Universidade Federal do Rio de Janeiro
Instituto de Matemática

Controle e estabilização dos sistemas Gear-Grimshaw e Boussinesq de ordem superior

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Tese
Orientador: Prof. Ademir Fernando Pazoto

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Tese de Doutorado apresentada ao Programa de Pós-graduação do Instituto de Matemática, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários para a obtenção do título de Doutor em Ciências.

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Resumo

Neste trabalho, consideramos dois sistemas dispersivos; inicialmente, consideramos um sistema derivado por Gear e Grimshaw para descrever a interação forte entre ondas longas fracamente não lineares. O modelo tem a estrutura de um par de equações de Korteweg-de Vries acopladas por efeitos dispersivos e não lineares. Nosso objetivo com este sistema é investigar suas propriedades de controlabilidade, num intervalo limitado, por meio de controles distribuídos. Quando a região de controle é uma vizinhança do ponto final direito do intervalo, provamos a controlabilidade exata local do problema não linear em uma classe de espaços L^2 com peso. Inicialmente, os resultados são estabelecidos para o sistema linearizado por meio de uma abordagem clássica de dualidade e depois são estendidos para o sistema completo por meio de um argumento de ponto fixo.

Em seguida, consideramos um sistema de Boussinesq que acopla duas equações do tipo Benjamin-Bona-Mahony linares de ordem superior. Inicialmente, investigamos as propriedades de controlabilidade do modelo linearizado em um intervalo limitado. Mais precisamente, por meio de controles que atuam no ponto extremo direito do intervalo, mostramos que o modelo é aproximadamente controlável, mas não espectralmente controlável. Isso significa que qualquer estado pode ser conduzido arbitrariamente próximo a outro estado, mas nenhuma combinação linear finita de autofunções, diferente de zero, pode ser conduzida a zero. Nossas provas dependem fortemente de uma análise espectral cuidadosa do operador associado às equações estacionárias. Também propomos vários mecanismos dissipativos que conduzem a sistemas para os quais todas as trajetórias são atraídas pela origem sempre que a propriedade de continuação única de soluções fracas seja verificada.

Palavras chave: Controlabilidade, estabilização, sistema de Gear-Grimshaw, equação de Korteweg-de Vries, sistema Boussinesq de ordem superior , equação de Benjamin-Bona-Mahony, propriedade de continuação única.

Abstract

In this work, we consider two dispersive systems; initially, we consider a system derived by Gear and Grimshaw to describe the strong interaction of weakly nonlinear long waves. It has the structure of a pair of Korteweg-de Vries equations coupled through both dispersive and nonlinear effects. Our purpose for this system is to investigate its controllability properties, when posed on a bounded interval, by means of distributed controls. When the control region is a neighborhood of the right end point of the interval, we prove the local exact controllability of the nonlinear problem in some well chosen weighted L^2-spaces. The results are first established for the linearized system through a classical duality approach and then extended for the full system via a fixed point argument.

Next, a Boussinesq system which couple two linearized higher-order Benjamin-Bona-Mahony type equations is considered. We first investigate the boundary controllability properties of the linearized model posed on a bounded interval. More precisely, by means of controls acting on the right endpoint of the interval, we show that the model is approximately controllable but not spectrally controllable. Our proofs relies strongly on a careful spectral analysis of the operator associated with the state equations. We also propose several dissipation mechanisms leading to systems for which all the trajectories are attracted by the origin provided that the unique continuation of weak solutions holds.

Key words: Controllability, stabilization, Gear-Grimshaw system, Korteweg-de Vries equation, higher order Boussinesq system, Benjamin-Bona-Mahony equation, unique continuation property.

Contents

Chapter 1

Introduction

The study of nonlinear wave phenomena is of broad scientific interest and pertains to a modern line of research which is important both scientifically and for potential applications. Progress in the development of new tools for modern applied mathematics resulted in a better scientific understanding of nonlinear waves in various and quite distinct fields. The mathematical models arising in nonlinear dispersive media are among the illustrations of successful outcomes resulting from the efforts to understand various nonlinear phenomena. Starting in the latter half of the 1960s, the mathematical theory for nonlinear dispersive wave equations came to the fore as a major topic within nonlinear analysis. Since then, physicists and mathematicians were led to derive sets of equations to describe the dynamics of the waves in some specific physical regimes and much effort has been expended on various aspects of the initial and boundary value problems.

The purpose of this work is present some mathematical results concerning two families of such systems. The first one is a model derived by Gear and Grimshaw [21] to describe strong interactions of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion. It can be written as

$$\begin{cases} u_t + uu_x + u_{xxx} + a_3 v_{xxx} + a_1 v v_x + a_2 (uv)_x = 0, \\ b_1 v_t + r v_x + v v_x + v_{xxx} + b_2 a_3 u_{xxx} + b_2 a_2 u u_x + b_2 a_1 (uv)_x = 0. \end{cases} \tag{1.1}$$

The parameters $r, a_1, a_2, a_3, b_1, b_2$ are real constants with b_1, b_2 positive, the unknowns u and v are real valued functions of the variables x and t and subscripts indicate partial differentiation. We also refer to [11] for an extensive discussion on the physical relevance of the model.

System (1.1) has the structure of a pair of Korteweg-de Vries (KdV) equations coupled through both dispersive and nonlinear effects and has been the object of research in recent years. It also appears as a special case of a broad class of nonlinear evolution equations that can also be interpreted as a coupled nonlinear version of generalized KdV equations of the form

$$\begin{aligned} u_t + u_{xxx} + f(u,v)_x = 0, \\ v_t + v_{xxx} + g(u,v)_x = 0, \end{aligned}$$

with f and g satisfying $f(u,v) = H_u(u,v)$ and $g(u,v) = H_v(u,v)$ for a smooth function H.

The other system that shall be study here is a higher-order generalization of the classical Boussinesq system introduced and studied by J. J. Bona, M. Chen and J.-C. Saut in [9, 10], more precisely,

$$\begin{cases} \eta_t + w_x & + a w_{xxx} - b \eta_{txx} + a_1 w_{xxxxx} + b_1 \eta_{txxxx} \\ & = -(\eta w)_x + b(\eta w)_{xxx} - \left(a + b - \frac{1}{3}\right)(\eta w_{xx})_x, \\ w_t + \eta_x & + c \eta_{xxx} - d w_{txx} + c_1 \eta_{xxxxx} + d_1 w_{txxxx} \\ & = -w w_x - c(w w_x)_{xx} - (\eta \eta_{xx})_x + (c + d - 1) w_x w_{xx} + (c+d) w w_{xxx}. \end{cases} \tag{1.2}$$

Here, the dependent variables $\eta = \eta(x,t)$ and $w = w(x,t)$ are real-valued functions of the variables x and t and subscripts indicate partial differentiation. The parameters $a, b, c, d, a_1, c_1, b_1, d_1$ are required to fulfill the relations

$$
\begin{aligned}
a+b &= \frac{1}{2}(\theta^2 - \frac{1}{3}), \quad c+d = \frac{1}{2}(1-\theta^2),\\
a_1 - b_1 &= -\frac{1}{2}(\theta^2 - \frac{1}{3})b + \frac{5}{24}(\theta^2 - \frac{1}{5})^2,\\
c_1 - d_1 &= \frac{1}{2}(1-\theta^2)c + \frac{5}{24}(1-\theta^2)(\theta^2 - \frac{1}{5}),
\end{aligned} \tag{1.3}
$$

where $\theta \in [0,1]$. Conditions (1.3) come from the physics of the problem and we tacitly assume them to hold throughout the entire paper. Depending on the problem under study, additional restrictions on the sign of these parameters will be imposed later on.

The original system was derived by Boussinesq to describe the two-way propagation of small-amplitude, long wavelength, gravity waves on the surface of water in a canal, but these systems arise also when modeling the propagation of long-crested waves on large lakes or the ocean and in other contexts. The variable, x, is proportional to the distance in the direction of propagation while t is proportional to elapsed time. The quantity $\eta(t,x) + h_0$ corresponds to the total depth of the liquid at the point x and at time t, where h_0 is the undisturbed water depth. The variable $w(t,x)$ represents the horizontal velocity at the point $(x,y) = (x, \theta h_0)$, at time t, where y is the vertical coordinate, with $y = 0$ corresponding to the channel bottom or sea bed. Thus, w is the horizontal velocity field at the height θh_0, where θ is a fixed constant in the interval $[0,1]$.

Notice that, when the parameters given in (1.3) are such that $a = a_1 = c = c_1 = 0$, the resulting system couples two higher order Benjamin-Bona-Mahony (BBM) type equations. If $b = b_1 = d = d_1 = 0$, we have a coupled system of two higher order Korteweg-de Vries (KdV) type equations.

1.1 Problems and main results

1.1.1 Controllability of the Gear-Grimshaw system in a weighted L^2-space

As it was pointed out in the previous section, system (1.1) appears as a special case of a broad class of nonlinear evolution equations that can also be interpreted as a coupled nonlinear version of generalized KdV equations of the form (1.2). Such mathematical formulations have received considerable attention in the past, and a satisfactory theory pertaining to the pure initial-value problem is available in the literature. However, the practical use of the waves systems and its relatives does not always involves such mathematical formulation. Therefore, it is also of interest to study the mathematical properties of the KdV family on a finite spatial interval.

In this chapter we shall be concerned with the study of an initial boundary value problem associated to (1.1) when $x \in [0,L]$ and $t \in \mathbb{R}^+$. Our main purpose is to address two mathematical issues connected to (1.1); well-posedness and controllability in a weighted Hilbert space. With this purpose, we introduce a function $\rho \in C^\infty([0,L])$ with

$$
\rho(x) = \begin{cases} 0 & \text{if } 0 < x < L - \nu, \\ 1 & \text{if } L - \dfrac{\nu}{2} < x < L, \end{cases} \tag{1.4}
$$

for some $\nu \in (0,L)$. Then, the control system reads as

$$
\begin{cases}
u_t + uu_x + u_{xxx} + a_3 v_{xxx} + a_1 v v_x + a_2 (uv)_x = (\rho(x) h_1)_x & \text{in } (0,T)\times(0,L),\\
b_1 v_t + r v_x + v v_x + v_{xxx} + b_2 a_3 u_{xxx} + b_2 a_2 u u_x & \\
\qquad + b_2 a_1 (uv)_x = (\rho(x) h_2)_x & \text{in } (0,T)\times(0,L),
\end{cases} \tag{1.5}
$$

with boundary conditions

$$\begin{cases} u(0,t) = u(L,t) = u_x(L,t) = 0 & \text{on} \quad (0,T), \\ v(0,t) = v(L,t) = v_x(L,t) = 0 & \text{on} \quad (0,T), \end{cases} \tag{1.6}$$

and initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \text{on} \quad (0,L). \tag{1.7}$$

In (1.5)-(1.7), the external forcing terms h_1 and h_2 are considered as control inputs. Their choice were motivated by the results on the controllability properties for the single KdV equation obtained in [14], from which we borrow some ideas. Our purpose is to see whether one can force the solutions of the system to have certain desired properties by choosing appropriate control inputs acting on a neighborhood of $x = L$. More precisely, we are mainly concerned with the following problem which are fundamental in control theory:

Given $T > 0$, initial states (u_0, v_0) and terminal states (u_1, v_1) in a certain space, can one find appropriate control inputs h_1 and h_2 (actually, $(\rho(x)h_1)_x$ and $(\rho(x)h_2)_x$), so that the system (1.5)-(1.7) admits a solution (u, v) which satisfies $(u(\cdot,0), v(\cdot,0)) = (u_0, v_0)$ and $(u(\cdot,T), v(\cdot,T)) = (u_1, v_1)$?

If one can always find a control input to guide the system described by (1.5)-(1.7) from any given initial state to any given terminal state, then the system (1.5)-(1.7) is said to be **exactly controllable**.

Inspired by the results obtained for the single KdV equation [12, 19, 39], significant progress has been made for system (1.1) on its boundary controllability properties (see, for instance, [6] for a nice review of the contributions). By contrast, the study of the internal controllability is still at its early stage. Recently, in [6], the authors proved the local null controllability property for system (1.1), posed on a finite interval, by means of a control supported on an interior open subset of the domain and acting on one equation only. The proof consists mainly on proving the controllability of the linearized system, which is done by getting a Carleman estimate for the adjoint system. A local inversion Theorem is then applied to get the result for the nonlinear system. By using the same approach, similar results were obtained in [16] for the Hirota-Satsuma system. Another related work is [13], where the authors consider the problem of controlling pointwise, by means of a time dependent Dirac measure supported at a given point, the linear system associated with (1.1) on the unit circle. In this case, the results are obtained by means of spectral analysis and Fourier expansion of the solutions.

As pointed before, the problem we address here was motivated by the analysis developed in [14] for the KdV equation. Indeed, after studying a single equation, it is natural and physically motivating to consider coupled systems of such equations. So, assuming that the control region is a neighborhood of the right end point of the interval $(0, L)$, we prove an exact controllability result in a weighted L^2-space. This is done under additional assumptions on some coefficients of the system (1.5). More precisely, we assume that

$$b_1, b_2 > 0 \quad \text{and} \quad 0 < a_3^2 b_2 < 1.$$

According to [11, 42], the parameters b_1 and b_2 are automatically positive and r is a non-dimensional parameter that could be assumed very small.

In order to state our main result, we need some notation: For any measurable function $w : (0,L) \to (0,+\infty)$ (not necessarily in $L^1(0,L)$), throughout the chapter we consider the weighted space

$$[L^2_{w(x)dx}]^2 = \left\{ (u_1, u_2) \in [L^1_{loc}(0,L)]^2; \int_0^L (u_1^2(x) + u_2^2(x)) w(x) dx < \infty \right\},$$

which is a Hilbert space endowed with the inner product

$$\begin{aligned}(\vec{u},\vec{v})_{[L^2_{w(x)dx}]^2} &= ((u_1,u_2),(v_1,v_2))_{[L^2_{w(x)dx}]^2}\\ &= \frac{b_2}{b_1}\int_0^L u_1(x)v_1(x)w(x)dx + \int_0^L u_2(x)v_2(x)w(x)dx.\end{aligned}$$

With the notation above, we introduce

$$H = [L^2_{(L-x)^{-1}dx}]^2 \quad \text{and} \quad V = \left\{\vec{u} \in [H^1_0(0,L)]^2, \vec{u}_x \in [L^2_{(L-x)^{-2}dx}]^2\right\},$$

endowed with the norms

$$||\vec{u}||^2_H := \frac{b_2}{b_1}||(L-x)^{-\frac{1}{2}}u_1||^2_{L^2} + ||(L-x)^{-\frac{1}{2}}u_2||^2_{L^2}$$

and

$$||\vec{v}||^2_V := \frac{b_2}{b_1}||(L-x)^{-1}v_{1,x}||^2_{L^2} + ||(L-x)^{-1}v_{2,x}||^2_{L^2},$$

where $\vec{u} = (u_1,u_2)$ and $\vec{v} = (v_1,v_2)$. Remark that b_1 and b_2 are positive.

We are now in position to state the main result of this chapter. It concerns the local exact controllability of (1.5)-(1.7) in the space H and can be summarized as follows:

Theorem 1.1.1. *Let $T > 0$. Then, there exists $\delta > 0$, such that, for any $(u_0,v_0),(u_1,v_1) \in H$ satisfying*

$$||(u_0,v_0)||_H \le \delta \quad \textit{and} \quad ||(u_1,v_1)||_H \le \delta,$$

one can find a control function $\vec{h} := (h_1,h_2) \in L^2(0,T;[L^2(0,L)]^2)$, such that the solution $(u,v) \in C([0,T];H) \cap L^2(0,T;V)$ of (1.5)-(1.7) satisfies $(u(\cdot,T),v(\cdot,T)) = (u_1,v_1)$ in $(0,L)$. Moreover, the forcing term $\vec{f} = (\rho(x)h_1,\rho(x)h_2))$ is a function in $L^2_{(T-t)dt}(0,T;[L^2(0,L)]^2)$ supported in $(0,T)\times(L-\nu,L)$, where $0<\nu<L$.

It will be also demonstrated that (1.5)-(1.7) is well-posed in $C([0,T];H) \cap L^2(0,T;V)$ when the initial data and the forcing terms h_1 and h_2 are small enough. As with other, dispersive wave equations, well-posedness seems to depend on the conservation laws, or energy-type inequalities satisfied by the solutions, and on the linear theory. Therefore, we first show that the corresponding linear problem generates a semigroup of continuous operator in H and $[L^2_{xdx}]^2$. This is done introducing an abstract framework, successfully applied in [14, 22], which combines Hille-Yosida theory and a generalized Lax-Milgram Theorem due to J.-L. Lions (see, for instance,[23, 28]). In particular, we also establish the so-called Kato smoothing effect, i.e., the solutions whose initial datum lies in H not only lie in $C([0,T];H)$ but also in $C([0,T];H) \cap L^2(0,T;V)$. This property made it possible to combine the Duhamel formula and a contraction mapping argument to prove directly the local well-posedness result. As it will become clear in our proofs, assumption $0 < a_3^2 b_2 < 1$ allows to obtain a priori estimates leading to the local global well-posedness results. Those a priori estimate are also useful to establish the well-posedness of the adjoint system in the dual space $[L^2_{(L-x)dx}]^2$, which is crucial to derive the controllability properties.

With the well-posedness established we investigate the controllability properties of (1.5)-(1.7). We combine the analysis of the linearized system and a contraction mapping argument for the full system. In order to analyze the linearized system, we follow a duality approach [20, 29], which reduces the exact controllability property to prove an *observability inequality* for the solutions of the corresponding adjoint system. Here, this is done combining multipliers and the so-called compactness-uniqueness argument, which leads one to apply a unique continuation result. Since we are dealing with a linear system, this has been shown to be true by means of Holmgren's uniqueness theorem. At that point we remark that some of the multipliers mentioned above were introduced in [39] to study the KdV equation and applied later in [34] in the context of the system (1.5)-(1.7). This justify in part the choice of the weight spaces and the assumption on the support of the function ρ.

The analysis described above is organized in three sections: In Section 2.1 we establish the well-posedness of the linear system and Section 2.2 is devoted to study its controllability properties. In both sections we split the results into several steps in order to make the reading easier. In Section 2.3, we prove the local well-posedness and the controllability of the full nonlinear system.

1.1.2 On the lack of controllability of a higher-order regularized long-wave system

Despite the success in studying dispersive models, the mathematical theory have been concerned with either the pure initial value problem posed on the entire real line or the periodic-initial value problem posed on the one-dimensional torus. A large body of literature has been concerned with the questions of existence, uniqueness and continuous dependence of solutions corresponding to initial data. The study of initial-boundary value problems with nonhomogeneous boundary conditions has not progressed to the same extent.

In this chapter, the goal is to advance the study of the initial-boundary value problems exploring the dynamics of dispersive equations using mathematical analysis from the controllability point of view. Consideration is given to an initial-boundary value problem associated to the linearized Boussinesq system (1.2) when the parameters given in (1.3) are such that $a_1 = c_1 = 0$. Our attention, in particular, is given to the following distributed control system:

$$\begin{cases} \eta_t + \omega_x + a\omega_{xxx} - b\eta_{txx} + b_1\eta_{txxxx} = 0 & \text{for } x \in (0,L),\ t>0, \\ \omega_t + \eta_x + c\eta_{xxx} - d\omega_{txx} + d_1\omega_{txxxx} = 0 & \text{for } x \in (0,L), t>0, \\ \eta(t,0) = 0, \quad \eta(t,L) = f_1(t) & \text{for } \ t \geq 0, \\ \omega(t,0) = 0, \quad \omega(t,L) = g_1(t) & \text{for } \ t \geq 0, \\ \eta_x(t,0) = 0, \quad \eta_x(t,L) = f_2(t) & \text{for } \ t \geq 0, \\ \omega_x(t,0) = 0, \quad \omega_x(t,L) = g_2(t) & \text{for } \ t \geq 0, \\ \eta(0,x) = \eta^0(x); \quad \omega(0,x) = \omega^0(x) & \text{for } x \in (0,L). \end{cases} \tag{1.8}$$

In (1.8), the external forcing terms f_i and g_i, $i = 1, 2$, are considered as control inputs. The purpose is to see whether one can force the solutions of the system to have certain desired properties by choosing appropriate control inputs acting at one end of the interval. More precisely, we are mainly concerned with the following problems which are fundamental in control theory:

Exact controllability: *Given $T > 0$, initial conditions (η^0, ω^0) and terminal states (η^1, ω^1) in a certain space H, there exist control inputs (f_1, g_1) and (f_2, g_2) so that the system (1.8) admits a solution (η, ω) which satisfies*

$$(\eta(0,\cdot), \omega(0,\cdot)) = (\eta^0, \omega^0) \text{ and } (\eta(T,\cdot), \omega(T,\cdot)) = (\eta^1, \omega^1).$$

If any given initial condition can be drive to $(0,0)$, the system is said to be ***null controllable****.*

Approximate controllability: *Given $T > 0$, $\epsilon > 0$, initial conditions (η^0, ω^0) and terminal states (η^1, ω^1) in a certain space H, there exist control inputs (f_1, g_1) and (f_2, g_2) so that the system (1.8) have a solution (η, ω) which satisfies*

$$||(\eta(T,\cdot), \omega(T,\cdot)) - (\eta^1, \omega^1)||_H < \epsilon.$$

Spectral controllability: *Any finite linear nontrivial combination of eigenvectors of the operator associated with the state equations can be driven to zero in finite time by using control inputs (f_1, g_1) and (f_2, g_2).*

Observe that exact controllability is essentially stronger notion than approximate controllability. In other words, exact controllability always implies approximate controllability. The converse statement is generally false.

Throughout this chapter, we assume that $b, d, b_1, d_1 > 0$ and consider the space

$$[H_0^2(0,L)]^2 = \left\{ (\varphi, \psi) \in [H^2(0,L)]^2 \,\middle|\, \frac{\partial^r \varphi}{\partial x^r}(0) = \frac{\partial^r \varphi}{\partial x^r}(L) = \frac{\partial^r \psi}{\partial x^r}(0) = \frac{\partial^r \psi}{\partial x^r}(L) = 0, \quad r = 0, 1 \right\}$$

endowed with the inner product

$$\begin{aligned} \left\langle \begin{pmatrix} \eta \\ \omega \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle = \int_0^L (\eta\varphi + \omega\psi)dx + & \int_0^L (b\eta_x\varphi_x + d\omega_x\psi_x)dx \\ & + \int_0^L (b_1\eta_{xx}\varphi_{xx} + d_1\omega_{xx}\psi_{xx})dx. \end{aligned} \tag{1.9}$$

The space $[H^{-2}(0,L)]^2$ is defined as the dual space of $[H_0^2(0,L)]^2$.

In what concerns system (1.8), our results can be summarized as follows:

- The approximate controllability holds for any $T > 0$. In more details, we prove that there exist control inputs $f_i, g_i \in H^1(0,T)$, $i = 1, 2$, such that the set of reachable states is dense in $[L^2(0,L)]^2$, for any $(\eta^0, \omega^0) \in [H^{-2}(0,L)]^2$ and $T > 0$.

  On the other hand, we give a negative result for the first problem introduced above.

- System (1.8) is not spectrally controllable if $(\eta^0, \omega^0) \in [H_0^2(0,L)]^2$.

Remark 1.1.2. *The following remarks are in order.*

(*i*) *When $(\eta^0, \omega^0) \in [H^{-2}(0,L)]^2$, the solution of (1.8) has to be understood in a weak sense. For instance, it can be defined by transposition. With this approach, we have to impose that $f_i, g_i \in H^1(0,T)$, $i = 1, 2$ in order to obtain a well-posedness result.*

(*ii*) *Throughout the work, it will become clear that the lack of exact controllability of the model comes from the existence of a limit point in the spectrum of the operator associated with the state equations, a phenomenon already noticed in [33] for the single BBM equation.*

By means of a series expansion of the solution in terms of the eigenvectors of the state operator, the approximate controllability is reduced to a unique continuation problem of the eigenvectors. In what concerns the lack of exact controllability, it is addressed through a spectral problem which is solved combining Paley-Wiener theorem and the asymptotic behavior of the eigenvalues. Such approach requires a careful spectral analysis of the operator associated to the state equation. Indeed, it provides important developments to justify the use of eigenvector expansions for the solutions, as well as, the asymptotic behavior of the eigenvalues. However, due to the the structure of the system, the eigenvalues can not be computed explicitly. To overcome this difficulty we prove that they are *close* to the eigenvalues of a well chosen differential operator. This is done by using less common two dimensional versions of the shooting method and Rouché's Theorem. Our approach was inspired by the techniques presented in [4] and [36]. In [36], the same strategy was successfully used to study the stabilization of a linear Boussinesq system of BBM-BBM type ($a = a_1 = c = c_1 = b_1 = d_1 = 0$) when a localized damping term acts on one equation only. By considering homogeneous Dirichlet boundary conditions, the authors prove that the energy associated to the model converges to zero as time goes to infinity. In the conservative case, i. e., in the absence of the damping term, the results obtained in [36] were properly adapted in [4] to study the controllability problems we address here. This approach does not apply directly in our case, since we are dealing with a higher order Boussinesq system. Therefore, further developments are required.

Before closing this section we emphasize that the problems we address here remain open for the corresponding nonlinear models, including for the single BBM equation. To our knowledge, the only result on the subject was obtained in [41] for the BBM equation on the torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. The authors show that, when an internal control acting on a moving interval is applied in the BBM equation, it is locally exactly controllable in $H^s(\mathbb{T})$, for any $s > 0$, and globally exactly controllable in $H^s(\mathbb{T})$, for any $s > 1$, in a sufficiently large time depending on the H^s-norms of the initial and terminal states. When (1.2) is posed on a periodic domain, the controllability problem has been addressed in [2]. General conditions are given to ensure both the well-posedness and the local exact controllability of the nonlinear problem by means of a control localized in the interior of the domain and acting on one equation only.

This chapter is organized as follows: In Section 3.1, we show that system (1.8) is globally well-posed. Additionally, the adjoint system associated to the homogeneous system is also presented. Section 3.2 is intended to show the controllability results. Finally, in Section 3.3, we develop the spectral analysis of the operator associated with the state equations which is used in our proofs. We choose this presentation in order to make the reading easier.

1.1.3 Asymptotic behavior of a linear higher-order Boussinesq system with damping

The study of fluid dynamics often leads to the study of equations that describe several physical situations, as the motion of the water waves under some physical regimes. Such equations can have a conservative nature, therefore, at least in that case, it is important to investigate the stability properties of the solutions by adding some dissipative effects.

It is well known that a good model to describe the physical phenomena concerning the unidirectional small amplitude long waves in nonlinear dispersive media is the Korteweg-de Vries (KdV) equation [27]

$$u_t + u_x + u_{xxx} + uu_x = 0.$$

As a rational alternative to the (KdV) equation is the so-called regularized long wave equation or Benjamin-Bona-Mahony (BBM) equation [7, 38]

$$u_t + u_x - u_{txx} + uu_x = 0.$$

On the other hand, in order to obtain a better agreement between models for the two-way propagation of waves and surface-wave experiments in a laboratory setting, field studies of wave-generated sediment transport, J. J. Bona, M. Chen and J.-C. Saut [9, 10] derived, from the classical Euler equations under a specific physical regime, the higher-order system (1.2) and the lower-order system

$$\begin{cases} \eta_t + w_x + aw_{xxx} - b\eta_{txx} = -(\eta w)_x, \\ w_t + \eta_x + c\eta_{xxx} - dw_{txx} = -ww_x. \end{cases} \tag{1.10}$$

As pointed out before, the dependent variables $\eta = \eta(x,t)$ and $w = w(x,t)$ are real-valued functions of the variables x and t and subscripts indicate partial differentiation. The parameters a, b, c and d obey the consistency conditions

$$a + b = \frac{1}{2}(\theta^2 - \frac{1}{3}), \quad c + d = \frac{1}{2}(\theta^2 - \frac{1}{2}) \geq 0, \quad \text{where } \theta \in [0,1].$$

Since it is important both scientifically and for potential applications, system (1.10) has been attracted the interests of mathematicians in recent years. Particularly, in what concerns the study of control and stabilization properties, it is important to consider the stabilization problem when dissipative effects are generated by internal and boundary damping. This issue becomes easier provided that the models have a strong intrinsic dissipative nature. Nevertheless, since the systems (1.2) and (1.10) are meant to replace the Euler equations, it is expected the preservation of energy, which suggests to include appropriated damping mechanisms for the decay of solutions.

The study of the stabilization and controllability properties for Boussinesq systems was initiated in [32] considering model (1.10). The work [32] deals with the internal controllability and stabilization of (1.10) on the torus and, when $b, d > 0$ and $a, c < 0$, the local exact controllability of the nonlinear system is shown to hold. As an application of the established exact controllability results, some simple feedback controls are constructed for particular choices of the parameters a, b, c and d, such that the resulting closed-loop systems are exponentially stable. Later on, in [37], the authors investigated the boundary stabilization of the Boussinesq system (1.10) of KdV-KdV type ($b = d = 0$) posed on a bounded interval. More precisely, they design a two-parameter family of feedback laws for which the solutions issuing from small data are globally defined and exponentially decreasing in the energy space. More recently, in [15], the exact boundary controllability of the Boussinesq system (1.10) of KdV-KdV type was studied. It was discovered that whether the associated linear system is exactly controllable or not depends on the length of the spatial domain. The extension of the exact controllability for the Boussinesq system (1.10) is derived in the energy space in the case of a control of Neumann type. It is obtained by incorporating a boundary feedback in the control in order to ensure the so-called Kato smoothing effect. In addition, proceeding as in [37], a local exponential stability result was also derived.

In the absence of the nonlinear terms and letting $a = c = 0$ and $b, d > 0$, the stabilization problem for the resulting linearized system of BBM-type was studied in [35] (see also [5, 18]). The authors consider the periodic case and, by introducing generalized damping operators in each equation, it is proved that whether the solutions of the system decay uniformly or not to zero depend on the parameters of the damping operators. When the model is posed on an bounded interval, the stabilization problem was studies in [36]. By means of a localized damping term acting in one equation and Dirichlet boundary conditions, it was proved that the energy associated to the model converges to zero as time tends to infinity.

Under the assumption of unique continuation property (UCP Conjecture) on the conservative scalar BBM equation, Rosier [40] answers affirmatively the issue. In fact, all the trajectories are attracted by the origin provided that some feedback laws (internal and boundary damping) are incorporated in the BBM equation. We remark that the unique continuation property for the BBM equation is still an open problem. Inspired by the ideas developed by Rosier, in [3] the authors propose several dissipation mechanisms for the linear system associated to (1.10) with the parameters a, b, c, d satisfying $a = c = 0$ and $b, d > 0$. They proved that the origin is asymptotically stable for the corresponding damped linear BBM-BBM model.

Motivated by the works mentioned above, in this chapter we investigate the stabilization properties for system (1.2) dropping the nonlinear terms and assuming that the parameters in (1.3) are such that $a_1 = c_1 = 0$ and $b, b_1, d, d_1 > 0$, $a, c < 0$ or $a = c \geq 0$. The resulting system couples two linear higher-order Benjamin-Bona-Mahony type equations, the so-called higher-order regularized long-wave system or higher-order BBM-system.

We first address the boundary stabilization problem. More precisely, we consider the system

$$\begin{cases} \eta_t + \omega_x + a\omega_{xxx} - b\eta_{txx} + b_1\eta_{txxxx} = 0 & \text{for } x \in (0, L),\ t > 0, \\ \omega_t + \eta_x + a\eta_{xxx} - d\omega_{txx} + d_1\omega_{txxxx} = 0 & \text{for } x \in (0, L), t > 0, \\ \eta(0, x) = \eta^0(x); \quad \omega(0, x) = \omega^0(x) & \text{for } x \in (0, L), \end{cases} \tag{1.11}$$

with the following boundary conditions

$$\begin{cases} b_1\eta_{txxx}(t,0) - b\eta_{tx}(t,0) = -(b+b_1)\eta(t,0) - a\omega_{xx}(t,0) - \dfrac{\omega(t,0)}{2} & \text{for } \ t \geq 0, \\ b_1\eta_{txxx}(t,L) - b\eta_{tx}(t,L) = (b+b_1)\eta(t,L) - a\omega_{xx}(t,L) - \dfrac{\omega(t,L)}{2} & \text{for } \ t \geq 0, \\ d_1\omega_{txxx}(t,0) - d\omega_{tx}(t,0) = -(d+d_1)\omega(t,0) - a\eta_{xx}(t,0) - \dfrac{\eta(t,0)}{2} & \text{for } \ t \geq 0, \\ d_1\omega_{txxx}(t,L) - d\omega_{tx}(t,L) = (d+d_1)\omega(t,L) - a\eta_{xx}(t,L) - \dfrac{\eta(t,L)}{2} & \text{for } \ t \geq 0, \end{cases} \tag{1.12}$$

$$\begin{cases} \eta_{txx}(t,0) = \eta_x(t,0) - a\dfrac{\omega_x(t,0)}{2b_1} & \text{for } \ t \geq 0, \\ \eta_{txx}(t,L) = -\eta_x(t,L) - a\dfrac{\omega_x(t,L)}{2b_1} & \text{for } \ t \geq 0, \\ \omega_{txx}(t,0) = \omega_x(t,0) - a\dfrac{\eta_x(t,0)}{2d_1} & \text{for } \ t \geq 0, \\ \omega_{txx}(t,L) = -\omega_x(t,L) - a\dfrac{\eta_x(t,L)}{2d_1} & \text{for } \ t \geq 0. \end{cases} \tag{1.13}$$

The natural energy associated to the Boussinesq system is given by

$$E(t) = \frac{1}{2}\int_0^L |\eta(t)|^2 + |\omega(t)|^2 + b|\eta_x(t)|^2 + d|\omega_x(t)|^2 + b_1|\eta_{xx}(t)|^2 + d_1|\omega_{xx}(t)|^2 dx$$

and, if we multiply the first (resp. second) equation in (**??**) by η (resp. ω), integrate by parts over $(0,L)$ and add the resulting equations, we obtain (at least formally)

$$\begin{aligned} \frac{dE(t)}{dt} =& -(b+b_1)(|\eta(t,L)|^2 + |\eta(t,0)|^2) - (d+d_1)(|\omega(t,L)|^2 + |\omega(t,0)|^2) \\ & - b_1(|\eta_x(t,L)|^2 + |\eta_x(t,0)|^2) - d_1(|\omega_x(t,L)|^2 + |\omega_x(t,0)|^2). \end{aligned} \tag{1.14}$$

Thus, the energy $E(t)$ is decreasing and the boundary conditions play the role of a feedback damping mechanism.

Next, we study the case in which a localized damping mechanism acts in one equation of the system:

$$\begin{cases} \eta_t + \omega_x + a\omega_{xxx} - b\eta_{txx} + b_1\eta_{txxxx} + \mathcal{B}\eta = 0 & \text{for } x \in (0,2\pi), \ t > 0, \\ \omega_t + \eta_x + c\eta_{xxx} - d\omega_{txx} + d_1\omega_{txxxx} = 0 & \text{for } x \in (0,2\pi), \ t > 0, \\ \eta(0,x) = \eta^0(x); \quad \omega(0,x) = \omega^0(x) & \text{for } x \in (0,2\pi), \end{cases} \tag{1.15}$$

where $\mathcal{B}$ is a linear bounded operator which will act only on an open subset $\Omega \subset (0,2\pi)$. In the next sections, we will specify the election of the feedback law $\mathcal{B}\eta$. In each case, the following boundary conditions will be imposed on the system (1.15):

$$\begin{cases} \dfrac{\partial^r \eta}{\partial x^r}(t,0) = \dfrac{\partial^r \eta}{\partial x^r}(t,2\pi), \quad \dfrac{\partial^r \omega}{\partial x^r}(t,0) = \dfrac{\partial^r \omega}{\partial x^r}(t,2\pi) \quad \text{for } t > 0, \quad r = 0,1. \end{cases}$$

Then, when $L = 2\pi$, the energy $E(t)$ defined above satisfies

$$\frac{dE(t)}{dt} = -\int_\Omega \mathcal{B}\eta(t)\eta(t)dx. \tag{1.16}$$

So, if $\int_\Omega \mathcal{B}\eta(t)\eta(t)dx > 0$, the energy decreases along the trajectories of the system. In both cases, (1.14) and (1.16), the question is whether $E(t)$ is asymptotically stable, as $t \to \infty$.

Following the approach developed in [40] and [3] we first prove the unique continuation property for solutions of the conservative system. The proof makes use of the explicit Fourier series expansion of the solution in terms of the eigenvectors of the differential operator associated to the space variable. In what concerns the systems above, the well-posedness is obtained by converting them into integral equations and applying the contraction-mapping principle. Then, by proving the convergence towards a solution which is null on a band, the unique continuation property implies that the origin is asymptotically stable. As it will become clear during our proofs, to ensure the global well-posedness of (1.11)-(1.13) in the energy space we will assume additional conditions on the parameters.

It is important to emphasize that the energy identities (1.14) and (1.16) do not imply any global in time a priori estimates for the nonlinear system. Thus, it does not conduct to the global existence of solutions in the energy space.

This chapter is outlined as follows. Section 4.1 is dedicated to prove the unique continuation property for solutions of the conservative system. In Section 4.2 it is studied the Boussinesq system with boundary damping and finally, in Section 4.3 we consider the system with a pair of internal damping terms.

1.1.4 Comments and perspectives

During the development of this work some natural questions came to the fore. Therefore, in this section we will mention a list of problems that we can study thereafter.

Chapter 2

Controllability of the Gear-Grimshaw system in a weighted L^2-space

In this chapter we are concerned with the controllability properties of system (1.5), posed on a bounded interval, by means of distributed controls. When the control region is a neighborhood of the right end point of the interval, we prove the local exact controllability of the nonlinear problem in some well chosen weighted L^2-spaces. The results are first established for the linearized system through a classical duality approach and then extended for the full system via a fixed point argument.

2.1 The linear system

In this section we prove the well-posedness of the linear system associated to (1.5)-(1.7). The results will be obtained for both homogeneous and nonhomogeneous systems.

Throughout the section, we consider the Hilbert spaces $[L^2_{w(x)dx}]^2$ defined before.

2.1.1 The homogeneous system

Let us first consider the homogeneous system

$$\begin{cases} u_t + u_{xxx} + av_{xxx} = 0 & \text{in } \ (0,T)\times(0,L), \\ cv_t + rv_x + v_{xxx} + bau_{xxx} = 0 & \text{in } \ (0,T)\times(0,L), \\ u(0,t) = u(L,t) = u_x(L,t) = 0 & \text{on } \ (0,T), \\ v(0,t) = v(L,t) = v_x(L,t) = 0 & \text{on } \ (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{on } \ (0,L). \end{cases} \tag{2.1}$$

The well-posedness results will be established in the spaces

$$[L^2_{xdx}]^2 \quad \text{and} \quad [L^2_{(L-x)^{-1}dx}]^2.$$

This is done by using the semigroup approach, combining Hille-Yosida Theorem and the following generalized Lax-Milgram Theorem due to J.-L. Lions (see, for instance, [22, 23]).

Theorem 2.1.1. *Let $W \subset V \subset H$ be three Hilbert spaces with continuous and dense embeddings. Let $a(u,v)$ be a bilinear form defined on $V \times W$ that satisfies the following properties:*

(i) (Continuity)

$$a(v,w) \leq M||v||_V||w||_W, \quad \forall v \in V, \ \forall w \in W$$

(ii) (Coercivity)

$$a(w,w) \geq m||w||_V^2, \quad \forall w \in W.$$

Then, for all $f \in V'$ (the dual of V), there exists $v \in V$, such that

$$a(v,w) = f(w), \quad \forall w \in W.$$

Assume that, in addition to (i) and (ii), $a(v,w)$ satisfies:

(iii) (Regularity) For all $g \in H$, any solution $v \in V$ of

$$a(v,w) = (g,w)_H, \quad \forall w \in W, \tag{2.2}$$

belongs to W. Then, equation (2.2) has a unique solution $v = v(g) \in W$. Let $D(A) := \{v(g); g \in H\} \subset W \subset H$ and set $Av := -g$, for $v \in D(A)$. (Note that there is a unique $g \in H$ satisfying (2.2).) Then, A is a maximal dissipative operator, and hence it generates a continuous semigroup of contractions in H.

Well-posedness in $[L^2_{xdx}]^2$

By using Theorem 2.1.1, we prove that the state operator associated to (2.1) generates a strongly continuous semigroup in $[L^2_{xdx}]^2$.

Theorem 2.1.2. *Let $\vec{u} = (u_1, u_2)$ and $A_1\vec{u} = (-u_{1,xxx} - au_{2,xxx}, -\frac{ab}{c}u_{1,xxx} - \frac{1}{c}u_{2,xxx} - \frac{r}{c}u_{2,x})$ with domain*

$$D(A_1) = \left\{\vec{u} \in [H^2(0,L) \cap H^1_0(0,L)]^2; \vec{u}_{xxx} \in [L^2_{xdx}]^2, \vec{u}_x(L) = \vec{0}\right\} \subset [L^2_{xdx}]^2.$$

Then, A_1 generates a strongly continuous semigroup in $[L^2_{xdx}]^2$.

Proof. We first introduce the spaces

$$H = [L^2_{xdx}]^2, \quad V = [H^1_0(0,L)]^2, \quad W = \left\{\vec{w} \in [H^1_0(0,L)]^2, \vec{w}_{xx} \in [L^2_{x^2dx}]^2\right\},$$

endowed with the norms

$$||\vec{u}||^2_H := \frac{b}{c}||\sqrt{x}u_1||^2_{L^2(0,L)} + ||\sqrt{x}u_2||^2_{L^2(0,L)}, \quad ||\vec{v}||^2_V := \frac{b}{c}||v_{1,x}||^2_{L^2(0,L)} + ||v_{2,x}||^2_{L^2(0,L)},$$

$$||\vec{w}||^2_W := \frac{b}{c}||xw_{1,xx}||^2_{L^2(0,L)} + ||xw_{2,xx}||^2_{L^2(0,L)},$$

where $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$.

Observe that $V \subset H$ with continuous and dense embeddings and $D(A_1) \subset W$. Moreover, using the following inequality proved in [14]

$$||w_x||_{L^2} \leq C||xw_{xx}||_{L^2}, \quad \forall\, w \in \left\{w \in H^1_0(0,L), w_{xx} \in L^2_{x^2dx}\right\}, \tag{2.3}$$

it follows that W is a Hilbert space and the embedding $W \subset V$ is dense and continuous.

Next, we define the bilinear form

$$a_\lambda(\vec{v}, \vec{w}) := a(\vec{v}, \vec{w}) + \lambda(\vec{v}, \vec{w})_H, \quad \text{with} \quad \lambda >> 1,$$

where

$$a(\vec{v}, \vec{w}) = \frac{b}{c}\int_0^L v_{1,x}\left[(xw_1)_{xx} + a(xw_2)_{xx}\right] dx$$
$$+ \frac{1}{c}\int_0^L v_{2,x}\left[ba(xw_1)_{xx} + (xw_2)_{xx} + r(xw_2)\right] dx, \quad \forall \vec{v} \in V,\ \vec{w} \in W,$$

to prove that conditions (i), (ii) and (iii) given by Theorem 2.1.1 are satisfied for any $\vec{v} \in V$ and $\vec{w} \in W$.

(i) Continuity

Combining Cauchy-Schwarz and Poincaré inequalities with (2.3) it follows that

$$\begin{aligned}
|a(\vec{v},\vec{w})| &\leq \frac{b}{c}||v_{1,x}||_{L^2}\left(||xw_{1,xx}+2w_{1,x}||_{L^2}+|a|.||xw_{2,xx}+2w_{2,x}||_{L^2}\right)\\
&\quad+\frac{1}{c}||v_{2,x}||_{L^2}\left(|ab|.||xw_{1,xx}+2w_{1,x}||_{L^2}+||xw_{2,xx}+2w_{2,x}||_{L^2}+|r|||xw_2||_{L^2}\right)\\
&\leq C_0(a,b,c,r)||\vec{v}||_V\left(||xw_{1,xx}||_{L^2}+||xw_{2,xx}||_{L^2}+||\vec{w}_x||_{[L^2]^2}+C||xw_{2,x}+w_2||_{L^2}\right)\\
&\leq C_1(a,b,c,r,L)||\vec{v}||_V\left(||\vec{w}||_W+||w_{2,x}||_{L^2}\right)\leq M||\vec{v}||_V||\vec{w}||_W,
\end{aligned}$$

where C_0, C_1 and M are positive constant. The above estimate allows us to conclude that $a(\cdot,\cdot)$, as well as $a_\lambda(\cdot,\cdot)$, are well defined and continuous on $V\times W$.

(ii) Coercivity

For any $\vec{w}=(w_1,w_2)\in[C^\infty([0,L])\cap H_0^1(0,L)]^2$, we have

$$\begin{aligned}
a(\vec{w},\vec{w}) &= \frac{b}{c}\int_0^L w_{1,x}\left[(xw_1)_{xx}+a(xw_2)_{xx}\right]dx+\frac{1}{c}\int_0^L w_{2,x}\left[ba(xw_1)_{xx}+(xw_2)_{xx}+r(xw_2)\right]dx\\
&= \frac{b}{c}\int_0^L w_{1,x}\left[xw_{1,xx}+2w_{1,x}\right]+aw_{1,x}\left[xw_{2,xx}+2w_{2,x}\right]dx\\
&\quad+\frac{1}{c}\int_0^L baw_{2,x}\left[xw_{1,xx}+2w_{1,x}\right]+w_{2,x}\left[xw_{2,xx}+2w_{2,x}\right]+\frac{r}{2}x[w_2^2]_x dx.
\end{aligned}$$

After integration by parts, $a(\vec{w},\vec{w})$ can be written as

$$\begin{aligned}
a(\vec{w},\vec{w}) &= \frac{1}{2}\frac{b}{c}\int_0^L x\left[w_{1,x}^2+2aw_{1,x}w_{2,x}+\frac{1}{b}w_{2,x}^2\right]_x dx\\
&\quad+2\frac{b}{c}\int_0^L\left[w_{1,x}^2+2aw_{1,x}w_{2,x}+\frac{1}{b}w_{2,x}^2\right]dx-\frac{r}{2c}\int_0^L w_2^2dx\\
&= \frac{L}{2}\frac{b}{c}\left[w_{1,x}^2(L)+2aw_{1,x}(L)w_{2,x}(L)+\frac{1}{b}w_{2,x}^2(L)\right]\\
&\quad+\frac{3}{2}\frac{b}{c}\int_0^L\left[w_{1,x}^2+2aw_{1,x}w_{2,x}+\frac{1}{b}w_{2,x}^2\right]dx-\frac{r}{2c}\int_0^L w_2^2dx.
\end{aligned}$$

Then,

$$\begin{aligned}
a(\vec{w},\vec{w}) &= \frac{L}{2}\frac{b}{c}\left[(w_{1,x}(L)+aw_{2,x}(L))^2+\left(\frac{1}{b}-a^2\right)w_{2,x}^2(L)\right]\\
&\quad+\frac{3}{2}\frac{b}{c}\int_0^L\left[w_{1,x}^2+2aw_{1,x}w_{2,x}+\frac{1}{b}w_{2,x}^2\right]dx-\frac{r}{2c}\int_0^L w_2^2dx.
\end{aligned}$$

For any $\epsilon_0>0$, an application of Young inequality gives

$$a(\vec{w},\vec{w})\geq\frac{3}{2}(1-\epsilon)\frac{b}{c}\int_0^L w_{1,x}^2dx+\frac{3}{2}\frac{b}{c}\left(\frac{1}{b}-\frac{a^2}{\epsilon}\right)\int_0^L w_{2,x}^2dx-\frac{r}{2c}\int_0^L w_2^2dx. \tag{2.4}$$

Thus, taking into account the assumptions on the coefficients of the system, we can choose $\epsilon_0>0$, such that $|a|\sqrt{b}<\sqrt{\epsilon_0}<1$ to obtain

$$a(\vec{w},\vec{w})\geq m_{\epsilon_0}||\vec{w}||_V^2-\frac{r}{2c}\int_0^L w_2^2dx \tag{2.5}$$

where $m_{\epsilon_0} := \min\left\{\frac{3}{2}(1-\epsilon_0), \frac{3}{2}\frac{b}{c}\left(\frac{1}{b}-\frac{a^2}{\epsilon_0}\right)\right\}$. Observe that, if $r=0$, the coercivity of $a(\cdot,\cdot)$ follows from (2.5), for any $L>0$. When $r\neq 0$, the result is obtained making use of the Poincaré inequality:

$$\int_0^L w^2 dx \leq \left(\frac{L}{\pi}\right)^2 \int_0^L w_x^2 dx.$$

Indeed, from (2.4) we get

$$a(\vec{w},\vec{w}) \geq \frac{3}{2}(1-\epsilon_0)\frac{b}{c}\int_0^L w_{1,x}^2 dx + \frac{3}{2}\frac{b}{c}\left(\frac{1}{b}-\frac{a^2}{\epsilon_0}-\frac{|r|L^2}{3b\pi^2}\right)\int_0^L w_{2,x}^2 dx \geq m_{L,\epsilon_0}||\vec{w}||_V^2,$$

for any $L>0$ satisfying

$$L < \pi\sqrt{\frac{3b}{|r|}\left(\frac{1}{b}-\frac{a^2}{\epsilon_0}\right)}$$

and $m_{L,\epsilon_0}>0$ given by

$$m_{L,\epsilon_0} := \min\left\{\frac{3}{2}(1-\epsilon_0), \frac{3}{2}\frac{b}{c}\left(\frac{1}{b}-\frac{a^2}{\epsilon_0}-\frac{|r|L^2}{3b\pi^2}\right)\right\}.$$

When $L \geq \pi\sqrt{\frac{3b}{|r|}\left(\frac{1}{b}-\frac{a^2}{\epsilon_0}\right)}$, instead of a, we consider the bilinear form

$$a_\lambda(\vec{v},\vec{w}) := a(\vec{v},\vec{w}) + \lambda(\vec{v},\vec{w})_H, \quad \text{with} \quad \lambda >> 1.$$

To prove the coercivity of a_λ, we need the following claim:

Claim: For any $\delta>0$,

$$||w||_{L^2}^2 \leq \delta||w||_{H_0^1}^2 + \delta^{-1}L||w||_{L^2_{xdx}}^2, \quad \forall\, w \in C([0,L])\cap H_0^1(0,L).$$

Combining the Claim and (2.5), it follows that

$$\begin{aligned} a(\vec{w},\vec{w}) &\geq m_{\epsilon_0}||\vec{w}||_V^2 - \frac{|r|}{2c}\delta||w_2||_{H_0^1}^2 - \frac{|r|L}{2c}\delta^{-1}||w_2||_{L^2_{xdx}}^2 \\ &\geq \left(m_{\epsilon_0}-\frac{|r|}{2c}\delta\right)||\vec{w}||_V^2 - \frac{|r|L}{2c}\delta^{-1}||\vec{w}||_H^2. \end{aligned}$$

Then, choosing $\delta_0 < m_{\epsilon_0}\frac{2c}{|r|}$ and $\lambda > L\frac{|r|}{2c}\delta_0^{-1}$, we get

$$a_\lambda(\vec{w},\vec{w}) \geq \left(m_{\epsilon_0}-\frac{|r|}{2c}\delta_0\right)||\vec{w}||_V^2 + \left(\lambda - \frac{|r|L}{2c}\delta_0^{-1}\right)||\vec{w}||_H^2.$$

Consequently, a_λ is continuous and coercive. Finally, since $[C^\infty([0,L])\cap H_0^1(0,L)]^2$ is dense in W, the result is valid for any $\vec{w}\in W$.

It remains to prove the Claim.

Proof of the Claim: From Cauchy-Schwarz and Hardy inequalities, we have

$$\begin{aligned} ||w||_{L^2}^2 &\leq ||x^{\frac{1}{2}}w||_{L^2}\left(\int_0^L x^{-1}w^2dx\right)^{\frac{1}{2}} = ||w||_{L^2_{xdx}}\left(\int_0^L xx^{-2}w^2dx\right)^{\frac{1}{2}} \\ &\leq \sqrt{L}||w||_{L^2_{xdx}}\left(\int_0^L [x^{-1}w]^2dx\right)^{\frac{1}{2}} \leq 2\sqrt{L}||w||_{L^2_{xdx}}\left(\int_0^L w_x^2dx\right)^{\frac{1}{2}} \\ &= 2\sqrt{L}||w||_{L^2_{xdx}}||w||_{H_0^1}. \end{aligned}$$

Then, from Young inequality it follows that

$$||w||^2_{L^2} \le \delta||w||^2_{H^1_0} + \delta^{-1}L||w||^2_{L^2_{xdx}}, \quad \forall\,\delta > 0.$$

Consequently, for given $\vec{g} = (g_1, g_2) \in H$, Theorem 2.1.1 guarantees the existence of a function $\vec{v} \in V$ which solves the problem

$$a_\lambda(\vec{v}, \vec{w}) = (\vec{g}, \vec{w})_H, \quad \forall \vec{w} \in W. \tag{2.6}$$

The next steps are devoted to prove that $\vec{v} \in D(A_1) \subset W$.

(iii) Regularity

We first write (2.6) as

$$\begin{aligned} a_\lambda(\vec{v}, \vec{w}) = \frac{b}{c}\int_0^L v_{1,x}\left[(xw_1)_{xx} + a(xw_2)_{xx}\right]dx + \frac{1}{c}\int_0^L v_{2,x}\left[ba(xw_1)_{xx} + (xw_2)_{xx} + r(xw_2)\right]dx \\ + \lambda\int_0^L x\left[\frac{b}{c}v_1w_1 + v_2w_2\right]dx = \int_0^L x\left[\frac{b}{c}g_1w_1 + g_2w_2\right]dx. \end{aligned} \tag{2.7}$$

Then, (2.7) allows us to conclude that

$$\begin{aligned} \frac{b}{c}\langle x[v_{1,xxx} + av_{2,xxx} + \lambda v_1], w_1\rangle_{\mathcal{D}',\mathcal{D}} + \frac{1}{c}\langle x[bav_{1,xxx} + v_{2,xxx} + rv_{2,x} + c\lambda v_2], w_2\rangle_{\mathcal{D}',\mathcal{D}} \\ = \frac{b}{c}\langle xg_1, w_1\rangle_{\mathcal{D}',\mathcal{D}} + \langle xg_2, w_2\rangle_{\mathcal{D}',\mathcal{D}}, \quad \forall \vec{w} \in [\mathcal{D}(0,L)]^2, \end{aligned}$$

where $\langle\cdot,\cdot\rangle_{\mathcal{D}',\mathcal{D}}$ denotes the duality between $[\mathcal{D}(0,L)]^2$ and its dual space. Consequently,

$$\begin{cases} v_{1,xxx} + av_{2,xxx} + \lambda v_1 = g_1, & \text{in } \mathcal{D}'(0,L), \\ bav_{1,xxx} + v_{2,xxx} + rv_{2,x} + c\lambda v_2 = cg_2, & \text{in } \mathcal{D}'(0,L). \end{cases} \tag{2.8}$$

From now on, we proceed in several steps:

- Step 1: $\vec{v} \in [H^3(\epsilon, L)]^2$, for all $\epsilon \in (0, L)$, and $\vec{v}_{xxx} \in H$.

  Indeed, from (2.8) it follows that

$$v_{2,xxx} = \left(1 - ba^2\right)^{-1}\left[\lambda(bav_1 - cv_2) - rv_{2,x} + cg_2 - bag_1\right], \text{ in } \mathcal{D}'(0,L). \tag{2.9}$$

Then, for any $\epsilon \in (0, L)$, we get

$$\int_\epsilon^L |v_{2,xxx}|^2 dx \le C||\vec{v}||^2_V + C'\epsilon^{-1}\int_\epsilon^L x[|g_1|^2 + |g_2|^2]dx \quad \le C||\vec{v}||^2_V + C'\epsilon^{-1}||\vec{g}||^2_H,$$

for some constants $C, C' > 0$. Hence, $v_2 \in H^3(\epsilon, L)$ and, from (2.8), we deduce that $v_1 \in H^3(\epsilon, L)$, which proves the first part of Step 1. On the other hand, from (2.9) we obtain the estimate

$$\int_0^L x|v_{2,xxx}|^2 dx \le C(L)\left(||\vec{v}||^2_V + ||\vec{g}||^2_H\right),$$

where $C(L) > 0$, which shows that $v_{2,xxx} \in L^2_{xdx}$. Thus, from (2.8) we also conclude that $v_{1,xxx} \in L^2_{xdx}$, and the proof of Step 1 ends.

- Step 2: $\vec{v} \in [H^2(0, L)]^2$, and hence $\vec{v} \in W$.

  Since $\vec{v}_{xxx} \in H$, we obtain a positive constant $C > 0$, such that, for any $\epsilon \in (0, L)$,

$$|v_{ixx}(\epsilon) - v_{ixx}(L)| \le \left(\int_\epsilon^L x|v_{ixxx}|^2 dx\right)^{\frac{1}{2}}\left(\int_\epsilon^L x^{-1}dx\right)^{\frac{1}{2}} \le C + C|ln(\epsilon)|, \quad i = 1, 2. \tag{2.10}$$

On the other hand,

$$
\begin{aligned}
|v_{ix}(\epsilon)-v_{ix}(L)| &\le \int_\epsilon^L |v_{ixx}(x)-v_{ixx}(L)|dx+(L-\epsilon)|v_{ixx}(L)| \\
&\le C\int_\epsilon^L |ln(x)|dx+(L-\epsilon)\left[|v_{ixx}(L)|+C\right], \quad i=1,2,
\end{aligned}
\tag{2.11}
$$

for some $C>0$. From (2.10) we deduce that $\vec{v}\in[H^2(0,L)]^2$ and, therefore, $\vec{v}_{xx}\in[L^2_{x^2dx}]^2$, which gives that $\vec{v}\in W$.

• Step 3: $\vec{v}\in D(A_1)$.

We first multiply the first equation in (2.8) by $\frac{b}{c}xw_1$ and integrate over (ϵ,L), where $w_1\in C^\infty([0,L])\cap H_0^1(0,L)$ and $\epsilon\in(0,L)$. After integrating by parts over (ϵ,L), we get

$$
\begin{aligned}
&\frac{b}{c}\left[xw_1(v_{1,xx}+av_{2,xx})-(xw_{1,x}+w_1)(v_{1,x}+av_{2,x})\right]\Big|_\epsilon^L \\
&+\frac{b}{c}\int_\epsilon^L [xw_1]_{xx}[v_{1,x}+av_{2,x}]dx+\lambda\frac{b}{c}\int_\epsilon^L xw_1v_1dx=\frac{b}{c}\int_\epsilon^L xw_1g_1dx.
\end{aligned}
\tag{2.12}
$$

Next, multiplying the second equation in (2.8) by $\frac{1}{c}xw_2$, with $w_2\in C^\infty([0,L])\cap H_0^1(0,L)$, and proceeding in a similar way, it follows that

$$
\begin{aligned}
&\frac{1}{c}\left[xw_2(bav_{1,xx}+v_{2,xx})-(xw_{2,x}+w_2)(bav_{1,x}+v_{2,x})\right]\Big|_\epsilon^L \\
&\quad+\frac{1}{c}\int_\epsilon^L [xw_2]_{xx}[bav_{1,x}+v_{2,x}]dx+\frac{r}{c}\int_\epsilon^L xw_2v_{2,x}dx+\lambda\int_\epsilon^L xw_2v_2dx \\
&=\int_\epsilon^L xw_2g_2dx.
\end{aligned}
\tag{2.13}
$$

Adding (2.12) and (2.13) hand to hand, we obtain the identity

$$
\begin{aligned}
&L\left[\frac{b}{c}w_{1,x}(L)(v_{1,x}(L)+av_{2,x}(L))+\frac{1}{c}w_{2,x}(L)\left(bav_{1,x}(L)+v_{2,x}(L)\right)\right] \\
&+\frac{b}{c}\left[\epsilon w_1(\epsilon)(v_{1,xx}(\epsilon)+av_{2,xx}(\epsilon))-(\epsilon w_{1,x}(\epsilon)+w_1(\epsilon))(v_{1,x}(\epsilon)+av_{2,x}(\epsilon))\right] \\
&+\frac{1}{c}\left[\epsilon w_2(\epsilon)\left(bav_{1,xx}(\epsilon)+v_{2,xx}(\epsilon)\right)-\left(\epsilon w_{2,x}(\epsilon)+w_2(\epsilon)\right)\left(bav_{1,x}(\epsilon)+v_{2,x}(\epsilon)\right)\right] \\
&=\frac{b}{c}\int_\epsilon^L v_{1,x}\left[(xw_1)_{xx}+a(xw_2)_{xx}\right]dx+\frac{1}{c}\int_\epsilon^L v_{2,x}\left[ba(xw_1)_{xx}+(xw_2)_{xx}+r(xw_2)\right]dx \\
&+\lambda\int_\epsilon^L x\left[\frac{b}{c}v_1w_1+v_2w_2\right]dx-\int_\epsilon^L x\left[\frac{b}{c}g_1w_1+g_2w_2\right]dx.
\end{aligned}
\tag{2.14}
$$

Observe that (2.10) and (2.11) allow us to pass (2.14) to the limit, as $\epsilon\to0$. Hence, from (2.7) we get

$$
\begin{aligned}
&-L\left[\frac{b}{c}w_{1,x}(L)(v_{1,x}(L)+av_{2,x}(L))+\frac{1}{c}w_{2,x}(L)\left(bav_{1,x}(L)+v_{2,x}(L)\right)\right] \\
&=\lim_{\epsilon\to0}\left\{\frac{b}{c}\left[\epsilon w_1(\epsilon)(v_{1,xx}(\epsilon)+av_{2,xx}(\epsilon))-(\epsilon w_{1,x}(\epsilon)+w_1(\epsilon))(v_{1,x}(\epsilon)+av_{2,x}(\epsilon))\right]\right. \\
&\left.+\frac{1}{c}\left[\epsilon w_2(\epsilon)\left(bav_{1,xx}(\epsilon)+v_{2,xx}(\epsilon)\right)-\left(\epsilon w_{2,x}(\epsilon)+w_2(\epsilon)\right)\left(bav_{1,x}(\epsilon)+v_{2,x}(\epsilon)\right)\right]\right\} \\
&=0.
\end{aligned}
$$

Since $\vec{w} = (w_1, w_2)$ is arbitrary, we conclude that

$$\begin{aligned} v_{1,x}(L) + a v_{2,x}(L) &= 0, \\ b a v_{1,x}(L) + v_{2,x}(L) &= 0. \end{aligned}$$

Solving the system above we obtain $\left(1 - ba^2\right) v_{2,x}(L) = 0$ and, due to the assumptions on the coefficients of the system, it follows that $v_{1,x}(L) = v_{2,x}(L) = 0$. Then, $\vec{v} \in D(A_1)$. Reciprocally, from the analysis developed above, it follows that the operator $A_1 - \lambda : D(A_1) \to H$ is onto. Then, $A_1 - \lambda$ generates a strongly semigroup of contractions in H.

■

Well-posedness in $[L^2_{(L-x)^{-1}dx}]^2$

Combining Hille-Yosida Theorem and (partially) Theorem 2.1.1, we prove that the state operator associated to (2.1) generates a strongly continuous semigroup in $[L^2_{(L-x)^{-1}dx}]^2$.

Theorem 2.1.3. *Let $\vec{u} = (u_1, u_2)$ and $A_2\vec{u} = (-u_{1,xxx} - a u_{2,xxx}, -\frac{ab}{c} u_{1,xxx} - \frac{1}{c} u_{2,xxx} - \frac{r}{c} u_{2,x})$ with domain*

$$D(A_2) = \left\{ \vec{u} \in [H^3(0,L) \cap H^1_0(0,L)]^2; \vec{u}_{xxx} \in [L^2_{(L-x)^{-1}dx}]^2, \vec{u}_x(L) = \vec{0} \right\} \subset [L^2_{(L-x)^{-1}dx}]^2.$$

Then, A_2 generates a strongly continuous semigroup in $[L^2_{(L-x)^{-1}dx}]^2$.

Proof. We first introduce the spaces

$$H = [L^2_{(L-x)^{-1}dx}]^2, \quad V = \left\{ \vec{u} \in [H^1_0(0,L)]^2, \vec{u}_x \in [L^2_{(L-x)^{-2}dx}]^2 \right\}, \quad W = [H^2_0(0,L)]^2, \tag{2.15}$$

endowed with the norms

$$||\vec{u}||^2_H := \frac{b}{c} ||(L-x)^{-\frac{1}{2}} u_1||^2_{L^2} + ||(L-x)^{-\frac{1}{2}} u_2||^2_{L^2}, \quad ||\vec{w}||^2_W := \frac{b}{c} ||w_{1,xx}||^2_{L^2(0,L)} + ||w_{2,xx}||^2_{L^2},$$

$$||\vec{v}||^2_V := \frac{b}{c} ||(L-x)^{-1} v_{1,x}||^2_{L^2} + ||(L-x)^{-1} v_{2,x}||^2_{L^2},$$

where $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$.

We first remark that:

(i) V is a Hilbert space and $V \subset H$ with continuous embedding.

Indeed, according to [22], the space $\mathcal{V} := \left\{ u \in H^1_0(0,L); u_x \in L^2_{(L-x)^{-2}dx} \right\}$, endowed with the norm $||u||_{\mathcal{V}} := ||(L-x)^{-1} u_x||_{L^2}$, is a Hilbert space and

$$||(L-x)^{-2} u||_{L^2} \leq \frac{2}{3} ||u||_{\mathcal{V}}, \quad \forall u \in \mathcal{V}. \tag{2.16}$$

Consequently, $\mathcal{V} \subset L^2_{(L-x)^{-1}dx}$ with continuous embedding, since

$$||u||_{L^2_{(L-x)^{-1}dx}} \leq \left(\int_0^L \frac{L^3}{(L-x)^4} |u|^2 dx \right)^{\frac{1}{2}} \leq \frac{2}{3} L^{\frac{3}{2}} ||u||_{\mathcal{V}}, \quad \forall\, u \in \mathcal{V}. \tag{2.17}$$

The results above allow us to deduce that V is a Hilbert space and, in addition, that the embedding $V \subset H$ is continuous as well.

(ii) W is a Hilbert space and $W \subset V$ with continuous embedding.

Poincaré inequality guarantees that the norms $||\cdot||_W$ and the H^2-norm are equivalent. Now, observe that, from Hardy inequality we have that

$$\int_0^L \frac{v^2}{(L-x)^2}dx \leq C\int_0^L v_x^2 dx, \quad \forall\, v \in H^1(0,L) \text{ with } v(L)=0. \tag{2.18}$$

Thus, for any $\vec{v}\in W$, it follows that $v_{1,x}$,$v_{2,x} \in H^1(0,L)$ and $v_{1,x}(L)=v_{2,x}(L)=0$. Then, from (2.18), we get

$$||\vec{v}||_V \leq C||\vec{v}||_W, \quad \forall\, \vec{v}\in W. \tag{2.19}$$

This shows that $W\subset V$ with continuous embedding. It is easily seen that $[\mathcal{D}(0,L)]^2$ is dense in H, V and W.

Next, we proceed as in the proof of Theorem 2.1.2 and define the bilinear form

$$a_\lambda(\vec{v},\vec{w}) = a(\vec{v},\vec{w}) + \lambda(\vec{v},\vec{w})_H,$$

where $\lambda>0$ will be defined later, and

$$\begin{aligned} a(\vec{v},\vec{w}) =& \frac{b}{c}\int_0^L v_{1,x}\left[\left(\frac{w_1}{L-x}\right)_{xx} + a\left(\frac{w_2}{L-x}\right)_{xx}\right]dx \\ &+ \frac{1}{c}\int_0^L v_{2,x}\left[ba\left(\frac{w_1}{L-x}\right)_{xx} + \left(\frac{w_2}{L-x}\right)_{xx} + r\left(\frac{w_2}{L-x}\right)\right]dx, \ \ \forall\,\vec{v}\in V,\ \vec{w}\in W. \end{aligned}$$

Then, in order to apply Theorem 2.1.1, the next steps are devoted to prove that $a_\lambda(\cdot,\cdot)$ is continuous and coercive.

(i) Continuity

First, observe that, for any $v=(v_1,v_2)\in V$ and $w=(w_1,w_2)\in W$

$$\begin{aligned} |a(\vec{v},\vec{w})| \leq& \frac{b}{c}\int_0^L |v_{1,x}|\left|\frac{w_{1,xx}}{L-x} + 2\frac{w_{1,x}}{(L-x)^2} + 2\frac{w_1}{(L-x)^3}\right|dx \\ &+ \frac{b|a|}{c}\int_0^L |v_{1,x}|\left|\frac{w_{2,xx}}{L-x} + 2\frac{w_{2,x}}{(L-x)^2} + 2\frac{w_2}{(L-x)^3}\right|dx \\ &+ \frac{b|a|}{c}\int_0^L |v_{2,x}|\left|\frac{w_{1,xx}}{L-x} + 2\frac{w_{1,x}}{(L-x)^2} + 2\frac{w_1}{(L-x)^3}\right|dx \\ &+ \frac{1}{c}\int_0^L |v_{2,x}|\left|\frac{w_{2,xx}}{L-x} + 2\frac{w_{2,x}}{(L-x)^2} + 2\frac{w_2}{(L-x)^3} + r\frac{w_2}{L-x}\right|dx. \end{aligned}$$

Then, from Cauchy-Schwarz inequality, (2.16), (2.17) and (2.19), we obtain a positive constant $C>0$, such that

$$\begin{aligned} &|a(\vec{v},\vec{w})| \\ &\leq \frac{b}{c}\left[||w_{1,xx}||_{L^2}\left|\left|\frac{v_{1,x}}{L-x}\right|\right|_{L^2} + 2\left|\left|\frac{w_{1,x}}{L-x}\right|\right|_{L^2}\left|\left|\frac{v_{1,x}}{L-x}\right|\right|_{L^2} + 2\left|\left|\frac{w_1}{(L-x)^2}\right|\right|_{L^2}\left|\left|\frac{v_{1,x}}{L-x}\right|\right|_{L^2}\right] \\ &+ \frac{b|a|}{c}\left[||w_{2,xx}||_{L^2}\left|\left|\frac{v_{1,x}}{L-x}\right|\right|_{L^2} + 2\left|\left|\frac{w_{2,x}}{L-x}\right|\right|_{L^2}\left|\left|\frac{v_{1,x}}{L-x}\right|\right|_{L^2} + 2\left|\left|\frac{w_2}{(L-x)^2}\right|\right|_{L^2}\left|\left|\frac{v_{1,x}}{L-x}\right|\right|_{L^2}\right] \\ &+ \frac{b|a|}{c}\left[||w_{1,xx}||_{L^2}\left|\left|\frac{v_{2,x}}{L-x}\right|\right|_{L^2} + 2\left|\left|\frac{w_{1,x}}{L-x}\right|\right|_{L^2}\left|\left|\frac{v_{2,x}}{L-x}\right|\right|_{L^2} + 2\left|\left|\frac{w_1}{(L-x)^2}\right|\right|_{L^2}\left|\left|\frac{v_{2,x}}{L-x}\right|\right|_{L^2}\right] \\ &+ \frac{1}{c}\left[||w_{2,xx}||_{L^2}\left|\left|\frac{v_{2,x}}{L-x}\right|\right|_{L^2} + 2\left|\left|\frac{w_{2,x}}{L-x}\right|\right|_{L^2}\left|\left|\frac{v_{2,x}}{L-x}\right|\right|_{L^2}\right] \\ &+ \frac{1}{c}\left|\left|\frac{v_{2,x}}{L-x}\right|\right|_{L^2}\left(2\left|\left|\frac{w_2}{(L-x)^2}\right|\right|_{L^2} + |r|||w_2||_{L^2}\right) \leq C||\vec{v}||_V||\vec{w}||_W, \end{aligned}$$

which shows that $a(\cdot,\cdot)$ and $a_\lambda(\cdot,\cdot)$ are well defined and continuous on $V\times W$.

(ii) Coercivity

For any $\vec{w}=(w_1,w_2)\in W$, we have

$$\begin{aligned}
a(\vec{w},\vec{w}) &= \frac{b}{c}\int_0^L w_{1,x}\left[\frac{w_{1,xx}}{L-x}+2\frac{w_{1,x}}{(L-x)^2}+2\frac{w_1}{(L-x)^3}\right]dx\\
&+\frac{ba}{c}\int_0^L w_{1,x}\left[\frac{w_{2,xx}}{L-x}+2\frac{w_{2,x}}{(L-x)^2}+2\frac{w_2}{(L-x)^3}\right]dx\\
&+\frac{ba}{c}\int_0^L w_{2,x}\left[\frac{w_{1,xx}}{L-x}+2\frac{w_{1,x}}{(L-x)^2}+2\frac{w_1}{(L-x)^3}\right]dx\\
&+\frac{1}{c}\int_0^L w_{2,x}\left[\frac{w_{2,xx}}{L-x}+2\frac{w_{2,x}}{(L-x)^2}+2\frac{w_2}{(L-x)^3}+r\frac{w_2}{L-x}\right]dx.
\end{aligned}$$

After integration by parts, if follows that

$$\begin{aligned}
a(\vec{w},\vec{w}) &= \frac{3}{2}\frac{b}{c}\int_0^L\left[w_{1,x}^2+2aw_{1,x}w_{2,x}+\frac{1}{b}w_{2,x}^2\right]\frac{1}{(L-x)^2}dx\\
&-3\frac{b}{c}\int_0^L\left[w_1^2+2aw_1w_2+\frac{1}{b}w_2^2\right]\frac{1}{(L-x)^4}dx-\frac{r}{2c}\int_0^L\frac{w_2^2}{(L-x)^2}dx.
\end{aligned}\tag{2.20}$$

Then, for any $\epsilon>0$, we can apply Young inequality to obtain

$$\begin{aligned}
a(\vec{w},\vec{w}) &\geq \frac{3}{2}\frac{b}{c}(1-\epsilon)\int_0^L\frac{w_{1,x}^2}{(L-x)^2}dx+\frac{3}{2}\frac{b}{c}\left(\frac{1}{b}-a^2\epsilon^{-1}\right)\int_0^L\frac{w_{2,x}^2}{(L-x)^2}dx\\
&-3\frac{b}{c}(1+\epsilon)\int_0^L\frac{w_1^2}{(L-x)^4}dx-3\frac{b}{c}\left(\frac{1}{b}+a^2\epsilon^{-1}\right)\int_0^L\frac{w_2^2}{(L-x)^4}dx\\
&-\frac{r}{2c}\int_0^L\frac{w_2^2}{(L-x)^2}dx.
\end{aligned}$$

Using (2.16), we can estimate the right hand side of the above inequality as follows:

$$\begin{aligned}
a(\vec{w},\vec{w}) &\geq \frac{1}{6}\frac{b}{c}(1-17\epsilon)\int_0^L\frac{w_{1,x}^2}{(L-x)^2}dx+\frac{1}{6}\frac{b}{c}\left(\frac{1}{b}-17a^2\epsilon^{-1}\right)\int_0^L\frac{w_{2,x}^2}{(L-x)^2}dx\\
&-\frac{r}{2c}\int_0^L\frac{w_2^2}{(L-x)^2}dx.
\end{aligned}\tag{2.21}$$

In order to estimate the last term in the above inequality, we apply Cauchy-Schwarz and Young inequalities and (2.16). More precisely, for any $\delta>0$, we get

$$\begin{aligned}
\int_0^L\frac{w_2^2}{(L-x)^2}dx &\leq ||(L-x)^{-\frac{3}{2}}w_2||_{L^2}||(L-x)^{-\frac{1}{2}}w_2||_{L^2}\\
&\leq \frac{2\sqrt{L}}{3}||(L-x)^{-1}w_{2,x}||_{L^2}||(L-x)^{-\frac{1}{2}}w_2||_{L^2}\\
&\leq \delta||(L-x)^{-1}w_{2,x}||_{L^2}^2+\frac{L}{9\delta}||(L-x)^{-\frac{1}{2}}w_2||_{L^2}^2.
\end{aligned}\tag{2.22}$$

Combining (2.22) and (2.21) it follows that

$$\begin{aligned}
a(\vec{w},\vec{w})+\frac{|r|L}{18c\delta}||(L-x)^{-\frac{1}{2}}w_2||_{L^2}^2 &\geq \frac{1}{6}(1-17\epsilon)\frac{b}{c}||(L-x)^{-1}w_{1,x}||_{L^2}^2\\
&+\left[\frac{1}{6}\frac{b}{c}\left(\frac{1}{b}-17a^2\epsilon^{-1}\right)-\delta\frac{|r|}{2c}\right]||(L-x)^{-1}w_{2,x}||_{L^2}^2.
\end{aligned}$$

Then, taking into account the assumptions on the coefficients of the system, we can choose $\epsilon > 0$, such that $|a|\sqrt{b} < \sqrt{\epsilon/17} < 1/17$. For this choice of ϵ we fix $\delta < \frac{1}{3|r|}\left(1 - 17ba^2\epsilon^{-1}\right)$ and define

$$m_{\epsilon,\delta} := \min\left\{\frac{1}{6}\left(1-17\epsilon\right), \left[\frac{1}{6}\frac{b}{c}\left(\frac{1}{b} - 17a^2\epsilon^{-1}\right) - \delta_1\frac{|r|}{2c}\right]\right\}.$$

Then, for any $\vec{w} \in [\mathcal{D}(0,L)]^2$ we have that

$$a(\vec{w},\vec{w}) + \frac{|r|L}{18c\delta_1}||\vec{w}||_H^2 \geq m_{\epsilon_1,\delta_1}||\vec{w}||_V^2. \tag{2.23}$$

By density, the result also holds for any $\vec{w} \in W$, which shows that the bilinear form

$$a_\lambda(\vec{v},\vec{w}) = a(\vec{v},\vec{w}) + \lambda(\vec{v},\vec{w})_H$$

is coercive for $\lambda > \frac{|r|L}{18c\delta}$.

Then, for given $\vec{g} = (g_1, g_2) \in H$, Theorem 2.1.1 guarantees the existence of a function $\vec{v} \in V$ which solves the problem

$$a_\lambda(\vec{v},\vec{w}) = (\vec{g},\vec{w})_H, \quad \forall \vec{w} \in W. \tag{2.24}$$

In what follows, we prove that $\vec{v} \in D(A_2)$.

(iii) Regularity

We proceed in several steps:

- Step 1: $\vec{v} \in [H^3(0,L)]^2$.

Arguing as in the proof of Theorem 2.1.2, from (2.24) we deduce that

$$\begin{cases} v_{1,xxx} + av_{2,xxx} + \lambda v_1 = g_1 & \text{in } \mathcal{D}'(0,L), \\ bav_{1,xxx} + v_{2,xxx} + rv_{2,x} + c\lambda v_2 = cg_2 & \text{in } \mathcal{D}'(0,L), \end{cases} \tag{2.25}$$

where $\mathcal{D}'(0,L)$ denotes the dual space of $\mathcal{D}(0,L)$. Moreover, since $\vec{g} \in [L^2(0,L)]^2$ and $\vec{v} \in [H^1(0,L)]^2$, it follows that $\vec{v}_{xxx} \in [L^2(0,L)]^2$ and hence $\vec{v} \in [H^3(0,L)]^2$.

- Step 2: $\vec{v}_x(L) = 0$.

Let us introduce the function $\vec{w}(x) = (x^2(L-x)^2\bar{w}_1(x), x^2(L-x)^2\bar{w}_2(x))$, where $\bar{w}_i \in C^\infty([0,L])$ are arbitrary chosen, for $i = 1, 2$. Observe that $\vec{w} \in W$ and $(L-x)^{-1}\vec{w} \in [H_0^1(0,L) \cap C^\infty([0,L])]^2$. Then, since $\vec{v} \in [H^3(0,L)]^2$, we can multiply the first equation in (2.25) by $bw_1/c(L-x)$ and integrate by parts in $(0,L)$ to obtain

$$\begin{aligned} &\frac{b}{c}\int_0^L \left(\frac{w_1}{L-x}\right)[v_{1,xxx} + av_{2,xxx} + \lambda v_1]dx = \frac{b}{c}[L^2\bar{w}_1(L)(v_{1,x}(L) + av_{2,x}(L))] \\ &\quad + \frac{b}{c}\int_0^L \left(\frac{w_1}{L-x}\right)_{xx}[v_{1,x} + av_{2,x}]dx + \lambda\frac{b}{c}\int_0^L \left(\frac{w_1}{L-x}\right)v_1 dx = \frac{b}{c}\int_0^L \left(\frac{w_1}{L-x}\right)g_1 dx. \end{aligned} \tag{2.26}$$

Analogously, if we multiply the second equation in (2.25) by $w_2/c(L-x)$ and integrate by parts in $(0,L)$, it follows that

$$\begin{aligned} &\frac{1}{c}\int_0^L \left(\frac{w_2}{L-x}\right)[bav_{1,xxx} + v_{2,xxx} + rv_{2,x} + c\lambda v_2]dx = \frac{1}{c}[L^2\bar{w}_2(L)(bav_{1,x}(L) + v_{2,x}(L))] \\ &\quad + \frac{1}{c}\int_0^L \left(\frac{w_2}{L-x}\right)_{xx}[bav_{1,x} + v_{2,x}]dx + \frac{r}{c}\int_0^L \left(\frac{w_2}{L-x}\right)v_{2,x}dx + \lambda\int_0^L \left(\frac{w_2}{L-x}\right)v_2 dx \\ &\quad = \int_0^L \left(\frac{w_2}{L-x}\right)g_2 dx. \end{aligned} \tag{2.27}$$

Adding identities (2.26) and (2.27) hand to hand, we get

$$\frac{b}{c}[L^2\bar{w}_1(L)(v_{1,x}(L)+av_{2,x}(L))]+\frac{1}{c}[L^2\bar{w}_2(L)(bav_{1,x}(L)+v_{2,x}(L))]+a_\lambda(\vec{v},\vec{w})=(\vec{g},\vec{w})_H.$$

Then, from (2.24), the following holds

$$\frac{b}{c}[L^2\bar{w}_1(L)(v_{1,x}(L)+av_{2,x}(L))]+\frac{1}{c}[L^2\bar{w}_2(L)(bav_{1,x}(L)+v_{2,x}(L))]=0.$$

Since w_1 and w_2 were arbitrary chosen, the identity above allows to conclude that

$$\begin{aligned} v_{1,x}(L)+av_{2,x}(L)=0,\\ bav_{1,x}(L)+v_{2,x}(L)=0. \end{aligned}$$

Solving the system above we deduce that $(1-a^2b)v_{2,x}(L)=0$ and, due to the assumptions on the coefficients of the systems, we obtain $v_{1,x}(L)=v_{2,x}(L)=0$.

- Step 3: $\vec{v}_{xxx}\in[L^2_{(L-x)^{-1}dx}]^2$.

We first prove that $v_{2,xxx}\in L^2_{(L-x)^{-1}dx}$. Indeed, from (2.25) it follows that

$$v_{2,xxx}=(1-ba^2)^{-1}[\lambda(bav_1-cv_2)+(cg_2-bag_1)-rv_{2,x}].$$

Since $g_i,v_i\in L^2_{(L-x)^{-1}dx}$, for $i=1,2$, we claim that $v_{2,x}\in L^2_{(L-x)^{-1}dx}$. Taking into account that $v_{2,x}\in H^1(0,L)$ and $v_{2,x}(L)=0$, we can apply (2.18) to obtain

$$\int_0^L\frac{v_{2,x}^2}{L-x}dx\le L\int_0^L\frac{v_{2,x}^2}{(L-x)^2}dx\le LC\int_0^L v_{2,xx}dx,$$

for some constant $C>0$, which proves the claim. Thus, from (2.25) we conclude that $v_{i,xxx}\in L^2_{(L-x)^{-1}dx}$, for $i=1,2$. Then, $\vec{v}\in D(A_2)\nsubseteq W$.

From the analysis developed above, it follows that, for $\lambda>\frac{|r|L}{18c\delta_1}$, the operator $A_2-\lambda: D(A_2)\to H$ is onto. Thus, in order to conclude the proof, is sufficient to show that $A_2-\lambda$ is dissipative in H. In fact, for any $\vec{w}\in D(A_2)$ we can integrate by parts to obtain

$$\begin{aligned}(A_2\vec{w},\vec{w})_H=&-\frac{1}{2L}\frac{b}{c}\left[w_{1,x}^2(0)+2aw_{1,x}(0)w_{2,x}(0)+\frac{1}{b}w_{2,x}^2(0)\right]\\ &-\frac{3}{2}\frac{b}{c}\int_0^L\left[w_{1,x}^2+2aw_{1,x}w_{2,x}+\frac{1}{b}w_{2,x}^2\right]\frac{1}{(L-x)^2}dx\\ &+3\frac{b}{c}\int_0^L\left[w_1^2+2aw_1w_2+\frac{1}{b}w_2^2\right]\frac{1}{(L-x)^4}dx+\frac{r}{2c}\int_0^L\frac{w_2^2}{(L-x)^2}dx.\end{aligned}$$

Proceeding as in the proof of the coercitivity of $a_\lambda(\cdot,\cdot)$ (see, for instance, (2.20)), we obtain a positive constant, denoted by m_{ϵ_1,δ_1}, such that

$$(A_2\vec{w}-\lambda\vec{w},\vec{w})_H\le -m_{\epsilon_1,\delta_1}||\vec{w}||_V^2\le 0,$$

which allows us to conclude that the operator $A_2-\lambda$ is maximal dissipative for $\lambda>\frac{|r|L}{18c\delta_1}$. Then, by Hille-Yosida theorem, it generates a strongly continuous semigroup in H. ■

The results proved above combined with the semigroup theory give us the global well-posedness for (2.1). Moreover, an additional regularity result for solutions of (2.1) is given by the next proposition.

Proposition 2.1.4. *Let H and V the Hilbert spaces defined in (2.15), and let $T>0$ be given. Then, for any $(u_0,v_0)\in H$, system (2.1) has a unique solution $(u,v)\in C([0,T];H)\cap L^2(0,T;V)$, such that*

$$||(u,v)||_{L^\infty(0,T;H)}+||(u,v)||_{L^2(0,T;V)}\leq C||(u_0,v_0)||_H, \tag{2.28}$$

where $C(T,L)$ is a positive constant.

Proof. With the notation introduced above, system (2.1) can be written in following equivalent form

$$\begin{cases} \vec{u}_t=A_2\vec{u}, \ t>0,\\ \vec{u}(0)=\vec{u}_0, \end{cases} \tag{2.29}$$

where $\vec{u}=(u,v), \vec{u}_t=(u_t,v_t)$ and $\vec{u}_0=(u_0,v_0)$. Since $D(A_2)$ is dense in H, it is sufficient to prove the result when $\vec{u}_0\in D(A_2)$. We first remark that the estimate

$$||\vec{u}||_{L^\infty(0,T;H)}\leq C||\vec{u}_0||_H \tag{2.30}$$

is a consequence of the semigroup theory. On the other hand, if $\vec{u}_0\in D(A_2)$, the solution $\vec{u}\in C([0,T];D(A_2))\cap C^1([0,T];H)$ and satisfies (2.29) in the classical sense. Then, taking the inner product in H with $\vec{u}$ and proceeding as in (2.23), we obtain positive constants m_{ϵ_1,δ_1} and δ_1, such that

$$(\vec{u}_t,\vec{u})_H=\frac{1}{2}\frac{d}{dt}||\vec{u}(t)||_H^2=-a(\vec{u},\vec{u})\leq -m_{\epsilon_1,\delta_1}||\vec{u}(t)||_V^2+\frac{|r|L}{18c\delta_1}||\vec{u}(t)||_H^2.$$

Since $V\subset H$ with continuous embedding, the following holds

$$\frac{d}{dt}||\vec{u}(t)||_H^2\leq -\left(2m_{\epsilon_1,\delta_1}-\beta\frac{|r|L}{9c\delta_1}\right)||\vec{u}(t)||_V^2,$$

for some $\beta>0$. Hence, choosing $\delta_2\leq\delta_1$ satisfying $2m_{\epsilon_1,\delta_2}-\beta\frac{|r|L}{9c\delta_2}>0$ and integrating the above estimate over $(0,T)$, we get

$$-||\vec{u}(t)||_H^2+||\vec{u}_0||_H^2\geq\left(2m_{\epsilon_1,\delta_1}-\beta\frac{|r|L}{9c\delta_1}\right)||\vec{u}||^2_{L^2(0,T;V)}.$$

Then,

$$||\vec{u}||^2_{L^2(0,T;V)}\leq C||\vec{u}_0||_H^2, \tag{2.31}$$

where C is a positive constant. From (2.30) and (2.31) we obtain (2.28). ■

2.1.2 The nonhomogeneous system

In this subsection, attention will be given to the nonhomogeneous system

$$\begin{cases} u_t+u_{xxx}+av_{xxx}=f_1 & \text{em} \ \ (0,T)\times(0,L),\\ cv_t+rv_x+v_{xxx}+bau_{xxx}=f_2 & \text{em} \ \ (0,T)\times(0,L),\\ u(0,t)=u(L,t)=u_x(L,t)=0 & \text{em} \ \ (0,T),\\ v(0,t)=v(L,t)=v_x(L,t)=0 & \text{em} \ \ (0,T),\\ u(x,0)=u_0(x), \ \ v(x,0)=v_0(x) & \text{em} \ \ (0,L). \end{cases} \tag{2.32}$$

We start with the following result:

Proposition 2.1.5. *For any $(u_0,v_0)\in[L^2_{xdx}]^2$ and $(f_1,f_2)\in L^2(0,T;[H^{-1}(0,L)]^2)$ system (2.32) has a unique solution $(u,v)\in C([0,T];[L^2_{xdx}]^2)\cap L^2(0,T;[H^1(0,L)]^2)$, such that*

$$||(u,v)||_{L^\infty(0,T;[L^2_{xdx}]^2)}+||(u,v)||_{L^2(0,T;[H^1(0,L)]^2)}\leq C\left(||(u_0,v_0)||_{[L^2_{xdx}]^2}+||(f_1,f_2)||_{L^2(0,T;[H^{-1}(0,L)]^2)}\right), \tag{2.33}$$

where $C(T,L)$ is a positive constant.

Proof. We first write system (2.32) as

$$\begin{cases} \vec{w}_t = A_1\vec{w} + \vec{f},\ t > 0, \\ \vec{w}(0) = \vec{w}_0, \end{cases} \tag{2.34}$$

where A_1 was introduced above, $\vec{w} = (u, v)$, $\vec{w}_0 = (u_0, v_0)$ and $\vec{f} = (f_1, f_2/c)$. Since $D(A_1)$ is dense in $[L^2_{xdx}]^2$, it is sufficient to prove the result when $\vec{w}_0 \in D(A_1)$ and $\vec{f} \in C([0,T]; D(A_1))$. In this case, the solution $\vec{w} \in C([0,T]; D(A_1)) \cap C^1([0,T]; [L^2_{xdx}]^2)$ and satisfies $\vec{w}_t = A_1\vec{w} + \vec{f}$ in the classical sense, which allows us to take the inner product in $[L^2(0,L)]^2$ with $(\frac{b}{c}xu, xv)$. Thus, after integration by parts over $(0,L)$, from the first and the second equations in (2.32) we obtain

$$\begin{aligned} &\frac{1}{2}\frac{b}{c}\frac{d}{dt}\int_0^L xu^2dx + \frac{3}{2}\frac{b}{c}\int_0^L u_x^2dx - \frac{ba}{c}\int_0^L xu_xv_{xx}dx - \frac{ba}{c}\int_0^L uv_{xx}dx \\ &= \frac{b}{c}\int_0^L xuf_1dx \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L xv^2dx + \frac{3}{2c}\int_0^L v_x^2dx - \frac{ba}{c}\int_0^L xv_xu_{xx}dx - \frac{ba}{c}\int_0^L vu_{xx}dx \\ &- \frac{r}{2c}\int_0^L v^2dx = \frac{1}{c}\int_0^L xvf_2dx, \end{aligned} \tag{2.36}$$

respectively. Adding identities (2.35) and (2.36) hand to hand and integrating over $(0,\tau)$, with $0 < \tau < T$, it follows that

$$\begin{aligned} &\frac{1}{2}\int_0^L x\left[\frac{b}{c}u^2 + v^2\right]dx - \frac{1}{2}\int_0^L x\left[\frac{b}{c}u_0^2 + v_0^2\right]dx \\ &+ \frac{3}{2}\frac{b}{c}\int_0^\tau\int_0^L \left[u_x^2 + \frac{1}{b}v_x^2\right]dxdt + 3\frac{ba}{c}\int_0^\tau\int_0^L u_xv_xdxdt \\ &- \frac{r}{2c}\int_0^\tau\int_0^L v^2dxdt = \frac{b}{c}\int_0^\tau\int_0^L xuf_1dxdt + \frac{1}{c}\int_0^\tau\int_0^L xvf_2dxdt. \end{aligned} \tag{2.37}$$

Since $H_0^1(0,L) \subset L^2(0,L) \subset H^{-1}(0,L)$, for all $\epsilon > 0$ we obtain a positive constant $C_\epsilon > 0$ satisfying

$$\int_0^\tau\int_0^L xw_if_idxdt = \int_0^\tau \langle f_i, xw_i\rangle_{H^{-1},H_0^1}dt \le \frac{\epsilon}{2}\int_0^\tau\int_0^L w_{i,x}^2dxdt + C_\epsilon\int_0^\tau ||f_i||^2_{H^{-1}}dt, \tag{2.38}$$

where we have set $w_1 = u$, $w_2 = v$, $i = 1,2$, and $\langle\ ,\ \rangle_{H^{-1},H_0^1}$ denotes the duality pairing between $H^{-1}(0,L)$ and $H_0^1(0,L)$. Moreover, from Young inequality, we get

$$\frac{ba}{c}\int_0^\tau\int_0^L u_xv_xdxdt \le \frac{b}{c}\left(\frac{\epsilon_1}{2}\int_0^\tau\int_0^L u_x^2dxdt + \frac{a^2}{2\epsilon_1}\int_0^\tau\int_0^L v_x^2dxdt\right), \tag{2.39}$$

for all $\epsilon_1 > 0$. Combining (2.37), (2.38) and (2.39) we obtain the following estimate

$$\begin{aligned} &\frac{1}{2}||\vec{w}(\tau)||^2_{[L^2_{xdx}]^2} - \frac{1}{2}||\vec{w}_0||^2_{[L^2_{xdx}]^2} \\ &+ \frac{3}{2}(1-\epsilon_1)\frac{b}{c}\int_0^\tau\int_0^L u_x^2dxdt + \frac{3}{2}\frac{b}{c}\left(\frac{1}{b} - \frac{a^2}{\epsilon_1}\right)\int_0^\tau\int_0^L v_x^2dxdt - \frac{r}{2c}\int_0^\tau\int_0^L v^2dxdt \\ &\le \frac{\epsilon}{2}\int_0^\tau\int_0^L \left[\frac{b}{c}u_x^2 + v_x^2\right]dt + C_\epsilon\int_0^\tau ||\vec{f}(t)||^2_{[H^{-1}]^2}dt. \end{aligned} \tag{2.40}$$

In order to conclude the proof, we apply an inequality proved in [14]: For $0 < \epsilon < L^2$, we have

$$\frac{1}{2}\int_0^\tau\int_0^L v^2 dxdt \leq \frac{\epsilon}{2}\int_0^\tau\int_0^L v_x^2 dxdt + \frac{1}{2\sqrt{\epsilon}}\int_0^\tau\int_0^L xv^2 dxdt.$$

The above inequality and (2.40) leads to

$$\begin{aligned}
&\frac{1}{2}||\vec{w}(\tau)||^2_{[L^2_{xdx}]^2} + \left[\frac{3}{2}(1-\epsilon_1) - \frac{\epsilon}{2}\right]\frac{b}{c}\int_0^\tau\int_0^L u_x^2 dxdt \\
&\qquad + \left[\frac{3}{2}\frac{b}{c}\left(\frac{1}{b} - \frac{a^2}{\epsilon_1}\right) - \frac{\epsilon}{2}\left(\frac{|r|}{c}+1\right)\right]\int_0^\tau\int_0^L v_x^2 dxdt \\
&\qquad \leq \frac{1}{2}||\vec{w}_0||^2_{[L^2_{xdx}]^2} + \frac{|r|}{2c\sqrt{\epsilon}}\int_0^\tau\int_0^L xv^2 dxdt + C_\epsilon\int_0^\tau ||\vec{f}(t)||^2_{[H^{-1}]^2}dt \\
&\qquad \leq \frac{1}{2}||\vec{w}_0||^2_{[L^2_{xdx}]^2} + \frac{|r|}{2c\sqrt{\epsilon}}\int_0^\tau ||\vec{w}(t)||^2_{[L^2_{xdx}]^2}dt + C_\epsilon\int_0^\tau ||\vec{f}(t)||^2_{[H^{-1}]^2}dt.
\end{aligned}$$

Hence, we can conclude that

$$\begin{aligned}
||\vec{w}(\tau)||^2_{[L^2_{xdx}]^2} + ||\vec{w}||^2_{L^2(0,T;[H^1(0,L)]^2)} &\leq C\left(||\vec{w}_0||^2_{[L^2_{xdx}]^2} + \int_0^\tau ||\vec{f}(t)||^2_{[H^{-1}]^2}dt\right) \\
&\quad + C'\int_0^\tau ||\vec{w}(t)||^2_{[L^2_{xdx}]^2}dt,
\end{aligned}$$

for some $C, C' > 0$. Applying Gronwall lemma we obtain (2.33). The uniqueness follows from the semigroup theory. ■

In order to obtain the controllability property we need a similar result in the spaces H and V, defined in (2.15), when $(f_1, f_2) = (\rho(x)\vec{h})_x$, with $\vec{h} := (h_1, h_2) \in L^2(0,T;[L^2(0,L)]^2)$.

Proposition 2.1.6. *For any $(u_0, v_0) \in H$ and $\vec{h} := (h_1, h_2) \in L^2(0,T;[L^2(0,L)]^2)$ set $(f_1, f_2) = (\rho(x)\vec{h})_x$, where $\rho \in C^\infty([0,L])$ is given by (1.4). Then, system (2.32) has a unique solution $(u,v) \in C([0,T];H) \cap L^2(0,T;V)$, such that*

$$||(u,v)||_{L^\infty(0,T;H)} + ||(u,v)||_{L^2(0,T;V)} \leq C\left(||(u_0,v_0)||_V + ||(h_1,h_2)||_{L^2(0,T;[L^2(0,L)]^2)}\right), \tag{2.41}$$

where $C(T,L)$ is a positive constant.

Proof. We proceed as in the proof of Proposition 2.1.5. By using the same notation, we first write system (2.32) as (2.34) considering A_2 instead of A_1. Since $D(A_2)$ is dense in H, it is sufficient to prove the result when $\vec{w}_0 \in D(A_2)$ and $\vec{h} \in C_0^\infty((0,T)\times(0,L))$, so that $\vec{f} \in C^1([0,T];H)$. In this case, the solution $\vec{w} \in C([0,T];D(A_2)) \cap C^1([0,T];H)$ and satisfies $\vec{w}_t = A_2\vec{w} + \vec{f}$ in the classical sense, which allows us to take the inner product in H with $\vec{w} = (u,v)$. Thus, arguing as in (2.23) we obtain

$$(\vec{w}_t, \vec{w})_H = -a(\vec{w},\vec{w}) + (\vec{f},\vec{w})_H \leq -m_{\epsilon_1,\delta_1}||\vec{w}||^2_V + \frac{|r|L}{18c\delta_1}||\vec{w}||^2_H + (\vec{f},\vec{w})_H, \tag{2.42}$$

where m_{ϵ_1,δ_1} and δ_1 are positive constants. On the other hand, Cauchy-Schwarz inequality gives us that

$$\begin{aligned}
|(\vec{f},\vec{w})_H| &= \left|\frac{b}{c}\int_0^L (\rho(x)h_1)_x\frac{u}{L-x}dx + \frac{1}{c}\int_0^L(\rho(x)h_2)_x\frac{v}{L-x}dx\right| \\
&\leq \frac{b}{c}\left|\int_0^L \rho(x)h_1\left(\frac{u_x}{L-x} + \frac{u}{(L-x)^2}\right)dx\right| + \frac{1}{c}\left|\int_0^L \rho(x)h_2\left(\frac{v_x}{L-x} + \frac{v}{(L-x)^2}\right)dx\right| \\
&\leq C||\vec{h}||_{[L^2(0,L)]^2}\left(||\vec{w}_x||_{[L^2(L-x)^{-2}]^2} + ||\vec{w}||_{[L^2(L-x)^{-4}]^2}\right),
\end{aligned}$$

for some $C > 0$. Moreover, from (2.16) we have

$$|(\vec{f}, \vec{w})_H| \leq C||\vec{h}||_{[L^2(0,L)]^2}||\vec{w}||_V \leq \frac{C}{2}\gamma||\vec{w}||_V^2 + C_\gamma||\vec{h}||^2_{[L^2(0,L)]^2}, \tag{2.43}$$

where $0 < \gamma < 2C^{-1}m_{\epsilon_1,\delta_1}$ and $C\gamma > 0$. Combining (2.43), (2.42) and integrating the resulting estimate over $(0, \tau)$, for $\tau \in (0, T)$, it follows that

$$||\vec{w}||^2_H + \int_0^\tau ||\vec{w}(t)||^2_V dt \leq ||\vec{w}_0||^2_H + C'' \left(\int_0^\tau ||\vec{w}(t)||^2_H dt + \int_0^\tau ||\vec{h}||^2_{[L^2(0,L)]^2} dt \right),$$

which allows us to conclude the result by applying Gronwall lemma. ■

2.2 Controllability of the linearized system

In this section we study the main controllability properties of the linearized model corresponding to (1.5)-(1.7). More precisely, we consider the following linear system with two distributed control inputs:

$$\begin{cases} u_t + u_{xxx} + av_{xxx} = f_1 = (\rho(x)h_1)_x & \text{em} \quad (0,T)\times(0,L), \\ cv_t + rv_x + v_{xxx} + bau_{xxx} = f_2 = (\rho(x)h_2)_x & \text{em} \quad (0,T)\times(0,L), \\ u(0,t) = u(L,t) = u_x(L,t) = 0 & \text{em} \quad (0,T), \\ v(0,t) = v(L,t) = v_x(L,t) = 0 & \text{em} \quad (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{em} \quad (0,L), \end{cases} \tag{2.44}$$

where ρ was defined in (1.4) and $h_1, h_2 \in L^2(0,T;L^2(0,L))$.

From now on we denote by

H the Hilbert space defined in (2.15) and by $H^* = [L^2_{(L-x)dx}]^2$ its dual space.

In order to characterize the controllability properties of the system above we use the Hilbert Uniqueness Method (HUM). Therefore, it is necessary to introduce the following adjoint system corresponding to (2.44):

$$\begin{cases} -\phi_t - \phi_{xxx} - a\psi_{xxx} = 0 & \text{in} \quad (0,T)\times(0,L), \\ -c\psi_t - r\psi_x - \psi_{xxx} - ba\phi_{xxx} = 0 & \text{in} \quad (0,T)\times(0,L), \\ \phi(0,t) = \phi(L,t) = \phi_x(0,t) = 0 & \text{in} \quad (0,T), \\ \psi(0,t) = \psi(L,t) = \psi_x(0,t) = 0 & \text{in} \quad (0,T), \\ \phi(x,T) = \phi_T(x), \quad \psi(x,T) = \psi_T(x) & \text{in} \quad (0,L). \end{cases} \tag{2.45}$$

We remark that, except for the coefficients, the change of variables $x \to L - x$ and $t \to T - t$ in (2.45) leads to the system (2.32) with $f_1 \equiv f_2 \equiv 0$. Then, in a similar way, the global well-posedness results obtained in the previous section can be proved for the adjoint system . In particular, an analogous estimate given by Proposition 2.1.5 remains valid for the solutions of (2.45):

Proposition 2.2.1. *For any $(\phi_T, \psi_T) \in H^*$ system (2.45) has a unique solution $(\phi, \psi) \in C([0,T];H^*) \cap L^2(0,T;[H^1(0,L)]^2)$, such that*

$$||(\phi,\psi)||_{L^\infty(0,T;H^*)} + ||(\phi,\psi)||_{L^2(0,T;[H^1(0,L)]^2)} \leq C||(\phi_T,\psi_T)||_{H^*}, \tag{2.46}$$

where C is a positive constant.

We can pass now to study the controllability properties of (2.44). The main result of this section reads as follows:

Theorem 2.2.2. *Let $T > 0$, $\nu \in (0, L)$ and $\rho(x)$ as in (1.4). Then, there exists a continuous operator $\Gamma : H \to L^2(0, T; L^2(0, L)) \cap L^2_{(T-t)dt}(0, T; [H^1(0, L)]^2)$, such that, for any $\vec{u}_0$, $\vec{u}_1 \in H$, the solution $\vec{w}$ of (2.44) with $\vec{h} = \Gamma(\vec{u}_1)$ satisfies $\vec{w}(T, x) = \vec{u}_1(x)$ in $(0, L)$. Note that the forcing term $\vec{f} = (\rho(x)I\vec{h})_x$ is actually a function in $L^2_{(T-t)dt}(0, T; L^2(0, L))$ supported in $(0, T) \times (L - \nu, L)$.*

Proof. Due to Proposition 2.1.6 we can assume that $(u_0, v_0) = (0, 0)$.

The proof will be done in several steps:

• Step 1: For for any $\vec{h} := (h_1, h_2) \in L^2(0, T; [L^2(0, L)]^2)$ and $\vec{\Phi}_T := (\phi_T, \psi_T) \in H^*$, it follows that

$$\langle \vec{w}(T, \cdot), \vec{\Phi}_T\rangle_{H,H^*} = -\int_0^T (\vec{h}, \rho(x)\vec{\Phi}_{c,x})_{[L^2(0,L)]^2} dt, \tag{2.47}$$

where $\vec{w} := (u, v)$ and $\vec{\Phi} := (\phi, \psi)$ denote the solutions of (2.44) and (2.45), respectively, $\vec{\Phi}_{c,x} := (\phi_x, \psi_x/c)$ and $\langle\cdot, \cdot\rangle_{H,H^*}$ denotes the duality pairing between H and H^*.

We start with more regular data $\vec{\Phi}_T \in [\mathcal{D}(0, L)]^2$ and $\vec{h} \in [\mathcal{D}(0, T) \times (0, L)]^2$. Next, we multiply the first equation in (2.44) by $\frac{b}{c}\phi$, the second one by $\frac{1}{c}\psi$ and integrate by parts over $(0, T) \times (0, L)$ to obtain

$$\begin{aligned} \frac{b}{c}\int_0^L u(T)\phi(T)dx + \frac{b}{c}\int_0^T\int_0^L (-u\phi_t - u\phi_{xxx} - va\phi_{xxx})dxdt& \\ = -\frac{b}{c}\int_0^T\int_0^L \rho(x)h_1\phi_x dxdt& \end{aligned} \tag{2.48}$$

and

$$\begin{aligned} \int_0^L v(T)\psi(T)dx + \int_0^T\int_0^L (-v\psi_t - v\frac{r}{c}\psi_x - v\frac{1}{c}\psi_{xxx} - u\frac{ba}{c}\psi_{xxx})dxdt& \\ = -\frac{1}{c}\int_0^T\int_0^L \rho(x)h_2\psi_x dxdt,& \end{aligned} \tag{2.49}$$

respectively. Adding (2.48) and (2.49) hand to hand, we get

$$(\vec{w}(T), \vec{\Phi}_T)_{[L^2(0,L)]^2} = -\int_0^T (\vec{h}, \rho(x)\vec{\Phi}_{c,x})_{[L^2(0,L)]^2} dt.$$

Then, Proposition 2.2.1 allows us to conclude the proof by using a density argument.

• Step 2: Let $\vec{\eta}_0 := (z_0, \omega_0) \in [L^2_{xdx}]^2$. Then, for all $\nu \in (0, L)$ there exists a positive constant $C = C(T, L, \nu)$, such that

$$||\vec{\eta}_0||^2_{[L^2_{xdx}]^2} \le C\int_0^T (||\vec{\eta}_x(t)||^2_{[L^2(0,\frac{\nu}{2})]^2} + ||\vec{\eta}(t)||^2_{[L^2(0,L)]^2})dt, \tag{2.50}$$

where $\vec{\eta} := (z, \omega)$ solves the problem

$$\begin{cases} z_t + z_{xxx} + a\omega_{xxx} = 0 & \text{in } (0, T) \times (0, L), \\ c\omega_t + r\omega_x + \omega_{xxx} + baz_{xxx} = 0 & \text{in } (0, T) \times (0, L), \\ z(0, t) = z(L, t) = z_x(L, t) = 0 & \text{in } (0, T), \\ \omega(0, t) = \omega(L, t) = \omega_x(L, t) = 0 & \text{in } (0, T), \\ z(x, 0) = z_0(x), \quad \omega(x, 0) = \omega_0(x) & \text{in } (0, L). \end{cases} \tag{2.51}$$

Moreover, if $\vec{\eta}_0 \in H^1_0(0, L)$, then $\vec{\eta} \in L^2(0, T; [H^2(0, L)]^2)$ and

$$\int_0^T ||\vec{\eta}(t)||^2_{[H^2(0,L)]^2} dt \le C||\vec{\eta}_0||^2_{[H^1_0(0,L)]^2}, \tag{2.52}$$

for some $C > 0$.

Before proving estimate (2.50), we remark that the existence of solutions is guaranteed by Proposition 2.1.5. Thus, we first recall an identity derived in [34]. For any $q \in C^\infty([0,T]\times[0,L])$ multiply the first equation in (2.51) by $\frac{b}{c}qz$ and the second one by $\frac{1}{c}q\omega$. After integrating by parts over $(0,T)\times(0,L)$ and adding the resulting identities we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^L q\left[\frac{b}{c}z^2+\omega^2\right]dx\Big|_0^T - \frac{1}{2}\int_0^T\int_0^L q_t\left[\frac{b}{c}z^2+\omega^2\right]dxdt\\
&+\frac{b}{2c}\int_0^T q\left[z_x^2+2az_x\omega_x+\frac{1}{b}\omega_x^2\right](0)dt - \frac{b}{2c}\int_0^T\int_0^L q_{xxx}\left[z^2+2az\omega+\frac{1}{b}\omega^2\right]dxdt\\
&+\frac{3b}{2c}\int_0^T\int_0^L q_x\left[z_x^2+2az_x\omega_x+\frac{1}{b}\omega_x^2\right]dxdt - \frac{r}{2c}\int_0^T\int_0^L q_x\omega^2dxdt = 0.
\end{aligned}\tag{2.53}$$

In particular, if we choose $q(x,t)=(T-t)p(x)$, where $p\in C^\infty([0,L])$ is nondecreasing and satisfies

$$p(x)=\begin{cases} x & \text{if } \ 0<x<\nu/4,\\ 1 & \text{if } \ \nu/2<x<L,\end{cases}$$

from (2.53) it follows that

$$\begin{aligned}
&\int_0^L p(x)\left[\frac{b}{c}z_0^2+\omega_0^2\right]dx \le \frac{1}{T}\int_0^T\int_0^L p(x)\left[\frac{b}{c}z^2+\omega^2\right]dxdt\\
&+3\frac{b}{c}\int_0^T\int_0^L p_x(x)\left[z_x^2+2az_x\omega_x+\frac{1}{b}\omega_x^2\right]dxdt - \frac{r}{c}\int_0^T\int_0^L p_x(x)\omega^2dxdt.
\end{aligned}$$

Moreover, from Young inequality we get

$$\int_0^L p(x)\left[\frac{b}{c}z_0^2+\omega_0^2\right]dx \le C(T)\int_0^T(||\vec{\eta}_x(t)||^2_{[L^2(0,\nu/2)]^2}+||\vec{\eta}(t)||^2_{[L^2(0,L)]^2})dt.$$

Hence,

$$\begin{aligned}
||\vec{\eta}_0||^2_{[L^2_{xdx}]^2} &\le C(L,\nu)\int_0^L p(x)\left[\frac{b}{c}z_0^2+\omega_0^2\right]dx\\
&\le C(T,L,\nu)\int_0^T(||\vec{\eta}_x(t)||^2_{[L^2(0,\frac{\nu}{2})]^2}+||\vec{\eta}(t)||^2_{[L^2(0,L)]^2})dt.
\end{aligned}$$

To prove (2.52) let us consider the operator $A : D(A)\subset[L^2(0,L)]^2]^2 \to [L^2(0,L)]]^2$ with domain

$$D(A)=\left\{\vec{\eta}\in[H^3\cap H_0^1(0,L)]^2; \vec{\eta}(0)=\vec{\eta}(L)=\vec{\eta}_x(L)=\vec{0}\right\}$$

and defined by

$$A\vec{\eta}=(-z_{xxx}-a\omega_{xxx}, -\frac{ab}{c}z_{xxx}-\frac{1}{c}\omega_{xxx}-\frac{r}{c}\omega_x).$$

Then, proceeding as in the proof of Proposition 2.1.5 (see also Theorem 2.2 in [34]) we obtain

$$\int_0^T||\vec{\eta}(t)||^2_{[H_0^1(0,L)]^2}dt \le C||\vec{\eta}_0||^2_{[L^2(0,L)]^2}.$$

On the other hand, if $\vec{\eta}_0\in D(A)$, Theorem 2.2 in [34] guarantees the existence of a unique solution $\vec{\eta}\in C([0,T];D(A))$ of (2.51). Let $\vec{\beta}=\vec{\eta}_t$. Then, $\vec{\beta}$ solves the system

$$\begin{cases}\vec{\beta}_t = A\vec{\beta}, \ t>0,\\ \vec{\beta}(0)=\vec{\beta}_0,\end{cases}$$

where $\vec{\beta}_0 := \vec{\beta}(0) = A\vec{\eta}_0 = (-z_{0,xxx} - a\omega_{0,xxx}, -\frac{ab}{c}z_{0,xxx} - \frac{1}{c}\omega_{0,xxx} - \frac{r}{c}\omega_{0,x}) \in [L^2(0,L)]^2$. Hence, by Theorem 2.2 in [34], it follows that there exists $C_0 > 0$, such that

$$||\vec{\beta}||_{L^2(0,T;[H_0^1(0,L)]^2)} \leq C_0||\vec{\beta}_0||_{[L^2(0,L)]^2}$$

and, therefore, $\vec{\eta} \in L^2(0,T;[H^4(0,L)]^2)$. Finally, by a standard interpolation argument (see [8]), we derive (2.52).

- Step 3: For any $\vec{\Phi}_T := (\phi_T, \psi_T) \in H^*$, the following observability inequality holds

$$||\vec{\Phi}_T||_{H^*}^2 \leq C\int_0^T ||\rho\vec{\Phi}_x||^2_{[L^2(0,L)]^2}dt, \tag{2.54}$$

where $\vec{\Phi} := (\phi, \psi)$ is the solution of (2.45) corresponding to $\vec{\Phi}_T$, and C is a positive constant.

If we set $\vec{\eta}(x,t) = \vec{\Phi}(L-x, T-t)$, inequality (2.54) is equivalent to

$$||\vec{\eta}_0||^2_{[L^2_{xdx}]^2} \leq C\int_0^T ||\rho(L-\cdot)\vec{\eta}_x||^2_{[L^2(0,L)]^2}dt, \tag{2.55}$$

where $\vec{\eta}$ solves (2.51). Therefore, we will focus on the proof of (2.55).

We argue by contradiction and suppose that (2.55) does not hold. Then, we can find a sequence $\{\vec{\eta}_{0,n}\} \subset [L^2_{xdx}]^2$, such that

$$1 = ||\vec{\eta}_{0,n}||^2_{[L^2_{xdx}]^2} \geq n\int_0^T ||\vec{\eta}_{n,x}||^2_{[L^2(0,\frac{\nu}{2})]^2}dt, \tag{2.56}$$

where, for each $n \in \mathbb{N}$, the function $\vec{\eta}_n$ denotes the solution of (2.51) with initial data $\vec{\eta}_{0,n}$. From (2.33) and (2.56) we deduce that $\{\vec{\eta}_n\}$ is bounded in $L^2(0,T;[H^1(0,L)]^2)$. Then, by (2.51) $\{\vec{\eta}_{n,t}\}$ is bounded is $L^2(0,T;[H^{-2}(0,L)]^2)$. Hence, Aubin-Lions lemma guarantees the existence of a subsequence, still denoted by the same index n, such that $\{\vec{\eta}_n\}$ converges strongly in $L^2(0,T;[L^2(0,L)]^2)$. On the other hand, (2.50) and (2.56) gives

$$||\vec{\eta}_{0,n}||^2_{[L^2_{xdx}]^2} \leq C\left[||\vec{\eta}_n||^2_{L^2(0,T;[L^2(0,L)]^2)} + \frac{1}{n}\right],$$

which shows that $\{\vec{\eta}_n\}$ is a Cauchy sequence in $[L^2_{xdx}]^2$. Denoting by $\vec{\eta}_0$ its strongly limit in $[L^2_{xdx}]^2$ we get

$$||\vec{\eta}_0||_{[L^2_{xdx}]^2} = 1.$$

Moreover, if $\vec{\eta}$ denotes the solution of (2.51) corresponding to $\vec{\eta}_0$, from Proposition 2.1.5 the following convergence holds

$$\vec{\eta}_n \to \vec{\eta} \text{ in } \ L^2(0,T;[H^1(0,L)]^2)$$

and then, from (2.56),

$$\vec{\eta}_{n,x} \to \vec{0} \text{ in } \ L^2(0,T;[H^1(0,\frac{\nu}{2})]^2).$$

The convergences above imply that $\vec{\eta}_x \equiv 0$ in $(0,T)\times(0,\nu/2)$. Hence $\vec{\eta}(x,t) = \vec{g}(t)$ in $(0,T)\times(0,\nu/2)$, for some function $\vec{g} = g(t)$. Since $\vec{\eta}$ satisfies (2.51), from the boundary condition $\vec{\eta}(0,t) = 0$ we deduce that $\vec{\eta} \equiv 0$ in $(0,T)\times(0,\nu/2)$. Then, from Holmgren theorem, $\vec{\eta} \equiv 0$ in $(0,T)\times(0,L)$, which implies that $\vec{\eta}(x,0) = 0$ contradicting $||\vec{\eta}_0||_{[L^2_{xdx}]^2} = 1$. This concludes the proof of Step 3.

It remains to apply the Hilbert Uniqueness Method (HUM). Let Λ denote the linear map

$$\begin{aligned}\Lambda : H^* &\longrightarrow H^* \\ \vec{\Phi}_T &\longmapsto (L-x)^{-1}I\vec{w}(\cdot,T),\end{aligned}$$

where $\vec{w}$ is the solution of (2.44) with $\vec{h}(x,t) = -\rho(x)(\phi_x, c\psi_x)$ and $\vec{u}_0 = (0,0)$. We first remark that Λ is continuous. Indeed, from (2.41) and (2.46) we have

$$\begin{aligned}||\Lambda(\vec{\Phi}_T)||_{H^*} = ||(L-x)^{-1}\vec{w}(\cdot,T)||_{H^*} &\leq ||\vec{w}||_{L^\infty(0,T;H)} \\ &\leq C||\vec{h}||_{L^2(0,T;[L^2(0,L)]^2)} \leq C||\vec{\phi}_x||_{L^2(0,T;[L^2(0,L)]^2)} \\ &\leq C||\vec{\Phi}_T||_{H^*}.\end{aligned}$$

Moreover, taking (2.47) and (2.54) into account it follows that

$$(\Lambda(\vec{\Phi}_T), \vec{\Phi}_T))_{H^*} = \langle \vec{w}(\cdot,T), \vec{\Phi}_T\rangle_{H,H^*} = \int_0^T ||\rho\vec{\phi}_x||^2_{[L^2(0,L)]^2} dt \geq C||\vec{\Phi}_T||^2_{H^*},$$

which shows that Λ is invertible and Λ^{-1} is continuous in H^*.

Hence, we can define the operator

$$\begin{aligned}\Gamma : H &\longrightarrow L^2(0,T;[L^2(0,L)]^2) \\ \vec{u}_1 &\longmapsto \vec{h} := -\rho(x)(\phi_x, c\psi_x),\end{aligned}$$

where $\vec{\Phi} = (\phi,\psi)$ solves the adjoint system (2.45) with $\vec{\Phi}_T = \Lambda^{-1}((L-x)^{-1}\vec{u}_1)$. Then, we have that Γ is continuous and the solution $\vec{w}$ of the problem (2.44) with $\vec{h}(x,t) = \Gamma(\vec{u}_1)$ and $\vec{u}_0 = (0,0)$ satisfies $\vec{w}(\cdot,T) = \vec{u}_1$. To show that Γ is continuous from H into $L^2_{(T-t)dt}(0,T;[H^1(0,L)]^2)$ it is sufficient to prove that

$$\int_0^T (T-t)||\vec{\Phi}||^2_{[H^2(0,L)]^2} dt \leq C||\vec{\Phi}_T||^2_{[L^2(L-x)dx]^2}, \tag{2.57}$$

for $\vec{\Phi}$ solution of (2.45). Indeed, if (2.57) holds the continuity of Λ^{-1} give us that

$$\begin{aligned}||\Gamma(u_1)||^2_{L^2_{(T-t)dt}(0,T;[H^1(0,L)]^2)} &= \int_0^T (T-t)||\rho(\phi_x, c\psi_x)^\top||^2_{[H^1(0,L)]^2} dt \\ &\leq C\int_0^T (T-t)||\vec{\Phi}||^2_{[H^2(0,L)]^2} dt \\ &\leq C||\vec{\Phi}_T||^2_{[L^2(L-x)dx]^2} \leq C||\vec{u}_1||^2_H,\end{aligned}$$

for some $C>0$. If we make the change of variables $x \longmapsto (L-x)$ and $t \longmapsto (T-t)$ estimate (2.57) is equivalent to

$$\int_0^T t||\vec{\eta}(t)||^2_{[H^2(0,L)]^2} dt \leq C||\vec{\eta}_0||^2_{[L^2_{xdx}]^2}, \tag{2.58}$$

where $\vec{\eta}$ solves (2.51). Estimate (2.58) can be proved as follows, combining (2.52) and Fubini theorem:

$$\int_0^T s||\vec{\eta}(s)||^2_{[H^2]^2} ds = \int_0^T \left(\int_t^T ||\vec{\eta}(s)||^2_{[H^2]^2} ds\right) dt \leq C\int_0^T ||\vec{\eta}(t)||^2_{[H^1_0(0,L)]^2} dt \leq C||\vec{\eta}_0||_{[L^2_{xdx}]^2}.$$

The proof is now complete. ■

2.3 Controllability of the nonlinear system

This section is devoted do analyze the local exact controllability properties of the full system in the space $H = [L^2_{(L-x)^{-1}dx}]^2$ defined in (2.15).

We note that the solutions $\vec{w} = (u,v)$ of (1.5)-(1.7) can be written as

$$\vec{w} = \vec{u}_L + \vec{y} + \vec{z},$$

where $\vec{u}_L$ is the solution of (2.1) with initial data $\vec{u}_0 \in H$, $\vec{y} = (y_1, y_2)$ solves the problem

$$\begin{cases} y_{1,t} + y_{1,xxx} + ay_{2,xxx} = (\rho(x)h_1)_x & \text{em} \quad (0,T) \times (0,L), \\ cy_{2,t} + ry_{2,x} + y_{2,xxx} + bay_{1,xxx} = (\rho(x)h_2)_x & \text{em} \quad (0,T) \times (0,L), \\ y_1(0,t) = y_1(L,t) = y_{1,x}(L,t) = 0 & \text{em} \quad (0,T), \\ y_2(0,t) = y_2(L,t) = y_{2,x}(L,t) = 0 & \text{em} \quad (0,T), \\ y_1(x,0) = 0, \quad y_2(x,0) = 0 & \text{em} \quad (0,L), \end{cases} \tag{2.59}$$

with $\vec{h} = (h_1, h_2) \in L^2(0,T; [L^2(0,L)]^2)$, and $\vec{z} = (z_1, z_2)$ is solution of

$$\begin{cases} z_{1,t} + z_{1,xxx} + az_{2,xxx} = G_1 & \text{em} \quad (0,T) \times (0,L), \\ cz_{2,t} + rz_{2,x} + z_{2,xxx} + baz_{1,xxx} = G_2 & \text{em} \quad (0,T) \times (0,L), \\ z_1(0,t) = z_1(L,t) = z_{1,x}(L,t) = 0 & \text{em} \quad (0,T), \\ z_2(0,t) = z_2(L,t) = z_{2,x}(L,t) = 0 & \text{em} \quad (0,T), \\ z_1(x,0) = 0, \quad z_2(x,0) = 0 & \text{em} \quad (0,L), \end{cases} \tag{2.60}$$

with $\vec{G} = (G_1, G_2) = (-uu_x - a_1 vv_x - a_2(uv)_x, -vv_x - ba_2 uu_x - ba_1(uv)_x)$.

The following result will be needed to study the solutions of (2.60). In order to prove it we recall that the space V defined in (2.15) can be written as $V = [\mathcal{V}]^2$, where the space $\mathcal{V} := \left\{ u \in H_0^1(0,L); u_x \in L^2_{(L-x)^{-2}dx} \right\}$, endowed with the norm $||u||_{\mathcal{V}} := ||(L-x)^{-1} u_x||_{L^2}$.

Proposition 2.3.1. *Let H and V be as in (2.15).*

(i) If $u, v \in L^2(0,T; \mathcal{V})$, then $(uu_x, vv_x, uv_x, u_x v) \in L^1(0,T; [L^2_{(L-x)^{-1}dx}]^4)$ and the map

$$(u,v) \in L^2(0,T;V) \to (uu_x, vv_x, uv_x, u_x v) \in L^1(0,T; [L^2_{(L-x)^{-1}dx}]^4)$$

is continuous. Moreover, there exists a constant $C > 0$, such that

$$||(uu_x, vv_x, uv_x, u_x v)||_{L^1(0,T;[L^2_{(L-x)^{-1}dx}]^4)} \le C ||u||_{L^2(0,T;\mathcal{V})} ||v||_{L^2(0,T;\mathcal{V})}.$$

(ii) For $\vec{G} \in L^1(0,T;H)$, the mild solution $\vec{z}$ of (2.60), given by

$$\vec{z}(x,t) = \int_0^t S_2(t-s) \vec{G}(s) ds,$$

where $(S_2(t))_{t \ge 0}$ is the semigroup given by Theorem 2.1.3, satisfies

$$\vec{z} \in C([0,T];H) \cap L^2(0,T;V) =: X$$

and the following estimate holds

$$||\vec{z}||_{L^\infty(0,T;H)} + ||\vec{z}||_{L^2(0,T,V)} \le ||\vec{G}||_{L^1(0,T;H)}.$$

Proof. For $u, v \in \mathcal{V}$ we first apply inequality (2.17) to obtain

$$||uv_x||_{L^2_{(L-x)^{-1}dx}} \le ||u||_{L^\infty} ||(L-x)^{-\frac{1}{2}} v_x||_{L^2} \le C ||u||_{\mathcal{V}} ||v||_{\mathcal{V}}.$$

The remaining terms can be estimated in a similar way and (i) follows.

To prove (ii), we first assume that $\vec{G} \in C^1([0,T];H)$. In this case, the solution $\vec{z} \in C([0,T]; D(A_2)) \cap C^1([0,T];H)$ and, from the classical semigroup theory, the following estimate holds

$$||\vec{z}||_{L^\infty(0,T;H)} \le ||\vec{G}||_{L^1(0,T;H)}. \tag{2.61}$$

On the other hand, if we write (2.60) as (2.34), with A_2 and $\vec{G}$ instead of A_1 and $\vec{f}$, we have that $\vec{z}_t = A_2\vec{z} + \vec{G}$ is satisfied in the classical sense. Then, we can take the inner product in H with $\vec{z}$ to obtain

$$(\vec{z}_t, \vec{z})_H = \frac{1}{2}\frac{d}{dt}||\vec{z}(t)||_H^2 \leq -C||\vec{z}(t)||_V^2 + C'||\vec{z}(t)||_H^2 + (\vec{G}, \vec{z})_H,$$

where C, C' are positive constants. Integrating the estimate above over $(0,T)$, from (2.61) it follows that

$$\begin{aligned}||\vec{z}||^2_{L^\infty(0,T;H)} + ||\vec{z}||^2_{L^2(0,T;V)} &\leq C\left(\int_0^t ||\vec{z}(s)||_H^2 ds + ||\vec{z}||_{L^\infty(0,T;H)}||\vec{G}||_{L^1(0,T;H)}\right)\\ &\leq C(T)||\vec{G}||^2_{L^1(0,T;H)},\end{aligned}$$

for some $C(T) > 0$, which proves the result when $\vec{G} \in C^1([0,T]; H)$. Then, by a density argument we obtain the result for $\vec{G} \in L^1(0,T;H)$. ■

We remark that, for all $\vec{u}, \vec{v} \in L^2(0,T;V)$ and $\vec{G}$ given in (2.60), Proposition 2.3.1 guarantees the existence of a constant $C > 0$, such that

$$||\vec{G}(\vec{u})||_{L^1(0,T;H)} \leq C||\vec{u}||^2_{L^2(0,T;V)}, \tag{2.62}$$

$$||\vec{G}(\vec{u}) - \vec{G}(\vec{v})||_{L^1(0,T;H)} \leq C\left(||\vec{u}||^2_{L^2(0,T;V)} + ||\vec{v}||^2_{L^2(0,T;V)}\right)||\vec{u} - \vec{v}||_{L^2(0,T;V)}. \tag{2.63}$$

Then, for some $R > 0$, to be defined latter, we introduce the ball

$$\mathcal{B}_R = \{\vec{u} \in L^2(0,T;V) : ||\vec{u}||_{L^2(0,T;V)} \leq R\}$$

and the operator $\mathcal{T} : L^2(0,T;V) \to X$, as follows

$$\mathcal{T}(\vec{u}) = S_2(t)\vec{u}_0 + \int_0^t S_2(t-s)\left[(\rho(x)I\vec{h})_x + \vec{G}(\vec{u}(s))\right] ds,$$

for X given in Proposition 2.3.1. Since $\mathcal{T}(\vec{u}) = \vec{u}_L + \vec{y} + \vec{z}$, for all $\vec{u}, \vec{v} \in \mathcal{B}_R$, from Propositions 2.1.4, 2.1.6, 2.3.1 and estimates (2.62)-(2.63), we obtain constants $C_1, C_2 > 0$ (which dos not depend on $||\vec{u}_0||_H$ and $||\vec{h}||_{L^2(0,T;[L^2(0,L)]^2)}$), such that

$$\begin{aligned}&||\mathcal{T}(\vec{u})||_{L^2(0,T;V)} \leq C_1\left(||\vec{u}_0||_H + ||\vec{h}||_{L^2(0,T;[L^2(0,L)]^2)}\right) + C_2R^2,\\ &||\mathcal{T}(\vec{u}) - \mathcal{T}(\vec{v})||_{L^2(0,T;V)} \leq 2C_2R||\vec{u} - \vec{v}||_{L^2(0,T;V)}.\end{aligned}$$

Choosing $R = (4C_2)^{-1}$ and $\delta_R = (16C_1C_2)^{-1}$, if $\vec{u}_0$ and $\vec{h}$ satisfy

$$||\vec{u}_0||_H \leq \delta_R, \quad ||\vec{h}||_{L^2(0,T;[L^2(0,L)]^2)} \leq \delta_R,$$

the corresponding solutions $\vec{u}, \vec{v} \in \mathcal{B}_R$ and

$$\begin{aligned}&||\mathcal{T}(\vec{u})||_{L^2(0,T;V)} \leq R,\\ &||\mathcal{T}(\vec{u}) - \mathcal{T}(\vec{v})||_{L^2(0,T;V)} \leq \frac{1}{2}||\vec{u} - \vec{v}||_{L^2(0,T;V)}.\end{aligned}$$

Then, by Banach fixed-point Theorem the operator $\mathcal{T}$ has a fixed point in $\mathcal{B}_R$.

From the discussion above, we obtain the following result:

Theorem 2.3.2. *There exists $\delta > 0$, such that for any $\vec{u}_0 \in H$ and $\vec{h} \in L^2(0,T;[L^2(0,L)]^2)$ satisfying*

$$||\vec{u}_0||_H \leq \delta \quad \text{and} \quad ||\vec{h}||_{L^2(0,T;[L^2(0,L)]^2)} \leq \delta,$$

problem (1.5)-(1.7) has a unique solution $\vec{w} \in X$.

We are now in position to show the main result of this work. Let us first introduce the operators $\Theta_1 : L^2(0,T;[L^2(0,L)]^2) \to X$, with $\Theta_1(\vec{h}) = \vec{y}$, and $\Theta_2 : L^1(0,T;H) \to X$, with $\Theta_2(\vec{G}) = \vec{z}$, where $\vec{y}$ and $\vec{z}$ are the solutions of (2.59) and (2.60), respectively. Due to Propositions 2.1.6 and 2.3.1 the operators Θ_1 and Θ_2 are well defined and continuous. Then, we have the following local exact controllability result:

Theorem 2.3.3. *Let $T > 0$. Then, there exists $\delta > 0$, such that, for any $\vec{u}_0, \vec{u}_1 \in H$ satisfying*

$$||\vec{u}_0||_H \leq \delta \quad \textit{and} \quad ||\vec{u}_1||_H \leq \delta,$$

one can find a control function $\vec{h} \in L^2(0,T;[L^2(0,L)]^2)$, such that the solution $\vec{w} \in X$ of (1.5)-(1.7) satisfies $\vec{w}(\cdot,T) = \vec{u}_1$ in $(0,L)$.

Proof. We apply the Banach fixed-point theorem. Let $\vec{G}(\vec{u})$ as in (2.60) and $\mathcal{F}$ the nonlinear map

$$\mathcal{F} : L^2(0,T;V) \to X,$$

defined by $\mathcal{F}(\vec{u}) = \vec{u}_L + \Theta_1 \circ \Gamma\left[\vec{u}_1 - \vec{u}_L(\cdot,T) - \Theta_2(\vec{G}(\vec{u}))(\cdot,T)\right] + \Theta_2(\vec{G}(\vec{u}))$. Here, $\vec{u}_L$ is the solution of (2.1) with initial data $\vec{u}_0 \in H$, Θ_1 and Θ_2 are defined as above and Γ is the control operator given by Theorem 2.2.2.

If $\vec{w}$ is a fixed point of $\mathcal{F}$, then $\vec{w}$ is a solution of (1.5)-(1.7) with control $\vec{h}$ given by $\vec{h} = \Gamma\left[\vec{u}_1 - \vec{u}_L(\cdot,T) - \Theta_2(\vec{G}(\vec{w}))(\cdot,T)\right]$, which satisfies

$$\vec{w}(\cdot,T) = \vec{u}_1.$$

For some $R > 0$, to be chosen later, we define the closed ball $\mathcal{B}_R$ in $L^2(0,T;V)$. Then, arguing as in the proof of Theorem 2.3.2 and using the continuity of the control operator Γ, we obtain positive constants $C'_1, C'_2 > 0$, such that, for all $\vec{u}, \vec{v} \in \mathcal{B}_R$, the following estimates holds

$$\begin{aligned}
||\mathcal{F}(\vec{u})||_{L^2(0,T;V)} &\leq C'_1\left(||\vec{u}_0||_H + ||\vec{u}_1||_H\right) + C'_2 R^2, \\
||\mathcal{F}(\vec{u}) - \mathcal{F}(\vec{v})||_{L^2(0,T;V)} &\leq 2C'_2 R||\vec{u} - \vec{v}||_{L^2(0,T;V)}.
\end{aligned}$$

Choosing $R = (4C'_2)^{-1}$ and $\delta_R = (16C'_1C'_2)^{-1}$ it follows that, for any $\vec{u}_0, \vec{u}_1 \in H$ satisfying

$$||\vec{u}_0||_H \leq \delta_R \;\; \text{e} \;\; ||\vec{u}_1||_H \leq \delta_R,$$

the operator $\mathcal{F}$ is a contraction which maps the closed ball $\mathcal{B}_R$ into itself. Then, by Banach fixed point theorem, $\mathcal{F}$ has a fixed point in $\mathcal{B}_R$. The proof is now complete. ∎

Chapter 3

On the lack of controllability of a higher-order regularized long-wave system

Considered here is a class of two higher-order Benjamin-Bona-Mahony type equations. Our aim is to investigate the controllability properties of the linearized model posed on a bounded interval. More precisely, we study whether the solutions can be driven to a given state at a given final time by means of controls acting on the right endpoint of the interval. We show that the model is approximately controllable but not spectrally controllable. This means that any state can be steered arbitrarily close to another state, but no finite linear combination of eigenfunctions, other than zero, can be steered to zero. Our proofs relies strongly on a careful spectral analysis of the operator associated with the state equations.

3.1 Global well-posedness

In this section we show the well-posedness of the homogeneous and non-homogeneous systems associated with (1.8).

3.1.1 The homogeneous system

Let us first consider the following homogeneous system

$$\begin{cases} \eta_t + \omega_x + a\omega_{xxx} - b\eta_{txx} + b_1\eta_{txxxx} = 0 & \text{for } x \in (0,L),\ t>0, \\ \omega_t + \eta_x + c\eta_{xxx} - d\omega_{txx} + d_1\omega_{txxxx} = 0 & \text{for } x\in(0,L), t>0, \\ \eta(t,0) = \eta(t,L) = 0 & \text{for } \ t \geq 0, \\ \omega(t,0) = \omega(t,L) = 0 & \text{for } \ t\geq 0, \\ \eta_x(t,0) = \eta_x(t,L) = 0 & \text{for } \ t\geq 0, \\ \omega_x(t,0) = \omega_x(t,L) = 0 & \text{for } \ t \geq 0, \\ \eta(0,x) = \eta^0(x); \quad \omega(0,x) = \omega^0(x) & \text{for } x\in(0,L). \end{cases} \tag{3.1}$$

System (3.1) can be written in the following vectorial form

$$\begin{pmatrix} \eta \\ \omega \end{pmatrix}_t (t) + \mathcal{A}\begin{pmatrix} \eta \\ \omega \end{pmatrix}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \eta \\ \omega \end{pmatrix}(0) = \begin{pmatrix} \eta^0 \\ \omega^0 \end{pmatrix},$$

where $\mathcal{A}$ is the operator belonging to $\mathcal{L}\left([H_0^2(0,L)]^2\right)$ defined by

$$\mathcal{A} = \begin{pmatrix} 0 & (1-b\partial_x^2 + b_1\partial_x^4)^{-1}(\partial_x + a\partial_x^3) \\ (1-d\partial_x^2+d_1\partial_x^4)^{-1}(\partial_x + c\partial_x^3) & 0 \end{pmatrix}.$$

Recall that, for $\alpha, \beta > 0$ the operator $(1-\alpha\partial_x^2+\beta\partial_x^4)^{-1}$ is defined in the following way:

$$(1-\alpha\partial_x^2+\beta\partial_x^4)^{-1}\phi = v \Leftrightarrow \begin{cases} v-\alpha v_{xx}+\beta v_{xxxx}=\phi & \text{in } (0,L) \\ \dfrac{\partial^r v}{\partial x^r}(0)=\dfrac{\partial^r v}{\partial x^r}(L), & r=0,1. \end{cases} \tag{3.2}$$

Then, if $\phi \in L^2(0,L)$, the elliptic equation (3.2) has an unique solution $v \in H^4(0,L)\cap H_0^2(0,L)$, the operator $(1-\alpha\partial_x^2+\beta\partial_x^4)^{-1}$ is a well-defined, compact operator in $L^2(0,L)$.

Remark 3.1.1. *Due to the regularizing effect of the operators*

$$(1-b\partial_x^2+b_1\partial_x^4)^{-1} \quad \text{and} \quad (1-d\partial_x^2+d_1\partial_x^4)^{-1}$$

it follows that $\mathcal{A}$ takes values in $[H^3(0,L)\cap H_0^2(0,L)]^2$ which is compactly embedded in $[H_0^2(0,L)]^2$. Hence $\mathcal{A}$ is compact.

From the classical semigroup theory, we have the following well-posedness result:

Theorem 3.1.1. *Let $b,d,b_1,d_1>0$ and $a=c>0$. For any $(\eta^0,\omega^0)\in[H_0^2(0,L)]^2$, system (3.1) has a unique classical solution $(\eta,\omega)\in C(\mathbb{R};[H_0^2(0,L)]^2)$. Moreover, $(\eta,\omega)\in C^\omega(\mathbb{R};[H_0^2(0,L)]^2)$, the class of analytic functions in $t\in\mathbb{R}$ with values in $[H_0^2(0,L)]^2$.*

Proof. We first show that $\mathcal{A}$ is a skew-adjoint operator in $[H_0^2(0,L)]^2$. For any $\varphi_i,\psi_i\in H_0^2\cap H^4(0,L)$, $i=1,2$, and some integrations by parts, we have from (1.9) that

$$\begin{aligned}
\left\langle \mathcal{A}\begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix}, \begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix}\right\rangle &= \left\langle \begin{pmatrix}(1-b\partial_x^2+b_1\partial_x^4)^{-1}(\partial_x+a\partial_x^3)\psi_1\\ (1-d\partial_x^2+d_1\partial_x^4)^{-1}(\partial_x+c\partial_x^3)\varphi_1\end{pmatrix}, \begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix}\right\rangle \\
&= \int_0^L(\partial_x+a\partial_x^3)\psi_1\varphi_2dx+\int_0^L(\partial_x+c\partial_x^3)\varphi_1\psi_2dx \\
&= -\int_0^L\psi_1(\partial_x+a\partial_x^3)\varphi_2dx-\int_0^L\varphi_1(\partial_x+c\partial_x^3)\psi_2dx \\
&= -\int_0^L\psi_1(1-d\partial_x^2+d_1\partial_x^4)(1-d\partial_x^2+d_1\partial_x^4)^{-1}(\partial_x+a\partial_x^3)\varphi_2dx \\
&\quad -\int_0^L\varphi_1(1-b\partial_x^2+b_1\partial_x^4)(1-b\partial_x^2+b_1\partial_x^4)^{-1}(\partial_x+c\partial_x^3)\psi_2dx \\
&= -\left\langle \begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix}, \begin{pmatrix}(1-b\partial_x^2+b_1\partial_x^4)^{-1}(\partial_x+c\partial_x^3)\psi_2\\ (1-d\partial_x^2+d_1\partial_x^4)^{-1}(\partial_x+a\partial_x^3)\varphi_2\end{pmatrix}\right\rangle \\
&= -\left\langle \begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix}, \mathcal{A}\begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix}\right\rangle.
\end{aligned}$$

By a density argument, the identity above holds for any $\varphi_i,\psi_i\in H_0^2(0,L)$, $i=1,2$. Then, Stone Theorem guarantees that $\mathcal{A}$ generates a group of isometries $\{S(t)\}_{t\in\mathbb{R}}$ in $[H_0^2(0,L)]^2$, which allows us to obtain the well-posedness result. The second part of the Theorem follows from the fact that $\mathcal{A}$ is a compact operator in $[H_0^2(0,L)]^2$ (see, for instance, [24, Theorem 11.4.1, Chap. XI]). ■

3.1.2 The nonhomogeneous system

In this subsection, attention will be given to the full system (1.8). We begin with the following result:

Theorem 3.1.2. *Let $b,d,b_1,d_1>0$ and $a=c>0$. For any $(\eta^0,\omega^0)\in[H_0^2(0,L)]^2$, and $(f_1,g_1),(f_2,g_2)\in[C_0^1(0,\infty)]^2$, system (1.8) has a unique classical solution $(\eta,\omega)\in C([0,\infty);[H_0^2(0,L)]^2)$.*

Proof. Let $\varphi_i, \psi_i \in C^\infty([0,L])$, $i = 1,2$, be functions, such that

$$\begin{aligned}&\varphi_1(0) = \psi_1(0) = \varphi_{1x}(0) = \psi_{1x}(0) = \varphi_{1x}(L) = \psi_{1x}(L) = 0,\\ &\phi_1(L) = \psi_1(L) = -1\end{aligned}$$

and

$$\begin{aligned}&\varphi_2(0) = \psi_2(0) = \varphi_2(L) = \psi_2(L) = \varphi_{2x}(0) = \psi_{2x}(0) = 0,\\ &\phi_{2x}(L) = \psi_{2x}(L) = -1.\end{aligned}$$

For instance,

$$\varphi_1(x) = \psi_1(x) = -\frac{3}{L^2}x^2 + \frac{2}{L^3}x^3 \text{ and } \varphi_2(x) = \psi_2(x) = \frac{1}{L}x^2 - \frac{1}{L^2}x^3$$

satisfy the conditions above. Then, if we consider the change of functions

$$\begin{pmatrix} z \\ m \end{pmatrix} = \begin{pmatrix} \eta \\ \omega \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(t)\varphi_1(x) + f_2(t)\varphi_2(x) \\ g_1(t)\psi_1(x) + g_2(t)\psi_2(x) \end{pmatrix}, \tag{3.3}$$

where $(u,v) \in C([0,\infty); [H_0^2(0,L)]^2)$ is the solution of the system

$$\begin{cases} u_t + v_x + av_{xxx} - bu_{txx} + b_1 u_{txxxx} = 0 & \text{for } x \in (0,L),\ t > 0,\\ v_t + u_x + cu_{xxx} - dv_{txx} + d_1 v_{txxxx} = 0 & \text{for } x \in (0,L), t > 0,\\ u(t,0) = u(t,L) = 0 & \text{for } \ t \geq 0,\\ v(t,0) = v(t,L) = 0 & \text{for } \ t \geq 0,\\ u_x(t,0) = u_x(t,L) = 0 & \text{for } \ t \geq 0,\\ v_x(t,0) = v_x(t,L) = 0 & \text{for } \ t \geq 0,\\ u(0,x) = \eta^0(x); \quad v(0,x) = \omega^0(x) & \text{for } x \in (0,L), \end{cases}$$

given by Theorem 3.1.1, the couple (z,m) solves the problem

$$\begin{cases} z_t + m_x + am_{xxx} - bz_{txx} + b_1 z_{txxxx} = F & \text{for } x \in (0,L),\ t > 0,\\ m_t + z_x + cz_{xxx} - dm_{txx} + d_1 m_{txxxx} = G & \text{for } x \in (0,L), t > 0,\\ z(t,0) = z(t,L) = 0 & \text{for } \ t \geq 0,\\ m(t,0) = m(t,L) = 0 & \text{for } \ t \geq 0,\\ z_x(t,0) = z_x(t,L) = 0 & \text{for } \ t \geq 0,\\ m_x(t,0) = m_x(t,L) = 0 & \text{for } \ t \geq 0,\\ z(0,x) = 0; \quad m(0,x) = 0 & \text{for } x \in (0,L), \end{cases} \tag{3.4}$$

with F and G given by

$$\begin{aligned} \begin{pmatrix} F(t,x) \\ G(t,x) \end{pmatrix} &= \begin{pmatrix} f_1'(t)[\varphi_1(x) - b\varphi_1^{(2)}(x) + b_1\varphi_1^{(4)}(x)] + g_1(t)[\psi_1'(x) + a\psi_1^{(3)}(x)] \\ g_1'(t)[\psi_1(x) - d\psi_1^{(2)}(x) + d_1\psi_1^{(4)}(x)] + f_1(t)[\varphi_1'(x) + a\varphi_1^{(3)}(x)] \end{pmatrix}\\ &+ \begin{pmatrix} f_2'(t)[\varphi_2(x) - b\varphi_2^{(2)}(x) + b_1\varphi_2^{(4)}(x)] + g_2(t)[\psi_2'(x) + a\psi_2^{(3)}(x)] \\ g_2'(t)[\psi_2(x) - d\psi_2^{(2)}(x) + d_1\psi_2^{(4)}(x)] + f_2(t)[\varphi_2'(x) + a\varphi_2^{(3)}(x)] \end{pmatrix}\\ &\in [C([0,\infty) \times [0,L])]^2, \end{aligned}$$

where (i), $i = 2,3,4$, denotes the derivative of order i. With the notation introduced in the previous section, system (3.4) can be written as an abstract evolution equation as follows

$$\begin{cases} W_t + \mathcal{A}W = \mathcal{H} \\ W(0) = 0, \end{cases}$$

where $W=(z,m)$ and $\mathcal{H}=\mathcal{A}_0(F,G)\in L^1(0,\infty;[H_0^2\cap H^4(0,L)]^2)$, being $\mathcal{A}_0:[L^2(0,L)]^2\longrightarrow[H_0^2\cap H^4(0,L)]^2$ defined by

$$\mathcal{A}_0=\begin{pmatrix} 0 & (1-b\partial_x^2+b_1\partial_x^4)^{-1} \\ (1-d\partial_x^2+d_1\partial_x^4)^{-1} & 0 \end{pmatrix}. \tag{3.5}$$

Since $\mathcal{A}$ generates a group of isometries in $[H_0^2(0,L)]^2$, we have that system (3.4) has a unique solution $W=(z,m)\in C([0,\infty);[H_0^2(0,L)]^2)$. Then, returning to (3.3), we conclude the proof. ■

Using the previous well-posedness results, we will study the existence of solutions of the system (1.8) in the sense of transposition:

Definition 3.1.2. *Let $(\eta^0,\omega^0)\in[H^{-2}(0,L)]^2$ and $(f_1,g_1),(f_2,g_2)\in[H^1(0,T)]^2$. A solution of system (1.8) is a couple $(\eta,\omega)\in C([0,T];[L^2(0,L)]^2)$, such that, for any $(h,k)\in L^1(0,T;[L^2(0,L)]^2)$, satisfies*

$$\begin{aligned}\int_0^T\int_0^L(\eta h+\omega k)dxdt &+ \left\langle\begin{pmatrix}\eta^0\\ \omega^0\end{pmatrix},\begin{pmatrix}u(0)\\ v(0)\end{pmatrix}\right\rangle_{[H^{-2}(0,L)]^2,[H_0^2(0,L)]^2}\\ &=\int_0^T f_1(t)[b_1u_{txxx}+cv_{xx}](t,L)dt\\ &+\int_0^T g_1(t)[d_1v_{txxx}+au_{xx}](t,L)dt\\ &-b_1\int_0^T f_2(t)u_{txx}(t,L)dt-d_1\int_0^T g_2(t)v_{txx}(t,L)dt,\end{aligned} \tag{3.6}$$

where (u,v) is solution of the adjoin system

$$\begin{cases} u_t+v_x+cv_{xxx}-bu_{txx}+b_1u_{txxxx}=h & \text{for } x\in(0,L),\ t>0,\\ v_t+u_x+au_{xxx}-dv_{txx}+d_1v_{txxxx}=k & \text{for } x\in(0,L), t>0,\\ u(t,0)=u(t,L)=0 & \text{for } \ t\geq 0,\\ v(t,0)=v(t,L)=0 & \text{for } \ t\geq 0,\\ u_x(t,0)=u_x(t,L)=0 & \text{for } \ t\geq 0,\\ v_x(t,0)=v_x(t,L)=0 & \text{for } \ t\geq 0,\\ u(T,x)=0;\quad v(T,x)=0 & \text{for } x\in(0,L). \end{cases} \tag{3.7}$$

The existence of solutions for system (3.7) can be proved following the arguments used in the proof of Theorem 3.1.2. Moreover, due to the regularizing effect of the operator $(1-\alpha\partial_x^2+\beta\partial_x^4)^{-1}$, with $\alpha,\beta>0$, we obtain the following result:

Theorem 3.1.3. *If $(h,k)\in L^1(0,T;[L^2(0,L)]^2)$, system (3.7) has a unique solution $(u,v)\in C([0,T];[H_0^2(0,L)]^2)$. Moreover,*

$$||(u,v)||_{L^1(0,T;[H_0^2\cap H^3(0,L)]^2)}+||(u_t,v_t)||_{L^1(0,T;[H_0^2\cap H^4(0,L)]^2)}\leq C||(h,k)||_{L^1(0,T;[L^2(0,L)]^2)}, \tag{3.8}$$

for some constant $C>0$.

Proof. System (3.7) can be written as an abstract evolution equation as follows

$$\begin{cases} W_t+\mathcal{A}W=\mathcal{F}\\ W(0)=0,\end{cases}$$

where $W=(u,v)$ and $\mathcal{F}=\mathcal{A}_0(h,k)\in L^1(0,T;[H_0^2\cap H^4(0,L)]^2)$, being $\mathcal{A}_0:[L^2(0,L)]^2\longrightarrow[H_0^2\cap H^4(0,L)]^2$ defined by (3.5). Since $\mathcal{A}$ generates a group of isometries in

$[H_0^2(0,L)]^2$, we have that system (3.7) has a unique solution $W=(u,v)\in C([0,T];[H_0^2(0,L)]^2)$. Moreover, using the equations in (3.7), we deduce that $(u_t,v_t)\in L^1(0,T;[H_0^2\cap H^3(0,L)]^2)$ and estimate (3.8) holds. Indeed, first, observe that each term of the equations in (3.7) belongs to $L^2(0,T;H^{-2}(0,L))$. Thus, scaling the first equation by u and the second by v we obtain

$$\frac{1}{2}\frac{d}{dt}||(u(t,\cdot),v(t,\cdot))||^2_{[H_0^2(0,L)]^2}=\int_0^L(hu+kv)dx. \tag{3.9}$$

Integrating the above identity from t up to T, from Young inequality it follows that

$$\begin{aligned}||(u(t,\cdot),v(t,\cdot))||^2_{[H_0^2(0,L)]^2}\leq{}& C\Big(||h||_{L^1(0,T;L^2(0,L))}||u||_{C([0,T];L^2(0,L))}\\&\qquad+||k||_{L^1(0,T;L^2(0,L))}||v||_{C([0,T];L^2(0,L))}\Big)\\ \leq{}& C\Big(\frac{1}{2\epsilon}||(h,k)||^2_{L^1(0,T;[L^2(0,L)]^2)}+\frac{\epsilon}{2}||(u,v)||^2_{C([0,T];[L^2(0,L)]^2)}\Big),\end{aligned} \tag{3.10}$$

for any $\epsilon>0$, where C is a positive constant. Then, by choosing $\epsilon>0$ sufficiently small in (3.10) we obtain

$$||(u,v)||_{C([0,T];[H_0^2(0,L)]^2)}\leq C||(h,k)||_{L^1(0,T;[L^2(0,L)]^2)}, \tag{3.11}$$

for some $C>0$. On the other hand, due to the regularizing effect of the operator $(1-\alpha\partial_x^2+\beta\partial_x^4)^{-1}$, $\alpha,\beta>0$, it follows

$$(1-b\partial_x^2+b_1\partial_x^4)^{-1}h(t,\cdot),(1-d\partial_x^2+d_1\partial_x^4)^{-1}k(t,\cdot)\in H^4(0,L)$$

and the operator $\mathcal{A}$ takes values in $[H_0^2\cap H^3(0,L)]^2$, which is compactly embedded in $[H_0^2(0,L)]^2$. Thus, combining (3.11) and the equations in (3.7), it follows that

$$\begin{aligned}&||(u_t(t,\cdot),v_t(t,\cdot))||_{[H^3(0,L)]^2}\leq\\&\qquad||((1-b\partial_x^2+b_1\partial_x^4)^{-1}(\partial_x+a\partial_x^3)u,(1-d\partial_x^2+d_1\partial_x^4)^{-1}(\partial_x+a\partial_x^3)v)||_{[H^3(0,L)]^2}\\&\qquad+C||((1-b\partial_x^2+b_1\partial_x^4)^{-1}h,(1-d\partial_x^2+d_1\partial_x^4)^{-1}k)||_{[H^4(0,L)]^2}\\&\qquad\leq C\left(||((\partial_x+a\partial_x^3)u,(\partial_x+a\partial_x^3)v)||_{[H^{-1}(0,L)]^2}+||(h,k)||_{[L^2(0,L)]^2}\right)\\&\qquad\leq C\left(||(u,v)||_{[H_0^2(0,L)]^2}+||(h,k)||_{[L^2(0,L)]^2}\right)\\&\qquad\leq C\left(||(u,v)||_{C([0,T];[H_0^2(0,L)]^2)}+||(h,k)||_{[L^2(0,L)]^2}\right).\end{aligned} \tag{3.12}$$

By integrating (3.12) on $(0,T)$ we get $(u_t,v_t)\in L^1(0,T;[H_0^2\cap H^3(0,T)]^2)$. On the other hand, since $(u(t,x),v(t,x))=(\int_0^t u_s(s,x)ds,\int_0^t v_s(s,x)ds)$, (3.12) allows us to deduce that $(u,v)\in L^1(0,T;[H_0^2\cap H^3(0,T)]^2)$, therefore, proceeding as in (3.12), it follows that

$$||(u_t(t,\cdot),v_t(t,\cdot))||_{[H^4(0,L)]^2}\leq C\left(||(u,v)||_{[H^3(0,L)]^2}+||(h,k)||_{[L^2(0,L)]^2}\right).$$

After integration over $(0,T)$ we deduce (3.8). ■

The next Theorem establishes the existence and uniqueness of solutions for system (1.8) in the sense of transposition.

Theorem 3.1.4. *Let $(\eta^0,\omega^0)\in[H^{-2}(0,L)]^2$ and $(f_1,g_1),(f_2,g_2)\in[H^1(0,T)]^2$. Then, there exists a unique solution $(\eta,\omega)\in C([0,T];[L^2(0,L)]^2)$ of system (1.8) which verifies (3.6).*

Proof. The result is proved in two steps. We first use the Riesz Representation Theorem to prove the existence of a solution in $L^1(0,T;[L^2(0,L)]^2)$. Then, the continuity in the time variable is proved by using density arguments.

We start by introducing the linear operator $\mathcal{T}: L^1(0,T;[L^2(0,L)]^2) \longrightarrow \mathbb{R}$ as follows

$$\begin{aligned}\mathcal{T}((h,k)) = &-\left\langle \begin{pmatrix} \eta^0 \\ \omega^0 \end{pmatrix}, \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \right\rangle_{[H^{-2}(0,L)]^2,[H_0^2(0,L)]^2} \\ &+ \int_0^T f_1(t)[b_1 u_{txxx} + cv_{xx}](t,L)dt + \int_0^T g_1(t)[d_1 v_{txxx} + au_{xx}](t,L)dt \\ &- b_1 \int_0^T f_2(t) u_{txx}(t,L)dt - d_1 \int_0^T g_2(t) v_{txx}(t,L)dt,\end{aligned}$$

where (u,v) is a solution of (3.7). We have that $\mathcal{T}$ is well defined and continuous. Indeed, proceeding as in the proof of Theorem 3.1.3, we obtain identity (3.9). Then, integrating over $(0,T)$, it follows that

$$||(u(0),v(0))||_{[H_0^2(0,L)]^2} \leq C||(h,k)||_{L^1(0,T;[L^2(0,L)]^2)}, \tag{3.13}$$

for some constant $C>0$. On the other hand, by using Cauchy-Schwarz inequality, the Sobolev embedding and estimate (3.8), the following estimate holds

$$\begin{aligned}\Bigg| \int_0^T f_1(t)[b_1 u_{txxx} + cv_{xx}](t,L)dt + \int_0^T g_1(t)[d_1 v_{txxx} + au_{xx}](t,L)dt \\ -b_1 \int_0^T f_2(t) u_{txx}(t,L)dt - d_1 \int_0^T g_2(t) v_{txx}(t,L)dt\Bigg| \\ \leq C\Big(||(f_1,g_1)||_{[H^1(0,T)]^2} + ||(f_2,g_2)||_{[H^1(0,T)]^2}\Big) ||(h,k)||_{L^1(0,T;[H^1(0,L)]^2)},\end{aligned} \tag{3.14}$$

where $C>0$. Finally, (3.13) and (3.14) allow us to conclude that $\mathcal{T} \in \mathcal{L}(L^1(0,T;[L^2(0,L)]^2);\mathbb{R})$. Then, from Riesz Representation Theorem, we obtain the existence a unique

$$(\eta,\omega) \in L^\infty(0,T;[L^2(0,L)]^2)$$

satisfying (3.6). Moreover,

$$\begin{aligned}||(\eta,\omega)||_{L^\infty(0,T;[L^2(0,L)]^2)} &= ||\mathcal{T}||_{\mathcal{L}(L^1(0,T;[L^2(0,L)]^2);\mathbb{R})} \\ &\leq C\left(||(\eta^0,\omega^0)||_{[H^{-2}(0,L)]^2} + ||(f_1,g_1)||_{[H^1(0,T)]^2} + ||(f_2,g_2)||_{[H^1(0,T)]^2}\right).\end{aligned} \tag{3.15}$$

By using density arguments, starting with more regular data, we can also get the regularity in the time variable. Indeed, since $(f_1,g_1),(f_2,g_2) \in [H^1(0,T)]^2$ and $(\eta^0,\omega^0) \in [H^{-2}(0,L)]^2$ there exist sequences $(f_{1,n},g_{1,n}),(f_{2,n},g_{2,n}) \in [\mathcal{D}(0,T)]^2$ and $(\eta_n^0,\omega_n^0) \in [\mathcal{D}(0,L)]^2$, such that

$$\begin{aligned}(f_{1,n},g_{1,n}) &\longrightarrow (f_1,g_1) \text{ in } [H^1(0,T)]^2, \\ (f_{2,n},g_{2,n}) &\longrightarrow (f_2,g_2) \text{ in } [H^1(0,T)]^2, \\ (\eta_n^0,\omega_n^0) &\longrightarrow (\eta^0,\omega^0) \text{ in } [H^{-2}(0,L)]^2,\end{aligned}$$

when $n \to \infty$. Let us denote by (η_n,ω_n) the solution of the system (1.8), corresponding to the data $(f_{1,n},g_{1,n}),(f_{2,n},g_{2,n})$ and (η_n^0,ω_n^0), given by Theorem 3.1.2. Then, $(\eta_n,\omega_n) \in C([0,T];[L^2(0,L)]^2)$ and, for each $n \in \mathbb{N}$, the solution (η_n,ω_n) satisfies (3.6). Thus, if (η,ω) is a solution by transposition of (1.8), it follows that $(\eta_n,\omega_n)-(\eta,\omega)$ is a solution by transposition with data

$(f_{1,n}, g_{1,n}) - (f_1, g_1)$, $(f_{2,n}, g_{2,n}) - (f_2, g_2)$ and $(\eta_n^0, \omega_n^0) - (\eta^0, \omega^0)$. Hence, by (3.15), we obtain

$$\begin{aligned}
&||(\eta_n, \omega_n) - (\eta, \omega)||_{L^\infty(0,T;[L^2(0,L)]^2)} \\
&\qquad \le C\Big(||(\eta_n^0, \omega_n^0) - (\eta^0, \omega^0)||_{[H^{-2}(0,L)]^2} + ||(f_{1,n}, g_{1,n}) - (f_1, g_1)||_{[H^1(0,T)]^2} \\
&\qquad\qquad + ||(f_{2,n}, g_{2,n}) - (f_2, g_2)||_{[H^1(0,T)]^2} \Big).
\end{aligned}$$

When $n \to \infty$, from the above inequality, we deduce that $(\eta_n, \omega_n) \to (\eta, \omega)$ in $L^\infty(0,T;[L^2(0,L)]^2)$ and, since $(\eta_n, \omega_n) \in C([0,T];[L^2(0,L)]^2)$, it follows that $(\eta, \omega) \in C([0,T];[L^2(0,L)]^2)$. ■

3.2 Controllability results

In this section we study some boundary controllability properties of the Boussinesq system. We begin with the following exact controllability problem:

Given $T > 0$ and an initial data $(\eta^0, \omega^0) \in [H^{-2}(0,L)]^2$, can we find control inputs $(f_1, g_1), (f_2, g_2) \in [H^1(0,T)]^2$, such that the solution (η, ω) of (1.8) satisfies

$$(\eta(T,x), \omega(T,x)) = (0,0) \quad \textit{for } x \in (0,L)?$$

We have the following characterization of a control driving system (1.8) to the rest.

Lemma 3.2.1. *The initial data $(\eta^0, \omega^0) \in [H^{-2}(0,L)]^2$ is controllable to zero in time $T > 0$ with controls $(f_1, g_1), (f_2, g_2) \in [H^1(0,T)]^2$ if and only if*

$$\begin{aligned}
&\left\langle \begin{pmatrix} \eta^0 \\ \omega^0 \end{pmatrix}, \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \right\rangle_{[H^{-2}(0,L)]^2, [H_0^2(0,L)]^2} = \\
&\qquad \int_0^T f_1(t)[b_1 u_{txxx} + c v_{xx}](t,L)dt + \int_0^T g_1(t)[d_1 v_{txxx} + a u_{xx}](t,L)dt \\
&\qquad - b_1 \int_0^T f_2(t) u_{txx}(t,L)dt - d_1 \int_0^T g_2(t) v_{txx}(t,L)dt,
\end{aligned} \tag{3.16}$$

for any solution (u, v) of the adjoin system

$$\begin{cases}
u_t + v_x + c v_{xxx} - b u_{txx} + b_1 u_{txxxx} = 0 & \textit{for } x \in (0,L),\ t \in (0,T), \\
v_t + u_x + a u_{xxx} - d v_{txx} + d_1 v_{txxxx} = 0 & \textit{for } x \in (0,L), t \in (0,T), \\
u(t,0) = u(t,L) = 0 & \textit{for } \ t \in (0,T), \\
v(t,0) = v(t,L) = 0 & \textit{for } \ t \in (0,T), \\
u_x(t,0) = u_x(t,L) = 0 & \textit{for } \ t \in (0,T), \\
v_x(t,0) = v_x(t,L) = 0 & \textit{for } \ t \in (0,T), \\
u(T,x) = u^T; \quad v(T,x) = v^T & \textit{for } x \in (0,L),
\end{cases} \tag{3.17}$$

with $(u^T, v^T) \in [H_0^2(0,L)]^2$.

Proof. Remark that the change of variables $t \to T - t$ and $x \to L - x$ reduces the system (3.17) to (1.8) with $f_i \equiv g_i \equiv 0$, for $i = 1, 2$. Then, we can apply to (u, v) the well-posedness results obtained in the previous section.

First, we prove the result for regular solutions. The less regular framework can be proved using density arguments as in the proof of Theorem 3.1.4. Let (η, ω) be a solution of (1.8) and

(u,v) solution of (3.17). After some integrations by parts, we have

$$\begin{aligned}
0=&\int_0^T\int_0^L u\left(\eta_t+\omega_x+a\omega_{xxx}-b\eta_{txx}+b_1\eta_{txxxx}\right)dxdt\\
&\quad+\int_0^T\int_0^L v\left(\omega_t+\eta_x+c\eta_{xxx}-d\omega_{txx}+d_1\omega_{txxxx}\right)dxdt\\
=&\int_0^L\left[u(T)\eta(T)-u(0)\eta(0)\right]dx+b\int_0^L\left[u_x(T)\eta_x(T)-u_x(0)\eta_x(0)\right]dx\\
&\quad+b_1\int_0^L\left[u_{xx}(T)\eta_{xx}(T)-u_{xx}(0)\eta_{xx}(0)\right]dx\\
&+\int_0^L\left[v(T)\omega(T)-v(0)\omega(0)\right]dx+d\int_0^L\left[v_x(T)\omega_x(T)-v_x(0)\omega_x(0)\right]dx\\
&\quad+d_1\int_0^L\left[v_{xx}(T)\omega_{xx}(T)-v_{xx}(0)\omega_{xx}(0)\right]dx\\
&+a\int_0^T u_{xx}(L)g_1dt-b_1\int_0^T u_{txx}(L)f_2dt+b_1\int_0^T u_{txxx}(L)f_1dt\\
&+c\int_0^T v_{xx}(L)f_1dt-d_1\int_0^T v_{txx}(L)g_2dt+d_1\int_0^T v_{txxx}(L)g_1dt.
\end{aligned}$$

By using the density of $H_0^2(0,T)$ in $H^{-2}(0,T)$, we can pass the identity above to the limit to obtain

$$\begin{aligned}
&\left\langle\begin{pmatrix}\eta^0\\ \omega^0\end{pmatrix},\begin{pmatrix}u(0)\\ v(0)\end{pmatrix}\right\rangle_{[H^{-2}(0,L)]^2,[H_0^2(0,L)]^2}=\left\langle\begin{pmatrix}\eta(T)\\ \omega(T)\end{pmatrix},\begin{pmatrix}u^T\\ v^T\end{pmatrix}\right\rangle_{[H^{-2}(0,L)]^2,[H_0^2(0,L)]^2}\\
&+\int_0^T f_1(t)[b_1u_{txxx}+cv_{xx}](t,L)dt+\int_0^T g_1(t)[d_1v_{txxx}+au_{xx}](t,L)dt\\
&-b_1\int_0^T f_2(t)u_{txx}(t,L)dt-d_1\int_0^T g_2(t)v_{txx}(t,L)dt.
\end{aligned}$$

Hence, (η^0,ω^0) is controllable to zero in time $T>0$ if and only if (3.16) holds. ■

The next result is devoted to show that system (1.8) is not spectrally controllable. This means that no nontrivial finite linear combinations of eigenvectors of the operator $\mathcal{A}$ defined in (4.4) can be driven to zero in finite time by using controls $(f_1,g_1),(f_2,g_2)\in[H^1(0,T)]^2$.

Theorem 3.2.1. *No eigenfunctions of the operator $\mathcal{A}$ can be driven to zero in finite time.*

Proof. We first note that, according to Theorem 3.3.2, the operator $\mathcal{A}$ has a sequence of purely imaginary eigenvalues $(\mu_n^j)_{n\in\mathbb{Z}^*,j\in\{1,2\}}$. Moreover, the corresponding eigenfunctions $(\Phi_n^j)_{n\in\mathbb{Z}^*,j\in\{1,2\}}$ form an orthogonal basis of $[H_0^2(0,L)]^2$. For each $k\neq 0$, let us consider

$$(\eta_k^0,\omega_k^0)=\Phi_k^j=(\varphi_k^j,\nu_k^j),\quad j=1,2,$$

eigenfunctions of the operator $\mathcal{A}$. In a similar way, if we consider

$$\begin{pmatrix}u_n^T\\ v_n^T\end{pmatrix}=\begin{cases}\Phi_n^j & n\neq k\\ 0 & n=k,\end{cases}$$

the corresponding solution of (3.17) can be written as

$$\begin{pmatrix}u_n\\ v_n\end{pmatrix}=e^{i\lambda_n^j(T-t)}\Phi_n^j,\quad\text{where}\quad i\lambda_n^j=-\frac{1}{\mu_n^j},$$

being μ_n^j, $(j = 1, 2)$ the eigenvalues of the operator $\mathcal{A}$ $(\mu_n^j \mathcal{A}\Phi_n^j = \Phi_n^j)$, given by Theorem 3.3.2. Moreover,

$$\lim_{|n|\to\infty} \lambda_n^j = 0.$$

On the other hand, since the sequence $(\Phi_n^j)_{n\in\mathbb{Z}^*, j\in\{1,2\}}$ forms an orthonormal basis of $[H_0^2(0, L)]^2$, we get

$$\left\langle \begin{pmatrix} \eta_k^0 \\ \omega_k^0 \end{pmatrix}, \begin{pmatrix} u_n(0) \\ v_n(0) \end{pmatrix} \right\rangle_{[H_0^2(0,L)]^2} = \delta_{n,k}^j e^{i\lambda_n^j T}, \quad j = 1, 2.$$

Thus, if (η_k^0, ω_k^0) is controllable to zero in time $T > 0$, from (3.16) it follows that

$$\begin{aligned} \int_0^T e^{i\lambda_n^j (T-t)} \Big[& f_1(t) \left(-i\lambda_n^j b_1 \varphi_{n,xxx}^j + a\nu_{n,xx}^j\right)(L) + g_1(t)\left(-i\lambda_n^j d_1 \nu_{n,xxx}^j + a\varphi_{n,xx}^j\right)(L) \\ & + b_1 f_2(t) i\lambda_n^j \varphi_{n,xx}^j(L) + d_1 g_2(t) i\lambda_n^j \nu_{n,xx}^j(L)\Big] = \delta_{n,k}^j e^{i\lambda_n^j T}, \quad j = 1, 2. \end{aligned} \tag{3.18}$$

For $j = 1$, the identity above can be written as follows

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} h(t) e^{i\lambda_n^1 (\frac{T}{2} - t)} dt = \delta_{n,k}^1 e^{i\lambda_n^1 T},$$

where

$$\begin{aligned} h(t) =& f_1\left(t + \frac{T}{2}\right)\left(-i\lambda_n^1 b_1 \varphi_{n,xxx}^1 + a\nu_{n,xx}^1\right)(L) + g_1\left(t + \frac{T}{2}\right)\left(-i\lambda_n^1 d_1 \nu_{n,xxx}^1 + a\varphi_{n,xx}^1\right)(L) \\ & + i\lambda_n^1 b_1 f_2 \left(t + \frac{T}{2}\right) \varphi_{n,xx}^1(L) + i\lambda_n^1 d_1 g_2\left(t + \frac{T}{2}\right)\nu_{n,xx}^1(L). \end{aligned}$$

Since $h \in L^2(-\frac{T}{2}, \frac{T}{2})$, if we define $F : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$F(z) = \int_{-\frac{T}{2}}^{\frac{T}{2}} h(t) e^{izt},$$

from Paley-Wiener Theorem, we have that F is an entire function. Moreover, since $\lim_{|n|\to\infty} \lambda_n^j = 0$, it follows that F is zero on a set with a finite accumulation point. Then, $F \equiv 0$ and, consequently,

$$\begin{aligned} f_1(t) \left(-i\lambda_n^1 b_1 \varphi_{n,xxx}^1 + a\nu_{n,xx}^1\right)(L) + g_1(t)\left(-i\lambda_n^1 d_1 \nu_{n,xxx}^1 + a\varphi_{n,xx}^1\right)(L) \\ + b_1 f_2(t) i\lambda_n^1 \varphi_{n,xx}^1(L) + d_1 g_2(t) i\lambda_n^1 \nu_{n,xx}^1(L) = 0, \end{aligned} \tag{3.19}$$

for all $t \in [0, T]$.

For $j = 2$, we can use (3.18) and proceed in a similar way to obtain

$$\begin{aligned} f_1(t) \left(-i\lambda_n^2 b_1 \varphi_{n,xxx}^2 + a\nu_{n,xx}^2\right)(L) + g_1(t)\left(-i\lambda_n^2 d_1 \nu_{n,xxx}^2 + a\varphi_{n,xx}^2\right)(L) \\ + b_1 f_2(t) i\lambda_n^2 \varphi_{n,xx}^2(L) + d_1 g_2(t) i\lambda_n^2 \nu_{n,xx}^2(L) = 0, \end{aligned} \tag{3.20}$$

for all $t \in [0, T]$.

Thus, by dividing (3.19) and (3.20) by $i\lambda_n^1$ and $i\lambda_n^2$, respectively, we deduce that (f_1, g_1) and (f_2, g_2) should satisfy the system

$$\begin{cases} f_1(t)A_n^1 + g_1(t)B_n^1 + f_2(t)C_n^1 + g_2(t)D_n^1 = 0 \\ \\ f_1(t)A_n^2 + g_1(t)B_n^2 + f_2(t)C_n^2 + g_2(t)D_n^2 = 0, \end{cases} \tag{3.21}$$

where

$$A_n^j = \frac{a}{i\lambda_n^j}\nu_{n,xx}^j(L) - b_1\varphi_{n,xxx}^j(L), \quad B_n^j = \frac{a}{i\lambda_n^j}\varphi_{n,xx}^j(L) - d_1\nu_{n,xxx}^j(L),$$
$$C_n^j = b_1\varphi_{n,xx}^j(L), \quad \text{and} \quad D_n^j = d_1\nu_{n,xx}^j(L), \quad \text{for } j = 1,2.$$

From the asymptotic behavior of the eigenvectors of $\mathcal{A}$ given by Lemma 3.3.8, we obtain that, for a subsequence, if necessary, the following holds:

$$\lim_{|n|\to\infty} C_n^j = \lim_{|n|\to\infty} D_n^j = \lim_{|n|\to\infty} A_n^2 = \lim_{|n|\to\infty} B_n^1 = 0, \quad j = 1,2, \tag{3.22}$$

$$\lim_{|n|\to\infty} A_n^1 = \lim_{|n|\to\infty} B_n^2 = \delta_0\frac{\sqrt{b_1d_1}}{L}, \quad \text{for some} \quad \delta_0 \in \mathbb{C}^*, \tag{3.23}$$

and

$$\begin{vmatrix} C_n^1 & D_n^1 \\ C_n^2 & D_n^2 \end{vmatrix} \sim \frac{-L^2b_1d_1}{[(2|n|+1)\pi - 2\varepsilon_n]^2 + 4} \neq 0, \quad \text{for all } n \in \mathbb{Z}^*, \tag{3.24}$$

where $\varepsilon_n \in (0,1)$. By using (3.22) and (3.23) in (3.21) we obtain

$$f_1(t)A_n^1 + g_1(t)B_n^1 + f_2(t)C_n^1 + g_2(t)D_n^1 \to \delta_0\frac{\sqrt{b_1d_1}}{L}f_1(t) = 0,$$
$$f_1(t)A_n^2 + g_1(t)B_n^2 + f_2(t)C_n^2 + g_2(t)D_n^2 \to \delta_0\frac{\sqrt{b_1d_1}}{L}g_1(t) = 0,$$

as $|n| \to \infty$. Then, $(f_1, g_1) \equiv (0,0)$ and the system (3.21) becomes simpler:

$$\begin{cases} f_2(t)C_n^1 + g_2(t)D_n^1 = 0 \\ \\ f_2(t)C_n^2 + g_2(t)D_n^2 = 0. \end{cases}$$

Hence, from (3.24) we deduce that $(f_1, g_1) \equiv (f_2, g_2) \equiv (0,0)$ is the unique solution of the system (3.21), which contradicts (3.18) and the proof ends. ■

Remark 3.2.2. *$\lambda = 0$ is not a eigenvalue of the operator $\mathcal{A}$. Indeed, if (φ,ν) satisfies $\mathcal{A}(\varphi,\nu) = 0$, then, it shall be solution of the uncoupled system*

$$\begin{cases} \nu_x + a\nu_{xxx} = 0 & \text{for } x \in (0,L), \\ \varphi_x + a\varphi_{xxx} = 0 & \text{for } x \in (0,L), \\ (\varphi(0),\nu(0)) = (\varphi(L),\nu(L)) = (0,0), & \\ (\varphi_x(0),\nu_x(0)) = (\varphi_x(L),\nu_x(L)) = (0,0). & \end{cases}$$

By setting $\tilde{\nu} = \nu_x$ we obtain $\tilde{\nu}(x) = c_1e^{\frac{i}{\sqrt{a}}x} + c_2e^{-\frac{i}{\sqrt{a}}x}$, for some constants c_1, c_2. Then, from the boundary condition $\tilde{\nu}(0) = 0$, we deduce that $\tilde{\nu}(x) = 2ic_1\sin\left(\frac{x}{\sqrt{a}}\right)$ and the boundary condition $\tilde{\nu}(L) = 0$ implies that $2ic_1\sin\left(\frac{L}{\sqrt{a}}\right) = 0$. Thus, if $L \neq \sqrt{a}\pi n$, with $n \in \mathbb{Z}^*$, we have that $c_1 = 0$ and $\nu \equiv const$. Then, from the boundary condition $\nu(0) = 0$ we conclude that $\nu \equiv 0$. On the other hand, if $L = \sqrt{a}\pi n$, for some $n \in \mathbb{Z}^*$, we have that $\nu(x) = -2i\sqrt{a}c_1\cos\left(\frac{x}{\sqrt{a}}\right)$ and the condition $\nu(L) = 0$ implies that $c_1 = 0$. Hence, $\nu \equiv 0$. Since the system is uncoupled, we can arguing as above to obtain $\varphi \equiv 0$.

Now, we pass to study the approximate controllability of the system (1.8). In order to do that, we introduce the following definition.

Definition 3.2.3. *System (1.8) is said to be approximately controllable in time $T > 0$ if, for every initial data $(\eta^0, \omega^0) \in [H^{-2}(0,L)]^2$, the set of reachable states*

$$R\left(\begin{pmatrix} \eta^0 \\ \omega^0 \end{pmatrix}, T\right) = \left\{ \begin{pmatrix} \eta(T,x) \\ \omega(T,x) \end{pmatrix} : \left(\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right) \in [H^1(0,T)]^2 \times [H^1(0,T)]^2 \right\}$$

is dense in $[L^2(0,L)]^2$.

The corresponding approximate controllability result reads as follows.

Theorem 3.2.2. *System (1.8) is approximately controllable in time $T > 0$ with controls $(f_1, g_1), (f_2, g_2) \in [H^1(0,T)]^2$.*

Proof. Due to the linearity of the system (1.8), it is sufficient to prove the result for any $T > 0$ and $(\eta^0, \omega^0) = (0,0)$. Thus, we will prove the density of the set $R\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, T \right)$ in $[L^2(0,L)]^2$.

Let $(\eta, \omega) \in C([0,T]; [L^2(0,L)]^2)$ the corresponding solution of (1.8) given by Theorem 3.1.4 and (u,v) solution of the adjoin system (3.17). Then, it follows that

$$\begin{aligned} &\left\langle \begin{pmatrix} \eta(T,x) \\ \omega(T,x) \end{pmatrix}, \begin{pmatrix} u^T \\ v^T \end{pmatrix} \right\rangle_{[H^{-2}(0,L)]^2, [H_0^2(0,L)]^2} \\ &= -\int_0^T f_1(t)[b_1 u_{txxx} + a v_{xx}](t,L)dt - \int_0^T g_1(t)[d_1 v_{txxx} + a u_{xx}](t,L)dt \\ &+ b_1 \int_0^T f_2(t) u_{txx}(t,L)dt + d_1 \int_0^T g_2(t) v_{txx}(t,L)dt. \end{aligned} \tag{3.25}$$

Assume that $R\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, T \right)$ is not dense in $[H_0^2(0,L)]^2$. In this case, there exists $(u^T, v^T) \neq (0,0)$ in $[H_0^2(0,L)]^2$, satisfying

$$\left\langle \begin{pmatrix} \eta(T,x) \\ \omega(T,x) \end{pmatrix}, \begin{pmatrix} u^T \\ v^T \end{pmatrix} \right\rangle_{[H^{-2}(0,L)]^2, [H_0^2(0,L)]^2} = 0,$$

for all $\left(\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right) \in [H^1(0,T)]^2 \times [H^1(0,T)]^2$. Consequently, from (3.25) we obtain

$$\begin{aligned} &-\left\langle \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} [b_1 u_{txxx} + a v_{xx}](t,L) \\ [d_1 v_{txxx} + a u_{xx}](t,L) \end{pmatrix} \right\rangle_{[L^2(0,T)]^2} \\ &\qquad + \left\langle \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}, \begin{pmatrix} b_1 u_{txx}(t,L) \\ d_1 v_{txx}(t,L) \end{pmatrix} \right\rangle_{[L^2(0,T)]^2} = 0, \end{aligned}$$

for all $\left(\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right) \in [H^1(0,T)]^2 \times [H^1(0,T)]^2$. Thus,

$$\begin{pmatrix} [b_1 u_{txxx} + a v_{xx}](t,L) \\ [d_1 v_{txxx} + a u_{xx}](t,L) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_1 u_{txx}(t,L) \\ d_1 v_{txx}(t,L) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \forall t \in (0,T). \tag{3.26}$$

Next, we want to write (3.26) as an infinite sum. From the proof of Theorem 3.1.1 we know that $\mathcal{A}$ is a skew adjoint operator in $[H_0^2(0,L)]^2$. Hence, it has a sequence of eigenvalues $(i\lambda_n)_{n\in\mathbb{Z}^*} \subset i\mathbb{R}$, each $i\lambda_n = (\mu_n)^{-1}$ with geometric multiplicity at most $\mathscr{M}_n$. The corresponding eigenfunctions form an orthonormal basis for $[H_0^2(0,L)]^2$, which we denote by

$$\bigcup_{n\in\mathbb{Z}^*} \{\Phi_n^k\}_{k=1}^{\mathscr{M}_n}.$$

Then, if $(u^T, v^T) \in [H_0^2(0,L)]^2$, we have

$$(u^T, v^T) = \sum_{n\in\mathbb{Z}^*} \sum_{k=1}^{\mathscr{M}_n} \alpha_n^k \Phi_n^k$$

and the corresponding solution (u, v) can be written as

$$(u, v) = \sum_{n\in\mathbb{Z}^*} \sum_{k=1}^{\mathscr{M}_n} \alpha_n^k \Phi_n^k e^{i\lambda_n(T-t)}. \tag{3.27}$$

Thus, from (3.26) and (3.27), it follows that

$$0 = u_{txx}(t, L) = \sum_{n\in\mathbb{Z}^*} -i\lambda_n \sum_{k=1}^{\mathscr{M}_n} \alpha_n^k \varphi_{n,xx}^k(L) e^{i\lambda_n(T-t)}.$$

Since (u, v) is analytic in time (see Theorem 3.1.1), we can integrate the identity above over $(-S, S)$, for any $S > 0$. Then, for each $m \in \mathbb{Z}^*$, we deduce that

$$0 = \lim_{s\to+\infty} \frac{1}{S} \int_{-S}^{S} u_{txx}(s, L) e^{i\lambda_m s} ds = -i\lambda_m \sum_{k=1}^{\mathscr{M}_m} \alpha_m^k \varphi_{m,xx}^k(L) e^{i\lambda_m T},$$

hence,

$$\sum_{k=1}^{\mathscr{M}_m} \alpha_m^k \varphi_{m,xx}^k(L) = 0. \tag{3.28}$$

Analogously, from $v_{txx}(t, L) = 0$, it results that

$$\sum_{k=1}^{\mathscr{M}_m} \alpha_m^k \nu_{m,xx}^k(L) = 0. \tag{3.29}$$

On the other hand, from (3.26)-(3.27) we have

$$0 = [b_1 u_{txxx} + a v_{xx}]\,(t, L) = \sum_{n\in\mathbb{Z}^*} \sum_{k=1}^{\mathscr{M}_n} \alpha_n^k \left[-i\lambda_n b_1 \varphi_{n,xxx}^k(L) + a\nu_{n,xx}^k(L)\right] e^{i\lambda_n(T-t)}$$

and

$$0 = [d_1 v_{txxx} + a u_{xx}]\,(t, L) = \sum_{n\in\mathbb{Z}^*} \sum_{k=1}^{\mathscr{M}_n} \alpha_n^k \left[-i\lambda_n d_1 \nu_{n,xxx}^k(L) + a\varphi_{n,xx}^k(L)\right] e^{i\lambda_n(T-t)}.$$

Next, we proceed as before and use (3.28) and (3.29) to obtain

$$\begin{aligned} 0 = \sum_{k=1}^{\mathscr{M}_m} & \alpha_m^k [-i\lambda_m b_1 \varphi_{m,xxx}^k(L) + a\nu_{m,xx}^k(L)] e^{i\lambda_m T} \\ &= \left[-i\lambda_m b_1 \sum_{k=1}^{\mathscr{M}_m} \alpha_m^k \varphi_{m,xxx}^k(L) + a \sum_{k=1}^{\mathscr{M}_m} \alpha_m^k \nu_{m,xx}^k(L)\right] e^{i\lambda_m T} \\ &= -i\lambda_m b_1 \sum_{k=1}^{\mathscr{M}_m} \alpha_m^k \varphi_{m,xxx}^k(L) e^{i\lambda_m T} \end{aligned}$$

and

$$0=\sum_{k=1}^{\mathscr{M}_m}\alpha_m^k[-i\lambda_m d_1\nu_{m,xxx}^k(L)+a\varphi_{m,xx}^k(L)]e^{i\lambda_m T}$$
$$=\left[-i\lambda_m d_1\sum_{k=1}^{\mathscr{M}_m}\alpha_m^k\nu_{m,xxx}^k(L)+a\sum_{k=1}^{\mathscr{M}_m}\alpha_m^k\varphi_{m,xx}^k(L)\right]e^{i\lambda_m T}$$
$$=-i\lambda_m d_1\sum_{k=1}^{\mathscr{M}_m}\alpha_m^k\nu_{m,xxx}^k(L)e^{i\lambda_m T},$$

respectively. Then,

$$\sum_{k=1}^{\mathscr{M}_m}\alpha_m^k\varphi_{m,xxx}^k(L)=\sum_{k=1}^{\mathscr{M}_m}\alpha_m^k\nu_{m,xxx}^k(L)=0. \tag{3.30}$$

Now, for each $m\in\mathbb{Z}^*$, we consider $\Phi_m=(\varphi_m,\nu_m)$ defined as follows

$$\Phi^m=\alpha_m^1\Phi_m^1+\ldots+\alpha_m^{\mathscr{M}_m}\Phi_m^{\mathscr{M}_m}.$$

Thus, from (3.28), (3.29) and (3.30) we have that

$$(\varphi_{m,xx}(L),\nu_{m,xx}(L))=(\varphi_{m,xxx}(L),\nu_{m,xxx}(L))=(0,0)$$

and $\Phi_m=(\varphi_m,\nu_m)$ solves the initial value problem

$$\begin{cases}-\varphi_m+b\varphi_{m,xx}-b_1\varphi_{m,xxxx}+(i\lambda_m)^{-1}\nu_{m,x}+a(i\lambda_m)^{-1}\nu_{m,xxx}=0 & \text{for } x\in(0,L),\\ -\nu_m+d\nu_{m,xx}-d_1\nu_{m,xxxx}+(i\lambda_m)^{-1}\varphi_{m,x}+a(i\lambda_m)^{-1}\varphi_{m,xxx}=0 & \text{for } x\in(0,L),\\ (\varphi_m(L),\nu_m(L))=(0,0), \\ (\varphi_{m,x}(L),\nu_{m,x}(L))=(0,0), \\ (\varphi_{m,xx}(L),\nu_{m,xx}(L))=(0,0), \\ (\varphi_{m,xxx}(L),\nu_{m,xxx}(L))=(0,0).\end{cases}$$

Then, by uniqueness,

$$\Phi_m=\alpha_m^1\Phi_m^1+\ldots+\alpha_m^{\mathscr{M}_m}\Phi_m^{\mathscr{M}_m}=(0,0).$$

Since $\{\Phi_m^k\}_{k=1}^{\mathscr{M}_m}$ are linearly independent, it follows that $\alpha_m^1=\ldots=\alpha_m^{\mathscr{M}_m}=0$ for all $m\in\mathbb{Z}^*$. Thus, from (3.27) it follows that $(u,v)=(0,0)$ and, in particular, $(u^T,v^T)=(0,0)$. This is a contradiction and the proof ends. ■

3.3 Spectral Analysis

This section is devoted to develop a spectral analysis of the operator $\mathcal{A}$ introduced above. We start by presenting some explicit formula and properties of a family of initial value problems depending on several parameters. These results allow us to obtain the asymptotic behavior of the eigenvalues and eigenfunctions of the differential operator associated to (1.8).

Study of some initial value problems

Firstly, we study the properties of the following simple initial value problem, where $\sigma\in\mathbb{C}^*$ is a complex nonzero parameter:

$$
\begin{cases}
a\sigma\nu_{xxx} - b_1\varphi_{xxxx} = f & \text{for } x \in (0,L),\\
a\sigma\varphi_{xxx} - d_1\nu_{xxxx} = g & \text{for } x \in (0,L),\\
(\varphi(0),\nu(0)) = (\varphi^0,\nu^0),\\
(\varphi_x(0),\nu_x(0)) = (\varphi^1,\nu^1),\\
(\varphi_{xx}(0),\nu_{xx}(0)) = (\varphi^2,\nu^2),\\
(\varphi_{xxx}(0),\nu_{xxx}(0)) = (\varphi^3,\nu^3).
\end{cases}
\tag{3.31}
$$

In (3.31) a, b_1 and d_1 are positive real numbers. We have the following result.

Lemma 3.3.1. *Given $(\varphi^0,\varphi^1,\varphi^2,\varphi^3,\nu^0,\nu^1,\nu^2,\nu^3) \in \mathbb{C}^8$ and $(f,g) \in [L^2(0,L)]^2$, there exists a unique solution (φ,ν) to the problem (3.31) given by the formula*

$$
\begin{pmatrix} \varphi(x) \\ \\ \nu(x) \end{pmatrix} =
$$

$$
\begin{pmatrix}
\frac{(b_1d_1)^{\frac{3}{2}}}{[a\sigma]^3}\left[\sinh(\frac{a\sigma x}{\sqrt{b_1d_1}}) - \frac{a\sigma x}{\sqrt{b_1d_1}}\right]\varphi^3 + \frac{b_1d_1^2}{[a\sigma]^3}\left[\left(\cosh(\frac{a\sigma x}{\sqrt{b_1d_1}}) - 1\right) - \frac{[a\sigma]^2}{b_1d_1}\frac{x^2}{2}\right]\nu^3 \\ \\
\frac{b_1^2d_1}{[a\sigma]^3}\left[\left(\cosh(\frac{a\sigma x}{\sqrt{b_1d_1}}) - 1\right) - \frac{[a\sigma]^2}{b_1d_1}\frac{x^2}{2}\right]\varphi^3 + \frac{(b_1d_1)^{\frac{3}{2}}}{[a\sigma]^3}\left[\sinh(\frac{a\sigma x}{\sqrt{b_1d_1}}) - \frac{a\sigma x}{\sqrt{b_1d_1}}\right]\nu^3
\end{pmatrix}
\tag{3.32}
$$

$$
+ \begin{pmatrix}
\varphi^2\frac{x^2}{2} + \varphi^1 x + \varphi^0 - \frac{1}{a\sigma}\int_0^x \bar{F}(s)ds \\ \\
\nu^2\frac{x^2}{2} + \nu^1 x + \nu^0 - \frac{1}{a\sigma}\int_0^x \bar{G}(s)ds
\end{pmatrix}
$$

where

$$
\bar{F}(x) = \int_0^x\int_0^s \left[\sqrt{\frac{d_1}{b_1}}\sinh(\frac{a\sigma(s-r)}{\sqrt{b_1d_1}})f(r) + \left(\cosh(\frac{a\sigma(s-r)}{\sqrt{b_1d_1}}) - 1\right)g(r)\right]drds,
$$

$$
\bar{G}(x) = \int_0^x\int_0^s \left[\left(\cosh(\frac{a\sigma(s-r)}{\sqrt{b_1d_1}}) - 1\right)f(r) + \sqrt{\frac{b_1}{d_1}}\sinh(\frac{a\sigma(s-r)}{\sqrt{b_1d_1}})g(r)\right]drds.
$$

Proof. By setting $(\varphi_{xxx},\nu_{xxx}) = (\tilde{\varphi},\tilde{\nu})$ we deduce that

$$
\begin{pmatrix} \tilde{\varphi}_x(x) \\ \\ \tilde{\nu}_x(x) \end{pmatrix} = \begin{pmatrix} 0 & \frac{a\sigma}{b_1} \\ \\ \frac{a\sigma}{d_1} & 0 \end{pmatrix}\begin{pmatrix} \tilde{\varphi}(x) \\ \\ \tilde{\nu}(x) \end{pmatrix} - \begin{pmatrix} \frac{f(x)}{b_1} \\ \\ \frac{g(x)}{d_1} \end{pmatrix}, \quad \begin{pmatrix} \tilde{\varphi}(0) \\ \\ \tilde{\nu}(0) \end{pmatrix} = \begin{pmatrix} \varphi^3 \\ \\ \nu^3 \end{pmatrix};
$$

consequently,

$$
\begin{pmatrix} \tilde{\varphi}(x) \\ \\ \tilde{\nu}(x) \end{pmatrix} = e^{Ax}\begin{pmatrix} \varphi^3 \\ \\ \nu^3 \end{pmatrix} - \int_0^x e^{A(x-s)}\begin{pmatrix} \frac{f(x)}{b_1} \\ \\ \frac{g(x)}{d_1} \end{pmatrix}ds,
\tag{3.33}
$$

where

$$
e^{Ax} = \begin{pmatrix} \cosh(\frac{a\sigma x}{\sqrt{b_1d_1}}) & \sqrt{\frac{d_1}{b_1}}\sinh(\frac{a\sigma x}{\sqrt{b_1d_1}}) \\ \\ \sqrt{\frac{b_1}{d_1}}\sinh(\frac{a\sigma x}{\sqrt{b_1d_1}}) & \cosh(\frac{a\sigma x}{\sqrt{b_1d_1}}) \end{pmatrix}.
$$

By integrating the equations in (3.31) we obtain

$$
\begin{pmatrix} \varphi_{xx}(x) \\ \\ \nu_{xx}(x) \end{pmatrix} = \begin{pmatrix} \varphi^2 - \frac{d_1}{a\sigma}\nu^3 + \frac{d_1}{a\sigma}\tilde{\nu}(x) + \frac{1}{a\sigma}\int_0^x g(s)ds \\ \\ \nu^2 - \frac{b_1}{a\sigma}\varphi^3 + \frac{b_1}{a\sigma}\tilde{\varphi}(x) + \frac{1}{a\sigma}\int_0^x f(s)ds \end{pmatrix}
\tag{3.34}
$$

and, from (3.33), it follows that

$$\begin{pmatrix} \varphi_{xx}(x) \\ \\ \nu_{xx}(x) \end{pmatrix} = \begin{pmatrix} \varphi^2 + \frac{\sqrt{b_1 d_1}}{a\sigma}\sinh(\frac{a\sigma x}{\sqrt{b_1 d_1}})\varphi^3 + \frac{d_1}{a\sigma}\left(\cosh(\frac{a\sigma x}{\sqrt{b_1 d_1}}) - 1\right)\nu^3 \\ \\ \nu^2 + \frac{b_1}{a\sigma}\left(\cosh(\frac{a\sigma x}{\sqrt{b_1 d_1}}) - 1\right)\varphi^3 + \frac{\sqrt{b_1 d_1}}{a\sigma}\sinh(\frac{a\sigma x}{\sqrt{b_1 d_1}})\nu^3 \end{pmatrix}$$

$$- \frac{1}{a\sigma}\begin{pmatrix} \int_0^x \left[\sqrt{\frac{d_1}{b_1}}\sinh(\frac{a\sigma(x-s)}{\sqrt{b_1 d_1}})f(s) + \left(\cosh(\frac{a\sigma(x-s)}{\sqrt{b_1 d_1}}) - 1\right)g(s)\right] ds \\ \\ \int_0^x \left[\left(\cosh(\frac{a\sigma(x-s)}{\sqrt{b_1 d_1}}) - 1\right)f(s) + \sqrt{\frac{b_1}{d_1}}\sinh(\frac{a\sigma(x-s)}{\sqrt{b_1 d_1}})g(s)\right] ds \end{pmatrix}.$$

After integration, we get

$$\begin{aligned} \begin{pmatrix} \varphi_x(x) \\ \\ \nu_x(x) \end{pmatrix} &= \begin{pmatrix} \varphi^1 + \frac{b_1 d_1}{[a\sigma]^2}\left(\cosh(\frac{a\sigma x}{\sqrt{b_1 d_1}}) - 1\right)\varphi^3 + \frac{d_1\sqrt{b_1 d_1}}{[a\sigma]^2}\sinh(\frac{a\sigma x}{\sqrt{b_1 d_1}})\nu^3 \\ \\ \nu^1 + \frac{b_1\sqrt{b_1 d_1}}{[a\sigma]^2}\sinh(\frac{a\sigma x}{\sqrt{b_1 d_1}})\varphi^3 + \frac{b_1 d_1}{[a\sigma]^2}\left(\cosh(\frac{a\sigma x}{\sqrt{b_1 d_1}}) - 1\right)\nu^3 \end{pmatrix} \\ &+ \begin{pmatrix} (\varphi^2 - \frac{d_1}{a\sigma}\nu^3)x - \frac{1}{a\sigma}\int_0^x F(s)ds \\ \\ (\nu^2 - \frac{b_1}{a\sigma}\varphi^3)x - \frac{1}{a\sigma}\int_0^x G(s)ds \end{pmatrix}, \end{aligned} \tag{3.35}$$

where

$$\begin{aligned} F(x) &= \int_0^x \left[\sqrt{\frac{d_1}{b_1}}\sinh(\frac{a\sigma(x-s)}{\sqrt{b_1 d_1}})f(s) + \left(\cosh(\frac{a\sigma(x-s)}{\sqrt{b_1 d_1}}) - 1\right)g(s)\right] ds, \\ G(x) &= \int_0^x \left[\left(\cosh(\frac{a\sigma(x-s)}{\sqrt{b_1 d_1}}) - 1\right)f(s) + \sqrt{\frac{b_1}{d_1}}\sinh(\frac{a\sigma(x-s)}{\sqrt{b_1 d_1}})g(s)\right] ds. \end{aligned}$$

Finally, by integrating (3.35), we obtain

$$\begin{aligned} \begin{pmatrix} \varphi(x) \\ \\ \nu(x) \end{pmatrix} &= \begin{pmatrix} \varphi^0 + \frac{(b_1 d_1)^{\frac{3}{2}}}{[a\sigma]^3}\sinh(\frac{a\sigma x}{\sqrt{b_1 d_1}})\varphi^3 + \frac{b_1 d_1^2}{[a\sigma]^3}\left(\cosh(\frac{a\sigma x}{\sqrt{b_1 d_1}}) - 1\right)\nu^3 \\ \\ \nu^0 + \frac{b_1^2 d_1}{[a\sigma]^3}\left(\cosh(\frac{a\sigma x}{\sqrt{b_1 d_1}}) - 1\right)\varphi^3 + \frac{(b_1 d_1)^{\frac{3}{2}}}{[a\sigma]^3}\sinh(\frac{a\sigma x}{\sqrt{b_1 d_1}})\nu^3 \end{pmatrix} \\ &+ \begin{pmatrix} \frac{1}{2}(\varphi^2 - \frac{d_1}{a\sigma}\nu^3)x^2 + (\varphi^1 - \frac{b_1 d_1}{[a\sigma]^2}\varphi^3)x - \frac{1}{a\sigma}\int_0^x \bar{F}(s)ds \\ \\ \frac{1}{2}(\nu^2 - \frac{b_1}{a\sigma}\varphi^3)x^2 + (\nu^1 - \frac{b_1 d_1}{[a\sigma]^2}\nu^3)x - \frac{1}{a\sigma}\int_0^x \bar{G}(s)ds \end{pmatrix}. \end{aligned} \tag{3.36}$$

Rearranging the terms in (3.36) we obtain (3.32). ∎

We define the set

$$Z = \left\{ z \in \mathbb{C} : |z| \geq \frac{1}{2}, |\Re(z)| \leq 1 \right\}$$

and show that the following estimates for the solution (φ, ν) of (3.31) hold if $\sigma \in Z$.

Lemma 3.3.2. *Let (φ, ν) be the solution of (3.31). There exists a positive constant $C > 0$, such that the following estimates hold for all $x \in [0, L]$ and $\sigma \in Z$:*

$$\sum_{i=0}^{2}\left|\frac{d^i\varphi}{dx^i}(x)\right| \leq |\varphi^0| + C\left(|\varphi^1| + |\varphi^2|\right) + \frac{C^2}{|\sigma|}\left[|\varphi^3| + |\nu^3| + \int_0^x |f(s)| + |g(s)|ds\right], \tag{3.37}$$

$$\sum_{i=0}^{2}\left|\frac{d^i\nu}{dx^i}(x)\right| \le |\nu^0| + C\left(|\nu^1|+|\nu^2|\right) + \frac{C^2}{|\sigma|}\left[|\varphi^3|+|\nu^3|+\int_0^x |f(s)|+|g(s)|ds\right], \tag{3.38}$$

$$\max\{|\varphi_{xxx}(x)|,|\nu_{xxx}(x)|\} \le C\left[|\varphi^3|+|\nu^3|+\int_0^x |f(s)|+|g(s)|ds\right]. \tag{3.39}$$

Proof. First, let us note that the following estimates hold for $(\tilde{\varphi},\tilde{\nu})$ given by (3.33):

$$\begin{aligned}
|\tilde{\varphi}(x)| &\le \left(|\varphi^3| + \sqrt{\frac{d_1}{b_1}}|\nu^3|\right) e^{|\Re(\sigma)|\frac{ax}{\sqrt{b_1d_1}}} + \int_0^x e^{|\Re(\sigma)|\frac{a(x-s)}{\sqrt{b_1d_1}}}\left[\frac{1}{b_1}|f(s)| + \frac{1}{\sqrt{b_1d_1}}|g(s)|\right] ds \\
&\le \left(|\varphi^3| + \sqrt{\frac{d_1}{b_1}}|\nu^3| + \int_0^x \left[\frac{1}{b_1}|f(s)| + \frac{1}{\sqrt{b_1d_1}}|g(s)|\right] ds\right) e^{|\Re(\sigma)|\frac{ax}{\sqrt{b_1d_1}}}
\end{aligned}$$

and

$$\begin{aligned}
|\tilde{\nu}(x)| &\le \left(\sqrt{\frac{b_1}{d_1}}|\nu^3| + |\varphi^3|\right) e^{|\Re(\sigma)|\frac{ax}{\sqrt{b_1d_1}}} + \int_0^x e^{|\Re(\sigma)|\frac{a(x-s)}{\sqrt{b_1d_1}}}\left[\frac{1}{\sqrt{b_1d_1}}|f(s)| + \frac{1}{d_1}|g(s)|\right] ds \\
&\le \left(\sqrt{\frac{b_1}{d_1}}|\nu^3| + |\varphi^3| + \int_0^x \left[\frac{1}{\sqrt{b_1d_1}}|f(s)| + \frac{1}{d_1}|g(s)|\right] ds\right) e^{|\Re(\sigma)|\frac{ax}{\sqrt{b_1d_1}}},
\end{aligned}$$

which allow us to deduce (3.39). Moreover, taking into account formulas (3.34), we obtain

$$\begin{aligned}
|\varphi_{xx}(x)| &\le |\varphi^2| + \frac{C}{|\sigma|}\left[|\varphi^3|+|\nu^3|+\int_0^x |f(s)|+|g(s)|ds\right], \\
|\nu_{xx}(x)| &\le |\nu^2| + \frac{C}{|\sigma|}\left[|\varphi^3|+|\nu^3|+\int_0^x |f(s)|+|g(s)|ds\right].
\end{aligned}$$

Then, from the first estimate above and by using that

$$\begin{aligned}
|\varphi_x(x)| &\le |\varphi^1| + \int_0^x |\varphi_{xx}(s)|ds, \\
|\varphi(x)| &\le |\varphi^0| + \int_0^x |\varphi_x(s)|ds,
\end{aligned}$$

for all $x\in[0,L]$, we obtain estimate (3.37). This argument also holds for the function v. Thus, we obtain estimate (3.38). ∎

Let us now consider the following slightly more complicated system,

$$\begin{cases}
-\xi + b\xi_{xx} - b_1\xi_{xxxx} + \sigma\zeta_x + a\sigma\zeta_{xxx} = 0 & \text{for } x\in(0,L), \\
-\zeta + d\zeta_{xx} - d_1\zeta_{xxxx} + \sigma\xi_x + a\sigma\xi_{xxx} = 0 & \text{for } x\in(0,L), \\
(\xi(0),\zeta(0)) = (\xi^0,\zeta^0), \\
(\xi_x(0),\zeta_x(0)) = (\xi^1,\zeta^1), \\
(\xi_{xx}(0),\zeta_{xx}(0)) = (\xi^2,\zeta^2), \\
(\xi_{xxx}(0),\zeta_{xxx}(0)) = (\xi^3,\zeta^3),
\end{cases} \tag{3.40}$$

for which we have the following result.

Proposition 3.3.3. *There exists a positive constant $C > 0$, such that*

$$\|(\xi,\zeta)\|_{[W^{2,\infty}(0,L)]^2} \leq C\left[\sum_{i=0}^{2}\left(|\xi^i|+|\zeta^i|\right)+\frac{1}{|\sigma|}\left(|\xi^3|+|\zeta^3|\right)\right], \tag{3.41}$$

for any $\sigma \in Z$ and any solution (ξ,ζ) of (3.40).

Proof. Let $\sigma \in Z$, and let (ξ,ζ) be a solution of (3.40). Then, (ξ,ζ) satisfies

$$\begin{cases} a\sigma\zeta_{xxx} - b_1\xi_{xxxx} = \xi - \sigma\zeta_x - b\xi_{xx} & \text{for } x \in (0,L), \\ a\sigma\xi_{xxx} - d_1\zeta_{xxxx} = \zeta - \sigma\xi_x - d\zeta_{xx} & \text{for } x \in (0,L), \\ (\xi(0),\zeta(0)) = (\xi^0,\zeta^0), \\ (\xi_x(0),\zeta_x(0)) = (\xi^1,\zeta^1), \\ (\xi_{xx}(0),\zeta_{xx}(0)) = (\xi^2,\zeta^2), \\ (\xi_{xxx}(0),\zeta_{xxx}(0)) = (\xi^3,\zeta^3). \end{cases} \tag{3.42}$$

Since (3.42) is a system of type (3.31) with $f = \xi - \sigma\zeta_x - b\xi_{xx}$ and $g = \zeta - \sigma\xi_x - d\zeta_{xx}$, we obtain from Lemma 3.3.2 a constant $C > 0$, such that

$$\begin{aligned}\sum_{i=0}^{2}\left|\frac{d^i\xi}{dx^i}(x)\right| \leq |\xi^0| + C\left(|\xi^1|+|\xi^2|\right) + \frac{C^2}{|\sigma|}\left[|\xi^3|+|\zeta^3|\right] \\ + \frac{C^2}{|\sigma|}\int_0^x 2|\sigma|\sum_{i=0}^{2}\left(\left|\frac{d^i\xi}{dx^i}(s)\right|+\left|\frac{d^i\zeta}{dx^i}(s)\right|\right)ds\end{aligned}$$

and

$$\begin{aligned}\sum_{i=0}^{2}\left|\frac{d^i\zeta}{dx^i}(x)\right| \leq |\zeta^0| + C\left(|\zeta^1|+|\zeta^2|\right) + \frac{C^2}{|\sigma|}\left[|\xi^3|+|\zeta^3|\right] \\ + \frac{C^2}{|\sigma|}\int_0^x 2|\sigma|\sum_{i=0}^{2}\left(\left|\frac{d^i\xi}{dx^i}(s)\right|+\left|\frac{d^i\zeta}{dx^i}(s)\right|\right)ds.\end{aligned}$$

By adding the estimates above we obtain

$$\begin{aligned}\sum_{i=0}^{2}\left(\left|\frac{d^i\xi}{dx^i}(x)\right|+\left|\frac{d^i\zeta}{dx^i}(x)\right|\right) \leq C\left[\sum_{i=0}^{2}(|\xi^i|+|\zeta^i|)\right] + \frac{C^2}{|\sigma|}(|\xi^3|+|\zeta^3|) \\ + C^2\int_0^x\sum_{i=0}^{2}\left(\left|\frac{d^i\xi}{dx^i}(s)\right|+\left|\frac{d^i\zeta}{dx^i}(s)\right|\right)ds,\end{aligned}$$

for every $x \in [0,L]$ and $\sigma \in Z$. Then, from Gronwall's inequality we have that (ξ,ζ) satisfies (3.41). ■

The following result compares solutions of (3.40) and (3.31).

Proposition 3.3.4. *There exists a positive constant $C > 0$, such that*

$$\|(\xi,\zeta)-(\varphi,\nu)\|_{[W^{2,\infty}(0,L)]^2} \leq \left(1+\frac{C^2}{|\sigma|}\right)\left[\sum_{i=0}^{2}\left(|\xi^i|+|\zeta^i|\right)+\frac{1}{|\sigma|}\left(|\xi^3|+|\zeta^3|\right)\right], \tag{3.43}$$

for any $\sigma \in Z$ and any initial data $(\xi^0,\xi^1,\xi^2,\xi^3,\zeta^0,\zeta^1,\zeta^2,\zeta^3) \in \mathbb{C}^8$, where (ξ,ζ) and (φ,ν) are the solutions, with precisely these initial data, of equations (3.40) and (3.31) with $f \equiv g \equiv 0$, respectively.

Proof. We define $\theta = \xi - \varphi$, $u = \zeta - \nu$ and note that (θ, u) is solution of

$$\begin{cases} a\sigma u_{xxx} - b_1\theta_{xxxx} = \xi - \sigma\zeta_x - b\xi_{xx} & \text{for } x \in (0, L), \\ a\sigma\theta_{xxx} - d_1 u_{xxxx} = \zeta - \sigma\xi_x - d\zeta_{xx} & \text{for } x \in (0, L), \\ (\theta(0), u(0)) = (0, 0), \\ (\theta_x(0), u_x(0)) = (0, 0), \\ (\theta_{xx}(0), u_{xx}(0)) = (0, 0), \\ (\theta_{xxx}(0), u_{xxx}(0)) = (0, 0). \end{cases}$$

Therefore, from Lemma 3.3.2 we obtain a constant $C > 0$, such that, for every $x \in [0, L]$ and $\sigma \in Z$,

$$\sum_{i=0}^{2} \left(\left| \frac{d^i\theta}{dx^i}(x) \right| + \left| \frac{d^i u}{dx^i}(x) \right| \right) \leq \frac{C^2}{|\sigma|} \bigg[\int_0^x (|\xi(s)| + |\xi_{xx}(s)| + |\zeta(s)| + |\zeta_{xx}(s)|) ds \\ + \int_0^x |\sigma| (|\xi_x(s)| + |\zeta_x(s)|) ds \bigg].$$

From the estimate above and (3.41) if follows that

$$\sum_{i=0}^{2} \left(\left| \frac{d^i\theta}{dx^i}(x) \right| + \left| \frac{d^i u}{dx^i}(x) \right| \right) \leq \frac{C^2}{|\sigma|} \left[\sum_{i=0}^{2} (|\xi^i| + |\zeta^i|) + \frac{1}{|\sigma|} (|\xi^3| + |\zeta^3|) \right] \\ + C^2 \sum_{i=0}^{2} (|\xi^i| + |\zeta^i|) + \frac{1}{|\sigma|} (|\xi^3| + |\zeta^3|).$$

Then, the solutions (ξ, ζ) and (φ, ν) satisfy (3.43). ■

Finally, we consider systems (3.31) and (3.40) with distinct parameters σ. The difference between the respective solutions are given by the following result.

Proposition 3.3.5. *Let (φ, ν) and (ξ, ζ) solutions of (3.31) with $\sigma = \mu$ and (3.40) with $\sigma = \tilde{\mu}$, respectively, and $f \equiv g \equiv 0$. Then, there exists a positive constant $C > 0$, such that*

$$\|(\xi, \zeta) - (\varphi, \nu)\|_{[W^{2,\infty}(0,L)]^2} \\ \leq C \left[\sum_{i=0}^{2} (|\xi^i - \varphi^i| + |\zeta^i - \nu^i|) + \frac{1}{|\mu|} (|\xi^3 - \varphi^3| + |\zeta^3 - \nu^3| + |\mu - \tilde{\mu}|(|\varphi^3| + |\nu^3|)) \right]. \tag{3.44}$$

Proof. We define $\theta = \xi - \varphi$, $u = \zeta - \nu$, and note that (θ, u) is solution of

$$\begin{cases} a\mu u_{xxx} - b_1\theta_{xxxx} = \xi - \mu\zeta_x - b\xi_{xx} + a(\tilde{\mu} - \mu)\nu_{xxx} & \text{for } x \in (0, L), \\ a\mu\theta_{xxx} - d_1 u_{xxxx} = \zeta - \mu\xi_x - d\zeta_{xx} + a(\tilde{\mu} - \mu)\varphi_{xxx} & \text{for } x \in (0, L), \\ (\theta(0), u(0)) = (\xi^0 - \varphi^0, \zeta^0 - \nu^0), \\ (\theta_x(0), u_x(0)) = (\xi^1 - \varphi^1, \zeta^1 - \nu^1), \\ (\theta_{xx}(0), u_{xx}(0)) = (\xi^2 - \varphi^2, \zeta^2 - \nu^2), \\ (\theta_{xxx}(0), u_{xxx}(0)) = (\xi^3 - \varphi^3, \zeta^3 - \nu^3). \end{cases}$$

Therefore, from Lemma 3.3.2 we obtain (3.44). ■

Spectral analysis of the operator $\mathcal{A}$

Given $b_1, d_1 > 0$, let us first introduce the operator $\mathcal{B} : (H_0^2(0,2\pi))^2 \to (H_0^2(0,2\pi))^2$ given by

$$\mathcal{B} = \begin{pmatrix} 0 & (b_1\partial_x^4)^{-1}(a\partial_x^3) \\ (d_1\partial_x^4)^{-1}(a\partial_x^3) & 0 \end{pmatrix}. \tag{3.45}$$

Recall that, for $\alpha > 0$, the operator $(-\alpha\partial_x^4)^{-1} : L^2(0,2\pi) \to L^2(0,2\pi)$ defined by

$$(-\alpha\partial_x^4)^{-1}\varphi = v \Leftrightarrow \begin{cases} -\alpha v_{xxxx} = \varphi \\ \dfrac{\partial^r v}{\partial x^r}(0) = \dfrac{\partial^r v}{\partial x^r}(L),\ r = 0,1, \end{cases}$$

is a well-defined, compact operator in $L^2(0,2\pi)$.

In this section, $\mu \in \mathbb{C}$ is called eigenvalue of the operator $\mathcal{A}(\mathcal{B})$ if exists a nontrivial vector $\Phi = (\varphi,\nu) \in [H_0^2(0,L)]^2$, called eigenfunction corresponding to μ, such that $\mu\mathcal{A}\Phi = \Phi$ ($\mu\mathcal{B}\Phi = \Phi$). The following two theorems are devoted to the spectral analysis of these operators.

Theorem 3.3.1. *The eigenvalues of the operator $\mathcal{B}$ defined by (3.45) are*

$$\tilde{\mu}_n = sgn(n)\frac{\sqrt{b_1d_1}}{aL}((2|n|+1)\pi - 2\varepsilon_n)\boldsymbol{i}, \tag{3.46}$$

where $\varepsilon_n \in (0,1)$, with $n \in \mathbb{Z}^*$. Each eigenvalue $\tilde{\mu}_n$ is double and has two independent eigenfunctions given by

$$\tilde{\Phi}_n^1 = \left[\frac{\sqrt{b_1d_1}}{a\tilde{\mu}_n}\right]^3 \begin{pmatrix} \mathcal{S}(\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}},x) \\ \sqrt{\frac{b_1}{d_1}}\mathcal{C}(\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}},x) \end{pmatrix}, \quad \tilde{\Phi}_n^2 = \left[\frac{\sqrt{b_1d_1}}{a\tilde{\mu}_n}\right]^3 \begin{pmatrix} \sqrt{\frac{d_1}{b_1}}\mathcal{C}(\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}},x) \\ \mathcal{S}(\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}},x) \end{pmatrix}, \tag{3.47}$$

where

$$\begin{aligned} \mathcal{S}\left(\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}},x\right) &= \sinh(\frac{a\tilde{\mu}_n x}{\sqrt{b_1d_1}}) - \frac{a\tilde{\mu}_n x}{\sqrt{b_1d_1}} + \left[\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}}\right]^3 L\left(\left[\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}}L\right]^2 - 4\right)^{-1} x^2, \\ \mathcal{C}\left(\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}},x\right) &= \left(\cosh(\frac{a\tilde{\mu}_n x}{\sqrt{b_1d_1}}) - 1\right) \\ &\quad - \left(\left[\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}}\right]^2 - \left[\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}}\right]^4 L^2\left(\left[\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}}L\right]^2 - 4\right)^{-1}\right)\frac{x^2}{2}. \end{aligned}$$

Moreover, the set $\left\{\tilde{\Phi}_n^j : n \in \mathbb{Z}^*, j \in \{1,2\}\right\}$ forms an orthogonal basis of $[H_0^2(0,L)]^2$.

Proof. By using Lemma 3.3.1, with $\varphi^0 = \varphi^1 = \nu^0 = \nu^1 = 0$ and $f \equiv g \equiv 0$, we deduce that (φ,ν) is a eigenfunction of $\mathcal{B}$ corresponding to the eigenvalue μ if and only if

$$\begin{aligned} \begin{pmatrix} \varphi(x) \\ \nu(x) \end{pmatrix} &= \frac{1}{\kappa^3}\begin{pmatrix} [\sinh(\kappa x) - \kappa x]\,\varphi^3 + \sqrt{\frac{d_1}{b_1}}\left[(\cosh(\kappa x) - 1) - \frac{[\kappa x]^2}{2}\right]\nu^3 \\ \sqrt{\frac{b_1}{d_1}}\left[(\cosh(\kappa x) - 1) - \frac{[\kappa x]^2}{2}\right]\varphi^3 + [\sinh(\kappa x) - \kappa x]\,\nu^3 \end{pmatrix} \\ &\quad + \begin{pmatrix} \varphi^2\frac{x^2}{2} \\ \nu^2\frac{x^2}{2} \end{pmatrix} \end{aligned} \tag{3.48}$$

and

$$\begin{pmatrix} \varphi(L) \\ \\ \nu(L) \end{pmatrix} = \begin{pmatrix} \varphi_x(L) \\ \\ \nu_x(L) \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}, \tag{3.49}$$

where $\kappa = a\mu/\sqrt{b_1 d_1}$. The data (φ^2, ν^2) can be written as function of κ and (φ^3, ν^3). Indeed, from (3.48) and (3.49) we obtain the following systems

$$\begin{cases} [\sinh(\kappa L) - \kappa L]\, \varphi^3 + \sqrt{\frac{d_1}{b_1}} \left[(\cosh(\kappa L) - 1) - \frac{[\kappa L]^2}{2}\right] \nu^3 + \kappa^3 \frac{L^2}{2} \varphi^2 = 0 \\ \\ \sqrt{\frac{b_1}{d_1}} \left[(\cosh(\kappa L) - 1) - \frac{[\kappa L]^2}{2}\right] \varphi^3 + [\sinh(\kappa L) - \kappa L]\, \nu^3 + \kappa^3 \frac{L^2}{2} \nu^2 = 0 \end{cases}$$

and

$$\begin{cases} (\cosh(\kappa L) - 1)\, \varphi^3 + \sqrt{\frac{d_1}{b_1}} (\sinh(\kappa L) - \kappa L)\, \nu^3 + \kappa^2 L \varphi^2 = 0 \\ \\ \sqrt{\frac{b_1}{d_1}} (\sinh(\kappa L) - \kappa L)\, \varphi^3 + (\cosh(\kappa L) - 1)\, \nu^3 + \kappa^2 L \nu^2 = 0. \end{cases}$$

Thus, we deduce that (φ^2, ν^2) should satisfy

$$\begin{pmatrix} \varphi^2 \\ \\ \nu^2 \end{pmatrix} = \frac{L}{[\kappa L]^2 - 4} \begin{pmatrix} 2 & \sqrt{\frac{d_1}{b_1}} L\kappa \\ \\ \sqrt{\frac{b_1}{d_1}} L\kappa & 2 \end{pmatrix} \begin{pmatrix} \varphi^3 \\ \\ \nu^3 \end{pmatrix}, \tag{3.50}$$

with $\kappa \neq \pm 2/L$. Replacing (3.50) in (3.48) we obtain

$$\begin{pmatrix} \varphi(x) \\ \\ \nu(x) \end{pmatrix} = \frac{1}{\kappa^3} \begin{pmatrix} \mathcal{S}(\kappa, x) & \sqrt{\frac{d_1}{b_1}} \mathcal{C}(\kappa, x) \\ \\ \sqrt{\frac{b_1}{d_1}} \mathcal{C}(\kappa, x) & \mathcal{S}(\kappa, x) \end{pmatrix} \begin{pmatrix} \varphi^3 \\ \\ \nu^3 \end{pmatrix}, \tag{3.51}$$

where

$$\begin{aligned} \mathcal{S}(\kappa, x) &= \sinh(\kappa x) - \kappa x + \frac{[\kappa^3 L]}{[\kappa L]^2 - 4} x^2 \\ \mathcal{C}(\kappa, x) &= (\cosh(\kappa x) - 1) - \left(\kappa^2 - \frac{[\kappa^4 L^2]}{[\kappa L]^2 - 4}\right) \frac{x^2}{2}. \end{aligned}$$

The next steps are devoted to obtain the eigenvalue associated to the eigenfunction given by (3.51). First, we note that $\mathcal{S}_x(\kappa, L) = \kappa \mathcal{C}(\kappa, L)$ and $\mathcal{C}_x(\kappa, L) = \kappa \mathcal{S}(\kappa, L)$. Then, from (3.51) and the boundary conditions (3.49) we have

$$\begin{pmatrix} \varphi(L) \\ \\ \nu(L) \end{pmatrix} = \frac{1}{\kappa^3} \begin{pmatrix} \mathcal{S}(\kappa, L) & \sqrt{\frac{d_1}{b_1}} \mathcal{C}(\kappa, L) \\ \\ \sqrt{\frac{b_1}{d_1}} \mathcal{C}(\kappa, L) & \mathcal{S}(\kappa, L) \end{pmatrix} \begin{pmatrix} \varphi^3 \\ \\ \nu^3 \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \varphi_x(L) \\ \\ \nu_x(L) \end{pmatrix} = \frac{1}{\kappa^2} \begin{pmatrix} \mathcal{C}(\kappa, L) & \sqrt{\frac{d_1}{b_1}} \mathcal{S}(\kappa, L) \\ \\ \sqrt{\frac{b_1}{d_1}} \mathcal{S}(\kappa, L) & \mathcal{C}(\kappa, L) \end{pmatrix} \begin{pmatrix} \varphi^3 \\ \\ \nu^3 \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}.$$

The systems above imply that κ is a root of the equation

$$\mathcal{C}(\kappa, L)^2 - \mathcal{S}(\kappa, L)^2 = 0,$$

which can be written as

$$\frac{4}{[\kappa L]^2-4}\left([\kappa L]\cosh\left(\frac{\kappa L}{2}\right)-2\sinh\left(\frac{\kappa L}{2}\right)\right)^2=0. \tag{3.52}$$

The following result allows us to localize the roots of (3.52).

Lemma 3.3.6. *The nontrivial roots* $(z_n)_{n\in\mathbb{Z}^*}$ *of*

$$f(z)=z\cosh\left(\frac{z}{2}\right)-2\sinh\left(\frac{z}{2}\right) \tag{3.53}$$

satisfy $z_n=\boldsymbol{i}y_n$*, where* $(y_n)_{n\in\mathbb{Z}^*}\subset\mathbb{R}$ *are the roots of the transcendental equation*

$$\tan\left(\frac{y}{2}\right)=\frac{y}{2}.$$

Proof. First, we show that (3.53) has no roots z with $\Re(z)\neq 0$: Indeed, if $z=x+iy$ we have that

$$f\,(x+iy)=f(x,y)=U(x,y)+iV(x,y)$$

where

$$\begin{aligned}
U(x,y)&=x\cosh\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)-\sinh\left(\frac{x}{2}\right)\left(2\cos\left(\frac{y}{2}\right)+y\sin\left(\frac{y}{2}\right)\right),\\
V(x,y)&=\cosh\left(\frac{x}{2}\right)\left(y\cos\left(\frac{y}{2}\right)-2\sin\left(\frac{y}{2}\right)\right)+x\sinh\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right).
\end{aligned}$$

For $y\in\mathbb{R}$ fixed, we define the nonnegative function $K_y(x):=|f(x,y)|^2$. Then,

- $K_y'(x)\Big|_{x=0}=x\cos(y)-x\cosh(x)+\frac{1}{2}(x^2+y^2)\sinh(x)\Big|_{x=0}=0,$
- $K_y''(x)=\frac{1}{2}(x^2+y^2-2)\cosh(x)+\cos(y)\geq 0$, for all $x\in\mathbb{R}$.

The statement above is proved by noting that $x\mapsto K_y''(x)$ is increasing (decreasing) for $x>0$ ($x<0$) and $K_y''(0)=\frac{1}{2}(y^2-2)+\cos(y)\geq 0$, for all $y\in\mathbb{R}$.

Both statements above imply that, for $y\in\mathbb{R}$ fixed, the convex function $x\mapsto|f(x,y)|^2$ has a global minimum value at $(0,y)$. This shows that (x_0,y_0) is root of (3.53) if and only if $x_0=0$ and y_0 is a root of the real valued function $g(y)=y\cos\left(\frac{y}{2}\right)-2\sin\left(\frac{y}{2}\right)$. Then,

$$y\cos\left(\frac{y}{2}\right)-2\sin\left(\frac{y}{2}\right)=0\Leftrightarrow\tan\left(\frac{y}{2}\right)=\frac{y}{2}.$$

∎

By analyzing the graphs of the functions $\tan(x)$ and x (see Figure 3.1), we deduce that the points of intersection $(x_n)_{n\in\mathbb{Z}^*}$, can be written as $x_n=\frac{(2n+1)}{2}\pi-\varepsilon_n$, $x_{-n}=-x_n$, where $\varepsilon_n\in(0,1)$, for all $n\geq 1$.

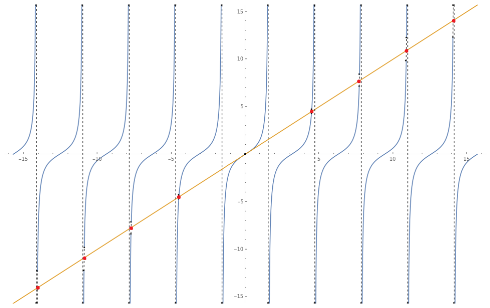

Figure 3.1: The distance ε_n between the root x_n of the equation $tan(x) = x$ and the asymptotic $x = sgn(n)\frac{(2|n|+1)}{2}\pi$ tends to 0, when $|n| \to \infty$.

From the analysis above, we conclude that the roots $(L\kappa_n)_{n\in\mathbb{Z}^*}$ of (3.52) satisfy $L\kappa_n \in i\mathbb{R}$ and $iL\kappa_n = -sgn(n)((2|n|+1)\pi - 2\varepsilon_n)$, for all $n \in \mathbb{Z}^*$. Then, the eigenvalues $(\tilde{\mu}_n)_{n\in\mathbb{Z}^*}$ satisfy $\tilde{\mu}_n = sgn(n)\frac{\sqrt{b_1 d_1}}{aL}((2|n|+1)\pi - 2\varepsilon_n)i$, where $\varepsilon_n \in (0,1)$, with $n \in \mathbb{Z}^*$. ∎

Remark 3.3.7. *If $\tilde{\mu}_n$ is an eigenvalue of the operator $\mathcal{B}$, from (3.50) we have that (φ^2, ν^2) satisfies*

$$\begin{pmatrix} \varphi^2 \\ \nu^2 \end{pmatrix} = \frac{b_1 d_1 L}{[aL\tilde{\mu}_n]^2 - 4b_1 d_1}\begin{pmatrix} 2 \\ \frac{aL\tilde{\mu}_n}{d_1} \end{pmatrix}\varphi^3 + \frac{b_1 d_1 L}{[aL\tilde{\mu}_n]^2 - 4b_1 d_1}\begin{pmatrix} \frac{aL\tilde{\mu}_n}{b_1} \\ 2 \end{pmatrix}\nu^3.$$

By using (3.46) we obtain

$$\begin{aligned}\left|\begin{pmatrix} \varphi^2 \\ \nu^2 \end{pmatrix}\right| &\le C'\left(\frac{1}{((2|n|+1)\pi - 2\varepsilon_n)^2} + \frac{1}{((2|n|+1)\pi - 2\varepsilon_n)}\right)(|\varphi^3| + |\nu^3|) \\ &\le \frac{\tau}{|\tilde{\mu}_n|}(|\varphi^3| + |\nu^3|),\end{aligned}$$

where τ and C' are positive constants.

We pass to analyze the spectral properties of the operator $\mathcal{A}$. The main difference with respect to $\mathcal{B}$ is that we do not have an explicit representation formula as (3.47) for the eigenfunctions of $\mathcal{A}$. Therefore, in order to prove the next theorem, we use a strategy which combines two dimensional versions of the shooting method and Rouché's Theorem.

Theorem 3.3.2. *The eigenvalues of the operator*

$$\mathcal{A} = \begin{pmatrix} 0 & (1 - b\partial_x^2 + b_1\partial_x^4)^{-1}(\partial_x + a\partial_x^3) \\ (1 - d\partial_x^2 + d_1\partial_x^4)^{-1}(\partial_x + a\partial_x^3) & 0 \end{pmatrix}$$

are purely imaginary numbers $(\mu_n^j)_{n\in\mathbb{Z}^*, j\in\{1,2\}}$ with the property that

$$\mu_n^j = \tilde{\mu}_n + \mathcal{O}\left(\frac{1}{|n|}\right) \quad (n \in \mathbb{Z}^*, j \in \{1,2\}). \tag{3.54}$$

Moreover, to each eigenvalue μ_n^j corresponds an eigenfunction Φ_n^j given by

$$\Phi_n^j = \tilde{\Phi}_n^j + \mathcal{O}\left(\frac{1}{|n|}\right) \quad (n \in \mathbb{Z}^*, j \in \{1,2\}), \tag{3.55}$$

with the property that the sequence $(\Phi_n^j)_{n\in\mathbb{Z}^*, j\in\{1,2\}}$ forms an orthogonal basis of $[H_0^2(0,L)]^2$.

Proof. According to the proof of Theorem 3.1.1, $\mathcal{A}$ is a compact skew-adjoint operator in $[H_0^2(0,L)]^2$. Then, it has a sequence of purely imaginary eigenvalues tending to infinity. In order to localize these eigenvalues, let us define, for given $\delta > 0$ and $N \in \mathbb{N}$, the sets

$$\begin{aligned}
&D_n(\delta) = \left\{(\mu,\gamma,\beta) \in \mathbb{C}^4 : |\mu - \tilde{\mu}_n|^2 + |\gamma|^2 < \frac{\delta^2}{n^2}, |\beta| < \frac{\tau}{|\mu|}\right\},\\
&\Gamma_n(\delta) = \partial D_n(\delta), \quad (|n| > N),
\end{aligned}$$

$$\begin{aligned}
&D_N = \left\{(\mu,\gamma,\beta) \in \mathbb{C}^4 : |\Re\mu| \le 1, |\Im\mu| \le \frac{\sqrt{b_1 d_1}}{aL}((2N+2)\pi - 2\varepsilon_N), |\gamma| \le 1, |\beta| \le \frac{\tau}{|\mu|}\right\},\\
&\Gamma_N = \partial D_N,
\end{aligned}$$

where τ is given in Remark 3.3.7 and $\beta \in \mathbb{C}^2$. Also, let us define the maps $F^j, G^j : \mathbb{C}^4 \to \mathbb{C}^4$, $j \in \{1,2\}$, by

$$\begin{aligned}
F^j(\mu,\gamma,\beta^j) &= \left(\begin{pmatrix} \varphi^j(\mu,\gamma,\beta^j,L) \\ \nu^j(\mu,\gamma,\beta^j,L) \end{pmatrix}, \begin{pmatrix} \varphi^j_x(\mu,\gamma,\beta^j,L) \\ \nu^j_x(\mu,\gamma,\beta^j,L) \end{pmatrix} \right),\\
G^j(\mu,\gamma,\beta^j) &= \left(\begin{pmatrix} \tilde{\varphi}^j(\mu,\gamma,\beta^j,L) \\ \tilde{\nu}^j(\mu,\gamma,\beta^j,L) \end{pmatrix}, \begin{pmatrix} \tilde{\varphi}^j_x(\mu,\gamma,\beta^j,L) \\ \tilde{\nu}^j_x(\mu,\gamma,\beta^j,L) \end{pmatrix} \right),
\end{aligned} \tag{3.56}$$

where $\beta^j = (\beta^j_1, \beta^j_2) \in \mathbb{C}^2$, for $j \in \{1,2\}$, and

$$\begin{pmatrix} \varphi^1(\mu,\gamma,\beta^1,\cdot) \\ \nu^j(\mu,\gamma,\beta^1,\cdot) \end{pmatrix}, \begin{pmatrix} \varphi^2(\mu,\gamma,\beta^2,\cdot) \\ \nu^2(\mu,\gamma,\beta^2,\cdot) \end{pmatrix}, \begin{pmatrix} \tilde{\varphi}^1(\mu,\gamma,\beta^1,\cdot) \\ \tilde{\nu}^j(\mu,\gamma,\beta^1,\cdot) \end{pmatrix}, \begin{pmatrix} \tilde{\varphi}^2(\mu,\gamma,\beta^2,\cdot) \\ \tilde{\nu}^2(\mu,\gamma,\beta^2,\cdot) \end{pmatrix}$$

are solutions of the initial values problems

$$\begin{cases}
-\varphi^1 + b\varphi^1_{xx} - b_1\varphi^1_{xxxx} + \mu\nu^1_x + a\mu\nu^1_{xxx} = 0 & \text{for } x \in (0,L),\\
-\nu^1 + d\nu^1_{xx} - d_1\nu^1_{xxxx} + \mu\varphi^1_x + a\mu\varphi^1_{xxx} = 0 & \text{for } x \in (0,L),\\
(\varphi^1(0), \nu^1(0)) = (0,0),\\
(\varphi^1_x(0), \nu^1_x(0)) = (0,0),\\
(\varphi^1_{xx}(0), \nu^1_{xx}(0)) = (\beta^1_1, \beta^1_2),\\
(\varphi^1_{xxx}(0), \nu^1_{xxx}(0)) = (1,\gamma),
\end{cases} \tag{3.57}$$

$$\begin{cases}
-\varphi^2 + b\varphi^2_{xx} - b_1\varphi^2_{xxxx} + \mu\nu^2_x + a\mu\nu^2_{xxx} = 0 & \text{for } x \in (0,L),\\
-\nu^2 + d\nu^2_{xx} - d_1\nu^2_{xxxx} + \mu\varphi^2_x + a\mu\varphi^2_{xxx} = 0 & \text{for } x \in (0,L),\\
(\varphi^2(0), \nu^2(0)) = (0,0),\\
(\varphi^2_x(0), \nu^2_x(0)) = (0,0),\\
(\varphi^2_{xx}(0), \nu^2_{xx}(0)) = (\beta^2_1, \beta^2_2),\\
(\varphi^2_{xxx}(0), \nu^2_{xxx}(0)) = (\gamma,1),
\end{cases} \tag{3.58}$$

$$\begin{cases}
a\mu\tilde{\nu}^1_{xxx} - b_1\tilde{\varphi}^1_{xxxx} = 0 & \text{for } x \in (0,L),\\
a\mu\tilde{\varphi}^1_{xxx} - d_1\tilde{\nu}^1_{xxxx} = 0 & \text{for } x \in (0,L),\\
(\tilde{\varphi}^1(0), \tilde{\nu}^1(0)) = (0,0),\\
(\tilde{\varphi}^1_x(0), \tilde{\nu}^1_x(0)) = (0,0),\\
(\tilde{\varphi}^1_{xx}(0), \tilde{\nu}^1_{xx}(0)) = (\beta^1_1, \beta^1_2),\\
(\tilde{\varphi}^1_{xxx}(0), \tilde{\nu}^1_{xxx}(0)) = (1,\gamma),
\end{cases}$$

$$
\begin{cases}
a\mu\tilde{\nu}^2_{xxx} - b_1\tilde{\varphi}^2_{xxxx} = 0 & \text{for } x \in (0, L), \\
a\mu\tilde{\varphi}^2_{xxx} - d_1\tilde{\nu}^2_{xxxx} = 0 & \text{for } x \in (0, L), \\
(\tilde{\varphi}^2(0), \tilde{\nu}^2(0)) = (0, 0), \\
(\tilde{\varphi}^2_x(0), \tilde{\nu}^2_x(0)) = (0, 0), \\
(\tilde{\varphi}^2_{xx}(0), \tilde{\nu}^2_{xx}(0)) = (\beta^2_1, \beta^2_2), \\
(\tilde{\varphi}^2_{xxx}(0), \tilde{\nu}^2_{xxx}(0)) = (\gamma, 1),
\end{cases}
$$

respectively.

According to Theorem 3.3.1 and Remark 3.3.7, we observe that $\tilde{\mu}$ is an eigenvalue of $\mathcal{B}$ if and only if $G^1(\tilde{\mu}, 0, \tilde{\beta}^1) = 0$, where $\tilde{\beta}^1 = (\tilde{\beta}^1_1, \tilde{\beta}^1_2)$ satisfies

$$
\tilde{\beta}^1_1 = \frac{2b_1d_1L}{(aL\tilde{\mu})^2 - 4b_1d_1} \quad \text{and} \quad \tilde{\beta}^1_2 = \frac{b_1aL^2\tilde{\mu}}{(aL\tilde{\mu})^2 - 4b_1d_1},
$$

or $G^2(\tilde{\mu}, 0, \tilde{\beta}^2) = 0$, where $\tilde{\beta}^2 = (\tilde{\beta}^2_1, \tilde{\beta}^2_2)$ satisfies

$$
\tilde{\beta}^2_1 = \frac{d_1aL^2\tilde{\mu}}{(aL\tilde{\mu})^2 - 4b_1d_1} \quad \text{and} \quad \tilde{\beta}^2_2 = \frac{2b_1d_1L}{(aL\tilde{\mu})^2 - 4b_1d_1}.
$$

Moreover, from the definition (3.56) and (3.57)-(3.58), we deduce that μ is an eigenvalue of $\mathcal{A}$ if and only if there exists $(\gamma, \beta) \in \mathbb{C}^3$, such that $F^1(\mu, \gamma, \beta) = 0$ or $F^2(\mu, \gamma, \beta) = 0$. Hence, we have reduced the problem of finding the eigenvalues of $\mathcal{A}$ to the problem of determining the zeros of the maps $(F^j)_{j=1,2}$. We analyze only the zeros of the map F^1, since the analysis of those of F^2 is similar. First, we note that the maps F^1 and G^1 are analytic and that

$$
\left|F^1(\mu, \gamma, \beta) - G^1(\mu, \gamma, \beta)\right| \leq \frac{C_1}{|\mu|} \quad \left(|\Re\mu| \leq 1, |\mu| \geq \frac{1}{2}, |\gamma| \leq 1, |\beta| \leq \frac{\tau}{|\mu|}\right), \tag{3.59}
$$

$$
\left|G^1(\mu, \gamma, \beta)\right| \geq \frac{\delta C_2}{|\mu|} \quad \left((\mu, \gamma, \beta) \in \Gamma_n(\delta)\right), \tag{3.60}
$$

for some positive constants C_1, C_2. Indeed, since $\mu \in Z, |\gamma| \leq 1$ and $|\beta| \leq \frac{\tau}{|\mu|}$, (3.59) is a direct consequence of Proposition 3.3.4. On the other hand, since $G^1(\tilde{\mu}_n, 0, \tilde{\beta}^1_n) = 0$, we can find $C_2 > 0$, such that

$$
|\mu||G^1(\mu, \gamma, \beta)| \geq \delta C_2,
$$

for $(\mu, \gamma, \beta) \in \Gamma_n(\delta)$ and we obtain (3.60). It follows from the multidimensional version of Rouché's Theorem [30, Theorem 1] (see, also, [31, Theorem 3]) that there exist $\delta > 0$ and $N > 0$, such that the maps F^1 and G^1 have the same number of zeros in $D_n(\delta)$, for each $|n| \geq N$. Since G^1 has exactly one zero $(\tilde{\mu}_n, 0, \tilde{\beta}^1_n)$ in $D_n(\delta)$, then F^1 has a unique zero $(\mu^1_n, \gamma^1_n, \beta^1_n)$ in $D_n(\delta)$. Thus, we have obtained the eigenvalues $(\mu^1_n)_{|n|\geq N}$ of $\mathcal{A}$ and proved the corresponding asymptotic estimate (3.54). Arguing as before, we get the existence of a family of zeros $(\mu^2_n, \gamma^2_n, \beta^2_n)_{|n|\geq N}$ for the map F^2. Then, we obtain the other sequence of eigenvalues $(\mu^2_n)_{|n|\geq N}$ of $\mathcal{A}$ and the corresponding asymptotic estimate. To obtain the remaining eigenvalues, we note that, since

$$
\mathcal{S}\left(\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}}, L\right) = \mathcal{C}\left(\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}}, L\right) = 0 \quad \text{for all} \quad 1 \leq |n| \leq N,
$$

then, there exist a positive constant C_3, such that

$$
\min\left\{\left|\mathcal{S}\left(\frac{a\mu}{\sqrt{b_1d_1}}, L\right)\right|, \left|\mathcal{C}\left(\frac{a\mu}{\sqrt{b_1d_1}}, L\right)\right|\right\} \geq C_3
$$

$$
\text{for } \mu \in \partial\left(|\Re\mu| \leq 1, |\Im\mu| \leq \frac{\sqrt{b_1d_1}}{aL}((2N+2)\pi - 2\varepsilon_N)\right).
$$

This implies that $\left|G^1(\mu,\gamma,\beta)\right| \geq \frac{\delta C_4}{|\mu|} \quad ((\mu,\gamma,\beta) \in \Gamma_N)$ for some $C_4 > 0$. Combining the last estimate with (3.59) and applying again the multidimensional Rouché's Theorem, we obtain the eigenvalues $(\mu_n^1)_{|n|\leq N}$ of $\mathcal{A}$ in D_N. From the analysis of the map F^2 we get the existence of the remaining eigenvalues $(\mu_n^2)_{|n|\leq N}$.

Let us pass to the analysis of the eigenfunctions. To each eigenvalue μ_n^j corresponds a unique normalized eigenfunction Φ_n^j satisfying (3.57) with $\gamma = \gamma_n^1$ and $\beta = \beta_n^1 = (\beta_{1,n}^1, \beta_{2,n}^1)$ or (3.58) with $\gamma = \gamma_n^2$ and $\beta = \beta_n^2 = (\beta_{1,n}^2, \beta_{2,n}^2)$, respectively. Since

$$|\gamma_n^j| \leq \frac{\delta}{|n|}, \quad |\mu_n^j - \tilde{\mu}_n| \leq \frac{\delta}{|n|} \quad \text{and} \quad |\beta_n^j - \tilde{\beta}_n^j| \leq \tau\left(\frac{1}{|\mu_n^j|} + \frac{1}{|\tilde{\mu}_n^j|}\right) \quad \text{for} \quad j = 1,2,$$

then, from Proposition 3.3.5, we deduce that (3.55) is verified. Finally, since $\mathcal{A}$ is a skew-adjoint operator, these eigenfunctions are orthogonal in $[H_0^2(0,L)]^2$. ■

The next steps are devoted to analyze carefully the asymptotic behavior of the eigenvalues of the operator $\mathcal{A}$. More precisely, we analyze the coefficients A_n^j, B_n^j, C_n^j and D_n^j $(j = 1,2)$ of the systems (3.21).

Lemma 3.3.8. *For a subsequence, if necessary, the following holds,*

$$\lim_{|n|\to\infty} C_n^j = \lim_{|n|\to\infty} D_n^j = \lim_{|n|\to\infty} A_n^2 = \lim_{|n|\to\infty} B_n^1 = 0, \quad j = 1,2,$$

$$\lim_{|n|\to\infty} A_n^1 = \lim_{|n|\to\infty} B_n^2 = \delta_0 \frac{\sqrt{b_1 d_1}}{L}, \quad \textit{for some} \quad \delta_0 \in \mathbb{C}^*,$$

and

$$\begin{vmatrix} C_n^1 & D_n^1 \\ C_n^2 & D_n^2 \end{vmatrix} \sim \frac{-L^2 b_1 d_1}{[(2|n|+1)\pi - 2\varepsilon_n]^2 + 4}, \quad \textit{for all} \quad n \in \mathbb{Z}^*.$$

Proof. In order to prove the result, we first consider the solutions of the following problems

$$\begin{cases} -a\tilde{\mu}_n \tilde{\nu}_{n,xxx}^1 - b_1 \tilde{\varphi}_{n,xxxx}^1 = 0 & \text{for } x \in (0,L), \\ -a\tilde{\mu}_n \tilde{\varphi}_{n,xxx}^1 - d_1 \tilde{\nu}_{n,xxxx}^1 = 0 & \text{for } x \in (0,L), \\ (\tilde{\varphi}^1(0), \tilde{\nu}^1(0)) = (0,0), \\ (\tilde{\varphi}_{n,x}^1(0), \tilde{\nu}_{n,x}^1(0)) = (0,0), \\ (\tilde{\varphi}_{n,xx}^1(0), \tilde{\nu}_{n,xx}^1(0)) = (\tilde{\beta}_{1,n}^1, \tilde{\beta}_{2,n}^1), \\ (\tilde{\varphi}_{n,xxx}^1(0), \tilde{\nu}_{n,xxx}^1(0)) = (1,0), \end{cases} \tag{3.61}$$

and

$$\begin{cases} -a\tilde{\mu}_n \tilde{\nu}_{n,xxx}^2 - b_1 \tilde{\varphi}_{n,xxxx}^2 = 0 & \text{for } x \in (0,L), \\ -a\tilde{\mu}_n \tilde{\varphi}_{n,xxx}^2 - d_1 \tilde{\nu}_{n,xxxx}^2 = 0 & \text{for } x \in (0,L), \\ (\tilde{\varphi}^2(0), \tilde{\nu}^2(0)) = (0,0), \\ (\tilde{\varphi}_{n,x}^2(0), \tilde{\nu}_{n,x}^2(0)) = (0,0), \\ (\tilde{\varphi}_{n,xx}^2(0), \tilde{\nu}_{n,xx}^2(0)) = (\tilde{\beta}_{1,n}^2, \tilde{\beta}_{2,n}^2), \\ (\tilde{\varphi}_{n,xxx}^2(0), \tilde{\nu}_{n,xxx}^2(0)) = (0,1). \end{cases} \tag{3.62}$$

For each $\tilde{\mu}_n = -sgn(n)\frac{\sqrt{b_1 d_1}}{aL}((2|n|+1)\pi - 2\varepsilon_n)i$ $(n \in \mathbb{Z}^*, \varepsilon_n \in (0,1))$, $(\tilde{\beta}_{1,n}^1, \tilde{\beta}_{2,n}^1)$ satisfying

$$\tilde{\beta}_{1,n}^1 = \frac{2b_1 d_1 L}{(aL\tilde{\mu}_n)^2 - 4b_1 d_1}, \quad \tilde{\beta}_{2,n}^1 = -\frac{b_1 a L^2 \tilde{\mu}_n}{(aL\tilde{\mu}_n)^2 - 4b_1 d_1} \tag{3.63}$$

and $(\tilde{\beta}^2_{1,n}, \tilde{\beta}^2_{2,n})$ satisfying

$$\tilde{\beta}^2_{1,n} = -\frac{d_1 a L^2 \tilde{\mu}_n}{(aL\tilde{\mu}_n)^2 - 4b_1 d_1}, \quad \tilde{\beta}^2_{2,n} = \frac{2b_1 d_1 L}{(aL\tilde{\mu}_n)^2 - 4b_1 d_1},$$

the solutions of (3.61) and (3.62) are given by formula (3.47) and will be denoted by

$$\tilde{\Phi}^1_n = \begin{pmatrix} \tilde{\varphi}^1_n \\ \\ \tilde{\nu}^1_n \end{pmatrix} \quad \text{and} \quad \tilde{\Phi}^2_n = \begin{pmatrix} \tilde{\varphi}^2_n \\ \\ \tilde{\nu}^2_n \end{pmatrix},$$

respectively. We set $\kappa_n = -\frac{a\tilde{\mu}_n}{\sqrt{b_1 d_1}}$. Then, from Theorem 3.3.1, we get $\mathcal{S}(\kappa_n, L) = \mathcal{C}(\kappa_n, L) = 0$, which implies that

$$\begin{aligned}
\sinh(\kappa_n L) &= \kappa_n L - \frac{[\kappa_n L]^3}{[\kappa_n L]^2 - 4} = -\frac{4[\kappa_n L]}{[\kappa_n L]^2 - 4},\\
\cosh(\kappa_n L) - 1 &= -\frac{2[\kappa_n L]^2}{[\kappa_n L]^2 - 4}.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{S}_{xx}(\kappa_n, L) &= \kappa_n^2 \left[\sinh(\kappa_n L) + \frac{2\kappa_n L}{[\kappa_n L]^2 - 4}\right] = -\kappa_n^3 \left[\frac{2L}{[\kappa_n L]^2 - 4}\right],\\
\mathcal{C}_{xx}(\kappa_n, L) &= \kappa_n^2 \left[(\cosh(\kappa_n L) - 1) + \frac{[\kappa_n L]^2}{[\kappa_n L]^2 - 4}\right] = -\kappa_n^3 \left[\frac{\kappa_n L^2}{[\kappa_n L]^2 - 4}\right].
\end{aligned}$$

Consequently, the functions $\tilde{\Phi}^j_n$, $j = 1, 2$, satisfy

$$\tilde{\Phi}^1_{n,xx}(L) = \begin{pmatrix} \tilde{\varphi}^1_{n,xx}(L) \\ \\ \tilde{\nu}^1_{n,xx}(L) \end{pmatrix} = \frac{-L}{[\kappa_n L]^2 - 4} \begin{pmatrix} 2 \\ \\ \kappa_n L \sqrt{\frac{b_1}{d_1}} \end{pmatrix} \tag{3.64}$$

and

$$\tilde{\Phi}^2_{n,xx}(L) = \begin{pmatrix} \tilde{\varphi}^2_{n,xx}(L) \\ \\ \tilde{\nu}^2_{n,xx}(L) \end{pmatrix} = \frac{-L}{[\kappa_n L]^2 - 4} \begin{pmatrix} \kappa_n L \sqrt{\frac{d_1}{b_1}} \\ \\ 2 \end{pmatrix}, \tag{3.65}$$

for $n \in \mathbb{N}^*$. Now, we pass to the study of the asymptotic behavior of the eigenvectors of the operator $-\mathcal{A}$. From the proof of Theorem 3.3.2 we have that, for each eigenvalue $-\mu^j_n$, the corresponding eigenfunctions $\Phi^1_n = (\varphi^1_n, \nu^1_n)$ and $\Phi^2_n = (\varphi^2_n, \nu^2_n)$ are solutions of

$$\begin{cases}
-a\mu^1_n \nu^1_{n,xxx} - b_1 \varphi^1_{n,xxxx} = \varphi^1_n - \mu^1_n \nu^1_{n,x} - b\varphi^1_{n,xx} & \text{for } x \in (0, L),\\
-a\mu^1_n \varphi^1_{n,xxx} - d_1 \nu^1_{n,xxxx} = \nu^1_n - \mu^1_n \varphi^1_{n,x} - d\nu^1_{n,xx} & \text{for } x \in (0, L),\\
(\varphi^1(0), \nu^1(0)) = (0, 0),\\
(\varphi^1_{n,x}(0), \nu^1_{n,x}(0)) = (0, 0),\\
(\varphi^1_{n,xx}(0), \nu^1_{n,xx}(0)) = (\beta^1_{1,n}, \beta^1_{2,n}),\\
(\varphi^1_{n,xxx}(0), \nu^1_{n,xxx}(0)) = (1, \gamma^1_n),
\end{cases} \tag{3.66}$$

and

$$\begin{cases}
-a\mu^2_n \nu^2_{n,xxx} - b_1 \varphi^2_{n,xxxx} = \varphi^2_n - \mu^2_n \nu^2_{n,x} - b\varphi^2_{n,xx} & \text{for } x \in (0, L),\\
-a\mu^2_n \varphi^2_{n,xxx} - d_1 \nu^2_{n,xxxx} = \nu^2_n - \mu^2_n \varphi^2_{n,x} - d\nu^2_{n,xx} & \text{for } x \in (0, L),\\
(\varphi^2(0), \nu^2(0)) = (0, 0),\\
(\varphi^2_{n,x}(0), \nu^2_{n,x}(0)) = (0, 0),\\
(\varphi^2_{n,xx}(0), \nu^2_{n,xx}(0)) = (\beta^2_{1,n}, \beta^2_{2,n}),\\
(\varphi^2_{n,xxx}(0), \nu^2_{n,xxx}(0)) = (\gamma^2_n, 1),
\end{cases} \tag{3.67}$$

respectively. We also note that, according to Theorem 3.3.2, the data in (3.66) and (3.67) satisfies

$$|\gamma_n^j| \le \frac{\delta}{|n|}, \quad |\mu_n^j - \tilde{\mu}_n| \le \frac{\delta}{|n|}.$$

Since $|\beta_n^j - \tilde{\beta}_n^j| \to 0$, as $|n| \to \infty$, for $j = 1, 2$, we can extract a subsequence, if necessary, such that

$$\begin{aligned} |\beta_{1,n}^1 - \tilde{\beta}_{1,n}^1| \le \frac{\delta}{|n|^2}, \quad |\beta_{2,n}^1 - \tilde{\beta}_{2,n}^1| \le \frac{\delta}{|n|}, \\ |\beta_{1,n}^2 - \tilde{\beta}_{1,n}^2| \le \frac{\delta}{|n|}, \quad |\beta_{2,n}^2 - \tilde{\beta}_{2,n}^2| \le \frac{\delta}{|n|^2}, \end{aligned} \tag{3.68}$$

for a given positive δ. Therefore, from Proposition 3.3.5, the eigenfunction (φ_n^1, ν_n^1) satisfies

$$\begin{aligned} &|\varphi_{n,xx}^1(L) - \tilde{\varphi}_{n,xx}^1(L)| + |\nu_{n,xx}^1(L) - \tilde{\nu}_{n,xx}^1(L)| \\ &\quad \le C\left[\left(|\beta_{1,n}^1 - \tilde{\beta}_{1,n}^1| + |\beta_{2,n}^1 - \tilde{\beta}_{2,n}^1|\right) + \frac{1}{|\mu_n^1|}\left(|\gamma_n^1| + |\mu_n^1 - \tilde{\mu}_n|(1 + |\gamma_n^1|)\right)\right] \\ &\quad \le C\left[\left(\frac{\delta}{|n|^2} + \frac{\delta}{|n|}\right) + \frac{1}{|\mu_n^1|}\left(\frac{\delta}{|n|} + \frac{\delta}{|n|}(1 + \frac{\delta}{|n|})\right)\right]. \end{aligned}$$

Similarly, the eigenfunction (φ_n^2, ν_n^2) satisfies

$$\begin{aligned} &|\varphi_{n,xx}^2(L) - \tilde{\varphi}_{n,xx}^2(L)| + |\nu_{n,xx}^2(L) - \tilde{\nu}_{n,xx}^2(L)| \\ &\quad \le C\left[\left(\frac{\delta}{|n|^2} + \frac{\delta}{|n|}\right) + \frac{1}{|\mu_n^2|}\left(\frac{\delta}{|n|} + \frac{\delta}{|n|}(1 + \frac{\delta}{|n|})\right)\right]. \end{aligned}$$

From the estimates above and (3.64)-(3.65), we conclude that

$$\begin{pmatrix} C_n^1 \\ \\ D_n^1 \end{pmatrix} = \begin{pmatrix} b_1\varphi_{n,xx}^1(L) \\ \\ d_1\nu_{n,xx}^1(L) \end{pmatrix} \sim \frac{-L}{[\kappa_n L]^2 - 4}\begin{pmatrix} 2b_1 \\ \\ \kappa_n L\sqrt{b_1 d_1} \end{pmatrix}$$

and

$$\begin{pmatrix} C_n^2 \\ \\ D_n^2 \end{pmatrix} = \begin{pmatrix} b_1\varphi_{n,xx}^2(L) \\ \\ d_1\nu_{n,xx}^1(L) \end{pmatrix} \sim \frac{-L}{[\kappa_n L]^2 - 4}\begin{pmatrix} \kappa_n L\sqrt{b_1 d_1} \\ \\ 2d_1 \end{pmatrix}.$$

Thus,

$$\begin{vmatrix} C_n^1 & D_n^1 \\ \\ C_n^2 & D_n^2 \end{vmatrix} \sim \frac{L^2 b_1 d_1}{[\kappa_n L]^2 - 4} \neq 0,$$

which gives the behavior of the coefficients C_n^j and D_n^j, for $j = 1, 2$.

On the other hand, by integrating the equations in (3.66) over $(0, L)$ we obtain the coefficients A_n^1 and B_n^1:

$$\begin{aligned} A_n^1 &= \left(-a\mu_n^1\nu_{n,xx}^1 - b_1\varphi_{n,xxx}^1\right)(L) = \int_0^L \varphi_n^1(x)dx + a\mu_n^1\beta_{2,n}^1 - b_1, \\ B_n^1 &= \left(-a\mu_n^1\varphi_{n,xx}^1 - d_1\nu_{n,xxx}^1\right)(L) = \int_0^L \nu_n^1(x)dx + a\mu_n^1\beta_{1,n}^1 - d_1\gamma_n^1. \end{aligned} \tag{3.69}$$

The next steps are devoted to study the term on the right hand side of the equations in (3.69). First, we note that, from Theorem 3.3.2,

$$\begin{aligned} \int_0^L \varphi_n^1(x)dx &= \int_0^L \tilde{\varphi}_n^1(x)dx + \mathcal{O}\left(\frac{1}{|n|}\right), \\ \int_0^L \nu_n^1(x)dx &= \int_0^L \tilde{\nu}_n^1(x)dx + \mathcal{O}\left(\frac{1}{|n|}\right). \end{aligned}$$

Then, from the formula (3.47) we conclude that

$$\lim_{|n|\to\infty}\int_0^L \varphi_n^1(x)dx = \lim_{|n|\to\infty}\int_0^L \nu_n^1(x)dx = 0. \tag{3.70}$$

On the other hand, from (3.68) we get

$$\begin{aligned} a\mu_n^1\beta_{1,n}^1 &= a\left(\tilde{\mu}_n^1 + \mathcal{O}\left(\frac{1}{|n|}\right)\right)\left(\tilde{\beta}_{1,n}^1 + \mathcal{O}\left(\frac{1}{|n|^2}\right)\right) \\ &= a\tilde{\mu}_n^1\tilde{\beta}_{1,n}^1 + a\tilde{\mu}_n^1\mathcal{O}\left(\frac{1}{|n|^2}\right) + a\tilde{\beta}_{1,n}^1\mathcal{O}\left(\frac{1}{|n|}\right) + a\mathcal{O}\left(\frac{1}{|n|^3}\right), \end{aligned} \tag{3.71}$$

and

$$\begin{aligned} a\mu_n^1\beta_{2,n}^1 &= a\left(\tilde{\mu}_n^1 + \mathcal{O}\left(\frac{1}{|n|}\right)\right)\left(\tilde{\beta}_{2,n}^1 + \mathcal{O}\left(\frac{1}{|n|}\right)\right) \\ &= a\tilde{\mu}_n^1\tilde{\beta}_{2,n}^1 + a\tilde{\mu}_n^1\mathcal{O}\left(\frac{1}{|n|}\right) + a\tilde{\beta}_{2,n}^1\mathcal{O}\left(\frac{1}{|n|}\right) + a\mathcal{O}\left(\frac{1}{|n|^2}\right). \end{aligned} \tag{3.72}$$

From (3.63), we note that the right side of (3.71) tends to 0 as $|n| \to \infty$, the last two terms on the right side of (3.72) tend to 0 as $|n| \to \infty$, and, finally, the first two terms satisfy

$$\lim_{|n|\to\infty} a\tilde{\mu}_n^1\tilde{\beta}_{2,n}^1 = -b_1 \quad \text{and} \quad \lim_{|n|\to\infty} a\tilde{\mu}_n^1\mathcal{O}\left(\frac{1}{|n|}\right) = \delta_0\frac{\sqrt{b_1 d_1}}{L},$$

for some $\delta_0 \in \mathbb{C}^*$. Then, from (3.70), (3.71) and (3.72), we conclude that

$$\lim_{|n|\to\infty} A_n^1 = \delta_0\frac{\sqrt{b_1 d_1}}{L} \quad \text{and} \quad \lim_{|n|\to\infty} B_n^1 = 0.$$

In order to conclude the proof, we integrate the equations in (3.67) over $(0, L)$ to obtain

$$\begin{aligned} A_n^2 &= \left(-a\mu_n^2\nu_{n,xx}^2 - b_1\varphi_{n,xxx}^2\right)(L) = \int_0^L \varphi_n^2(x)dx + a\mu_n^2\beta_{2,n}^2 - b_1\gamma_n^2, \\ B_n^2 &= \left(-a\mu_n^2\varphi_{n,xx}^2 - d_1\nu_{n,xxx}^2\right)(L) = \int_0^L \nu_n^2(x)dx + a\mu_n^2\beta_{1,n}^2 - d_1. \end{aligned}$$

Then, by arguing as in the previous steps, we deduce that

$$\lim_{|n|\to\infty} A_n^2 = 0 \quad \text{and} \quad \lim_{|n|\to\infty} B_n^2 = \delta_0\frac{\sqrt{b_1 d_1}}{L}.$$

■

Chapter 4

Asymptotic behavior of a linear higher-order Boussinesq system with damping

We introduce several mechanisms to dissipate the energy associated to a linear higher-order Benjamin-Bona-Mahony-type system. We consider either a distributed (localized) feedback law, or a boundary feedback law. In each case, we prove the global well-posedness of the system and the convergence towards a solution which is null on a band. If the Unique Continuation Property holds for the conservative model, this implies that the origin is asymptotically stable for the corresponding damped one.

4.1 Unique Continuation Property

The aim of this section is to study some unique continuation properties for the following higher-order system

$$\begin{cases} \eta_t + \omega_x + a\omega_{xxx} - b\eta_{txx} + b_1\eta_{txxxx} = 0 & \text{for } x \in (0,2\pi),\ t>0, \\ \omega_t + \eta_x + c\eta_{xxx} - d\omega_{txx} + d_1\omega_{txxxx} = 0 & \text{for } x \in (0,2\pi), t>0, \\ \frac{\partial^r \eta}{\partial x^r}(t,0) = \frac{\partial^r \eta}{\partial x^r}(t,2\pi), \quad \frac{\partial^r \omega}{\partial x^r}(t,0) = \frac{\partial^r \omega}{\partial x^r}(t,2\pi) & \text{for } t>0, \quad r=0,1, \\ \eta(0,x) = \eta^0(x); \quad \omega(0,x) = \omega^0(x) & \text{for } x \in (0,2\pi), \end{cases} \tag{4.1}$$

where $b, b_1, d, d_1 > 0$ and $a, c < 0$ or $a = c \geq 0$.

We first introduce a few notations. Given any $v \in L^2(0,2\pi)$ and $k \in \mathbb{Z}$, we denote by $\widehat{v}_k$ the $k-$Fourier coefficient of v,

$$\widehat{v}_k = \frac{1}{2\pi}\int_0^{2\pi} v(x)e^{-ikx}\,\mathrm{d}x.$$

Then, for any $s \in \mathbb{R}$, we define the Hilbert space

$$H_p^s(0,2\pi) = \left\{ v = \sum_{k\in\mathbb{Z}} \widehat{v}_k e^{ikx} \in L^2(0,2\pi) \;\middle|\; \sum_{k\in\mathbb{Z}} |\widehat{v}_k|^2(1+k^2)^s < \infty \right\}$$

with respect to the inner product

$$(v,w)_s = \sum_{k\in\mathbb{Z}} k\widehat{v}_k\overline{\widehat{w}_k}(1+k^2)^s.$$

We denote by $\|\cdot\|_s$ the corresponding norm to the inner product given above, more precisely,

$$||v||_s = \left(\sum_{k\in\mathbb{Z}} |\widehat{v}_k|^2(1+k^2)^s\right)^{1/2}.$$

Under the considerations above, for $\alpha, \beta > 0$ we can define the operator $(1-\alpha\partial_x^2+\beta\partial_x^4)_p^{-1}$ in the following way:

$$(1-\alpha\partial_x^2+\beta\partial_x^4)_p^{-1}\phi = v \Leftrightarrow \begin{cases} v-\alpha v_{xx}+\beta v_{xxxx}=\phi & \text{in} \quad (0,2\pi) \\ \\ \frac{\partial^r v}{\partial x^r}(0)=\frac{\partial^r v}{\partial x^r}(2\pi) & r=0,1,2,3. \end{cases} \tag{4.2}$$

Since for any $\varphi \in L^2(0,2\pi)$, the elliptic equation (4.2) has an unique solution $v \in H_p^4(0,2\pi)$, the operator $(1-\alpha\partial_x^2+\beta\partial_x^4)_p^{-1}$ is a well-defined, compact operator in $L^2(0,2\pi)$.

Given $s \in \mathbb{R}$, let us introduce the Hilbert space

$$V^s = H_p^s(0,2\pi)\times H_p^s(0,2\pi),$$

endowed with the inner product defined by

$$\langle (f_1,f_2),(g_1,g_2)\rangle = (f_1,g_1)_s + (\mathcal{H}f_2,\mathcal{H}g_2)_s,$$

and the operator $\mathcal{H}$ defined in the following way

$$\mathcal{H}\left(\sum_{k\in\mathbb{Z}} \hat{a}_k e^{ikx}\right) = \sum_{k\in\mathbb{Z}} \sqrt{\frac{w_1}{w_2}}\,\hat{a}_k e^{ikx},$$

where, for $b, b_1, d, d_1 > 0$ and $a, c < 0$ or $a = c \geq 0$, we set

$$w_1 = w_1(k) = \frac{1-ak^2}{1+bk^2+b_1k^4} \quad \text{and} \quad w_2 = w_2(k) = \frac{1-ck^2}{1+dk^2+d_1k^4}. \tag{4.3}$$

Thus, $w_1w_2, w_1/w_2 > 0$.

System (4.1) can be written in the following vectorial form

$$\begin{pmatrix} \eta \\ \omega \end{pmatrix}_t (t) + A\begin{pmatrix} \eta \\ \omega \end{pmatrix}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \eta \\ \omega \end{pmatrix}(0) = \begin{pmatrix} \eta^0 \\ \omega^0 \end{pmatrix},$$

where A is the linear compact operator in V^s defined by

$$A = \begin{pmatrix} 0 & (1-b\partial_x^2+b_1\partial_x^4)_p^{-1}(\partial_x+a\partial_x^3) \\ (1-d\partial_x^2+d_1\partial_x^4)_p^{-1}(\partial_x+c\partial_x^3) & 0 \end{pmatrix}. \tag{4.4}$$

Thus, if we assume that the initial date in (4.1) is given by

$$(\eta^0,\omega^0) = \sum_{k\in\mathbb{Z}} (\hat{\eta}_k^0,\hat{\omega}_k^0)e^{ikx},$$

then, at least formally, the solution of (4.1) can be written as

$$(\eta,\omega)(t,x) = \sum_{k\in\mathbb{Z}} (\hat{\eta}_k(t),\hat{\omega}_k(t))e^{ikx},$$

where $(\hat{\eta}_k(t),\hat{\omega}_k(t))$ fulfill

$$\begin{cases} (1+bk^2+b_1k^4)(\hat{\eta}_k)_t + ik(1-ak^2)\hat{\omega}_k = 0, & t\in(0,T) \\ (1+dk^2+d_1k^4)(\hat{\omega}_k)_t + ik(1-ck^2)\hat{\eta}_k = 0, & t\in(0,T) \\ \hat{\eta}_k(0)=\hat{\eta}_k^0, \quad \hat{\omega}_k(0)=\hat{\omega}_k^0. \end{cases} \tag{4.5}$$

Hence, we have the following result:

Lemma 4.1.1. *Let $\sigma(k) = \sqrt{w_1 w_2}$. The eigenvalues of the operator A defined by (4.4) are given by*

$$\lambda_k^{\pm} = \pm i|k|\sigma(k) \quad (k \in \mathbb{Z}^*). \tag{4.6}$$

The solution $(\hat{\eta}_k(t), \hat{\omega}_k(t))$ of (4.5) is given by

$$\begin{cases} \hat{\eta}_k(t) = \cos[k\sigma(k)t]\hat{\eta}_k^0 - i\sqrt{\frac{w_1}{w_2}}\sin[k\sigma(k)t]\hat{\omega}_k^0, \\ \hat{\omega}_k(t) = \cos[k\sigma(k)t]\hat{\omega}_k^0 - i\sqrt{\frac{w_2}{w_1}}\sin[k\sigma(k)t]\hat{\eta}_k^0. \end{cases} \tag{4.7}$$

Proof. System (4.5) can be written in the following equivalent form

$$\begin{pmatrix} \hat{\eta}_k \\ \hat{\omega}_k \end{pmatrix}_t (t) + A(k) \begin{pmatrix} \hat{\eta}_k \\ \hat{\omega}_k \end{pmatrix}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \hat{\eta}_k \\ \hat{\omega}_k \end{pmatrix}(0) = \begin{pmatrix} \hat{\eta}_k^0 \\ \hat{\omega}_k^0 \end{pmatrix},$$

where

$$A(k) = \begin{pmatrix} 0 & ikw_1 \\ & \\ ikw_2 & 0 \end{pmatrix}.$$

Hence, we deduce that the solution of (4.5) is given by

$$\begin{pmatrix} \hat{\eta}_k \\ \hat{\omega}_k \end{pmatrix}(t) = e^{-A(k)t} \begin{pmatrix} \hat{\eta}_k^0 \\ \hat{\omega}_k^0 \end{pmatrix} \tag{4.8}$$

and it is easy to see that the eigenvalues $\lambda_k^{\pm}$ of the matrix $A(k)$ are given by (4.6). Thus, in order to obtain (4.7), we will make use of the following result from [1]:

Proposition 4.1.2. *Let A a 2×2 matrix with eigenvalues $\lambda_1 \neq \lambda_2$. If*

$$Q_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}; \quad Q_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1},$$

then

1. $A = \lambda_1 Q_1 + \lambda_2 Q_2$;
2. $Q_1^2 = Q_1; \quad Q_2^2 = Q_2; \quad Q_2 Q_1 = Q_1 Q_2 = 0$;
3. $A^k = \lambda_1^k Q_1 + \lambda_2^k Q_2, \quad \forall k \in \mathbb{N}$;
4. $e^{At} = e^{\lambda_1 t} Q_1 + e^{\lambda_2 t} Q_2$.

Moreover, if $\lambda_1 = \lambda_2 = \lambda_0$ and $Q = A - \lambda_0 I$, then $e^{At} = (I + tQ)e^{\lambda_0 t}$.

Then, we have that

$$e^{-A(k)t} = e^{-\lambda_k^+ t} Q_1(k) + e^{-\lambda_k^- t} Q_2(k), \tag{4.9}$$

where

$$Q_1(k) = \frac{A(k) - \lambda_k^- I}{\lambda_k^+ - \lambda_k^-} = \frac{1}{2} \begin{pmatrix} 1 & sgn(k)\sqrt{\frac{w_1}{w_2}} \\ sgn(k)\sqrt{\frac{w_2}{w_1}} & 1 \end{pmatrix} \tag{4.10}$$

and

$$Q_2(k) = \frac{A(k) - \lambda_k^+ I}{\lambda_k^- - \lambda_k^+} = \frac{1}{2} \begin{pmatrix} 1 & -sgn(k)\sqrt{\frac{w_1}{w_2}} \\ -sgn(k)\sqrt{\frac{w_2}{w_1}} & 1 \end{pmatrix}. \tag{4.11}$$

Then, from (4.8)-(4.11) we obtain

$$\begin{pmatrix} \hat{\eta}_k \\ \hat{\omega}_k \end{pmatrix}(t) = \frac{1}{2} \begin{bmatrix} (e^{-\lambda_k^+ t} + e^{-\lambda_k^- t})\hat{\eta}_k^0 + sgn(k)\sqrt{\frac{w_1}{w_2}}(e^{-\lambda_k^+ t} - e^{-\lambda_k^- t})\hat{\omega}_k^0 \\ sgn(k)\sqrt{\frac{w_2}{w_1}}(e^{-\lambda_k^+ t} - e^{-\lambda_k^- t})\hat{\eta}_k^0 + (e^{-\lambda_k^+ t} + e^{-\lambda_k^- t})\hat{\omega}_k^0 \end{bmatrix}. \tag{4.12}$$

By using (4.6) and the Euler's formula, we see that (4.12) is equivalent to (4.7). ∎

Using Lemma 4.1.1 we can prove that the operator A generates an analytic group in V^s.

Theorem 4.1.1. *The family of linear operators* $(S(t))_{t\geq 0}$ *defined by*

$$S(t)(\eta^0,\omega^0)=\sum_{k\in\mathbb{Z}}(\hat{\eta}_k(t),\hat{\omega}_k(t))e^{ikx},\quad (\eta^0,\omega^0)\in V^s,$$

where the coefficients $(\hat{\eta}_k(t),\hat{\omega}_k(t))$ *are given by (4.7), is a group of isometries in* V^s*, for any* $s\in\mathbb{R}$.

Proof. First, we prove that $S(t)$ is a well-defined linear and continuos operator for any $t\in\mathbb{R}$. If $(\eta^0,\omega^0)=\sum_{k\in\mathbb{Z}}(\hat{\eta}_k^0,\hat{\omega}_k^0)e^{ikx}\in V^s$, then, we claim that the series $\sum_{k\in\mathbb{Z}}(\hat{\eta}_k,\hat{\omega}_k)e^{ikx}$ converges in $C([0,\infty),V^s)$. This is equivalent to say that the sequence

$$\mathcal{P}=\left(\sum_{|k|\leq N}(\hat{\eta}_k,\hat{\omega}_k)e^{ikx}\right)_{N\geq 1}$$

is a Cauchy sequence in $C([0,\infty),V^s)$. From (4.7), we obtain

$$\begin{aligned}\sup_{t\in[0,\infty)}\left\|\sum_{N\leq|k|\leq N+p}(\hat{\eta}_k,\hat{\omega}_k)e^{ikx}\right\|_{V^s}^2&=\sup_{t\in[0,\infty)}\sum_{N\leq|k|\leq N+p}\left(|\hat{\eta}_k(t)|^2+\frac{w_1}{w_2}|\hat{\omega}_k(t)|^2\right)(1+k^2)^s\\&=\sum_{N\leq|k|\leq N+p}\left(|\hat{\eta}_k^0|^2+\frac{w_1}{w_2}|\hat{\omega}_k^0|^2\right)(1+k^2)^s.\end{aligned}$$

Thus, $\mathcal{P}$ is a Cauchy sequence in $C([0,\infty),V^s)$. Hence, the operator $S(t)$ is well-defined in V^s and $S(\cdot)(\eta^0,\omega^0)\in C([0,\infty),V^s)$. Moreover, since

$$\left\|\sum_{|k|\leq N}(\hat{\eta}_k,\hat{\omega}_k)e^{ikx}\right\|_{V^s}^2=\sum_{|k|\leq N}\left(|\hat{\eta}_k^0|^2+\frac{w_1}{w_2}|\hat{\omega}_k^0|^2\right)(1+k^2)^s,$$

we have that $(S(t))_{t\in\mathbb{R}}$ is a family of linear and continuous operators which are also isometries. It is easy to see that $S(0)=I$. On the other hand, we have

$$\begin{aligned}&S(s)\circ S(t)(\eta^0,\omega^0)=S(s)\sum_{k\in\mathbb{Z}}(\hat{\eta}_k(t),\hat{\omega}_k(t))e^{ikx}\\&=S(s)\sum_{k\in\mathbb{Z}}\left(\cos[k\sigma(k)t]\hat{\eta}_k^0-i\sqrt{\frac{w_1}{w_2}}\sin[k\sigma(k)t]\hat{\omega}_k^0,\cos[k\sigma(k)t]\hat{\omega}_k^0-i\sqrt{\frac{w_2}{w_1}}\sin[k\sigma(k)t]\hat{\eta}_k^0\right)e^{ikx}\\&=\sum_{k\in\mathbb{Z}}\Bigg(\cos[k\sigma(k)s]\cos[k\sigma(k)t]\hat{\eta}_k^0-i\sqrt{\frac{w_1}{w_2}}\cos[k\sigma(k)s]\sin[k\sigma(k)t]\hat{\omega}_k^0\\&\qquad-\sin[k\sigma(k)s]\sin[k\sigma(k)t]\hat{\eta}_k^0-i\sqrt{\frac{w_1}{w_2}}\sin[k\sigma(k)s]\cos[k\sigma(k)t]\hat{\omega}_k^0,\\&\qquad\cos[k\sigma(k)s]\cos[k\sigma(k)t]\hat{\omega}_k^0-i\sqrt{\frac{w_2}{w_1}}\cos[k\sigma(k)s]\sin[k\sigma(k)t]\hat{\eta}_k^0\\&\qquad-\sin[k\sigma(k)s]\sin[k\sigma(k)t]\hat{\omega}_k^0-i\sqrt{\frac{w_2}{w_1}}\sin[k\sigma(k)s]\cos[k\sigma(k)t]\hat{\eta}_k^0\Bigg)e^{ikx}\\&=\sum_{k\in\mathbb{Z}}\Bigg(\cos[k\sigma(k)(s+t)]\hat{\eta}_k^0-i\sqrt{\frac{w_1}{w_2}}\sin[k\sigma(k)(s+t)]\hat{\omega}_k^0,\\&\qquad\qquad\cos[k\sigma(k)(s+t)]\hat{\omega}_k^0-i\sqrt{\frac{w_2}{w_1}}\sin[k\sigma(k)(s+t)]\hat{\eta}_k^0\Bigg)e^{ikx}\\&=\sum_{k\in\mathbb{Z}}(\hat{\eta}_k(s+t),\hat{\omega}_k(s+t))e^{ikx}=S(s+t)(\eta^0,\omega^0),\end{aligned}$$

for any $t, s \in \mathbb{R}$ and, in addition,

$$\begin{aligned}
&||S(t)(\eta^0, \omega^0) - (\eta^0, \omega^0)||_{V^s}^2 \\
&= \sum_{k\in\mathbb{Z}} \left((\cos[k\sigma(k)t] - 1)^2 + \sin^2[k\sigma(k)t]\right) \left[|\hat{\eta}_k^0|^2 + \frac{w_1}{w_2}|\hat{\omega}_k^0|^2\right](1+k^2)^s \\
&= 4\sum_{k\in\mathbb{Z}} \sin^2\left[\frac{k\sigma(k)t}{2}\right]\left[|\hat{\eta}_k^0|^2 + \frac{w_1}{w_2}|\hat{\omega}_k^0|^2\right](1+k^2)^s.
\end{aligned}$$

Consequently $\lim_{t\to 0} S(t)(\eta^0, \omega^0) = (\eta^0, \omega^0)$ in V^s. ■

Theorem 4.1.2. *The infinitesimal generator of the group $(S(t))_{t\in\mathbb{R}}$ is the bounded operator $(D(-A), -A)$, where $D(-A) = V^s$ and A is given by (4.4).*

Proof. We will show that

$$\lim_{t\to 0} \frac{S(t)(\eta^0, \omega^0) - (\eta^0, \omega^0)}{t} = -A(\eta^0, \omega^0),$$

if and only if $(\eta^0, \omega^0) \in V^s$. This is equivalent to show that the derivative in zero of the series $\sum_{k\in\mathbb{Z}}(\hat{\eta}_k(t), \hat{\omega}_k(t))e^{ikx}$, where $(\hat{\eta}_k(t), \hat{\omega}_k(t))$ is given by (4.7), is convergent to $-A(\eta^0, \omega^0)$ in V^s, if and only if $(\eta^0, \omega^0) \in V^s$. If we denote by

$$\mathcal{S}_N(t) = \sum_{|k|\le N} (\hat{\eta}_k(t), \hat{\omega}_k(t))e^{ikx},$$

a partial sum of the series, a straightforward computation which takes into account (4.5) shows that

$$[\mathcal{S}_N]_t(0) = -A(\mathcal{S}_N)(0). \tag{4.13}$$

Now, let $(D(B), B)$ be the infinitesimal generator of the group $(S(t))_{t\in\mathbb{R}}$. If $(\eta^0, \omega^0) \in D(B)$, from (4.13) we obtain that

$$\begin{aligned}
B(\eta^0, \omega^0) &= \lim_{t\to 0} \frac{S(t)(\eta^0, \omega^0) - (\eta^0, \omega^0)}{t} = \left[\sum_{k\in\mathbb{Z}}(\hat{\eta}_k(t), \hat{\omega}_k(t))e^{ikx}\right]_t (0) \\
&= \lim_{N\to\infty} [\mathcal{S}_N]_t(0) = \lim_{N\to\infty} -A(\mathcal{S}_N)(0) = -A(\eta^0, \omega^0).
\end{aligned}$$

Hence, $(\eta^0, \omega^0) \in D(-A) = V^s$ and $B(\eta^0, \omega^0) = -A(\eta^0, \omega^0)$, for any $(\eta^0, \omega^0) \in D(B)$. On the other hand, if we take $(\eta^0, \omega^0) \in D(-A) = V^s$, then we have to show that the series

$$\left[\sum_{k\in\mathbb{Z}}(\hat{\eta}_k(t), \hat{\omega}_k(t))e^{ikx}\right]_t (0)$$

is convergent. This is equivalent to show that

$$[\mathcal{S}_N]_t(0) = \left[\sum_{k\le N}(\hat{\eta}_k(t), \hat{\omega}_k(t))e^{ikx}\right]_t (0)$$

is a Cauchy sequence. Indeed,

$$||[\mathcal{S}_{N+p}]_t(0) - [\mathcal{S}_N]_t(0)||_{V^s}^2 = \sum_{N\le|k|\le N+p} \left(|\hat{\eta}_{k,t}(0)|^2 + \frac{w_1}{w_2}|\hat{\omega}_{k,t}(0)|^2\right)(1+k^2)^s. \tag{4.14}$$

From (4.3) and the equations in (4.5) we deduce that

$$|\hat{\eta}_{k,t}(0)|^2 = k^2 w_1^2 |\hat{\omega}_k(0)|^2 \quad \text{and} \quad |\hat{\omega}_{k,t}(0)|^2 = k^2 w_2^2 |\hat{\eta}_k(0)|^2. \tag{4.15}$$

Then, from (4.15) we have that

$$\begin{aligned} |\hat{\eta}_{k,t}(0)|^2 + \frac{w_1}{w_2}|\hat{\omega}_{k,t}(0)|^2 &= k^2 w_1^2 |\hat{\omega}_k(0)|^2 + k^2 w_1 w_2 |\hat{\eta}_k(0)|^2 \\ &= k^2 w_1 w_2 \left[|\hat{\eta}_k(0)|^2 + \frac{w_1}{w_2}|\hat{\omega}_k(0)|^2 \right] \\ &\leq M \left[|\hat{\eta}_k(0)|^2 + \frac{w_1}{w_2}|\hat{\omega}_k(0)|^2 \right], \end{aligned} \tag{4.16}$$

where M is a positive constant depending only on a, b, c, d, b_1 and d_1. Thus, from (4.14) and (4.16) we obtain the following estimate

$$\begin{aligned} ||[\mathcal{S}_{N+p}]_t(0) - [\mathcal{S}_N]_t(0)||_{V^s}^2 &\leq M \sum_{N \leq |k| \leq N+p} \left(|\hat{\eta}_k(0)|^2 + \frac{w_1}{w_2}|\hat{\omega}_k(0)|^2 \right)(1+k^2)^s \\ &= M \left|\left| \sum_{|k|\leq N+p} (\hat{\eta}_k^0, \hat{\omega}_k^0) e^{ikx} - \sum_{|k|\leq N} (\hat{\eta}_k^0, \hat{\omega}_k^0) e^{ikx} \right|\right|_{V^s} \end{aligned}$$

and, since $(\eta^0, \omega^0) \in D(-A) = V^s$, we have that $([\mathcal{S}_N]_t(0))_{N\geq 0}$ is a Cauchy sequence. Consequently,

$$\begin{aligned} -A(\eta^0, \omega^0) = \lim_{N\to\infty} -A(\mathcal{S}_N)(0) = \lim_{N\to\infty} [\mathcal{S}_N]_t(0) &= \left[\sum_{k\in\mathbb{Z}} (\hat{\eta}_k(t), \hat{\omega}_k(t)) e^{ikx} \right]_t (0) \\ = \lim_{t\to 0} \frac{S(t)(\eta^0, \omega^0) - (\eta^0, \omega^0)}{t} &= B(\eta^0, \omega^0). \end{aligned}$$

Hence, $(\eta^0, \omega^0) \in D(B)$ and $-A(\eta^0, \omega^0) = B(\eta^0, \omega^0)$, for any $(\eta^0, \omega^0) \in D(-A) = V^s$. ■

Remark 4.1.3. *In fact, much more can be said about the regularity of solutions of (4.1). Since (4.1) is linear and $-A$ is a bounded operator, we can easily deduce that $(\eta, \omega) \in \mathcal{C}^\omega([0,\infty); V^s)$, where $\mathcal{C}^\omega([0,\infty); V^s)$ represents the class of the analytic functions defined in $[0,\infty)$ with values in V^s. Indeed, for $t_0 \in [0,\infty)$*

$$\begin{aligned} \left|\left| \sum_{n=0}^{\infty} \frac{d^n}{dt^n}(\eta,\omega)(t_0) \frac{(t-t_0)^n}{n!} \right|\right|_{V^s} &\leq \sum_{n=0}^{\infty} \frac{|t-t_0|^n}{n!} \left|\left| \frac{d^n}{dt^n}(\eta,\omega)(t_0) \right|\right|_{V^s} \\ &\leq ||(\eta,\omega)(t_0)||_{V^s} \sum_{n=0}^{\infty} \frac{|t-t_0|^n}{n!} ||A||^n_{\mathcal{L}(V^s)} \leq \infty. \end{aligned}$$

Hence, the series $\sum_{n=0}^{\infty} \frac{d^n}{dt^n}(\eta,\omega)(t_0)\frac{(t-t_0)^n}{n!}$ is absolutely convergent and

$$\begin{aligned} (\eta,\omega)(t) = exp(-A(t-t_0))(\eta,\omega)(t_0) &= \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!}(-A)^n(\eta,\omega)(t_0) \\ &= \sum_{n=0}^{\infty} \frac{d^n}{dt^n}(\eta,\omega)(t_0)\frac{(t-t_0)^n}{n!}. \end{aligned}$$

From Theorem 4.1.2 and the semigroup theory, we obtain the following global well-posedness result:

Theorem 4.1.3. *Let $T > 0$ and $s \in \mathbb{R}$. For each $(\eta^0, \omega^0) \in V^s$ and $(f, g) \in L^1(0, T; V^s)$, there exists a unique solution $(\eta, \omega) \in W^{1,1}([0, T]; V^s)$ of the system*

$$\begin{pmatrix} \eta \\ \omega \end{pmatrix}_t (t) + A \begin{pmatrix} \eta \\ \omega \end{pmatrix} (t) = \begin{pmatrix} f \\ g \end{pmatrix}, \quad \begin{pmatrix} \eta \\ \omega \end{pmatrix} (0) = \begin{pmatrix} \eta^0 \\ \omega^0 \end{pmatrix},$$

which verifies the constant variation formula

$$\begin{pmatrix} \eta \\ \omega \end{pmatrix} (t) = S(t) \begin{pmatrix} \eta^0 \\ \omega^0 \end{pmatrix} + \int_0^t S(t-s) \begin{pmatrix} f \\ g \end{pmatrix} (s) ds.$$

Moreover, if $(f, g) \equiv (0, 0)$ it follows that $(\eta, \omega) \in C^\omega(\mathbb{R}; V^s)$, the class of analytic functions in $t \in \mathbb{R}$ with values in V^s.

The main result of this section reads as follows:

Theorem 4.1.4. *Let (η, ω) solution of system (4.1) given by Theorem 4.1.3. Suppose that there exist an open set $\Omega \subset [0, 2\pi]$ and $T > 0$, such that*

$$\eta(t, x) = 0, \quad \forall (t, x) \in (0, T) \times \Omega. \tag{4.17}$$

Then,

$$(\eta, \omega) = (0, 0) \quad \text{in} \quad \mathbb{R} \times (0, 2\pi).$$

Proof. We note that (4.12) can be written as

$$\begin{cases} \hat{\eta}_k(t) = a_k^+ e^{-\lambda_k^+ t} + a_k^- e^{-\lambda_k^- t}, \\ \hat{\omega}_k(t) = b_k^+ e^{-\lambda_k^+ t} + b_k^- e^{-\lambda_k^- t}, \end{cases} \tag{4.18}$$

where $a_k^\pm = \frac{1}{2}\left(\hat{\eta}_k^0 \pm sgn(k)\sqrt{\frac{w_1}{w_2}}\hat{\omega}_k^0\right)$ and $b_k^\pm = \frac{1}{2}\left(\hat{\omega}_k^0 \pm sgn(k)\sqrt{\frac{w_2}{w_1}}\hat{\eta}_k^0\right)$. Since the solution (η, ω) is an analytic function of t, from (4.17) we deduce that

$$\eta(t, x) = 0, \quad \forall (t, x) \in \mathbb{R} \times \Omega.$$

Consequently, for any $S > 0$ and $x \in \Omega$, if we multiply $\eta(t, x)$ by $e^{\lambda_k^+ t}$, and integrate between $-S$ and S, from (4.18) we obtain

$$\begin{aligned} 0 &= \lim_{S \to \infty} \frac{1}{2S} \int_{-S}^{S} \left(\sum_{j \in \mathbb{Z}} \left(a_j^+ e^{-\lambda_j^+ t} + a_j^- e^{-\lambda_j^- t} \right) e^{ijx} \right) e^{\lambda_k^+ t} dt \\ &= a_k^+ e^{ikx} + a_{-k}^+ e^{-ikx} \quad \text{in} \quad \Omega. \end{aligned} \tag{4.19}$$

Indeed, we have that

$$\begin{aligned} 0 = \eta(t, x) e^{\lambda_k^+ t} &= \left(\sum_{j \in \mathbb{Z}} \left(a_j^+ e^{-\lambda_j^+ t} + a_j^- e^{-\lambda_j^- t} \right) e^{ijx} \right) e^{\lambda_k^+ t} \\ &= \left(a_k^+ + a_k^- e^{2\lambda_k^+ t} \right) e^{ikx} + \left(a_{-k}^+ + a_{-k}^- e^{2\lambda_{-k}^+ t} \right) e^{-ikx} \\ &\quad + \sum_{j \in \mathbb{Z} \setminus \{-k, k\}} \left(a_j^+ e^{(-\lambda_j^+ + \lambda_k^+)t} + a_j^- e^{(-\lambda_j^- + \lambda_k^+)t} \right) e^{ijx} \quad \text{in} \quad \Omega. \end{aligned} \tag{4.20}$$

Then, if we integrate $\eta(t, x) e^{\lambda_k^+ t}$ on $[-S, S]$ and by using the fact that, for $\alpha \in \mathbb{R}$ the following holds

$$\lim_{S \to \infty} \frac{1}{2S} \int_{-S}^{S} e^{i\alpha t} dt = \lim_{S \to \infty} \frac{\sin(\alpha S)}{\alpha S} = 0,$$

from (4.20) we obtain

$$0=\lim_{S\to\infty}\frac{1}{2S}\int_{-S}^{S}\eta(t,x)e^{\lambda_k^+ t}=a_k^+e^{ikx}+a_{-k}^+e^{-ikx}\quad\text{in}\quad\Omega.$$

On the other hand, if we multiply $\eta(t,x)$ by $e^{\lambda_k^- t}$, similar computations yields

$$\begin{aligned}0&=\lim_{S\to\infty}\frac{1}{2S}\int_{-S}^{S}\left(\sum_{j\in\mathbb{Z}}\left(a_j^+e^{-\lambda_j^+ t}+a_j^-e^{-\lambda_j^- t}\right)e^{ijx}\right)e^{\lambda_k^- t}dt\\&=a_k^-e^{ikx}+a_{-k}^-e^{-ikx}\quad\text{in}\quad\Omega.\end{aligned}\tag{4.21}$$

Since both functions on the right hand side of (4.19) and (4.21) are analytic in x, it follows that

$$a_k^{\pm}e^{ikx}+a_{-k}^{\pm}e^{-ikx}=0\quad\text{in}\quad[0,2\pi].$$

By using the orthogonality of $\{e^{ikx}\}_{k\in\mathbb{Z}}$ and $\{e^{-ikx}\}_{k\in\mathbb{Z}}$ in $[0,2\pi]$, we deduce that $a_k^{\pm}=a_{-k}^{\pm}=0$. This implies directly that $\hat{\eta}_k^0=\hat{\omega}_k^0=0$ for any $k\in\mathbb{Z}$. Hence, $(\eta,\omega)=(0,0)$ in $\mathbb{R}\times(0,2\pi)$. ∎

As consequence of Theorem 4.1.4, we have the following result:

Theorem 4.1.5. *Let (η,ω) solution of system (4.1) given by Theorem 4.1.4. Suppose that there exist an open set $\Omega\subset[0,2\pi]$ and $T>0$ such that*

$$\eta_x(t,x)=0,\quad\forall(t,x)\in(0,T)\times\Omega.\tag{4.22}$$

Then,

$$(\eta,\omega)=(c_1,c_2)\quad in\quad\mathbb{R}\times(0,2\pi),$$

for some constants c_1 and c_2.

Proof. From the Lemma 4.1.1, we have that

$$\begin{aligned}\eta_x(t,x)&=\sum_{k\in\mathbb{Z}}ik\hat{\eta}_k(t)e^{ikx}=\sum_{k\in\mathbb{Z}}\left(ika_k^+e^{-\lambda_k^+ t}+ika_k^-e^{-\lambda_k^- t}\right)e^{ikx}\\\omega_x(t,x)&=\sum_{k\in\mathbb{Z}}ik\hat{\omega}_k(t)e^{ikx}=\sum_{k\in\mathbb{Z}}\left(ikb_k^+e^{-\lambda_k^+ t}+ikb_k^-e^{-\lambda_k^- t}\right)e^{ikx}.\end{aligned}\tag{4.23}$$

Then, proceeding as in the proof of Theorem 4.1.4, from (4.22) and (4.23) the following identities holds

$$ka_k^{\pm}e^{ikx}+ka_{-k}^{\pm}e^{-ikx}=0\quad\text{in}\quad[0,2\pi],$$

for any $k\in\mathbb{Z}^*$. From the orthogonality of $\{e^{ikx}\}_{k\in\mathbb{Z}}$ and $\{e^{-ikx}\}_{k\in\mathbb{Z}}$ in $[0,2\pi]$, it follows that $a_k^{\pm}=a_{-k}^{\pm}=0$. This implies directly that $\hat{\eta}_k^0=\hat{\omega}_k^0=0$ for any $k\in\mathbb{Z}^*$. Hence, $(\eta,\omega)=(c_1,c_2)$ in $\mathbb{R}\times(0,2\pi)$, for some $c_1,c_2\in\mathbb{R}$. ∎

4.2 Boundary Stabilization

In this section we are concerned with the study of the boundary stabilization of the higher-order Boussinesq system posed on a bounded domain. More precisely, we consider the following problem

$$\begin{cases}\eta_t+\omega_x+a\omega_{xxx}-b\eta_{txx}+b_1\eta_{txxxx}=0 & \text{for } x\in(0,L),\ t>0,\\\omega_t+\eta_x+a\eta_{xxx}-d\omega_{txx}+d_1\omega_{txxxx}=0 & \text{for } x\in(0,L), t>0,\\\eta(0,x)=\eta^0(x);\quad\omega(0,x)=\omega^0(x) & \text{for } x\in(0,L),\end{cases}\tag{4.24}$$

with the following boundary conditions

$$
\begin{cases}
b_1\eta_{txxx}(t,0) - b\eta_{tx}(t,0) = -(b+b_1)\eta(t,0) - a\omega_{xx}(t,0) - \dfrac{\omega(t,0)}{2} & \text{for} \quad t \geq 0, \\
b_1\eta_{txxx}(t,L) - b\eta_{tx}(t,L) = (b+b_1)\eta(t,L) - a\omega_{xx}(t,L) - \dfrac{\omega(t,L)}{2} & \text{for} \quad t \geq 0, \\
d_1\omega_{txxx}(t,0) - d\omega_{tx}(t,0) = -(d+d_1)\omega(t,0) - a\eta_{xx}(t,0) - \dfrac{\eta(t,0)}{2} & \text{for} \quad t \geq 0, \\
d_1\omega_{txxx}(t,L) - d\omega_{tx}(t,L) = (d+d_1)\omega(t,L) - a\eta_{xx}(t,L) - \dfrac{\eta(t,L)}{2} & \text{for} \quad t \geq 0,
\end{cases}
\tag{4.25}
$$

$$
\begin{cases}
\eta_{txx}(t,0) = \eta_x(t,0) - a\dfrac{\omega_x(t,0)}{2b_1} & \text{for} \quad t \geq 0, \\
\eta_{txx}(t,L) = -\eta_x(t,L) - a\dfrac{\omega_x(t,L)}{2b_1} & \text{for} \quad t \geq 0, \\
\omega_{txx}(t,0) = \omega_x(t,0) - a\dfrac{\eta_x(t,0)}{2d_1} & \text{for} \quad t \geq 0, \\
\omega_{txx}(t,L) = -\omega_x(t,L) - a\dfrac{\eta_x(t,L)}{2d_1} & \text{for} \quad t \geq 0.
\end{cases}
\tag{4.26}
$$

If we multiply the first equation in (4.24) by η, the second one by ω and integrate by parts over $(0,L)$, we obtain (at least formally)

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}||(\eta(t),\omega(t))||^2_{[H^2(0,L)]^2} = &- (b+b_1)(|\eta(t,L)|^2 + |\eta(t,0)|^2) \\
&- (d+d_1)(|\omega(t,L)|^2 + |\omega(t,0)|^2) \\
&- b_1(|\eta_x(t,L)|^2 + |\eta_x(t,0)|^2) - d_1(|\omega_x(t,L)|^2 + |\omega_x(t,0)|^2).
\end{aligned}
\tag{4.27}
$$

Hence, $||(\eta(t),\omega(t))||_{[H^2(0,L)]^2}$ is nonincreasing and the boundary conditions play the role of a feedback damping mechanism. Before to establish the stabilization result, we first show the following well-posedness theorem for (4.24)-(4.26):

Theorem 4.2.1. *Let $s \in (5/2, 7/2)$. For any $(\eta^0,\omega^0) \in [H^s(0,L)]^2$, there exist $T > 0$ and a unique solution (η,ω) of (4.24)-(4.26) in the class $C([0,T];[H^s(0,L)]^2)$. Moreover, the map*

$$
\begin{aligned}
\mathcal{F} : [H^s(0,L)]^2 &\longrightarrow C([0,T];[H^s(0,L)]^2) \\
(\eta^0,\omega^0) &\longmapsto (\eta,\omega)
\end{aligned}
$$

is Lipschitz continuous.

Proof. In order to simplify the notation, we will denote the operator

$$
\mathcal{L}_{\alpha,\beta} := (1 - \alpha\partial_x^2 + \beta\partial_x^4)
$$

where $\alpha,\beta > 0$. The proof will be done by using a fixed point argument. Therefore, in order to write the problem as an integral equation, we set $(\hat{\eta},\hat{\omega}) = (\eta_t,\omega_t)$ and remark that $(\hat{\eta},\hat{\omega})$ solves the elliptic problem

$$
\Big(\mathcal{L}_{b,b_1}\hat{\eta}, \mathcal{L}_{d,d_1}\hat{\omega}\Big) = \Big(- (1 - a\partial_x^2)\partial_x\omega, -(1 - a\partial_x^2)\partial_x\eta\Big) \tag{4.28}
$$

$$
(b_1\hat{\eta}_{xxx}(0) - b\hat{\eta}_x(0), d_1\hat{\omega}_{xxx}(0) - d\hat{\omega}_x(0)) = (b_1B_9 - bB_1, d_1B_{10} - dB_2) \tag{4.29}
$$

$$
(b_1\hat{\eta}_{xxx}(L) - b\hat{\eta}_x(L), d_1\hat{\omega}_{xxx}(L) - d\hat{\omega}_x(L)) = (b_1B_{11} - bB_3, d_1B_{12} - dB_4) \tag{4.30}
$$

$$
(\hat{\eta}_{xx}(0), \hat{\omega}_{xx}(0)) = (B_5, B_6) \tag{4.31}
$$

$$
(\hat{\eta}_{xx}(L), \hat{\omega}_{xx}(L)) = (B_7, B_8). \tag{4.32}
$$

where we have set

$$
\begin{aligned}
(B_1, B_2) &= \left(\frac{\omega(t,0)}{2b} + \eta(t,0), \frac{\eta(t,0)}{2d} + \omega(t,0)\right)\\
(B_3, B_4) &= \left(\frac{\omega(t,L)}{2b} - \eta(t,L), \frac{\eta(t,L)}{2d} - \omega(t,L)\right)\\
(B_5, B_6) &= \left(\eta_x(t,0) - a\frac{\omega_x(t,0)}{2b_1}, \omega_x(t,0) - a\frac{\eta_x(t,0)}{2d_1}\right)\\
(B_7, B_8) &= \left(-\eta_x(t,L) - a\frac{\omega_x(t,L)}{2b_1}, -\omega_x(t,L) - a\frac{\eta_x(t,L)}{2d_1}\right)\\
(B_9, B_{10}) &= \left(-\eta(t,0) - a\frac{\omega_{xx}(t,0)}{b_1}, -\omega(t,0) - a\frac{\eta_{xx}(t,0)}{d_1}\right)\\
(B_{11}, B_{12}) &= \left(\eta(t,L) - a\frac{\omega_{xx}(t,L)}{b_1}, \omega(t,L) - a\frac{\eta_{xx}(t,L)}{d_1}\right).
\end{aligned}
$$

Now, we set the polynomials $g_{i,[\eta,\omega]}(x)$, $i = 1, 2$ defined as

$$
\begin{aligned}
g_{1,[\eta,\omega]}(x) = g_1(x) = {} & B_1 x + \frac{B_5}{2}x^2 + \frac{B_9}{6}x^3\\
& + \left[\frac{5}{2L^3}(B_3 - B_1) - \frac{1}{L^2}(B_7 + \frac{3}{2}B_5) + \frac{1}{8L}(B_{11} - 3B_9)\right]x^4\\
& + \left[-\frac{3}{L^4}(B_3 - B_1) + \frac{1}{5L^3}(7B_7 + 8B_5) - \frac{1}{5L^2}(B_{11} - \frac{3}{2}B_9)\right]x^5\\
& + \left[\frac{1}{L^5}(B_3 - B_1) - \frac{1}{2L^4}(B_7 + B_5) + \frac{1}{12L^3}(B_{11} - B_9)\right]x^6,
\end{aligned}
$$

and

$$
\begin{aligned}
g_{2,[\eta,\omega]}(x) = g_2(x) = {} & B_2 x + \frac{B_6}{2}x^2 + \frac{B_{10}}{6}x^3\\
& + \left[\frac{5}{2L^3}(B_4 - B_2) - \frac{1}{L^2}(B_8 + \frac{3}{2}B_6) + \frac{1}{8L}(B_{12} - 3B_{10})\right]x^4\\
& + \left[-\frac{3}{L^4}(B_4 - B_2) + \frac{1}{5L^3}(7B_8 + 8B_6) - \frac{1}{5L^2}(B_{12} - \frac{3}{2}B_{10})\right]x^5\\
& + \left[\frac{1}{L^5}(B_4 - B_2) - \frac{1}{2L^4}(B_8 + B_6) + \frac{1}{12L^3}(B_{12} - B_{10})\right]x^6.
\end{aligned}
$$

Thus, we have that $(g_1(x), g_2(x))$ satisfies the following boundary conditions

$$
\begin{aligned}
(g_{1,x}(0), g_{2,x}(0)) &= (B_1, B_2)\\
(g_{1,x}(L), g_{2,x}(L)) &= (B_3, B_4)\\
(g_{1,xx}(0), g_{2,xx}(0)) &= (B_5, B_6)\\
(g_{1,xx}(L), g_{2,xx}(L)) &= (B_7, B_8)\\
(g_{1,xxx}(0), g_{2,xxx}(0)) &= (B_9, B_{10})\\
(g_{1,xxx}(L), g_{2,xxx}(L)) &= (B_{11}, B_{12}).
\end{aligned}
$$

Consequently, (g_1, g_2) will satisfy the boundary conditions (4.29)-(4.32). Then, the solution $(\hat{\eta}, \hat{\omega})$ of (4.28)-(4.32) can be written as

$$
(\hat{\eta}, \hat{\omega}) = (h_1 + g_1, h_2 + g_2),
$$

where

$$
(h_1, h_2) = \left(\mathcal{L}^{-1}_{b,b_1,N}(-(1 - a\partial_x^2)\partial_x\omega - \mathcal{L}_{b,b_1}g_1), \mathcal{L}^{-1}_{d,d_1,N}(-(1 - a\partial_x^2)\partial_x\eta - \mathcal{L}_{d,d_1}g_2)\right)
$$

is solution of

$$(\mathcal{L}_{b,b_1}h_1, \mathcal{L}_{d,d_1}h_2) = \Big(-(1-a\partial_x^2)\partial_x\omega - \mathcal{L}_{b,b_1}g_1, -(1-a\partial_x^2)\partial_x\eta - \mathcal{L}_{d,d_1}g_2 \Big),$$

with boundary conditions

$$\left(\frac{d^ih_1}{dx^i}(0), \frac{d^ih_2}{dx^i}(0)\right) = \left(\frac{d^ih_1}{dx^i}(L), \frac{d^ih_2}{dx^i}(L)\right) = (0,0) \quad \text{for} \quad i = 1,2,3,$$

where, for any $\alpha, \beta > 0$, $\mathcal{L}_{\alpha,\beta,N} := (1-\alpha\partial_x^2 + \beta\partial_x^4)_N$ denotes the elliptic operator with homogeneous Neumann boundary conditions. Thus,

$$\eta_t = \hat{\eta} = -\mathcal{L}_{b,b_1,N}^{-1}(1-a\partial_x^2)\partial_x\omega + (1-\mathcal{L}_{b,b_1,N}^{-1}\mathcal{L}_{b,b_1})g_1, \tag{4.33}$$

and

$$\omega_t = \hat{\omega} = -\mathcal{L}_{d,d_1,N}^{-1}(1-a\partial_x^2)\partial_x\eta + (1-\mathcal{L}_{d,d_1,N}^{-1}\mathcal{L}_{d,d_1})g_2. \tag{4.34}$$

We remark that

$$(1-\alpha\partial_x^2+\beta\partial_x^4)_N^{-1}(1-a\partial_x^2)\partial_x(H^s(0,L)) \subset H^s(0,L) \tag{4.35}$$

and

$$||(1-\alpha\partial_x^2+\beta\partial_x^4)_N^{-1}(1-a\partial_x^2)\partial_x f||_{H^s(0,L)} \le C||f||_{H^s(0,L)}, \tag{4.36}$$

for $s \in (\frac{5}{2}, \frac{7}{2})$ and any $\alpha, \beta > 0$, where C is a positive constant.

The linear and bounded operator $(1-\mathcal{L}_{\alpha,\beta,N}^{-1}\mathcal{L}_{\alpha,\beta})$ will be denoted by $\mathcal{B}_{\alpha,\beta}$. Then, motivated by the above considerations, for any $(\eta^0, \omega^0) \in [H^s(0,L)]^2$, we define the following operator

$$\Gamma(\eta,\omega)(t) := (\Gamma_1\eta(t), \Gamma_2\omega(t)),$$

where

$$\Gamma_1\eta(t) := \eta^0 + \int_0^t -\mathcal{L}_{b,b_1,N}^{-1}(1-a\partial_x^2)(\partial_x\omega)(\tau)d\tau + \int_0^t \mathcal{B}_{b,b_1}g_1(x)(\tau)d\tau,$$

$$\Gamma_2\omega(t) := \omega^0 + \int_0^t -\mathcal{L}_{d,d_1,N}^{-1}(1-a\partial_x^2)(\partial_x\eta)(\tau)d\tau \int_0^t \mathcal{B}_{d,d_1}g_2(x)(\tau)d\tau,$$

with

$$\begin{aligned}
&\int_0^t \mathcal{B}_{b,b_1}g_1(x)(\tau)d\tau \\
&= \int_0^t \mathcal{B}_{b,b_1}\left[\frac{\omega(\tau,0)}{2b} + \eta(\tau,0)\right] x d\tau + \int_0^t \mathcal{B}_{b,b_1}\left[\eta_x(\tau,0) - a\frac{\omega_x(\tau,0)}{2b_1}\right]\frac{x^2}{2}d\tau \\
&+ \int_0^t \mathcal{B}_{b,b_1}\left[-\eta(\tau,0) - a\frac{\omega_{xx}(\tau,0)}{b_1}\right]\frac{x^3}{6}d\tau \\
&+ \int_0^t \mathcal{B}_{b,b_1}\Bigg[\frac{5}{2L^3}\left(\frac{\omega(\tau,L)}{2b} - \eta(t,L) - \frac{\omega(\tau,0)}{2b} - \eta(\tau,0)\right) \\
&\qquad - \frac{1}{L^2}\left(-\eta_x(\tau,L) - a\frac{\omega_x(\tau,L)}{2b_1} + \frac{3}{2}\left(\eta_x(\tau,0) - a\frac{\omega_x(\tau,0)}{2b_1}\right)\right) \\
&\qquad + \frac{1}{8L}\left(\eta(\tau,L) - a\frac{\omega_{xx}(\tau,L)}{b_2} - 3\left(-\eta(\tau,0) - a\frac{\omega_{xx}(\tau,0)}{b_1}\right)\right)\Bigg]x^4 d\tau
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \mathcal{B}_{b,b_1}\Bigg[-\frac{3}{L^4}\left(\frac{\omega(\tau,L)}{2b}-\eta(t,L)-\frac{\omega(\tau,0)}{2b}-\eta(\tau,0)\right)\\
&\qquad+\frac{1}{5L^3}\left(7\left(-\eta_x(\tau,L)-a\frac{\omega_x(\tau,L)}{2b_1}\right)+8\left(\eta_x(\tau,0)-a\frac{\omega_x(\tau,0)}{2b_1}\right)\right)\\
&\qquad-\frac{1}{5L^2}\left(\eta(\tau,L)-a\frac{\omega_{xx}(\tau,L)}{b_2}-\frac{3}{2}\left(-\eta(\tau,0)-a\frac{\omega_{xx}(\tau,0)}{b_1}\right)\right)\Bigg]x^5d\tau\\
&+\int_0^t \mathcal{B}_{b,b_1}\Bigg[\frac{1}{L^5}\left(\frac{\omega(\tau,L)}{2b}-\eta(t,L)-\frac{\omega(\tau,0)}{2b}-\eta(\tau,0)\right)\\
&\qquad-\frac{1}{2L^4}\left(-\eta_x(\tau,L)-a\frac{\omega_x(\tau,L)}{2b_1}+\eta_x(\tau,0)-a\frac{\omega_x(\tau,0)}{2b_1}\right)\\
&\qquad+\frac{1}{12L^3}\left(\eta(\tau,L)-a\frac{\omega_{xx}(\tau,L)}{b_2}+\eta(\tau,0)+a\frac{\omega_{xx}(\tau,0)}{b_1}\right)\Bigg]x^6d\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t \mathcal{B}_{d,d_1}g_2(x)(\tau)d\tau\\
&=\int_0^t \mathcal{B}_{d,d_1}\left[\frac{\eta(\tau,0)}{2d}+\omega(\tau,0)\right]xd\tau+\int_0^t \mathcal{B}_{d,d_1}\left[\omega_x(\tau,0)-a\frac{\eta_x(\tau,0)}{2d_1}\right]\frac{x^2}{2}d\tau\\
&+\int_0^t \mathcal{B}_{d,d_1}\left[-\omega(\tau,0)-a\frac{\eta_{xx}(\tau,0)}{d_1}\right]\frac{x^3}{6}d\tau\\
&+\int_0^t \mathcal{B}_{d,d_1}\Bigg[\frac{5}{2L^3}\left(\frac{\eta(\tau,L)}{2d}-\omega(t,L)-\frac{\eta(\tau,0)}{2d}-\omega(\tau,0)\right)\\
&\qquad-\frac{1}{L^2}\left(-\omega_x(\tau,L)-a\frac{\eta_x(\tau,L)}{2d_1}+\frac{3}{2}\left(\omega_x(\tau,0)-a\frac{\eta_x(\tau,0)}{2d_1}\right)\right)\\
&\qquad+\frac{1}{8L}\left(\omega(\tau,L)-a\frac{\eta_{xx}(\tau,L)}{d_2}-3\left(-\omega(\tau,0)-a\frac{\eta_{xx}(\tau,0)}{d_1}\right)\right)\Bigg]x^4d\tau\\
&+\int_0^t \mathcal{B}_{d,d_1}\Bigg[-\frac{3}{L^4}\left(\frac{\eta(\tau,L)}{2d}-\omega(t,L)-\frac{\eta(\tau,0)}{2d}-\omega(\tau,0)\right)\\
&\qquad+\frac{1}{5L^3}\left(7\left(-\omega_x(\tau,L)-a\frac{\eta_x(\tau,L)}{2d_1}\right)+8\left(\omega_x(\tau,0)-a\frac{\eta_x(\tau,0)}{2d_1}\right)\right)\\
&\qquad-\frac{1}{5L^2}\left(\omega(\tau,L)-a\frac{\eta_{xx}(\tau,L)}{d_2}-\frac{3}{2}\left(-\omega(\tau,0)-a\frac{\eta_{xx}(\tau,0)}{d_1}\right)\right)\Bigg]x^5d\tau\\
&+\int_0^t \mathcal{B}_{d,d_1}\Bigg[\frac{1}{L^5}\left(\frac{\eta(\tau,L)}{2d}-\omega(t,L)-\frac{\eta(\tau,0)}{2d}-\omega(\tau,0)\right)\\
&\qquad-\frac{1}{2L^4}\left(-\omega_x(\tau,L)-a\frac{\eta_x(\tau,L)}{2d_1}+\omega_x(\tau,0)-a\frac{\eta_x(\tau,0)}{2d_1}\right)\\
&\qquad+\frac{1}{12L^3}\left(\omega(\tau,L)-a\frac{\eta_{xx}(\tau,L)}{d_2}+\omega(\tau,0)+a\frac{\eta_{xx}(\tau,0)}{d_1}\right)\Bigg]x^6d\tau.
\end{aligned}
$$

Then, we seek (η,ω) as a fixed point of the integral equation

$$
\Gamma(\eta(t),\omega(t))=(\eta(t),\omega(t)). \tag{4.37}
$$

By using the Sobolev embedding $H^s(0,L)\hookrightarrow C([0,L])$ for $s>5/2$, we have that

$$
\begin{aligned}
&|\eta(t,0)|,|\eta(t,L)|\le \sup_{x\in[0,L]}|\eta(t,x)|\le C||\eta||_{H^s(0,L)},\\
&|\eta_x(t,0)|,|\eta_x(t,L)|\le \sup_{x\in[0,L]}|\eta_x(t,x)|\le C||\eta_x||_{H^{s-1}(0,L)}\le C||\eta||_{H^s(0,L)},\\
&|\eta_{xx}(t,0)|,|\eta_{xx}(t,L)|\le \sup_{x\in[0,L]}|\eta_{xx}(t,x)|\le C||\eta_{xx}||_{H^{s-2}(0,L)}\le C||\eta||_{H^s(0,L)},
\end{aligned}
\tag{4.38}
$$

for some constant $C>0$. From (4.38) we obtain that

$$
|\mathcal{B}_{b,b_1}g_1(x)|\,,|\mathcal{B}_{d,d_1}g_2(x)|\le C||(\eta(t),\omega(t))||_{[H^s(0,L)]^2}. \tag{4.39}
$$

Let $T>0$ to be chosen later. Then, for each $(\eta^1,\omega^1),(\eta^2,\omega^2)\in C([0,T'];[H^s(0,L)]^2)$, from (4.36) and (4.39) it follows that

$$
\begin{aligned}
&||\Gamma(\eta^1,\omega^1)-\Gamma(\eta^2,\omega^2)||_{C\left([0,T];[H^s(0,L)]^2\right)}=\sup_{0\le t\le T'}||\Gamma(\eta^1,\omega^1)(t)-\Gamma(\eta^2,\omega^2)(t)||_{[H^s(0,L)]^2}\\
&\le\int_0^{T'}\Big(||\mathcal{L}^{-1}_{b,b_1}(1-a\partial_x^2)\partial_x(\omega^1-\omega^2)(\tau)||_{H^s(0,L)}+||\mathcal{B}_{b,b_1}(g_{1,[\eta^1,\omega^1]}-g_{1,[\eta^2,\omega^2]})(\tau)||_{H^s(0,L)}\Big)d\tau\\
&\quad+\int_0^{T'}\Big(||\mathcal{L}^{-1}_{d,d_1}(1-a\partial_x^2)\partial_x(\eta^1-\eta^2)(\tau)||_{H^s(0,L)}+||\mathcal{B}_{d,d_1}(g_{2,[\eta^1,\omega^1]}-g_{2,[\eta^2,\omega^2]})(\tau)||_{H^s(0,L)}\Big)d\tau\\
&\le CT'||(\eta^1,\omega^1)-(\eta^2,\omega^2)||_{C\left([0,T];[H^s(0,L)]^2\right)},
\end{aligned}
$$

where C is a positive constant. Choosing $T>0$ satisfying $CT\le\frac{1}{2}$, from the estimate above we obtain

$$
||\Gamma(\eta^1,\omega^1)-\Gamma(\eta^2,\omega^2)||_{C\left([0,T];[H^s(0,L)]^2\right)}\le\frac{1}{2}||(\eta^1,\omega^1)-(\eta^2,\omega^2)||_{C\left([0,T];[H^s(0,L)]^2\right)}. \tag{4.40}
$$

Let $(\eta,\omega)\in\mathcal{B}_R(0)=\{(\eta,\omega)\in C([0,T];[H^s(0,L)]^2):||(\eta,\omega)||_{C([0,T];[H^s(0,L)]^2)}\le R\}$, where $R=2||(\eta^0,\omega^0)||_{[H^s(0,L)]^2}$. From (4.40), we obtain the following estimate

$$
\begin{aligned}
||\Gamma(\eta,\omega)||_{C([0,T];[H^s(0,L)]^2)}&\le||(\eta^0,\omega^0)||_{[H^s(0,L)]^2}+||\Gamma(\eta,\omega)-\Gamma(0,0)||_{C([0,T];[H^s(0,L)]^2)}\\
&\le||(\eta^0,\omega^0)||_{[H^s(0,L)]^2}+\frac{1}{2}||(\eta,\omega)||_{C([0,T];[H^s(0,L)]^2)}\le R,
\end{aligned}
\tag{4.41}
$$

which allows us to conclude that $\Gamma(\mathcal{B}_R(0))\subset\mathcal{B}_R(0)$. Hence, from (4.40) and (4.41) it follows that $\Gamma:\mathcal{B}_R(0)\to\mathcal{B}_R(0)$ is a contraction. Consequently, by Banach fixed-point Theorem, there exists a unique solution $(\eta,\omega)\in\mathcal{B}_R(0)$ of the integral equation (4.37) for all $t\in(0,T)$.

Finally, in order to prove that the map $\mathcal{F}$ is Lipschitz continuous, we proceed as in the proof of (4.41). In fact, for any $(\eta^{0,1},\omega^{0,1})$ and $(\eta^{0,2},\omega^{0,2})\in[H^s(0,L)]^2$ if we consider the corresponding solutions

$$
(\eta^1,\omega^1)\in C([0,T_1];[H^s(0,L)]^2)\ \text{ and }\ (\eta^2,\omega^2)\in C([0,T_2];[H^s(0,L)]^2),
$$

respectively, it follows that

$$
\begin{aligned}
&||\mathcal{F}(\eta^{0,1},\omega^{0,1})-\mathcal{F}(\eta^{0,2},\omega^{0,2})||_{C([0,T];[H^s(0,L)]^2)}\\
&=||(\eta^1,\omega^1)-(\eta^2,\omega^2)||_{C([0,T];[H^s(0,L)]^2)}\\
&\le||(\eta^{0,1},\omega^{0,1})-(\eta^{0,2},\omega^{0,2})||_{[H^s(0,L)]^2}+\frac{1}{2}||(\eta^1,\omega^1)-(\eta^2,\omega^2)||_{C([0,T];[H^s(0,L)]^2)},
\end{aligned}
\tag{4.42}
$$

where $T=\min\{T_1,T_2\}$. Since $(\eta^1-\eta^2,\omega^1-\omega^2)$ also solves the system (4.24) with initial conditions $(\eta^{0,1}-\eta^{0,2},\omega^{0,1}-\omega^{0,2})$, we deduce that

$$
||\mathcal{F}(\eta^{0,1},\omega^{0,1})-\mathcal{F}(\eta^{0,2},\omega^{0,2})||_{C([0,T];[H^s(0,L)]^2)}\le2||(\eta^{0,1},\omega^{0,1})-(\eta^{0,2},\omega^{0,2})||_{[H^s(0,L)]^2}.
$$

The proof is complete.

■

Remark 4.2.1. *Theorem 4.2.1 does not cover the well-posedness in the energy space $[H^2(0,L)]^2$. Indeed, in order to estimate the boundary terms involving the second derivatives, we have to assume that $s > 5/2$. Therefore, to prove the stabilization result, we assume that $a = c = 0$ and consider a higher-order Boussinesq system of BBM-type. Then, from the proof of the Theorem 4.2.1, we note that such boundary term do not appear and (4.35)-(4.36) hold for $s \in (3/2, 5/2)$. Then, by using the Sobolev embedding $H^s(0,L) \hookrightarrow C([0,L])$ for $s > 3/2$, we can prove that the respective integral equation (4.37) has a fixed point in $C([0,T];[H^s(0,L)]^2)$, for $s \in (3/2,5/2)$. Thus, we obtain the following result that guarantees the well-posedness in the energy space $[H^2(0,L)]^2$.*

Corollary 4.2.2. *Let $a = c = 0$ and $s \in (3/2, 5/2)$. For any $(\eta^0, \omega^0) \in [H^s(0,L)]^2$, there exist $T > 0$ and a unique solution (η, ω) of (4.24)-(4.26) in the class $C([0,T];[H^s(0,L)]^2)$. Moreover, the map*

$$(\eta^0, \omega^0) \in [H^s(0,L)]^2 \longrightarrow (\eta, \omega) \in C([0,T];[H^s(0,L)]^2)$$

is Lipschitz continuous.

Now, we will prove the stabilization result of this section.

Theorem 4.2.2. *For any $(\eta^0, \omega^0) \in \left[H^2(0,L)\right]^2$, the solution (η, ω) of (4.24)-(4.26) given by Corollary 4.2.2 satisfies*

$$\begin{aligned}
&(\eta(t), \omega(t)) \to (0,0) \quad \textit{weakly in} \quad \left[H^2(0,L)\right]^2, \\
&(\eta(t), \omega(t)) \to (0,0) \quad \textit{strongly in} \quad \left[H^s(0,L)\right]^2, \quad \textit{for all} \quad s < 2,
\end{aligned}$$

as $t \to \infty$.

Proof. From Corollary 4.2.2 and equations (4.33)-(4.34) with $a = c = 0$, we deduce that $(\eta_t, \omega_t) \in C([0,T];[H^5(0,L)]^2)$, then (4.27) holds. Consequently, the solution is global in time and the map $t \to ||(\eta(t), \omega(t))||_{[H^2(0,L)]^2}$ is nonincreasing and has a nonnegative limit, as $t \to \infty$. Moreover, we obtain the existence of $(\tilde{\eta}^0, \tilde{\omega}^0) \in [H^2(0,L)]^2$ and a sequence $(t_n)_{n\geq 0}$, such that $t_{n+1} - t_n \geq T$ and

$$(\eta(t), \omega(t)) \to (\tilde{\eta}^0, \tilde{\omega}^0) \quad \text{weakly in} \quad \left[H^2(0,L)\right]^2, \tag{4.43}$$

$$(\eta(t), \omega(t)) \to (\tilde{\eta}^0, \tilde{\omega}^0) \quad \text{strongly in} \quad \left[H^s(0,L)\right]^2, \tag{4.44}$$

and

$$(\eta(t_n + \cdot), \omega(t_n + \cdot)) \to (\tilde{\eta}, \tilde{\omega}) \quad \text{in} \quad C([0,T];[H^s(0,2\pi)]^2),$$

for $s < 2$, where $(\tilde{\eta}, \tilde{\omega}) \in C([0,T];\left[H^2(0,2\pi)\right]^2)$ denotes the solution of (4.24)-(4.26) with initial data $(\tilde{\eta}^0, \tilde{\omega}^0)$.

On the other hand, from (4.27) we get

$$\begin{aligned}
&||(\eta(t_{n+1}), \omega(t_{n+1}))||^2_{[H^2(0,L)]^2} - ||(\eta(t_n), \omega(t_n))||^2_{[H^2(0,L)]^2} = \\
&- 2(b + b_1) \int_{t_n}^{t_{n+1}} (|\eta(t,L)|^2 + |\eta(t,0)|^2) dt - 2(d + d_1) \int_{t_n}^{t_{n+1}} (|\omega(t,L)|^2 + |\omega(t,0)|^2) dt \\
&- 2b_1 \int_{t_n}^{t_{n+1}} (|\eta_x(t,L)|^2 + |\eta_x(t,0)|^2) dt - 2d_1 \int_{t_n}^{t_{n+1}} (|\omega_x(t,L)|^2 + |\omega_x(t,0)|^2) dt,
\end{aligned}$$

which allows us to conclude that

$$\begin{aligned}
\lim_{n\to\infty} \Bigg(&(b + b_1) \int_{t_n}^{t_{n+1}} (|\eta(t,L)|^2 + |\eta(t,0)|^2) dt + (d + d_1) \int_{t_n}^{t_{n+1}} (|\omega(t,L)|^2 + |\omega(t,0)|^2) dt \\
&+ b_1 \int_{t_n}^{t_{n+1}} (|\eta_x(t,L)|^2 + |\eta_x(t,0)|^2) dt + d_1 \int_{t_n}^{t_{n+1}} (|\omega_x(t,L)|^2 + |\omega_x(t,0)|^2) dt \Bigg) = 0.
\end{aligned}$$

Thus,

$$
\begin{aligned}
(b+b_1)\int_0^T(|\tilde{\eta}(t,L)|^2+|\tilde{\eta}(t,0)|^2)dt+(d+d_1)\int_0^T(|\tilde{\omega}(t,L)|^2+|\tilde{\omega}(t,0)|^2)dt\\
+b_1\int_0^T(|\tilde{\eta}_x(t,L)|^2+|\tilde{\eta}_x(t,0)|^2)dt+d_1\int_0^T(|\tilde{\omega}_x(t,L)|^2+|\tilde{\omega}_x(t,0)|^2)dt=0
\end{aligned}
$$

and therefore

$$
\tilde{\eta}(t,L)=\tilde{\eta}(t,0)=\tilde{\eta}_x(t,L)=\tilde{\eta}_x(t,0)=\tilde{\omega}(t,L)=\tilde{\omega}(t,0)=\tilde{\omega}_x(t,L)=\tilde{\omega}_x(t,0)=0,\quad t\in(0,T).
$$

Let us consider $(\bar{\eta},\bar{\omega})$ the extended function by zero of $(\tilde{\eta},\tilde{\omega})$ for $x\in(-l,l)\backslash(0,L)$, where $(0,L)\subset(-l,l)$ is an interval. Then, $(\bar{\eta},\bar{\omega})$ solves

$$
\begin{cases}
\bar{\eta}_t+\bar{\omega}_x-b\bar{\eta}_{xxt}+b_1\bar{\eta}_{txxxx}=0 & \text{for } x\in(-l,l),\ t\in(0,T),\\
\bar{\omega}_t+\bar{\eta}_x-d\bar{\omega}_{xxt}+d_1\bar{\omega}_{txxxx}=0 & \text{for } x\in(-l,l), t\in(0,T),\\
\frac{\partial^r\bar{\eta}}{\partial x^r}(t,-l)=\frac{\partial^r\bar{\eta}}{\partial x^r}(t,l)=\frac{\partial^r\bar{\omega}}{\partial x^r}(t,-l)=\frac{\partial^r\bar{\omega}}{\partial x^r}(t,l)=0 & \text{for } t\in(0,T),\quad r=0,1,\\
\bar{\eta}(0,x)=\bar{\eta}^0(x);\quad \bar{\omega}(0,x)=\bar{\omega}^0(x) & \text{for } x\in(-l,l).
\end{cases}
$$

and satisfies

$$
(\bar{\eta}(t,x),\bar{\omega}(t,x))=(0,0)\ \text{ for }\ (t,x)\in(0,T)\times((-l,l)\backslash(0,L)),
$$

where

$$
\bar{\eta}^0(x)=\begin{cases}\tilde{\eta}^0(x) & x\in(0,L)\\ 0 & x\in(-l,l)\backslash(0,L)\end{cases}
$$

and

$$
\bar{\omega}^0(x)=\begin{cases}\tilde{\omega}^0(x) & x\in(0,L)\\ 0 & x\in(-l,l)\backslash(0,L).\end{cases}
$$

We remark that Theorems 4.1.3 and 4.1.4 can be proved for a domain of the form $(-l,l)$. Therefore, since $(\bar{\eta}^0,\bar{\omega}^0)\in[H_0^2(-l,l)]^2$, from Theorem 4.1.3 it follows that $(\bar{\eta},\tilde{\omega})\in C^\omega([0,T];[H_0^2(-l,l)]^2)$, and from Theorem 4.1.4 we deduce that $(\bar{\eta}^0,\bar{\omega}^0)=(0,0)$. Hence, $(\tilde{\eta}^0,\tilde{\omega}^0)=(0,0)$. Finally, from (4.43) and (4.44) we have that, as $t\to\infty$,

$$
\begin{aligned}
&(\eta(t),\omega(t))\to(0,0)\ \text{ weakly in }\ \left[H^2(0,L)\right]^2,\\
&(\eta(t),\omega(t))\to(0,0)\ \text{ strongly in }\ \left[H^s(0,L)\right]^2, \text{for all }\ s<2.
\end{aligned}
$$

■

4.3 Internal Stabilization

In this section, we are interested in the asymptotic behavior of the solutions of the following higher-order system

$$
\begin{cases}
\eta_t+\omega_x+a\omega_{xxx}-b\eta_{txx}+b_1\eta_{txxxx}+\mathcal{B}\eta=0 & \text{for } x\in(0,2\pi),\ t>0,\\
\omega_t+\eta_x+c\eta_{xxx}-d\omega_{txx}+d_1\omega_{txxxx}=0 & \text{for } x\in(0,2\pi), t>0,\\
\frac{\partial^r\eta}{\partial x^r}(t,0)=\frac{\partial^r\eta}{\partial x^r}(t,2\pi),\quad \frac{\partial^r\omega}{\partial x^r}(t,0)=\frac{\partial^r\omega}{\partial x^r}(t,2\pi) & \text{for } t>0,\quad r=0,1,\\
\eta(0,x)=\eta^0(x);\quad \omega(0,x)=\omega^0(x) & \text{for } x\in(0,2\pi),
\end{cases}
\tag{4.45}
$$

where $a=c\geq 0,\quad b,d,b_1,d_1>0$ and $\mathcal{B}:H_p^s(0,2\pi)\longrightarrow H_p^s(0,2\pi)$ is a bounded operator. More precisely, let

$$
\begin{cases}
\rho\in C_p^\infty(0,2\pi)\ \text{ a nonnegative function on }\ (0,2\pi)\\
\text{with }\ \rho(x)>0\ \text{ on a given open set }\ \Omega_1\subset(0,2\pi).
\end{cases}
\tag{4.46}
$$

We analyze the following cases for the operator $\mathcal{B}$:

$$
\mathcal{B}\varphi=\rho(x)\varphi\ \text{ and }\ \mathcal{B}\varphi=(\rho(x)\varphi_x)_x\,.
$$

Internal Stabilization with the Feedback $\mathcal{B}\varphi = \rho(x)\varphi$

First, by using a fixed point argument, we prove that the system is globally well-posed, therefore we write the solution of (4.45) in its integral form

$$\begin{aligned} \eta(t) &= \eta^0 - \int_0^t (1 - b\partial_x^2 + b_1\partial_x^4)^{-1}(\partial_x\omega + a\partial_x^3\omega + \mathcal{B}\eta)(\tau)d\tau, \\ \omega(t) &= \omega^0 - \int_0^t (1 - d\partial_x^2 + d_1\partial_x^4)^{-1}(\partial_x\eta + a\partial_x^3\eta)(\tau)d\tau, \end{aligned} \tag{4.47}$$

where $(1 - \alpha\partial_x^2 + \beta\partial_x^4)^{-1}f$ denotes, for $f \in L^2(0, 2\pi)$ and $\alpha, \beta > 0$, the unique solution $v \in H_p^4(0, 2\pi)$ of the elliptic equation $(1 - \alpha\partial_x^2 + \beta\partial_x^4)v = f$. Moreover, for any $s \geq 0$,

$$\begin{aligned} &||(1 - \alpha\partial_x^2 + \beta\partial_x^4)^{-1}f||_{H_p^s(0,2\pi)} \leq C||f||_{H_p^s(0,2\pi)}, \\ &||(1 - \alpha\partial_x^2 + \beta\partial_x^4)^{-1}\partial_x f||_{H_p^s(0,2\pi)} \leq C||f||_{H_p^s(0,2\pi)}, \\ &||(1 - \alpha\partial_x^2 + \beta\partial_x^4)^{-1}\partial_x^3 f||_{H_p^s(0,2\pi)} \leq C||f||_{H_p^s(0,2\pi)}, \end{aligned} \tag{4.48}$$

for all $\alpha > 0$, where C is a positive constant. Then we have the following result:

Theorem 4.3.1. *Let $s \geq 0$. For any $(\eta^0, \omega^0) \in \left[H_p^s(0, 2\pi)\right]^2$, there exist $T > 0$ and a unique solution (η, ω) of (4.45) with $\mathcal{B}\varphi = \rho(x)\varphi$ in the class $C([0, T]; \left[H_p^s(0, 2\pi)\right]^2)$. If $s = 2$, the solution exists for every $T > 0$. Moreover, the map $\mathcal{F}$ defined as follows*

$$\begin{aligned} \mathcal{F} : \left[H_p^s(0, 2\pi)\right]^2 &\longrightarrow C([0, T]; \left[H_p^s(0, 2\pi)\right]^2) \\ (\eta^0, \omega^0) &\longmapsto (\eta, \omega) \end{aligned}$$

is Lipschitz continuous.

Proof. Motivated by (4.47), for any $(\eta^0, \omega^0) \in \left[H_p^s(0, 2\pi)\right]^2$, we introduce the operator

$$\Gamma(\eta, \omega)(t) := (\eta^0, \omega^0) - \begin{pmatrix} \int_0^t \mathcal{L}_{b,b_1}^{-1}(\partial_x\omega + a\partial_x^3\omega + \rho(x)\eta)(\tau)d\tau \\ \int_0^t \mathcal{L}_{d,d_1}^{-1}(\partial_x\eta + a\partial_x^3\eta)(\tau)d\tau \end{pmatrix}^t.$$

Let $0 < T' < T$, to be chosen later. Then, for each $(\eta^1, \omega^1), (\eta^2, \omega^2) \in C([0, T']; \left[H_p^s(0, 2\pi)\right]^2)$, from (4.48) it follows that

$$\begin{aligned} &||\Gamma(\eta^1, \omega^1) - \Gamma(\eta^2, \omega^2)||_{C\left([0,T'];\left[H_p^s(0,2\pi)\right]^2\right)} \\ &= \sup_{0 \leq t \leq T'} ||\Gamma(\eta^1, \omega^1)(t) - \Gamma(\eta^2, \omega^2)(t)||_{\left[H_p^s(0,2\pi)\right]^2} \\ &\leq \int_0^{T'} \Big(||\mathcal{L}_{b,b_1}^{-1}\partial_x(\omega^1 - \omega^2)(\tau)||_{H_p^s(0,2\pi)} + ||\mathcal{L}_{b,b_1}^{-1}a\partial_x^3(\omega^1 - \omega^2)(\tau)||_{H_p^s(0,2\pi)} \\ &\qquad + ||\mathcal{L}_{b,b_1}^{-1}\rho(x)(\eta^1 - \eta^2)(\tau)||_{H_p^s(0,2\pi)} \Big) d\tau \\ &+ \int_0^{T'} \Big(||\mathcal{L}_{d,d_1}^{-1}\partial_x(\eta^1 - \eta^2)(\tau)||_{H_p^s(0,2\pi)} + ||\mathcal{L}_{d,d_1}^{-1}a\partial_x^3(\eta^1 - \eta^2)(\tau)||_{H_p^s(0,2\pi)} \Big) d\tau \\ &\leq CT' \Big(||(\rho(\eta^1 - \eta^2), \omega^1 - \omega^2)||_{C\left([0,T'];\left[H_p^s(0,2\pi)\right]^2\right)} + ||(\eta^1 - \eta^2, \omega^1 - \omega^2)||_{C\left([0,T'];\left[H_p^s(0,2\pi)\right]^2\right)} \Big) \\ &\leq CT'||(\eta^1, \omega^1) - (\eta^2, \omega^2)||_{C\left([0,T'];\left[H_p^s(0,2\pi)\right]^2\right)}, \end{aligned}$$

where C is a positive constant. Choosing $T' > 0$ satisfying $CT' \leq \frac{1}{2}$, from the estimate above we obtain

$$||\Gamma(\eta^1, \omega^1) - \Gamma(\eta^2, \omega^2)||_{C\left([0,T'];\left[H_p^s(0,2\pi)\right]^2\right)} \leq \frac{1}{2}||(\eta^1, \omega^1) - (\eta^2, \omega^2)||_{C\left([0,T'];\left[H_p^s(0,2\pi)\right]^2\right)}. \tag{4.49}$$

Let $(\eta, \omega) \in \mathcal{B}_R(0) = \{(\eta, \omega) \in C([0, T']; [H_p^s(0, 2\pi)]^2) : ||(\eta, \omega)||_{C([0,T'];[H_p^s(0,2\pi)]^2)} \leq R\}$, where $R = 2||(\eta^0, \omega^0)||_{[H_p^s(0,2\pi)]^2}$. From (4.49), we obtain the following estimate

$$\begin{aligned}||\Gamma(\eta, \omega)||_{C([0,T'];[H_p^s(0,2\pi)]^2)} &\leq ||(\eta^0, \omega^0)||_{[H_p^s(0,2\pi)]^2} + ||\Gamma(\eta, \omega) - \Gamma(0, 0)||_{C([0,T'];[H_p^s(0,2\pi)]^2)} \\ &\leq ||(\eta^0, \omega^0)||_{[H_p^s(0,2\pi)]^2} + \frac{1}{2}||(\eta, \omega)||_{C([0,T'];[H_p^s(0,2\pi)]^2)} \leq R,\end{aligned} \tag{4.50}$$

which allows us to conclude that $\Gamma(\mathcal{B}_R(0)) \subset \mathcal{B}_R(0)$. Hence, from (4.50) and (4.49) it follows that Γ is a contraction. Consequently, by Banach fixed-point Theorem, there exists a unique solution $(\eta, \omega) \in \mathcal{B}_R(0)$ of the integral equation (4.47) for all $t \in (0, T')$.

Let (η, ω) a smooth solution of (4.45), then, we multiply the first (resp. second) equation in (4.45) by η (resp. ω), integrate by parts over $(0, 2\pi)$, and add the two obtained equations to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} |\eta(t)|^2 + |\omega(t)|^2 + b|\eta_x(t)|^2 + d|\omega_x(t)|^2 + b_1|\eta_{xx}(t)|^2 + d_1|\omega_{xx}(t)|^2 dx \\ + \int_0^{2\pi} \rho(x)\eta^2 dx = 0,\end{aligned}$$

which implies that

$$||(\eta(t), \omega(t))||_{[H_p^2(0,2\pi)]^2} \leq ||(\eta^0, \omega^0)||_{[H_p^2(0,2\pi)]^2},$$

for any $t \geq 0$. The estimate above, can be extended for any $(\eta^0, \omega^0) \in [H_p^2(0, 2\pi)]^2$ by a density argument. Consequently, for every $T > 0$ and $(\eta^0, \omega^0) \in [H_p^2(0, 2\pi)]^2$, (4.45) admits a unique solution $(\eta, \omega) \in C([0, T]; [H_p^2(0, 2\pi)]^2)$. Proceeding as in the proof of (4.42), from (4.49) we obtain the Lipschitz continuity of the flow map $\mathcal{F}$:

$$||\mathcal{F}(\eta^{0,1}, \omega^{0,1}) - \mathcal{F}(\eta^{0,2}, \omega^{0,2})||_{C\left([0,T];\left[H_p^s(0,2\pi)\right]^2\right)} \leq 2||(\eta^{0,1}, \omega^{0,1}) - (\eta^{0,2}, \omega^{0,2})||_{\left[H_p^s(0,2\pi)\right]^2}. \tag{4.51}$$

and the proof ends. ■

We are now concerned with the respective stabilization result:

Theorem 4.3.2. *For any $(\eta^0, \omega^0) \in \left[H_p^2(0, 2\pi)\right]^2$, the solution (η, ω) of (4.45) given by Theorem 4.3.1 satisfies*

$$\begin{aligned}&(\eta(t), \omega(t)) \to (0, 0) \quad \textit{weakly in} \quad \left[H_p^2(0, 2\pi)\right]^2, \\ &(\eta(t), \omega(t)) \to (0, 0) \quad \textit{strongly in} \quad \left[H_p^s(0, 2\pi)\right]^2, \quad \textit{for all} \quad s < 2,\end{aligned}$$

as $t \to \infty$.

Proof. When $s = 2$, we can use Theorem 4.3.1 and the equations of the system (4.45) to deduce that

$$\eta_t = -\mathcal{L}_{b,b_1}^{-1}(1 - a\partial_x^2)\partial_x\omega - \mathcal{L}_{b,b_1}^{-1}\rho(x)\eta \quad \text{and} \quad \omega_t = -\mathcal{L}_{d,d_1}^{-1}(1 - a\partial_x^2)\partial_x\eta$$

belong to $C([0, T]; H_p^3(0, 2\pi))$. Thus, each term in (4.45) belongs to $L^2(0, T; H_p^{-1}(0, 2\pi))$. Consequently, scaling the first (resp. second) equation in (4.45) by η (resp. w), we obtain

$$\frac{d}{dt}||(\eta(t), \omega(t))||^2_{[H_p^2(0,2\pi)]^2} + 2\int_0^{2\pi} \rho(x)|\eta(t, x)|^2 dx = 0. \tag{4.52}$$

Integrating (4.52), we get

$$||(\eta(t),\omega(t))||^2_{[H^2_p(0,2\pi)]^2} - ||(\eta^0,\omega^0)||^2_{[H^2_p(0,2\pi)]^2} + 2\int_0^t\int_0^{2\pi}\rho(x)|\eta(s,x)|^2dxds = 0. \tag{4.53}$$

Identity (4.52) shows that the map $t\mapsto ||(\eta(t),\omega(t))||_{[H^2_p(0,2\pi)]^2}$ is nonincreasing and

$$||(\eta(t),\omega(t))||_{[H^2_p(0,2\pi)]^2} \le ||(\eta^0,\omega^0)||_{[H^2_p(0,2\pi)]^2}, \quad \text{for all} \quad t\ge 0. \tag{4.54}$$

Hence, there exist $l\in\mathbb{R}^+$, such that

$$\lim_{t\to+\infty} ||(\eta(t),\omega(t))||_{[H^2_p(0,2\pi)]^2} = l.$$

Moreover, from (4.54) we infer the existence of a sequence $t_n\to+\infty$, such that $t_{n+1}-t_n\ge T$ and

$$(\eta(t_n),\omega(t_n)) \to (\tilde{\eta}^0,\tilde{\omega}^0) \quad \text{weakly in} \quad [H^2_p(0,2\pi)]^2, \tag{4.55}$$

for some $(\tilde{\eta}^0,\tilde{\omega}^0)\in[H^2_p(0,2\pi)]^2$. Then, proceeding as in the proof of (4.53) we obtain

$$\begin{aligned} ||(\eta(t_{n+1}),\omega(t_{n+1}))||^2_{[H^2_p(0,2\pi)]^2} - &||(\eta(t_n),\omega(t_n))||^2_{[H^2_p(0,2\pi)]^2} \\ &+ 2\int_{t_n}^{t_{n+1}}\int_0^{2\pi}\rho(x)|\eta(t,x)|^2dxdt = 0. \end{aligned}$$

Consequently,

$$\lim_{n\to+\infty}\int_{t_n}^{t_{n+1}}\int_0^{2\pi}\rho(x)|\eta(t,x)|^2dxdt = 0. \tag{4.56}$$

On the other hand, from (4.55) and the Sobolev embedding, for $s\in[0,2)$ we obtain the following convergence

$$(\eta(t_n),\omega(t_n)) \to (\tilde{\eta}^0,\tilde{\omega}^0) \quad \text{strongly in} \quad [H^s_p(0,2\pi)]^2. \tag{4.57}$$

Since the couple $(\eta(t_n+t,x),\omega(t_n+t,x))$ solves the system (4.45) with initial data $(\eta(t_n),\omega(t_n))$, from (4.51) and (4.57) we get

$$(\eta(t_n+\cdot),\omega(t_n+\cdot)) \to (\tilde{\eta},\tilde{\omega}) \quad \text{in} \quad C([0,T];[H^s_p(0,2\pi)]^2), \quad \text{as} \quad n\to+\infty,$$

where $(\tilde{\eta},\tilde{\omega})\in C([0,T];[H^s_p(0,2\pi)]^2)$ denotes the solution with initial data $(\tilde{\eta}^0,\tilde{\omega}^0)$. The convergence above combined with (4.56) yields

$$\begin{aligned} 0 = \lim_{n\to+\infty}\int_{t_n}^{t_{n+1}}\int_0^{2\pi}\rho(x)|\eta(t,x)|^2dxdt &\ge \lim_{n\to+\infty}\int_0^T\int_0^{2\pi}\rho(x)|\eta(t+t_n,x)|^2dxdt \\ &= \int_0^T\int_0^{2\pi}\rho(x)|\tilde{\eta}(t,x)|^2dxdt = 0. \end{aligned} \tag{4.58}$$

Thus, $(\tilde{\eta},\tilde{\omega})\in C([0,T];[H^s_p(0,2\pi)]^2)$ solves the following system

$$\begin{cases} \tilde{\eta}_t + \tilde{\omega}_x + a\tilde{\omega}_{xxx} - b\tilde{\eta}_{xxt} + b_1\tilde{\eta}_{txxxx} = 0 & \text{for } x\in(0,2\pi),\ t>0, \\ \tilde{\omega}_t + \tilde{\eta}_x + a\tilde{\eta}_{xxx} - d\tilde{\omega}_{xxt} + d_1\tilde{\omega}_{txxxx} = 0 & \text{for } x\in(0,2\pi), t>0, \\ \frac{\partial^r\tilde{\eta}}{\partial x^r}(t,0) = \frac{\partial^r\tilde{\eta}}{\partial x^r}(t,2\pi), \quad \frac{\partial^r\tilde{\omega}}{\partial x^r}(t,0) = \frac{\partial^r\tilde{\omega}}{\partial x^r}(t,2\pi) & \text{for } t>0, \quad r=0,1, \\ \tilde{\eta}(0,x) = \tilde{\eta}^0(x); \quad \tilde{\omega}(0,x) = \tilde{\omega}^0(x) & \text{for } x\in(0,2\pi), \end{cases}$$

and (4.58) allows us to conclude that

$$\tilde{\eta}(t,x) = 0, \quad \text{in} \quad (t,x)\in(0,T)\times\Omega_1,$$

where Ω_1 was defined in (4.46). Finally, from Theorem 4.1.4 we have $(\tilde{\eta}^0,\tilde{\omega}^0)=(0,0)$ and, as $t\to\infty$, the following holds

$$\begin{aligned} &(\eta(t),\omega(t)) \to (0,0) \quad \text{weakly in} \quad \left[H^2_p(0,2\pi)\right]^2, \\ &(\eta(t),\omega(t)) \to (0,0) \quad \text{strongly in} \quad \left[H^s_p(0,2\pi)\right]^2, \quad \text{for all} \quad s<2, \end{aligned}$$

which completes the proof. ∎

Internal Stabilization with the Feedback $\mathcal{B}\varphi = (\rho(x)\varphi_x)_x$

The well-posedness of the system (4.45) is proved by arguing as in the proof of Theorem 4.3.1:

Theorem 4.3.3. *Let $s \geq 0$. For any $(\eta^0, \omega^0) \in \left[H_p^s(0,2\pi)\right]^2$, there exist $T > 0$ and a unique solution (η, ω) of (4.45) with $\mathcal{B}\varphi = (\rho(x)\varphi_x)_x$ in the class $C([0,T]; \left[H_p^s(0,2\pi)\right]^2)$. If $s = 2$, the solution exists for every $T > 0$. Moreover, the map $\mathcal{F}$ defined as follows*

$$\begin{aligned} \mathcal{F} : \left[H_p^s(0,2\pi)\right]^2 &\longrightarrow C([0,T]; \left[H_p^s(0,2\pi)\right]^2) \\ (\eta^0, \omega^0) &\longmapsto (\eta, \omega) \end{aligned}$$

is Lipschitz continuous.

Proof. Proceeding as in the proof of the Theorem 4.3.1, for any $(\eta^0, \omega^0) \in \left[H_p^s(0,2\pi)\right]^2$ we introduce the operator

$$\Gamma(\eta,\omega)(t) := (\eta^0, \omega^0) - \begin{pmatrix} \int_0^t \mathcal{L}_{b,b_1}^{-1}(\partial_x\omega + a\partial_x^3\omega + (\rho(x)\eta_x)_x)(\tau)d\tau \\ \int_0^t \mathcal{L}_{d,d_1}^{-1}(\partial_x\eta + a\partial_x^3\eta)(\tau)d\tau \end{pmatrix}^t.$$

Then, by using the following estimative:

$$||(1-\alpha\partial_x^2 + \beta\partial_x^4)^{-1}(\rho u_x)_x||_{H_p^s(0,2\pi)} \leq C||u||_{H_p^s(0,2\pi)},$$

for $s \geq 0$, $\alpha, \beta > 0$, where C a positive constant, it can be showed that Γ contracts in a ball of the space $C([0,T]; \left[H_p^s(0,2\pi)\right]^2)$. Therefore, we omit the details. ■

Remark 4.3.1. *From Theorem 4.3.3 and by integrating on $(0, 2\pi)$ the equations in system (4.45) we obtain the following conservations laws*

$$\frac{d}{dt}\int_0^{2\pi} \eta(t,x)dx = 0 \quad \text{and} \quad \frac{d}{dt}\int_0^{2\pi} \omega(t,x)dx = 0.$$

This implies that

$$\int_0^{2\pi} \eta(t,x)dx = \int_0^{2\pi} \eta^0(x)dx \quad \text{and} \quad \int_0^{2\pi} \omega(t,x)dx = \int_0^{2\pi} \omega^0(x)dx$$

We are now in a position to prove the stabilization result:

Theorem 4.3.4. *For any $(\eta^0, \omega^0) \in \left[H_p^2(0,2\pi)\right]^2$, the solution (η, ω) of (4.45) given by Theorem 4.3.3 satisfies*

$$\begin{aligned} &(\eta(t), \omega(t)) \to ([\eta^0], [\omega^0]) \quad \text{weakly in} \quad \left[H_p^2(0,2\pi)\right]^2, \\ &(\eta(t), \omega(t)) \to ([\eta^0], [\omega^0]) \quad \text{strongly in} \quad \left[H_p^s(0,2\pi)\right]^2, \quad \text{for all} \quad s < 2, \end{aligned}$$

as $t \to \infty$, where $[f] := \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(x)dx$.

Proof. We first remark that, if $\varphi \in H_p^2(0,2\pi)$, from (4.46) we have that $(\rho\varphi_x)_x \in L^2(0,2\pi)$. Thus, we can proceed as in the proof of (4.53) to obtain

$$\frac{d}{dt}||(\eta(t),\omega(t))||^2_{[H_p^2(0,2\pi)]^2} + 2\int_0^{2\pi} \rho(x)|\eta_x(t,x)|^2 dx = 0. \tag{4.59}$$

Moreover, arguing as in the proof of Theorem 4.3.2, we obtain $(\tilde{\eta}^0, \tilde{\omega}^0) \in [H_p^2(0, 2\pi)]^2$ and a sequence $t_n \to +\infty$, such that $t_{n+1} - t_n \geq T$ and

$$(\eta(t), \omega(t)) \to (\tilde{\eta}^0, \tilde{\omega}^0) \;\text{ weakly in }\; \left[H_p^2(0, 2\pi)\right]^2, \tag{4.60}$$

$$(\eta(t), \omega(t)) \to (\tilde{\eta}^0, \tilde{\omega}^0) \;\text{ strongly in }\; \left[H_p^s(0, 2\pi)\right]^2, \tag{4.61}$$

and

$$(\eta(t_n + \cdot), \omega(t_n + \cdot)) \to (\tilde{\eta}, \tilde{\omega}) \;\text{ in }\; C([0, T]; \left[H_p^s(0, 2\pi)\right]^2), \tag{4.62}$$

for any $s < 2$, where $(\tilde{\eta}, \tilde{\omega}) \in C([0, T]; \left[H_p^2(0, 2\pi)\right]^2)$ denotes the solution of (4.45) with initial data $(\tilde{\eta}^0, \tilde{\omega}^0)$. From (4.62) it follows that

$$(\eta(t_n + \cdot), \omega(t_n + \cdot)) \;\text{ is bounded in }\; L^2(0, T; \left[H_p^s(0, 2\pi)\right]^2).$$

Then, we can extract a subsequence (if necessary), satisfying

$$(\eta(t_n + \cdot), \omega(t_n + \cdot)) \to (\tilde{\eta}, \tilde{\omega}) \;\text{ weakly in }\; L^2(0, T; \left[H_p^2(0, 2\pi)\right]^2). \tag{4.63}$$

On the other hand, from (4.59) we get

$$\begin{aligned} ||(\eta(t_{n+1}), \omega(t_{n+1}))||^2_{[H_p^2(0,2\pi)]^2} - ||(\eta(t_n), \omega(t_n))||^2_{[H_p^2(0,2\pi)]^2} & \\ + 2\int_{t_n}^{t_{n+1}} \int_0^{2\pi} \rho(x)|\eta_x(t, x)|^2 dx dt = 0, & \end{aligned}$$

which leads to

$$\lim_{n\to\infty} \int_{t_n}^{t_{n+1}} \int_0^{2\pi} \rho(x)|\eta_x(t, x)|^2 dx dt = 0, \tag{4.64}$$

since $|| \cdot ||_{[H^2(0,2\pi)]^2}$ is nonincreasing, and therefore has a limit, as $t \to \infty$. (See (4.59)). By combining (4.63) and (4.64), we deduce that

$$\int_0^T \int_0^{2\pi} \rho(x)|\tilde{\eta}_x(t, x)|^2 dx dt \leq \liminf_{n\to\infty} \int_{t_n}^{t_{n+1}} \int_0^{2\pi} \rho(x)|\eta_x(t, x)|^2 dx dt = 0. \tag{4.65}$$

Thus, we have that $(\tilde{\eta}, \tilde{\omega})$ solves

$$\begin{cases} \tilde{\eta}_t + \tilde{\omega}_x + a\tilde{\omega}_{xxx} - b\tilde{\eta}_{xxt} + b_1\tilde{\eta}_{txxxx} = 0 & \text{for } x \in (0, 2\pi),\ t > 0, \\ \tilde{\omega}_t + \tilde{\eta}_x + a\tilde{\eta}_{xxx} - d\tilde{\omega}_{xxt} + d_1\tilde{\omega}_{txxxx} = 0 & \text{for } x \in (0, 2\pi), t > 0, \\ \frac{\partial^r \tilde{\eta}}{\partial x^r}(t, 0) = \frac{\partial^r \tilde{\eta}}{\partial x^r}(t, 2\pi), \quad \frac{\partial^r \tilde{\omega}}{\partial x^r}(t, 0) = \frac{\partial^r \tilde{\omega}}{\partial x^r}(t, 2\pi) & \text{for } t > 0, \quad r = 0, 1, \\ \tilde{\eta}(0, x) = \tilde{\eta}^0(x); \quad \tilde{\omega}(0, x) = \tilde{\omega}^0(x) & \text{for } x \in (0, 2\pi), \end{cases}$$

and (4.65) allows us to conclude that

$$\tilde{\eta}_x(t, x) = 0, \quad \forall (t, x) \in (0, T) \times \Omega_1,$$

for Ω_1 defined in (4.46). Thus, from Theorem 4.1.5 we have that $(\tilde{\eta}, \tilde{\omega}) = (c_1, c_2)$ on $(0, T) \times (0, 2\pi)$ for some $c_1, c_2 \in \mathbb{R}$. From the Remark 4.3.1 and (4.60)-(4.61) it follows that

$$(c_1, c_2) = ([\eta^0], [\omega^0])$$

and

$$(\eta(t), \omega(t)) \to ([\eta^0], [\omega^0]) \;\text{ weakly in }\; \left[H_p^2(0, 2\pi)\right]^2,$$

$$(\eta(t), \omega(t)) \to ([\eta^0], [\omega^0]) \;\text{ strongly in }\; \left[H_p^s(0, 2\pi)\right]^2, \;\text{ for all }\; s < 2,$$

■

Chapter 5

Comments and perspectives

During the development of this work some natural questions came to the fore. Therefore, in this section we will mention a list of problems that we can study thereafter.

5.1 One control only

The problem we address in Chapter 2 is open when the control acts in one equation only ($h_1 \equiv 0$ or $h_2 \equiv 0$) or the control region does not contain a neighborhood of the right end point of the interval. This is probably a purely technical problem that could be overcome by proving unique continuation results. But, as far as we know, this remains to be done, including for the single KdV equation. On the other hand, following the approach in [14] one may expect null controllability results in the classical Sobolev spaces H^s, which also remains to be done.

The results obtained in Chapter 2 were published in

On the controllability of a nonlinear dispersive system in a weighted L^2-space, Differential and Integral Equations, Volume 34, Number 3-4 (2021), 127-164.

5.2 Higher order KdV terms and asymptotic behavior

In Chapter 3, the conditions on the coefficients of the highest order BBM terms ($b_1 > 0$ and $d_1 > 0$) provide a regularizing effect which is very useful for the well-posedness of the system (1.8). On the other hand, from the controllability point of view, KdV type models are known to have a much better behavior (see, for instance, [33, 39]). Therefore, it is an interesting issue to study what can be done in the presence of the highest KdV terms ($a_1 > 0$ and $c_1 > 0$), including the full system (1.2).

In the spirit of the problem mention above, the controllability problem also remains open when $b_1 = d_1 = 0$ and $a_1, c_1 > 0$, i.e, in the absence of the highest BBM terms. The KdV terms should provide good controllability properties, but in order to prove the well-posedness of the resulting nonlinear system, more regularity of the solutions is needed.

The spectral analysis developed in Chapter 3 also leads to the study of the stabilization problem when the time t is sufficiently large. By considering homogeneous Dirichlet boundary conditions and a damping term acting in one equation of (1.8), the asymptotic behavior of the energy associated to the model can be studied. Indeed, proceeding as in Section 3.3, a similar spectral analysis can be developed to construct a Riesz basis of $[H_0^2(0, L)]^2$ consisting of generalized eigenvalues of the corresponding differential operator. Then, by using arguments similar to those developed in [36], we can conclude that $||(\eta(\cdot, x), \omega(\cdot, x))||_{[H_0^2(0,L)]^2} \to 0$, as $t \to \infty$.

The results obtained in Chapter 3 are available in

On the lack of controllability of a higher-order regularized long-wave system, preprint (2021).

5.3 Another dissipative effects

In Chapter 4, we introduce some damping mechanisms that make the energy associated to the linear higher-order Boussinesq system converge to zero. However, our results do not provide any decay rate. In this sense, the results obtained in [18] for the lower order Boussinesq system could be extended for the full system (1.2), posed in $\mathbb{R}$, when *complete* and *partial* dissipations are considered.

Bibliography

[1] D. K. Arrowsmith and C. M. Place, *Dynamical Systems: Differential Equations, Maps, and Chaotic Behavior*, Chapman and Hall Mathematics Series. Chapman & Hall, 1992.

[2] G. J. Bautista, S. Micu and A. F. Pazoto, *On the controllability of a model system for long waves in nonlinear dispersive media*, Nonlinearity, To appear.

[3] G. J. Bautista and A. F. Pazoto, *Large-Time Behavior of a Linear Boussinesq System for the Water Waves*, J. Dynam. Differential Equations 31 (2019), 959–978.

[4] G. J. Bautista and A. F. Pazoto, *On the controllability of a Boussinesq system for two-way propagation of dispersive waves*, J. Evol. Equ. 20 (2020), 607–630.

[5] G. J. Bautista and A. F. Pazoto, *Decay of solutions for a dissipative higher-order Boussinesq system on a periodic domain*, Commun. Pure Appl. Anal. 19 (2020), 747–769.

[6] J. A. Bárcena-Petisco, S. Guerrero and A. F. Pazoto, *Local null controllability of a model system for strong interaction between internal solitary waves*, Commun. Contemp. Math., To appear.

[7] T. Benjamin, J.L. Bona and J. Mahony, *Model equations for long waves in nonlinear, dispersive media*, Philos. Trans. Royal Soc. London Series A 272 (1972), 47–78.

[8] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Grundlehren der Mathematisehen Wissenschaften, No. 223. Springer-Verlag, Berlin New York, 1976.

[9] J.3L. Bona, M. Chen and J.-C. Saut, *Boussinesq equations and other systems for smallamplitude long waves in nonlinear dispersive media. I: Derivation and linear theory,* J. Nonlinear Sci. 12 (2002), 283–318.

[10] J.L. Bona, M. Chen and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II: Nonlinear theory,* Nonlinearity 17 (2004), 925–9052.

[11] J.L. Bona, G. Ponce, J.-C. Saut and M. Tom, *A model system for strong interaction between internal solitary waves*, Comm. Math. Phys. 143 (1992), 287–313.

[12] M. Caicedo, R. A. Capistrano-Filho and B.-Y. Zhang, *Neumann boundary controllability of the Korteweg-de Vries equation on a bounded domain*, SIAM J. Control Optim. 55 (2017), 3503–3532.

[13] R. A. Capistrano-Filho, V. Komornik and A. F. Pazoto, *Pointwise control of the linearized Gear-Grimshaw system*, Evol. Equ. Control Theory 9 (2020) 693-719.

[14] R. A. Capistrano-Filho, A. F. Pazoto and L. Rosier, *Internal controllability of the Korteweg-de Vries equation on a bounded domain*, ESAIM Control Optim. Calc. Var. 21 (2015), 1076–1107.

[15] R. A. Capistrano-Filho, A. F. Pazoto and L. Rosier, *Control of Boussinesq system of KdV-KdV type on a bounded interval*, ESAIM Control Optim. Calc. Var. 25 (2019), Art. 58, 55 pp.

[16] N. Carreño, E. Cerpa, and E. Crépeau, *Internal null controllability of the generalized Hirota-Satsuma system*, ESAIM Control Optim. Calc. Var. 26 (2020), Art. 75, 22 pp.

[17] T. Cazenave and A. Haraux, *An introduction to semilinear evolution equations*, Oxford Lecture Series in Mathematics and its Applications, 13, Oxford University Press, New York, 1998.

[18] M. Chen and O. Goubet, *Long-time asymptotic behavior of two-dimensional dissipative Boussinesq systems*, Discrete Contin. Dyn. Syst. Ser. S 2 (2009), 37–53.

[19] E. Cerpa, I. Rivas and B.-Y. Zhang, *Boundary Controllability of the Korteweg-de Vries Equation on a Bounded Domain*, SIAM J. Control Optim. 51 (2013), 2976–3010.

[20] S. Dolecki and D. L. Russell, *A general theory of observation and control*, SIAM J. Control Optimization, 15 (1997), 185–220.

[21] J. A. Gear and R. Grimshaw, *Weak and strong interactions between internal solitary waves*, Stud. Appl. Math. 70 (1984), 235–258.

[22] O. Goubet and J. Shen *On the dual Petrov-Galerkin formulation of the KdV equation on a finite interval*, Adv. Differential Equations 12 (2007), 221–239.

[23] P. Grisvard, *Elliptic problems in nonsmooth domains*, Monographs and Studies in Applied Mathematics, Volume 24, Pitman, Boston, MA, 1985.

[24] E. Hille and R. S. Phillips. *Functional analysis and semi-groups,* volume 31 of AMS Colloquium Publications. Providence, R. I., 1957.

[25] R. Hirota and J. Satsuma, *Soliton solutions of a coupled Korteweg-de Vries equation*, Phys. Lett. A 85 (1981), 407–408.

[26] R. Hirota and J. Satsuma, *A Coupled KdV equation is one case of the four-reduction of the KP hierarchy*, J. Phys. Soc. Japan 51 (1982), 3390–3397.

[27] D. J. Korteweg, G. de Vries, *On the change of the form of long waves advancing in a rectangular canal and on a new type of long stationary waves*, Philos. Mag. 39 (1895) 422443.

[28] J.-L. Lions, *Sur Les Problèmes Aux Limites Du Type Dèrivèe Oblique*, Annals of Mathematics, 64 (1956), 207-239.

[29] J.-L. Lions, *Contrôlabilité exacte et stabilisation de systèmes distribués*, Recherches en Mathématiques Appliquées, Volume I, Masson, Paris, 1988.

[30] N. G. Lloyd, *On analytic differential equations*, Proc. London Math. Soc. 3 (1975), 430–444.

[31] N. G. Lloyd, *Remarks on generalising Rouché's theorem*, J. London Math. Soc. 2 (1979), 259–272.

[32] S. Micu, J. H. Ortega, L. Rosier and B.-Y. Zhang, *Control and stabilization of a family of Boussinesq systems*, Discrete Contin. Dyn. Syst. 24 (2009), 273–313.

[33] S. Micu, *On the controllability of the linearized Benjamin-Bona-Mahony equation.* SIAM J. Cont. Optim 39 (2001), 1677-1696.

[34] S. Micu, J. H. Ortega and A. F. Pazoto *On the controllability of a coupled system of two korteweg-de vries equations*, Commun. Contemp. Math. 11 (2009), 779–827.

[35] S. Micu and A. F. Pazoto, *Stabilization of a Boussinesq system with generalized damping*, Systems Control Lett. 105 (2017) 62–69.

[36] S. Micu and A. F. Pazoto, *Stabilization of a Boussinesq system with localized damping*, J. Anal. Math. 137 (2019), 291–337.

[37] A. F. Pazoto and L. Rosier, *Stabilization of a Boussinesq system of KdV-KdV type*, Systems Control Lett. 57 (2008), 595–601.

[38] D. Peregrine, *Calculations of the development of an undular bore*, J. Fluid Mech. 25 (1966), 321–330.

[39] L. Rosier, *Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain*, ESAIM Control Optim. Calc. Var. 2 (1997), 33–55.

[40] L. Rosier, *On the Benjamin-Bona-Mahony equation with a localized damping*, J. Math. Study (2016), 195–204.

[41] L. Rosier and B.-Y. Zhang, *Unique continuation property and control for the Benjamin-Bona-Mahony equation on a periodic domain*, J. Differential Equations 254 (2013), 141–178.

[42] J.-C. Saut and N. Tzvetkov, *On a model system for the oblique interaction of internal gravity waves*, M2AN Math. Model. Numer. Anal. 34 (2000), 501–523.