Entropy Theory of Expansive Systems

Elias Ferraz Rego

Tese de Doutorado apresentada ao Programa de pós-graduação do Instituto de Matemática, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Doutor em Matemática.

Orientador: Alexander Eduardo Arbieto Mendoza
Coorientador: Alfonso Artigue Carro

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Aprovada por:

Alexander Arbieto - IM/UFRJ (Presidente)

Alfonso Artigue - UDELAR/Uruguai

Maria José Pacífico - IM/UFRJ

Jaqueline Siqueira - IM/UFRJ

Daniel Smania - ICMC/USP

Dante Carrasco Oliveira - Universidad del Bío Bío/Chile

Welington Cordeiro - Uniwersitet Mikolaja Kopernika/Polônia

Isaia Nisoli - IM/UFRJ (Suplente)

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Entropy Theory of Expansive Systems

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Coorientador: Alfonso Artigue Carro

Resumo

Nesta tese investigamos as implicações da expansivadade para a teoria de entropia de sistemas dinâmicos. Precisamente, investigamos condições para garantir que sistemas dinâmicos com algum tipo de expansividade tenham entropia positiva em algum sentido. Nós investigamos este problema em 4 contextos distintos, são eles, fluxos singulares R-expansivos, fluxos singulares com pontos expansivos, ações expansivas de grupos finitamente gerados em espaços de dimensão positiva e ações expansivas localmente livres de grupos de Lie conexos.

No primeiro contexto, nós determinamos condições para que fluxos R-expansivos tenham conjutos estáveis e instáveis não trivias. Em seguida usamos esses conjuntos para obter entropia topológica positiva para fluxos Komuro-expansivos. No segundo contexto, nós encontramos condições pontuais para a existência de ferraduras topológicas em fluxos singulares. Para ações de grupos finitamente gerados, mostramos que toda tal ação tem entropia geométrica positiva se o espaço de fase não for totalmente desconexo. No quarto contexto, nós introduzimos o conceito de ações expansivas de grupos de Lie conexos em variedades fechadas. Este conceito estende os conceitos de fluxos expansivos não singulares e ações expansivas de $\mathbb{R}^k$. Estudamos problemas relacionados a existência de tais ações, suas simetrias e por fim provamos que qualquer ação expansiva localmente livre desse tipo tem entropia geométrica positiva.

Nós também introduzimos um novo conceito de entropia topológica: A R-entropia topológica. Mostramos que essa entropia é de fato um conceito novo que pode ser útil para o estudo de fluxos em superfícies.

Entropy Theory of Expansive Systems

Elias Ferraz Rego

Advisor: Alexander Eduardo Arbieto Mendoza
Coadvisor: Alfonso Artigue Carro

Abstract

In this thesis we investigate the implications of expansiveness for the entropy theory of dynamical systems. We are precisely looking for conditions to ensure that dynamical systems with some kind of expansiveness have positive entropy in some sense. We have investigated this problem in 4 different contexts. They are singular R-expansive flows, singular flows with expansive points, expansive actions of finitely groups generated in spaces non totally disconnected and expansive locally-free actions of connected Lie groups.

In the first context, we found conditions for R-expansive flows to have non-trivial stable and unstable sets. We then use these sets to obtain positive topological entropy for Komuro-expansive flows. In the second context we find specific conditions for the existence of topological horseshoes in singular flows. For finitely generated group actions, we show that all such action has a positive geometric entropy if the phase space is not totally disconnected. In the fourth context, we introduce the concept of expansive action of connected Lie group on closed manifolds, extending the concepts of non-singular expansive flows and expansive actions of $\mathbb{R}^k$. We study problems related to the existence of such actions, their symmetries and finally we prove that any locally free expansive action of this type has positive geometric entropy.

We have also introduced a new type of topological entropy: Topological R-entropy. We show that this entropy is in fact a new concept that can be useful for the study of flows on surfaces.

Contents

Introduction

Entropy theory is a very strong research topic in dynamical systems. The origin of the modern concept of entropy goes back to Shannon's work in information theory, but entropy was quickly perceived as a strong instrument to study dynamical systems. Initially the entropy for dynamical systems was defined in the measurable context by Kolmogorov ([Kol]) and Sinai ([Si]). Later, its topological version was defined in [AKM] by Adler, Konheim and McAndrew.

Topological entropy can be understood as a measure of complexity for dynamical systems. Roughly speaking, it is a non-negative number (possibly $+\infty$) which measures the mean exponential growth of distinct possible states as times passes. Positive entropy is closely associated with chaotic behavior. While several chaotic systems such as Horseshoes, Anosov diffeormorphisms and the Lorenz Attractor have positive entropy(see [Rob]), also positive topological entropy implies some type of chaotic behavior. For instance, we can cite instance the work of Sumi ([Su]) where it is proved that a C^2 diffeomorphism with positive entropy is chaotic in the Li-Yorke sense. Previous considerations lead us naturally to the following question:

"How to ensure that a given dynamical system has positive topological entropy?"

This question has attracted the interest of many researchers for decades. In fact, there are many studies in the direction of relate positive entropy to other dynamical properties, such as uniform hyperbolicity, partial hyperbolicity, Lyapunov exponents, among others.

A property that is usually related to positive topological entropy is expansiveness. The expansiveness theory is of great importance on dynamical systems theory. Expansiveness is a purely topological property and was introduced by Utz in [U] in the context of homeomorphisms. It is deeply related with the hyperbolic theory introduced by S. Smale in [Sma]. Expansive systems share many chaotic features. In fact, many expansive systems exhibit complex behavior.

In [Ka] and [Fa] it was proved that expansive homeomorphisms always have positive entropy, if the phase space is sufficiently rich. The main techniques used in the homeomorphism scenario were extended to continuous flows in [KS] and later it was proved in [ACP] the positiveness of topological entropy for cw-expansive non-singular flows. We remark that the results in [ACP] includes the expansive case. In this thesis we propose to pursuit the problem of obtaining positive topological entropy for dynamical systems that present the expansiveness property in some form.

This text is divided in two parts. The first part deals with singular flows. In the second part we investigate more general expansive group actions. We have chosen to make this division due to the great distinction of the techniques used in each scenario. We believe this can help the reader to have a better understanding of the subjects treated here. It also makes possible to the reader to study each topic separately. Next we will describe the content of each part.

Part I

Part I is devoted to study the entropy theory of expansive singular flows. Despite the well-established results concerning entropy of expansive non-singular flows ([KS] and [ACP]), there are not similar results for the case of singular flows. This is because the presence of singularities accumulated by regular orbits increases the complexity of the context. Some aspects that obstruct the use of the same techniques of the non-singular case to treat singular flows are:

- Not every point admits a cross-section.
- We may not control the size of the existent cross-sections near the singularities.
- The presence of singularities can distorts a lot reparametrization of orbits close to the singularities .

In particular, the previous obstructions turns make very challenging the task of obtain stable and unstable sets for expansive flows without additional ergodic or differentiable strucuturestructure, such as, dominated decomposition.

To overcome these difficulties we work with the rescaled techniques introduced on [WW] by L. Wen and X. Wen. These techniques are inspired in the ideas of S. Liao on rescaled-tubular neighborhoods for singular flows (see [L1] and [L2]). In this text any rescaled-property will be called a R-property. In [WW] the authors also introduced a new concept of expansiveness called R-expansiveness, which is suitable for R-flow boxes. Our first main goal on this thesis is to construct a theory of local stable and local unstable sets suitable to this type of expansiveness. Actually we obtained the following:

Theorem A. *Let ϕ be a R-expansive C^r-flow($r \geq 1$) on a closed manifold M with dim(M) $>$ 1. Let $K \subset M$ be a non-singular compact invariant set, then there is some point $y \in K$ such that y has a non-trivial connected local R-stable or R-unstable set.*

It is desirable to improve previous result in order to obtain non-trivial connected R-stable and R-unstable sets for any point on K. To do this we define R-stable points and R-Unstable points and extend part of the theory developed by M. Paternain on [Pa]. But now in the singular case, it is not clear if R-expansiveness forbids the existence of R-stable of R-unstable points. As a consequence, we obtained the following result:

Theorem B. *Let ϕ be a R-expansive C^r-flow($r \geq 1$) on a closed manifold M with dim(M) $>$ 1. Let $K \subset M$ be non-singular compact invariant set. If K do not contain stable points, then every $p \in O(x)$ has non-trivial connected local R-stable and R-unstabe sets.*

Once we have the existence of R-stable or R-unstable sets, we can use them to study the topological entropy of flows. Indeed, we obtained the following criteria to R-expansive flows to possess positive topological entropy.

Theorem C. *Let ϕ be a R-expansive flow. If there is some non-singular Lyapunov stable set $\Gamma \subset M$ such that one can finds a point with a non-trivial piece of connected local R-unstable set, then $h(\phi) > 0$.*

As consequence of Theorem B, we obtain the following:

Corollary D. *Let ϕ be a R-expansive flow. If there is some non-singular Lyapunov stable set $\Gamma \subset M$ without R-stable and R-unstable points, then $h(\phi) > 0$.*

Since attractors are examples of Lyapunov Stable sets, then we have the following results:

Theorem E. *Let ϕ be a R-expansive flow. If there exists a non-periodic attractor $\Gamma \subset M \setminus Sing(\phi)$, then $h(\phi) > 0$.*

Combining the previous theorem with the techniques contained on [Art2] we obtain the two following results.

Theorem F. *Let k^*-expansive flow with non-degenerated singularities. If there exists a non-periodic attractor $\Gamma \subset M \setminus Sing(\phi)$, then $h(\phi) > 0$.*

Theorem G. *Let k^*-expansive flow with hyperbolic singularities. If there exists a non-periodic attractor $\Gamma \subset M \setminus Sing(\phi)$, then $h(\phi) > 0$.*

In the spirit of the R-properties, we define a new version of topological entropy, the so called R-topological entropy. This concept of entropy is based on ideas of [WW]. We study some of its properties and how R-topological entropy is related to the classical topological entropy. For instance, we exhibit an example of R-expansive surface flow with positive R-entropy. This is an interesting fact because there are not surface flows with classical positive topological entropy, so R-entropy can be useful to capture some kind of complexity that are not perceived by topological entropy.

Another approach we use in this work to deal with flows is pointwise dynamics. By pointwise dynamics we mean to study the evolution of dynamical systems requiring the existence of points on the phase space with nice dynamical properties. See for instance the work of W. Reddy on pointwise expansive homeomorphisms ([R]), where a point x is said to be expansive if any other point y must to be separated from x by the dynamics at some instant of time. Notice that in this case the expansiveness is considered only with respect to x and not globally on the whole phase space.

Pointwise dynamics has attracted much interest on last years. We can mention the works [Moo], [Mor], [YZ] [AV] where sensitive points, entropy points and shadowable points were considered.

On a previous work we have introduced the theory of uniformly expansive points for homeomorphisms and have established a criteria to obtain positive entropy from these points ([AR]). In this thesis, we extend these results to the setting of continuous flows. This extension is not a trivial task, since the techniques used here are totally distinct from the ones for homeomorphisms. Next we state the main results we have obtained in this direction.

Theorem H. *Let ϕ be a non-singular continuous flow on a compact metric space. If there is some point $x \in \Omega(\phi) \setminus Per(\phi)$ which is uniformly expansive and shadowable, then $h(\phi) > 0$.*

To treat the singular case, only pointwise shadowableness is not enough to obtain a result similar to the previous one. This is due to the temporal distortion caused by the singularities on the shadow reparametrizations. In [K2], M. Komuro exposed the distinction between the behavior of shadowing in the singular and non-singular cases. To recover some properties from the non-singular case to singular flows, he introduced a stronger form of shadowing property. Here we introduce its pointwise version and use it to obtain the following result:

Theorem I. *Let ϕ be a continuous flow. Suppose there is some point $x \in \Omega(\phi) \setminus Crit(\phi)$ uniformly expansive and strongly shadowable, then there is some compact and ϕ-invaritant set $Y \subset M$ such that $\phi|_Y$ is semiconjugated to a suspension of subshift with positive entropy.*

An interesting remark about the two previous results is that they can be used in low dimensional scenario, in contrast with the other results of this part.

Part II

The second part of this thesis aims to extend the expansiveness theory for systems which are more general than homeomorphisms and flows. Specifically, we consider group actions on compact spaces and investigate how expansive behaviour rules their dynamics. In particular, we are interested in the influences of expansiveness to the entropy of such systems.

Similarly to the case of homeomorphisms and flows, we need to use distinct definitions of expansiveness to treat group actions depending on the nature of the acting group. There are some known efforts in to study expansive group actions. In [Hur] and [BDS], the authors studied expansive actions of countable and finitely generated groups. For connected groups we have the definition introduced by W. Bonomo, J. Rocha and P. Varandas introduced in [BRV], where the symmetry properties of such actions were considered.

One of our goals is to extend this definition to actions of general connected Lie groups and study its consequences, as well as, study the entropy theory for the finitely generated case. Next we state our main results on this part.

For the entropy of actions of finitely generated groups, S. Hurder proved in [Hur] that any expansive action of the circle has positive geometric entropy. Here we extend this result to expansive actions on any space which is not totally disconnected.

Theorem K. *Suppose that Φ is a continuous action of a finitely generated group G by homemomphisms on a compact metric space . If the topological dimension of X is positive, then $h(\phi, K) > 0$ for any generator K.*

When the acting group is connected the study of dynamics is closely related to the study of foliations. Since singular foliations can be very complicated, we restrict ourselves to locally-free actions, in order to obtain a foliated action. In this scenario we prove a generalization of a result of [LG] and [Fl]which states that there are not non-singular expansive flows on surfaces.

Theorem L. *There are not codimesion one locally-free C^r expansive actions of nilpotent and connected Lie groups.*

We also deal with the the question of positiveness of entropy in the setting of actions of connected Lie groups. We study the geometric entropy of expansive actions which is a concept closely related to the entropy for pseudo-groups and foliations introduced by A. Bis in [B]. In [IT] T. Inaba and N. Tsuchiya introduced a concept of expansiveness for foliations and proved that expansive codimension one foliations have positive geometric entropy. On the general condimensional case they obtained positiveness of geometric entropy assuming a stronger form of expansiveness on the foliation. Here we improve their results to any codimensional context.

Theorem N. *Any C^r expansive foliation has positive geometric entropy.*

We will see that the orbit foliation of an expansive foliation must be expansive and this as has as consequence the following result.

Corollary O. *Any expansive locally-free action of a connected Lie group on a closed manifold has positive geometric entropy.*

We also studied the symmetries of expansive actions. There are many efforts in the direction of understanding the symmetries of C^r-actions. We reefer the reader to the works of D. Obata, M. Leguil and B. Santiago for actions of $\mathbb{Z}$ and $\mathbb{R}$ ([LOS] and [O]). In the setting of expansive homeomorphisms P. Walters proved in [W] that such systems has discrete centralizer. Here, we extended this result to the finitely generated case obtaining:

Theorem J. *The centralizer of any expansive C^0-action of a finitely generated group G on a closed manifold is a discrete set on the space of C^0-actions of G on M.*

If the group is connected then this question is challenging. In [BRV] the authors proved that expansive R^k-actions have quasi-trivial centralizers. Here we extend the definition of quasi-triviality for expansive actions of more general Lie group action and extend their results. But we need an additional hypothesis on G.

Theorem M. *Any expansive C^r-action of an exponential Lie group on a closed manifold has quasi-trivial centralizer.*

This text is divided as follows.

- On chapter 1 we investigate the Lewowicz's Theorem and study in details its proof. In this chapter This chapter intends to introduce to the reader the basic tools of the theory of expansive systems and to establish the guidelines for the results we seek on this work.

-Chapter 2 starts the first part of this work where we deal with singular flows. In this chapter we give the basic definitions and results needed to obtain our results.

- In Chapter 3 we introduce the rescaled-theory of singular flows. We present the concept of R-expansiveness and introduce our definition of R-entropy. We study some of the properties of R-topological entropy and exhibit an examples that gives a clear distinction between R-topological entropy and classical topological entropy.

- In Chapter 4 we study the R-stable and R-unstable sets of R-expansive flows. We also prove theorems A, B, C, D, E,F and G.

- Chapter 5 is devoted to introduce to the reader the setting of pointwise dynamics. We define expansive points, uniformly-expansive points and shadowable points for flows.

- In chapter 6 we prove our mains theorem on the pointwise setting, namely, Theorem H and I.

- Chapter 7 introduces the second part of this work, where we deal with expansive group actions. We introduce these systems and study the finitely generated case. In particular we prove Theorems J and K.

- In Chapter 8 we introduce the concept of expansive actions of connected Lie groups and study its properties. In this section we prove Theorems L, M, N and O.

- The final Chapter 9 is devoted to state some problems and questions derived from the results developed on this thesis. With this chapter we intend to offer the reader some possible research directions for future works on the subject treated here.

- Additionally, we include two appendix, where the reader can find some basic background on group and foliations theory.

Chapter 1

Topological Entropy of Expansive Homeomorphism

This initial chapter aims to introduce the reader to the central problem of this work, which is to explore the relationship between expansiveness and entropy. Maybe one of the first efforts in this direction was the classical work of the Uruguayan mathematician Jorge Lewowicz on expansive homeomorphisms. Although it was widely known at that time that hyperbolicity has strong implications on entropy, Lewowicz's results relates purely expansiveness with entropy. Actually, Lewowicz proved that expansiveness alone is enough to imply positive topological entropy for homeomeorphism whose phase spaces are not too poor. Although we will call this result as Lewowicz's Theorem through this text, we stress that other authors also have proved this result independently, for instance, see the works of Kato in [Ka] and Fathi in [Fa].

The above quoted result has deeply influenced us to pursuit the main problems dealt on this thesis. Because of this, we decided to write this chapter in order to introduce the reader to the Lewowicz's work. We also intend to make clear to the reader the philosophy and the techniques we will be inspired in future chapters.

This chapter is constructed in the following way: first we establish all the notation and concepts needed to work in the setting of discrete-time systems. Next we introduce the reader to Lewowicz's Theorem and then we will give its proof.

1.1 Discrete-Time Systems

In this section we define the basic concepts of discrete-time dynamical systems. A discrete-time dynamical system is a pair (M, f) where M is a set with some additional structure and $f : M \to M$ is a invertible transformation preserving this structure. Dynamical sytems are classified according this structure. For instance if M is a topological space and f is a homeomorphism, then (M, f) is a topological dynamical system. If M is a smooth manifold and f is a diffeormorphism, then (M, f) is a smooth dynamical system. We will often replace the notation (M, f) by f when the phase space M is understood. Through out this chapter, we will work in the setting of topological dynamics, therefore f we will always be a self-homeomorphisms on a compact metric space M.

A central scope in dynamical systems theory is to understand the long-term behavior of systems. For a given homeomorphism f the concept of time is played by the iterates of f. To precise this idea let us set some notation. Let $f : M \to M$ be a homeomorphism. For a given $n \in \mathbb{Z}$, we denote

$$f^n = \begin{cases} Id_M, \ n = 0 \\ f \circ \cdots \circ f, \text{ n-times}, n > 0 \\ f^{-1} \circ \cdots \circ f^{-1}, \text{ n-times}, n < 0 \end{cases} .$$

With previous definition we can understand time as follows: if x is some point on the space X, then the state of x after n units of time is the point $f^n(x)$.

Now let us define the orbit of x to be the set

$$O(x) = \{f^n(x); n \in \mathbb{Z}\}.$$

The set $O(x)$ contains the information about the point x in any instant of time. So, study the time-behavior of some point is the same as study its orbit.

There are two central concepts on this work, namely, expansiveness and topological entropy. We will postpone a more detailed discussion on the former to Section 3 of this chapter. Now we spend some time defining topological entropy and explaining what it represents. As mentioned before, we are seeking for study the behavior of points in X through time, but for some homeomorphisms these behaviors are quite simple. To see this, let us investigate some examples.

Example 1.1.1 (Identity Map). *Let M be some compact metric space and let $f : M \to M$ be the identity map of X. Since $f^n(x) = x$ for any $n \in \mathbb{Z}$, we have that $O(x) = \{x\}$ for any $x \in S^1$.*

Example 1.1.2 (Rational Rotation). *Let X be the unitary circle S^1. Here we see S^1 as the quotient space $\mathbb{R}/\mathbb{Z}$ under the equivalence relation $x \sim y$ if, and only if, $x - y \in \mathbb{Z}$. Let $\alpha \in \mathbb{R}$ and consider the rotation homeomorphism $R_\alpha : S^1 \to S^1$ defined by $R_\alpha(x) = (x + a) mod(\mathbb{Z})$.*

Suppose that $\alpha \in \mathbb{Q}$, then there are $q, r \in \mathbb{Z}$ such that $\alpha = \frac{q}{r}$. Thus $f^r(x) = x + \frac{qr}{r} mod(\mathbb{Z}) = x$ and then $O(x)$ is finite set for any $x \in S^1$

Example 1.1.3 (Irrational Rotation). *Now let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and consider $R_\alpha : S_1 \to S_1$. We claim that $\overline{O(x)} = S_1$ for every $x \in S_1$. Indeed, fix some $x \in S^1$ and $\varepsilon > 0$. To prove the claim we need to observe three facts.*

1. *R_α is an isometry and therefore $d(R_\alpha^{n+1}(x), R_\alpha^n(x)) = d(R_\alpha(x), x)$ for every $n \in \mathbb{Z}$.*
2. *For any $n \in \mathbb{Z}$ the arcs $(R_\alpha^{n+1}(x), R_\alpha^n(x))$ and $(R_\alpha^n(x), R_\alpha^{n-1}(x))$ are disjoint.*
3. *Since $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then the orbit of x is an infinite set.*

Now suppose that $O(x) \cap B_\varepsilon(y) = \emptyset$. Then combining the above facts we have that the length of S^1 should be infinite, but this contradicts the compactness of S^1.

A feature shared by the three above examples is that for each of then, all the orbits have the same behavior. While this is a good thing for understanding dynamical systems, for a huge number dynamical system, this is not the case. Indeed, many dynamical systems have, at same time, orbits with the three above behaviors and this can make things more complex. An attempt to quantify this complexity is made by computing the system's topological entropy.

Before to introduce topological entropy for homeomorphisms we need first to introduce some definitions. Fix some $\varepsilon > 0$ and some natural number n. We say that a pair of points is **n-ε-separated** if there is some $0 \leq i \leq n$ such that $d(f^i(x), f^i(y)) > \varepsilon$, i.e. if the pair does not spend more than n units of time to be ε-apart.

Let us make some remarks about separated sets. First notice that if we define a new metric d_n on M by $d_n(x, y) = \max_{0 \leq i \leq n}\{d(f^i(x), f^i(y))\}$ then d_n is a metric equivalent to d and therefore they induce the same topology at M. But now a pair $x, y \in M$ is n-ε-separated if, and only if, $d_n(x, y) > \varepsilon$.

Consider $K \subset M$ and $E \subset K$. We say that E is a n-ε-separated subset of K if any pair of distinct points of E is n-ε-separated. Let $S_\varepsilon(n, K)$ be the maximal cardinality of a n-ε-separated subset of K. This number is always finite due the compactness of M and the equivalence between the metrics d and d_n.

We define the **topological entropy** of f on K to be the quantity:

$$h(f, K) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S_\varepsilon(n, K)$$

and finally, the topological entropy of f is the quantity $h(f) = h(f, M)$

An intuitive way to see what topological entropy means is to think on ε as an error to distinguish distinct orbits. If we fix ε and n, then $S_\varepsilon(n)$ counts how many orbits we can distinguish up to time n. Then the first limit measures the exponential growth of this number as n goes to infinity. The second limit refines the error allowed in counting distinct orbits. So entropy is a way to measure the exponential growth of distinct orbits of f as time passes.

Above discussion shows us that for some homeomorphism to have positive entropy we need that many points be separated as time grows. Since all previous examples are isometries, no pair of points is separated by these systems and this implies that their entropy vanishes.

Next example displays a homeomorphism with positive entropy.

Example 1.1.4 (The Shift Map). *Let A be a finite alphabet with n elements. Denote $\Sigma_n = A^{\mathbb{Z}}$ for the bilateral sequences formed by the elements of A. Define on Σ_n the following metric:*

- *$d(s, s') = 0$ if, and only if $s = s'$.*

- $d(s, s') = \frac{1}{2^i}$ *if* $i > 0$ *is the maximal integer such that* $s_k = s'_k$*, for every* $|k| \leq i$*.*

The metric d *turns* Σ_n *into a compact metric space.*

Define the shift map $\sigma : \Sigma_n \to \Sigma_n$ *as* $\sigma(s_k) = s_{k+1}$*. We say that* $s \in \Sigma_n$ *is a periodic sequence if there is some* $k \in \mathbb{N}$ *such that* $s_{i+k} = s_i$ *for any* $i \in \mathbb{Z}$*. Notice that the periodic sequences are exactly the points whose orbits are periodic for* σ*.*

We claim that $h(\sigma) > 0$*. Indeed, first notice that by a combinatorial argument, for each* $k > 1$ *we can find exactly* n^k *periodic sequences of period* k*.*

Moreover, if s *and* s' *are two distinct periodic sequences of period* k*, then they are* k*-*ε*-separated by* σ *for any* $\varepsilon < \frac{1}{2}$*. To see this, let* $i > 0$ *denote the first positive integer such that* $s_i \neq s'_i$*. Thus* $d(\sigma^i(s), \sigma^i(s')) = \frac{1}{2}$*. Now this implies that*

$$h(\sigma) = \lim_{\varepsilon \to 0} \limsup_{k \to \infty} \frac{1}{k} \log(S_\varepsilon(k)) \geq \log n$$

.

Actually, it is easy to see that in fact $h(\sigma) = \log n$*.*

1.2 The Lewowicz's Theorem

In this section we introduce the theorem that we call on this text as Lewowicz's Theorem. This remarkable was proved independently by Fathi and Lewowicz, and later it was generalized by Kato to the context of cw-expansivity. Here we will expose the techniques developed by Lewowicz to prove this result. These techniques will furnish us the guidelines for the kind of problems treated in this work. For instance, we will prove more general versions of this theorem on the second part of this thesis. Lewowicz's Theorem explores the relationship between expansiveness and entropy. Since we already have discussed about topological entropy on previous section, we now introduce and explore expansiveness.

Definition 1.2.1. *A homeomorphism* $f : M \to M$ *is called expansive if there is some* $e > 0$ *such that if* $d(f^n(x), f^n(y)) < e$ *for every* $n \in \mathbb{Z}$*, then* $x = y$*. The constant* e *is called the expansiveness constant of* f *and we will also refer to* f *as an* e*-expansive homeomorphism.*

Now we are able to state the main result of this chapter.

Theorem 1.2.2 (Lewowicz's Theorem)**.** *Let* $f : M \to M$ *be an expansive homeomorpism. If* $\dim(M) > 0$*, then* $h(f) > 0$

Lewowicz's Theorem essentially says that expansiveness implies positive topological entropy if the phase space of f is sufficiently rich. Indeed, the result is not valid for zero dimensional space as the following example shows:

Example 1.2.3. *Let M be a finite set and define f in a way that M is a union of periodic orbits for f. Then f is clearly expansive, we just need to consider*

$$e < \min\{d(x, y); x, y \in M, x \neq y\}.$$

Furthermore, since the phase space is finite, then f cannot have positive entropy.

In order to have a better understanding for the ideas on the proof of Theorem 1.2.2, let us think about the following informal meaning of expansiveness:

"Any two distinct point of X must be e-apart in some time"

The above informal definition provides us clues that expansiveness can be related with the positiveness of topological entropy. Indeed, a homeomorphism has positive topological entropy if f does not take so long to separate a large amount of points. So a way to try to obtain positive entropy from expansiveness, is to force that expansiveness separates many points in controlled time. Next result precisely guarantees this.

Theorem 1.2.4. *(Uniformly Expansiveness) A homeormphism is e-expansive if, and only if, for any $\delta > 0$ there is some $n \in \mathbb{N}$ such that if $d(x, y) \geq \delta$, then $d(f^i(x), f^i(y)) > e$, for some $-n \leq i \leq n$.*

Proof. Suppose f is e-expansive. Suppose that there is some $\delta > 0$ such that one can find sequences $x_n, y_n \in M$ and a sequence of integers $m_n \to \infty$ satisfying $d(x_n, y_n) \geq \delta$ and $d(f^i(x_n), f^i(y_n)) < e$ for every $n \in N$ and $-m_n \leq i \leq m_n$.

The compactness of M allows us to assume that $x_n \to x$ and $y_n \to y$. Therefore we have $d(x, y) \geq \delta$ and by the continuity of f one has $d(f^n(x), f^n(y)) < e$ for ever $n \in \mathbb{Z}$. But this is impossible since f is e-expansive.

The converse is obvious. □

Before presenting the proof of Lewowicz's Theorem, we provide some ideas to help the reader to have a more clear intuition about the steps of the proof . We will need to find a huge number of points that are separated, so we will follow the steps bellow.

Step 1- First we have by uniform expansiveness that if two point are at least at distance $\delta \leq \frac{e}{4}$ apart, then they will not spend much time to be e-separated.

Step 2- Since $dim(X) > 0$ some point has non-trivial connected component.

Step 3- Step 2 will allow us to find a connected curve C satisfying the following: Any arc of C with length at least δ will not spend more than N units of time in the future or in the past to achieve length e. Previous property will be valid for the images of C at any time n.

Step 4- We take the curve C and apply f until the image of C has length $\frac{e}{2}$.

Step 5- Subdivide this image on two disjoint subarcs of length at least δ.

step 6 - Apply again step 4 on the arcs obtained on step 5 and repeat the steps 4 through 6 inductively. This process will generate the separated sets we are looking for.

In next section we will execute precisely all above steps and then prove Lewowicz's Theorem.

1.3 Proof of Lewowicz's Theorem

Now we will prove Lewowicz theorem. To this we need to obtain more ingredients for the proof. We begin this chapter defining stable and unstable sets. Actually, the unstable sets will contain the n-ε-separated sets used to estimate topological entropy. Next we define these sets precisely.

Definition 1.3.1. *For any point $x \in M$ and $\gamma > 0$ its γ-local stable set is defined as follows:*

$$W^s_\gamma(x) = \{y \in M; d(f^i(x), f^i(y)) \leq \gamma, \forall i \geq 0\}$$

The γ-local unstable set $W^u_\gamma(x)$ of x is defined as follows:

$$W^s_\gamma(x) = \{y \in M; d(f^i(x), f^i(y)) \leq \gamma, \forall i \leq 0\}$$

Next we state a powerful tool on expansive systems theory.

Theorem 1.3.2. *Suppose that f is an e-expansive homeomorphism. For any $\gamma > 0$, there exists $N > 0$ such that:*

$$f^n(W^s_e(x)) \subset W^s_\gamma(f^n(x)) \text{ and } f^{-n}(W^u_e(x)) \subset W^u_\gamma(f^{-n}(x))$$

For every $n \geq N$ and any $x \in M$.

Proof. We will only prove the theorem for the stables sets, the proof for the unstable case is analogous.

Fix $\gamma > 0$ and let $N > 0$ be given by the uniform expansiveness of f with respect to γ. We claim that this N satisfies the wanted condition. Indeed, if not there exists some $x \in X$ and $y \in W^s_e(x)$ such that for some $k > 0$ one has $d(f^{k+N}(y), f^{k+N}(x)) > \gamma$. But this implies that for some $-N \leq i \leq N$ we must have $d(f^{N+k+i}(x), f^{N+k+i}(y)) > e$. But this is impossible since $y \in W^s_e(x)$. Therefore we conclude that the result holds. □

The previous result is a key tool for proving Lewowicz's Theorem since it allows us to obtain that unstable local sets expands in controlled positive time. This is necessary, since in the definition of topological entropy we need to find points which are separated in the future.

The following result is the last ingredient and it says that some point at X must have in its unstable set a non-trivial piece of connected set containing x. Let us denote $C(A, x)$ for the connected component of A containing x and let us denote $CW^s_\gamma(x) = C(W^s_\gamma(x), x)$.

Let us define $K(M)$ to be the set of compact subsets of M and $C(M)$ to be the set of continuum subset of M. Recall that a continuum is a compact and connected set. We can regard $K(M)$ and $C(M)$ into metric spaces by endowing them with the following Hausdorff metric:

$$D(A,B) = \max\{\sup_{x\in A}\inf_{y\in B} d(x,y), \sup_{y\in B}\inf_{x\in A} d(x,y)\}$$

Theorem 1.3.3 ([Her]). *The hyperspace $K(M)$ and the continuum hyperspace $C(M)$ are compact for the Hausdorff metric.*

Now we can prove the following

Theorem 1.3.4. *Let f be an e-expansive homeomorphism and suppose $\dim(M) > 0$. For any $0 < \gamma \le e$ there exists some $x \in M$ such that*

$$CW^s_\gamma(x) \neq \{x\} \text{ or } CW^u_\gamma(x) \neq \{x\}.$$

Proof. Since $\dim(M) > 0$, there exists some $y \in X$ such that $C(M,y) \neq \{y\}$. Fix $\gamma > 0$ and suppose that $W^s_\gamma(x) = \{x\}$ for every $x \in X$. Set $B_0 = \overline{(B_\gamma(y)} \cap C(M,y)$. Then B_0 is a non-trivial continuum containing y. Now since $W^s_\gamma(y) = \{y\}$, there is some $n > 0$ such that $diam(f^n(B_0)) \ge \gamma$. Let n_0 be the minimal natural satisfying previous condition. Set $B_1 = f^{n_0}(B_0)$. By the continuity of f we have that B_1 is a compact and connected set containing $y_1 = f^n(y)$. Again, since $W^s_\gamma(y_1) = \{y_1\}$, for some $n > 0$ one has $diam(f^n(B_1)) \ge \gamma$. We set n_1 to be the minimal natural number satisfying previous condition. Inductively, we can construct a sequence of non-trivial continuum sets B_k and a sequence of times n_k such that:

- $diam(B_k) \ge \gamma$ and $diam(f^i(B_k)) < \gamma$ for $i < n_k$.
- $f^{n_k}(B_{k-1}) \subset B_k$
- $f^{n_k}(y_{k-1}) \in B_{k+1}$

Then by the compactness of the continuum hyperspace we can assume that $B_k \to B$. Now B is a nontrivial continuum containing $x = \lim y_k$. We claim that $B \subset W^u_\gamma(x)$. Indeed, take $z \in B$ and fix $i \le 0$. Thus we have that

$$d(f^i(x), f^i(z)) \le d(f^i(x), f^i(y_k)) + d(f^i(y_k), f^i(z_k)) + d(f^i(z_k), f^i(z))$$

Where where $z_k \in B_k$ and $z_k \to z$. Now taking k large enough we can conclude that

$$d(f^i(x), f^i(z)) \le \gamma.$$

□

Now we are able to prove lewowicz theorem.

Proof of Lewowicz's Theorem. We shall follow the steps of on previous section.

Step 1 - Fix some $0 < \delta < \frac{e}{4}$ and let N_δ be given by the uniform $\frac{e}{2}$-expansiveness of f.

Step 2 - Since $\dim(X) > 0$ there is some $x \in X$ such that $C(M, x) \neq \{x\}$.

Step 3 - Now use Theorem 1.3.4 to obtain a point $x_0^0 \in X$ such that $diam(CW_\delta^u(x_0^0)) \geq \delta$. Thus we can find a closed curve $[x_0^0, x_1^0]$ on $CW_\delta^u(x_0^0)$ such that $d(x_0^0, x_1^0) \geq \delta$.

Step 4 - Uniformly expansiveness implies that

$$E_1 = \{x_0^0, x_1^0\}$$

is an N-$\frac{e}{2}$-separated set.

Step 5 - Call $f^{i_0^0}(x_0) = x_0^1$ and $f^{i_0^0}(x_1) = x_3^1$. By Step 4, we have that $d(x_0^1, x_3^1) > e$. This allows us to divide the arc $[x_0^1, x_3^1]$ in two disjoint sub arcs $[x_0^1, x_1^1]$ and $[x_2^1, x_3^1]$ with $d(x_0^1, x_1^1) \geq \delta$ and $d(x_2^1, x_3^1) \geq \delta$. Now uniformly expansiveness of f implies that there are $0 \leq i_0^1, i_1^1 \leq N$ such that $d(f^{i_0^1}(x_0^1), f^{i_0^1}(x_1^1)) \geq \delta$ and $d(f^{i_1^1}(x_2^1), f^{i_1^1}(x_3^1) \geq \delta$. But this implies that

$$E_2 = f^{-i_0^0}(\{x_0^1, x_1^1, x_2^1, x_3^1\})$$

is a $2N$-$\frac{e}{2}$-separated set.

Step 6 - Now applying recursively the Steps 4 and 5 on the segments

$$[f^{i_0^1}(x_0^1), f^{i_0^1}(x_1^1)] = [x_0^2, x_3^2] \text{ and } [f^{i_1^1}(x_2^1), f^{i_1^1}(x_3^1)] = [x_4^2, x_7^2],$$

we obtain the set

$$E_3 = f^{-i_0^0}(f^{-i_0^1}(\{x_0^2, x_1^2, x_2^2, x_3^3\}) \cup f^{-i_1^1}(\{x_4^2, x_5^2, x_6^2, x_7^2\}))$$

which is a $3N$-$\frac{e}{2}$-separated set for f. Inductively, for each $n > 0$ there is some set $E_n = \{x_0, ...x_{2^n-1}\}$ which is nN-$\frac{e}{2}$-separated set for f.

Now we can estimate the entropy of f using the sets E_n. Indeed, we have:

$$h(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log S_\varepsilon(n, X) \geq \lim_{n\to\infty} \frac{1}{nN} \log \#E_n = \lim_{n\to\infty} \frac{1}{nN} \log \#E_n = \lim_{n\to\infty} \frac{1}{nN} \log 2^n$$

But this implies

$$h(f) \geq \lim_{n\to\infty} \frac{1}{nN} \log 2^n \geq \frac{\log 2}{N} > 0$$

Thus the Lewowicz's Theorem is proved. □

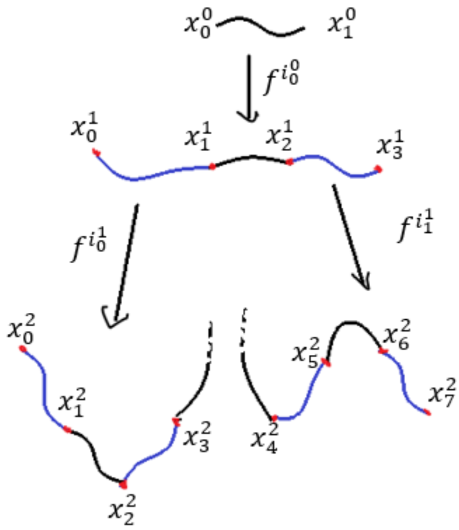

Figure 1.1: Proof of Lewowicz's Theorem

Part I

Entropy of Expansive Flows

Chapter 2

General Results of Flows Theory

This chapter begins the first part of this thesis where we investigate the relationship of expansiveness and topological entropy for continuous-time systems. We will see that already exist versions of Lewowicz's Theorem for flows, but these versions only take of non-singular flows into account. For the singular case, the problem is much harder, and because of this it remains open. There are some interesting results about entropy of expansive singular flows, but considering strong assumptions, such as dominated decomposition, non-uniform hyperbolicity, multi-singular hyperbolicity. But there are not results of this nature only supposing expansiveness. Much of this lack of results with few structure is due to the fact that it is not so easy to develop a theory of stable or unstable sets without additional hypothesis. In this work we propose a rescaled-theory of stable and unstable sets which is suitable for singular flows with some kind of expansiveness. We also propose some pointwise techniques which allows us to obtain positive entropy, even the phase space is low-dimensional.

Before to begin, let us state some concepts and results that will be often used in this text. Through out this text M will denote the phase space of the systems under consideration. The nature of M will depend on the nature of the studied system, but in general, M will be a compact metric space. When we are dealing with differentiable systems, then M denotes a closed manifold, that is, compact and boundaryless. We say that a map is C^r if it is r-times differentiable and all of its derivatives are continuous until order r.

2.1 Preliminaires

In this section we define flows (continuous-time dynamical systems) and study some of their basic features. The theory of flows emerged from the classical problems of solving ordinary differential equations. Since the age of I. Newton and his models to understand nature it has attracted attention from scientists. But the modern approach on flows theory raised from the works of H. Poincaré on the qualitative theory of differential equations. The precise definition of flow given is as follows:

Definition 2.1.1. *A flow ϕ on M is a C^r-map $\phi : \mathbb{R} \times M \to M$ satisfying the following conditions:*

1. $\phi(0, x) = x$, *for every* $x \in M$.
2. $\phi(t + s, x) = \phi(t, \phi(s, x)))$, *for every* $t, s \in \mathbb{R}$ *and every* $x \in M$.

The notation ϕ_t stands for the map $\phi(t, \cdot) : M \to M$, when t is fixed. The previous definition has some immediate consequences. For instance, $\phi_0 = Id_M$, $\phi_{-t} = \phi_t^{-1}$ and $\phi_{t+s} = \phi_{s+t}$. Previous features implies that for any $t \in M$, the maps ϕ_t are homeomorphisms if $r = 0$ and diffeormorphisms if $r \geq 1$. Thus another way to define flows is through a family of homeomormphisms (diffeormorphisms if $r > 0$) $\{\phi_t : M \to M\}_{t\in\mathbb{R}}$ such that $\phi_0 = Id_M$ and $\phi_{t+s} = \phi_t \circ \phi_s$. Analogously to the case of discrete-time systems, we define the orbit of a point x under ϕ to be the set

$$O(x) = \{\phi_t(x); t \in \mathbb{R}\}.$$

As mentioned before, flows are related to the problem of solving a ordinary differential equation. To see this, suppose initially that ϕ is C^r with $r > 0$ and fix $x \in M$. Then $\phi(t, x) = \alpha(t)$ is a C^r-curve on M such that $\alpha(0) = x$. For any $x \in M$ let us denote $X(x) = \alpha'(0) \in T_xM$. Then we can define a vector field $X : M \to TM$ which is called the vector field associated to ϕ. This vector field is called in this way because the generated curves of the flow are the solutions of the following ordinary differential equation:

$$X(x) = \frac{d\phi_t}{dt}(x)|_{t=0}$$

Conversely, if we have some vector field $X : M \to TM$ which is locally-Lipschitz, then by the Picard-Lindelöff Theorem we can find local unique solution curves for the above ODE. Moreover, if M is closed these solutions are globally defined on M and generate a flow ϕ on M. The class of differentiability of ϕ is the same as the class of differentiability of the vector field X. Hereafter, we always denote X for the vector field generated by ϕ.

We say that a point $x \in M$ is a *singularity* for ϕ if $\phi_t(x) = x$ for every $t \in \mathbb{R}$. If the flow is smooth, last condition is equivalent to $X(x) = 0$. A point x is *regular* if it is not singular. We say that a flow is *non-singular* if all of its points are regular. A regular point x is said to be periodic if $O(x)$ is a compact set. The *period* $\pi(x)$ of a periodic point x is defined to be smallest positive real number such that $\phi_{\pi(x)}(x) = x$. Let us denote $Per_t(\phi)$ for the set of periodic points of ϕ with period smaller than t and $Per(\phi) = \cup_{t>0} Per_t(\phi)$. We define the set of *critical points* of ϕ as

$$Crit(\phi) = Per(\phi) \cup Sing(\phi).$$

A useful tool to study flows is the concept of cross-section through a regular point of ϕ.

Definition 2.1.2. *Let x be a regular point for ϕ. A compact set S_x is a cross section of time $\eta > 0$ for ϕ trough x if $x \in S_x$, $\phi_{[-\eta,\eta]}(S_x)$ is a neighborhood of x and $\phi_{[-\eta,\eta]}(y) \cap S_x = \{y\}$, for any $y \in S_x$.*

If ϕ is a continuous flow, then any regular point x has a cross section through x (See [Whi]). Moreover, in [BW] it is proved that if ϕ is non-singular , then there is some $\xi > 0$ such that for any $0 < \delta < \xi$, one can find a finite family of cross section $\{S_{x_1}, ..., S_{x_n}\}$ with time ξ and satisfying:

1. $diam(S_{x_i}) \leq \delta$, for any $i = 1, ..., n$.
2. $M = \bigcup_{i=1}^n \phi_{[-\delta,0]}(S_{x_i}) = \bigcup_{i=1}^n \phi_{[0,\delta]}(S_{x_i})$

Later, the above result was improved in [KS] obtaining the following:

Theorem 2.1.3. *If ϕ is a non-singular continuous flow, there is some $\xi > 0$ such that:*

For any $0 < \delta < \xi$, there are two families of finite compact cross-sections $\{S_i\}_{i=1}^n$ and $\{T_i\}_{i=1}^n$ such that $T_i \subset S_i^*$, $diam(T_i) \leq \delta$ and

$$M = \bigcup_{i=1}^n \phi_{[-\delta,0]}(T_{x_i}) = \bigcup_{i=1}^n \phi_{[0,\delta]}(T_{x_i}) = \bigcup_{i=1}^n \phi_{[-\delta,0]}(S_{x_i}^*) = \bigcup_{i=1}^n \phi_{[0,\delta]}(S_{x_i}^*)$$

Where $S_i^* = Int(\phi|_{[\delta,\delta]}(S_i)) \cap S_i$, for every $0 \leq i \leq n$.

Remark: Denote S^+ for $\cup_{i=1}^n S_{x_i}$. If we put $\beta = \sup\{t > 0; \forall x \in S^+, \phi^t(x) \notin S^+\}$, then $0 < \beta < \xi$ and once a point $x \in X$ crosses a cross section $S \in \mathcal{S}$, it takes at most time β to cross another cross section.

If the flow is smooth, there is a natural choice of cross-sections for regular points. To see this, let $x \in M$ be a regular point. Thus we have that $X(x) \neq 0$. The normal space of x in T_xM is the set

$$\mathcal{N}_x = \{v \in T_xM; v \perp X(x)\}.$$

Let us denote $\mathcal{N}_x(r) = \mathcal{N}_x \cap B_r(0)$, where $B_r(0)$ is the ball in T_xM of radius r and centered at 0.

The tubular flow theorem for smooth flows asserts that for any regular point x there are $\varepsilon_x > 0$ and $r_x > 0$ such that the set

$$N_x(r) = \exp_x(\mathcal{N}_x(r_x))$$

is a cross section of time ε_x through x. Moreover, any $y \in N_x(r_x)$ is regular.

Now if $-\varepsilon_x < t < \varepsilon_x$, the continuity of ϕ implies that for some $\delta > 0$ we have that the points in $N_x(\delta)$ meet the cross section $N_{\phi_t(x)}(r_{\phi_t(x)})$ in a time close to t. Thus we define the holonomy map between $N_x(r_x)$ and $N_{\phi_t(x)}(r_{\phi_t(x)})$ to be the map:

$$P_{x,t} : \mathbb{N}_x(\delta) \to N_{\phi_t(x)}(r_{\phi_t(x)})$$

defined by $P_{x,t}(y) = \phi_t(y)$, where t is the only $-\varepsilon_x < t < \varepsilon_x$ such that $\phi_t(y) \in N_{\phi_t(x)}(r_{\phi_t(x)})$.

One of the main difficulties in the use of cross-sections for singular flows is the fact that the radius r_x can goes to zero when x approaches some singularity. To overcome this difficulty S. Liao developed a theory that allows us to have a control over how r_x collapses. Later these ideas were widely used to work with singular flows in several contexts. Next we state some results that will be used later.

Theorem 2.1.4 ([WW]). *Suppose that X is a C^1-vector field and let ϕ be the flow induced by X. Then there exist $L > 0$ and a small β_0 such that for any $0 < \beta < \beta_0$, $t > 0$ and $x \in M \setminus Sing(\phi)$ we have:*

1. *The set $\phi|_{[-\beta\|X(x)\|,\beta\|X(x)\|]}(N^r_\beta(x))$ is a flow box, in particular it does not contain singularities.*
2. *The ball $B_{\beta\|X(x)\|}(x)$ is contained on $\phi|_{[-\beta\|X(x)\|,\beta\|X(x)\|]}(N^r_\beta(x))$*
3. *The holonomy map $P_{x,t}$ is well defined and injective from $N_x(\frac{\beta}{L^t}\|X(x)\|)$ to $N_{\phi_t(x)}(\beta\|X(\phi_t(x))\|)$. Moreover, for any $y \in N_x(\frac{\beta}{L^t}\|X(x)\|)$ its orbit segment $\phi_{[0,t]}(y)$ is entirely contained in the β-scaled tubular neighborhood of $\phi_{[0,t]}(x)$. The same statement is valid for $t < 0$.*

Here, the β-scaled flow box of $\phi_{[0,t]}(x)$ is the set

$$\bigcup_{s\in[0,t]} N_{\phi_s(x)}(\beta\|X(\phi_s(x))\|).$$

Since the holonomy maps and rescaled-tubular neighborhoods will be often used through this text, we will present a figure to illustrate them and improve the understanding of the reader.

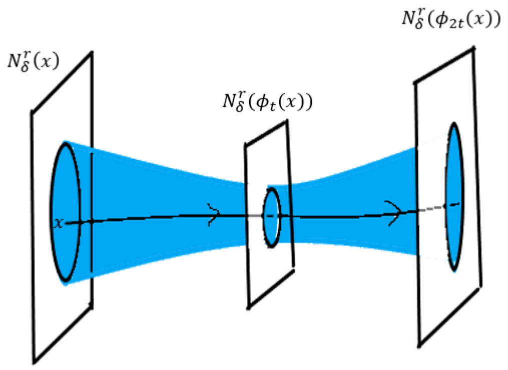

Figure 2.1: Rescaled Flow Boxes

By the previous theorem, if $0 < \beta \le \beta_0$ we can define for any $x \in M \setminus Sing(\phi)$ and $t \in \mathbb{R}$ a family of injective holonomy maps $\{P^r_{x,nt}\}_{n\in\mathbb{Z}}$, where

$$P^r_{x,nt} = P_{\phi_{nt}(x),t} : N_{\phi_{nt}(x)}\left(\frac{\beta}{L^t}\|X(\phi_{nt}(x)\|\right) \to N_{\phi_{(n+1)t}(x)}(\beta\|X(\phi_{(n+1)t}(x))\|)$$

Let us denote $C_\phi(M)$ for the set of non-negative functions $f : M \to [0, \infty)$ such that $f(x) = 0$ if, and only if $x \in Sing(\phi)$. Next result will help us to find continuity properties for the R-holonomy maps.

Theorem 2.1.5 ([JNY]). *Let ϕ be a continuous flow on M.*

1. *For any $e \in C_\phi(M)$ and $T > 0$ we can find $r \in C_\phi(M)$ such that if $d(x, y) \le r(x)$, then $d(\phi_t(x), \phi_t(y)) \le e(\phi_t(x))$, for every $t \in [-T, T]$*
2. *For any $e \in C_\phi(M)$ there is some $r \in C_\phi(M)$ such that $r(x) \le \max\{e(y); y \in B_{r(x)}(x)\}$.*

Note that the functions $\delta_X = \delta\|X(x)\| \in C_\phi(M)$. This allow us to apply the previous theorem for these functions.

Suppose that ϕ is a C^r-flow and let $\Lambda \subset M$ be a compact invariant set. We say that Λ is a *hyperbolic set* if there are $C, \lambda > 0$ and a decomposition $T\Lambda = E^s \oplus < X > \oplus E^u$ of the tangent bundle such that:

- $< X >$ is the subspace generated by the velocity direction of ϕ.
- $\|D\phi_t|_{E^s}\| \le Ce^{-\lambda t}$, for $t > 0$.
- $\|D\phi_t|_{E^u}\| \le Ce^{\lambda t}$, for $t < 0$.

  A classical fact about hyperbolic sets is that above decomposition varies continuously on Λ. Since $< X >= 0$ on singularities, this implies that any singularity on Λ cannot be accumulated by regular orbits of Λ.

Let $f : M \to M$ be a homeomorphism and let $r : M \to \mathbb{R}^+$ be a continuous map. Let us consider the quotient space

$$M_r = \{(t, x); 0 \le t \le r(x), x \in M\}/(r(x), x) \sim (0, f(x)).$$

One can endow M_r with a metric and consider it as a compact metric space (See [BW] for details).

Definition 2.1.6. *We define the suspension flow of f with roof r as the flow $\phi_t^\sigma : M_r \to M_r$ induced by the quotient projection on the time translation flow $T_t(s, x) = (t + s, x)$.*

During this text all the suspension flows in consideration will be taken with the function r being constant and equal to 1.

2.2 Expansiveness

In this section we discuss expansiveness for flows. Recall form chapterr one that a homeomorphism is expansive if it separates any distinct pair of points by a uniform distance in some time. If we move from the discrete-time to the continuous-time setting, we cannot define expansiveness in the same way. Next proposition illustrates this fact.

Proposition 2.2.1 ([BW]). *Let ϕ be a continuous flow, then for any $\delta > 0$ there is $\varepsilon > 0$ such that if $y = \phi_s(x)$ with $|s| \leq \varepsilon$ then $d(\phi_t(x), \phi_t(y)) \leq \delta$ for every $t \in \mathbb{R}$*

Proof. Since ϕ is continuous, it is a continuous mapping $\phi : M \times \mathbb{R} \to M$. Now fix $\delta > 0$, since ϕ is continuous on $\mathbb{R}$ and M is compact, there is some $\varepsilon > 0$ such that if $|s| < \varepsilon$ then $d(\phi_s(x), x) \leq \delta$ for every $x \in M$. Now suppose that $y = \phi_s(x)$. Then for any $t \in \mathbb{R}$, the choice of ε implies

$$d(\phi_t(x), \phi_t(y)) = d(\phi_t(x), \phi_s(\phi_t(x)) \leq \delta$$

This concludes the proof. □

By the previous proposition, given $\varepsilon > 0$ we can find $\delta > 0$ such that if two points are in the same orbit and are at most ε-close in time, we will never see a separation between then. So a way to try to generalize expansiveness is to require that points which are not in a temporally-small piece of orbit be separated by ϕ. But only this is not enough, because when we work with flows, reparametrizations play an important role in the theory.

Considering these facts, R. Bowen and P. Walters defined in [BW] the first version of expansiveness for flows.

Definition 2.2.2 (BW-Expansiveness). *We say that a continuous flow is BW-Expansive if for every $\varepsilon > 0$, there is $\delta > 0$ such that the following holds: If $x, y \in M$, $\rho : \mathbb{R} \to \mathbb{R}$ is a continuous function with $\rho(0) = 0$ and $d(\phi_t(x), \phi_{\rho(t)}(y)) \leq \delta$ for every $t \in \mathbb{R}$, then $y = \phi_s(x)$ for some $|s| \leq \varepsilon$.*

An easy consequence of above definition is the following.

Proposition 2.2.3 ([BW]). *Let ϕ be a continuous flow and let $x \in Sing(\phi)$. If ϕ is BW-expansive, then x is an isolated point of M.*

Proof. Fix $\varepsilon > 0$ and let $\delta > 0$ be given by the expansiveness of ϕ. Now take any $y \in B^M_\delta(x)$ and consider the constant function $\rho(x) = 0$. Then we have

$$d(\phi_t(x), \phi_{h(t)}(y)) \leq \delta$$

for any $t \in \mathbb{R}$. Now expansiveness implies that $y = \phi_s(x) = x$. Thus $B^M_\delta(x) = \{x\}$ and the proof is complete. □

BW-expansiveness is a successful tool to study non-singular flows such as Axiom A and Anosov flows, but the previous proposition shows that it is not suitable to deal with flows containing singularities accumulated by regular orbits such as the Lorenz Attractor. Later, M. Komuro gave in [K1] a definition of expansiveness which is suitable for this attractor, the so called k^*-expansiveness. Let us fix the following notation.

$$Rep(\mathbb{R}) = \{\rho : \mathbb{R} \to \mathbb{R}; \rho \text{ is an increasing homemorphism and } \rho(0) = 0\}$$

Definition 2.2.4 (k^*-Expansiveness). *We say that a continuous flow ϕ is k^*-expansive if for every $\varepsilon > 0$, there is some $\delta > 0$ such that the following holds:*

If $x, y \in M$, $\rho \in Rep(\mathbb{R})$ and we have $d(\phi_t(x), \phi_{\rho(t)}(y)) \leq \delta$ for every $t \in \mathbb{R}$, then there is some $t_0 \in \mathbb{R}$ such that $y = \phi_{t_0+s}(x)$ for some $|s| \leq \varepsilon$.

In the absence of singularities k^*-expansiveness and BW-expansiveness are equivalent, and then we will aways refer to a K^*-expansive flow only as expansive.

2.3 Shadowing Property

The shadowing property is the second main topological feature of hyperbolic systems. Indeed, it is vastly used to prove many important properties of smooth systems, such as generic properties, stability and entropy properties. In the topological dynamics setting, it is also of great interest.

The shadowing property is a feature of approximating flavor. It allows one to obtain a real orbit of the flow, approximating a set formed by pieces of orbits which behaves almost as a real orbit, but with some approximation errors. Hereafter we will work in order to state precisely this concept.

Let $S = (x_i, t_i)_{i=a}^{b}$ be a sequence with $-\infty \leq a < b \leq \infty$, $x_i \in M$ and $t_i \in \mathbb{R}$. We say that S is a (δ, T)-pseudo-orbit for ϕ, if $t_i \geq T$ and

$$d(\phi^{t_i}(x_i), x_{i+1}) < \delta$$

for every $a \leq i \leq b$.

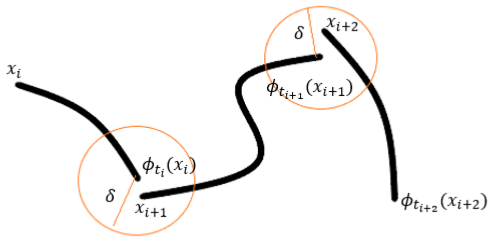

Figure 2.2: Pseudo-Orbits

Let $(x_i, t_i)_{i=a}^{b}$ be a (δ, T)-pseudo-orbit. Define s_i to be

$$s_i = \begin{cases} \sum_{n=0}^{i-1} t_n, i \geq 0 \\ \sum_{n=i}^{-1} t_n, i < 0. \end{cases}$$

We say that S is ε-T-shadowed if there exists a point z and $h \in Rep(\mathbb{R})$ such that $d(\phi^{h(t)}(z), \phi^{t-s_i}(x_i)) < \varepsilon$ for every i and $s_i \leq t \leq s_{i+1}$.

Definition 2.3.1. *We say that ϕ has the shadowing property if for every $\varepsilon > 0$ there exists $\delta > 0$ such that any $(\delta, 1)$-pseudo-orbit is ε-shadowed.*

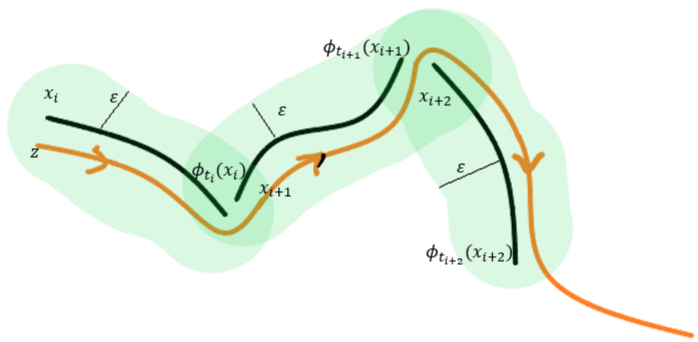

Figure 2.3: Shadowing Property

The next proposition shows that there is not difference between use (δ, T) or $(\delta, 1)$ pseudo-orits to define shadowing.

Proposition 2.3.2 ([K2]). *A flow ϕ has the shadowing property if, and only if, for any $\varepsilon > 0$ and any $T > 0$ there is some $\delta > 0$ such that every (δ, T)-pseudo-orbit is ε-shadowed.*

Similarly to the case of expansiveness, there are some distinctions between shadowing for the singular and the non-singular case. For instance, next theorem exhibits one of these distinctions.

Theorem 2.3.3 ([K2]). *A non-singular flow ϕ has the shadowing property if, and only if, for every $\varepsilon > 0$, there is some $\delta > 0$ such that every finite δ-T-pseudo-orbit S is ε-shadowed.*

To see some examples of singular flows for which the previous result is not valid, we refer the reader to [K2]. In the same work, M. Komuro introduced a stronger form of shadowing, in order to recover Theorem 2.3.3. We now introduce this concept. First we define the following set:

$$Rep_{\varepsilon}(\phi) = \left\{\rho \in Rep(\phi); \left|\frac{\rho(s) - \rho(t)}{s - t} - 1\right| \leq \varepsilon, \forall s, t \in \mathbb{R}\right\}$$

We say that a δ-T-pseudo orbit $S = (x_i, t_i)$ is ε-strongly-shadowed if there exists a point x and $h \in Rep_{\varepsilon}(\phi)$ such that $d(\phi^{h(t)}(x), \phi^{t-s_i}(x_i)) < \varepsilon$ for every i and $s_i \leq t \leq s_{i+1}$.

Definition 2.3.4. *A flow ϕ has the strongly shadowing property if for every $\varepsilon > 0$, there is some $\delta > 0$ such that every δ-T-pseudo orbit is ε-strongly shadowed.*

This stronger version of shadowing allows us to recover a version of the Theorem 2.3.3 and will be crucial to prove Theorem I in Chapter 6.

2.4 Topological Entropy

In this section we will define the topological entropy of flows. We follow the same ideas as in the case of homeomorphisms and define entropy using separated and generator sets. We also provide to the reader some classical results that will be used in further sections.

Fix $\varepsilon > 0$ and $t > 0$. We say that a pair of points is *t-ε-separated by ϕ* if there is some $0 \leq s \leq t$ such that $d(\phi_s(x), \phi_s(y)) > \varepsilon$. Let $K \subset M$. We say that $E \subset K$ is a t-ε-separated set if any pair of distinct point of E is t-ε-separated. Let $s_t(\varepsilon, K)$ denote the maximal cardinality of a t-ε-separated subset of K. This number is finite due to the compactness of M. Now we will define the generator sets. We say that K is *t-ε-generated by F* if for any point $x \in K$, there is some $y \in F$ such that $d(\phi_s(x), \phi_s(y)) \leq \varepsilon$ for any $0 \leq s \leq t$. Let $r_t(\varepsilon, K)$ denote the minimal cardinality of an t-ε-generator set for K. This quantity is also finite by compactness of M. We define the *topological entropy of ϕ on K* to be the number $h(\phi, K)$ defined by

$$h(\phi, K) = \lim_{\varepsilon \to 0} \limsup_{t \in \infty} \frac{1}{t} \log s_t(\varepsilon, k) = \lim_{\varepsilon \to 0} \limsup_{t \in \infty} \frac{1}{t} \log r_t(\varepsilon, k).$$

We need to make two comments on the above definition. First, this limits always exists (it can be infinity), since the lim sup always exists and that $s_t(\varepsilon)$ is monotone as ε goes to 0. The second point is that it does not matter if we are using separated or generator sets, the result will be the same. This is pretty similar to the case of homeomorphisms. Finally we can define:

Definition 2.4.1. *The topological entropy $h(\phi)$ of ϕ is defined to be $h(\phi) = h(\phi, M)$.*

Since if $K_0 \subset K_1$ we have that $h(\phi, K_0) \leq h(\phi, K_1)$ it is easy to see that

$$h(\phi) = \sup_{K \subset M} \{h(\phi, K)\}.$$

Next we will prove an elementary result that states that the topological entropies of homeomorphisms and flows are related.

Theorem 2.4.2. *Let ϕ be a continuous flow and ϕ_1 be its time one homeomorphism. Then $h(\phi) = h(\phi_1)$.*

Proof. First notice that $h(\phi) \geq h(\phi_1)$. Indeed, for any $\varepsilon > 0$ and $t \in \mathbb{N}$, we have

$$s_t(\varepsilon, \phi) \geq s_t(\varepsilon, \phi_1).$$

To prove the other inequality we begin by fixing $\varepsilon > 0$ and choosing some $0 \leq \delta \leq \varepsilon$ such that if $d(x, y) \leq \delta$, then $d(\phi_t(x), \phi_t(y)) \leq \varepsilon$ with $t \in [0, 1]$. Now let F be an n-δ-generator set for ϕ_1 with minimal cardinality. Then by the choice of δ we have that E is an n-ε-generator set for ϕ. Since $\delta \to 0$ as $\varepsilon \to 0$, we conclude $h(\phi) \leq h(\phi_1)$. $\square$

The topological entropy of flows and homeomorphisms shares many interesting properties. For instance both are invariant by conjugacy.

Theorem 2.4.3. *If ϕ and ψ are conjugated flows then $h(\phi) = h(\psi)$.*

Proof. Let $\phi_t : M \to M$ and $\psi : M \to N$ be two conjugated flows on M and fix $\varepsilon > 0$. Let $h : M \to N$ be the conjugacy homeomorphism. Let $0 < \delta \leq \varepsilon$ be given such that if $d(x, y) \leq \delta$, then $d(h(x), h(y)) \leq \varepsilon$. Fix $T > 0$ and let $E \subset M$ be a T-δ-generator subset for ϕ, with minimal cardinality.

We claim that $h(E)$ is an T-ε-generator set for ψ. Indeed, since $d(\phi_t(x), \phi_t(y)) \leq \delta$ for $t \in [0, T]$, then

$$\psi_t(h(x)), \psi_t(h(y))) = d(h(\phi_t(x), h(\phi_t(y))) \leq \varepsilon$$

For $t \in [0, T]$. Thus we have that $r_t(\varepsilon, \psi) \leq r_t(\delta, \phi)$, and since $\delta \to 0$ as $\varepsilon \to 0$, we have that $h(\psi) \leq h(\phi)$.

The reverse inequality is proved in a analogous way. □

Although topological entropy is invariant by conjugacy, the same is not true for times changes.

Proposition 2.4.4. *Let ϕ be a flow, fix $k \in \mathbb{R}$ and define $\psi_t = \phi_{kt}$. Then $h(\psi) = |k|h(\phi)$.*

Proof. Fix $k \geq 0$, $\varepsilon > 0$ and $T > 0$. By the definition of ψ we have that $S_T(\varepsilon, \psi) = S_{kT}(\varepsilon, \phi)$. Therefore

$$\frac{1}{k}h(\psi) = \frac{1}{k}\limsup_{t\to\infty}\lim_{\varepsilon} S_{kt}(\varepsilon, \phi) = h(\phi).$$

Thus we conclude that $h(\psi) = kh(\phi)$.

The proof for $k \leq 0$ relies on the following claiming:

Claim: $h(\phi_t) = h(\phi_{-t})$.

To prove the claiming, fix $\varepsilon, T > 0$ ad let $E \subset M$ be a T-ε-generator set with minimal cardinality. Clearly, we have that $\phi_T(E)$ is a T-ε-separated set for ϕ_{-t} and this proves the claiming.

Finally, if $k \leq 0$, we have that

$$\frac{1}{|k|}h(\psi_t) = h(\phi_{|k|t}) = h(\phi_{kt}).$$

and the proof is complete. □

By the previous proposition, we have that the topological entropy is not preserved by time-changes. More generally, we have that the topological entropy is not preserved by topological equivalences.

Definition 2.4.5. *Two flows $\phi_t : M \to M$ and $\phi : N \to N$ are said to be topologically equivalent if there is some homeomorphism $h : M \to N$ satisfying:*

1. *h maps orbits of ϕ into orbits of ψ*
2. *h preserves the orientation of the orbits.*

In [SYZ] it is proved that the positiveness of the entropy of two non-singular flows is preserved by topological equivalence. This result will be useful to us, so we will state it precisely.

Theorem 2.4.6. *[[SYZ]] Let ϕ and ψ be two topologically equivalent non-singular flows. If $h(\phi) > 0$, then $h(\psi) > 0$.*

We mention that the above result is false in the presence of singularities. We refer the reader to the works [SYZ], [SV] and [SZ] for examples of flows whose positiveness of topological entropy is no preserved by topological equivalence.

2.5 Topological Entropy of Non-Singular Expansive Flows

In this section we will discuss some extensions of Lewowicz's Theorem. In particular, we will see some generalizations to non-singular continuous flows. The first of its generalization is due to Kato for CW-expansive homeomorphisms.

Definition 2.5.1. *A homeomorphism is CW-expansive if there is $e > 0$ such that for any non-degenerate continuum $C \subset M$, there is $n \in \mathbb{Z}$ such that $diam(f^n(C)) > e$*

Actually, Kato realized that assuming that the space is not totally disconnected, to obtain a proof for Lewowicz's theorem, we just need to have that continuums expands in uniformly controlled time. He obtained the following:

Theorem 2.5.2 ([Ka])**.** *Suppose that $dim(M) > 0$. If $f : M \to M$ is a CW-expansive homeomorphism, then $h(f) > 0$.*

Once we have results relating positive entropy to expansiveness for discrete-time systems, a natural question arises:

"Is it possible to extend these results to continuous-time systems?"

The problem of generalizing some of the techniques used in Lewowicz Theorem's proof such as uniformly expansiveness and existence of non-trivial stable sets was first addressed by H.B. Keynes and M. Sears in [KS]. But this problem is not simple since the structures can be more complicated while we are working with flows. As we have see in previous sections, expansiveness is not immediately generalized to the context of continuous flows. Moreover, there are distinct ways to define expansiveness for flows depending on the kind o flow that is being considered. Keynes and Sears dealt with the non-singular case.

Later, following the ideas of Kato and the techniques developed in [KS], a version of Lewowicz's Theorem was established for CW-expansive flows by A. Arbieto, W. Cordero and M. J. Pacífico in [ACP].

Theorem 2.5.3 ([ACP])**.** *Let ϕ be a continuous flow and suppose* $\dim(M) > 1$*. If ϕ is CW-expansive, then $h(\phi) > 0$.*

We remark that the previous result also applies to expansive non-singular flows. Once the above results are in the setting of non-singular flows, it is worth to mention that there are not similar results for the singular case. One of our goals in this thesis is to explore the relationship between expansiveness and entropy for the singular case. In next chapters we will see some results on this direction. In particular, some of our results will allow us to obtain positive entropy for systems with some kind of expansiveness, even if they are defined in low dimensional phase spaces.

Chapter 3

Rescaled Properties of Flows

In this section we will discuss a new family of dynamical properties, the so called the rescaled-properties of flows (for simplicity, we will always replace the prefix "rescaled" by "R" through this text). These are very recent dynamical features that are deeply related with the Liao's ideas of R-tubular neighborhoods for flows. We will study R-expansiveness which is a new dynamical feature introduced by L. Wen and X. Wen in [WW]. We will also introduce a new concept of entropy, the so called R-entropy and illustrate its relation with the classical topological entropy for flows.

We would like to stress that from this chapter until the end of this work every time we state some result which is not an original result of this work, we will cite the reference where the reader can find all the details and proofs. In this way, every result in this and in the next chapters which are not referenced are original results of our work.

3.1 R-Expansiveness

Expansiveness theory for singular flows is quite complicated. As we have discussed before, since there is no trivial extension of the definition from the discrete-time setting to flows, many distinct definitions of expansiveness exist. In the previous chapter we have studied some basic features of BW-expansive and k^*-expansive flows. But there exist other distinct definitions of expansiveness. For instance, we can cite kinematic-expansiveness and geometric-expansiveness (See [Art1] for the details). In this chapter we are interested in other kind of expansiveness introduced by L. Wen and X. Wen in [WW], the so called rescaled-expansiveness (R-expansiveness for short).

The concept R-expansiveness is very close to the BW-expansiveness, but the distance of separation of the orbits is "resized" by the size of the vector field. Next we precise this idea.

Definition 3.1.1. *A C^r-flow ϕ on M is said to be R-expansive if for every $\varepsilon > 0$, there is some $\delta > 0$ such that the following is satisfied:*

"If $x, y \in M$, $\rho \in Rep(\phi)$ and $d(\phi_t(x), \phi_{\rho(t)}(y)) \leq \delta||X(\phi_t(x))||$ for every $t \in \mathbb{R}$, then $\phi_{\rho(t)}(y) \in \phi|_{[-\varepsilon,\varepsilon]}(x)$ for any $t \in \mathbb{R}$."

Although the previous definition has a high level of similarity with the previous versions of expansiveness that we have studied before, there are crucial distinctions here. For instance, it is surprising that some highly non-expansive flows (in the classical sense), satisfies R-expansiveness.

Example 3.1.2. *The identity flow is R-expansive. Indeed, if $\phi_t = Id_M$ for any $t \in \mathbb{R}$, then $||X(x)|| = 0$ for any $x \in M$. Therefore, for any $\delta > 0$ we have that $d(\phi_t(x), \phi_{\rho(t)}(y)) \leq \delta||X(x)||$ is satisfied if, and only if, $x = y$. Thus ϕ is trivially R-expansive.*

Beside the above trivial example, many highly non-trivial flows are R-expansive. Indeed, in [WW] it is proved the following resut:

Theorem 3.1.3 ([WW]). *If ϕ is a multi-singular hyperbolic flow. Then $\phi|_{\Omega(\phi)}$ is R-expansive*

In face of previous result, we can the following question:

"How can we interpret previous theorem and example?"

Or in other words

"How can R-expansiveness includes such a trivial example as well as many examples with highly chaotic behaviour?"

To answer the previous questions we need to have in mind that the behind the idea of the rescaled-features of dynamical systems there is a fight between the geometric distance of M and the velocity of ϕ. Indeed, when the generating vector-field of ϕ is away from zero, then the separation of R-expansiveness is big in the geometric sense. On the other hand, when the velocity field is close to zero, then it distorts a lot the separation distances, then one see just small separation in the geometric meaning.

In Example 3.1.2, the influence caused by the vector field on the separation distances is the biggest possible, since $X(x) = 0$ for every x. On the other hand, in multi-singular examples, such as the Lorenz's attractor, the influence caused by the vector field is not big enough to prevent the appearance of geometric separation.

Besides the results in [WW], A. Artigue studied in [Art2] another criteria to conclude that a flow is R-expansive. First notice that it is easy to show that if ϕ is non-singular, then R-expansiveness is equivalent to BW-expansiveness. So, the hard case is exactly the singular one.

One of the central goals of Artigue in [Art2] was to establish under which hypothesis k^*-expansiveness is related with R-expansiveness. Next we state some of his main results which will be used by us on next chapters.

Theorem 3.1.4. *Let ϕ be a k^*-expansive flow. If $Sing(\phi)$ contains only non-degenerate singularities, then ϕ is R-expansive.*

On above result, non-degenerate means that any $p \in Sing(\phi)$ satisfies

$$DX(p) : T_pM \to T_pM$$

is invertible for any $t \in \mathbb{R}$. An easy corollary of previous theorem is the following result:

Theorem 3.1.5. *If ϕ is k^*-expansive flow such that $Sing(\phi)$ is formed by hyperbolic singularities, then ϕ is R-expansive.*

3.2 R-Topological Entropy

In this section we introduce a new concept for flows which is a rescaled version of topological entropy, the so called R-topological entropy. The idea behind the R-topological entropy is to study how the flow separates points until some time, but requiring the separation distance to be scaled by the norm of X.

Given a point $x \in M$, we define the t-ε-r-dynamical ball centered at x to be the set

$$B^r(x, t, \varepsilon) = \{y \in M; d(\phi_s(x), \phi_s(y)) \leq \varepsilon \|X(\phi_s(x)\|, 0 \leq s \leq t\}$$

Note that if $x \in Sing(\phi)$ then $B^r(x, t, \varepsilon) = \{x\}$ for every $\varepsilon, t > 0$. On the other hand, Theorem 2.1.4 implies that if x is regular we can take $\varepsilon > 0$ small enough to guarantee that the t-ε-r-dynamical balls avoid singularities. Moreover, this small ε can be taken uniformly in $M \setminus Sing(\phi)$.

Let $P \subset M$ and $P \subset A$. We say that A t-ε-r-generates P if

$$P \subset \cup_{x \in A} B^r(x, t, \varepsilon)$$

If $p \in P$ is a singularity, then it can only be r-generated by the dynamical ball $\{p\}$. This fact implies that we cannot generate P with a finite set if $Sing(\phi) \cap P$ is not isolated on P. On the other hand, if $Sing(\phi) \cap P$ is isolated in P then it is a finite set and therefore it does not add information to the computation of R-entropy for P.

The previous facts make us to consider only compact sets without singular points to compute the topological R-entropy. Let $K_0(M)$ denote the set of compact subsets of M without singular points.

We can also define a R-version of separated sets. We say that two points $x, y \in M$ are t-ε-r-separated if $y \notin B^r(x, t, \varepsilon)$ or $x \notin B^r(y, t, \varepsilon)$.

Let us take $P \in K_0(M)$. Consider $R_\varepsilon(P, t)$ the minimal cardinality of a t-ε-R-generator set of X and $S_\varepsilon(P, t)$ the maximal cardinality of a t-ε-R-seperated subset for P. Since P is compact and has not singularities, then $R_\varepsilon(P, t), S_\varepsilon(P, t) < \infty$ for any t and $\varepsilon > 0$.

Proposition 3.2.1. *If $P \in K_0(M)$, then for any $t > 0$ and $\varepsilon > 0$ one has*

$$R(P, t, \varepsilon) \leq S(P, t, \varepsilon) \leq R\left(P, t, \frac{\varepsilon}{2}\right)$$

Proof. Let E be a t-ε-R-separated set for P with maximal cardinality. We claim that E is a t-ε-R-generator set for P. Indeed, if the claim does not hold, we can find $z \in P \setminus \cup_{x\in E} B^r(x,t,\varepsilon)$. But this implies that z and x are t-ε-R-separated for any $x \in E$. Then $S(P,t,\varepsilon) \leq R(P,t,\varepsilon)$.

To prove the second inequality, fix E a t-ε-R-separated subset of P with maximal cardinality and let F be a set t-$\frac{\varepsilon}{2}$-R-generator set for P. Then for any $x \in E$ let $x' \in F$ be a point such that $x \in B^r(x',t,\frac{\varepsilon}{2})$. Therefore, this choice is made injectively. This implies that $S(P,t,\varepsilon) \leq R(P,t,\frac{\varepsilon}{2})$ and the proposition is proved.

□

Now we can define the R-topological entropy of P to be the number:

$$h^r(\phi,P) = \lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log S_\varepsilon(P,t) = \lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log R_\varepsilon(P,t)$$

Finally, the topological R-entropy of ϕ is defined to be

$$h^r(\phi) = \sup_{P\in K_0(M)}\{h^r(P,\phi)\}.$$

Let $P,Q \in K_0(M)$ and suppose that $P \subset Q$. Let $E \subset M$ be an t-ε-r-generator with minimal cardinality for Q. Then

$$P \subset \bigcup_{x\in E} B^r(x,t,\varepsilon)$$

and it implies $h^r(P,\phi) \leq h^r(Q,\varepsilon)$.

Our first result exhibits a relation between topological entropy and topological r-entropy.

Theorem 3.2.2. *If X is a C^1 vector field, then $h(\phi,P) \leq h^r(\phi,P)$ for any $P \in K_0(M)$. Moreover, if X is non-singular these quantities coincide.*

Proof. Let $K = \sup_{x\in M}\{\|X(x)\|\}$. By definition of the dynamical balls, we have that $B^r(x,t,\varepsilon) \subset B(x,t,K\varepsilon)$ for any point $x \in M$ and $\varepsilon, t > 0$. This implies $r_{K\varepsilon}(P,t) \leq R_\varepsilon(P,t)$ for any $P \in K_0(M)$. Taking the limits in both sides of previous inequality we obtain $h(P,\phi) \leq h^r(P,\phi)$

If X is non-singular we have that $K' = \inf_{x\in M}\{\|X(x)\|\} > 0$. By definition $B(x,t,K\varepsilon) \subset B^r(x,t,\varepsilon)$ and then $R_\varepsilon(P,t) \leq r_{K'\varepsilon}(P,t)$. Then we obtain $h^r(P,\phi) \leq h(P,\phi)$. □

Previous result says us that R-topological entropy gives us more chaotic information about the flow than topological entropy.

Next example shows that these two quantities may not coincide if the flow has singularities.

Example 3.2.3. *We start defining a periodic flow ψ on $\mathbb{T}^2$ with constant velocity equal to one. Here we will see the torus as a square S with sides of length 4 in the plane R^2 with vertices at the points $(-2,0),(2,0),(-2,4)$ and $(2,4)$ and identifying first the lower and upper sides and then identifying the right and left sides. Let us consider on S the vector field X constant and equal to $(1,0)$. Thus X generates the flow ψ desired.*

Now we modify this flow to obtain a flow with positive R-entropy. First consider the function ρ on S satisfying the following conditions:

1. *ρ is constant along the vertical segments $x \times [0,4]$.*

2. *$\rho((x,y)) = 1$, if $(x,y) \in [-2,-1] \times [0,4]$ or $(x,y) \in [1,2] \times [0,4]$.*

3. *$\rho((x,y)) = 0$, if $(x,y) \in \{0\} \times [0,4]$.*

4. *$\rho((x,y)) = -x$, if $p \in [-1,0] \times [0,4]$*

5. *$\rho((x,y)) = x$, if $p \in [0,1] \times [0,4]$.*

Let ϕ be the flow generated by the field ρX (see the figure).

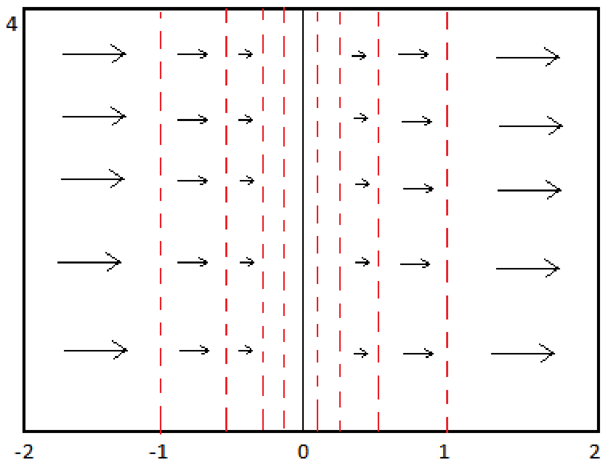

Figure 3.1: Flow in $\mathbb{T}^2$

We claim that $h^r(\phi) > 0$. Indeed, consider the circle C on the torus transversal to the orbit segments and represented on S by the vertical segment $\{-1\} \times [0,4]$. Notice that the orbit of any point on C converges to a singularity when $t \to \infty$. Now fix a point $x \in X$. By the choice of the function ρ it is easy to see that $\|\rho X(\phi_n(x))\| = e^{-n}$. Keeping this in mind, we construct a n-ε-r-separated set for ϕ as follow:

For each $n \geq 1$ choose a set $P_n \subset C$ with 2^n points dividing the circle on 2^n segments of equal lenght. If we denote L for lenght of C, then if we choose two distinct points $x, y \in P^n$ we have that $d(\phi_t(x), \phi_t(y)) \geq \frac{L}{2^n}$ for every $t \in \mathbb{R}$.

Now fix $\varepsilon > 0$ and set $N > 0$ such that $L2^{-n} \geq \varepsilon e^{-n}$ for every $n \geq N$. This implies that if $n \geq N$ then for any two point in $x, y \in P_n$ we have that

$$d(\phi_n(x), \phi_n(y)) \geq L2^{-n} \geq \varepsilon e^{-n} = \varepsilon \max\{\|\rho X(\phi_n(x))\|, \|\rho X(\phi_n(y)\|\}.$$

This implies that P_n is a n-ε-r-separated set for ϕ and then:

$$h^r(\phi) \geq \lim_{\varepsilon\to 0} \limsup_{t\to\infty} \frac{1}{t} \log S(t, \varepsilon, C) \geq \limsup_{n\to\infty} \frac{1}{n} \log \#P_n = \log 2.$$

On the other hand $h(\phi) = 0$ since the topological entropy of any surface flow vanishes (see [LG]).

Now we extend a classical result about the finiteness of topological entropy for smooth systems to the context of R-topological entropy.

Theorem 3.2.4. *If X is a Lipschitz vector feld, then $h^r(\phi) \leq d \log(L)$, where L is the Lipschitz constant of X and $d = \dim(M)$.*

Proof. Let $P \in K_0(M)$ and $\varepsilon > 0$. Since P is compact we can cover it with a family $\{f_i, U_i\}_{i=1}^n$ of charts satisfying the following assumptions:

- $f_i(U_i) = B_2(0) \subset \mathbb{R}^d$, for $i = 1, ..., n$
- $P \subset \cup_{i=1}^n f_i^{-1}(B_1(0))$
- The maps f_i^{-1} are A-Lipschitz. for $i = 1, ..., n$.

Now fix $t > 0$, the Lipchitz property of f_i^{-1} implies that

$$f_i^{-1}(B_{\frac{\varepsilon}{AL^t}}(x)) \subset B^M_{\frac{\varepsilon}{L^t}}(f_i^{-1}(x))$$

if $B_{\frac{\varepsilon}{AL^t}}(x)) \subset U_i$.

By a classical result of geometry of $\mathbb{R}^d$ the number $N(\varepsilon, t)$ of balls r needed to cover the ball $B_1(0)$ is at most $C\frac{5}{r}^d$, where C is given by the riemannian metric of $\mathbb{R}^r$.

Let $P(r, t, i) = \{x_1^i, ..., x_N^i\}$ denote the image of f_i^{-1} of the centers of these r-balls. If $r = \frac{\varepsilon}{AL^t}$, then Theorem 2.1.4 implies that $P(r, t, i)$ is a t-ε-r-generator set for U_i. Then the r-entropy of K is estimated as follows

$$h^r(\phi, K) \leq \lim_{t\to\infty} \frac{1}{t} \log n\#P(r, t, i) \leq \lim_{t\to\infty} \frac{1}{t} \log nC\frac{5}{r}^d \leq \lim_{t\to\infty} \frac{1}{t} \log nC \left(\frac{5}{\frac{\varepsilon}{AL^t}}\right)^d = d \log L$$

and this concludes the proof.

□

Chapter 4

Topological Entropy of Expansive Flows

In this chapter we discuss the relationship between expansiveness and topological entropy for singular flows. We obtain here our first main results. Section 4.1 is devoted to develop a new theory of stable sets based on Liao's ideas and rescaled holonomy maps. These new stable sets will share some interesting features with the stable sets defined for non-singular expansive flows. They will also be crucial to prove the main theorems of Section 4.2, which are criteria for obtain positive entropy for some R-expansive flows. It will also imply positive topological entropy for some classes of k^*-expansive flows.

4.1 R-Stable and R-Unstable Sets for R-Expansive Flows

This section is intended to construct a theory of R-stable and R-unstable sets for singular flows. This construction is directly inspired by the R-techniques which were introduced in the works of by S. Liao and later used in [WW]. Hereafter, ϕ denotes a C^r-flow and $x \in M$ denotes a regular point. We use the estimatives from Theorem 2.1.4 on the size of the cross-sections to obtain our versions of stable sets for flows.

Fix some regular point $x \in M$. As discussed in Chapter 2, the tubular flow theorem gives us some $\varepsilon(x) > 0$ and $\delta(x) > 0$ such that the set

$$N^r_x(\delta(x)) = \exp_x(B_{\delta(x)}(0) \cap < X(x) >^{\perp})$$

is a cross section of time $\varepsilon(x)$ through x. If X is non-singular we can obtain that $\varepsilon(x), \delta(x) > C > 0$ for any $x \in M$. If X has singularities, we may have $\varepsilon(x), \delta(x) \to 0$ when $x \to Sing(\phi)$. On the other hand, Theorem 2.1.4 gave us a "uniform" control on how these constants collapse. Actually, for any $\beta > 0$ sufficiently small the set

$$N^r_\beta(x) = \exp_x(B_{\beta||X(x)||}(0) \cap < X(x) >^{\perp})$$

is a cross section of time $\beta||X(x)||$ for the flow. Moreover, the holonomy map $P_{x,t}$ is well defined on $N^r_\beta(x)$ and if $y \in N^r_{\frac{\beta}{L^t}}(x)$, the orbit segment between y and $P_{x,t}(y)$ belongs to the β-rescaled tubular neighborhood of $O(x)$. This gives us a way to guarantee that the holonomy maps are well defined. Let us fix $x \in M \setminus Sing(\phi)$, $t > 0$ and $\beta > 0$.

Definition 4.1.1. *The β-r-stable and β-r-unstable local sets of x are respectivelly*

$$W^{r,s}_{\beta,t}(x) = \left\{ y \in N^r_{\frac{\beta}{L^t}}(x)\,; d(P_{x,nt}(x), P_{x,nt}(y)) \le \frac{\beta}{L^t}\|X(P_{x,nt}(x))\|, \forall n \in \mathbb{N} \right\}$$

and

$$W^{r,u}_{\beta,t}(x) = \left\{ y \in N^r_{\frac{\beta}{L^t}}(x)\,; d(P_{-nt,x}(x), P_{x,-nt}(y)) \le \frac{\beta}{L^t}\|X(P_{x,-nt}(x))\|, \forall n \in \mathbb{N} \right\}$$

Notice that these sets are well defined if β is small enough. An interesting consequence of the definition is that we can use these sets to characterize R-expansiveness.

Theorem 4.1.2. *The flow ϕ is R-expansive if, and only if, there exists $\delta > 0$ such that for any regular point $x \in M$ and any $t > 0$, one has $W^{r,s}_{\delta,t}(x) \cap W^{u,s}_{\delta,t}(x) = \{x\}$.*

Proof. Fix x a regular point, $\beta > 0$ small enough and let $0 < \varepsilon < \beta$. Let $0 < \delta < \varepsilon$ be given by the r-expansiveness of ϕ related to ε. Now suppose that $y \in W^{r,s}_{\delta,t}(x) \cap W^{r,u}_{\delta,t}(x)$. Since $y \in W^{r,s}_{\delta}(x)$ we have

$$d(P_{x,n}(x), P_{x,n}(y)) < \frac{\delta}{L^t}\|X(P_{x,n}(x))\|$$

for any non-negative integer n. By the Theorem 2.1.4 we obtain that $d(\phi_s(x), \phi_s(y)) \le \delta\|(X(\phi_t(x))\|$ for every $s \ge 0$. On the other hand, since $y \in W^{r,u}_{\delta,t}(x)$, a similar argument shows that $d(\phi_s(x), \phi_s(y)) \le \delta\|(X(\phi_t(x))\|$ for every $s \le 0$. Now R-expansiveness implies that $y \in \phi_{[-\varepsilon,\varepsilon]}(x)$, but since $x, y \in N^r_{\frac{\delta}{L^t}}(x)$, we have that $y = x$.

Conversely suppose that $P = \sup_{x\in M}\{\|X(x)\|\}$, fix $0 < \varepsilon < \beta$ and let $0 < \delta < \varepsilon$ such that $W^{r,s}_{\frac{\delta}{P},t}(x) \cap W^{r,u}_{\frac{\delta}{P},t}(x) = \{x\}$ for any regular point x and any $t > 0$. Fix $t > 0$ such that $L^t > 1$ and suppose there exist a reparametrization h and two points x, y satisfying $d(\phi_s(x), \phi_{h(s)}(y)) \le \frac{\delta}{PL^t}\|X(\phi_s)(x))\|$ for $s \in \mathbb{R}$. This implies in particular that

$$d(x, y) < \frac{\delta}{PL^t}\|X(x)\|.$$

Since $\delta < \varepsilon < \beta$, Theorem 2.1.4 implies that there exists some

$$|s_0| < \frac{\delta}{PL^t}\|X(x)\| \le \delta \le \varepsilon$$

such that $y_0 = \phi_{s_0}(y) \in N^r_{\delta}(x)$. More generally, since $d(\phi_{nt}(x), \phi_{h(nt)}(y)) < \frac{\delta}{PL^t}\|X(\phi_{nt}(x)\|$, for any $n \in \mathbb{Z}$, there exists $|s_n| < \varepsilon$ such that $y_n = \phi_{h(nt)+s_n}(y) \in N^r_{\delta}(\phi_{nt}(x))$. But last fact implies that the set $\{y_n\}$ is the orbit of y_0 under the holonomy maps $\{P_{x,nt}\}$. In additon, one has that $y_0 \in W^{r,s}_{\delta,t}(x) \cap W^{r,u}_{\delta,t}(x)$ and therefore we must have $y_0 = x$. Then $\phi_{s_0}(y) = x$ and the flow ϕ is R-expansive. $\square$

An interesting fact about the above characterization is that we do not need to concern about reparametrizations, since we are only working with the holonomy maps of ϕ.

For the remaining of this section, we are assuming that the flows in consideration are R-expansive and the constant δ given by previous result will be called a constant of R-expansiveness of ϕ. Next we work in order to obtain versions of some well know results about non-singular expansive flows for the R-expansive case. We begin with a version of uniform expansiveness.

Theorem 4.1.3. *Let $K \subset M$ be a compact and invariant set without singular points and let $\delta > 0$ be a constant of R-expansiveness for ϕ. Denote $A = \inf_{x\in K}\{\|X(x)\|\}$. Then for any $0 < \eta \le \delta A$ and $t > 0$, there exists $N_\eta > 0$ such that if $x \in K$ and $y \in N^r_\delta(x)$ with $d(x, y) > \eta$, then there is some $-N_\eta \le i \le N_\eta$ such that $d(P_{x,it}(x), P_{x,it}(y)) \ge \delta\|X(P_{x,it}(x))\|$.*

Proof. If the result is false, we can find some $\eta > 0$, sequences $x_n \in K$, $y_n \in N^r_\delta(x_n)$, $m_n \to \infty$ and $t > 0$, such that $d(P_{x_n,it}(x_n), P_{x_n,it}(y_n)) \le \delta\|X(P_{x_n,it}(x_n)\|$ for $-m_n \le i \le m_n$. Now by compactness of K we can suppose that $x_n \to x \in K$, $y_n \to y \in M$. Furthermore, since $\|X(x)\| > A > 0$ for every $x \in K$, we have that $diam(N^r_\delta(x_n)) > C > 0$ for any x_n. Since X is a C^r vector field, the normal direction of X also varies continuously with x, so we have that $y \in N^r_\delta(x)$. But now, the continuity of the holonomy maps implies that $d(P_{x,it}(x), P_{x,it}(y)) \le \delta\|X(P_{x,it}(x)\|$ for every $i \in \mathbb{Z}$ and then $x = y$, a contradiction, since $d(x, y) > \eta$.

□

Next we show that R-stable (R-unstable) sets need to contract in the future (in the past).

Theorem 4.1.4. *Let $K \subset M$ be a compact and invariat set without singular points. Then For any $0 < \eta < \delta$, there is $N_\eta > 0$ such that*

$$P_{x,nt}(W^{r,s}_{\delta,t}(x)) \subset W^{r,s}_{\gamma,t}(P_{x,nt}(x)) \text{ and } P_{x,-nt}(W^{r,u}_{\delta,t}(x)) \subset W^{r,u}_{\gamma,t}(P_{x,-nt}(x))$$

for every $n \ge N_\eta$ and every $x \in K$ and every $t > 0$.

Proof. Let us fix $0 < \eta < \delta \inf_{x\in K}\{\|X(x)\|\}$. Let N be given by the previous theorem. Now suppose that there exists $x \in K$ and $t > 0$ such that $P_{x,nt}(W^{r,s}_{\delta,t}(x)) \not\subset W^{r,s}_{\gamma,t}(P_{x,nt}(x))$. Then there is some $y \in W^{r,s}_{\delta,t}(x))$ and $n > N$ satisfying $d(P_{x,nt}(x), P_{x,nt}(y)) > \eta$. Now by the choice of N we must have $d(P_{x,(n+i)t}(x), P_{x,(n+i)t}(y)) > \delta\|X(P_{x,(n+i)t}(x))\|)$ for some $-N \le i \le N$. But this is impossible since $n > N$.

□

Corollary 4.1.5. *Let x be a periodic point with period $\pi(x) = t$. For every $\gamma > 0$ there exists N such that:*

$$P_{x,nt}(W^{r,s}_{\delta,t}(x)) \subset W^{r,s}_{\gamma,t}(P_{x,nt}(x)) \text{ and } P_{x,-nt}(W^{r,u}_{\delta,t}(x)) \subset W^{r,u}_{\gamma,t}(P_{x,-nt}(x))$$

for every $n \ge N$.

Hereafter we work in order to obtain non-trivial pieces of connected R-stable and R-unstable sets for R-expansive flows. We denote $S^r_\delta(x) = \exp(S_{\delta(x)\|X(x)\|}(x) \cap \mathcal{N}(x))$, where $S_\varepsilon(x) = \{v \in T_xM; \|v\| = \varepsilon\}$.

Theorem A. *Let ϕ be a R-expansive flow, $K \subset M$ be a compact invariant set without singularities and suppose that* $\dim(M) > 1$. *Then for every $0 < \gamma < \delta \inf_{x\in K}\{\|X(x)\|\}$ there is some $p \in K$ such that $CW^{r,s}_{\delta,t}(p) \cap S^r_\gamma(p) \neq \emptyset$ or $CW^{r,u}_{\delta,t}(p) \cap S^r_\gamma(p) \neq \emptyset$.*

Proof. Since $\dim(M) > 1$, $N^r_\eta(x)$ is connected. Fix some $x \in K$ and suppose that there exists some $\gamma > 0$ such that $W^{r,u}_{\gamma,t}(y) \cap S^r_\gamma = \emptyset$ for every $y \in K$. Denote $K_0 = \overline{(N^r_\gamma(x))}$. Since K_0 is connected, there exists some $y \in (K_0 \cap S^r_\gamma(x))$ which is not it $W^{r,u}_{\gamma,t}(x)$. This implies that we can take a minimal $m_0 > 0$ such that $diam(P_{x,-m_0t}(K_0)) > \delta\|X(P_{x,m_0t}(x))\|$. Since K_0 is connected, we have that $P_{x,-m_0t}(K_0)$ is also connected and then $P_{x,-m_0t}(x) \cap S^r_\gamma(P_{x,-m_0t}(x)) \neq \emptyset$. Thus we define K_1 to be the closure of the connected component of x in $P_{-m_0t,x}(K_0) \cap N^r_\gamma(x)$. Thus we can repeat the previous steps to find m_1 and a continuum $K_2 \subset P_{x,-m_1t,}(K_1) \cap \overline{(N^r_\gamma(P_{x,-(m_0+m_1)t}(x))}$. Inductivelly, we can find a sequence of times $\{m_k\}$ and a sequence of continuum sets $\{K_k\}$ sucht that the following is valid:

- $P_{x,-m_kt}(K_k) \supset K_{k+1}$
- $K_k \cap S^r_\gamma(P_{x,-m_kt}(x)) \neq \emptyset$
- $P_{x,-nt}(K_k) \subset N^r_\delta(P_{x,-nt}(x))$, if $0 \leq n \leq m_k$

By the compactness of the continuum hyperspace, we can assume that the sequence K_k converges to a continuum K. Now we have that

$$P_{x,(-\sum_{i=0}^k m_i)t}(x) \to p \in K$$

and therefore we have $K \subset W^{r,s}_{\delta,t}(p)$, by the continuity of the holonomy maps. Moreover, since $diam(K_k) \geq \gamma$ we have that $K \cap S^r_\gamma(x) \neq \emptyset$ and this concludes the proof.

□

An immediate consequence of the previous result is the following corollary:

Corollary 4.1.6. *If $x \in M$ is a periodic point, then for any $t, \varepsilon > 0$ we have*

$$CW^{r,s}_{\delta,t}(p) \cap S^r_\gamma(p) \neq \emptyset \text{ or } CW^{r,u}_{\delta,t}(p) \cap S^r_\gamma(p)$$

for every $p \in O(x)$.

It would be great if all points of K actually have both non-trivial connected R-stable and R-stable sets. Next we proceed in order to obtain this, but first we need to introduce the concept of R-stable and R-unstable points. Let us set some notation. Recall from previous chapter that we denoted $B^r(x, t, \varepsilon)$ for the t-ε-R-dynamical centered at x for ϕ. Here we will denote $N^r_t(x, n, \varepsilon)$ for the n-ε-R-dynamical ball for $\{P_{x,nt}\}$ centered at x.

Definition 4.1.7. *We say that $x \in M \setminus Sing(\phi)$ is an R-stable (R-unstable) point of ϕ if for every $t > 0$, the set $\{W^{r,s}_{\varepsilon,t}(x)\}_{\varepsilon>0}$ ($\{W^{r,u}_{\varepsilon,t}(x)\}_{\varepsilon>0}$) is a neighborhood basis for x on $N^r_\delta(x)$. In other words, if for every $\varepsilon > 0$, there is some $\eta > 0$ such that if $y \in N^r_\eta(x)$ and $d(x,y) \le \frac{\eta}{L^t}\|X(x)\|$, then*

$$d(P_{x,nt}(x), P_{x,nt}(y)) \le \varepsilon\|X(P_{x,nt}(x))\|$$

for every $n \ge 0$ ($n \le 0$).

Next theorem is a trivial consequence of the definitions and then we shall omit its proof.

Theorem 4.1.8. *If $\overline{O(x)} \cap Sing(\phi) = \emptyset$, then are equivalent:*

1. *x is an R-stable point.*
2. *$W^{r,s}_{\delta,t}(x)$ is a neighborhood of x on $N^r_\delta(x)$.*
3. *There is some $0 < \varepsilon_0 < \delta$ such that for any $0 < \varepsilon < \varepsilon_0$ and $t > 0$ we have*

$$W^{r,s}_{\varepsilon,t}(x) = N^r_t(x, N_\varepsilon, \varepsilon).$$

Hereafter we will always suppose $x \in \Lambda$, where Λ is a compact invariant set without singularities. An easy corollary of Theorem 4.1.4 is the following proposition.

Proposition 4.1.9. *If for some $t > 0$, we have $y \in W^{r,s}_{\varepsilon,t}(x)$, then $\omega(x) = \omega(y)$.*

Before to prove the proposition, let us make some remarks that will be used on next results. By Theorem 4.1.4, if $y \in W^{r,s}_{\varepsilon,t}(x)$, then $d(P_{x,nt}(x), P_{x,nt}(y)) \to 0$ as $n \to \infty$. In addition, Theorem 2.1.4 implies that $d(\phi_t(x), \phi_t(y)) \to 0$ as $t \to \infty$.

Proof. Now we prove the proposition. Let $z \in \omega(x)$ and suppose that $y \in W^{r,s}_{\varepsilon,t}(x)$. If $t_k \to \infty$ is such that $\phi_{t_k}(x) \to z$, then previous remarks implies that $\phi_{t_k}(y) \to z$ and therefore $z \in \omega(y)$. The contrary inclusion is analogous. □

As a consequence of previous proposition we obtain the following.

Theorem 4.1.10. *Suppose that x is R-stable point which is recurrent. Then x is a periodic point.*

Proof. Suppose that x is an recurrent R-stable point and fix $\eta > 0$ such that $N^r_\eta(x) \subset W^{r,s}_{\varepsilon,t}(x)$. Since x is a recurrent point, we can find a sequence $t_k \to \infty$ such that $\phi_{t_k}(x) \to x$. In particular, if we chose k big enough, we have that $\phi_{t_k}(x) \in B^r_\delta(x)$. Since Theorem 2.1.4 implies that $B^r_\delta(x)$ is contained on the R-flow box of $N^r_\delta(x)$, we find a sequence of times $n_k t \to \infty$ such that $P_{x,n_k t}(x) \to x$.

Let $r \in C_\phi(M)$ be a function given by item 2 of Theorem 2.1.5 such that $0 < r(x) \le \frac{\eta}{4}\|X(x)\|$. Let us fix a sequence n_k with $n_k > N_{r(x)}$ such that $P_{x,n_k t}(x) \in N_{r(x)}(x)$. Then by Theorem 4.1.4 we have that $(P_{x,n_kt}(N^r_\eta(x)) \cap N_{r(x)}(x)) \subset W^{r,s}_{\frac{\eta}{4},t}(P_{x,n_kt}(x))$. Theorem 2.1.5 implies that

$$P_{x,n_kt}(N^r_\eta(x)) \cap N_{r(x)}(x)) \subset N^r_{\frac{\eta}{2}}(x).$$

Now, if we apply again P_{x,n_kt} to $(P_{x,n_kt}(N^r_\eta(x)) \cap N_{r(x)}(x))$, we obtain that

$$P_{x,2n_kt}(P_{x,n_kt}(N^r_\eta(x)) \cap N_{r(x)}(x))) \subset N^r_{\frac{\eta}{2}}(x).$$

Finally, Theorem 4.1.4 implies that

$$\bigcap_{j=1}^{\infty} P_{x,jn_kt}(\overline{N^r_\eta(x)}) = \{z\}$$

and by construction we have that z is periodic for $\{P_{x,nt}\}$. This implies that z is periodic for ϕ and by the previous proposition, we have that $x \in \omega(x) = \omega(z) = O(z)$. This finishes the proof.

□

Theorem 4.1.11. *If ϕ is R-expansive and $x \in M$ such that $\overline{O(x)} \cap Sing(\phi) = \emptyset$. If x is an R-stable point, then there is a neighborhood of x on $N^r_\delta(x)$ formed by R-stable points.*

Proof. To prove this, suppose that x is an R-stable set and fix $0 < 4\varepsilon < \delta$ such that

$$\left(\bigcup_{t\geq 0} \overline{N^r_\varepsilon(\phi_t(x)))}\right) \cap Sing(\phi) = \emptyset$$

Since x is R-stable, then there is some $0 < \eta < \varepsilon$ such that $N^r_\eta(x) \subset W^{r,s}_{\varepsilon,t}(x)$. This imples that if $y \in N^r_\eta(x)$ then $\inf_{t\geq 0}\{\|X(\phi_t(y))\| > A > 0$. Now fix some $\nu > 0$ and set $0 < \gamma \leq \nu A$. Fix some $y \in N^r_\eta(x)$. Theorem 4.1.4 combined with Theorem 2.1.4 implies that we can find some N_η such that

$$d(P_{y,nt}(y), P_{y,nt}(z)) \leq \frac{\gamma}{L^t}$$

for any $z \in B_\eta(x)$ and any $n \geq N_\eta$. Finally, the continuity of the holonomy maps allows us to find some $\mu > 0$ (Theorem 2.1.5) such that if $z \in N^r_\eta(x)$ and $d(z, y) < \mu$, then

$$d(P_{y,nt}(y), P_{y,nt}(z)) \leq \frac{\gamma}{L^t}$$

for $0 \leq n \leq N_\eta$. But this says that $N^r_\mu(y) \subset W^{r,s}_{\nu,t}(y)$ and therefore, y is R-stable.

□

Theorem 4.1.12. *Let ϕ be a R-expansive flow and $K \subset M$ be a compact invariant set without singularities. If $x \in K$ is a R-stable or R-unstable point, then x is periodic.*

Proof. Suppose that $x \in K$ is a R-stable point and let $\delta > 0$ be the R-expansiveness constant of ϕ. Before to continue, let us set some notation. Let $F_x(N^r_\gamma(x))$ be an R-flow box. Suppose that, $A \subset F_x(N^r_\gamma(x))$. Define $A(x)$ to be the set $\{\phi_{t_y}(y)\}$ where $y \in A$ and t_y is the unique t satisfying $|t| \leq \gamma\|X(x)\|$ and $\phi_{t_y}(y) \in N^r_\gamma(x)$. For any γ denote $A^z_\gamma = B_\gamma(z)$.

Claim: Suppose that $z \in \alpha(x)$. Then there are a sequence $n_k \to \infty$ and $\gamma > 0$ such that $P_{x,-n_kt}(x) \to z$ and $A^z_\gamma(P_{x,-n_kt}(x)) \subset W^{r,s}_{\delta,t}(P_{x,-n_kt}(x))$, for every $k > 0$.

If the claiming if false, we can find a subsequence n_k such that

$$P_{x,-n_kt}(x) \subset A^z_{\frac{1}{k}}(P_{x,-n_kt}(x)) \text{ and } A^z_{\frac{1}{k}}(P_{x,-n_kt}(x)) \not\subset W^{r,s}_{\delta,t}(P_{x,-n_kt}(x)).$$

For any $k \geq 1$.

Fix some $\varepsilon > 0$ as in intem (3) of Theorem 4.1.8. Since $A^z_{\frac{1}{k}}(P_{x,-n_kt}(x)) \not\subset W^{r,s}_{\varepsilon,t}(P_{x,-n_kt}(x))$, we can find a point $P_{x,-n_kt}(y_k) \in A^z_{\frac{1}{k}}(P_{x,-n_kt}(x)) \cap \partial W^{r,s}_{\varepsilon,t}(P_{x,-n_kt}(x))$ But this implies:

$$\sup_{n \geq n_k} d(P_{x,nt}(x), P_{x,nt}(y_k)) = d(P_{x,(m_k-n_k)t}(x), P_{x,(m_k-n_k)t}(y_k)) = \varepsilon$$

for some $m_k > 0$. The continutity of $\{P_{x,nt}\}$ implies that $m_k \to \infty$.

Suppose that $P_{x,(m_k-n_k)t}(x) \to x*$ and $P_{x,(m_k-n_k)t}(y_k) \to y*$. We have that $d(x*, y*) = \varepsilon$

But now, we have that for any $i \in \mathbb{Z}$

$$d(P_{x*,it}(x*), P_{x*,it}(y*)) = \lim_{k\to\infty} d(P_{x,(i+m_k-n_k)t}(x), P_{x,(i+m_k-n_k)t}(y_k)) \leq \sup_{n>-n_k} d(P_{x,nt}(x), P_{x,nt}(y_k)) = \varepsilon,$$

since $l + m_k$ is positive if we suppose k big enough. This contradicts R-expansiveness by Theorem 4.1.2 and then the claim is valid.

Now fix $\varepsilon > 0$, $z \in \alpha(x)$, n_k and let γ as in the claiming. Since $A^z_\gamma(P_{x,-n_kt}(x)) \subset W^{r,s}_{\varepsilon,t}(P_{x,-n_kt}(x))$, then $P_{x,n_1t}(y) \in B^r_\varepsilon(x)$, for every $y \in A^z_\gamma(P_{x,-n_kt}(x))$. In particular, any $y \in A^z_\gamma(P_{x,-n_kt}(x))$ satisfies $d(P_{x,(n_1-n_k)t}(y), x) \leq 2\varepsilon$. Since $\phi_{-n_kt}(x) \to z$, this implies that $\phi_{(-n_k+n_1)t}(x) \to x$ and then $x \in \alpha(x)$. Finally, x is periodic due to Theorem 4.1.10

Theorem B. *Let ϕ be a R-expansive flow with expansiveness constant $\delta > 0$ and $K \subset M$ be a non-singular compact invariant set. If Γ does not contains R-stable or R-unstable points, then for any $0 < \varepsilon < \delta$, $t > 0$ and any $x \in K$ we have:*

$$CW^{r,s}_{\varepsilon,t}(x) \neq \{x\} \text{ and } CW^{r,u}_{\varepsilon,t}(x) \neq \{x\}.$$

Proof. The proof is based on the following claiming:

Claim: For every $0 < \varepsilon < \delta$, and $\eta > 0$, there is some $K = K_{\varepsilon,\eta}$ such that

$$N^r_\eta(x) \not\subset N^r_t(x, K, \varepsilon) \text{ and } N^r_\eta(x) \not\subset N^r_{-t}(x, K, \varepsilon)$$

for every $x \in K$.

If the claim is false, we can find $\varepsilon > 0$ and $\eta > 0$ and a sequence of points $x_k \in K$ such that $N^r_\eta(x) \subset N^r_t(x_k, k, \varepsilon)$ for any $k > 0$. Now, if we suppose that $x_k \to x$, then x must an R-stable point of K and this is a contradiction. The case of R-unstable points is analogous and the claim is proved.

Now fix $x \in K$, $0 < \varepsilon < \delta$ and let N_ε be given by Theorem 4.1.3. Let $\eta > 0$ be such that if $d(x, y) \leq \eta$, then $d(P_{x,nt}(x), P_{x,nt}(y)) \leq \varepsilon$ if $|n| \leq N_\varepsilon$. Fix some $n \geq \max\{N_\varepsilon, K_{\varepsilon,\eta}\}$. By the claiming, we have that

$$P_{x,-nt}(N^r_\eta(P_{x,nt}(x)) \not\subset C(N^r_t(x, n, \varepsilon), x).$$

Thus there is some

$$y_0 \in P_{x,-nt}(N^r_\eta(P_{x,nt}(x)) \cap \partial C(N^r_t(x, n, \varepsilon), x).$$

In particular, this implies that for some $0 \leq k \leq n$, we have that

$$d(P_{x,kt}(x), P_{x,kt}(y_0)) = \varepsilon.$$

But now, $k \notin [n - N_\varepsilon, n - 1]$, by the choice of η. Also $k \notin [N_\varepsilon, n - N_\varepsilon]$, otherwise there should exists some $0 \leq j \leq n$ such that $d(P_{x,jt}(x), P_{x,jt}(y_0)) > \delta$ contradicting the $y_0 \in N^r_t(x, n, \varepsilon)$. Thus $k \in [0, N_\varepsilon]$ and therefore $d(x, y_0) > \eta$, by the choice of η.

Finally, we have that for any $n \geq \max\{N_\varepsilon, K_{\varepsilon,\eta}\}$ we have that $C(N^r_t(x, n, \varepsilon), x)$ is a connected set with diameter greater than η. Thus by the compactness of the continuum hyperspace of M we have that the set

$$\bigcap_{n>0} \overline{C(N^r_t(x, n, \varepsilon), x)}$$

is connected set contained on $W^{r,s}_{\varepsilon,t}(x)$ with diameter greater than η. Since the case for the R-unstable sets is analogous, the theorem is proved.

□

The above results tell us that under assumptions of expansiveness, one can always find non-trivial connected R-stable and R-unstable sets for points whose orbit does not accumulate on $Sing(\phi)$. The same is not valid for points approaching arbitrarily singularities. To see this, recall Example 3.2.3. In that case, if $x \in \mathbb{T}^2$ is any regular point for ϕ, then for any $\varepsilon > 0$ the only point whose orbit remains $\varepsilon\|X(\phi_t(x))\|$-close to x for any positive time is x itself. Thus $W^{r,s}_{\varepsilon,t}(x) = W^{r,u}_{\varepsilon,t}(x) = \{x\}$ for any $x \in \mathbb{T}^2 \setminus Sing(\phi)$. Since we are dealing here with a surface flow, it could be that this is a simple pathology due to the low dimension of $\mathbb{T}^2$. But this is not true, as next example shows.

Example 4.1.13. *Let $D = \{x \in \mathbb{R}^2; \|x\| \leq 1\}$ and consider $M = S^1 \times D$ the solid torus. Let ϕ' be a periodic translation flow such that there is $p \in S^1$ such that $\{p\} \times D$ is a global cross-section for ϕ'. If X' is the vector field induced by ϕ', we define a new vector field $X = \rho X'$ where ρ is a real C^∞-function satisfying the following:*

1. *There is a neighborhood U of $\{p\} \times D$ such that $\rho|_{\{p\}\times D} = 0$.*

2. $\rho|_{M\setminus U} = 1$ *and* $0 < \rho(x) < 1$ *for any* $x \in U \setminus (\{p\} \times D)$

3. ρ *decreases as* x *approaches to* $\{p\} \times D$.

This flow ϕ *induced by* X *has a behavior analogous to that on Example 3.2.3, but here the flow is higher dimensional.*

4.2 Topological Entropy of k^*-expansive flows

In this section we apply the techniques developed on previous section to the study of the topological entropy of expansive flows. We are aiming to obtain positive entropy for expansive flows. As well as in the non-singular case, stable and stable sets play an important role here. Let us begin our exposition recalling the definition of Lyapunov stable sets.

Definition 4.2.1. *We say that a compact and invariant set* Λ *is Lyapunov Stable if for any* $\varepsilon > 0$*, there is some* $\delta > 0$ *such that if* $x \in B_\delta(\Lambda)$*, then* $\phi_t(x) \in B_\varepsilon(\Lambda)$*, for every* $t \geq 0$.

Combining the results of the previous section we obtain the following result:

Theorem C. *Let* ϕ *be a R-expansive flow. If there exists a non-singular Lyapunov stable set* $\Gamma \subset M$ *, containing a point with a non-trivial piece of connected local R-unstable set, then* $h(\phi) > 0$.

Proof. Fix $\gamma > 0$ the constant of R-expansiveness of ϕ given by Theorem 4.1.2. Let Γ be a non-singular Lyapunov stable set. Fix $0 < \varepsilon \leq \gamma$ such that $\overline{B_\varepsilon(\Gamma)} \cap Sing(\phi) = \emptyset$

Let $\delta > 0$ be given by the Lyapunov stability of Γ with respect to ε and fix $x \in \Gamma$ such that for some $0 < \eta < \delta$ and $t > 0$, the set $CW^{r,u}_{\eta,t}(x)$ is a non-trivial connected set. Now let $y \in CW^{r,u}_{\eta,t}(x)$ be such that $y \neq x$. Then Theorem 4.1.4 implies that $d(P_{x,-nt}(y), P_{x,-nt}(x)) \to 0$. But Theorem 2.1.4 will imply that in fact $d(\phi_{-t}(x), \phi_{-t}(y)) \to 0$. Now the Lyapunov stability of Γ guarantees that $\phi_t(y) \in B_\varepsilon(\Gamma)$ for any $t \geq 0$.

Last facts imply that

$$\Lambda = \overline{\bigcup_{t\in\mathbb{R}} \phi_t(\overline{CW^{r,u}_{\eta,t}(x)})} \subset \overline{B_\varepsilon(\Gamma)}$$

But since $\overline{B_\varepsilon(\Gamma)}$ does not contain singularities, then $\phi|_\Lambda$ is a R-expansive non-singular flow. In particular, it is BW-expansive and have dimension greater than one, since it contains $O(x)$ and $W^{r,u}_{\eta,t}(x)$. So, we conclude by Theorem 2.5.3 that $h(\phi) > 0$.

□

As a corollary we have the of previous theorem and Theorem B we have the following:

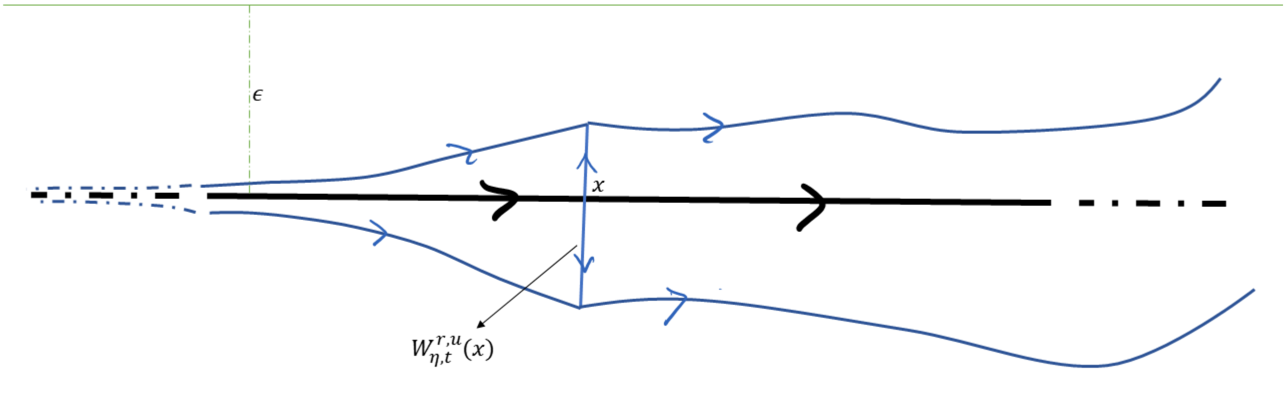

Figure 4.1: Idea of Theorem C

Corollary D. *Let ϕ be a R-expansive flow. If there exists a non-singular Lyapunov stable set $\Gamma \subset M$ without R-stable and R-unstable points, then $h(\phi) > 0$.*

Next we recall the definition of Attractor sets

Definition 4.2.2. *Let Γ be a compact and invariant set. We say that Γ is an attractor if:*

- *$\phi|_\Gamma$ is transitive.*
- *There is a neighborhood U of Γ satisfying $\overline{\phi_t(U)} \subset U$ for any $t > 0$.*
- *$\Gamma = \cap_{t\geq 0}\phi_t(U)$.*

The neighborhood on above definition is called the isolating neighborhood of Γ. We say that Γ is a non-periodic attractor if it is not a periodic orbit.

Theorem 4.2.3. *If $\Gamma \subset M \setminus Sing(\phi)$ is a non-periodic atractor of ϕ, then it does not contain R-stable and R-unstable points.*

Proof. Since Γ is an attractor, then it has some point with dense orbit. In particular, we have that $x \in \omega(x)$. Now, suppose that Γ contains a stable point p. Therefore, there is some sequence $t_k \to \infty$ such that $\phi_{t_k}(x) \to p$. But now Theorem 4.1.11 implies that x is also an R-stable point and therefore Theorem 4.1.12 implies that x is a periodic orbit. But it is a contradiction, since $\overline{O(x)} = \Gamma$.

□

It is easy to show that attractor are examples of Lyapunov stable sets. Therefore as an easy consequence, we can obtain another criterion for positive entropy of R-expansive flows. Precisely, we have:

Theorem E. *Let ϕ be a R-expansive flow. If there exists a non-periodic attractor $\Gamma \subset M \setminus Sing(\phi)$, then $h(\phi) > 0$.*

Once we have established the previous theorem, we can use the results in [Art2] to obtain the following results.

Theorem F. *Let ϕ be a k^*-expansive flow such that the derivative $D_\sigma X : T_\sigma M \to T_\sigma M$ is invertible for any $\sigma \in Sing(\phi)$. If there exists a non-periodic attractor $\Gamma \subset M \setminus Sing(\phi)$, then $h(\phi) > 0$.*

Proof. Let ϕ be a C^r-flow as in the statement. Thus Theorem 3.1.4 implies that ϕ is R-expansive. So the result is a consequence of Theorem E □

Theorem G. *Let ϕ be a k^*-expansive flow such that $Sing(\phi)$ is a hyperbolic set. If there exists a non-periodic attractor $\Gamma \subset M \setminus Sing(\phi)$, then $h(\phi) > 0$.*

Proof. The result is a consequence of Theorem F combined with Theorem 3.1.5.

□

Chapter 5

Pointwise Dynamics

In this chapter we start to study how pointwise dynamics can be used to help us to investigate the implications of expansiveness for the entropy theory of flows. Here we will use only a local version of expansiveness, so we will need to assume other hypothesis on the flows, but this will allow to relax the dimensional hypothesis of Lewowicz's Theorem. Indeed, the techniques developed in this and in the next chapter can be used to obtain positive entropy for flows in dimension one.

5.1 Preliminaries

Recall that in Chapter 1, it was mandatory the phase space to have a non-trivial connected set to obtain Lewowicz's Theorem. Indeed, it was fundamental to find the unstable connected sets and then uniform expansiveness implied the existence of separated sets with many points. In the zero-dimensional case the space may be too poor and then Lewowiz's Theorem is not valid. But we can find examples of zero-dimensional expansive systems with positive entropy. Maybe the most famous one is the shift map introduced on Example 1.1.4. The shift illustrates that expansive systems can have positive entropy, even on zero dimensional spaces. In this example, the phase space is not too poor, since it is a cantor set.

On a previous work derived from my master's thesis, we proved that for a given homeomorphim f and under pointwise dynamical assumptions, one can guarantees that f has a subsystem equivalent to a shift map. In particular, this implies positiveness of topological entropy. Precisely, we proved the following:

Theorem 5.1.1. *[AR] Let f be a homeomorphism. Suppose that there exists $x \in \Omega(f) \setminus Per(f)$ an uniformly-expansive and shadowable point, then there are $n > 0$ and a compact invariant set $Y \subset M$ such that $f^n|_Y$ is conjugated to the full shift. In particular $h(f) > 0$*

On the previous result, we can obtain positiveness of entropy for spaces of any dimension. Hereafter, our goal is to extend the previous result for flows.

5.2 Expansive Points for Flows

In this section we start to study the pointwise dynamics of flows. In particular, we will define a pointwise version of expansiveness. But before begin, let us save some time to explain what is pointwise dynamics. By pointwise dynamics we mean the study of dynamical systems through properties defined in terms of points. To illustrate this concept, let us recall the definition of topological transitivity.

Definition 5.2.1. *A homeomorphism f is said to be topologically transitive if for any pair of non-empty open sets $U, V \subset M$, there is some time $t \in \mathbb{Z}$ such that $f^{-n}(U) \cap V \neq \emptyset$.*

This is a global property in the sense that we need to know information about all opens sets to verify topological transitivity. On the other hand, a classical result result states that topological transitivity is equivalent to the existence of points with dense orbit.

Theorem 5.2.2. *If M has not isolated points, then f is topologically transitive if, and only if, there is some point $x \in M$ such that $\overline{O(x)} = M$.*

Previous result relates transitivity with a pointwise property. This is the coeur of pointwise dynamics: To study the behaviour of dynamical systems assuming the existence of points in the phase space with nice dynamical properties.

Now we begin to investigate aspects of pointwise expansiveness for flows. The first definition of pointwise expansivity is due to Reddy in [R] in the setting of homeomorphisms. He required for any point x in the phase space the existence of a positive number $c(x)$ such that the dynamic ball centered at x and with radius $c(x)$ contains only x. Then we can try to adapt this definition to the time continuous case.

Definition 5.2.3. *A point x is called an expansive point of ϕ if for every $\varepsilon > 0$ there exists $c(x) > 0$ such that if there exists $y \in M$ and a reparametrization h such that $d(\phi^t(x), \phi^{h(t)}(y)) < c(x)$ for every $t \in \mathbb{R}$, then $y = \phi^s(z)$ with $z \in O(x)$ and $|s| \leq \varepsilon$. A flow ϕ is pointwise expansive if every point of x is expansive.*

Suppose ϕ is pointwise expansive. An easy consequence of the definition is that given $\varepsilon > 0$, if we can find $c > 0$ such that $c(x) \geq c$ for every x, then ϕ is expansive. As in the homeomorphism case, $Sing(\phi)$ is finite if ϕ is pointwise expansive. Indeed, if there are infinitely many fixed points, then they must accumulate in some singularity x. Then $c(x)$ must be 0 and ϕ cannot be pointwise expansive.

Another difficult about this definition is to relate pointwise expansivity with expansivity. As in the homeomorphism case, pointwise expansiveness does not implies expansiveness. The following example shows this fact.

Example 5.2.4. *In [CC] B. Carvalho and W. Cordeiro give an example of 2-expansive homeomorphism with the shadowing property which is not expansive. In their example all the points are expansive points. If one consider a suspension flow of f, one will obtain an example of non-expansive flow whose points are expansive.*

In order to define a pointwise kind of expansivity which allows us to recover global expansivity , we proceed in the same way as in [AR] and define uniformly-expansive points.

Definition 5.2.5. *We say that x is an uniformly-expansive point of ϕ if there is a neighborhood U of x with the following property: For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $y, z \in U$ and there exists a reparametrization h satisfying $d(\phi^t(y), \phi^{h(t)}(z)) < \delta$ for every $t \in \mathbb{R}$ then $y = \phi^s(w)$ with $w \in O(z)$ and $|s| < \varepsilon$.*

We denote $Exp(\phi)$ for the set of uniformly-expansive points of ϕ.

Proposition 5.2.6. *$Exp(\phi)$ is an invariant set.*

Proof. Consider $x \in Exp(\phi)$ and let U be its expansivity neighborhood. Fix $\varepsilon > 0$ and let $\delta > 0$ be given by the expansivity and such that $B_\delta(x) \subset U$. If $s \in \mathbb{R}$, let $0 < \eta < \delta$ be such that $d(\phi_{-s}(u), \phi_{-s}(v)) \leq \delta$ if $d(u, v) \leq \eta$.

Then $\phi^s(B_\eta(x))$ is a neighborhood of $\phi^s(x)$. Suppose there are $z, y \in \phi^s(B_\eta(x))$ and $h \in Rep(\phi)$ such that $d(\phi^t(z), \phi^{h(t)}(y)) < \eta$ for every $t \in \mathbb{R}$. Then $\phi^{-s}(z), \phi^{-s}(y) \in B_\eta(x)$ and by the chose of η, we have for any $t \in \mathbb{R}$

$$d(\phi_{-s+t}(y), \phi_{-s+h(t)}(z)) \leq \delta.$$

Thus $\phi_{-s} = \phi_{t_0}(z')$ with $z' \in O(z)$ and $|t_0| < \varepsilon$. Thus $\phi^s(x)$ is an uniformly-expansive point of ϕ with expansivity neighborhood $\phi^s(B_\eta(x))$. □

Next we show that it is possible to recover expansiveness from uniformly-expansive points.

Theorem 5.2.7. *A flow ϕ expansive if, and only if , every point is uniformly-expansive.*

Proof. Obviously, we just need to prove the converse. Suppose that every point of X is uniformly-expansive and fix $\varepsilon > 0$. Since X is compact, we can cover X with a finite number of open sets $U_{x_1}, ..., U_{x_n}$ given by the expansiveness of the points $x_1, ..., x_n$. Set $\delta = \min\{\delta_{x_1}, ..., \delta_{x_n}, \eta\}$, where $\delta(x_i)$ is the expansivity constant of x_i and η is the Lebesgue number of the cover. If there are points $x, y \in X$ and a reparametrization h such that $d(\phi^t(x), \phi^{\rho(t)}(y)) < \delta$ for every t. Then $x, y \in U_{x_i}$ for some i and therefore $y = \phi^t(z)$ with $z \in O(x)$ and $|t| < \varepsilon$. Then we conclude that ϕ is expansive. □

To show that uniformly-expansive points are actually a new concept, we need to show some examples of flows ϕ such that $Exp(\phi) \neq \emptyset$ and $Exp(\phi) \neq M$. We postpone these examples to chapter 6, because these examples will also be under hypothesis of the theorems of that chapter.

5.3 Shadowable Points For Flows

Shadowable points were introduced in [Mor] by C.A. Morales in the setting of homeomorphisms. Later, they were used to define a measurable version of shadowing property. Its flows' version was first considered in [AV]. Next we proceed to defined these points and then we state some basic results which will be used by us on Chapter 6.

Fix $x \in M$. We say that an ε-T-pseudo orbit $S = (x_i, t_i)$ is through x if $x_0 = x$.

Definition 5.3.1. *A point x is a shadowable point of ϕ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that every $(\delta, 1)$-pseudo-orbit through x is ε-shadowed.*

We can also define a pointwise version of strong-shadowing.

Definition 5.3.2. *A point x is a strongly-shadowable point of ϕ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that every $(\delta, 1)$-pseudo-orbit through x is ε-strongly-shadowed.*

In [AV] the authors showed that shadowing property for flows is equivalent to all points be shadowable. It is obvious that the same can be done for strongly-shadowable points. On the other hand, there are examples of flows with shadowable points that do not satisfy shadowing property (See [AV]).

Chapter 6

Entropy of Flows With Uniformly Expansive Points

6.1 Expansiveness, Shadowing and Subshifts

In this section we will study the implications of global expansiveness and shadowing to the entropy of flows. We will see that these two features combined can allow us to find a symbolic subsystem for flows. In other terms, we will be able to obtain a topological horseshoe on the phase space. Our first result deals with global dynamics and furnishes a way to construct subflows that are equivalent to topological horseshoes.

Theorem 6.1.1. *Let ϕ be a non-singular continuous flow. In addition, suppose that ϕ is expansive and has the shadowing property. If $\Omega(\phi) \cap Crit(\phi) \neq \emptyset$, then there is some compact and invariant set $Y \subset M$ such that $\phi|_Y$ is conjugated to a suspension of a subshift.*

We will break the proof of Theorem 6.1.1 into three lemmas to improve the readability.

Lemma 6.1.2. *It is possible to construct Y as in the statement of Theorem 6.1.1.*

Proof. Let us begin choosing $p \in \Omega(\phi) \setminus Per(\phi)$ and fix $\xi > 0$ as in Theorem 2.1.3. For this ξ let $8e > 0$ be the expansivity constant of ϕ. Now let $\{T_1, ..., T_l\}$ and $\{S_1, ..., S_k\}$ be two families of cross sections of time ξ satisfying the following conditions:

1. $S_i \subset T_i^*$ for $i = 1, 2, ..., k$
2. $diam(T_i) < e$ for $i = 1, 2, .., k$
3. $X = \phi^{[-e,0]}(T^+) = \phi^{[0,e]}(T^+) = \phi^{[-e,0]}(S^+) = \phi^{[0,e]}(S^+)$

Consider $\beta = \sup\{t > 0; x \in T^+ \Rightarrow \phi^t(x) \notin T^+\}$ and $0 < 2\rho < \beta$. Let us define the natural projection $P_i : \phi^{[-\rho,\rho]}(T_i) \to T_i$ by $P_i(y) = \phi^t(y) \in T_i$ with $|t| < \rho$. Since each S_i is compact, we can choose $0 < 2\varepsilon < e$ such that $B_\varepsilon(S_i) \in \phi^{[-\rho,\rho]}(T_i^*)$ for $i = 1, 2, .., k$.

For $\varepsilon > 0$, let $T_1 > 0$ and $\delta_1 > 0$ such that every δ_1-T_1-pseudo orbit is ε-strongly-shadowed.

We will use the shadowing property twice in this proof, so let $0 < 2\delta_2 < \delta_1$ and $T_2 > 0$ be such that every δ_2-T_2-pseudo orbit is δ_1-strongly-shadowable. Let $0 < 2\eta < \delta_2$ be such that if $d(x, y) < \eta$ then $d(\phi^t(x), \phi)^t(y)) < \delta_2$ for any $x, y \in X$ and $t \in [-T_2, T_2]$.

Since p is a non-wandering point, there are $x_a \in B_\eta(p)$ and $t_a > T_2$ such that $\phi^{t_a}(x_a) \in B_\eta(p)$. Next, we define the following set:

$$A = \{..., (p, T_2), (\phi^{T_2}(x_a), t_a - T_2), (p, T_2), (\phi^{T_2}(x_a), t_a - T_2), ...\}.$$

We claim that A is a δ_2-T_2-pseudo-orbit. Indeed, by the choice of η we have $d(\phi^{T_2}(p), \phi^{T_2}(x_a)) < \delta_2$ and $d(\phi^{t_a}(x_a), p) < \eta < \delta_2$.

The shadowing property implies the existence of a point $a \in B_{\delta_1}(b)$ which δ_1-shadows A. More precisely, there exists a reparametrization $h \in Rep_{\delta_1}(\phi)$ such that:

- If $t \in [it_0, it_0 + 1]$, then $d(\phi^{h(t)}(a), \phi^{t'}(p)) \leq \delta_1$ with $t' = t - it_0$.
- If $t \in [it_0 + 1, (i + 1)t_0]$, then $d(\phi^{h(t)}(a), \phi^{t'}(x_a) \leq \delta_1$ with $t' = t - (i + 1)t_0$.

Notice that $d(\phi^{h(t)}(a), \phi^{h(t+t_a)}(a)) < 2\delta_1 < e$ for $t \in \mathbb{R}$. Then expansivity implies $\phi^{h(t_a)} \in O(a)$. Thus a is a periodic point and it must be different from p.

Set $\varepsilon' = d(p, O(a))$ and let $0 < 2\delta_3 < \varepsilon'$ and $T_3 > T_2$ be given by the ε'-strong-shadowing .

Now, let $T > T_3$ be such that $d(\phi^T(a), \phi^T(p)) > 8e$ by expansivity.

Let $0 < 2\eta' < \delta_3$ be such that $d(\phi^t(x), \phi^t(y)) < \delta_3$ if $d(y, z) < \eta'$ and $t \in [-T, T]$.

Since x is a non-wandering point, we can choose $x_b \in B_{\eta'}(p)$ such that there exists $t_b > T$ satisfying $\phi^{t_b} \in B_{\eta'}(p)$.

If we repeat the steps to construct a, we can find a periodic point $b \in B_\varepsilon(x)$ different from x and a which satisfies $d(\phi^T(a), \phi^T(b)) > 7e$. We notice that $d(a, b) \leq d(a, p) + d(p, b) \leq \varepsilon$.

Let $\pi(a)$ and $\pi(b)$ be the periods of a and b, respectively.

For each $s \in \Sigma_2 = \{0, 1\}^{\mathbb{Z}}$ we define the sequence $\{(x_i, t_i)\}_{i \in \mathbb{Z}}$ putting $(x_i, t_i) = (a, \pi(a))$ if $s_i = 0$ and $(x_i, t_i) = (b, \pi(b))$ if $s_i = 1$.

It is easy to see that each A_s is an δ_1-T_1-pseudo orbit and therefore there exists a point y_s which ε-shadows A_s. Moreover, each shadow is unique by expansivity. Let us define $W = \cup y_s$ with $s \in \Sigma_2$ and

$$Y = \cup_{t \in \mathbb{R}} \phi^t(W)$$

□

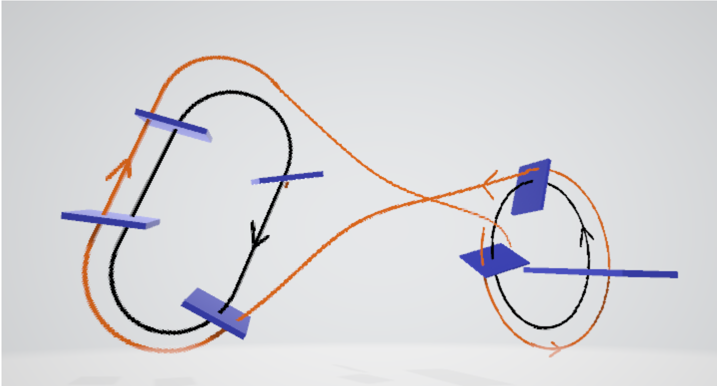

Figure 6.1: Construction of Y

Lemma 6.1.3. *Y is a compact set.*

Proof. To show that Y is closed suppose that $y_n \to y$ with $y_n \in Y$. For each y_n there exists a sequence $s_n \in \Sigma_2$ such that y_n ε-shadows A_{s_n}. To prove our assertion, we have to obtain $s \in \Sigma_2$ and $h \in Rep_\varepsilon(\phi)$ such that y ε-shadows A_s. Since Σ_2 is compact, we can assume that $s_n \to s$. Moreover, each reparametrization of y_n belongs to $Rep_\varepsilon(\phi)$, then $\{h_n\}$ is an equicontinuous sequence. Thus we can assume that $h_n \to h \in Rep_\varepsilon(\phi)$.

Fix $T > 0$ and let $\rho > 0$ such that $d(x, y) < \rho$ implies $d(\phi^t(x), \phi^t(y)) < \varepsilon$. Let us take $y_n \in B_\rho(x)$. Thus the chioce of ρ implies $d(\phi^{h(t)}(y), \phi^{h_n(t)}(y_n)) < \varepsilon$, if n is large enough. This inequality combined with the fact that the sequences s_n have the first entries equal to the first entries of s if n is large, gives us that y ε-shadows A_s until time T. Now, a straigthforward limit calculation proves that y ε-strongly-shadows A_s and therefore Y is closed. □

Lemma 6.1.4. *$\phi|_Y$ is topologically conjugated to a subshift.*

Proof. Since Y is closed we can take new families of cross sections $\mathcal{T} = \{T'_1, ..., T'_{k'}\}$ and $\mathcal{S} = \{S'_1, ..., S'_{k'}\}$ where $T'_i = T_i \cap Y$ and $S'_i = S_i \cap Y$. Notice that if we define $T^{*\prime}_i = T^*_i \cap Y$ then these families satisfies the properties (1), (2) and (3) of the original families.

Consider the point a and let t^a_0 be the smallest $t \geq 0$ such that $\phi^t(a) \in S'^+$ and consider the pair (S^a_0, t^a_0) such that $\phi^{t^a_0}(a) \in S^a_0$ with $S^a_0 = \{S'_1, ..., S'_{k'}\}$. Then define in the same manner the pair (S^a_i, t^a_i) where is the i-th smallest positive time such that $\phi^t(a) \in S'^+$. For negative time one can define these pairs in an analogous way, but using the greatest negative times.

Notice that $\{(S^a_i, t^a_i)\}_{i \in \mathbb{Z}}$ codify the order and the times in which the orbit of a intersect the cross sections in the family $\mathcal{S} = \{S'_1, ..., S'_{k'}\}$. Analogously, we can obtain a similar sequence $\{(S^b_i, t^b_i)\}_{i \in \mathbb{Z}}$ for b. Since the orbits of a and b are periodic, the previous sequences are also periodic. To be more specific, for $z = a, b$ there exists k_z such that $S^z_{i+k_z} = S^z_i$ and $t^z_{i+k_z} = t^z_i + t^z_{imod(k_z)}$.

Now, since the orbit of any point $y \in Y$ ε-shadows the pseudo-orbits obtained concatenating the orbits of a and b, then by the choice of ε, the analogous sequence

$\{(S_i^y, t_i^y)\}$ defined for y is obtained concatenating the sequences $(S_0^a, ..., S_{k_a}^a)$ and $(S_0^b, ..., S_{k_b}^b)$ in the first entry, and with times close to the times of crossing for a and b.

In [BW], Bowen and Walters showed that we can construct a suspension of a subshift of $\Sigma_{\mathcal{T}}$ for which $\psi = \phi|_Y$ is a factor. To conclude our proof we will brieflyl describe the construction used in their proof and discuss the main difference with our case. (For more precise details see [BW])

The family $\mathcal{S}$ codify the path described by orbits of ψ in the following way: Since any point $y \in Y$ needs to cross some $S_0^y \in \mathcal{S}$ at most in time ξ, we construct for y the sequence $\{(S_i^y, t_i^y)\}$ as above. The desired subshift is defined as the set of Σ_ψ of the sequences s of $\Sigma_{\mathcal{S}}$ for which there exists a point y_s which crosses the cross sections of $\mathcal{S}$ in the order defined by s and the roof function r of the suspension is given at each point as the time spent to cross the cross sections. Then the factor map identify a point $s \in \Sigma_\psi^r$ with of a point $y \in Y$ in a way that s represents the order and the times such that y crosses the sections in $\mathcal{T}$. In their case, there is not a reason for two different orbits of Σ_ψ^r be related to different orbits of Y. The same does not occur in our case. Indeed, by the choice of ε the sequences $\{(S_i^y, t_i^y)\}$ are in a one-to-one correspondence with the orbits of ψ. So the factor map defined in [BW] is a homeomorphism in our case.

□

6.2 Expansive Points and The Entropy of Non-Singular Flows

In this section we prove Theorem H. After we have proved Theorem 6.1.1 in the last section, the proof presented here is easier to understand. This is because much of the construction performed here is similar to that on the previous section. The main difference here is that now shadowing and expansiveness are of local matter. Precisely, we prove the following:

Theorem H. *Let ϕ be a non-singular flow. If there is some point $x \in \Omega(\phi) \setminus Crit(\phi)$ uniformly-expansive and shadowable, then $h(\phi) > 0$.*

Proof. The proof will be divided in two steps.

Step 1: Constructing an invariant set Y.

Let p be under the hypothesis of the theorem and let U be its expansivity neighborhood. Fix $0 < \kappa < 1$ and let $8e > 0$ be an expansivity constant of p with respect to κ and such that $B_{8e}(p) \subset U$.

Let $0 < 2\varepsilon < e$ be given by the e-shadowing though p. In [AV] the authors proved that any point in $B_\varepsilon(p)$ is $2e$-shadowable. We will use the shadowing property twice in this proof, so let $0 < 2\delta < \varepsilon$ be such that every δ-pseudo orbit trough p is ε-shadowable.

Let $0 < 2\eta < \delta$ be such that if $d(x,y) < \eta$ then $d(\phi^t(x), \phi^t(y)) < \delta$ for any $x, y \in X$ and $t \in [-1,1]$.

Since p is a non-wandering point, there are $x_a \in B_\eta(p)$ and $t_a > 1$ such that $\phi^{t_0}(x_a) \in B_\delta(p)$. Next, we define the following set:

$$A = \{..., (p,1), (\phi^1(x_a), t_a - 1), (p,1), (\phi^1(x_a), t_a - 1), ...\}.$$

We claim that A is a δ-1-pseudo-orbit through x. Indeed, by the choice of η we have $d(\phi^1(p), \phi^1(x_a)) < \delta$ and $d(\phi^{t_a}(x_a), p) < \eta < \delta$

The shadowing property through p implies the existence of a point $a \in B_\varepsilon(b)$ which ε-shadows A. More precisely, there exists a reparametrization $h \in Rep(\phi)$ such that:

- If $t \in [it_0, it_0 + 1]$, then $d(\phi^{h(t)}(a), \phi^{t'}(p)) \leq \delta$ with $t' = t - it_0$.
- If $t \in [it_0 + 1, (i+1)t_0]$, then $d(\phi^{h(t)}(a), \phi^{t'}(x_a) \leq \delta$ with $t' = t - (i+1)t_0$.

Notice that $d(\phi^{h(t)}(a), \phi^{h(t+t_a)}(a)) < 2\varepsilon < e$ for $t \in \mathbb{R}$. Thus expansitivty implies that a is a periodic point and it must be different from p.

Now, let T be such that $d(\phi^T(a), \phi^T(p)) > 8e$. Set $\varepsilon' = d(p, O(a))$ and let $0 < 2\delta' < \varepsilon$ be given by the ε'-shadowing through p. Fix $0 < 2\eta' < \delta'$ be such if $d(x,y) < \eta'$ then $d(\phi^t(x), \phi^t(y)) < \delta'$ for every $x, y \in X$ and every $t \in [-1,1]$.

Let $0 < \eta' < \eta$ be such that $d(\phi^t(x), \phi^t(y)) < \delta$ if $d(y,z) < \eta'$ and $t \in [-T, T]$.

Since x is a non-wandering point, we can choose $x_b \in B_{\eta'}(p)$ such that there exists $t_b > T$ satisfying $\phi^{t_b} \in B_{\eta'}(p)$.

If we repeat the steps to construct a, we can find a periodic point $b \in B_\varepsilon(x)$ different form x and a satisfying $d(\phi^T(a), \phi^T(b)) > 7e$. We notice that $d(a,b) \leq d(a,p) + d(p,b) \leq \varepsilon$.

Let $\pi(a)$ and $\pi(b)$ be the períods of a and b, respectively.

For each $s \in \Sigma_2 = \{0,1\}^{\mathbb{Z}}$ we define the sequence $\{(x_i, t_i)\}_{i \in \mathbb{Z}}$ putting $(x_i, t_i) = (a, \pi(a))$ if $s_i = 0$ and $(x_i, t_i) = (b, \pi(b))$ if $s_i = 1$.

It is easy to see that each A_s is an ε-1-pseudo orbit trough a or b and therefore there exists a point y_s which $2e$-shadows A_s. Moreover, each shadow is unique by expansivity. Let us define $W = \cup_{s \in \Sigma_2} y_s$ and $Y = \cup_{t \in \mathbb{R}} \phi^t(W)$

Notice that although global shadowing property is equivalent to global strongly shadowing property in the non-singular case(see [K2]), we cannot obtain directly the same for their pointwise versions. This implies that the set Y is not necessarily closed. In addition, it is easy to see that Y is an invariant subset and $\phi|_Y$ is expansive, but we cannot guarantee that the same for $\phi|_{\overline{Y}}$. This is because the set of uniformly expansive points is not necessarily closed. Thus we cannot use Theorem 6.1.1 to conclude the proof. Instead of this, we compute the entropy of ϕ directly.

Step 2: Estimating the Entropy of ϕ.

We begin considering the set $\overline{Y}$ which is a compact ϕ-invariant set. Hence by Theorem 2.1.3 we can obtain a finite family of cross sections $\mathcal{T}$ of diameter at most e which "generates" $\overline{Y}$. Let us also choose $\beta > 0$ as in the remark of that theorem.

Let us make some considerations on the periodic pseudo-orbits A_s. By a periodic pseudo-orbits of period i, we mean the pseudo-orbits A_s for which there exists $i \geq 0$ such that $s_{n+i} = s_n$ for every $n \in \mathbb{Z}$. Notice that the same argument used to prove that a and b are periodic points in the construction of Y can be used to prove that each y_s, e-shadow of A_s is periodic, if A_s is periodic. For each n denote $B_n = \{s_1^n, ..., s_{2^n}^n\}$ the set of 2^n distinct periodic sequences of period n in Σ_2. We claim that its respective $y_{s_1^n}, ..., y_{s_{2^n}}^n$ shadows are also distinct. Indeed, let $s, s' \in B_n$ be two distinct sequences. Consider i_0 the minimal $i \geq 0$ such that $s_i \neq s_i'$. Let (x_i) and (x_i') be the sequences in $\{O(a), O(b)\}^{\mathbb{Z}}$ corresponding to order in which y_s and $y_{s'}$ shadows the orbits of a and b. We have that $x_i = x_i'$ for $0 \leq i \leq i_0$ and $x_{i_0} = x_{i_0}'$. Since there exists a point $a' \in O(a)$ and a point in $b' \in O(b)$ which are at least $7e$ apart, then $y_s \neq y_{s'}$. Otherwise, y_s $2e$-shadow simultaneously $O(a)$ and $O(b)$. Set $t_n = \max\{\pi(y_1^n), ..., \pi(y_{2^n}^n)\}$.

Claim: $t_n \to \infty$ as $n \to \infty$.

Indeed, to prove the claim we consider in B_n a sequence y_s such its corresponding sequence $s \in \Sigma_2$ satisfies $s_i \neq s_{i+1}$ for every $i \in \mathbb{Z}$. Then when y_s is $2e$-shadowing $O(a)$ it needs to cross some cross section near to a'. The same occurs when y_s shadows $O(b)$. Thus y_s spends at least time β to stop shadowing $O(a)$ and begins to shadow $O(b)$. Then $\pi(y_s) \geq n\beta$. Thus t_n converges monotonically to infinity.

Let us define a reparametrization h setting $h(t_n) = n$, $h(-t_n) = -n$, $h(0) = 0$ and mapping linearly the intervals $[t_n, t_{n+1}]$ in $[n, n+1]$. Consider the flow $\psi^t = \phi^{h(t)}$. For each n the expansiveness of ϕ in Y gives us that the set $B_n = \{y_1^n, ..., y_{2^n}^n\}$ is t_n-α-separated for if α is small enough. Thus B_n is a n-α-separated set of ψ. Hence

$$h(\psi) \geq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \# B_n = \log 2 > 0.$$

Finally, ψ is a time change of ϕ and its entropy does not vanishes, then we can conclude the same for ϕ by Theorem 2.4.6.

□

Example 6.2.1. *In order to obtain a non-trivial example of flows for which previous the theorem applies, let us consider a linear Anosov diffeomorphism f of $\mathcal{T}^2$. Then we can blow up the fixed point of f into a closed disc D and extend f to D as the identity map, obtaining a homeomorphism $\overline{f}$. If we consider the suspension flow of $\overline{f}$, then the suspension of all non-wandering points away from the disc D are under the hypothesis of the previous Theorem. The same is not true for the suspension of the points in D.*

6.3 Expansive Points and The Entropy of Singular Flows

This section is devoted to prove a version of Theorem H for singular flows. Since we are now in the non-singular scenario, only shadowableness is not enough to obtain the same result. Indeed, we need a stronger assumption on shadowing. Precisely, we want to prove the following:

Theorem I. *Let ϕ be a continuous flow. Suppose there is some point $x \in \Omega(\phi) \setminus Crit(\phi)$ uniformly expansive and strongly shadowable, then there is some compact and ϕ-invaritant set $Y \subset M$ such that $\phi|_Y$ is semiconjugated to a suspension of subshift with positive entropy.*

The main difference here is that strong shadowing allows us to construct Y as closed non-singular set. Then the conclusion will be a consequence of Theorem 6.1.1.

Proof. In order to establish this result, we will proceed in a very similar way as in Theorem H. We begin constructing an invariant set. Let p be a non-critical, non-wandering, shadowable and uniformly-expansive point of ϕ. Let U be the expansiveness neighborhood of p. Since in the previous theorem we did not use the fact that ϕ is non-singular in the construction of Y, we can do exactly the same construction. So let $Y \subset X$ obtained exactly as in Theorem H. The only difference here is that we use the strong-shadowableness of p. Then every reparametrization related to the points y_s are in $Rep_{2e}(\phi)$. We will see that this fact allows us to construct Y "away" from singularities.

Claim: Y is a closed set.

To show that Y is closed suppose that $y_n \to y$ with $y_n \in Y$. For each y_n there exists a sequence $s_n \in \Sigma_2$ such that y_n $2e$-shadows A_{s_n}. To prove our assertion, we have to obtain $s \in \Sigma_2$ and $h \in Rep_{2e}(\phi)$ such that y e-shadows A_s. Since Σ_2 is compact, we can assume that $s_n \to s$. Moreover, each reparametrization of y_n belongs to $Rep_{2e}(\phi)$, then $\{h_n\}$ is an equicontinuous sequence. Thus we can assume that $h_n \to h \in Rep_{2e}(\phi)$.

Fix $T > 0$ and let $\rho > 0$ such that $d(x, y) < \rho$ implies $d(\phi^t(x), \phi^t(y)) < e$. Let us take $y_n \in B_\rho(x)$. Thus the chioce of ρ implies $d(\phi^{h(t)}(y), \phi^{h_n(t)}(y_n)) < e$, if n is large enough. This inequality combined with the fact that the sequences s_n have the first entries equal to the first entries of s if n is large, gives us that y $2e$-shadows A_s up tol time T. Now, a straigthfoward limit computation proves that y $2e$-strongly-shadows A_s and therefore Y is closed.

First, notice that the previous construction implies that $Y \cap Sing(\phi|_Y) = \emptyset$. Hence $\psi = \phi|_Y$ is an expansive non-singular flow with the shadowing property. Thus theorem 6.1.1 gives us the conjugacy. Furthermore, for ψ we are under the hypothesis of Theorem H. Then we know that $h(\phi) > 0$.

□

Example 6.3.1. *To obtain a non-trivial example of flow whose previous theorem applies, we consider the flow $\phi_t : M_r \to M_r$ of Example 6.2.1 and let X be its generated vector field. The* *phase space of* ϕ *contains a invariant solid torus* T *such that* $\phi|_T$ *is a translation. Now fix* $p \in Int(T)$*, and let* $\rho : M_r \to \mathbb{R}$ *be a smooth function such that:*

- $\rho|_{M_r \setminus B_\varepsilon(p)} = 1$ *for some neighborhood* $B_\varepsilon(p) \subset T$*.*
- $0 \leq \rho|_{B_\varepsilon(p)} \leq 1$.
- $\rho(x) = 0$ *if, and only if* $x = p$*.*

Now let ψ *be the flow induced by the vector field* ρX*. Thus the only fixed point of* ψ *is* p *and* $\psi|_{M_r \setminus T} = \phi$*. Thus any interior point of* $M_r \setminus T$ *is under the hypothesis of the previous theorem.*

Part II

Entropy of Expansive Actions

Chapter 7

Finitely Generated Expansive Group Actions

This chapter begins the second part of this thesis, which is devoted to the study of dynamical systems that are, in some sense, generalizations of discrete-time and continuous-time systems. To be specific, we will be interested in understand the behaviour of groups actions.

7.1 Preliminaries

We start by discussing some ideas about what is exactly a dynamical system. Suppose we are in the discrete-time setting and the systems in question is some homeomorphism f. As we have seen on Chapter 1, the time here is represented by the iterates of f. Other interesting fact is that if one wants to define anything that depends on time, then the future states need depend directly on the past states. This is perfectly captured by the fact that we are taking iterates of f. Indeed, if we start at some point x on the phase space, then to know where the point is after two units of time, we need compute twice the image of x. This says that we cannot achieve the state of x two units of time in the future if we do not compute the state of x one unity of time first.

If we are considering a flow ϕ, this concept is slightly more complicated. Indeed, when time can take any real value, the concept of iterate is lost. On the other hand, the key property that translates the concept of dependence on times is the group property of flows, i.e. $\phi_{t+s}(x) = \phi_t \circ \phi_s(x)$. Actually, this property is satisfied in discrete-time systems for the set of iterations $\{f^n\}$. Last points make us realize that what is behind of the concept of time dependence is this group-like property.

Thus in order to generalize the concept of dynamical systems we need some group G, a family of transformations indexed for the elements of G and satisfying a group property similar to that of flows. This is an informal definition of group action.

Let us now define group actions precisely. Let G be a topological group, i.e. a group with the structure of topological space and such that the group operation is continuous. Let M be some topological space.

Definition 7.1.1. *A group action of G on M is a map $\Phi : G \times M \to M$ satisfying the following conditions:*

- $\Phi(e, x) = x$ *for any $x \in M$, where e denotes the identity element of G.*
- $\Phi(gh, x) = \Phi(g, \Phi(h, x))$ *for any $g, h \in G$ and $x \in M$*

We will denote $\Phi_g : M \to M$ for the map induced by the action, if we fix the element g. It is straightforward to check that the group property implies that the maps Φ_g are homeomorphisms for any $g \in G$. If M is a smooth manifold, we say that Φ is C^r if its induced homeomorphisms are in fact C^r diffeomorphisms. We denote $\mathcal{A}^r(G, M)$ for the set of C^r-actions of G on M.

Another way to describe a C^r-group action is to consider the space $Diff^r(M)$ of C^r diffeormorphisms on M (or the set $H(M)$ of homeomorphisms of M if Φ is C^0). The set $Diff^r(M)$ is a group whose operation is the composition of maps. Thus the action Φ can be seen as a group homomorphism $\varphi : G \to Diff^r(M)$, where $\Phi(g) = \Phi_g$. In this text, we will often denote gx for $\Phi(g, x)$ when there is not risk of confusion.

The orbit of the point $x \in M$ under the action Φ is the set

$$O_\Phi(x) = \{gx; g \in G\}$$

We shall omit the subscript on above definition when there is not confusion. Let us fix $x \in M$. The isotropy group of x is the set $G_x \subset G$ defined by $G_x = \{g \in G; gx = x\}$. It is easy to show that G_x is in fact a subgroup of G. Note that if $G = \mathbb{Z}$ or $G = \mathbb{R}$, then a point is periodic if, and only if, $G_x \neq \{e\}$. When considering general groups though , there is not a precise definition for periodic orbits and periods. Maybe the closest concept related to periodic orbits is the concept of compact orbit, but in this case there is not a well defined concept of period.

In the aforementioned sense, group actions are generalizations of the classical dynamical systems. For group actions the role of time is played by the group. Previous fact brings several difficulties, making, in particular, the study of actions strongly dependent on the kind of group under consideration. To illustrate this, for actions of $\mathbb{Z}$ and $\mathbb{R}$ the meaning of future and past is well defined. However, if $G = \mathbb{R}^2$ we can go to infinity in infinite distinct ways, so we can not define α-limits or ω-limits in the same way for these actions. Other difference is that the structure of the orbits is entirely dependent on G. For instance, compact groups has only compact orbits, non-compact groups can have both compact and non compact orbits and for finite groups there is not asymptotic behaviour for the orbits. Because of this, we need to impose some assumptions on the structure of G to obtain results.

One of the central subjects of this thesis is the expansiveness. Hereafter we will discuss the definition of expansiveness for group actions. The first fact we need to be careful about, is that the definition of expansiveness strongly depends of the kind of group acting on M. To illustrate this, recall that the definition of expansiveness for actions of $\mathbb{Z}$ and $\mathbb{R}$ are totally distinct. The main reason for this distinction is that $\{e\}$ is an isolated set for $G = \mathbb{Z}$, but the same is not true if $G = \mathbb{R}$. Actually, this distinction forbids one to just extend the same definition of expansiveness for actions of $\mathbb{Z}$ to actions of $\mathbb{R}$. In the work [BDS], the authors proposed a definition of expansiveness that quite surprisingly a priori could be used for actions of any topological group. However, this definition is only suitable for actions of finitely generated groups and therefore their results hold only in this context. Because of this distinction, we will divide our study of expansiveness in two scenarios. Namely, actions of finitely generated groups and actions of connected Lie groups. In the next sections we will treat the former and in the subsequent chapter we will treat the case of actions of connected Lie groups.

7.2 Expansive Actions of Finitely Generated Groups

In this section we will study expansive actions of finitely generate groups. We will see some examples, give some characterizations and study properties of topological entropy and the centralizer of such actions. Through this section Φ will denote an action of a finitely generated group G.

The concept of expansiveness for finitely generated actions is exactly the same as that for actions of $\mathbb{Z}$

Definition 7.2.1 ([Hur],[BDS]). *A group action Φ is expansive if there is some $e > 0$ such that if $x \neq y$ there is some $g \in G$ such that $d(gx, gy) > e$.*

We begin our exposition of such systems with the following question:

"What do we need to guarantee that an action is expansive?"

Clearly, any expansive homeomorphism is an expansive $\mathbb{Z}$-action. Next result gives us an easy criterion to construct more examples.

Proposition 7.2.2. *Suppose that there exists an element $g \in G$ such that its induced diffeomorphism f_g is expansive, then Φ is expansive.*

Proof. Let $g \in G$ and suppose that f_g is expansive. Let $e > 0$ be its expansiveness constant. If we take $x \neq y$ in M then there exists n such that

$$d(g^n x, g^n y) = d(f_g^n(x), f_g^n(y)) > e$$

and therefore e is an expansive constant for Φ. □

By the previous proposition, if one finds an expansive element $g \in G$ then ϕ is expansive. A natural question is: "Do a dynamical property of an action implies a similar property for its generators?". Next example shows that this answer is negative for expansivity.

Example 7.2.3. *We begin defining two diffeomorphims of the torus $\mathbb{T}^2$. Let f be a translation of the torus $\mathbb{T}^2$. Let T be a linear Anosov map on $\mathbb{T}^2$ and let p be its fixed point. Blow up p in to a small disc D and define f on $\mathbb{T}^2$ as a C^1 extension of f to D as the identity map on D. It is clear that f and g are not expansive. Le Φ be the action on $\mathbb{T}^2$ of the group generated by f and g.*

We claim that ϕ is expansive. Notice that there exists $e > 0$ such that any two distinct points in $\mathbb{T}^2 \setminus D$ are e apart at some time under the action of g. So we just need to consider the case when $x, y \in D$. To do that just notice that there exists $n > 0$ such that at least one of $f^n(x)$ and $f^n(y)$ is outside of D. Now we can apply g until $f^n(x)$ and $f^n(y)$ be e-apart. This proves that e is an expansive constant for Φ.

In [Hur] S. Hurder studied expansive actions induced by circle homeomorphisms. Next example is one of these actions and illustrates that expansiveness can be obtainned in a way that none of its induced homeomorphisms is expansive.

Example 7.2.4 ([Hur])**.** *Let us consider the homeomorphisms $f_1, f_2 : S^1 \to S^1$ such that f_1 is a irrational rotation and f_2 is a morse-smale homeomorphism with exactely two fixed points, a source p_1 and a sink p_2. Now let Φ be the action on S^1 generated by f_1 and f_2. It is easy to see that Φ is exapansive. Indeed, let $0 < e < \frac{diam(S^1)}{4}$. Notice p_1 and p_2 divide the circle in two distinct connected arcs and if two points x, y are in distinct arcs, then they will be e-apart at some time by the iteration of f_2 or f_2^{-1}. If x and y are in the same arc, then we can apply f_1 on x and y until they belong to distinct arcs. Thus we just need to apply f_2 or f_2^{-1} some times to see the separation. In this example, none of the homeomorphisms induced by Φ can be expansive since they are define on the circle.*

It would be good if one obtain an criterion for characterize the expansiveness of actions. Since we cannot achieve this through the expansiveness of the generated homeomorphisms, we need to find answers for this question with other approach.

Next result is a characterization due to A. Berzanouni, M. S. Divandar and E. Shah. Before to state it let us first recall the concept of syndeticness. If G is a topological group, we say that a subset of $H \subset G$ is *syndetic* if there is some compact set $K \subset G$ such that $KH = G$.

Proposition 7.2.5 ([BDS])**.** *An action Φ of a finitely generated group G on M is expansive if, and only if, the restriction of Φ to any syndetic subgroup of G is expansive.*

Proof. If the the restriction of Φ to any non-trivial subgroup of G is expansive, then it is obvious that Φ is expansive.

Now suppose that Φ is expansive and let H be a syndetic subgroup of G. Since G is discrete, then any compact subset of G is finite. Then there is a finite set $K \subset G$ such that $KH = G$. Fix $0 < c \leq e$, where e is the expansiveness constant of Φ. Since K is finite, we can find $\delta > 0$ such that if $d(x, y) \leq \delta$, then $d(gx, gy) \leq c$ for any $g \in K$. Now if Φ_H is not expansive we can find a pair of distinct points $x, y \in M$ such that $d(hx, hy) \leq \delta$ for every $h \in H$. But this implies that $d(ghx, ghy) \leq c$ for any $g \in K$ and any $h \in H$. This contradicts the expansiveness of Φ, since $c \leq e$.

□

In what follows we will proceed on distinct way to characterize expansive actions in terms of time varying homeomorphisms or non-autonomous discrete-time dynamical systems. In the sequel we define this concept precisely.

Definition 7.2.6. *Let $\mathcal{F} = \{f_i\}_{n\geq 0}$ be a sequence of self-homeomorphisms of some compact metric space X. The time varying homeomorphism (TVH for short) generated by $\mathcal{F}$ is the sequence of homeomorphims $F = \{F_n\}_{n\in\mathbb{Z}}$ satisfying:*

- $F_0 = Id_X$.
- $F_n = f_n \circ \cdots \circ f_1$.
- $F_{-n} = f_n^{-1} \circ \cdots \circ f_1^{-1}$

We say that F is expansive, if there exists $e > 0$ such that for any pair of distinct point $x, y \in M$ one can find $n \in \mathbb{Z}$ such that $d(F_n(x), F_n(y)) > e$.

TVH's are often called non-autonomous discrete-time systems on the literature. Let G a be finitely generated group acting by homeomorphisms on a compact metric space X. Let $K = \{g_1, ..., g_i\}$ be a set of generators of G. Consider the sequence space $K^{\mathbb{N}}$ of sequences in K and fix $s \in K^{\mathbb{N}}$. Then each coordinate s_i of s is an element g_{s_i} of K and it has an associated homemorphism f_{s_i}. Consider $\mathcal{F}_s = \{f_{s_i}\}$ the sequence of homeomorphisms associated to s. It generates a TVH F_s. Our next result characterizes an action Φ its generated time varying maps.

Theorem 7.2.7. *A group action Φ is expansive if, and only if, some of its generated TVH is expansive.*

Proof. If some of the time varying maps generated by Φ is expansive then it is obvious that Φ is expansive. Conversely, suppose that Φ is expansive and let K be a generator of G. Consider any sequence s of $K^{\mathbb{N}}$ containing the homeomorphisms induced by all finite combinations of elements of K. Then it is clear that F_s is expansive, since for any $g \in G$ there is some $n \in \mathbb{Z}$ such that F^n is the homeomorphism induced by g. □

Next we will study the symmetries of expansive actions, which is one of the great questions on dynamical systems theory. Indeed, the study of the the centralizers of diffeormorphisms and vector fields is an important research topic on dynamical systems theory. To motivate the problem we will treat here, we will describe it algebraically. Recall that an action of G on M can be seen as group homomorphism $\rho : G \to Diff^r(M)$. Since the image of a group homomorphism is always a subgroup of its range, then another way to see a group action is as a subgroup of the group $Diff^r(M)$. In this sense a group action has a algebraic flavor.

One of the fundamental questions of group theory is to study the symmetries of elements and subgroups. Let us precise it. Let $g \in G$, we say that h commutes with g if

$gh = hg$. More generally, the set of elements of G which commutes with all the elements of some subset $K \subset G$ is called the centralizer of K on G (or the set of symmetries of K on G) and it is denote by $C(K)$.

If we now return to the algebraic meaning of group actions, then one could ask what are the subgroups of $Diff^r(M)$ which commutes with the subgroup $\rho(G)$.

Now let us come back and translate above discussion to our first concept of group actions.

Definition 7.2.8. *We say that two actions Φ and Ψ of G on M commute if $\Phi_g \circ \Psi_h = \Psi_h \circ \Phi_g$, for any $g, h \in G$.*

Let Φ be an action of G on M, we define the C^r centralizer of Φ to be the set:

$$C^r(\Phi) = \{\Psi \in \mathcal{A}^r(G, M); \Psi \text{ commutes with } \Phi\}$$

We denote d_r for the C^r-distance on the space $\mathcal{A}^r(G, M)$, precisely, if K is a finite generator of G, then

$$d_r(\Phi, \Psi) = \max_{g_i \in K} \sup_{x \in M} \{d(\Phi_{g_i}(x), \Psi_{g_i}(x))\}$$

The next definition is a generalization of the concept of discrete centralizer for homeomorphisms in [W].

Definition 7.2.9. *We say that an action Φ has discrete C^R-centralizer if $C^r(\Phi)$ is a discrete subset of $\mathcal{A}^r(G, M)$ on the C^r-topology.*

Now we have:

Theorem J. *The centralizer $C^0(\Phi)$ of an expansive C^0-action Φ is a discrete subset of $\mathcal{A}^0(G, M)$.*

Proof. Let Φ be an expansive action of G on M and let $\Phi \in C^0(\Phi)$. Let $e > 0$ be the expansiveness constant of Φ. Suppose that $\Psi' \in \mathcal{A}^0(G, M)$ and $d_0(\Psi', \Psi) < e$. Here d_0 is the C^0-distance on $\mathcal{A}^0(G, M)$. Fix $g \in G$ and chose some $h \in G$. Then we have that

$$d(\Phi_h(\Psi_g(x)), \Phi_h(\Psi'_g(x))) = d(\Psi_g(\Phi_h(x)), \Psi'_g(\Phi_h(x))) \leq e.$$

Notice that previous inequality is true for any $h \in G$ and then the expansiveness of Φ implies that $\Psi_g(x) = \Psi'_g(x)$ for any $x \in M$ and any $g \in G$. Thus we have that $\Psi = \Psi'$. □

In the next chapter we will prove a similar result for actions of connected Lie groups and we will see that in that case the problem is highly delicate.

7.3 Topological Entropy of Expansive Actions of Finitely Generated Groups

In this section we will study the geometric entropy for expansive actions of finitely generated groups. In particular we will prove a version of Lewowicz Theorem for these systems.

We will begin defining geometric entropy for finitely generated group actions. Here we will follow the definition of geometric entropy given by Bis in [B]. To start, let us consider a finitely generated group G and let G_1 be a set of generators of G.

For each $n \geq 0$ we define G_n to be the set of elements of G which are combinations of at most n elements of G_1. We say that x and y are n-ε-G_1-separated by the action Φ if there is some $g \in G_n$ such that $d(gx, gy) \geq \varepsilon$. If $K \subset M$, we say that $E \subset P$ is a n-ε-separated subset of P if any pair of distinct points are n-ε-G_1-separated by φ.

Let $S(n, \varepsilon, G_1, P)$ denote the maximal cardinality of an n-ε-separated subset of P.

Definition 7.3.1. *The geometric entropy $h(\varphi, G_1, P)$ of φ on P with respect to G_1 is the number:*

$$h(\Phi, G_1, P) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S(n, \varepsilon, G_1, P)$$

And the topological entropy of Φ with respect to G_1 is the number $h(\phi, G_1) = h(\Phi, G_1, M)$.

We would like to point out some distinctions between geometric entropy and the classical topological entropy for homeomorphisms. The main difference between both definition is that for topological entropy of homeomorphsims its mandatory that time goes to infinity in future direction. As we have seen on Section 7.1, the concept of future is not well defined on more general groups. This distinction makes us realize that the classical topological entropy is not suitable to treat the case of entropy of actions of more general groups. An interesting fact about geometric entropy is the following result:

Theorem 7.3.2 ([B])**.** *Let Φ be a finitely generated group action. If $h(\Phi, G_1) > 0$, then $h(\Phi, G_1') > 0$ for any other finite generator G_1'.*

Since it does not matter what direction on time we are taking to compute geometric entropy, we do not need to look for the existence of connected unstable sets to obtain a version of Lewowicz Theorem for geometric entropy. Indeed, the only main ingredient we need is uniform expansiveness.

Theorem 7.3.3. *[BDS] Φ is an expansive action if, and only if, for every $\delta > 0$ there is some $n \geq 0$ such that if $d(x, y) \geq \delta$, then there is some $g \in G_n$ such that $d(gx, gy) > e$.*

Proof. Suppose Φ is expansive. If we can find $\delta > 0$ and sequences $x_n, y_n \in M$ and sequence $m_n \to \infty$ such that $d(x_n, y_n) \geq \delta$ and $d(gx_n, gy_n) \leq e$ for any $g \in G_{m_n}$. By compactness of M we can assume that $x_n \to x$, $y_n \to y$. Now for each $g \in G$ we have that

$$d(gx, gy) \leq d(gx, gx_n) + d(gx_n.gy_n) + d(gy_n, gy)$$

But if $n \to \infty$ we conclude that $d(gx, gy) \leq e$. Thus Φ cannot be expansive.

The converse is obvious

□

Now we can state and prove the main result of this section.

Theorem K. *Let M be a compact metric space with positive topological dimension. If Φ is an expansive action of a finitely generated group G on M, then $h(\Phi, G_1) > 0$ for any generator G_1.*

Proof. Suppose that Φ is an e-expansive action and G_1 is some generator of G. Let us fix $0 < \delta \leq \frac{e}{2}$ and let N be given by the uniform expansiveness with respect to δ. Since $\dim(M) > 0$ we can find some point $x \in M$ such that $C(X, x) \neq \{x\}$. Then we can find some arc $[x, y] \subset M$.

Since $x' \neq y'$, by expansiveness of Φ there is some $g \in G$ such that $[gx', gy'] = [x_0^0, x_1^0]$ has length $\frac{e}{2}$. Since $d(x_0^0, x_1^0) \geq \delta$ there is some $g \in G_n$. We can find some $g \in G_N$ such that $d(gx_0^0, gx_1^0) > e$. The map induced by g_0 is a homeomorphism, then we have that $[g_0 x_0^0, g_0 x_1^0] = [x_0^1, x_3^1]$ is a connected arc with length ate least e contained in the g_0-image of $[x_0, x_1]$. Thus the set $E_1 = \{x_0^0, x_1^0\}$ is n-ε-G_1-separated set.

Now we can subdivide $[x_0^1, x_3^1]$ in two disjoint arcs $[x_0^1, x_1^1]$ and $[x_2^1, x_3^1]$ satisfying $d(x_0^1, x_1^1) \geq \delta$ and $d(x_2^1, x_3^1) \geq \delta$. Again, uniform expansiveness of Φ implies that the set $E_2 = \{x_0^0, x_1^0, g_0^{-1}x_1^1, g_0^{-1}x_2^1, \}$ is a $2N$-ε-G_1-separated set.

If we repeat these steps we can inductively find for each natural n and nN-ε-G_1-separeted set $E - n$ with 2^n elements. Therefore we have:

$$h(\Phi, G_1) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S(n, \varepsilon, G_1) \geq \lim_{n \to \infty} \frac{1}{nN} \log \# E_n = \frac{1}{N} \log 2$$

And this concludes the proof.

□

Chapter 8

Expansive Actions of Connected Lie Groups

In this chapter we start to investigate expansive actions of connected non-trivial Lie Groups. As we have discussed before this study must be made separated from the case of actions of finitely generated groups. Indeed, this is know for the case of $\mathbb{R}$-actions since the 70's when R. Bowen and W. Walters gave the first definition of expansive flows in [BW]. Later an extension to the setting of $\mathbb{R}^k$-actions was given by W. Bonomo, J. Rocha and P. Varandas in [BRV]. Our goal in here is to extend these notions for actions of more general connecte Lie groups. As we will see on next sections, the study of these actions is closely related to the study of foliations on manifolds.

8.1 Definition and First Results

Let M be a closed Riemannian manifold. Recall that a Lie Group is a smooth manifold with group structure, for which the product and inversion operations are smooth. Some simple examples of Lie groups are S^1, $\mathbb{R}^d$, $\mathbb{Z}^d$, $\mathbb{C}^d$ and $GL_n(\mathbb{R})$. It is easy to see that $\mathbb{Z}^d$ is a finitely generated Lie group. Since Lie Groups are manifolds, then finitely generated Lie groups are discrete groups. Then the expansiveness theory for these groups is the same as that studied in previous chapter.

On the other hand, $\mathbb{R}^d$ and S^1 are examples of connected Lie groups and for this reason, the expansiveness theory for actions of these groups is more complicated than the theory for finitely generated actions. To see this, consider $G = \mathbb{R}$. In this case, the action is a flow and as we have seen on Part I of this text, expansiveness for flows is a tough topic. Next we define the main concept of this chapter.

Definition 8.1.1. *We say that a C^r-action $\Phi : G \times M \to M$ is expansive if the following is satisfied: For every $\varepsilon > 0$, there is some $\delta > 0$ such that if we can find $x, y \in M$ and a continuous map $\rho : G \to G$ with $\rho(e) = e$ satisfying $d(gx, h(g)y) \leq \delta$ for any $g \in G$, then $y = gx$ for some $g \in G$ with $|g| \leq \varepsilon$.*

The first elementary result we will see about these kind of systems is something which is well know for the case of flows.

Theorem 8.1.2. *If M is a connected manifold and Φ is expansive, then Φ has no fixed points.*

Proof. The proof is completely analogous to that for BW-expansive flows. For reader's convenience, we will furnishes the proof. Supose that p is fixed for Φ. Take $\varepsilon > 0$ and let $\delta > 0$ be given by the expansiveness. If we consider h to be the constant map equal to e. Then $d(gx, h(g)y) < \delta$ for any $g \in G$ and $y \in B_\delta(x)$. Expansiveness implies that $B_\delta(x) = \{x\}$ which is impossible since M is conneceted. □

For the case of $\mathbb{R}$-actions, non-singularity is automatically satisfied by expansiveness. On the other hand, in higher dimension the analogous concept is that of locally-free actions.

Definition 8.1.3. *Let G be a Lie group, we say that an action Φ of G on M is locally-free if G_x is a discrete subgroup of G, for any $x \in M$.*

Although expansive $\mathbb{R}$-actions are locally-free, next example shows that it is not valid in higher dimensions.

Example 8.1.4. *Let M be a C^r manifold and X be a C^r vector field generating an BW-expansive flow with non-trivial centralizer. Let Y be a non-trivial vector filed commuting with X. In [BRV] it is proved that the centralizer of X is quasi-trivial, then Y generates a flow with the same orbits of X. Then the $\mathbb{R}^2$-action generated by X and Y is an expansive action with orbits of dimension one and the action cannot be locally free.*

In the case of flows, expansive systems are essentially divided in two scenarios: The singular and the non-singular one. The non-singular case is deeply related to foliations theory. Indeed, if some C^r-flow ϕ has not singularities, then its orbits form a foliation of the manifold in one dimensional submanifolds. In particular, any BW-expansive C^r-flow determines a C^r-foliation of M.

In the higher dimensional case when Φ is not locally-free, the set of orbits may decomposed M into submanifolds of distinct dimension and this may make things much more harder. To avoid these difficulties, in this text we will always deal with locally-free actions. Next result is classical and states that locally-free action generates a foliation for M. For the remainder of this chapter, G will always denote a connected Lie Group.

Theorem 8.1.5 ([Walczak]). *Φ is a C^r-locally-free action of G on M if, and only if, the set $\{O(x)\}_{x\in M}$ is a foliation of M such that $dim(\mathcal{F}) = dim(G)$.*

Next we prove a proposition which will be useful on next sections

Proposition 8.1.6. *If Φ is a locally-free action of G on M, then there exists $\delta > 0$ such that $G_x \cap B^G_\delta(e) = \{e\}$ for any $x \in M$.*

Proof. This is a trivial consequence of the fact that Φ is a foliated action. Indeed, if the result is false, we can obtain a sequence of points $x_n \in M$ such that

$$G_{x_n} \cap B^G_{\frac{1}{n}}(e) \neq \{e\}$$

Assume that $x_n \to x$. Since Φ is foliated, then there is some $\eta > 0$ such that $U_x = \Phi(B_\eta(e), (T_x))$ is a foliated neighborhood of $\Phi(B_\eta(e), (x))$. Using the continuity of the action, we can find n big enough such that $\frac{1}{n} \leq \eta$, $x_n \in U_x$, $x_n = g_n x_n$ for some $g_n \neq e$ with $|g_n| \leq \frac{1}{n}$ and $\Phi(\gamma_n, x_n) \subset U_x$, where γ_n is the geodesic on G connecting e and g_n. But this is impossible, since U_x is a foliated neighborhood and the orbit of x_n intersects the transversal T_x twice in time smaller than ε. □

Next we will prove that expansiveness is indeed a dynamical property. Recall that two actions Φ and Ψ of G on M and N, respectively, are conjugated if there exists a homeomorphism $h : M \to N$ satisfying for any $g \in G$ the following conjugacy equation:

$$h \circ \Phi_g = \Psi_g \circ h$$

Theorem 8.1.7. *Any action conjugated to an expansive action is expansive.*

Proof. Suposse that Φ is an expansive action conjugated to Ψ and fix $\varepsilon > 0$. Let $\delta_1 > 0$ be given by the expansivity of Φ. Let $f : N \to M$ be the conjugacy homeomorphism. Then one can chose $\delta > 0$ such that $d(f(x), f(y)) < \delta_1$ if $d(x, y) < \delta$ for every $x, y \in N$. We claim that δ is an expansivity constant of Ψ. Indeed, if there are $x, y \in N$ and a continuous map $\rho : G \to G$ fixing e such that $d(\Psi_g(x), \Psi_{\rho(g)}(y)) < \delta$ for every $g \in G$, then $d(\Phi_g(f(x)), \Phi_{\rho(g)}(f(y))) < \delta_1$ for every g. Then $f(y) \in \Psi(B_\varepsilon(e), f(x))$ and therefore $y \in \Psi(B_\varepsilon(e), x)$. This completes the proof. □

8.2 The Codimension One Scenario

In this section we study codimension one expansive actions of connected Lie groups. A classical result due to H. Lianfa, S. Guozhuo in [LG] states that there are not *BW*-expansive flows on closed surfaces. This section is devoted to extend this result for actions of higher dimensional groups. A translation of the concept of surface flow for actions of higher dimensional groups is the concept of codimension one actions

Definition 8.2.1. *An action Φ of G on M is said have condimension one if*

$$\dim(M) = \dim(G) + 1.$$

As we have seen on last section, the orbits of any locally-free action generates a foliation of M. An interesting fact about expansiveness for actions is that the generated foliation has an expansive behavior. To precise what we mean on last sentence, we need to define the concept of expansive foliation due to Inaba and Tsuchiya ([IT]).

Let $\mathcal{F}$ be a folitation of M and fix $\mathcal{T}$ a complete transversal to $\mathcal{F}$. For any $\varepsilon > 0$ denote $D_\epsilon(x)$ for the transversal disc at x with radius ε. An $\mathcal{F}$-curve is a curve contained in some leaf of $\mathcal{F}$. Fix some $\mathcal{F}$-curve α and let N be some disc of $\mathcal{T}$ containing $\alpha(0)$.

Definition 8.2.2. *A fence F along α is a continuous map $F : [0,1] \times N \to M$ such that:*

1. *$F|_{\{t\}\times N}$ is an embedding of a disc $D_\varepsilon(\alpha(t))$ for any $t \in [0,1]$*
2. *$F|_{[0,1]\times\{x\}}$ is a $\mathcal{F}$-curve for any $x \in N$*
3. *There exists $x_0 \in N$ such that $F|[0,1] \times \{x_0\} = \alpha$.*

We define expansiveness for foliations as follows.

Definition 8.2.3 ([IT]). *$\mathcal{F}$ is said to be expansive if there exists $\delta > 0$ such that for any $x \in M$ and $y \in D_\delta(x) \setminus \{x\}$ there exist a $\mathcal{F}$-curve $\alpha : [0,1] \to M$ with $x = \alpha(0)$, and a fence F along α such that that $F(1,y) \notin D_\delta(\alpha(1))$.*

We note that that previous definition is equivalent to the action of the pseudo group of holonomy of $\mathcal{F}$ to have a expansive behavior. Now we can prove the following:

Theorem 8.2.4. *The orbit foliation of a locally-free expansive action is expansive.*

Proof. Suppose Φ is an expansive locally-free action and fix some complete tranversal $\mathcal{T}$ to the orbit foliation of Φ. Propostion 8.1.6 allows us to chose $\varepsilon > 0$ such that for any $x \in M$ one has $\Phi(B_\varepsilon(e), x) \cap T(x) = \{x\}$. Let $\delta > 0$ be the expansive constant related to ε. Now fix $x \in M$ and let take $y \in D_\delta(x)$. By the choose of ε, $y \notin \varphi(B_\varepsilon(e), x)$.

Then expansivity of Φ gives us a $g_0 \in G$ such that $d(g_0x, g_0y) > \delta$. Now let $\gamma : [0,1] \to G$ be a geodesic connecting e and g_0. Define the map $F([0,1] \times D_\delta(x)) \to M$ by

$$F(t,p) = \Phi(\gamma(t), p)$$

.

Previous map is clearly a fence satisfying $F(1,y) \notin D_\delta(x)$ and therefore the orbit foliation of Φ is expansive. $\square$

Next we define some concepts and state some known results that will be used in the proof of the main theorem of this section. Let $\mathcal{F}$ be a codimension one folitation of M and $\mathcal{T}$ be a complete transversal to $\mathcal{F}$.

Definition 8.2.5. *A leaf $L \in \mathcal{F}$ is said to be resilient if there exist $x \in L$ and a L-loop α containing x such that for the holonomy map $h : U \to U$ induced by α, we can find $y \in T(x) \setminus \{x\}$ such that $h^n(y) \to x$.*

Essentially, a resilient leaf contains some point whose holonomy contracts tranversally and which self-spirals. This behavior is quite complex and forbids L to have trivial holonomy group. Indeed, any resilient leaf has exponential growth for its holonomy pseudo-group.

Other consequence of the definition is that L is a non-proper leaf. Recall that L is a proper leaf if for any $x \in L$ and any foliated chart $\xi : U \to \mathbb{R}^k$, with $x \in U$ the plaque $P(x)$ of x in U satisfies $P(x) = L \cap U$. The existence of y in the definition of resilience forbids L to be proper and this implies that L cannot be compact (See [Walczak]). Next result guarantees the existence of resilient leaves:

Theorem 8.2.6 ([IT]). *Any codimension one expansive foliation contains a resilient leaf.*

The last main ingredient we need is a result due to G. Hector, E. Ghys and Y. Moriyama. Before to state it, we need to introduce the following concept:

Definition 8.2.7. *We say that a foliation $\mathcal{F}$ is almost without holonomy if the holonomy of any non-compact leaf is trivial.*

Theorem 8.2.8. *[HGM] If Φ is a codimension one locally-free action of a nilpotent group G, then the orbit foliation of Φ is almost without holonomy.*

Next we will proceed in order to show that previous theorem together with the results of [IT] and [HGM] will imply the non-existence of expansive actions of codimension one for some groups. Precisely, we have the following:

Theorem L. *There are not expansive codimension-one actions of nilpotent groups G on M.*

Proof. Let Φ be a locally free expansive action of a nilpotent Lie group of G on M. By Theorem 8.2.4 the orbit foliation $\mathcal{F}_\Phi$ of Φ is expansive. But Theorem 8.2.6 implies that $\mathcal{F}_\Phi$ contains a resilient leaf F. On the other hand, Theorem 8.2.8 implies that all non-compact leaf must have trivial-holonomy group. But F is a non-compact leaf whose holonomy group grows exponentially. This is a contradiction, therefore Φ cannot be expansive and theorem is proved.

□

8.3 Centralizer of Expansive Actions

In this section we investigate the symmetries of actions of connected Lie Groups. In last chapter, we studied this problem for the finitely generated case, but now things are harder. For the case when $G = \mathbb{R}^d$, we quote the work [BRV] of W. Bonomo, J. Rocha and P. Varandas, where it is proved the following:

Theorem 8.3.1. *The centralizer of any expansive $\mathbb{R}^k$-action is quasi-trivial.*

Our main goal here is to extended that result to more general Lie groups. We begin introducing the concept of quasi-triviality on this scenario.

Definition 8.3.2. *An action Φ of G on M has quasi-trivial C^r-centralizer if any $\Psi \in C^r(\Phi)$ satisfies the following condition: There is some map $\xi : M \to End(G)$ constant along the orbits of Φ and such that $\Phi(\xi(g), \cdot) = \Psi(g, \cdot)$ for any $g \in G$*

Essentially, an action Φ has quasi-trivial centralizer if any action commuting with Ψ has the same orbits as Φ, but the time is reparametrized by endomorphisms which only vary transversally to the orbits. We remark that previous definition naturally generalizes the respective definition of quasi-triviality for flows and $\mathbb{R}^d$-actions.

Hereafter, we will proceed to obtain Theorem M. Let us discuss the ideas behind this proof. In [BRV], the author started with an expansive $\mathbb{R}^k$-action Φ and given any other action Ψ commuting with ϕ it was possible to find a local group homomorphism which locally reparametrizes $\mathbb{R}^k$. The hard task there is to extend this local homomorphism to a global endomorphism of $\mathbb{R}^k$. This extension was strongly supported on the vector space structure of $\mathbb{R}^k$.

Now if we are working with general Lie groups we do not have an available vector space structure for G. But we have a natural vector space associated to G, namely the Lie algebra $\mathfrak{G}$ of G. Recall that $\mathfrak{G}$ is isomorphic to T_eM. Thus we introduce the class of Lie groups which will be suitable to our generalization.

Definition 8.3.3. *We say that a Lie group is exponential if its exponential map* $\exp : \mathfrak{G} \to G$ *is a surjective group homomorphism.*

Here we are denoting exp for the exponential map at the identity element of G. Clearly $\mathbb{R}$ is an exponential group, but there are other examples of exponential groups such as cylinders and more general products of an abelian compact lie groups with some $\mathbb{R}^k$.

The idea behind our generalization is that starting with an expansive action of G one can obtain a related expansive action of $\mathbb{R}^k$, if the group G is exponential.

Suppose that G is an exponential Lie group. Recall that by the group isomorphism theorem we have that ${}^{\mathfrak{G}}\!/_{Ker(\exp_e)}$ is isomorphic to G. Let us denote ρ for this isomomorphism and recall that ρ is the factor map of exp.

$$\begin{tikzcd}
\mathfrak{G} \arrow[rr, "\exp_e"] \arrow[dr, "\pi"'] & & G \arrow[dl, shift left, "\rho"] \\
& {}^{\mathfrak{G}}\!/_{Ker(\exp_e)} \arrow[ur, shift left] &
\end{tikzcd}$$

Figure 8.1: Isomorphism Theorem

Now, given an action $\Phi : G \times M \to M$, we can use ρ to induce an action $\Phi' : {}^{\mathfrak{G}}\!/_{Ker(\exp)} \times M \to M$ as follows:

$$\Phi'(v, x) = \Phi(\rho(v), x)$$

Next proposition is an elementary consequence of the definitions.

Proposition 8.3.4. *If Φ is expansive, then Φ' is expansive.*

Proof. Suppose that Φ is expansive. Fix $\varepsilon > 0$ small enough to the exponential map be a local isometry on $B_\varepsilon(0)$ and let $\delta > 0$ be given by the expansiveness of Φ. Suppose that there are $x, y \in \mathfrak{G}$ and a continuous $\eta : {}^{\mathfrak{G}}\!/_{Ker(\exp)} \to {}^{\mathfrak{G}}\!/_{Ker(\exp)}$ satisfying $\eta(0)$ such that

$$d(\Phi'_v(x), \Phi'_{\eta(v)}(y)) \le \delta$$

for any $v \in {}^{\mathfrak{G}}\!/_{Ker(\exp)}$.

Since ρ is a group isomorphism, then

$$\rho' = \rho \circ \eta \circ \rho^{-1} : G \to G$$

is a continuous map fixing e. Moreover, we have that

$$d(\Phi_g(x), \Phi_{\rho(\eta(\rho^{-1}(g)))}(y)) = d(\Phi'_{\rho^{-1}(g)}(x), \Phi'_{\eta(\rho^{-1}(g)}(y)) \le \delta$$

for any $g \in G$.

Thus, there is some $g_0 \in B_\varepsilon(e)$ such that $\Phi_{g_0}(y) = x$. But this implies that $\Phi'_{\rho^{-1}(g_0)}(x) = y$ and then Φ' is expansive. □

Next suppose that Φ and Ψ are two actions of G on M which commute. Then for any $v, u \in \mathfrak{G}$, we have the following:

$$\Phi'_v \circ \Psi'_u = \Phi_{\rho(v)} \circ \Psi_{\rho(u)} = \Psi_{\rho(u)} \circ \Phi_{\rho(v)} = \Psi'_u \circ \Phi'_v$$

But previous observations easily imply the following result:

Proposition 8.3.5. *For any $r \ge 0$ one has $\Psi \in C^r(\Phi)$ if, and only if $\Psi' \in C^r(\Phi')$*

□

Now already have all the necessary elements to prove the main theorem of this section.

Theorem M. *If Φ is a C^r locally-free expansive action of an exponential Lie group G on M, then Φ has quasi-trivial centralizer.*

Proof. Let Φ be an expansive action and suppose that G is an exponential group. Fix $\Phi' \in C^r\Phi$. Let

$$\rho : {}^{\mathfrak{G}}\!/_{Ker(\exp)} \to G$$

be the factor isomorphism of exp. Let Ψ' be the action induced in ${}^{\mathfrak{G}}\!/_{Ker(\exp)}$ by ρ.

Recall that ${}^{\mathfrak{G}}\!/_{Ker(\exp)}$ is a finite dimensional real vector space, then it is isomorphic to some $\mathbb{R}^n$. Therefore, Φ' and Ψ' can be seen as actions of $\mathbb{R}^n$ on M. By Propositions 8.3.4 and 8.3.5 Φ' is expansive and $\Psi' \in \Phi'$. Now Theorem 8.3.1 implies that for any

$x \in M$ there is a group-endomorphism η_x of $\mathfrak{G}$ such that $\Psi'_{\eta(v)}(x) = \Phi'_v(x)$ for any $x \in M$ and $v \in \mathfrak{G}$, satisfying $\eta_x = \eta_y$ for any $y \in O_{\Phi'}(x)$.

Define a family of endomorphism of G by

$$\eta'_x = \rho \circ \eta_x \circ \rho^{-1}.$$

Now, this implies that

$$\Psi_g(x) = \Psi'_{\rho^{-1}(g)}(x) = \Phi'_{\eta(\rho^{-1}(g))}(x) = \Phi_{\rho(\eta_x(\rho^{-1}(g)))}$$

For every $x \in M$ and $g \in G$. It is clear that $\eta'_x = \eta'_y$ for any $y \in O_\Phi(x)$ and this concludes the proof.

□

8.4 Entropy of Expansive Actions of Connected Lie Groups

Now we begin to investigate the relationship between expansiveness and topological entropy for connected Lie Group Actions. Remember that if $\mathcal{F}$ is a foliation of M there is a pseudo-group $\mathcal{G}$ naturally associated to $\mathcal{G}$. Namely, the holonomy pseudo-group of $\mathcal{F}$. When M is compact, we have that $\mathcal{G}$ is finitely generated. Next we will describe a natural way to obtain a finite generator for holonomy pseudo-group of the orbit foliation of an locally-free action. Let Φ be a locally-free action on M and fix some point $x \in M$. Let $\mathcal{T}$ be complete tranversal to the orbit foliation of Φ. Since the action is locally free for every $\varepsilon > 0$ we can find $\delta_x > 0$ such that $T_x = B_{\delta_x}(x) \cap T(x)$ is a local cross-section of time ε for the action Φ through x. Precisely, we have the following property: If $y \in T_x$ then $\Phi(B_\varepsilon(e), y) \cap T_x = \{y\}$. By compactness of M we can find $\{x_1, ..., x_n\} \in M$ such that

$$\bigcup_{i=1}^{n} \Phi(B_\varepsilon, T_{x_i}) = M$$

Last condition, implies that the holonomy maps between the cross-section T_{x_i} generates the holonomy pseudo-group of the orbit foliation of Φ. Note that these conditions are totally analogous to the techniques of cross-sections developed by R. Bowen and P. Walters to study BW-expansive flows.

In [IT], the authors proved that any expansive codimension one foliation has positive entropy. This is a consequence of the existence of resilient leaves. For higher codimensional expansive foliations, they proved the same result under stronger assumptions on the expansiveness of $\mathcal{F}$. Our main goal on this section is to weaken this stronger hypothesis and obtain positive entropy only assuming the expansiveness of the foliation. In particular, it will imply that expansive actions of connected Lie groups do have positive geometric entropy. First let us recall, the definition of geometrical entropy for pseudo-groups introduced by Bis in [B].

Let $\mathcal{G}$ be a finitely generated pseudo-group with generator G. Let $g \in \mathcal{G}$. We say that g has size k and denote $\#g = k$ if $g = g_{i_1} \circ ... \circ g_{i_k}$, with $g_{i_1}, ..., g_{i_k} \in G$ or $g = Id_M$. Fix some $\varepsilon > 0$ and some natural n. We say that a pair of points $x, y \in M$ is n-ε-G-separated by $\mathcal{G}$ if there exists $g \in \mathcal{G}$ such that $x, y \in D_g$, $\#g \leq n$ and $d(g(x), g(y)) > \varepsilon$. A subset $E \subset M$ is n-ε-G-separated if any pair of its distinct points is n-ε-G-separated by $\mathcal{G}$. Let $S(n, \varepsilon, G)$ denote the maximal cardinality of a n-ε-G-separated subset of M.

Definition 8.4.1. *The topological entropy of $\mathcal{G}$ with respect to G is defined to be*

$$h(\mathcal{G}, G) \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S(n, \varepsilon, G)$$

As in the case of finitely generated groups, if some pseudo-group has positive entropy with respect to some finite generator G, then it also has positive entropy with respect to any other finite generator. If $\mathcal{G}$ is pseudo-group with finite generator K, we denote

$$\eta = \min_{g \in K}\{diam(D_g)\}.$$

Next concept is a notion of expansiveness for actions of pseudo-groups.

Definition 8.4.2. *A finitely generated pseudo-group $\mathcal{G}$ is expansive if there is some $0 < e < \eta$ such that for any pair of distinct points x, y, there is some $g \in \mathcal{G}$ such that $x, y \in D_g$ and $d(g(x), g(y)) > e$.*

It is immediate that expansiveness for a foliation is equivalent to the expansiveness for its holonomy pseudo-group (See [Walczak]). Then by Theorem O the holonomy group of the orbit folitaion of an expansive action is expansive.

Next theorem states that expansive pseudo-groups are uniformly expansive.

Theorem 8.4.3. *Let $\mathcal{F}$ be a folitation of a compact M and let $\mathcal{T}$ be a finite and compact complete transversal to $\mathcal{F}$. Then $\mathcal{F}$ is expansive if, and only if, for any $\delta > 0$ there is some $N \in \mathbb{N}$ such that if x and y are in the same element of $\mathcal{T}$ and satisfy $d(x, y) \geq \delta$, then there is $g \in \mathcal{G}$ with $\#g \leq N$, such that $d(g(x), g(y)) > e$.*

Proof. Let $\mathcal{T}$ be a finite complete transversal to $\mathcal{T}$. Suppose that $\mathcal{G}$ is expansive and we can find $\delta > 0$ with the following property. For every $n \in \mathbb{N}$, there are $x_n, y_n \in T_n \in \mathcal{T}$ such that $d(x_n, y_n) \geq \delta$ and $\{x_n, y_n\}$ is not n-e-G-separated. Since $\mathcal{T}$ is finite we can assume that $T_n = T$ for every $n \in \mathbb{N}$. Now compactness allows us to assume that $x_n \to x \in T$ and $y_n \to y \in T$. Thus we have that $d(x, y) \geq \delta$ and $d(g(x), g(y)) \leq e$ for any $g \in \mathcal{G}$. But this is impossible, since $\mathcal{G}$ is expansive.

The converse is obvious.

□

In next theorem we improve the result on [IT] about the entropy of expansive foliations.

Theorem N. *Let $\mathcal{F}$ an expansive foliation of some compact manifold M. Then its holonomy pseudo-group has positive topological entropy.*

Proof. Even the one codimensional case was proven by Inaba and Tshuya in [IT], we remark that our techniques cover any codimensional case. Let $\mathcal{T}$ be a complete transversal to $\mathcal{F}$. Since M is compact, we can assume that $\mathcal{T}$ is finite and therefore $\mathcal{F}$ is uniformly expansive. Now fix $0 < \eta \leq \frac{\varepsilon}{4}$ and let $N \in \mathbb{N}$ be given by the uniform expansiveness with respect to η.

Fix $T_0 \in \mathcal{T}$. Since the codimension of $\mathcal{F}$ is positive we can find on T_0 a connected arc $[a, b]$ with $d(a, b) = \eta$. But now, uniformly expansiveness implies that we can find $g \in \mathcal{G}$ such that $a, b \in D_g$, $\#g \leq N$ and $d(g(x), g(y)) > \frac{\varepsilon}{2}$. This implies that $C_1 = [g(a), g(b)]$ is connected arc containing $g(a), g(b)$ and contained in some transversal T_1. Also, the set $E_1 = \{a, b\}$ is N-$\frac{\varepsilon}{2}$-G-separated by $\mathcal{G}$. Now subdivide the arc C_1 in two disjoint arcs $C_1' = [g(a), a_1]$ and $C_1'' = [b_1, g(b)]$ with diameters at least η and such that $g(a)$ and $g(b)$ are are end point of C_1' and C_1'' respectively.

Now uniformly expansiveness implies that there are maps g', g'' such that $\#g', \#g'' \leq N$, $g(a), a_1 \in D_{g'}$, $b_1, g(b) \in D_{g''}$, $d(g'(g(a), g'(a_1)) \geq \frac{\varepsilon}{2}$ and $d(g''(b_1), g''(g(b)) \geq \frac{\varepsilon}{2}$. But this implies that the set $E_2 = \{a, g'^{-1}(a_1), g''^{-1}(b_1), b\}$ is a $2N$-$\frac{\varepsilon}{2}$-G-separated set.

Proceeding exactly as in Theorems 1.2.2 and K, we prove inductively that for every $n \in \mathbb{N}$, there is some nN-$\frac{\varepsilon}{2}$-G-separated set E_n with 2^n elements.

Finally, the toplogical entropy of $\mathcal{G}$ with respect to G satisfies the following:

$$h(\mathcal{G}, G) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S(n, \varepsilon, G) \leq \lim_{n \to \infty} \frac{1}{nN} \log \#E_n = \frac{\log 2}{N}$$

And the proof is complete □

Since the topological entropy of the holonomy pseudo-group of a foliation $\mathcal{F}$ coincides with the geometric entropy of $\mathcal{F}$, Theorem O is an obvious consequence of Theorem N. Recall that the geometric entropy of an locally-free action is the geometric entropy of its orbit foliation.

Theorem O. *The geometric entropy of any expansive locally-free action is positive.*

Proof. Suppose that Φ is a locally-free expansive action of a connected Lie Group G on M. Since Φ is locally free, we have that the orbits of Φ generates a foliation $\mathcal{F}$ of M. By Theorem 8.2.4, $\mathcal{F}$ is expansive. Now Theorem N implies that $\mathcal{F}$ has positive geometric entropy and the Theorem is proved. □

Chapter 9

Some Problems and Questions

9.1 Singular Flows

In this section we expose some questions and problems about the theory of singular expansive flows that are related with the topics that we have exposed. We have constructed a theory of R-stable sets and R-unstable sets for some points on the phase space of singular flows. These sets are important to develop a theory of entropy for such systems. We only proved the existence of such sets for points whose orbits do not accumulate on $Sing(\phi)$. Moreover, we exposed some examples of R-expansive flows such that no point has non-trivial connected R-stable sets. For these examples, every points accumulate on $Sing(\phi)$. We state below some problems related to the above discussion:

Question.
- *What assumptions do we need to obtain the existence of non-trivial connected R-stable sets for regular points whose orbits accumulate on $Sing(\phi)$?*
- *Additional structure on $Sing(\phi)$?*
- *Dominated Decomposition?*
- *Sectional Hyperbolicity or Asymptotic Sectional Hyperbolicity?*
- *Volume preserving?*

Question. *How do the assumptions of the previous question help us to obtain positive topological entropy?*

Question. *How can we use R-stable and R-unstable sets to study the ergodic theory of expansive flows?*

Recall that Examples 3.2.3 and 4.1.13 are examples of flows for which all non-singular points have trivial R-stable and R-unstable sets. Moreover, with no much more effort one can see that these flows satisfy a kind o uniform R-expansiveness. Some questions derived from this observation are:

Question. *Does any regular point of a singular uniformly R-expansive flow have trivial connected stable or unstable sets? Does every regular point of an uniformly R-expansive must accumulates on singularities?*

Question. *What structure on $Sing(\phi)$ forbids the existence of uniformly R-expansive flows?*

In contrast with the non-singular case where there are no stable points, we have seen that singular R-stable can only be periodic. But it would be desirable that following question can be answer negatively.

Question. *Do exist some R-expansive flow on a connected manifold with R-stable or R-unstable points?*

In [JNY] the author introduced a rescaled form of shadowing property. So we can state the following questions:

Question. *Is it possible to define R-shadowable points? What are the implications of these points to the dynamics of ϕ?*

Finally we propose to investigate in more details the applications of R-topological entropy to understand the behavior of surface flows.

Question. *How can we use the R-topological entropy to improve our knowledge about surface flows?*

These problems are fundamental to study the entropy theory of singular expansive flows and seems to be quite challenging. We believe that they can motivate some future works on the entropy theory of such systems.

9.2 Actions of Connected Lie Groups

The theory of expansive group actions is a very recent subject of the study of dynamical systems as we have seen on this text. This there are many unexplored possible directions on this subject. In this section we propose some questions and invite the reader to think about them. May this questions guide us in the study of expansive systems for future explorations.

To begin, let us recall that through out this work we always dealt with locally-free Lie group action. Because of this assumption we could work with regular foliation. It could be possible to obtain a foliation for any expansive Lie group action? In other words

Question. *Do the orbits of an expansive Lie groups action always form a foliation of M?*

It would be great to see the intersection of the theory of expansive actions with the theory of smooth action. Since BW-expansiveness is the model of expansiveness for Anosov flows, the following question is quite natural.

Question. *Is every Anosov action an expansive action?*

In [BRV] the authors affirmed that the only manifolds that supports an Expansive $\mathbb{R}^k$ action are the torus $\mathbb{T}^n$. Unfortunately , it is false. Indeed, in [PT] it was proved that if a 3-dimensional manifold M supports an Anosov flow, then its fundamental group has exponential growth. In particular, M cannot be $\mathbb{T}^3$. Later this result was generalized to expansive flows by M. Paternain in [Pa]. In particular, if we take any toral expansive homeomorphism, its suspension flow is not defined on $\mathbb{T}^3$. This makes us wonder carefully about what manifolds support expansive actions.

Question. *What are the manifolds support expansive actions of connected Lie Groups? Do they must have exponential growth of fundamental groups?*

A related question is the following:

Question. *Which groups can act locally-free on closed manifolds?*

In the Chapter 7 we proved the non-existence of codimension one locally-free expansive actions of nilpotent Lie groups. An immediate question is the following:

Question. *Can we relax the nilpotent assumption on Theorem L?*

Appendix A

Foliations

In this appendix we define and state some classical results of foliations theory that are used through this text. We refer the reader to [Walczak] to find a more complete exposition on this subject.

Definition A.0.1. *Let M be a smooth manifold. A p-dimensional C^r foliation $\mathcal{F}$ is a decomposition of M in to connected C^r-manifolds (called leaves) such that for any $x \in M$, there is a C^r-differentiable chart $\xi = (\xi', \xi'') : U \to \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ defined on a neighborhood of U and satisfying:*

(i) - For any leaf L, each connected component of $L \cap U$ (called plaque) satisfies $\xi'' = const$.

Charts satisfying (i) are called distinguished charts. An atlas such that all charts are distinguished is called a foliated atlas. The number q on previous definition is said to be the codimension of $\mathcal{F}$. The topology of the leaves as manifolds is in general stronger than that induced by the topology of M. We say that a leaf L is proper if these two topologies coincide.

Remark: Compact leafs are always proper, but the convers does not hold.

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Definition A.0.2. *A foliated atlas $\mathcal{A}$ is nice if it satisfies:*

- *The covering $\{D_\xi\}_{\xi\in\mathcal{A}}$ is locally finite.*
- *For any $\xi \in \mathcal{A}$, we have that $\xi(D_\xi)$ is an open cube of $\mathbb{R}^n$*
- *If $\xi_1, \xi_2 \in \mathcal{A}$ and $D_{\xi_1} \cap D_{\xi_2} \neq \emptyset$, then there exist a chart $\xi_3 \in \mathcal{A}$ distinguished $\mathcal{F}$ such that D_{ξ_3} contains the closure of $D_{\xi_1} \cup D_{\xi_2}$ and $\xi_3|_{D_{\xi_1}} = \xi_1$.*

Theorem A.0.3. *There always exists nice atlases for a foliation $\mathcal{F}$.*

Let M be a foliated manifold and let $\mathcal{U}$ be a nice covering for M. For any $U \in \mathcal{U}$, let T_U denotes the plaques of U. Define on U the equivalence relation $x \sim y$ if, and only if

x and y are in the same plaque of U. Then $T_U = {}^{U}\!/_{\sim}$ can be equipped with the quotient topology. T_U is C^R-diffeomorphic to an open cube of $\mathbb{R}^q$, where q is the codimention of $\mathcal{F}$.

Definition A.0.4. *The disjoint union*

$$T = \bigcup_{U \in \mathcal{U}} T_U$$

is a complete transversal for $\mathcal{F}$.

Definition A.0.5. *Let X be a topological space. A peseudo-group* $\mathcal{G}$ *on X is a family*

$$\{h : D_h \to R_h\}$$

of local homeomorphism of X satisfying the followinng conditions:

1. *If* $h, g \in \mathcal{G}$ *and* $D_g \subset R_h$, *then* $g \circ h \in \mathcal{G}$
2. *If* $g \in \mathcal{G}$, *then* $g^{-1} \in \mathcal{G}$
3. *If* $g \in \mathcal{G}$ *and* $U \subset D_g$, *then* $g|_U \in \mathcal{G}$
4. *If g is a local homeomorphism of X and* $\mathcal{U}$ *is an open cover of* D_g *such that* $g|_U \in \mathcal{G}$, *then* $g \in \mathcal{G}$.
5. $Id_X \in \mathcal{G}$.

Definition A.0.6. *We say that a pseudo-group* $\mathcal{G}$ *is finitely generated if there are* $g_1, ..., g_k \in \mathcal{G}$ *such that*

$$\bigcup_{i=1}^{n} D_{g_i} = \bigcup_{i=1}^{n} R_{g_i} = X$$

Any foliation brings with itself a natural pseudo-group.

Definition A.0.7. *Let* $\mathcal{U}$ *be a nice covering for a foliated manifold M. If* $U, V \in \mathcal{U}$ *and* $U \cap V \neq \emptyset$, *then define a map* $h_{UV} : D|_{h_{UV}} \to T_V$, *where* D_{UV} *is the open set of* T_U *such that any* $P \in D_{UV}$ *intersects V by*

$$h_{UV}(P) = P' \text{ if } P \cap P' \neq \emptyset$$

Theorem A.0.8. *If* $\mathcal{U}$ *is a nice covering for M, then the family* $\mathcal{H} = \{h_{UV}; U, V \in \mathcal{U}\}$ *is a pseudo-group called the holonomy pseudo group of* $\mathcal{F}$.

We note that $\mathcal{H}$ is well defined, since the holonomy pseudo groups of any two nice coverings of a foliated manifold are isomorphic.

Theorem A.0.9. *If M is a compact foliated manifold, then* $\mathcal{H}$ *is finitely generated.*

Let $\gamma : [0, 1] \to M$ be a curve contained on a leaf L. Then we can find a $0 = t_0 < t_1 < \ldots < t_k = 1$ and a chain o plaques $P_0, \ldots, P_k$, such that $\gamma|_{[t_i, t_{i+1}]} \subset P_i$, and $P_i \cap P_{i+1}$. Thus we can define a map $h_\gamma = h_{U_0 U1} \circ \cdots \circ h_{U_{k-1} U_k}$ from an open set of T_{U_0} to containing $\gamma(t_0)$. h_γ is called the holonomy map of γ. holonomy maps of homotopic curves are equal.

Fix $x \in L$. We define the holonomy the homomorphism $\Phi_l : \pi_1(L, x) \to \mathcal{H}$ by $\Phi([\gamma]) = h_\gamma$. The holonomy group of L is the set $\mathcal{H}_L = Img(\Phi_L)$

Then we have:

Theorem A.0.10. *Holonomy groups of L corresponding to different points, different charts and different nice coverings are isomorphic.*

Appendix B

Elements of Group Theory

In this section we give some elementary results of group theory that are be used trough this text. For the proof of the facts presentes here and more details about group theory we refer the reader to [H] and [Ro].

Definition B.0.1. *A non-empty set G is called a group is equipped with a binary operation*

$$(\cdot) : G \times G \to G$$

such that:

1. *There is some element $e \in G$ such that $e \cdot g = g \cdot e = g$, for any $g \in G$. Such e is called the identity of G.*

2. $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$*, for any* $g_1, g_2, g_3 \in G$.

3. *For any $g \in G$, there is some element $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$, for any $g \in G$. Such g^{-1} is called the inverse of g.*

We stand the notation gh for $g \cdot h$. The operation of G will often be called the product of G.

G is said to be Abelian if $gh = hg$ for every $g, h \in G$.

Definition B.0.2. *A subgroup H of G is a subset $H \subset G$ such that H is itself a group if its equipped whit the operation of G.*

Definition B.0.3. *Let G and G' be groups. A map $\rho : G \to H$ is a group homomorphism if for any $g, h \in G$ we have $\rho(gh) = \rho(g)\rho(h)$. ρ is said to be an endomorphism if $G = G'$ and an isomorphism if is bejective.*

Group isomorphisms are the notion of equivalence between groups. The Kernel of a homomorphism ρ is the set

$$Ker(\rho) = \{g \in G; \rho(g) = e \in G'\}$$

.

$Ker(\rho)$ is always a subgroup of G. A subgroup $H \subset G$ is said to be normal if $gH = Hg$ for every $g \in G$. Here, the coset gH is defined as $gH = \{gh; h \in H\}$. We note that $Ker(\rho)$ is always a normal subgroup of G. If H is a normal subgroup, then the realation $g \sim h$ is and only if $gh^{-1} \in H$ is an equivalence relation, where gH are the equivalence classes. The quotient space $G/_\sim = G/_H$ can be made a group with a product induced by the product of G and the natural projection $\pi : G \to G/_H$.

Theorem B.0.4 (Isomorphism Theorem)**.** *Let $\rho : G \to G'$ be group homomorphism. Then $G/_{Ker(\rho)}$ is isomorphic to $Im(\rho)$.*

Given $K \subset G$, we define $< K >$ to be the subset generated by K, i.e., $< K >$ is the subgroup formed by all finite products of elements in K and their inverses. We say that G is finitely generated if there is a finite set $K \subset G$ such that $< K >= G$.

Let H be a subgroup of G. Denote $K_{G,H} = g^{-1}h^{-1}gh \in G; g \in G, h \in H$. The commutator subgroup $[G; H]$ is defined to be the subgroup $< K_{G,H} >$.

Definition B.0.5. *A group is said to be nilpotent if there exists a finite sequence of subgroups*

$$\{e\} = H_0 \subset \cdots H_n = G$$

such that:

For any $0 \leq i < n-1$, H_i is a normal subgroup of H_{i+1}.

For any $0 \leq i \leq n-1$, $[H_{i+1} : H_i]$ is a subgroup of H_i.

Abelian groups are always nilpotent. Although above definition is very abstract, we can think about a non-abelian nilpotent as group which is as close as possible to be abelian. Next we define and state some facts of Lie Groups Theory.

Definition B.0.6. *A Lie group is a group G with the an additional structure of smooth manifold such that the group multiplication and the group inversion operations are smooth maps*

Fix some element $g \in G$ and consider the Left transformation map $L_g : G \to G$ defined by $L_g(h) = gh$, for any $h \in G$. For any Lie group there exists Riemmanian metric $<,>$ which is left invariant, i.e, for any $u, v \in T_gM$ we have $< u, v >=< DL_g(u), DL_g(v) >$. Let d be the metric induced by previous Riemannian metric. We denote $|g|$ for $d(g, e)$ and $|g - h|$ for $d(g, h)$. Next we define Lie algebras

Definition B.0.7. *A Lie algebra $\mathfrak{A}$ is a pair $(A, [\cdot,\cdot])$, where A is a vector space and $[\cdot,\cdot] : A \times A \to A$ is a smooth map called the Lie bracket of A. In addition, the Lie bracket must satisfies the following axioms:*

- $[u + v, w] = [u, w] + [v, w]$ *and* $[u, v + w] = [u, w] + [v, w]$ *for any* $u, v, w \in A$.
- $[u, u] = 0$, *for any* $u \in A$.

- $[u,[v,w]] + [w,[u,v]] + [v,[w,u]] = 0$, *for any* $u, v, w \in A$.
- $[u,v] = -[v,u]$ *for any* $u, v \in A$.

We say that a vector field X of G is left invariant if $DL_g(X(h)) = X(gh)$ for any $g, h \in G$. The set $\mathfrak{L}$ of left invariant vector fields of G is a vector space isomorphic to T_eG.

Definition B.0.8. *The Lie algebra $\mathfrak{G}$ of G is defined to be the set $\mathfrak{G} = (\mathfrak{L}, [\cdot,\cdot])$ where the bracket operation is define as the comutator operation on $\mathfrak{L}$, i.e., $[X,Y] = XY - YX$.*

Due to the isomorphism between $\mathfrak{G}$ and T_eG, we can see the Lie algebra of a group G as an infinitesimal generator of G, in the sense that any element $g \in G$ sufficiently close to the identity is generated by the action of the map $\exp_e$ to a linear combination of infinitesimal elements of $\mathfrak{G}$. Precisely, fix a basis $\{v_1, ..., v_n\}$ of T_eG. If $g \in G$ is such that $|g|$ is small enough, then there are $a_1, ...a_n \in \mathbb{R}$, such that $|a_i|$ is small and $g = \exp_e(a_1v_1 + ... + a_nv_n)$.

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