ x &\mapsto f(x(1)) - f(x(0)) - \int x^*\lambda + \int H \circ x. \end{aligned}$$

The additional term $f(x(1)) - f(x(0))$ when compared to the action functional in standard Floer (co)homology (i.e., of periodic orbits) comes from the freedom of the boundary condition, and is required for recovering the fact that the critical points are the Hamiltonian chords.

Theorem 3.2.1. A path is a critical point for the action functional if and only if it is a Hamiltonian chord.

Proof. By essentially repeating the proof of the case of the free loop space, we arrive at the following formula:

$$
\begin{aligned}
dA_H(Y) = \frac{d}{ds}A_h(x_s) &= \frac{d}{ds}f(x_s(1)) - \frac{d}{ds}f(x_s(0)) - \int_0^1 j_0^*\mathcal{L}_{\partial_s}v^*\lambda + \int_0^1 dH(Y)dt \\
&= \lambda(Y_1) - \lambda(Y_0) - \int_0^1 j_0^*(d\iota_{\partial_s}v^*\lambda - \iota_{\partial_s}v^*\omega) + \int_0^1 dH(Y)dt \\
&= \lambda(Y_1) - \lambda(Y_0) - \int_0^1 d(j_0^*\iota_{\partial_s}v^*\lambda) - \int_0^1 j_0^*\iota_{\partial_s}v^*\omega + \int_0^1 dH(Y)dt \\
&= \lambda(Y_1) - \lambda(Y_0) - (\lambda(Y_1) - \lambda(Y_0)) - \int_0^1 j_0^*\iota_{\partial_s}\omega(v_*\cdot, v_*\cdot) + \int_0^1 dH(Y)dt \\
&= -\int_0^1 j_0^*\omega(\partial_s v, v_*) + \int_0^1 dH(Y)dt \\
&= \int_0^1 -\omega(Y,\dot{x}) + dH(Y)dt,
\end{aligned}
$$

from which the result follows. □

We now define the end-point Floer cohomology in very much the same way by following the procedure outlined below.

(i) We define the spaces $\widehat{\mathcal{W}}(x_-, x_+)$ of the solutions of the Floer equation

$$\partial_s u + J(\partial_t u - X) = 0,$$

this time with Lagrangian boundary conditions (i.e., $u(\cdot, 0)$ and $u(\cdot, 1)$ both belonging to $\widehat{L}$), which converge in $s \to \pm\infty$ to $x_\pm$ and then consider the moduli space $\mathcal{W}(x_-, x_+) := \widehat{\mathcal{W}}(x_-, x_+)/\mathbb{R}$, where the $\mathbb{R}$-action is given by translation in the s-coordinate.

(ii) Given a field $\mathbb{K}$, we define $CW^*(L, H)$ as the $\mathbb{K}$-vector space generated by the Hamiltonian chords.

(iii) We define a differential in $CW^*(L, H)$ given by the counting with orientation sign of the isolated trajectories connecting the elements of the basis. Explicitly, we define d on a basis element y by

$$dy = \sum_{u \in \mathcal{W}_0(x,y)} \epsilon_u x,$$

where $\mathcal{W}_0(x, y)$ consists of the isolated points of $\mathcal{W}(x, y)$, except the constant solutions.

Remark 3.2.2. The change of defining

$$dy = \sum_{u \in \mathcal{W}_0(x,y)} \epsilon_u x$$

instead of

$$dx = \sum_{u \in \mathcal{W}_0(x,y)} \epsilon_u y$$

is what defines the difference between cohomology and homology respectively, at least as far as basic definitions go. We opt for cohomology because this seems more natural for our applications, since spectral invariants are defined via the (singular) cohomology of the Lagrangian L. (Recall that singular cohomology has a distinguished element, the unit $1 \in H^0$.)

Transversality and compactness for a generic time-dependent perturbation of J both hold true, as in the Floer case. Indeed, the wrapped Floer homology is a special case of Lagrangian Floer homology. We make a brief commentary in the next section.

We will not enter the discussion of the orientation signs, since it suffices to consider the homology with $\mathbb{Z}_2$ coefficients in our applications. We will also not use any grading on the homology. This is possible if the relative Chern class $c_1(M, L)$ vanishes, similarly to what occurs in the Floer case. (Of course, there it is the Chern class of the manifold in question only.)

3.3 Lagrangian Floer Homology

Although we have explored the fundamentals of Floer homology through its focus on periodic and brake orbits, there is an alternate more general Lagrangian approach which encompasses both. We discuss it briefly since it will be useful in order to talk about the index in the wrapped context and also because it is another way to be at ease with the analytical aspects of the wrapped Floer homology instead of thinking it in terms of reproducing everything done for the periodic case.

Let (M, ω) be a symplectic manifold with two compact Lagrangians L_0, L_1 that intersect transversally. We again make an assumption of asphericity on M. We can consider the space of paths

$$\Omega(L_0, L_1) := \{\gamma : [0,1] \to M \mid \gamma(0) \in L_0, \gamma(1) \in L_1\}.$$

Fixed some $\alpha \in \Omega(L_0, L_1)$, we take $\Omega(L_0, L_1; \alpha)$ to mean the connected component that contains α. Consider the universal covering $\widetilde{\Omega}(L_0, L_1; \alpha)$ of $\Omega(L_0, L_1; \alpha)$,. By using the usual construction of the universal covering, we can see elements of $\widetilde{\Omega}(L_0, L_1; \alpha)$ as homotopy classes, and thus an element of $\widetilde{\Omega}(L_0, L_1; \alpha)$ can be represented by a pair $[\![(\gamma, h)]\!]$, where $h : [0,1] \times [0,1] \to M$ is such that $h(s, \cdot) \in \Omega(L_0, L_1; \gamma)$ for all s, $h(0, \cdot) = \alpha$ and $h(1, \cdot) = \gamma$.

The action functional is then defined as the symplectic area of w. More specifically,

$$\begin{aligned} A : \widetilde{\Omega}(L_0, L_1; \alpha) &\to \mathbb{R} \\ [\![(\gamma, h)]\!] &\to \int h^* \omega. \end{aligned}$$

Assuming for simplicity that the Lagrangians are simply connected, it is relatively straightforward to conclude that A is well-defined. [1] If we want to compute the critical points, we

[1]If this were not the case, then we would need to make an adaptation for the space $\widetilde{\Omega}(L_0, L_1; \alpha)$.

see that after fixing some compatible structure g, J we get

$$\begin{aligned} dA_{(\gamma,h)}(Y) &= \int_0^1 \iota_Y \omega(\partial_t \gamma, \cdot) dt \\ &= \int_0^1 \omega(\partial_t \gamma, Y) dt \\ &= \int_0^1 g(\partial_t \gamma, -JY) dt. \\ &= \int_0^1 g(J\partial_t \gamma, Y) dt, \end{aligned}$$

so that $[\![(\gamma, h)]\!]$ is a critical point if and only if $J\partial_t\gamma = 0$, which is to say that γ is constant, meaning that γ is actually an intersection point between L_0 and L_1. Not only that, but the above computation shows us that the L^2-gradient of the actional functional is given by

$$J\partial_t\gamma,$$

so that the equation for the minus gradient flow becomes

$$\partial_s u + J\partial_t u = 0,$$

which is just a Cauchy-Riemann equation, contrasted to the Floer equation, which is a perturbed Cauchy-Riemann equation.

The solutions of this Cauchy-Riemann equation with the appropriate boundary requirements are called *J-holomorphic strips*, and are the analogues of the Floer equation's solutions in the periodic case.

We now establish the relationship between this construction and the traditional (periodic) Floer homology, and also the wrapped Floer homology.

Perhaps simpler is the relationship with the wrapped Floer homology. Indeed, it is equivalent to the Lagrangian Floer homology with the Lagrangians L and $\phi_H^1(L)$, where ϕ_H^1 is the time-one flow map generated by the Hamiltonian flow of H.

The (periodic) Floer homology can be recovered by taking the Lagrangians on $M \times M$ instead, with one Lagrangian being the diagonal and the other being the graph of the time-one flow, i.e. the set $\{(x, \phi_H^1(x)) \mid x \in M\}$.

3.4 Grading

We will present two alternatives for the index: one is more directly related to the idea of winding number of the previous chapter, and the other is related to the counting of regular crossings.

The first one is as follows: First we consider the case of a compact surface with boundary S and a pseudo-holomorphic curve $u : S \to M$ which takes its boundary components C_i to Lagrangians L_i, we consider the pull-back bundle u^*TM. This is a complex bundle. The Maslov indexes of the restriction of this bundle to the boundary components impose a quantifiable condition on whether the subbundles u^*TL_i over the boundary components C_i are trivializable or not. To be explicit, let C_i be a boundary component. Then a trivialization of u^*TM yields a map $\widetilde{u}_i : C_i \to \mathcal{L}(\mathbb{C}^n)$ given by taking x to the representation of u^*TL_i on this trivialization. This yields a loop on $\mathcal{L}(\mathbb{C}^n)$, and the integer associated to it under the isomorphism $\mu : \pi_1(\mathcal{L}(\mathbb{C}^n)) \to \pi_1(S^1)$ is the index which we are associating to C_i.

An holomorphic strip is a map from $S = \mathbb{R} \times [0, 1]$ to M, and thus we are not exactly in the same situation as above. We adapt the index to this situation as follows: letting the boundaries $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ be denoted by B_0 and B_1 respectively, it is clear that both are lines instead of circles as per the case we mentioned above. Choosing a trivialization of u^*TM such that the induced ones of u^*TL_0 and u^*TL_1 are constant for $|s| > R$ for some big enough R, we can build a loop on $\mathcal{L}(\mathbb{C}^n)$ in the following way:

(i) Truncate the map u to some region $|s| < R'$ such that $R' > R$. Effectively, what we are doing is restricting to a manifold with boundary (actually, corners) in order to adapt the situation described above.

(ii) Once done so, there are two obvious paths of Lagrangians: the restrictions of u to the top interval and the bottom interval. What is left to be done is an interpolation between both in order to give a loop on $\mathcal{L}(\mathbb{C}^n)$.

(iii) In order to do this interpolation, we take the pseudo-complex structure J on M in such a way that $JT_xL_0 = T_xL_1$ in T_xM and $JT_yL_0 = T_yL_1$ in T_yM, where x, y are the asymptoptic points of the holomorphic strip. Then, we just interpolate via

$$\exp((-t\pi/2)J)T_xL_1$$

in the left side of the strip (the one that converges to x) and

$$\exp((t\pi/2)J)T_xL_0$$

in the right side (the one that converges to y).

With this done, we have a loop on $\mathcal{L}(\mathbb{C}^n)$, and we assign to a given holomorphic strip u with asymptoptics x, y the index $\mu(x, y, u)$ being the Maslov index of this loop. (c.f. [19].)

Setting aside the technical aspects of dependency on the holomorphic strip u, note that this construction does *not* assing an index to a given intersection point. It assigns an index having chosen two points, so this is only responsible a priori for indirectly computing the dimension of the moduli space. In other words, it gives a relative index. In order to grade the homology properly, we of course need some index-like function ν such that $\nu(x) - \nu(y) = \mu(x, y)$.

In order for this to be possible, the dependency on u must be addressed. It turns out that by taking different u's, the index $\mu(x, y, u)$ changes by factors depending on the Maslov classes of the two Lagrangians and the first Chern class of TM. So under the hypothesis that all those vanish, then we can grade the homology. (For more information, c.f. [20].)

Now let us briefly discuss the alternate point of view related to crossings. This is useful since it touches upon the concept of the *Robin-Salamon index*, which allows us to talk about indices for a great range of paths which can allow degeneracies, for example, and are applicable to other situations. Recalling that we wish to assign to a path of Lagrangians on $\mathbb{C}^n$ an index, we proceed as follows:

(i) Let Λ_0, Λ_1 be Lagrangians and suppose without loss of generality that $\Lambda_0 = \mathbb{R}^n \times \{0\}$. Then Λ_1, which we assume to complement Λ_0 in the sense of $\mathbb{C}^n = L_0 \oplus L_1$, can be written as

$$\Lambda_1 = \{(Bx, x) \mid x \in \mathbb{R}^n\}$$

for some real symmetric linear map $B : \mathbb{R}^n \to \mathbb{R}^n$.

(ii) Let now $\Lambda : (-\epsilon, \epsilon) \to \mathcal{L}(\mathbb{C}^n)$ denote a path of Lagrangians and $A : (-\epsilon, \epsilon) \to \mathrm{Sym}(n)$ be a path of symmetric linear operators on $\mathbb{R}^n$ such that

$$\Lambda(t) = \{(x, A(t)x) \mid x \in \mathbb{R}^n\}.$$

Since $\mathbb{C}^n = \Lambda_0 \oplus \Lambda_1$, it follows that for each $(x, 0) \in \Lambda_0$ there is a unique $x_t \in \Lambda(t)$ such that $x = (x, 0) + x_t$ if ϵ is small enough.

(iii) Using the above, we can define the functional

$$Q_t((x, 0)) = \omega((x, 0), x_t),$$

where ω is the canonical symplectic form on $\mathbb{C}^n$.

(iv) In a similar way to item (ii), we can find using item (i) for each t a unique $y_t \in \mathbb{R}^n$ such that $x_t = (By_t, y_t)$, so that

$$(x, 0) + x_t = (x + By_t, y_t),$$

and thus by the defining property of A we have the equation $y_t = A(t)(x + By_t)$. Differentiating this with respect to t we get that $\dot{y}_0 = \dot{A}(0)x$. Also, we have

$$\begin{aligned} Q_t((x, 0)) &= \omega((x, 0), x_t) \\ &= \omega((x, 0), (By_t, y_t)) \\ &= \omega((x, 0), (y_t - A(t)x, y_t)) \\ &= \langle (0, x), (y_t - A(t)x, y_t) \rangle \\ &= \langle x, y_t \rangle, \end{aligned}$$

so that the derivative of Q_t with respect to t at 0 is $\langle x, \dot{A}(0)x \rangle$. We will call this value Ω_Λ, noting that it only depends on Λ.

(v) Now, given a Lagrangian $V \in \mathcal{L}(\mathbb{C}^n)$ and a Lagrangian path L, we define the *crossing form*

$$C(L, V, t) := \Omega_{L(t+\cdot)}|_{V \cap L(t)}.$$

The *Robbin-Salamon index* of the path L is then defined as

$$\mu_{RS}(L, V) = \frac{1}{2}(\mathrm{sgn}(C(L, V, 0)) + \mathrm{sgn}(C(L, V, T))) + \sum_{0<t<T} \mathrm{sgn}(C(L, V, t))$$

in case all crossings are regular, so that the sum in the above expression is finite.

Returning to the context of wrapped Floer homology, given a contractible brake orbit c connecting x and y, we can consider a capping half-disk, i.e. a map $u : (D^+, D^+_{\mathbb{R}}) \to (\widehat{X}, \widehat{L})$. where D^+ is the closed upper half disk of the complex plane and $D^+_{\mathbb{R}}$ is its intersection with the real numbers and the mapping notation means, as usual, that $D^+_{\mathbb{R}}$ is sent to L, such that u restricted to the upper half circle is x. We then trivialize $c^*T\widehat{X}$ such that the Lagrangian T_xL_0 corresponds to $\mathbb{R}^n \subset \mathbb{C}^n$ and then, if x is nondegenerate, define the index of x as

$$\mu(x) = \mu_{RS}(\Psi_c T_x L_0, T_x L_0),$$

where Ψ_c is the path of symplectic matrices on the trivialization given by the linearized Hamiltonian flow, and we are making a slight abuse of notation in reusing T_xL_0 to mean its realization in the trivialization.

One immediate question is: in the case where we have a symmetric Hamiltonian H, as is our case of interest, how does the index of the brake orbit relate to the index of the associated periodic orbit? The answer is given by the following proposition proved in [16], [13]. (Proposition 4.3. of [13] .)

Proposition 3.4.1. If c is a nondegenerate brake orbit connecting x and y and α is its associated periodic orbit, then

$$\mu(\alpha) = \mu(\Psi T_x L_0, T_x L_0) + \mu(\Psi_{-c} T_y L_1, T_y L_1).$$

3.5 Examples

We will have two main examples in mind in this dissertation regarding the Liouville domain X: that of the sublevel set of a Hamiltonian of the form kinectic energy plus potential energy on the cotangent bundle and L being the zero section, and the cotangent unit disk bundle with L being the conormal bundle of a given submanifold of the base manifold. To be more specific, in this last example we want to focus on the conormal bundle of fixed point sets of involutions of the base manifold. As we show next, this lifts to the conormal bundle being the fixed point set of an anti-symplectic involution of the cotangent bundle.

Proposition 3.5.1. Let $K \subset M$ be the fixed point set of a smooth involution $g : M \to M$ and assume that K is non-empty. Then $\widetilde{g} = \mathrm{flip} \circ g_\# : T^*M \to T^*M$ [2] is an anti-symplectic involution which has as fixed-point set the conormal bundle of K, where $g_\#$ denotes the lift map of a diffeomorphism g given by $g_\#(x, v) = (g(x), (dg^*)^{-1}v)$.

Proof. The map $\widetilde{g}$ is given by

$$(x, dg^*\xi) \mapsto (g(x), -\xi).$$

It is clear that $\widetilde{g}$ is an anti-symplectic involution. It is also evident that if (x, p) is a fixed-point, the first component must be an element of K.

On the other hand, ξ must satisfy

$$\xi = -\xi(dg\cdot).$$

[2] The map flip is defined by $\mathrm{flip} : T^*M \to T^*M$, $(x, p) \mapsto (x, -p)$.

Since g is an involution, dg is an involution, hence can be diagonalizable since its characteristic polynomial is $X^2 - 1$. Moreover, all the eigenvalues are either 1 or -1, with the 1's being associated with the elements of $T_x K$.

We will show that $\xi|_{T_q K}$ must be zero. Indeed, if $v \in T_q K$ then $\xi(v) = -\xi(dgv)$ implies $\xi(v) = -\xi(v)$, hence $\xi(v) = 0$.

Conversely, if $\xi|_{T_q K} = 0$, then letting v_i^+ denote the eigenvectors with eigenvalue $+1$ and v_j^- denote the eigenvectors with eigenvalue -1, we get

$$
\begin{aligned}
\xi(\sum a_i v_i^+ + \sum a_j v_j^-) &= \xi(\sum a_j v_j^-) \\
&= -\xi(\sum a_j dg(v_j^-)) \\
&= -\xi(dg(\sum a_j v_j^-)) \\
&= -\xi(\sum a_i v_i^+ + dg(\sum a_j v_j^-)) \\
&= -\xi(\sum a_i dg(v_i^+) + dg(\sum a_j v_j^-)) \\
&= -\xi(dg(\sum a_i v_i^+ + \sum a_j v_j^-)),
\end{aligned}
$$

so that the equation required of ξ is satisfied. This establishes the proof. □

This case is of geometric interest since, for example, if we let the Hamiltonian be the kinectic energy, then brake orbits in this example constitute geodesics which have projections on the base manifold leaving and arriving at K orthogonally.

In the context of the previous discussion, we will refer to the conormal bundle of K as $\mathcal{N}^*K$.

Chapter 4

Symmetric Capacity

Different notions of capacities are of central importance in symplectic topology. As explained in the introduction of this dissertation, one key example is the Hofer-Zehnder capacity. The symmectric capacity is essentially just an adaptation of this notion that takes into consideration the symmetry φ of a symmetric symplectic manifold (M, φ).

In this chapter, we introduce its definition and also investigate the case of closed surfaces. Furthermore, we make sure to check that the fundamental result that relates finiteness of the capacity of a thickening with the existence of a full-measure set of values possessing brake orbits in its level-sets is true, which is an adaptation of the well-known case of closed orbits.

Let us establish the following notation:

Definition 4.0.1. A *symmetric symplectic manifold* is a quadruple (M, L, ω, φ), where (M, ω) is a symplectic manifold and φ is an anti-symplectic involution with non-empty fixed-point set L.

4.1 Definition and basic properties

We have the following definition given in [15].

Definition 4.1.1. For a symmetric symplectic manifold (M, L, ω, φ) , possibly with boundary, we define the set of *φ-admissible Hamiltonians*, denoted here by $\mathcal{H}_a(M, L, \omega, \varphi)$, to be the set of C^2 functions satisfying the following properties.

(i) There is a compact set $K \subset M$ depending on H such that $K \subset \operatorname{int}(M)$ and H is constant equal to its maximum outside K.

(ii) There exists an open set O, also depending on H, which intersects L and where H is constant and equal to 0.

(iii) $H \geq 0$.

(iv) $H \circ \varphi = H$.

(v) H is *φ-slow*, which is to say that all brake orbits are either constant or have minimal period greater than 1.

We can then define the *symmetric capacity* of (M, L, ω, φ) to be

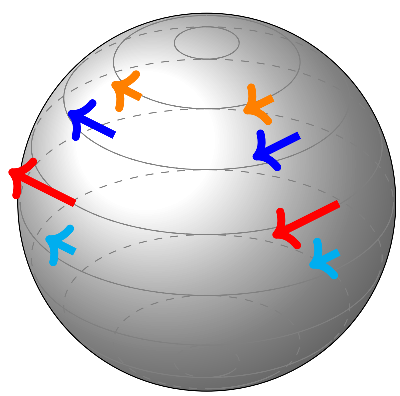
$$c_\varphi(M, L, \omega, \varphi) := \sup\{\max(H) \mid H \in \mathcal{H}_a(M, L, \omega, \varphi)\}.$$

Note that there are mainly three main factors which distinguish the symmetric capacity as defined above from the Hofer-Zehnder capacity as commonly defined in the literature. (Fcf. [11], for instance.)

The more subtle one is perhaps that we demand that the open set where H attains its minimum intersects L. The two other ones are more natural: we demand H to be invariant under φ and for it to be slow relative to brake orbits.

We should note that this definition, as is, attributes to every closed symmetric manifold (M, L, ω, φ) an infinite symmetric capacity.

For example, consider the unit sphere S^2 with its canonical symplectic form given by its volume form as a submanifold of $\mathbb{R}^3$, with $\varphi(x, y, z) = (x, y, -z)$. Then, the Hamiltonian $H(x, y, z) = \lambda(z)z^2$, where $\lambda(z)$ is a smooth real function which is zero at a neighbourhood of zero, symmetric and non-decreasing in the positive reals, is such that all the closed orbits are given by circles of latitude, and the equator and a small neighbourhood of it consists of constant orbits.



As such, H is trivially φ-slow, since there are no brake orbits except the constant ones. It is also clear that H satisfies all other requirements. The same holds true for aH, where a is any positive real number. It follows that $c_\varphi(S^2, S^1, \omega, \varphi) = +\infty$.

We now prove that this is true for any closed symmetric manifold.

Theorem 4.1.2. If M is a closed symmetric manifold, then $c_\varphi(M, L, \omega, \varphi) = +\infty$.

Proof. By Weinstein's Lagrangian neighbourhood theorem[1], there is a neighbourhood U of L which is symplectomorphic via some g to a neighbourhood U' of the zero section of T^*L and L is taken to the zero-section under g. Define

$$\begin{aligned} K : U' &\to \mathbb{R}; \\ (x, p) &\mapsto \lambda(p)\|p\|^2, \end{aligned}$$

where λ is real smooth function which is non-decreasing in $[0, +\infty)$, symmetric, 0 at a small neighbourhood of 0 and constant equal to 1 outside a small enough interval.

$K \circ g$ defines a function on U which can be extended smoothly to a function $\widetilde{H}$ on M by making it constant equal to 1 U. Then, letting $H := \widetilde{H} + \widetilde{H} \circ \varphi$ we still have that $H^{-1}(0) = L$ and L consists of critical points of H. Since orbits stay in a level-set of a time-independent Hamiltonian, it follows that an orbit in any other level-set of H except $H^{-1}(0)$ will not intersect L, and thus there are no brake orbits there. On $H^{-1}(0)$, all points are critical points. Therefore, H is slow. It trivially satisfies all other requirements for being a φ-admissible Hamiltonian. Likewise, aH is also φ-admissible for every $a > 0$, and it follows that $c_\varphi(M, L, \omega, \varphi) = +\infty$. □

Note that in the examples we treat in this dissertation (i.e., Liouville domains), we are dealing with manifolds with boundary. The further assumption that we make that $\partial L = \partial M \cap L$ makes it so that the maximum of a φ-admissible Hamiltonian is attained on L. Since the Hofer-Zehnder capacity hopes to find orbits by increasing the oscillation of the Hamiltonian, it is sensible in our case of brake orbits to require that the maximum is attained on L in case it is not automatic, as it is the case for a compact manifold with no boundary. So we propose the following definition.

Definition 4.1.3. For a symmetric symplectic manifold (M, L, ω, φ), possibly with boundary, we define the set of *φ-strongly-admissible Hamiltonians*, denoted here by $\mathcal{H}_a^{str}(M, L, \omega, \varphi)$, to be the set of C^2 functions satisfying the following properties.

(i) There is a compact set $K \subset M$ depending on H such that $K \subset \mathrm{int}(M)$ and H is constant equal to its maximum outside K.

(ii) There exists an open set O, also depending on H, which intersects L and where H is constant and equal to 0.

(iii) $H \geq 0$.

(iv) $H \circ \varphi = H$.

(v) $\max|_L H = \max H$ and $H(L) = [0, \max H]$.

(vi) H is *φ-slow*, which is to say that all brake orbits are either constant or have minimal period greater than 1.

Likewise, we define the strong symmetric capacity as

$$c_\varphi^{str}(M, L, \omega, \varphi) = \sup\{m(H) \mid H \in \mathcal{H}_a^{str}(M, L, \omega, \varphi)\}.$$

[1] We actually only need the tubular neighbourhood theorem, since the fact that the diffeomorphism is a symplectomorphism is irrelevant to the argument.

This definition makes it so that the case of a closed manifold does not degenerate as we have seen in Theorem 4.1.2.

Just as in the non-symmetric case, we can filter the capacity by relative homotopy class. Namely, we can make the following definition:

Definition 4.1.4. Given a relative homotopy class of paths $[\![\gamma]\!]$ in (M, L), we define

$$c_\varphi(M, L, \omega, \varphi, [\![\gamma]\!]) = \sup\{m(H) \mid H \in \mathcal{H}_a(M, L, \omega, \varphi, [\![\gamma]\!])\},$$

where $H \in \mathcal{H}_a(M, L, \omega, \varphi, [\![\gamma]\!])$ is the space of those Hamiltonians which satisfy all required properties to be $\varphi-$admissible, except that the φ-slow property is only required to hold on paths which are in the same homotopy class as $[\![\gamma]\!]$.

Likewise, we define $c_\varphi^{str}(M, L, \omega, \varphi, [\![\gamma]\!])$ in a similar way.

Using the fact that the symmetric capacity of the unit disk is π (c.f. [15], [26]) helps us define in a relevant way a symmetric Gromov capacity as follows.

Definition 4.1.5. The *symmetric Gromov capacity* c_φ^{gr} of a symmetric manifold (M, L, ω, φ) is defined as the supremum of the capacities of the balls $B^{2n}(r)$ of the same dimension which can be equivariantly simpletically embedded in M.

In this case, there is a filtration of the symmetric Gromov capacity that does not exist in the non-symmetric case: we can filtrate it by the connected components of L. Since an equivariant embedding f must preserve fixed points and the fixed-point set of the involution on the disk is connected, f must send the fixed-point set of the involution on the disk to some connected component of L. This allows us to make the following definition.

Definition 4.1.6. Given a connected component L_i of a symmetric manifold (M, L, ω, φ), we define the *symmetric Gromov capacity with respect to* L_i, c_{φ,L_i}^{gr}, as the supremum of the capacities of the balls $B^{2n}(r)$ of the same dimension which can be equivariantly simpletically embedded in M such that the fixed point set of the involution in $B^{2n}(r)$ is taken into L_i.

It might be worthy to point out that it is always possible to embed some ball equivariantly around any given connected component L_i. In the non-symmetric case this follows directly from Darboux's theorem. In the symmetric case this follows from the Weinstein's Lagrangian neighbourhood theorem.

Similarly to the standard case, finiteness of the symmetric capacity of a neighbourhood of a regular energy-level also establishes the existence of brake orbits in a dense set. (Actually a full-measure set.) We prove this in the next propositions, which follow closely the ideas in [29], which were discovered by Macarini and Schlenk. ([17].). The proof of this next proposition can also be found in [15].

Proposition 4.1.7. Let (M, L, φ, ω) be a symmetric manifold and $H : M \to \mathbb{R}$ be a C^2 symmetric Hamiltonian such that 1 is a regular value for H such that $H^{-1}(1) \cap L \neq \emptyset$ transversely. Let $S_\lambda := H^{-1}(\lambda)$. Given a small enough neighbourhood U of $H^{-1}(1)$ such that the modified gradient flow (for some chosen metric) foliates U with regular energy levels with energies belonging in some interval $I = (1 - \epsilon, 1 + \epsilon)$, if $c_\varphi(U, U \cap L) < \infty$, then there is a dense set D for which every $\lambda \in D$ is such that S_λ possesses a brake orbit.

Proof. Let $\delta < \epsilon$ and choose a smooth real function $f : \mathbb{R} \to \mathbb{R}$ for which

$$\begin{aligned}
f(s) &= c_\varphi(U, U \cap L) + 1 && \text{for } s \le 1-\delta \text{ and } s \ge 1+\delta \\
f(s) &= 0 && \text{for } 1-\delta/2 \le s \le 1+\delta/2 \\
f'(s) &< 0 && \text{for } 1-\delta < s < 1-\delta/2 \\
f'(s) &> 0 && \text{for } 1+\delta/2 < s < 1+\delta.
\end{aligned}$$

Letting $F := f \circ H$, it is clear that F satisfies properties $(i), (ii), (iii), (iv)$ for an admissible Hamiltonian. Since $\max f > c_\varphi(U, U \cap L)$, we must have that f does not satisfy property (v), i.e. it is not φ-slow, therefore admitting a non-constant brake orbit x with minimal period lesser than or equal to 1.

The Hamiltonian vector field X_F for F satisfies

$$X_F(x) = f'(H(x)) \cdot X_H(x)$$

and we have that $H(x(t))$ is constant, say equal to some λ. Indeed,

$$\begin{aligned}
(H \circ x)'(t) &= dH_{x(t)}(x'(t)) \\
&= \omega(x'(t), X_H(x(t)) \\
&= \omega(X_F(x(t)), X_H(x(t)) \\
&= \omega(f'(H(x))X_H(x(t)), X_H(x(t))) \\
&= 0.
\end{aligned}$$

Since $X_F(x) = f'(H(x)) \cdot X_H(x)$ and x is not a constant solution, we must have $f'(H(x(t))) \neq 0$, and thus $f'(\lambda) \neq 0$, which tells us that λ is either on $(1-\delta, 1-\delta/2)$ or $(1+\delta/2, 1+\delta)$ by our requirements on f, which implies that $|\lambda - 1| < \delta$. Letting $\tau := f'(\lambda)$ we can now reparametrize x in order to get a brake orbit for H: Define

$$y(t) := x(t/\tau).$$

Then

$$\begin{aligned}
y'(t) &= \frac{1}{\tau} x'(t/\tau) \\
&= \frac{1}{\tau} f'(H(x(t/\tau)) X_H(x(t/\tau)) \\
&= X_H(x(t/\tau)) \\
&= X_H(y(t)),
\end{aligned}$$

thus establishing that y is indeed an orbit of the Hamiltonian flow of H. Note that a reparametrization of a brake orbit is a brake orbit. So, y is a brake orbit on S_λ. Since $|\lambda - 1| < \delta$ and $\delta < \epsilon$ is arbitrary, the result follows.

□

As a corollary, we have that given a regular value λ of H for which $H^{-1}(\lambda) \cap L \neq \emptyset$ transversely, then there is a sequence $\lambda_j \to \lambda$ which consists of values for which their level sets have brake orbits. This sets up the next proposition, which is intended to understand when those brake orbits can give rise to a brake orbit in $H^{-1}(\lambda)$ itself.

Proposition 4.1.8. Consider the local thickening as in Proposition 4.1.7 and λ a regular value of H. Take a sequence $\lambda_j \to \lambda$ which consists of values for which their level set has brake orbits. Choose for each such λ_j a brake orbit x_j with period T_j. If these periods are uniformly bounded by some C, then there is a brake orbit in S_λ which has period $T \leq C$.

Proof. Let g be a Riemannian metric on M. Since X_H is non-zero with constant norm on $H^{-1}(1)$, it follows that we can shrink U in order to guarantee that there exists a constant C for which

$$C^{-1} \leq \|X_H(x)\| \leq C$$

for all $x \in U$. Letting L be the length of a path according to g, we get that

$$C^{-1} \cdot T_j \leq L(x_j) \leq C \cdot T_j$$

for all j. Let $y_j(t) := x_j(T \cdot t)$, i.e. the orbits x_j with normalized period equal to 1. Note then that

$$\dot{y}_j(t) = T_j X_H(y_j(t)). \tag{4.1}$$

The hypothesis implies then that the family y_j is equicontinuous. It is also pointwise relatively compact, since S is compact and the y_j lies from some point forward in a small tubular neighbourhood of S. It follows from the Arzelà-Ascoli theorem that there exists a subsequence of y_j that converges uniformly to some y. By passing to subsequences successively, we can find a further subsequence of y such that the periods T_j converge to some T. In order not to overload indices, we will call this subsequence y_j.

Then, note that by equation 4.1 we have locally that

$$\begin{aligned}\|\dot{y}_n(t) - \dot{y}_m(t)\| &= \|T_n X_H(y_n(t)) - T_m X_H(y_m(t))\| \\ &= \|T_n X_H(y_n(t)) - T_n X_H(y_m(t)) + T_n X_H(y_m(t)) - T_m X_H(y_m(t))\| \\ &\leq T_n \|X_H(y_n(t)) - X_H(y_m(t))\| + \|X_H(y_m(t))\| \cdot |T_n - T_m|.\end{aligned}$$

Therefore, $\dot{y}_j$ also converges uniformly and it follows that the convergence of y_j to y is actually C^1. Since $y_j(1) = y_j(0)$, it follows that $y(1) = y(0)$ and thus y has period 1. Also, considering the C^1 convergence and equation 4.1, we have

$$\dot{y}(t) = T X_H(y(t)).$$

So it suffices to show that $T \neq 0$, so that we can reparametrize y to give an orbit for X_h.

Suppose $T = 0$. If that were the case, then y is a constant orbit since $\dot{y} = 0$. Let y^* denote the point representing such constant orbit. Let $V = X_H(y^*)$. The fact that S is a regular energy level implies that $V \neq 0$. Since $X_H(y_j(t)) \to X_H(y^*)$, we have that given $1 > \epsilon > 0$, in local coordinates

$$\langle X_H(y_j), V \rangle \geq (1 - \epsilon)\|V\|^2$$

for all sufficiently large j. Fix any such j. Therefore

$$T_j^{-1} \langle \dot{y}_j, V \rangle \geq (1 - \epsilon)\|V\|^2.$$

Since given a path connecting two different points of a manifold there is a chart containing the path, we can pick a chart containing the path y_j up to a time k arbitrarily close to $t = T_j$. Then, by integrating up to k, we get

$$\begin{aligned} \int_0^k T_j^{-1}\langle \dot{y}_j, V\rangle &\geq k(1-\epsilon)\|V\|^2 \\ \implies T_j^{-1}\langle y_j(k) - y_j(0), V\rangle &\geq k(1-\epsilon)\|V\|^2. \end{aligned}$$

Since this holds for every $k < T_j$, by computing the limit as $k \to T_j$ we get

$$0 \geq T_j(1-\epsilon)\|V\|^2,$$

which is an absurd since $V \neq 0$. □

We now have what is needed to refine Proposition 4.1.7, concluding that we can have D as in the conclusion of that proposition to be a set of full measure. This result has been proved in the non-symmetric case by Macarini and Schlenk ([17]) and the symmetric case follows the exact same proof, *mutatis mutandis*.

Proposition 4.1.9. Consider the thickening as in Proposition 4.1.7 and $S = \{x \in M \mid H(x) = 0\}$.[2] If $c_\varphi(U, U \cap L) < +\infty$, then there exists a subset $\Lambda \subset I$ of full Lebesgue measure such that for every $\lambda \in \Lambda$, S_λ carries a brake orbit.

Proof. Since we will use the capacity several times, we will denote $c_\varphi(U, U \cap L)$ simply by $c_\varphi(U)$.

Let $\Lambda_n \subset I$ be the set of values on I for which S_λ carries a periodic solution with period $0 < T \leq n$. Proposition 4.1.8 implies that Λ_n is closed, thus measurable. Therefore it follows that Λ is also measurable, since it is the (countable) union of all Λ_n's. Let

$$U_\alpha := \bigcup_{|\lambda|<\alpha} S_\lambda \cap U = \{x \in U \mid -\alpha < H(x) < \alpha\}.$$

The function

$$\begin{aligned} \mathcal{C} : I &\to \mathbb{R} \\ \alpha &\mapsto c_\varphi(U_\alpha, U_\alpha \cap L) \end{aligned}$$

is monotone due to the monotonicity property of the symmetric capacity. It follows that $\mathcal{C}$ is differentiable almost everywhere. Therefore, for almost every $\alpha^* \in I$, we have that there exists K' and a sufficiently small neighbourhood $V_{\alpha^*} = (\alpha^* - \delta, \alpha^* + \delta)$ such that

$$c_\varphi(U_\alpha) - c_\varphi(U_{\alpha^*}) \leq K'(\alpha - \alpha^*).$$

for all $\alpha \in V_{\alpha^*}$, and therefore

$$c_\varphi(U_\alpha) \leq K'(\alpha - \alpha^*) + c_\varphi(U_{\alpha^*}).$$

[2]The value being zero is of course just for simplification. If the value was something else, we could just translate H.

Since $\mathcal{C}$ is bounded, it follows that

$$K := \max\{K', \sup_{\alpha \geq \alpha^* + \delta} \{\frac{c_\varphi(U_\alpha) - c_\varphi(U_{\alpha^*})}{\alpha - \alpha^*}\}\},$$

is a real number. Therefore, for $\alpha \in V_{\alpha^*}$,

$$c_\varphi(U_\alpha) \leq \; K'(\alpha - \alpha^*) + c_\varphi(U_{\alpha^*}) \leq K(\alpha - \alpha^*) + c_\varphi(U_{\alpha^*})$$

and for $\alpha \geq \alpha^* + \delta$,

$$\frac{c_\varphi(U_\alpha) - c_\varphi(U_{\alpha^*})}{\alpha - \alpha^*} \leq K$$
$$\Longrightarrow c_\varphi(U_\alpha) \leq K(\alpha - \alpha^*) + c_\varphi(U_{\alpha^*}).$$

So, we have that for every $\alpha > \alpha^* - \delta$,

$$c_\varphi(U_\alpha) \leq K(\alpha - \alpha^*) + c_\varphi(U_{\alpha^*}).$$

It is useful to recall here that K and δ depend on α^*, which is a point where c_φ is differentiable.

We will show in a separate lemma (Lemma 4.1.10), using the above estimate, that if c_φ is differentiable at α^* and B is an interval with α^* as one of its endpoints, then either α^* or the other end of the interval B belong to Λ.

Assume $m(\Lambda) < m(I)$, and thus $m(I \backslash \Lambda) > 0$. Let $A := I \backslash \Lambda$, and consider $f = \mathbf{1}_A$. By Lebesgue's differentiation theorem (c.f. [24]), the set of those x for which

$$\lim_{J \to 0} \frac{\int_J f}{\mu(J)} \neq f(x),$$

where the J's are intervals around x, is of null measure. Therefore, since $\mu(A) \neq 0$, there is some $x \in A$ for which the limit is equal to $f(x)$, which is 1.

It follows that for every $\delta > 0$, we can find a small enough interval J such that

$$\frac{\int_J f}{\mu(J)} \geq 1 - \delta,$$

i.e.

$$\mu(A \cap J) \geq (1 - \delta)\mu(J).$$

So, consider now c_J to be the midpoint of J and ψ to be reflection with respect to this midpoint. By the preceding observations, the measure of the set

$$\{x \in J \cap (c_J, +\infty) \mid x \text{ or } \psi(x) \text{ belong to } \Lambda\}$$

is precisely half of the measure of J, so

$$m(J \cap \Lambda) \geq \frac{m(J)}{2}.$$

Choosing δ sufficiently small (for example, less than 1/3), we then have that

$$m(J) = m(J \cap A) + m(J \cap \Lambda) \geq (1 - \delta + 1/2)m(J) > m(J),$$

a contradiction.

$\square$

Let us now show the required lemma of the previous proof.

Lemma 4.1.10. With the notation as in the proof of Proposition 4.1.9, if $\mathcal{C}$ is differentiable at α^*, then α^* or $-\alpha^*$ belong to Λ.

Likewise, the same result holds for symmetry around another point instead of 0. Explicitly, if $\alpha^* - c$ and $\alpha^* - 2c$ belong to I, then either α^* or $\alpha^* - 2c$ belong to Λ.

Proof. As in the previous proposition, we omit the Lagrangian $U \cap L$ in the notation of the symmetric capacity.

Recall that if $\mathcal{C}$ is differentiable at α^*, then there exists $\delta > 0$ and K such that

$$c_\varphi(U_\alpha) \leq c_\varphi(U_{\alpha^*}) + K(\alpha - \alpha^*) \tag{4.2}$$

for all $\alpha > \alpha^* - \delta$.

Fix α^*. For $\alpha > \alpha^*$, we have by definition of the symmetric capacity that there exists some symmetric-Hofer-Zehnder-admissible Hamiltonian G on U_{α^*} for which

$$\max G > c(U_{\alpha^*}) - (\alpha - \alpha^*).$$

Choose now a smooth function $f : \mathbb{R} \to \mathbb{R}$ for which

$$f(x) = \begin{cases} c(U_\alpha) + (\alpha - \alpha^*), & s \leq -\alpha \text{ and } s \geq \alpha, \\ m(G), & s \in [-\alpha^*, \alpha^*] \end{cases}$$

and such that it has negative derivative on $(-\alpha, \alpha^*)$ and positive derivative on (α^*, α). Moreover, we require that $|f'(s)| < K + 3$, which can be done and due to (4.2) this same requirement on the modulus of f' can be done for any $\alpha > \alpha^*$. Now, define $F : M \to \mathbb{R}$ by

$$F(x) = \begin{cases} f \circ H(x), & x \notin U_{\alpha^*} \\ G(x), & x \in U_{\alpha^*}. \end{cases}$$

Then F is obviously symmetric-Hofer-Zehnder admissible and $m(F) = c_0(U_\alpha) + (\alpha - \alpha^*) > c_0(U_\alpha)$. It follows that F has some fast brake orbit x, i.e. a non-constant brake orbit with period less than or equal to 1. G on the other hand does not have a fast brake orbit, so the fast brake orbit x of F must lie on $U_\alpha \backslash U_{\alpha^*}$. Since there $F = f \circ H$, we have that

$$X_F = (f' \circ H) X_H$$

on that region. As a consequence, H is constant on x, so x must lie on S_λ or $S_{-\lambda}$ for some $\lambda \in (\alpha^*, \alpha)$ and $|f' \circ H(x(t))|$, which is some constant τ, is such that $0 < \tau < K + 3$. Reparametrizing x by $y = x(t/\tau)$, we have that y is a brake orbit of H with period smaller than or equal to $K + 3$.

So we found, for a given $\alpha > \alpha^*$, a brake orbit on S_λ or $S_{-\lambda}$ for some $\lambda \in (\alpha, \alpha^*)$ which has period less than $K + 3$. Since $\alpha > \alpha^*$ is arbitrary, we can take a sequence of such $\lambda_j \to \alpha^*$. We can now pick a subsequence of that sequence for which all of its terms are such that the brake orbit is either on S_{λ_j} or $S_{-\lambda_j}$. So, by Proposition 4.1.8, it follows that either α^* or $-\alpha^*$ belong to Λ.

In order to verify the last assertion of the proposition, it suffices to analyze change our Hamiltonian H to $H - \alpha^* + c$ and apply the theorem for c instead of α^*. Indeed, we have that either c or $-c$ belongs to Λ, and those values give $H(x) - \alpha^* + c = c \implies H(x) = \alpha^*$ and $H(x) - \alpha^* + c = -c \implies H(x) = \alpha^* - 2c$. $\square$

4.2 Surfaces

The case of surfaces, as in the non-symmetric case, allows for significant reduction from the symplectic to the topological.

The first manifestation of this can be seen by the fact that a orientation-reversing involution is always anti-symplectic. Before proving this, we will show some examples of symmetric symplectic surfaces (M, L, ω, φ).

Example 4.2.1 (Sphere)**.** Take $M = S^2, \omega$ the standard area form on S^2 and $\varphi = (x, y, z) \mapsto (x, y, -z)$. Then $L = S^1$ as in Figure 4.1

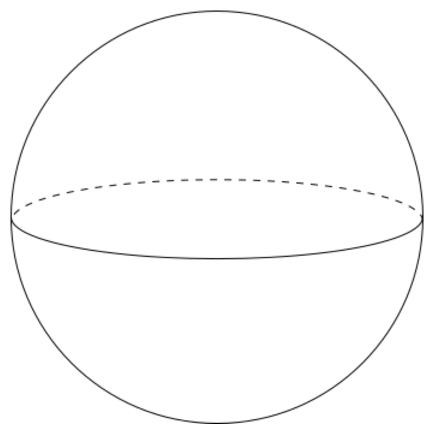


Figure 4.1: Sphere with vertical reflection as involution.

Example 4.2.2 (Torus)**.** Take M as an embedding of the torus in $\mathbb{R}^3$ as in Figure 4.2, with ω being the induced area form, together with φ being $\varphi = (x, y, z) \mapsto (x, -y, z)$. Then L is the union of two circles.

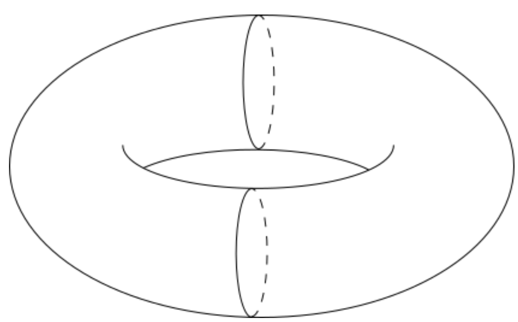


Figure 4.2: Torus with horizontal reflection as involution.

Example 4.2.3 (Rhombic torus)**.** Taking M again as the torus, but instead consider it as the square $I \times I$ with opposite sides identified as usual, ω the standard area form as inherited by $\mathbb{R}^2$ and take φ as the orthogonal reflection along this diagonal. Then L is just one circle, and in the usual visualization as a doughnut we have Figure 4.3.

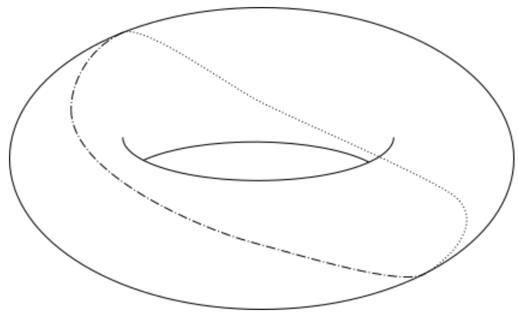


Figure 4.3: Rhombic torus seen as a doughnut.

Example 4.2.4 (Higher genus surfaces)**.** The reflections obviously work well for higher genus surfaces embedded in $\mathbb{R}^3$ in symmetric ways. For instance, we can take a vertical (as in Figure 4.4) or horizontal (as in Figure 4.5) reflection on a genus-two surface.

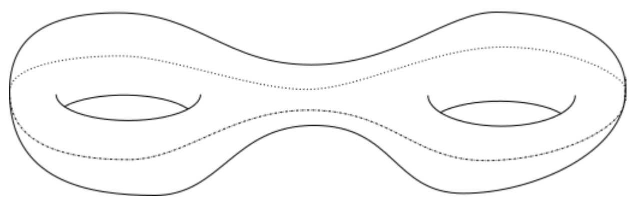


Figure 4.4: Genus-two surface with vertical reflection.

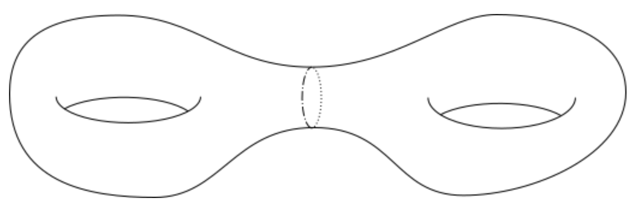


Figure 4.5: Genus-two surface with horizontal reflection.

Proposition 4.2.5. Let (M, φ) be a compact connected surface together with an orientation-reversing involution φ. Then φ is anti-symplectic.

Proof. We have that $\varphi^*\omega_p = \lambda_{\varphi(p)}\omega_{\varphi(p)}$ for some real number $\lambda_{\varphi(p)}$ for each p. Since $\varphi^2 = Id$, then

$$\omega = \lambda^2\omega.$$

Since ω is nondegenerate and M is connected, it follows that $\lambda = \pm 1$. Since φ reverses orientation, it is -1.

□

In the same vein of reducing the symplectic case to the topological, we introduce the following specialized version of the Moser theorem.

Theorem 4.2.6. Let α, β be volume forms in a symmetric compact manifold M such that $\varphi^*\alpha = -\alpha$ and $\varphi^*\beta = -\beta$ and

$$\int_M \alpha = \int_M \beta.$$

Then, there exists a self-diffeomorphism f for which $f^*\beta = \alpha$ and

$$f \circ \varphi = \varphi \circ f.$$

Proof. Since $\int_M \beta - \alpha = 0$, we have that $\alpha - \beta$ is exact, i.e. $\beta - \alpha = d\theta$ for some $n-1$-form θ. The usual proof of the Moser theorem proceeds by taking the vector field X_t that represents $-\theta$ considering the family of volume forms $\eta_t := (1-t)\alpha + t\beta$ in the sense that

$$\iota_{X_t}\eta_t = -\theta$$

and integrates it to find that at time 1 it satisfies what is needed. We adapt this proof by changing θ to

$$\Omega := \frac{1}{2}\left(\theta - \varphi^*\theta\right).$$

Note that applying φ^* to both sides of the equation

$$\beta - \alpha = d\theta$$

yields that

$$-\beta + \alpha = d\varphi^*\theta,$$

therefore

$$\begin{aligned} d\Omega &= \frac{1}{2}\left(d\theta - d\varphi^*\theta\right) \\ &= \frac{1}{2}\left(\beta - \alpha + \beta - \alpha\right) \\ &= \beta - \alpha \end{aligned}$$

and it is immediate that $\varphi^*\Omega = -\Omega$. So, let X_t be defined by

$$\iota_{X_t}\eta_t = -\Omega.$$

Due to Cartan's formula, letting ϕ^t be the flow of X_t, we have that

$$\begin{aligned} \frac{d}{dt}(\phi^t)^*\eta_t &= (\phi^t)^*\left(\mathcal{L}_{X_t}\eta_t + \frac{d}{dt}\eta_t\right) \\ &= (\phi^t)^*\left(d\iota_{X_t}\eta_t + \beta - \alpha\right) \\ &= 0, \end{aligned}$$

and we have that $(\phi^1)^*\beta = \alpha$. Letting $f := \phi^1$, we just need to prove that $f \circ \varphi = \varphi \circ f$.

It suffices to show that the path $\varphi(\phi^t(x))$ satisfies

$$\eta_t(d\varphi(X_t), \cdot, \cdots, \cdot) = -\Omega.$$

Since $\eta_t = (1-t)\alpha + t\beta$, we have that the left side is

$$
\begin{aligned}
(1-t)\alpha(d\varphi(X_t), \cdot, \cdots, \cdot) + t\beta(d\varphi(X_t), \cdot, \cdots, \cdot) &= -(1-t)\alpha(X_t, d\varphi(\cdot), \cdots, d\varphi(\cdot)) \\
&\quad - t\beta(X_t, d\varphi(\cdot), \cdots, d\varphi(\cdot)) \\
&= -\eta_t(X_t, d\varphi(\cdot), \cdots, d\varphi(\cdot)) \\
&= \Omega(d\varphi(\cdot), \cdots, d\varphi(\cdot)) \\
&= \varphi^*\Omega \\
&= -\Omega,
\end{aligned}
$$

as we wanted. □

This version allows us to simplify the problem of existence of equivariant symplectomorphisms to finding just equivariant diffeomorphisms between compact surfaces. More precisely, we have the following:

Corollary 4.2.7. Let $(M_1, \varphi_1, \omega_1)$ and $(M_2, \varphi_2, \omega_2)$ be symmetric compact surfaces such that their area coincides and there exists a diffeomorphism $f : M_1 \to M_2$ such that $f \circ \varphi_1 = \varphi_2 \circ f$. Then there exists a symplectomorphism g such that $g \circ \varphi_1 = \varphi_2 \circ f$.

Proof. Consider M_1 equipped with $f^*\omega_2$. We will show that this is a symmetric manifold with respect to φ_1. Indeed,

$$
\begin{aligned}
\varphi_1^*(f^*\omega_2) &= (f \circ \varphi_1)^*\omega_2 \\
&= (\varphi_2 \circ f)^*\omega_2 \\
&= f^*\varphi_2^*\omega_2 \\
&= f^*(-\omega_2) \\
&= -f^*\omega_2.
\end{aligned}
$$

The above establishes that f is an equivariant symplectomorphism between $(M_1, \varphi_1, f^*\omega_2)$ and $(M_2, \varphi_2, \omega_2)$. Since the specialized version of the Moser theorem implies that $(M_1, \varphi_1, f^*\omega_2)$ will be symplectomorphic to (M_1, φ_1, ω), the result follows from transitivity.

□

With the previous results, it is clear that the existence of an equivariant diffeomorphism plays a central role. We show some results in that direction.

First, we show that if the fixed-point set of an orientation-reversing involution on the sphere is non-empty, then it is a single circle.

Theorem 4.2.8. Let $\varphi : S^2 \to S^2$ be an orientation-reversing involution. If the fixed-point set is non-empty, then it is a circle.

Remark 4.2.9. We know that the fixed point set of an orientation-reversing involution on a closed surface is a union of circles (possibly empty). Indeed, such fixed-point set is a Lagrangian submanifold, thus one-dimensional. It is also closed as a subset of the manifold, hence compact, and thus it is a union of circles by the characterization of one-dimensional manifolds.

Proof. Suppose the fixed-point set has more than one component. Choose one of the components, say c_1, such that all other components are either in one component of the complement of c_1 or on the other. That this is possible follows from the Jordan curve theorem. We know by the Schoenflies theorem, which is a strengthened version of the Jordan curve theorem for the two-dimensional case via the Riemann mapping theorem (c.f. [8]), that the complement of c_1 consists of two disks D_1, D_2. On the disk which has all other connected components of the fixed point set, let us say it is D_2, consider a diffeomorphism with $\mathbb{R}^2$ and denote by c_2 the largest curve in this representation.

The unbounded component of the complement of c_2 is the one that φ sends D_1 to. Since φ is supposed to be a diffeomorphism and the unbounded component is not simply-connected but the disk is, we have a contradiction. □

This result obviously does not hold in general. Indeed, given a suitable embedding of the surface of genus two in $\mathbb{R}^3$, the reflection around the plane xy has as its fixed-point set three circles, whereas the reflection around the plane xz has only one.

Likewise, there are involutions in the torus with two circles as fixed-point set by embedding it in $\mathbb{R}^3$ in the obvious way and also only one by looking at the torus as $S^1 \times S^1$ and considering the reflection around the diagonal circle.

The next proposition is a result to shed some light on this issue and, together with the Moser theorem and its corollary, gives us nice criteria to know when two symmetric surfaces are equivariantly symplectomorphic.

Proposition 4.2.10. Let M be a closed (compact without boundary) orientable surface. If φ_1, φ_2 are two orientation-reversing involutions such that their their fixed-point sets have the same number of connected components and the quotient spaces by such involutions (after removing small annuli around the fixed circles) are either both orientable or both non-orientable, then there is an equivariant self-diffeomorphism with respect to those involutions, i.e. a diffeomorphism $f : M \to M$ such that

$$f \circ \varphi_1 = \varphi_2 \circ f.$$

Proof. The core idea is passing to the quotient and using the classification of surfaces to induce a homeomorphism there, which then lifts to the original spaces since we either have the orientation double cover or a trivial double cover. Since the quotient is a manifold with boundary, some care is necessary.

We proceed as follows to implement the strategy above: remove small annuli around the circles which constitute the fixed-point set; do this separately for each involution, and consider the quotient maps π_1, π_2 with respect to the orbit spaces obtained when making the quotient by the involutions. We have that these orbit spaces are manifolds with boundary which have Euler characteristic given by half of the original space. By classification of surfaces and the fact that we are assuming both to be orientable or non-orientable, we have that they are diffeomorphic. Since the cover is an orientation double cover or a trivial double cover, this diffeomorphism lifts to a diffeomorphism between the manifold M without the annuli. This lift is obviously equivariant. We can now attach annuli back equivariantly in order to finish our construction. □

Proposition 4.2.10 also holds if we allow M to have boundary. Since we do not need the result in that generality and there is some minor technicality arising from the fact that removing anulli can result in manifolds with boundary when the fixed-point set has arcs connecting the boundary, we will not give more details.

However, we need the result in the case of the disk and a single arc as the fixed-point set. We state this below as a consequence of the Riemann mapping theorem.

Proposition 4.2.11. Let φ_1, φ_2 be two involutions on the closed disk for which the fixed-point sets are individually a single arc connecting the boundary to itself. Then there is a self-diffeomorphism which is equivariant with respect to those involutions.

Proof. First, we prove that D minus an arc connecting the boundary consists of two components which are simply-connected. Embed D as the bottom hemisphere of S^2, take L to be the closed upper hemisphere together with the arc of D. The Poincaré-Alexander-Lefschetz duality (c.f. [8]) implies that

$$H^1(S^n, L) \simeq H_1(D - L).$$

and

$$H^2(S^n, L) \simeq H_0(D - L).$$

The long exact reduced cohomology sequence of the pair (S^n, L) contains

$$0 \simeq \widetilde{H}^0(L) \to H^1(S^2, L) \to \widetilde{H}^1(S^2) \simeq 0.$$

Therefore $H^1(S^2, L) = 0$. Consequently, by the aforementioned duality, $H_1(D - L) = 0$. Since $D - L$ is homeomorphic to an open subset of the plane and an open subset of the plane must have a free fundamental group, it follows that $\pi_1(D - L) = 0$, since $H_1(D - L)$ is the abelianization of $\pi_1(D - L)$.

The long exact cohomology sequence of the pair (S^n, L) also contains

$$0 \simeq H^1(S^2) \to H^1(L) \to H^2(S^2, L) \to H^2(S^2) \simeq \mathbb{Z} \to H^2(L) \simeq 0.$$

Since $H^1(L) = \mathbb{Z}$ (it is a disk with an arc attached on the boundary, pictorically a bucket), we have that the above sequence can be simplified to

$$0 \to \mathbb{Z} \to H^2(S^2, L) \to \mathbb{Z} \to 0.$$

Since $Ext(\mathbb{Z}, \mathbb{Z}) = 0$, we have that $H^2(S^2, L) = \mathbb{Z} \oplus \mathbb{Z}$, and consequently due to duality so is $H_0(D - L)$, establishing that we have two connected components.

The result now follows from removing an annuli around the arcs, applying the Riemann mapping theorem and then reattaching the annuli equivariantly as before. □

Recall that the Hofer-Zehnder capacity equals the area when we are dealing with two-dimensional manifolds. (c.f. [11].) In a similar note, we prove the following statement.

Theorem 4.2.12. Let (M, L, ω, φ) be a two-dimensional closed symmetric manifold. Then

$$c_\varphi^{str}(M, L, \omega, \varphi) \leq \left| \int_M \omega \right|.$$

If an equivariant version of the Dacorogna-Moser theorem holds for disks and if there exists a φ-invariant set of measure zero such that, when removed from M, we are left with an open disk such that the intersection of the fixed-point set with this disk consists only of a single arc, then

$$c_\varphi^{str}(M, L, \omega, \varphi) \geq \left| \int_M \omega \right|.$$

Remark 4.2.13. Some remarks are in order.

First, the hypothesis of existence of the invariant set of measure zero satisfying the required condition is not a strong requirement: all examples we gave satisfy this. Indeed, we are not aware of a case which does not satisfy this hypothesis.

We suspect that the equality may hold in general, but the argument for the second inequality needs to be refined for that since the approach via Gromov capacity from below may not necessarily work. Of course, the equivariant Dacorogna-Moser also needs to hold in order for equality to hold.

Let us also be precise about what we mean by "equivariant Dacorogna-Moser":

Conjecture 4.2.14. Let $(D_1, \omega_1), (D_2, \omega_2)$ be two symplectic disks equipped with anti-symplectic involutions φ_1 and φ_2, respectively, such that their symplectic area coincide. Then there exists a symplectomorphism $f : D_1 \to D_2$ such that $f \circ \varphi_1 = \varphi_2 \circ f$.

Note that we have a similar statement in Proposition 4.2.6, the difference being the presence of a boundary. Note that the proof does not follow through since we do not have completeness of the flow as we do in the closed (compact, without boundary) case. Indeed, the Dacorogna-Moser theorem is much more analytically subtle. (c.f. [9].)

Proof of Theorem 4.2.12. The proof follows an adaptation of the idea on [11].

First, we prove that

$$c_\varphi^{str}(M, L, \omega, \varphi) \leq \left| \int_M \omega \right|.$$

Due to conformality, we can assume the area to be positive. Fix $\epsilon > 0$ and choose now $H \in \mathcal{H}_a^{str}(M, L, \omega, \varphi)$ such that

$$m(H) \geq \int_M \omega + \epsilon.$$

If we manage to prove that there is some brake-orbit of H which is fast, i.e. has period ≤ 1, we conclude that $c_\varphi^{str} \leq \int_M \omega + \epsilon$. Since $\epsilon > 0$ is arbitrary, the result will follows.

So we must be able to find a fast orbit. Due to Sard's theorem, the set of critical values of H has measure zero. Since it is also compact due to the fact that H is C^1 and the manifold is closed, we can find disjoint intervals $I_j = [a_j, b_j]$ of regular values for which

$$\sum b_j - a_j \geq m(H) - \epsilon/2 \geq \int_M \omega + \epsilon/2.$$

Given h a regular value, we have that $H^{-1}(h)$ consists of disjoint embedded circles. Since $H(L) = [0, m(H)])$, at least one of them intersects L and is therefore a brake orbit. Choose for each j one of the connected components of $H^{-1}(I_j)$ which intersects L and call it A_j.

First, we prove that if $H^{-1}(h') \cap A_j$ intersects L for some $h' \in I_j$, then $H^{-1}(h) \cap A_j$ intersects L for all $h \in I_j$. To see this, it suffices to note that any intersection of $H^{-1}(h) \cap L$ is transverse, since a brake orbit is either constant, which is not the case since otherwise h would not be a regular value of H, or intersects L transversely. With this established, we have that each A_j is such that all $H^{-1}(h)$ for $h \in A_j$ are brake orbits.

Fixing a transversal line of initial points with respect to the Hamiltonian vector field, we can define the diffeomorphism

$$\psi_j : (t, h) \to \gamma(t, h),$$

where $\gamma(t, h)$ is the flow of the Hamiltonian vector field of responsible for the circle on $H^{-1}(h) \cap A_j$ and $0 < t < T(h)$, where $T(h)$ is the period of said circle.

Since $H(\gamma(t, h)) = h$, it follows that $\omega(\partial_t \psi_j, \partial_h \psi_j) = dH(\partial_h \psi_j) = 1$. Therefore,

$$\begin{aligned}
\psi_j^* \omega(\xi, \eta) &= \omega(d\psi_j(\xi), d\psi_j(\eta)) \\
&= \omega(\xi_1 \partial_t \psi_j + \xi_2 \partial_h \psi_j, \eta_1 \partial_t \psi_j + \eta_2 \partial_h \psi_j) \\
&= \xi_1 \eta_2 - \xi_2 \eta_1 \\
&= (dt \wedge dh)(\xi, \eta).
\end{aligned}$$

By the change of variables theorem, we have that

$$\int_{A_j} \omega = \int_{\psi_j^{-1}(A_j)} \psi_j^* \omega = \int_{a_j}^{b_j} T(h) dh.$$

Since all of those periods pertain to brake orbits, assuming that $T(h) > 1$ for all $h \in R$ yields that

$$\int_M \omega \geq \sum_j \int_{A_j} \sum_j (b_j - a_j) \geq \int_M \omega + \epsilon/2,$$

a contradiction. This establishes that there must exist a fast brake orbit somewhere, as we wanted to show.

We now show the next part. For that, it suffices to prove that under those hypotheses,

$$c_\varphi^{str}(M, L, \omega, \varphi) \geq \left| \int_M \omega \right|.$$

Let $\widetilde{D}$ be the disk given by the hypothesis. Given $\epsilon > 0$, we can pick a large enough set D in the interior of such disk which is diffeomorphic to a closed disk D_0 on the plane such that

$$\left| \int_D \omega \right| \geq \left| \int_M \omega \right| - \epsilon.$$

We can scale D_0 in order for its area to agree with $\left| \int_D \omega \right|$, and we assume this has been done. By assumption (the equivariant Dacorogna-Moser), D_0 is equivariantly symplectomorphic to D. Due to monotonicity of the symmetric capacity and the fact that the capacity of the disk is its area, the result follows.

□

Theorem 4.2.12 allows us to establish the strong symmetric capacity for a large class of compact surfaces. For example, the sphere and all obvious embeddings of surfaces of genus g with the involutions being reflections along planes. From these cases, the specialized Moser theorem (or rather, its corollary) establishes all topologically similar cases.

Chapter 5

Spectral invariants on wrapped Floer Homology

We now discuss the adaptation of the spectral invariant and the arguments given in [12] for the case of wrapped Floer homology. The main interest is the result that if a certain cohomological class associated to a given filtration of the action by a value a vanishes, then the symmetric capacity is bounded above by a.

5.1 Brake orbits

Proposition 5.1.1. Let x be a brake orbit in the sense of beginning and ending on L. Then, by continuing with $x(1+t) = \varphi(x(1-t))$, we have that x is a periodic orbit satisfying $\varphi(x(t)) = x(2-t)$.

Proof. We need to show that

$$\omega((\varphi(x(1-t)))', \cdot) = dH(\cdot).$$

By the chain rule, this is equivalent to

$$\omega(d\varphi(-x'(1-t)), \cdot) = dH(\cdot) = dH(d\varphi^{-1}(\cdot)),$$

since $H \circ \varphi^{-1} = H$. Since $\varphi^*\omega = -\omega$, this is equivalent to

$$\omega(x'(1-t), d\varphi^{-1}\cdot) = dH(d\varphi^{-1}(\cdot)),$$

which is equivalent to the fact that x itself is an orbit of the Hamiltonian flow. Thus, the result follows once we show that this continuation is smooth at time 0 and 1. The argument for both is the same: we just need the fact that if γ is a continuous path on $\mathbb{R}^n$ such that $\lim_{t\to a^+} \dot\gamma(t) = \lim_{t\to a^-} \dot\gamma(t)$, then γ is differentiable at a and $\dot\gamma(a)$ is equal to those limits. This is a straightforward result from the mean value theorem. □

Corollary 5.1.2. Paths beginning and ending on L at time 0 and 1 respectively are in correspondence with periodic orbits x of period 2 satisfying $\varphi(x(t)) = x(-t)$.

Proof. We only need to verify that restricting a periodic orbit of period 2 satisfying $\varphi(x(t)) = x(-t)$ will yield a path beginning and ending on L at time 0 and 1 respectively.

Plugging 0 in, we have that $\varphi(x(0)) = x(0)$, so that $x(0)$ is a fixed point of φ, hence an element of L. Likewise, $\varphi(x(1)) = x(-1)$ and since x has period 2, we have that $\varphi(x(1)) = x(-1+2) = x(1)$, showing that $x(1)$ is also a fixed point and hence an element of L. □

On a similar note, we have the following:

Proposition 5.1.3. It holds that

$$\phi_H^t \circ \varphi = \varphi \circ \phi_H^{-t}$$

for all t for which the flow is defined.

Proof. We have that $\phi_H^t(\varphi(x)) = c_{\varphi(x)}(t)$, where c is such that $dH = \omega(c'_{\varphi(x)}(t), \cdot)$ and $c_{\varphi(x)}(0) = \varphi(x)$. These properties define $c_{\varphi(x)}$, so it suffices to prove that $\gamma : t \mapsto \varphi(\phi_H^{-t}(x))$ also satisfies these properties.

Note that $\gamma(0) = \varphi(\phi_H^{-0}(x)) = \varphi(x)$. Now, we have that

$$\begin{aligned}
\omega(\gamma'(t), \cdot) &= \omega(d\varphi(-c'_x), \cdot) \\
&= -\omega(-c'_x, d\varphi \cdot) \\
&= \omega(c'_x, d\varphi \cdot) \\
&= dH(d\varphi \cdot) \\
&= d(H \circ \varphi) = dH,
\end{aligned}$$

□

5.2 Preliminary results and definitions

We will use the conventions and the definitions of Chapter 3. Specifically, those pertaining to the construction of the wrapped Floer cohomology.

We define for $I, I' \subset \mathbb{R}$ the intervals $I_+ := (-\infty, \inf I] \cup I$ and $I_- := I_+ \backslash I$. We then have the following.

Proposition 5.2.1. If I, I' are non-empty intervals such that $I_\pm \subset I'_\pm$, there exists a natural homomorphism $\Psi_H^{II'} : \mathrm{WFH}^{I',\alpha} \to \mathrm{WFH}^{I,\alpha}$.

Proof. The condition that $I_\pm \subset I'_\pm$ just states that all elements x of the complex WC_* which are in the filtered complex WC_*^I are such that $\iota\partial x$ and $\partial\iota x$ either persist together or vanish together, where $\iota : \mathrm{WC}_*^I \to \mathrm{WC}_*^{I'}$ is the map that on the chain level takes x to x or to 0 in case the action of x is not on I'. See figure 5.1 below.

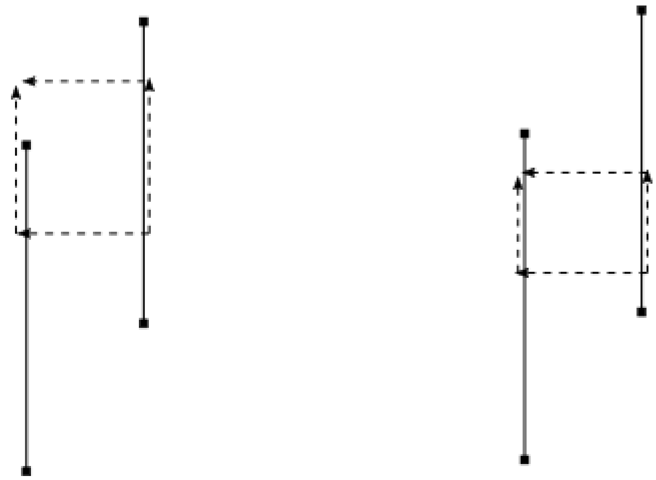


Figure 5.1: Illustration of the maps ι and ∂. (Horizontal and vertical arrows, respectively.)

It follows that ι is a chain map. The map Ψ is the induced map on cohomology.

$\square$

From the above proposition, it follows that we have the following triangle on cohomology.

$$\begin{tikzcd}
 & \mathrm{WFH}^{(a,c]}(H) \arrow[dl, "\Psi"'] & \\
\mathrm{WFH}^{(a,b]}(H) \arrow[rr, "\partial"] & & \mathrm{WFH}^{(b,c]}(H), \arrow[ul, "\Psi"']
\end{tikzcd} \tag{5.1}$$

where $-\infty \leq a < b < c \leq +\infty$. We prove next that it is exact.

Proposition 5.2.2. The triangle 5.1 is exact.

Proof. First we prove exactness at $\mathrm{WFH}^{(a,b]}(H)$. To prove that $\partial\Psi([\![x]\!]) = 0$, we look at the chain level. Either $\partial x = 0$ and we are already done, or its action lives in $(-\infty, b]$ and we are also done, or it lives in $(b, c]$. In this last case, we must have $\partial x = 0$, otherwise x would not be an element of $\mathrm{WFH}^{(a,c]}(H)$. (i.e., $\widetilde{\partial}\,[\![x]\!]$ would not be zero.)

To prove that $\ker\partial = \operatorname{Im}\Psi$, note that if $\partial\,[\![x]\!] = 0$, then ∂x is either 0 or its action lives on $[c, +\infty)$. In any case, then it is already an element of $\mathrm{WFH}^{(a,c]}(H)$ because $\widetilde{\partial}\,[\![x]\!]$ will be zero on that chain complex.

Similar reasonings hold for the exactness elsewhere.

$\square$

The usual arguments of proving invariance under changing the Hamiltonian through a map induced by a homotopy of Hamiltonians yield the following proposition.

Proposition 5.2.3. Let H, H' be two admissible Hamiltonians and assume that $a_H \leq a_{H'}$. Let

$$\Delta := \int_{S^1} \max(H_t - H'_t)dt,$$

and let $I, I' \subset \mathbb{R}$ be intervals satisfying $I_\pm + \Delta \subset I'_\pm$. Then there exists a natural homomorphism $\Psi^{II'}_{HH'} : \mathrm{WFH}^{I,\alpha}(H) \to \mathrm{WFH}^{I',\alpha}(H')$. Furthermore, the following properties hold:

- If $H = H'$, then $\Psi^{II'}_{HH} = \Psi^{II'}_{H}$, i.e., the map induced by inclusion as described above.
- Suppose that H, H' and H'' are all admissible Hamiltonians and satisfy $a_H \le a_{H'} \le a_{H''}$ and let I, I', I'' be nonempty intervals. Then, if the maps below are defined, we have that the diagram is commutative.

$$\begin{tikzcd}
& \mathrm{WFH}^{I',\alpha}(H') \arrow[dr, "\Psi^{II'}_{H'H''}"] & \\
\mathrm{WFH}^{I'',\alpha}(H) \arrow[ur, "\Psi^{II''}_{HH'}"] \arrow[rr, "\Psi^{II''}_{HH''}"] & & \mathrm{WFH}^{I,\alpha}(H'').
\end{tikzcd}$$

Remark 5.2.4. It might be reasonable to expect some term concerning the primitive f in Δ since it appears in the action functional. But Δ comes from the energy estimate which comes from integration of $\|\partial_s u\|^2$ of a Floer solution, which does not involve f. If this is not clear, see the computation in Theorem 3.2.1.

We obtain the following corollary.

Corollary 5.2.5. Let (X, λ) be a Liouville domain and H, H' be admissible Hamiltonians. Then

- If $a_H = a_{H'}$, then the natural homomorphism described above is indeed an isomorphism in the case $I = I'$.
- If $H_t \le H'_t$ for every $t \in S^1$, then there exists a natural homomorphism $\mathrm{WFH}^{I,\alpha}(H) \to \mathrm{WFH}^{I,\alpha}(H')$ for any nonempty interval I.

5.3 Truncated Floer cohomology

Let now $\mathcal{H}^{neg}_{ad}(X, \lambda)$ denote the space of admissible Hamiltonians which are negative on X, and define

$$\mathrm{WFH}^{I,\alpha}(X,\lambda) := \varinjlim_{H \in \mathcal{H}^{neg}_{ad}(X,\lambda)} \mathrm{WFH}^{I,\alpha}(H).$$

Lemma 5.3.1. Given an admissible H, there exists a natural isomorphism $\Psi_H : \mathrm{WFH}^{\alpha}(H) \to \mathrm{WFH}^{>-a_H,\alpha}(X, \lambda)$.

Proof. Consider the following isomorphisms.

1. $\mathrm{WFH}^{\alpha}(H) \to \varinjlim_{\substack{G \in \mathcal{H}_{ad} \\ a_G \le a_H}} \mathrm{WFH}^{\alpha}(G)$;
2. $\varinjlim_{\substack{G \in \mathcal{H}^{neg}_{ad} \\ a_G \le a_H}} \mathrm{WFH}^{\alpha}(G) \to \varinjlim_{\substack{G \in \mathcal{H}_{ad} \\ a_G \le a_H}} \mathrm{WFH}^{\alpha}(G)$;

3. $\varinjlim_{\substack{G \in \mathcal{H}_{ad}^{neg} \\ a_G \le a_H}} \mathrm{WFH}^{>-a_H,\alpha}(G) \to \varinjlim_{\substack{G\mathcal{H}_{ad}^{neg} \\ a_G \le a_H}} \mathrm{WFH}^{\alpha}(G)$;

4. $\varinjlim_{\substack{G \in \mathcal{H}_{ad}^{neg} \\ a_G \le a_H}} \mathrm{WFH}^{>-a_H,\alpha}(G) \to \varinjlim_{G \in \mathcal{H}_{ad}^{neg}} \mathrm{WFH}^{>-a_H,\alpha}(G) = \mathrm{WFH}^{>-a_H,\alpha}(X,\lambda)$

The first isomorphism comes from the maps $\Psi'_H s$. The second isomorphism is just a matter of shifting the Hamiltonians. The third isomorphism is obtained directly by the universal property of the direct limit. And the last isomorphism is trivial.

□

We will now need the fact that the wrapped Floer cohomology of a pair (X, L) reduces itself to the Morse cohomology of L when the Hamiltonian is sufficiently C^2-small on X and has sufficiently low inclination at infinity. This is similar to the case of (periodic) Floer homology, where Floer solutions "degenerate" to Morse trajectories and the only periodic orbits are the critical points. Intuitively, since our trajectories are Hamiltonian paths beginning and ending on L, we could expect that taking a C^2-small Hamiltonian would lead us to critical points on L and therefore we would have an isomorphism with the cohomology of L. However, this line of thought has some issues. For instance, we could be missing important critical points of $H|_L$ for the computation of the cohomology, since our argument was getting critical points of H, as a function on M, that are contained in L.

Nevertheless, the isomorphism is true by using the fact that $\mathrm{WFH}^*(H) \simeq HF^*(\phi_H^1(L), L)$ (see Chapter 3), where H is C^2-small and HF^* is Lagrangian Floer cohomology. (It is worth noticing that H being C^2-small with low inclination reduces the problem to a compact scenario. For more information, see [23].) The latter is known to be isomorphic to $H^*(L)$. (c.f. [10].)

Proposition 5.3.2. For δ sufficiently small, we have the following isomorphisms:

$$\bigoplus_* H^*(L) \cong \mathrm{WFH}^{(-\delta,\delta)}(X,\lambda) \cong \mathrm{WFH}^{(-\delta,+\infty)}(X,\lambda) \cong \mathrm{WFH}^{(-\delta,+\infty),c_X}(X,\lambda),$$

Proof. Let δ be smaller than the length of all Reeb chords. Now, let H be a C^2-small homology-admissible Hamiltonian. Note that we still can get rid of the chords in the collar when filtering for $(-\delta,\delta)$ even though there is a term $f(x(1)) - f(x(0))$ in the action functional. Indeed, since the action in the collar is given by

$$f(x(1)) - f(x(0)) - Rh'(R) + h(R)$$

and f is locally constant outside L (therefore bounded), it suffices to require h' to grow sufficiently fast in order for those orbits to disappear of the filtered complex. (Recall that the only thing that matters for the cohomology is the slope at infinity.)

Then, the filtered complex $WC^{(-\delta,\delta)}(H)$ consists only of chords on X, as also does $WC^{(-\delta,+\infty)}(H)$. Therefore,

$$\mathrm{WFH}^{(-\delta,\delta)}(H) \cong \mathrm{WFH}^{(-\delta,+\infty)}(H) \cong \mathrm{WFH}(H^\delta),$$

where H^δ coincides with H on X but has δ as its inclination at infinity. As was observed, we have that this last group is isomorphic to $H^*(L)$. Since this is not dependent on the Hamiltonian H (and note that δ does not depend on the Hamiltonian either), we have that $\mathrm{WFH}^{(-\delta,\delta)}(H) \cong \mathrm{WFH}^{(-\delta,\delta)}(X, \lambda)$.

□

It follows that for every $0 < a \leq \infty$, one can define a natural homomorphism

$$\iota_a : \bigoplus_* H^*(L) \simeq \mathrm{WFH}^{>-\delta, c_X}(X, \lambda) \to \mathrm{WFH}^{>-a, c_X}(X, \lambda)$$

if we take a sufficiently small δ. If δ is sufficiently small, it is clear that this does not depend on δ. We will refer to the image of the canonical class $1 \in H^0(L)$ under this map by F_a. Note that $F_a \in \mathrm{WFH}^{>-a,c_X}(X, \lambda)$ and that it only depends on a.

For any admissible Hamiltonian H, we define F_H as $\Psi^{-1}(F_{a_H}) \in \mathrm{WFH}(H)$.

5.4 Spectral invariants

As a particular instance of Triangle 5.1, the triangle 5.2 is exact, where a is any real number.

$$\begin{tikzcd}
 & \mathrm{WFH}(H) \arrow[dl, "j^a"'] & \\
\mathrm{WFH}^{\leq -a}(H) \arrow[rr, "\partial"] & & \mathrm{WFH}^{>-a}(H), \arrow[ul, "i^a"']
\end{tikzcd} \tag{5.2}$$

We now recall that the natural homomorphism $\Psi_H : \mathrm{WFH}(H) \to \mathrm{WFH}^{>-a_H}(X, \lambda)$ is in fact an isomorphism. Therefore, for a given class $x \in \mathrm{WFH}^{>-a_H}(X, \lambda)$, we can define the *spectral invariant* $\rho(H : x)$ by

$$\rho(H : x) := \inf\{a \in \mathbb{R} \mid \Psi_H^{-1}(x) \in \operatorname{im} i^a\}.$$

This definition represents when (i.e., the "minimal" a for which) the class x effectively appears, in cohomology, as an element of the inclusion of the complex $\mathrm{WFH}_*^{>-a}(H) \to \mathrm{WFH}_*(H)$. Note that $\rho(H : 0) = -\infty$, and also that if $x \neq 0$, then $\rho(H : x)$ is indeed a real number. Furthermore, due to exactness, we also have that $\rho(H : x) = \{a \in \mathbb{R} \mid j^a(\Psi_H^{-1}(x)) = 0\}$.

We summarize some properties of the spectral invariant below.

Lemma 5.4.1. (1) For any admissible Hamiltonian and any nonzero $x \in \mathrm{WFH}^{>-a_H,\alpha}(X, \lambda)$,

$$\rho(H : x) \in \mathrm{Spec}^\alpha(H).$$

(2) If H, K are admissible Hamiltonians such that the support of $H - K$ is compact, then for any nonzero $x \in \mathrm{WFH}(X, \lambda)$ it holds that

$$|\rho(H : x) - \rho(K : x)| \leq \|H - K\|,$$

where the norm is the *Hofer-norm* given by $\|H\| := \int_{S^1}(\max H_t - \min H_t)dt$.

(3) Let H be an admissible Hamiltonian and $2a_H \notin \operatorname{Spec}(X, \lambda)$. If K is linear at $+\infty$ and $\partial_t^r H|_{t=0} = \partial_t^r K_{t=0}$ for any integer $r \geq 0$, then, for any $x, y \in \mathrm{WFH}^{>-a_H}(X, \lambda)$,

$$\rho(H * K : x * y) \leq \rho(H : x) + \rho(K : y).$$

Proof. (1) Suppose $\rho(H : x)$ does not belong to $\operatorname{Spec}^\alpha(H)$. Since $\operatorname{Spec}^\alpha(H)$ is closed and x is nonzero, there exists an interval around $-\rho(H : x)$ with radius ϵ such that there is no orbit with action in that interval. Therefore, triangle 5.1 tells us that $\mathrm{WFH}^{>-\rho(H:x)-\epsilon}(H) \to \mathrm{WFH}^{>-\rho(H:x)+\epsilon}$ is an isomorphism, since

$$\mathrm{WFH}^{(-\rho(H:x)-\epsilon, -\rho(H:x)+\epsilon]}(H) = 0,$$

being the cohomology of an empty complex. It follows that

$$\operatorname{Im}(\mathrm{WFH}^{>-\rho(H:x)-\epsilon,\alpha}(H) \to \mathrm{WFH}^\alpha(H)) = \operatorname{Im}(\mathrm{WFH}^{>-\rho(H:x)+\epsilon}(H) \to \mathrm{WFH}^\alpha(H)).$$

Since the spectral invariant is by definition the infimum of the set of those a's such that $\Psi_H^{-1}(x) \in \operatorname{Im} i^a$, it follows that $\Psi_H^{-1}(x)$ is not in $\operatorname{Im} i^{-\rho(H:x)+\epsilon}$, and thus by what we have seen above it cannot be in $\operatorname{Im} i^{-\rho(H:x)-\epsilon)}$ as well, a contradiction.

(2) Due to proposition 5.2.3, there is, for every $a \in \mathbb{R}$, a monotonicity homomorphism $\mathrm{WFH}^{>-a}(H) \to \mathrm{WFH}^{>-a-\|H-K\|}(K)$. The commutativity of the diagram

$$\begin{tikzcd}
\mathrm{WFH}^{>-a}(H) \arrow[r] \arrow[d, "i^a"] & \mathrm{WFH}^{>-a-\|H-K\|}(K) \arrow[d, "i^{a+\|H-K\|}"] \\
\mathrm{WFH}(H) \arrow[r] & \mathrm{WFH}(K)
\end{tikzcd}$$

shows us that if $\Psi_H^{-1}(x) \in \operatorname{Im} i^a$, then $\Psi_K^{-1}(x) \in \operatorname{Im} i^{a+\|H-K\|}$. Therefore,

$$\rho(K : x) \leq \rho(H : x) + \|H - K\|.$$

The same argument with the roles reversed yield that

$$\rho(H : x) \leq \rho(K : x) + \|K - H\|$$

and we can conclude the inequality as stated.

□

5.5 Bounding the capacity

Before proceeding with our discussion, we make the following conventions:

- For any $H \in C_0^\infty(\operatorname{int} X)$, $a \in \mathbb{R}$ and $\nu \in C^\infty([1, \infty))$, we define $H_{a,\nu} : S^1 \times \widehat{X} \to \mathbb{R}$ by

$$H_{a,\nu}(t, x) := \begin{cases} aH(x), & x \in \operatorname{int} X; \\ \nu(r), & x = (z, r) \in \partial X \times [1, \infty). \end{cases}$$

- For any $K \in C^\infty(S^1 \times \widehat{X})$ which is linear at ∞ and $a_K \notin \mathrm{Spec}(X, \lambda)$., we abbreviate $\rho(K : F_{a_K})$ as $\rho(K)$.

We now aim to prove the result in which we are able to bound the symmetric capacity by a when $F_a = 0$. We temporarily alter the definition of Symmetric-Hofer-Zehnder admissibility for the purposes of the end of this chapter in order to facilitate the proofs. Instead of a non-negative Hamiltonian which is equal to its maximum outside a compact set as in Definition 4.1.3, we will define it as a non-positive Hamiltonian which is equal to zero outside a compact set. Then, we set the capacity as the sup of the values $-\min H$. Note that by translation these two definitions give the same value for the capacity; the only thing being different is the set of admissible Hamiltonians.

For that, we need the following proposition.

Proposition 5.5.1. Let (X, L, λ) be a Liouville domain equipped with an anti-symplectic involution φ for which L is its (non-empty) fixed-point set satisfying the properties described in Section 3.2, $H \in C_0^\infty(\mathrm{int}(X))$, $\nu \in C^\infty([1, \infty))$. Suppose that:

(1) There exists $r_0 > 1$ such that $\nu(r) \equiv 0$ on $[1, r_0]$.

(2) There exists $r_1 > 1$ and $a_\nu \in (0, -\min H)$, where this a_ν is not in $\mathrm{Spec}(X, \lambda)$, such that $\nu'(r) \equiv a_\nu$ on $[r_1, \infty)$.

(3) $S(\nu) := \inf_{r \geq 1}(-r\nu'(r) + \nu(r)) > \min H$.

(4) f is constant on $\widehat{L} \backslash L$.[1]

Under these assumptions, if H is Symmetric-Hofer-Zehnder admissible, [2] with respect to c_X, then $\rho(H_{1,\nu}) = -\min H$.

It will be useful for the proof of Proposition 5.5.1 to have the next lemma at hand.

Lemma 5.5.2. Given $r > 1$, suppose that K is an admissible Hamiltonian which is time independent and linear on $\partial X \times [r, \infty)$. If the C^2-norm of $K|_{X(r)}$ is sufficiently small and K has a unique point of mimimum, then $\rho(K) = -\min K$.

Proof. The fact that K is time independent allows us to treat K as a function on $\widehat{X}$. Having sufficiently small C^2-norm, we know that it reduces the wrapped Floer complex to the Morse complex of L, and thus we have $H^*(L) \simeq \mathrm{WFH}^*(K)$. By definition, F_K is represented by the canonical element 1 under this isomorphism, which is to say that it is given by the point of minimum of K.

For each critical point q, we have that, if seen as a brake orbit,

$$A_K(q) = \int K \circ q dt = K(q).$$

Note now that for the minimum to appear as the image of $i^a : \mathrm{WFH}^{>-a}(K) \to \mathrm{WFH}(K)$, we then need $-a < \min K$, and thus $a > -\min K$. Since any $a > -\min K$ will do, that is to say that $\rho(K : F_K) = -\min K$.

[1]For example, this happens if ∂L is connected, or if $\theta|_L = 0$ as per the notation in Section 3.2.

[2]For emphasis, recall that this is *not* the same as being homology-admissible.

As a last observation, take note of the fact that we are still not using the assumption over the primitive f at this moment. Indeed, since q is a constant path, $f(x(1)) - f(x(0)) = f(q) - f(q) = 0$, thus making f disappear in the computation of the action that was done above. □

We are now on grounds for proving Proposition 5.5.1. We proceed as follows.

Proof. Suppose $H \in C_0^\infty(\operatorname{int} X)$ is symmetric-Hofer-Zehnder-admissible. We will show that for any $\epsilon > 0$, there exists an equivariant $K \in C_0^\infty(\operatorname{int} X)$ such that

- Any nonconstant contractible periodic orbit of X_K has period larger than one.
- $\min K < \min H$.
- $\min K$ is isolated in the set of critical values of K.
- $\min K|_L$ is attained by a unique point in L.

To see this, take $g : [0, +\infty) \to \mathbb{R}$ a smooth function which satisfies the following requirements:

(1) g is C^2-close to zero on $(0, \epsilon/2)$;

(2) $0 \leq g' \leq 1$;

(3) $g'(x) = 1$ for every $x > \epsilon$.

Let $K_1 = -g\circ(-H)$. Note that this is still equivariant. By making a C^2-small perturbation of K_1 on $K^{-1}((0, \epsilon/2))$ such that it is still equivariant but now Morse there [3], we know that low values of K_1 are isolated and no nonconstant periodic orbits are being added. We consider a negative bump function ρ supported on a neighbourhood of some point p which achieves the minimum on the restriction to L. Then, by letting $K_2 := K_1 + \rho + \rho \circ \varphi$, we have that K satisfies our requirements.

With such available K's at hand, note that it is then enough to show that $\rho(K_{1,\nu}) = \max K$, due to our result about continuity of ρ with respect to the Hofer-norm. (Note that $\|H - K\| \leq |H - K|_{C_0}$.)

We will divide the proof in steps.

Step 1 . *There exist* $0 < \epsilon_0 < 1$ *and* $0 < \delta_0 < \min \operatorname{Spec}(X, \lambda)/a_\nu$ *such that* $\rho(K_{\epsilon,\delta\nu}) = \epsilon \max K$ *for any* $\epsilon \in (0, \epsilon_0]$ *and* $\delta \in (0, \delta_0]$.

It is clear that for ϵ, δ sufficiently small, the C^2-norm of $K_{\epsilon,\delta\nu}|_{X(r_1)}$ is also small. So it suffices to approximate this $K_{\epsilon,\delta\nu}$ by admissible Hamiltonians.

Step 2 . *It is true that* $\rho(K_{1,\delta\nu}) = -\min K$ *for any* $0 < \delta < \min\{\delta_0, (-\epsilon_0 \min K/S(\nu)\}$.

For any $\epsilon \in (0, 1]$, $\mathcal{P}^{cx}(\epsilon K)$ consists of only constant loops at critical points of K, since nonconstant contractible periodic orbits of X_K have period larger than 1. If

[3] This is possible due to an analogue of the standard result of density of Morse functions, but taking into account invariance under a compact Lie group action. For more information, see [28]. Here, the group action is given by applying the involution, thus giving rise to a $\mathbb{Z}_2$-action.

we compute the action of an element $x \in \mathcal{P}(K_{\epsilon,\delta\nu})$ which is not contained in X, we get

$$\begin{aligned}\mathcal{A}_{K_{\epsilon,\delta\nu}}(x) &= f(x(1)) - f(x(0)) - \int x^*\lambda + \int K_{\epsilon,\delta\nu}(x(t))dt \\ &= -\int \lambda(\dot{x}) + \int_0^1 \delta\nu(r(t))dt \\ &= \int -r\delta\nu'(r(t)) + \delta\nu(r(t))dt \\ &\geq S(\nu)\delta,\end{aligned}$$

where $f(x(1)) = f(x(0))$ since we are assuming f is constant in the collar.

It follows that $\mathrm{Spec}^{cX}(K_{\epsilon,\delta\nu}) \subset [\delta S(\nu), +\infty) \cup \epsilon \,\mathrm{Cr}\, V(K)$, where $\mathrm{Cr}\, V(K)$ refers to the set of critical values of K. Since $\delta a_\nu < \min \mathrm{Spec}(X, \lambda)$, we have that $F_{\delta a_\nu}$ is nonzero, and therefore $\rho(K_{\epsilon,\delta\nu}) \in (-\infty, -\delta S(\nu)] \cup -\epsilon\,\mathrm{Cr}\, V(K)$.

Let $I := \{\epsilon \in [\epsilon_0, 1] \mid \rho(K_{\epsilon,\delta\nu}) = -\epsilon \min K\}$. Since $\rho(K_{\epsilon,\delta\nu})$ depends continuously on ϵ, we have that I is closed. That I is open follows from the fact that $-\delta S(\nu) < -\epsilon_0 \min K$ and that the set of critical values is nowhere dense. Noting that Step 1 says that $\epsilon_0 \in I$, we can conclude that $I = [\epsilon_0, 1]$. In particular, we conclude that $\rho(K_{\epsilon,\delta\nu}) = -\min K$.

Step 3 . *It is true that* $\rho(K_{1,\nu}) = -\min K$*.*

Since $S(\nu) > \min H > \min K$, it follows that $\mathrm{Spec}(K_{1,\nu}) \subset [\min K, +\infty)$ by the computation of the action on Step 2. Thus, $\rho(K_{1,\nu}) \leq -\min K$.

In order to show that $-\min K \leq \rho(K_{1,\nu})$, we proceed as follows. Take δ such that $0 < \delta < \min\{\delta_0, (-\epsilon_0 \min K/S(\nu)\}$. Take $c > 0$ such that $\mathrm{Cr}\, V(K) \cap (-\infty, \min K + c] = \{\min K\}$ and $S(\nu) > \min K + c$. Consider now the following commutative diagram:

$$\begin{array}{ccc} \mathrm{WFH}(K_{1,\delta\nu}) & \longrightarrow & \mathrm{WFH}^{\leq \min K + c}(K_{1,\delta\nu}) \\ \downarrow & & \downarrow \\ \mathrm{WFH}(K_{1,\nu}) & \longrightarrow & \mathrm{WFH}^{\leq \min K + c}((K_{1,\nu}), \end{array}$$

where the horizontal arrows are just the maps induced by inclusion and the vertical arrows are the monocity homomorphisms. Since $S(\nu) > \min K + c$, the only elements of the chain complexes that generate the cohomology on the right is the critical point which realizes $\min K$. Since $K_{1,\delta\nu}|_{S^1\times X} = K_{1,\nu}|_{S^1\times X}$, we have that both coincide near the critical point, so that the monocity homomorphism is just the identity and thus the induced map (the right arrow) is an isomorphism.

By the previous step, $\rho(K_{1,\delta\nu}) = -\min K$. Since the top map is $j^{-\min K - c}$, it follows that $F_{K_{1,\delta\nu}}$ does not vanish. Since the right map is an isomorphism, it continues to be non-zero down in the bottom right corner. Therefore, by commutativity, it must be non-zero down in the bottom-left corner. Since it is $F_{1,\nu}$ there we conclude that $\rho(K_{1,\nu}) \geq -\min K - c$. Since this holds for every $c > 0$ small enough, we conclude that $\rho(K_{1,\nu}) \geq -\min K$.

$\square$

We are now able to bound the symmetric capacity when $F_a = 0$ for some a. (In particular, when the wrapped Floer cohomology vanishes.)

Corollary 5.5.3. Let (X, L, λ) be a Liouville domain equipped with an anti-symplectic involution φ for which L is its (non-empty) fixed-point set satisfying the properties described in Section 3.2 and such that f is constant on $\widehat{L} \backslash L$. If $F_a = 0$, then $c_{sym}(\operatorname{int} X, d\lambda : \{c_X\}) \leq a$.

Proof. If $c_{sym} > a$, then there exists a Hamiltonian H which is symmetric-Hofer-Zehnder admissible with respect to c_X such that $-\min H > a$. Taking ν that satisfies the hypothesis of Proposition 5.5.1 and $a_\nu = a$, we then have that $\rho(H_{1,\nu}) = -\min H$. We have reached a contradiction, since $F_a = 0$ implies $\rho(H_{1,\nu}) = -\infty$. $\square$

Chapter 6

Local coefficients

6.1 Introduction

In [5] we can see the idea that even though the (in their case, symplectic) homology may not vanish, introducing a suitable system of local coefficients can make it possible to get a vanishing (symplectic) homology, and effectively employ the result obtained in chapter 5 in order to guarantee finiteness of the Hofer-Zehnder capacity. This allows us to expand the finiteness of the symmetric capacity to larger classes of domains.

Since the concept of a local system of coefficients may effectively not be common knowledge, we introduce it in this section. The core idea is that we have at each point of our topological space an R-module and, given a path between x, y, a way of sending the R-module over x to the one over y in a way that is consistent under homotopy. Formally, we have the following definition.

Definition 6.1.1. Let X be a path-connected topological space which admits a universal cover and R be a commutative ring with unit. In summary, a *local system of coefficients* on X is a collection of R-modules M_x for each $x \in X$ and a natural transformation $\mathcal{T}$ between the category of points of X with morphisms being homotopy classes of paths connecting them, and the category of the modules M_x with morphisms being module homomorphisms.

Expanded, it is a collection of R-modules M_x as above, together with a collection of homomorphisms $T_{[\alpha]} : M_x \to M_y$ for every $x, y \in X$ sand homotopy class of paths connecting x, y such that

$$T_{[cnst]} = \mathrm{Id}$$

and

$$T_{[\alpha]\cdot[\beta]} = T_{[\beta]}T_{[\alpha]},$$

where $[\alpha] \cdot [\beta]$ is concatenation of paths.

An alternative more abstract point of view of the above definition can be done by clumping down the entire data as an element of $\mathrm{Hom}(\pi_1(X), G)/G$, where G is the group of automorphisms of the module M. This equivalence arises from the fact that by choosing a basepoint and an isomorphism $M_x \simeq M$, where M is a fixed module which is isomorphic to all M_x's (note that by the functoriality of the definition and the fact that we can always reverse the path, all homomorphisms are in fact isomorphisms), we can associate to every class $\alpha \in \pi_1(X, x)$ an automorphism of M. Thus, the data gives us an element of $\mathrm{Hom}(\pi_1(X), \mathrm{Aut}(M))$. Since changing the isomorphism $M_x \to M$ changes M up to an inner automorphism, we get an element of $\mathrm{Hom}(\pi_1(X), \mathrm{Aut}(M))/\mathrm{Aut}(M)$ as mentioned before. That these two viewpoints are indeed equivalent can be seen in [27].

Local systems can be useful for a variety of reasons. For instance, we will present two examples before proceeding: a computation which shows that we can make the homology of a non-acyclic manifold to vanish after introducing local systems (in the spirit of the introduction to this section), and a computation which shows that an orientation local system can save Poincaré duality in a non-orientable manifold. But before that, it might be useful to recall how orientations are taken into account in Morse homology.

6.2 Orientations in Morse homology

Let us recall that the boundary map in Morse homology is given by

$$\partial x = \sum_{\substack{y, \\ \mathrm{ind}(y)=\mathrm{ind}(x)-1}} \sum_{u \in \mathcal{M}(x,y)} \epsilon(u, x, y) y,$$

where $\mathcal{M}(x, y)$ is the space of unparametrized paths connecting x, y asymptotically (i.e., the quotient of the space of paths by the $\mathbb{R}$-action induced by translation), and $\epsilon(u, x, y)$ is either ± 1, with the signs determined after the following procedure:

1. We orient each unstable manifold arbitrarily.
2. This determines an orientation on each intersection of unstable and stable manifolds, as shall be explained.
3. If the orientation determined above coincides with the direction of the gradient flow for the given flow line $u \in \mathcal{M}(x, y)$, then $\epsilon(u, x, y) = 1$. Otherwise, it is -1.

So the point to be explained is item (2). Establishing some notation, let $T^u_x(p), T^s_x(p)$ be, respectively, the tangent spaces of the unstable and stable manifolds $W^u(x)$ and $W^s(x)$ at p, and let $T^{u,s}_{x,y}(p)$ be the tangent space of the intersection of the unstable manifold of x with the stable manifold of y some point p.

Fix some Riemannian metric on M and let x, y be two critical points such that $\mathrm{index}(y) = \mathrm{index}(x)-1$. Given a point $p \in W^u(x) \cap W^s(y)$, we have that $T^u_x(p) = T^{u,s}_{x,y}(p) \oplus N^{u,s}_y(p)$, where $N^{u,s}_y(p)$ is a complement of $T^{u,s}_{x,y}(p)$ coming from parallel transportation of the orthogonal complement of $T^s_y(y)$ along the compactified curve given by the connected component of $W^u(x) \cap W^s(y)$ where p lies. Since this orthogonal complement at y is precisely $T^u_y(y)$, we have orientations both for $T^u_x(p)$ and $N^{u,s}_y(p)$, and therefore this induces an orientation of $T^{u,s}_{x,y}(p)$. Note that none of this required the manifold M to be orientable.

For instance, consider the following image.

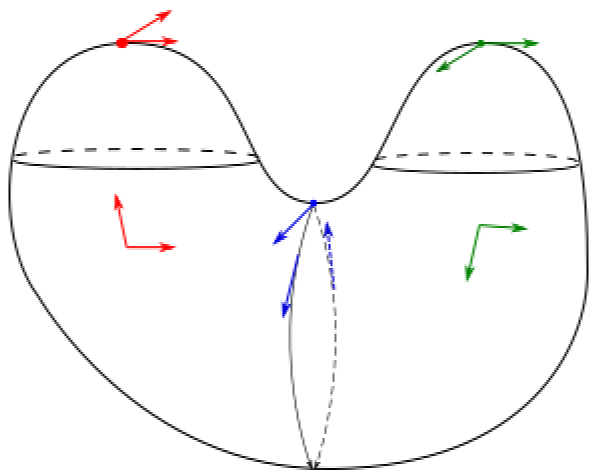


Figure 6.1: Orienting the unstable manifolds.

In Figure 6.2, we have done the first step of the procedure as outlined previously: we have oriented all unstable manifolds. In the cases of the red and green pairs, we are considering the first vector of the orientation to be the horizontal one.

By proceeding with the second step as explained, we should obtain the following induced orientations on the intersections of unstable and stable manifolds (represented in pink, yellow and purple):

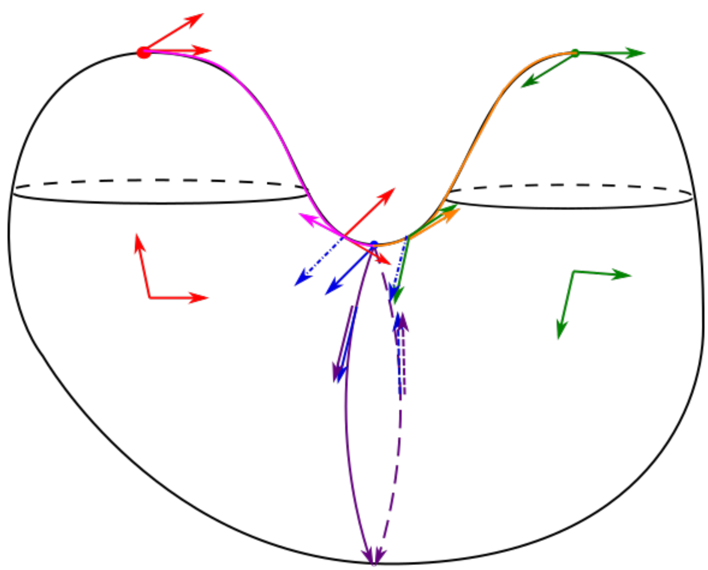


Figure 6.2: Orienting the intersections of unstable and stable manifolds.

Note that the orientations of the flow line coincide with the orientation given by the second step only in the case of the intersection of the unstable manifold leaving the critical point of index 1 and the stable manifold of the bottom critical point. By representing each generator of the Morse homology with its color and letting $F(A)$ denote the free abelian group generated by the set A, we then get that the map is the following.

$$F(\{\textcolor{red}{\bullet}, \textcolor{green}{\bullet}\}) \overset{\partial_2}{\to} F(\{\textcolor{blue}{\bullet}\}) \overset{\partial_1}{\to} F(\{\bullet\}),$$

where

$$\partial_2 = \begin{cases} \textcolor{red}{\bullet} \mapsto -\textcolor{blue}{\bullet}, \\ \textcolor{green}{\bullet} \mapsto -\textcolor{blue}{\bullet} \end{cases}$$

and

$$\partial_1(\textcolor{blue}{\bullet}) = \bullet - \bullet = 0.$$

As expected, we get $H_2^{Morse}(M) = \mathbb{Z} = H_0^{Morse}(M)$ and 0 for all other degrees.

6.3 Examples of local coefficients in Morse homology

In what follows, we will employ Morse homology with local coefficients. Since our objective is to use symplectic and wrapped Floer homology with local coefficients, Morse homology is more close to this goal than, say, singular homology. For more elaboration on this twisted version of Morse homology, see [7].

In order to compute Morse homology with local coefficients, we will restrict ourselves to local coefficients being $M \simeq \mathbb{Z}$ and the difference in the complex being the free modules $\bigoplus_{q \in \mathrm{Crit}_k} \mathbb{Z}_q$ and the boundary map is

$$\partial g_x = \sum_y \sum_{u \in \mathcal{M}_0(x,y)} \epsilon(u) \tau_u(g_p)_y.$$

Note that if the local system is trivial, then this reduces itself to the common Morse homology.

First example: Circle.

Pick f to be the height function on the unit circle, and the local coefficients $\mathcal{T}$ to be determined by $\rho = -1 \in \mathrm{Hom}(\pi_1(S^1), \mathbb{Z}_2) \simeq \mathbb{Z}_2$. So if x is the top point and y the bottom one, the boundary map becomes

$$\partial 1_x = \tau_{u_1}(1_x)_y - \tau_{u_2}(1_x)_y,$$

where u_1 is the right half-circle and u_2 is the left half-circle, both oriented pointed downwards. On a trivial local system, this would vanish and we would recover the fact that x represents a non-trivial homology class on $H_1(S^1)$. However, we must have that $\tau_{u_1}(1_x) = -\tau_{u_2}(1_x)$, since $-u_2 \circ u_1$ is a generator of π_1, hence $\tau_{u_2}^{-1} \circ \tau_{u_1} = -1$. Therefore, we have

$$\partial 1_x = 2_y$$

and it follows that

$$H_0^{Morse}(S^1, \mathcal{T}) \simeq \mathbb{Z}_2$$

and

$$H_1^{Morse}(S^1, \mathcal{T}) \simeq 0.$$

Note that if instead of coefficients being based on $\mathbb{Z}$ we considered $\mathbb{R}$ instead, then the homology would vanish completely.

Second example: $\mathbb{R}P^2$

In this case, it is convenient to consider $X = \mathbb{R}P^2$ as an ellipsoid with different semi-axes a, b, c and identified antipodes, and the Morse function coming from the function $x \mapsto \|x\|^2$ on the quotient.

We can consider the local coefficients to be determined by $\rho = -1 \in \mathrm{Hom}(\pi_1(\mathbb{R}P^2), \mathbb{Z}_2) \simeq \mathbb{Z}_2$. This local system has the interpretation of being the identity if a loop preservers orientation or minus the identity if it reverses.

The Morse complex then is composed of only the fiber $\mathbb{Z}_x$ over each critical point x. Let p, q, r stand for the maximum, the saddle and the minimum respectively. Then, after fixing orientations for the unstable manifolds (c.f. [18] for example), the boundary maps are

$$\begin{aligned} 1_p &\mapsto \tau_{u_1}(1_p)1_q + \tau_{u_2}(1_p)1_q \\ 1_q &\mapsto \tau_{v_1}(1_q)1_r - \tau_{v_1}(1_q)1_r. \end{aligned}$$

By a similar reason as in the case of the circle, those maps are

$$\begin{aligned} \partial 1_p &\mapsto 0 \\ \partial 1_q &\mapsto 2_r, \end{aligned}$$

so that

$$H_*^{Morse}(\mathbb{R}P^2, \mathcal{T}) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 2 \\ \mathbb{Z}_2 & \text{if } * = 1 \\ 0 & \text{otherwise.} \end{cases}$$

6.4 Interlude explaining the thought process

As we have mentioned previously, our main purpose behind using local coefficients is to find some system for which the wrapped Floer homology vanishes so that we can apply the results of chapter 5. Specifically, we want to apply the following result, which is a direct consequence of corollary 5.5.3.

Corollary 6.4.1. Let (X, L, φ) be a Liouville quadruple such that $\mathrm{WFH}(X, L) = 0$. Then $c_{sym}(\mathrm{int}\, X, d\lambda : \{c_X\}) < +\infty$.

All algebraic constructs done in that chapter are still possible for the case of wrapped Floer homology with coefficients, which will be introduced in the following section. However, the key factor that relates the capacity with the homology is done by Lemma 5.5.2 and an observation is necessary: In order for it to hold, since for a small enough Hamiltonian the wrapped Floer homology reduces itself to the Morse homology of the Lagrangian, we need the critical points to effectively appear in this homology. So, for the argument to go through, the safest assumption would be that the local system is trivial on the base manifold, so that the Morse homology of L with this local system is simply the standard Morse homology of L and the arguments go on unaltered. With this is mind, in the same spirit of [5], we conclude the following.

Proposition 6.4.2. Let (X, L, φ) be a Liouville quadruple and $\mathcal{T}$ be a local system of coefficients on $\Omega(\overline{X}, \overline{L})$ such that $\mathrm{WFH}(X, L; \mathcal{T}) = 0$ and $\mathcal{T}|_L$ is trivial.[1] Then $c_{sym}(\mathrm{int}\, X, d\lambda : \{c_X\}) < +\infty$.

The objective of this chapter is to explain symplectic homology with coefficients and attempts to provide organic situations where we have such a situation as described above.

6.5 Symplectic cohomology with local coefficients

A way to introduce local coefficients in symplectic cohomology is by considering *Novikov fields* attached at every point of the free loop space $\Lambda(\widehat{X})$, where the *Novikov field* over a field $\mathbb{K}$ is defined as the $\mathbb{K}$-algebra of formal series

$$N := \{\sum_{k=0}^{\infty} k_j t^{a_j} \mid k_j \in \mathbb{K}, a_j \in \mathbb{R}, a_j \to +\infty\}.$$

Now, in order to build the maps between those fields, we fix a real singular 1-cocycle α of $\Lambda(\widehat{X})$. Given two loops $x, y \in \Lambda(\widehat{X})$, we define

$$\begin{aligned} T_{x,y,[u]} : \Lambda_x &\to \Lambda_y \\ s &\mapsto t^{\alpha(u)} s. \end{aligned}$$

It is easy to see that if we change α to another representative of its cohomological class, then we get an isomorphic local system. We will denote this local system associated to α as N_α. Also worth noticing is that if α is zero, then the local system is trivial, since $T_{x,y,[u]}$ becomes $s \mapsto t^0 s = s$.

The symplectic homology with local coefficients is then built upon using this data by just changing the algebra of the boundary and continuation maps in the following way:

$$\begin{aligned} C_*(H)_a = C_*(H; N_\alpha) &:= \text{Free } N\text{-module generated by the periodic orbits;} \\ \partial x &= \sum_{\mathcal{M}_0(x,y)} \epsilon(u) t^{\alpha(u)} y; \\ SH_*(H)_\alpha &= SH_*(H; N_\alpha) = H_*(C_*(H; \Lambda_\alpha); \partial); \\ \Psi_{H_-,H_+}(x_-) &= \sum_{v \in \mathcal{M}^{H_s}(x_-, x+)} \epsilon(v) t^{\alpha(v)} x_+. \end{aligned}$$

As before,

$$SH_*(X)_\alpha = SH_*(X; N_\alpha) := \lim_{\longleftarrow} SH_*(H)_\alpha.$$

It would be useful to insert twists by cochain classes of the manifold instead of its loop (or path) spaces. We can do this by looking at some relationships between the homology of the path spaces of a base space and of the space itself. For instance, a 2-cochain class of $\widehat{X}$ can be traced to a 1-cochain class in the loop space by going through the maps defined in homology

[1] Here, we are considering M as the space of constant paths.

by the evaluation map $\mathrm{ev} : \Lambda\widehat{X} \times S^1 \to \widehat{X}$ and the projection on the first coordinate of the Kunneth isomorphism. The intuition for this construction is of course reminiscent of the underlying principle that paths in the free loop space are homotopies between two paths, and hence define a two-dimensional object when seen from the perspective of the base space.

If we trace out the maps defined above, we can see that it corresponds to the following map

$$\eta \mapsto \left(u \mapsto \int u^*\eta \right),$$

where we are picking a smooth representative of the singular class of u, seen as a cylinder $u : [0,1] \times S^1 \to \overline{M}$. This composition is usually called *transgression* in the literature, and denoted by τ. Note that any form obtained by transgression will yield a trivial local system when restricted to $\widehat{X}$ as the set of constant loops, since $\int u^*\eta = \int \eta(\partial_s u, \partial_t u) ds dt$ will always be zero since u will be a time-independent map. This is an important observation, since we are searching for non-trivial local systems that restrict to trivial ones on the base manifold.

It is worth noticing that τ is not an isomorphism in general. If M is simply connected, it is. If it is not, we can instead look at the universal cover $\widetilde{M}$.[2] Then the composition defined above, usually called *transgression* in the literature, becomes an isomorphism $\tau : H^2(\widetilde{M}) \to H^1(\Lambda\widetilde{M})$. Note that we also have the following chain of isomorphisms

$$H^1(\Lambda\widetilde{M}) \simeq \mathrm{Hom}(H_1(\Lambda\widetilde{M}), \mathbb{Z}) \simeq \mathrm{Hom}(\pi_1(\Lambda\widetilde{M}), \mathbb{Z}) \simeq \mathrm{Hom}(\pi_2(\widetilde{M}), \mathbb{Z}),$$

the first arising from the universal coefficients theorem, the second due to the fact that π_1 is the abelianization of H_1 and $\mathbb{Z}$ is abelian, the last due to the relationship between the fundamental groups of a manifold and its free loop space.

6.6 Local system of coefficients in wrapped Floer homology

Having done the symplectic case, the wrapped case consists of minor adaptations. Evidently, we do not introduce a local system on $\Lambda(\widehat{X})$, but on $\Omega(\widehat{X}, \widehat{L})$, i.e. the space of paths with endpoints lying on $\widehat{L}$. With this distinction in mind, the procedure is similar: we pick a real singular relative 1-cocycle $\alpha \in H^1(\widehat{X}, \widehat{L}) \simeq H^1(X, L)$ and we then have a local system of coefficients T on $\Omega(\widehat{X}, \widehat{L})$. Analogously, we define

$$\begin{aligned}
WC_*(H)_a = WC_*(H; N_\alpha) &:= \text{Free } N\text{-module generated by the brake orbits;} \\
\partial x &= \sum_{\mathcal{M}_0(x,y)} \epsilon(u) t^{\alpha(u)} y; \\
\mathrm{WFH}_*(H)_\alpha = \mathrm{WFH}_*(H; N_\alpha) &= H_*(C_*(H; \Lambda_\alpha); \partial); \\
\Psi_{H_-,H_+}(x_-) &= \sum_{v \in \mathcal{M}^{H_s}(x_-,x+)} \epsilon(v) t^{\alpha(v)} x_+; \\
\mathrm{WFH}_*(X)_\alpha = \mathrm{WFH}_*(X; N_\alpha) &:= \varprojlim \mathrm{WFH}_*(H)_\alpha.
\end{aligned}$$

[2]Note the difference between the notation for the universal cover, a tilde, and the notation for the symplectization, a hat.

Given a class $\eta \in H^2(\widehat{X}, \widehat{L}; \mathbb{R}) \simeq H^2(X, L; \mathbb{R})$, this defines a transgressed class $\tau\eta$ in the same way as before, i.e. by defining

$$\tau\eta(u) = \int_C u^*\eta,$$

where u is seen as a map from the cylinder $C = [0,1] \times S^1$ to M. This is an isomorphism if $\pi_1(X/L) = 1$, in particular if M is simply-connected.

By the exact sequence of a pair (X, L), we have a map

$$H^2(X, L; \mathbb{R}) \to H^2(X; \mathbb{R}),$$

which is just the pull-back by the inclusion $\iota : X \to (X, L)$. This allows us to associate to a given $\eta \in H^2(X, L; \mathbb{R})$, a class $\overline{\eta} \in H^2(X; \mathbb{R})$. In our present context, this gives us a way to retrieve a local system of coefficients for the symplectic homology given one for the wrapped Floer homology.

The next result, which can be seen in [23], is then useful for our purposes.

Proposition 6.6.1. If $SH^*(X)_{\overline{\eta}} = 0$, then $\mathrm{WFH}(X, L)_\eta = 0$.

This is a direct consequence of the algebraic structure involving these homologies, more specifically the fact that the wrapped Floer homology is a module over the symplectic homology. This structure arises from the aforementioned map $H^2(X, L) \to H^2(X)$ and the TQFT structure on those homologies. Since the detailed algebraic construction would perhaps be too much of a sidetrack, we refer the reader to the given reference for the details.

From [21] and [23], we have the next theorem.

Theorem 6.6.2. If M is a closed manifold of finite type[3] and $\eta \in H^2(M; \mathbb{R})$ is a class such that $\tau\eta \neq 0 \in H^1(\Lambda M)$, then

$$SH^*(T^*M)_\eta \simeq H_{n-*}(\Lambda M; N_{\tau\eta}) = 0$$

Putting Proposition 6.6.1 and theorem 6.6.2 together, we get the following theorem.

Theorem 6.6.3. Let $\eta \in H^2(M, L; \mathbb{R})$ be such that $\tau\overline{\eta} \neq 0$. Then

$$\mathrm{WFH}(D^*TM, \mathcal{N}^*L)_\eta = 0.$$

A concrete way to check that $\tau\eta \neq 0$ is accomplished by using the fact that $\tau\eta \neq 0$ if and only if $\pi^*\eta \neq 0$, where π stands for the universal cover. In particular, if M is simply-connected, then $\tau\eta \neq 0$ if and only if $\eta \neq 0$.

So, with these facts in mind, we obtain the following result.

Theorem 6.6.4. Let M be a manifold equipped with a smooth involution φ such that its fixed-point set is L. Suppose there is $\eta \in H^2(M, L; \mathbb{R})$ such that $\tau\overline{\eta} \neq 0$. Then $c_{sym}(D^*TM, \mathcal{N}^*L, \widetilde{\varphi}) < +\infty$.

[3]A topological space is said to be of *finite type* if all its homotopy groups, with perhaps the exception of the fundamental group, are finitely generated.

In particular, we have the following result.

Corollary 6.6.5. Let M be a simply-connected manifold equipped with a smooth involution φ such that its fixed-point set L is also simply-connected. Suppose $H^2(M,L;\mathbb{R}) \neq 0$. Then $c_{sym}(D^*TM, \mathcal{N}^*L, \widetilde{\varphi}) < +\infty$.

Proof. By the universal coefficients theorem we have that $H^1(L;\mathbb{R}) = 0$ since L is simply-connected, and therefore the map $H^2(M,L;\mathbb{R}) \to H^2(M,\mathbb{R})$ is injective. Therefore, if we take $\eta \neq 0$ in $H^2(M,L;\mathbb{R})$, then $\overline{\eta} \neq 0$. This is equivalent to $\tau\overline{\eta} \neq 0$ since M is simply-connected, so the result follows from theorem 6.6.4. ☐

Another class of examples is that of ALE spaces[4] , as given by the next result.

Proposition 6.6.6. Let M be an ALE space and $\varphi : M \to M$ be an anti-symplectic involution with fixed-point set being L. Then $c_{sym}(M,L,\varphi) < +\infty$.

Proof. By [22], we have that $SH^*(M)_{\overline{\eta}} = 0$ for a generic $\overline{\eta} \in H^2(M;\mathbb{R})$. Since M is four-dimensional, L is a two-dimensional compact manifold with boundary, thus $H^2(L;\mathbb{R}) = 0$. Therefore, by the exact sequence of the pair (M,L) we have that $\overline{\eta}$ lifts to a non-zero element $\eta \in H^2(M,L;\mathbb{R})$, and thus we can apply Proposition 6.6.1 to infer that the wrapped Floer homology vanishes. ☐

6.7 Path spaces and local systems

As always, let $\Omega(M,L)$ denote the space of paths on M with endpoints on L. We assume that M and L are connected.

Note that $\Omega(M,L)$ is the pull-back by the inclusion $i : L \times L \to M \times M$ of the fibration $M^{[0,1]} \to M^{\{0,1\}}$ given by evaluation at the endpoints. Therefore, the evaluation map

$$\begin{aligned} \mathrm{ev} : \Omega(M,L) &\to L \times L \\ c &\mapsto (c(0), c(1)) \end{aligned}$$

is also a fibration. (That these maps are indeed fibrations is a consequence of foundational results about fibrations which can be found in [8].) It follows that we have an exact sequence of fundamental groups

$$\cdots \to \pi_n(\Omega(M,\{l_1,l_2\})) \to \pi_n(\Omega(M,L)) \to \pi_n(L \times L) \to \cdots$$

Note that $\pi_n(\Omega(M,\{l_1,l_2\})) \simeq \pi_n(\Omega(M))$, the standard based loop space.

Comparing with [5], we should observe that there they analyze

$$0 \to \pi_1(\Omega_0(M)) \to \pi_1(L_0M) \xrightarrow{\pi} \pi_1(M) \to 1.$$

This exact sequence emerges due to the existence of the map $s : M \to L_0M$ embedding M as constant loops on L_0M. This map gives a section to π in each dimension, thus showing that

[4]In summary, M is said to be an *asymptotically locally Euclidean space* (ALE space) if M is a simply-connected hyperkähler 4-manifold which at infinity looks like $\mathbb{C}^2/G$ for a finite subgroup G of $SL(2,\mathbb{C})$

π is surjective and hence the long exact sequence of this fibration fragments itself as short exact sequences of each dimension.

Of course, in the present case we do not have a straightforward way to include $L \times L$ into $\Omega(M, L)$. We could hope that taking (p, q) to a path connecting p to q could provide us with such a map, but this can depend heavily on the chosen path.

Therefore, we want the existence of a section

$$s : L \times L \to \Omega(M, L),$$

i.e. a map such that $\mathrm{ev} \circ s = \mathrm{Id}$. This can be guaranteed, for instance, if the inclusion $\iota : L \to M$ is null-homotopic, as the next proposition states.

Proposition 6.7.1. Suppose the inclusion $\iota : L \to M$ is null-homotopic. Then the evaluation $\mathrm{ev} : \Omega(M, L) \to L$ admits a section s.

Proof. Fix $a, b \in L$ and let $H : L \times I \to M$ be the null-homotopy. Then the map

$$s(a, b) = t \mapsto \begin{cases} H(a, 2t), & t \in [0, 1/2] \\ H(b, 2 - 2t), & t \in [1/2, 1], \end{cases}$$

or in other words $s(a, b) = H|_{a\times I} * H|_{b\times I}^{-1}$, is a section for ev. □

Corollary 6.7.2. Suppose the inclusion $\iota : L \to M$ is null-homotopic. Then there is the following exact sequence

$$0 \to \pi_1(\Omega_0(M)) \xrightarrow{f} \pi_1(\Omega_0(M, L)) \xrightarrow{g} \pi_1(L \times L) \to 1.$$

Moreover, the sequence splits as a direct product, i.e.

$$\pi_1(\Omega_0(M, L)) \simeq \pi_1(\Omega_0(M)) \times \pi_1(L \times L).$$

Proof. Note that, in principle, $\pi_1(\Omega(M, L))$ splits as a semi-direct product. However, since the inclusion is null-homotopic, the action of $\pi_1(L \times L)$ on $\pi_1(\Omega(M))$ given by

$$\begin{aligned} (\pi_1(L \times L), \pi_1(\Omega_9(M))) &\to \pi_1(\Omega_0(M)) \\ (\alpha, \gamma) &\mapsto f^{-1}(s_*(\alpha) f(a) s_*(\alpha)^{-1}) \end{aligned}$$

is the identity. □

Illustrative examples of the inclusion $\iota : L \to M$ being null-homotopic is given by inclusions $\iota : S^m \to S^n$ with $m < n$. The case of the inclusion of S^1 into S^2 as an equator admits a good visualization of the previous proof. In this case we can take the null-homotopy as in Figure 6.3, collapsing the equator to a point by moving it up in the sphere. In this scenario, the section constructed in 6.7.1 is given by the concatenation of fragments of geodesics connecting the respective points to the north pole, as seen in Figure 6.4.

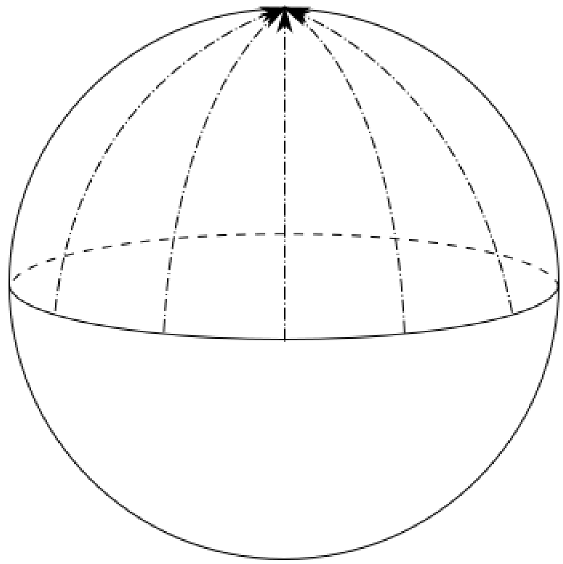


Figure 6.3: Null-homotopy of the inclusion $S^1 \hookrightarrow S^2$ as an equator.

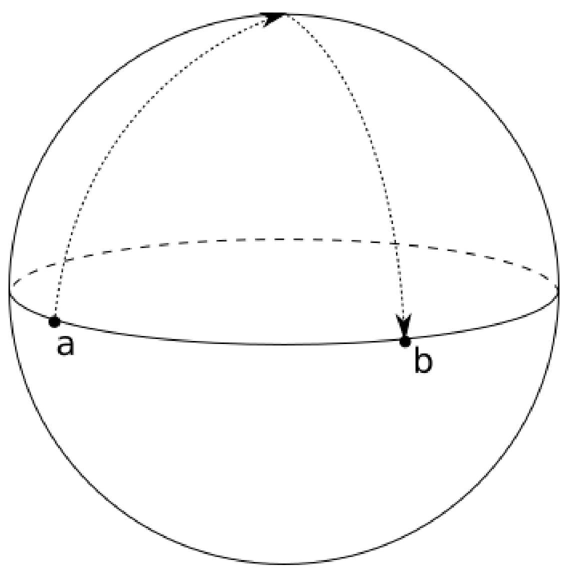


Figure 6.4: The section $s : L \times L \to \Omega(M, L)$.

Recall that if we have an exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

that splits as a direct product with a section $s : C \to B$, then B is the inner direct sum

$$B = \iota(A) \oplus s(C).$$

In the previous case, we have $\pi_1(L \times L)$ representing $s_*(\pi_1(L \times L))$. When we take the Hom of both sides, we get

$$\mathrm{Hom}(\pi_1(\Omega_0(M, L)); \mathbb{Z}_l) \simeq \mathrm{Hom}(\pi_1(\Omega_0(M)); \mathbb{Z}_l) \times \mathrm{Hom}(\pi_1(L \times L); \mathbb{Z}_l) \qquad (6.1)$$

The projection on the second factor then corresponds to the restriction of the local system associated to $\alpha \in \mathrm{Hom}(\pi_1(\Omega_0(M,L);\mathbb{Z}_l))$ to the space $L \times L$. More precisely, it is the local system given by the pull-back via s. If we consider the diagonal map $d : L \to L \times L$, then the pull-back via $(d \circ s)$ provides us with the restriction of the local system to L, as seen in $\Omega(M,L)$ as constant paths. If the pull-back via s is trivial, then the pull-back via $(d \circ s)$ also is due to functoriality. In particular, it follows that a non-trivial element $\eta \in \mathrm{Hom}(\pi_1(\Omega_0(M,L));\mathbb{Z}_l)$ which is, under isomorphism 6.1, trivial in the second coordinate provides us with a non-trivial local system on $\Omega_0(M,L)$ which restricts to a trivial one in L.

We also have the following chain of isomorphisms.

$$\mathrm{Hom}(\pi_1(\Omega_0(M));\mathbb{Z}_l) \simeq \mathrm{Hom}(\pi_2(M));\mathbb{Z}_l) \simeq \mathrm{Hom}(\pi_2(\widetilde{M});\mathbb{Z}_l) \simeq H^2(\widetilde{M};\mathbb{Z}_l).$$

The first one is due to the fact that $\pi_1(\Omega_0(M)) \simeq \pi_2(M)$, as can be seen from the long exact sequence given by the fibration $(M^{[0,1]}, x_0) \to M$ given by evaluation at 1, where $(M^{[0,1]}, x_0)$ stands for paths in M starting at x_0. Since this space is contractible the result follows.

The second one can be seen by looking at the cover $\widetilde{M} \to M$ as a fibration. Since the fibers are discrete, their fundamental groups for $n \geq 1$ vanish and we get that $\pi_i(M) \simeq \pi_i(\widetilde{M})$ for $i \geq 2$.

The last isomorphism is a result of the universal coefficients theorem together with the fact that $H_1(\widetilde{M};\mathbb{Z}_l) = 0$ due to Hurewicz's theorem and the fact that the universal cover is simply connected.

It follows from this discussion that non-trivial local systems on $\Omega(M,L)$ which are pulled-back via s to trivial local systems on $L \times L$ (and thus restrict to trivial local systems on L) are in bijection with elements of $\mathrm{Hom}(\pi_1(\Omega_0(M));\mathbb{Z}_l) \simeq H^2(\widetilde{M};\mathbb{Z}_l)$.

We will need some results mentioned in [5]. The following is a variant of the Leray-Serre spectral sequence.

Proposition 6.7.3. Let $F \xrightarrow{i} E \xrightarrow{j} B$ be a fibration over a CW-complex and let $\mathcal{T}$ be a local system on E. Then there is a spectral sequence $(E^r, d^r), r \geq 2$, converging to $H_*(E;\mathcal{T})$ which has as second page

$$E^2_{p,q} \simeq H_p(B;\mathcal{H}_q(F;i^*\mathcal{T})), \quad p,q \geq 0,$$

where $\mathcal{H}_q(F;i^*\mathcal{T})$ stands for the local system on B given by the q-th homology (with local system given by $\mathcal{T}$ of the fiber F).

This next proposition can also be seen in [5].

Proposition 6.7.4. Let M be a path-connected space fixed basepoint. Let R be a ring and $G \subset R^*$ be a subgroup such that

$$1 - g \in R^*$$

for any $g \in G$ with $g \neq 1$. Then any non-trivial G-local system $\mathcal{T}$ of rank one R-modules on $\Omega_0 M$ has the property that $H_*(\Omega_0 M;\mathcal{T}) = 0$.

With Propositions 6.7.3 and 6.7.4, we have the following.

Theorem 6.7.5. Let M be a manifold with submanifold L such that the inclusion $\iota : L \to M$ is null-homotopic and $H^2(\widetilde{M}; \mathbb{Z}_l) \neq 0$. Then for any rank-one local system $\mathcal{C}$ on $L \times L$ there exists a local system $\mathcal{T}$ on $\Omega(M, L)$ which restricts via s to a local system isomorphic to $\mathcal{C}$ on $L \times L$ and such that

$$H_*(\Omega(M, L); \mathcal{T}) = 0.$$

Proof. By the discussion following Corollary 6.7.2, we have the existence of a non-trivial $\mathbb{Z}_l$-local system $\mathcal{S}$ on $\Omega_0(M, L)$ which restricts to the trivial local system on $L \times L$ by s, and thus to the trivial local system on L. By Proposition 6.7.4, it follows that $H_*(\Omega_0 M; i^*\mathcal{S}) = 0$.

Consider now again the fibration given by the evaluation $\mathrm{ev} : \Omega(M, L) \to L \times L$, and take $\mathcal{T} := \mathcal{S} \otimes \mathrm{ev}^*\mathcal{C}$. We have that this local system restricted to the fiber $\Omega_0(M)$ is isomorphic to $i^*\mathcal{S}$, and thus by Proposition 6.7.3 it follows that $H_*(\Omega(M, L); \mathcal{T}) = 0$. Furthermore, $s^*\mathcal{T} = \mathcal{C}$, showing that $\mathcal{T}$ indeed satisfies the conclusion of the theorem.

□

With Theorem 6.7.5 in hand, we can now find a local system which restricts to the trivial on M but is so that the symplectic homology with such local system vanishes. In fact, we get a slightly more general result:

Theorem 6.7.6. Let M be a manifold with submanifold L such that the inclusion $\iota : L \to M$ is null-homotopic and $H^2(\widetilde{M}; \mathbb{Z}_2) \neq 0$. Then for any rank-one local system $\mathcal{C}$ on $L \times L$ there exists a local system $\mathcal{L}$ on $\Omega(T^*M, \mathcal{N}^*L)$ which restricts by s to $\mathcal{C}$ and such that $\mathrm{WFH}(T^*M, \mathcal{N}^*L; \mathcal{L}) = 0$.

Proof. By Theorem 6.7.5, there is a rank-one local system $\mathcal{T}$ on $\Omega(M, L)$ which restricts to $\mathcal{C}$ for which $H_\cdot(\Omega_0(M, L); \mathcal{T}) = 0$. The projection $\pi : \Omega_0(T^*M, \mathcal{N}^*L) \to \Omega_0(M, L)$ gives rise to the extension (i.e., pull-back) of the local system $\mathcal{T}$ to $\pi^*\mathcal{T}$.

We know that

$$\mathrm{WFH}_*(T^*M; \pi^*\mathcal{T}) \simeq H_*(\Omega_0(M, L); \mathcal{T})$$

due to [1], and hence we have the result by taking $\mathcal{L} := \pi^*\mathcal{T}$.

□

We also state the case of $\mathbb{Z}_l$ for other primes l as a conjecture. The reason for this is that we did not give much attention to the issue of orientation when defining the homology since most of the thesis does not need to do so, but this is now relevant since the isomorphism between the homology of the path space and the symplectic homology of the cotangent bundle is more subtle when orientations are involved. We elaborate on the Appendix, but to summarize: the isomorphism between those homologies comes from the fact that we can define a Morse complex on the relevant path space of the manifold in such a way that it corresponds to the symplectic-homology complex. But the boundary maps in those two theories are not coherent. More precisely, the natural isomorphism between those complexes is not a chain map. The way to fix this is by introducing a suitable local system of coefficients. Since $\mathbb{Z}_2$-coefficients are blind to this orientation problem, this issue does not show up in the first part of the proof. This "fix" is known to work for the periodic case, but in our case one would need to check whether the local system can be implemented successfully and this might require some care.

Conjecture 6.7.7. Let M be a manifold with submanifold L such that the inclusion $\iota : L \to M$ is null-homotopic and $H^2(\widetilde{M}; \mathbb{Z}_l) \neq 0$. Then for any rank-one local system $\mathcal{C}$ on $L \times L$ there exists a local system $\mathcal{L}$ on $\Omega(T^*M, \mathcal{N}^*L)$ which restricts by s to $\mathcal{C}$ and such that $\mathrm{WFH}(T^*M, \mathcal{N}^*L; \mathcal{L}) = 0$.

"Proof". Due to Theorem 6.7.5, we can pick a rank-one local system $\mathcal{T}$ on $\Omega(M, L)$ which restricts to $\mathcal{C} \otimes o_{L\times L}$, where $o_{L\times L}$ is the orientation local system of $L \times L$ (i.e. the $\mathbb{Z}_2$-local system that assigns to each loop either ± 1 whether it reverses orientation or not) for which $H_\cdot(\Omega_0(M, L); \mathcal{T}) = 0$.

If the result mentioned in the appendix is valid for the endpoint case, we know that there is a local system η on the $\Omega(T^*M, \mathcal{N}^*L)$ which restricts to $o_{L\times L}$ and such that

$$\mathrm{WFH}_*(T^*M; \pi^*\mathcal{T} \otimes \eta) \simeq H_*(\Omega(M, L); \mathcal{T}).$$

Since $\pi^*\mathcal{T} \otimes \eta$ restricts to $(\mathcal{C} \otimes o_{L\times L}) \otimes o_{L\times L} \simeq \mathcal{C} \otimes (o_{L\times L} \otimes o_{L\times L}) \simeq \mathcal{C}$, we have our result. □

As expected, we can now apply Theorem 6.7.6 to get a new range of examples where the symmetric capacity is finite. Explicitly, we have the following theorem.

Theorem 6.7.8. Let M be a manifold with an involution $\varphi : M \to M$ with non-empty fixed point set L such that the inclusion $\iota : L \to M$ is null-homotopic and $H^2(\widetilde{M}; \mathbb{Z}_2) \neq 0$. Then $c_{sym}(D^*TM, \mathcal{N}^*L, \widetilde{\varphi}) < +\infty$.

Proof. The theorem follows directly by using Theorem 6.7.6 with $\mathcal{C}$ being the trivial $\mathbb{Z}$ local system together with Proposition 6.4.2. □

Chapter 7

Appendix - Symplectic homology of the cotangent bundle

7.1 Introduction

This appendix is intended mainly to explain the isomorphism between the wrapped Floer homology of the pair $(T^*M, \mathcal{N}^*L)$, as in Chapter 6 and the (singular) homology of the path-space $\Omega(M, L)$. The main idea is to make an equivalence between the wrapped Floer homology and a Morse homology on such path-space. It turns out, however, that under this equivalence, the orientations that the Floer homology and the Morse homology dictate for their boundary maps disagree. In order to fix for this issue, we must consider a suitable local system of coefficients that corrects for this defect.

Of course, if we are taking $\mathbb{Z}_2$ coefficients, this point is moot as explained before. However, in order to phrase Theorem 6.7.8 in more generality, we must consider the case of other coefficients, and thus this issue must be addressed.

Although the relationship between the Floer homology of the cotangent bundles and the homology of the free loop space of the base manifold was established by Abbondandolo and Schwarz in [2] (for the endpoint case, c.f. [1]), this mismatch of the orientations was not accounted for, and it ended up being solved relatively recently by Abouzaid (c.f. [4]).

We will present the main ideas behind the isomorphism and show where the correction must take place, but we will not enter in much detail. In particular, we will not show how to solve the issue. For a detailed exposition, please see the above references. In order to be coherent with the aforementioned literature, we will make the exposition of the periodic case. The endpoint case is treated in [1], although with a different point of view. There, the boundary conditions are codified as the conormal bundle of a submanifold $Q \subset M \times M$. This contains the case of $(T^*M, \mathcal{N}^*L)$ in particular, by taking $Q := L \times L$.

We will follow the notation and exposition of [2].

7.2 Morse-theoretic approach

Morse theory has been known to work for some particular action functionals in infinite dimension for quite some time. Perhaps the most simple example is the energy functional, capturing closed geodesics. (c.f. [14].) However, such functionals must be well-behaved in order for the theory to be comfortable and resemble the finite-dimensional case. Explicitly, they must satisfy a condition known in the literature as “Condition (C) of Palais-Smale”, which essentially emulates compactness for sequences which have bounded energy and have their gradients converging to zero in the $W^{1,2}$-sense. The action functional we have seen for Floer homology is not well-behaved (for instance, it is not bounded by below), and thus Morse homology does not go through, which is essentially the entire reason behind the shift towards the Floer-theoretical approach.

However, in the case of the cotangent bundle, we can choose the Hamiltonian involved in the definition of $SH(T^*M, H)$ to be the Legendre transform of a Lagrangian $\mathcal{L}$ which is strongly convex in the fibers of the cotangent bundle, i.e:

$$H(t, q, p) = \sup_{v \in T_q M} (p(v) - \mathcal{L}(t, q, v)).$$

The assumption of convexity is related to the transform being well-defined. As usual in Lagrangian mechanics, we then have the correspondence of the periodic orbits of the Hamiltonian flow with the critical points of the functional

$$\mathcal{E}(c) := \int_0^1 L(t, q(t), \dot{q}(t))dt.$$

This functional is more well-behaved than the functional $\mathcal{A}_H$ and can be shown to be suitable for Morse theory. Moreover, it can be shown that the Morse index of an orbit coincides with the Conley-Zehnder index, and thus we have readily a graded bijection between the generators of the Morse complex and the generators of the Floer complex. This bijection, as expected, does not necessarily go down to homology as a chain isomorphism. In order to do so, we need essentially to account for the fact that in one hand we have the negative gradient flow of the action functional related to $\mathcal{L}$, and on the other we have Floer solutions related to H. The way this is done is by considering a moduli space which accounts for this change.

Explicitly, we consider the space $\Gamma_{x,y}$ of maps $\gamma : (-\infty, 0) \to \Lambda(M)$ which converge as $s \to -\infty$ to some $q \in C_*(L)$ and, for $s = 0$, lift to a loop in T^*M which is $u(0, \cdot)$ of a solution $u(s, t)$ of the Floer equation associated to H which converges to some $y \in C_*(H)$, where we are denoting the Morse chain complex by $C_*(L)$ and the Floer chain complex by $C_*(H)$. For a generic choice of the almost-complex structure, the space of such maps is a compact smooth manifold of dimension $m(q) - \mu_{CZ}(y)$, and thus when the indexes coincide we have a finite number of points. We define the chain isomorphism to be

$$\Theta(q_k) = \sum_{y_k \in C_*(H)} \sum_{\gamma \in \Gamma_{q_k, y_k}} \epsilon(\gamma, q_k, y_k) y_k,$$

where the subscript indicates the index and ϵ is dependent on the orientation conventions

In case we have $\mathbb{Z}_2$ coefficients, this map would be just counting the number of elements in Γ_{x_k,y_k}. Explicitly,

$$\Theta(q_k) = \sum_{y_k \in C_*(H)} (\#\Gamma_{q_k,y_k}) y_k.$$

It is possible to prove the following estimate relating the base loop to its lifts:

$$\mathcal{A}_H(x) \leq \mathcal{E}(q), \tag{7.1}$$

with equality holding only if x and q are related via the Legendre transform, where $x(t) = (q(t), p(t))$ represents a loop in T^*M. By using 7.1, which is of theoretical interest, we can prove that this map is an isomorphism. Indeed, this estimate establishes that the moduli space $\Gamma_{q,y}$ is empty if $\mathcal{E}(q) \leq \mathcal{A}(x)$, unless q and x correspond to the same orbit, and in such case $\Gamma_{q,y}$ consists of only one point: the stationary solution. Therefore, $\Theta(q) = \pm q$ in such cases. Thus, by taking ordered basis with increasing action, we see that Θ is lower triangular and has ± 1 entries in the diagonals. It follows that Θ is an isomorphism.

The orientation problem enters precisely at the next step: we would wish for Θ to be a chain map, so that it would be a chain isomorphism and then would descend to homology. But this is not necessarily the case. This is directly related to how we define the boundary maps in each homology, since being a chain map is, explicitly, satisfying

$$\partial_1 \Theta = \Theta \partial_2.$$

And, of course, those boundary maps are intrinsically related to how we orient each moduli space in each homology theory. There are different ways to solve this. As expected, we can "manually" change the definition of either boundary operator in order to account for the defect. Or we can introduce a suitable local system in one of the homologies. Of course, all of them are equivalent, their difference being a just a matter of personal taste. In the corrigendum of [2] (c.f. [3]), they adopt the point of view of changing ∂_2. In [4], it is done by introducing a local system of coefficients on the loop space for the Morse homology. The advantage of this later point of view is that it comes naturally equipped to deal with other local systems as well. More precisely, Remark 4.1.2 of [4] states that not only there is an isomorphism

$$\mathcal{V} : SH^*(T^*M; \mathbb{Z}) \to H_{-*}(\Lambda(M); \eta),$$

where η is a suitable local system of coefficients, but the proof also extends directly to show that there is also an isomorphism

$$\mathcal{V} : SH^*(T^*M; \nu) \to H_{\pm *}(\Lambda(M); \eta \otimes \nu),$$

for any local system ν on ΛM, where $\pm *$ indicates a possible grading reversal.

The endpoint version of this last isomorphism is precisely what is needed for the argument of Theorem 6.7.8 and is known to be true with $\mathbb{Z}_2$ coefficients by [1]. The case of $\mathbb{Z}_l$ coefficients would need to be checked for possible subtleties in the adaptation of the orientations for the endpoint case.

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