Generalized Fractional Benney type Systems

by

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To my parents, Libia and Juan who always support me in difficult times of my career, to my brothers Alberto and Victor and also to my nephews Anthony, Diego, Kira.
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RESUMO

Sistema fracionário generalizado de tipo Benney

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Benney introduziu em [4] uma estratégia geral para derivar sistemas de equação diferenciais parciais não lineares associadas a soluções de ondas longas e curtas. Motivados pela teoria geral de Benney, propomos novos modelos para interações de ondas curtas e longas quando ondas curtas são descritas pela equação fracionária de Schrödinger e ondas longas pela equação fracionária de meios porosos. Estabelecemos a existência global de soluções fracas para o Problema Cauchy acoplado,

$$\begin{cases} i\partial_t u - (-\Delta)^s u = \alpha\, v\, u + |u|^2 u, & x \in \mathbb{R}, \quad t > 0, \\ \partial_t v + (-\Delta)^{s/2} g(v) = \beta\, (-\Delta)^{s/2} |u|^2, & x \in \mathbb{R}, \quad t > 0, \end{cases}$$

onde $0 < s < 1$, e α, β são constantes reais. A função de valor complexo $u(t, x)$ é a incógnita da equação fracionaria de Schrödinger, que descreve a onda curta, e a função de valor real $v(t, x)$ é a incógnita da equação fracionaria do meio poroso fracionario, que descreve a onda longa. A função $g \in C^1(\mathbb{R})$ é assumida não decrescente (zonas degeneradas serão consideradas). Aqui $(-\Delta)^s$ denota o Laplaciano fracionario usual em $\mathbb{R}^n$,que caracterizam efeitos de difusão não locais e de longo alcance e pode ser definido por $\mathcal{F}\{(-\Delta)^s f\}(\xi) = |\xi|^{2s}\mathcal{F}\{f\}(\xi)$, onde $\mathcal{F}$ é a transformada de Fourier.

Palavras-chaves: Sistemas fracionário do tipo Benney, equação de Schrödinger fracionário, equação do meio poroso fracionário, problema de Cauchy.

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ABSTRACT

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Benney introduced in [4] a general strategy for deriving systems of nonlinear partial differential equations associated with long and short-wave solutions. Motivated by Benney's general theory, we propose new models for short wave-long wave interactions when the short waves are described by fractional Schrödinger equation and the long waves by a fractional equation of porous media. We have established the global existence of weak solutions to the Cauchy problem coupled,

$$\begin{cases} i\partial_t u - (-\Delta)^s u = \alpha\, v\, u + |u|^2 u, & x \in \mathbb{R}, \quad t > 0, \\ \partial_t v + (-\Delta)^{s/2} g(v) = \beta\, (-\Delta)^{s/2} |u|^2, & x \in \mathbb{R}, \quad t > 0, \end{cases}$$

where $0 < s < 1$, and α, β are real constants. The complex value function $u(t,x)$ is the unknown of the fractional Schrödinger equation, which describes the short wave, and the real valued function $v(t,x)$ is the unknown of the fractional porous medium equation, which describes the long wave. The function $g \in C^1(\mathbb{R})$ is assumed to be increasing (degenerated zones will be considered). Here $(-\Delta)^s$ denotes the usual fractional Laplacian in $\mathbb{R}^n$, which characterize nonlocal, long-range diffusion effects and can be defined by $\mathcal{F}\{(-\Delta)^s f\}(\xi) = |\xi|^{2s}\mathcal{F}\{f\}(\xi)$, where $\mathcal{F}$ is the Fourier Transform.

Keywords: Fractional Benney type systems, fractional Schrödinger equation, fractional porous medium equation, Cauchy problem.

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Chapter 1

Introduction

The main issue of this thesis is to introduce the generalized fractional Benney type systems, and to study the existence of solutions for them. First, we consider the following Cauchy problem

$$\begin{cases} i\,\partial_t u - (-\Delta)^s u = \alpha\, v\, u + \gamma\, |u|^2 u, & x \in \mathbb{R},\ t > 0, \\ \partial_t v + (-\Delta)^{s/2} g(v) = \beta\, (-\Delta)^{s/2} |u|^2, & x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where α, β and γ are real constants. The complex value function $u(t,x)$ is the unknown of the fractional Schrödinger equation, which describes the short wave, and the real value function $v(t,x)$ is the unknown of the fractional porous medium equation, which describes the long wave. Here $(-\Delta)^s$, $(0 < s < 1)$, denotes the usual fractional Laplacian in $\mathbb{R}^n$, which characterize nonlocal, long-range diffusion effects and can be defined by $\mathcal{F}\{(-\Delta)^s f\}(\xi) = |\xi|^{2s}\mathcal{F}\{f\}(\xi)$, where $\mathcal{F}$ is the Fourier Transform. The function $g \in C^1(\mathbb{R})$ is assumed to be increasing, hence degenerated zones for $v(t,x)$ are not considered. Although, we also study the following system

$$\begin{cases} i\,\partial_t u - (-\Delta)^s u = \alpha\, v\, u + \gamma\, |u|^2 u, & x \in \mathbb{R},\ t > 0, \\ \partial_t v = \beta\, (-\Delta)^{s/2} |u|^2, & x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

which could be seen as a particular case of (1.1), (e.g. $g \equiv$ constant), but it has its own interest.

The theory of evolutionary equations modeling the interaction between short waves and long waves goes back to Benney [4]. Indeed, in that paper Benney propose a general system (see equations (3.27), (3.28) in that paper), and we recall below the closer one studied by Bekiranov, Ogawa, Ponce [6], that is to say

$$\begin{cases} i\,\partial_t S - (-\Delta)S + i\,C_S\,\nabla S = \alpha S\,L + \gamma\,|S|^2 S, & x\in\mathbb{R},\ \ t>0,\\ \partial_t L + C_L\,\nabla L + \nu P(D_x)L + \lambda\,\nabla L^2 = \beta\,\nabla|S|^2, & x\in\mathbb{R},\ \ t>0,\end{cases} \tag{1.3}$$

where $C_S, \alpha, \gamma, C_L, \nu, \lambda$ and β are real constants. Moreover, $P(D_x)$ is a linear differential operator with constant coefficients. Applying a proper gauge transformation and a scaling of the variables, the system (1.3), when $\nu = 0$, is equivalent to

$$\begin{cases} i\,\partial_t u - (-\Delta)u = \alpha\ v\ u + \gamma\ |u|^2 u,\\ \quad \partial_t v + C\ \nabla v = \beta\ \nabla|u|^2,\end{cases} \tag{1.4}$$

where $C = \pm 1$. In fact, the authors in [6] claim that, the system (1.4) is the most typical case in the theory of wave interaction. In particular, for $s = 1$ and $g(v) = v$, the system (1.1) recalls (1.4), since we have the following equivalence

$$\|(-\Delta)^{1/2} f\|_{L^2(\mathbb{R}^n)} = \|\nabla f\|_{L^2(\mathbb{R}^n)}, \quad \text{for each } f\in H^1(\mathbb{R}^n).$$

Inspired by the system (1.4), we formulate a general fractional framework for short wave and long wave interactions described in the Cauchy Problem (1.1). We stress that, the exponent of the fractional Laplacian in the long wave is proposed as a half of the short wave. This is convenient when $s = 1$ (as observed above), but also mathematically justified during the main estimates, see Lemma 3.2 and Theorem 3.1. Somehow, this relation between the exponents in (1.1) indicates how these waves interact and affect each other.

Benney in [4] also consider the following system (see equations (3.8), (3.9) in that paper)

$$\begin{cases} iS_t - (-\Delta)S + i\,C_g\,\nabla S = \alpha\ LS + \gamma\ |S|^2 S, & x\in\mathbb{R},\ \ t>0,\\ \qquad L_t + C_l\ \nabla L = \beta\ \nabla|S|^2, & x\in\mathbb{R},\ \ t>0.\end{cases} \tag{1.5}$$

In particular, when $C_g = C_l$ long waves and short waves are resonant, and in this case Tsutsumi and Hatano in [33] proved that, the transformation: $x \mapsto y = x - C_g\,t$ eliminates the first x-derivative terms in (1.5), hence we have

$$\begin{cases} iu_t - (-\Delta)u = \alpha\ vu + \gamma\ |u|^2 u,\\ \qquad v_t = \beta\ \nabla|u|^2.\end{cases} \tag{1.6}$$

Again, inspired by the system (1.6) we formulated the fractional short wave and long wave system as considered in the Cauchy Problem (1.2). We observe that, the degeneration makes the second equation in the system (1.2) much different from the non-degenerate one, i.e. (1.1), (in which g' is strictly away from zero). Moreover, the main difficult in dealing with the systems (1.1) and (1.2) at the same time, that is g is assumed non-decreasing, is that the regions of nondegeneration and degeneration are glued together in such a way that depends on the solution itself. Therefore, there is no hope of obtaining a correct formulation by simply taking into account separately the degenerate and the non-degenerate zones concurrently.

It is very important to highlight the physical background or motivations to consider the generalized fractional Benney type systems proposed in this thesis. Indeed, the short (transversal) wave described by the Schrödinger equation may represent a signal (wave packets), that is $u(t, x)$ is a function that conveys information to control, for instance, some underwater equipment. This information propagates in a generalized medium, where long (longitudinal) waves are described by the porous medium equation. The fractional Laplacian introduces the long-range interactions in both equations, which are coupled by the α, β constants, that is to say, the signal is affected by medium where it propagates. Clearly, how lower are these constants less coupled are the equations. In fact, the constant α makes the difference concerning the global in time existence, see Theorem 1.1 (Main Theorem) below. If the perturbation is too weak ($|\alpha| << 1$), then there exist global in time solutions. Another very important point is the energy input to the signal, i.e. $\|u_0\|_{L^2}$. As far as the information has to be sent, more energy is needed. Again, the statement of the Main Theorem show us that, the global in time solvability depends on the amount of energy given to the signal. Similar discussion follows to applications in Synthetic Aperture Radar (see [3]), and atmospheric internal gravity waves (see [29], [35]), which also represent complex anomalous systems (better modeled by Fractional Laplacians).

Since Benney introduced the models of nonlinear equations to study the interaction between short and long waves, there is a variety of articles on this subject. We list some papers which are close to our context. We also mention the case when the long wave is described by a scalar conservation law, which can also be seen related to (1.1).

Bekiranov, Ogawa, Ponce [6] applying Bourgain arguments, (see Remark 5 in that paper), were able to show local well-posedness of (1.4), with initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s-1/2}(\mathbb{R})$, $(s \geq 0)$.

Tsutsumi, Hatano (see [32], [33]) showed local well-posedness of (1.6) where the initial data $(u_0, v_0) \in H^{1/2}(\mathbb{R}) \times L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $\gamma = 0$, and for $\gamma \neq 0$ $(u_0, v_0) \in H^{j-1/2}(\mathbb{R}) \times H^{j-1}(\mathbb{R})$, ($j \geq 2$ be an integer). They also obtained global well-posedness in similar spaces [33].

Dias, Figueira [13] considered for the first time, a scalar conservation law coupled with a semilinear Schrödinger equation modeling the interaction phenomenon between short wave and long wave. More precisely, they tackle the following system

$$\begin{cases} i\,\partial_t u - (-\Delta)u = v\,u + |u|^2 u, & x \in \mathbb{R}, \quad t > 0, \\ \partial_t v + \nabla g(v) = \nabla |u|^2, & x \in \mathbb{R}, \quad t > 0, \end{cases} \tag{1.7}$$

where the flux function $g(v) = av^2 - bv^3$, with $a \in \mathbb{R}$ and $b > 0$. Using the general approach known as the vanishing viscosity method and arguments from the compensated compactness theory, they showed existence of entropy solutions where the initial data $(u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$.

Dias, Figueira, Frid [14] studied the following system

$$\begin{cases} i\,\partial_t u - (-\Delta)u = \alpha\, h(v)\, u + |u|^2 u, & x \in \mathbb{R},\ \ t > 0, \\ \partial_t v + \nabla g(v) = \beta\, \nabla((h'(v)|u|^2), & x \in \mathbb{R},\ \ t > 0, \end{cases} \tag{1.8}$$

which is a generalization of (1.7). The novelty is the function $h(v)$, called interaction function, and the flux $g(v)$ is no more a polynomial. Under some nonlinear conditions for $g, h \in C^3(\mathbb{R})$, and also a weak interaction $(0 < \alpha \leq \alpha_0)$, they show existence of entropy solutions, with initial data $(u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$, applying the vanishing viscosity method and compensated compactness theory.

Statement of the Main Result. Hereafter, we fix $\gamma = 1$, and without loss of generality $g(0) = 0$.

The following definition tells us in which sense a pair $(u(t, x), v(t, x))$ is a weak solution to the Cauchy problem (1.1). The Cauchy problem (1.2) (degenerate case) is considered in Chapter 4.

Definition 1.1. *Given an initial data* $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s/2}(\mathbb{R})$, $(\frac{1}{2} < s < 1)$*, and any* $T > 0$ *fixed, a pair*

$$(u, v) \in L^\infty(0, T; H^s(\mathbb{R})) \times L^2(0, T; H^{s/2}(\mathbb{R}))$$

is called a weak solution of the Cauchy problem (1.1)*, when it satisfies:*

$$\begin{aligned} i\int_0^T \int_{\mathbb{R}} \Big(u(t,x)\, \partial_t \overline{\varphi}(t,x) + (-\Delta)^{s/2} u(t,x)\, (-\Delta)^{s/2} \overline{\varphi}(t,x)\Big) dxdt + i\int_{\mathbb{R}} u_0(x)\, \overline{\varphi}(0,x) dx \\ + \alpha \int_0^T \int_{\mathbb{R}} v(t,x)\, u(t,x)\, \overline{\varphi}(t,x) dxdt + \int_0^T \int_{\mathbb{R}} |u(t,x)|^2\, u(t,x)\, \overline{\varphi}(t,x) dxdt = 0, \end{aligned} \tag{1.9}$$

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} v(t,x)\, \partial_t \psi(t,x) - g(v(t,x))\, (-\Delta)^{s/2} \psi(t,x) dxdt + \int_{\mathbb{R}} v_0(x)\, \psi(0,x) dx \\ + \beta \int_0^T \int_{\mathbb{R}} |u|^2(t,x)\, (-\Delta)^{s/2} \psi(t,x) dxdt = 0, \end{aligned} \tag{1.10}$$

for each test function $\varphi, \psi \in C_c^\infty((-\infty, T) \times \mathbb{R})$*, with* φ *being complex-valued and* ψ *real-valued.*

Now, we state plainly the main result of this thesis.

Theorem 1.1 (Main Theorem)**.** *Let* $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s/2}(\mathbb{R})$, $(\frac{1}{2} < s < 1)$*, and* $g \in C^1(\mathbb{R})$ *satisfying*

$$0 < m \leq g'(\cdot) \leq M < \infty.$$

Then, there exist $\alpha_0 > 0$ *and* $E_0 > 0$*, such that, if* $|\alpha| \leq \alpha_0$ *or* $\|u_0\|_{L^2(\mathbb{R})} \leq E_0$*, then there exists a weak solution*

$$(u, v) \in L^\infty(0, T; H^s(\mathbb{R})) \times L^2(0, T; H^{s/2}(\mathbb{R}))$$

of the Cauchy problem (1.1).

One recalls that most of the long wave and short wave Benney type systems considered in the literature are posed in one space dimension (i.e. $n = 1$). Albeit, the models proposed here by (1.1) and (1.2) are well defined for $n \geq 1$. In fact, this is also one of the motivations to introduce them. The results of existence of solutions are established in one space dimension for generality with respect to the initial data, since the space $H^s(\mathbb{R})$ is an algebra for $s > 1/2$. For the multidimensional case, ($n \geq 2$), where the embedding $H^s(\mathbb{R}^n)$ in $L^\infty(\mathbb{R}^n)$ for $0 < s \leq 1$ does not hold, we need more specific estimates and a decay assumption on the initial data u_0, see for instance Hayashi, Nakamitsu, Tsutsumi [21]. We should also mention another multidimensional framework for long wave and short wave systems, as considered by Frid, Marroquin, Pan [18].

Finally, we recall that the fractional Schrödinger equation appears in the water wave models in [22]. In fact, the fractional Schrödinger equation was introduced in the theory related to fractional quantum mechanics associated to s-stable Lévy process (see for instance [23]). There are not many works in this field, but the theory is developing fast, hence jointly with [22] we address the reader to the following papers [11], [10] and [19], which start to develop the theory of fractional Schrödinger equations. On the other hand, the fractional porous medium equations has been widely studied in the last years. For instance, we address Vázquez [34] (and references there in), where is described the physical and mathematical background related to nonlinear diffusion equations involving nonlocal effects. Concerning systems of wave interaction, where both the long wave and short wave are fractional, to the best of our knowledge this is the first work is this direction.

We now briefly describe ours models of short wave-long wave interaction (1.1) and (1.2). In the chapter 3, we show the existence of weak solution, as given by definition 1.1 of the system (1.1). We assume that $g' \geq m > 0$. First, in section 3.1, we perturb the system, adding Laplacian terms in both equations of (1.1), with different velocities of perturbation. In section 3.2 we show inequalities that will allow us to show existence of solutions of the perturbed problem. To follow, in section 3.3, applying the Banach fixed point theorem we obtain a unique local solution in time $(u^\varepsilon, v^\varepsilon)$, of the Cauchy problem perturbed in the space $C([0, T_0]; H^1(\mathbb{R}) \times C([0, T_0]; H^1(\mathbb{R})$. After, in section 3.4, we get estimates for $\|u_0\|_{L^2(\mathbb{R})} << 1$, or $|\alpha| << 1$. We establish the existence and uniqueness of global solutions $(u^\varepsilon, v^\varepsilon)$ for the system (1.6), where $0 < \varepsilon < 1$, in the space $C([0,T]; H^1(\mathbb{R}) \times C([0,T]; H^1(\mathbb{R})$ for all $T > 0$. Finally, in section 3.5, we show the existence of weak solutions to the Cauchy problem coupled (1.1). The perturbed model has a parameter ε and an interaction parameters α, β. we prove that the sequence of solutions $(u^\varepsilon, v^\varepsilon)$ converge as $\varepsilon \to 0$ to a weak solution of system (1.1) in the space $C([0,T]; H^s(\mathbb{R}) \times C([0,T]; H^{s/2}(\mathbb{R})$ for all $T > 0$, while the interaction parameters α, β is kept fixed.

In chapter 4, we show the existence of the weak solution as given by Definition 1.1 of system (1.2). This equation is completely degenerate, that can be obtained from (1.1) by setting $g' = 0$. But, Benney (see [4]) presented a general theory to derive nonlinear partial differential equations in which both, long and short wave solutions, coexist and interact each other non linearly. In [4], Benny proposed two pairs of coupled equations, one of them was (1.5). Approaching by a succession of Cauchy problems of type (1.1), that is

$$\begin{cases} i\,\partial_t u^\varepsilon - (-\Delta)^s u^\varepsilon = \alpha\, v^\varepsilon\, u^\varepsilon + |u^\varepsilon|^2 u^\varepsilon, \\ \partial_t v^\varepsilon + \varepsilon\,(-\Delta)^{s/2} v^\varepsilon = \beta\,(-\Delta)^{s/2} \left|u^\varepsilon\right|^2, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x),\;\; v^\varepsilon(0,x) = v_0^\varepsilon(x), \end{cases} \tag{1.11}$$

where $u_0^\varepsilon \in H^s(\mathbb{R})$, $v_0^\varepsilon \in H^{s/2}(\mathbb{R})$ is a approximation of the initial data (u_0, v_0). The existence of weak solutions of equation (1.8) is guaranteed by the previous case, with $g(v) = v$. Here we describe, formally, the main basic estimates, which are required to show existence of weak solutions to the (1.2). Moreover, we prove that the solution of (1.2) is obtained as the limit of solutions of the approximate problem nondegenerate.

Chapter 2

Notation and background

In this section we fix the notations, and collect some preliminary results. Most of the material is well-known or a direct extension of existing work, for instance we refer to [12], [26], and [31]. We mainly provide the proofs of the new results.

First, we denote by dx, $d\xi$, etc. the Lebesgue measure on $\Omega \subset \mathbb{R}^n$ an open set (possibly non-smooth), and by $L^p(\Omega)$ ($p \in [1, +\infty)$) the set of (real or complex) p-summable functions with respect to the Lebesgue measure.

As usual the Schwartz space is denoted by $\mathcal{S}(\mathbb{R}^n)$, which consists of rapidly decaying C^∞ functions in $\mathbb{R}^n$. Moreover, the topology of this space is generated by the semi-norms

$$P_N(\varphi) = \sup_{x\in\mathbb{R}^n}(1 + |x|)^N \sum_{|\alpha|\leq N} |D^\alpha\varphi(x)|, \quad (N = 0, 1, 2, \ldots),$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Let $\mathcal{S}'(\mathbb{R}^n)$ be the set of all tempered distributions, that is the topological dual of $\mathcal{S}(\mathbb{R}^n)$. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we denote by

$$\mathcal{F}\,\varphi(\xi) = \widehat{\varphi}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi.x}\,\varphi(x)\,dx$$

and

$$\mathcal{F}^{-1}\,\varphi(x) = \check{\varphi}(x) = \hat{\varphi}(-x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi.x}\,\varphi(\xi)\,d\xi,$$

the Fourier transform and inverse Fourier transform of φ and we recall that one can extend $\mathcal{F}, \mathcal{F}^{-1}$ from $\mathcal{S}$ to $\mathcal{S}'(\mathbb{R}^n)$.

Lemma 2.1. *Let $\varphi \in \mathcal{S}$, then $\widehat{\overline{\varphi}}(\xi) = \overline{\hat{\varphi}(-\xi)}$ for all $\xi \in \mathbb{R}^n$.*

2.1 Functional spaces

2.1.1 The space $W^{s,p}(\Omega)$

The Sobolev spaces $W^{s,p}(\Omega)$, where a real $p \geqslant 1$ is the integrability index and a real $s \geqslant 0$ is the smoothness index, more precisely, for $s \in (0, 1)$, $p \in [1, +\infty)$, is the (fractional) Sobolev space of order s with Lebesgue exponent p, defined by

$$W^{s,p}(\Omega) := \Big\{u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dxdy < +\infty\Big\},$$

endowed with norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |u|^p dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy\right)^{\frac{1}{p}}.$$

Moreover, for $s > 1$ we write $s = m + \sigma$, where m is an integer and $\sigma \in (0, 1)$. In this case, the space $W^{s,p}(\Omega)$ consists of those equivalence classes of functions $u \in W^{m,p}(\Omega)$ whose distributional derivatives $D^\alpha u$, with $|\alpha| = m$, belong to $W^{\sigma,p}(\Omega)$, that is

$$W^{s,p}(\Omega) = \Big\{u \in W^{m,p}(\Omega) : \sum_{|\alpha|=m} \|D^\alpha u\|_{W^{\sigma,p}(\Omega)} < \infty\Big\},$$

and this is a Banach space with respect to the norm

$$\|u\|_{W^{s,p}(\Omega)} = \Big(\|u\|^p_{W^{m,p}(\Omega)} + \sum_{|u|=m} \|D^\alpha u\|^p_{W^{\sigma,p}(\Omega)}\Big)^{\frac{1}{p}}.$$

Clearly, if $s = m$ is an integer, the space $W^{s,p}(\Omega)$ coincides with the Sobolev space $W^{m,p}(\Omega)$. Also, it is very interesting the case when $p = 2$, i.e. $W^{s,2}(\Omega)$. In this case, the (fractional) Sobolev space is also a Hilbert space, and we can consider the inner product

$$\langle u, v\rangle_{W^{s,2}(\Omega)} = \langle u, v\rangle + \int_\Omega \int_\Omega \frac{(u(x) - u(y))}{|x - y|^{\frac{n}{2}+s}} \, \frac{(v(x) - v(y))}{|x - y|^{\frac{n}{2}+s}} \, dxdy,$$

where $\langle \cdot, \cdot\rangle$ is the inner product in $L^2(\Omega)$.

2.1.2 The space $H^s(\mathbb{R}^n)$

Now, following Tartar [31] we take into account an alternative definition of the space $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ via Fourier Transform. Precisely, we may define

$$H^s(\mathbb{R}^n) := \Big\{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \, |\mathcal{F}u(\xi)|^2 \, d\xi < \infty\Big\} \tag{2.1}$$

and we observe that the above definition, is valid also for any real $s \geq 1$. We may also use an analogous definition for the case $s < 0$ by setting

$$H^s(\mathbb{R}^n) = \Big\{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \, \big|\mathcal{F}u(\xi)\big|^2 d\xi < \infty\Big\}.$$

$H^s(\mathbb{R}^n)$ is a Hilbert space with the scalar product

$$(u, v)_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \, \widehat{u}(\xi) \, \overline{\widehat{v}}(\xi) \, d\xi.$$

The equivalence of the above definitions is stated in the following

Proposition 2.1. *Let $0 < s < 1$. Then the above definitions are equivalent. In particular, for any $u \in H^s(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left|u(x)-u(y)\right|^2}{\left|x-y\right|^{n+2s}}\,dx\,dy = 2\,C_{n,s}^{-1}\int_{\mathbb{R}^n}\left|\,|\xi|^s\,\mathcal{F}u(\xi)\right|^2 d\xi \tag{2.2}$$

where

$$C_{n,s}^{-1} = \int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{|\zeta|^{n+2s}}d\zeta$$

Proof. To Prove the Proposition see E. Di Nezza, G. Palatucci, E. Valdinoci [12] Theorem 3.4, page 532-533. □

Next some density results that will allow to show the important inequalities, we also enunciate results of continuous and compact immersion and that the spaces $H^s(\mathbb{R}^n)$ are an algebra for $s > n/2$.

Lemma 2.2. *For any $s \in \mathbb{R}$, $C_c^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$*

Proof. To Prove the Lemma see Hajer Bahouri, Jean-Yves Chemin, Raphel Danchin [1] Theorem 1.61, page 41. □

Lemma 2.3. *For any $s \in \mathbb{R}$, $\mathcal{S}$ is dense in $H^s(\mathbb{R}^n)$*

Proof. To Prove the Lemma see Gerald B. Folland [1], page 302. □

Lemma 2.4. *For any $s_1 < s_2$, $H^{s_2}(\mathbb{R}^n)$ is a dense subspace of $H^{s_1}(\mathbb{R}^n)$ in the topology of $H^{s_1}(\mathbb{R}^n)$, and $\|\,.\,\|_{s_1} \leq \|\,.\,\|_{s_2}$.*

Proof. To Prove the Lemma see Gerald B. Folland [1], page 302. □

Lemma 2.5. *For any $s_1 < s_2$, the embedding of $H^{s_2}(\mathbb{R}^n)$ into $H^{s_1}(\mathbb{R}^n)$ is locally compact.*

Proof. To Prove the Lemma see Hajer Bahouri, Jean-Yves Chemin, Raphel Danchin [1] Theorem 1.69, page 47. □

Lemma 2.6. *For any $s \in \mathbb{R}$, $(H^s(\mathbb{R}^n))^* = H^{-s}(\mathbb{R}^n)$*

Proof. To Prove the Lemma see Gerald B. Folland [17] Proposition 9.16, page 302-303. □

Lemma 2.7. *Let $s > \frac{n}{2}$, then $H^s(\mathbb{R}^n) \hookrightarrow C_0(\mathbb{R}^n)$ (functions vanishing at ∞).*

Proof. To Prove the Lemma see Michael E. Taylor [30] Proposition 1.3, page 272. □

Lemma 2.8. *Let $s > \frac{n}{2}$, then $H^s(\mathbb{R}^n))$ is an algebra with respect to the product of functions. Moreover, there exists a constant $C = C(s) > 0$, such that for any $f, g \in H^s(\mathbb{R}^n)$*

$$\|f\,g\|_{H^s(\mathbb{R}^n)} \leq C\,\|f\|_{H^s(\mathbb{R}^n)}\|g\|_{H^s(\mathbb{R}^n)}. \tag{2.3}$$

Proof. To Prove the Lemma see F. Linares and G. Ponce [25] Theorem 3.5, page 51. □

2.2 Fractional Laplacian operator

Let $s > 0$. The fractional Laplacian operator can be defined on smooth functions in $\mathbb{R}^n$ using Fourier Transform by

$$\widehat{(-\Delta)^s f}(\xi) = |\xi|^{2s} \hat{f}(\xi) \tag{2.4}$$

and extended in a natural way to functions in the Sobolev space $H^{2s}(\mathbb{R}^n)$. Hence the fractional Laplacian is a pseudo-differential operator with principal symbol $|\xi|^{2s}$. For $0 < s < 1$, the fractional Laplacian can also be described using singular integrals (see [12]) in the following way

$$\begin{aligned} (-\Delta)^s f(x) &= C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+2s}} dy \\ &= C_{n,s} \lim_{\varepsilon \to 0^+} \int_{\complement B_\varepsilon(x)} \frac{f(x) - f(y)}{|x-y|^{n+2s}} dy. \end{aligned} \tag{2.5}$$

Moreover, its inverse, that is to say, $\mathcal{K}_s := (-\Delta)^{-s}$, $(0 < s < 1)$, is given by convolution with the Riesz kernel $K_s(x) = C_{n,s} |x|^{2s-n}$, that is

$$\mathcal{K}_s f = K_s * f.$$

It follows from (2.1), (2.2) and (2.4) that, there exist positive constants m_s, M_s, such that, for each $f \in H^s(\mathbb{R}^n)$

$$m_s(\|f\|_{L^2(\mathbb{R}^n)} + \|(-\Delta)^{s/2} f\|_{L^2(\mathbb{R}^n)}) \leq \|f\|_{H^s(\mathbb{R}^n)} \leq M_s(\|f\|_{L^2(\mathbb{R}^n)} + \|(-\Delta)^{s/2} f\|_{L^2(\mathbb{R}^n)}). \tag{2.6}$$

Lemma 2.9. *For any $s > 0$ and $u \in \mathcal{S}$, then $(-\Delta)^s u \in C^\infty(\mathbb{R}^n)$.*

Proof. To Prove the Proposition see Elias M. Stein and Rami Shakarchi [28], page 217. □

2.2.1 Bilinear form

In order to study the fractional diffusion term, it will be important to associate a bilinear form to the operator $\mathcal{K}_s$ in the space $H^s(\mathbb{R}^n)$, $0 < s < 1$, which is given for any pair $v, w \in H^s(\mathbb{R}^n)$ by

$$\mathcal{B}_s(v, w) := C_{n,s} \iint_{\mathbb{R}^{2n}} (v(x) - v(y)) \frac{1}{|x-y|^{n+2s}} (w(x) - w(y))\ dxdy. \tag{2.7}$$

The bilinear form $\mathcal{B}_s$ was introduced in [8] as an auxiliary tool in the study of regularity properties of solutions to the fractional type porous medium equation.

Lemma 2.10. *If v is given by $v = G(w)$, with $G' \geq 0$, then, $\mathcal{B}_s(v, w) \geq 0$. Furthermore, for every $v, w \in H^1(\mathbb{R}^n)$ we have the characterization*

$$\mathcal{B}_s(v, w) = C \iint_{\mathbb{R}^{2n}} \nabla v(x) \frac{1}{|x-y|^{n-2+2s}} \nabla w(y)\ dxdy, \tag{2.8}$$

where C is a positive constant.

Proof. To Prove the Proposition see L. Caffarelli, Fernando Soria and J. L. Vázquez [8] Corollary 3.1 and Proposition 3.2, page 7. □

2.3 Auxiliary kernels

The usual strategy to show the solvability of Benney type systems is to regularize with a Laplacian operator, see [13, 14]. Here, since we are dealing with fractional operators (in both long and short wave equations), we consider new ideas of perturbation.

• **Unitary group for the Schrödinger equation**.

For each $\varepsilon > 0$, we consider the following Cauchy problem for $u(t,x) \in \mathbb{C}$, driven by the linear fractional perturbed Schrödinger equation

$$\begin{cases} i\,\partial_t u - (-\Delta)^s u - \varepsilon^a(-\Delta)u = 0, \;\; x \in \mathbb{R}^n \;\; t \in \mathbb{R}, \\ \qquad\qquad u(0,x) = u_0(x), \;\; x \in \mathbb{R}^n, \end{cases} \tag{2.9}$$

where $a \in \mathbb{R}$ is a fixed parameter chosen a posteriori. Applying the Fourier transform in the spatial variable, we have

$$\begin{cases} i\,\partial_t \widehat{u}(t,\xi) - |\xi|^{2s}\,\widehat{u}(t,\xi) - \varepsilon^a\,|\xi|^2\,\widehat{u}(t,\xi) = 0, \;\; \xi \in \mathbb{R}^n \;\; t \in \mathbb{R}, \\ \qquad\qquad \widehat{u}(0,\xi) = \widehat{u_0}(\xi), \;\; \xi \in \mathbb{R}^n, \end{cases}$$

whose solution is given by $\widehat{u}(t,\xi) = e^{-i\,\left(|\xi|^{2s}+\varepsilon^a\,|\xi|^2\right)t}\widehat{u_0}(\xi)$. Therefore, it follows that

$$u(t,x) = \mathcal{F}^{-1}\Big\{e^{-i\,\left(|\xi|^{2s}+\varepsilon^a\,|\xi|^2\right)t}\mathcal{F}u_0(\xi)\Big\}(x)$$

solves the Cauchy problem (2.9). For $u_0 \in L^2(\mathbb{R}^n)$, ($\mathcal{F}u_0 \in L^2(\mathbb{R}^n)$), then

$$e^{-i\,\left(|\xi|^{2s}+\varepsilon^a\,|\xi|^2\right)t}\mathcal{F}u_0(\xi) \in L^2(\mathbb{R}^n).$$

Now, we define for each $t \in \mathbb{R}$ the operator

$$u \mapsto U_\varepsilon(t)u := \mathcal{F}^{-1}e^{-i\,\left(|\xi|^{2s}+\varepsilon^a\,|\xi|^2\right)t}\mathcal{F}u, \tag{2.10}$$

which is bounded in $L^2(\mathbb{R}^n)$ for each $u \in L^2(\mathbb{R}^n)$. Indeed, we have

$$\begin{aligned} \|U_\varepsilon(t)u\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |U_\varepsilon(t)u(x)|^2\,dx = \int_{\mathbb{R}^n} |\widehat{U_\varepsilon(t)}u(\xi)|^2\,d\xi \\ &= \int_{\mathbb{R}^n} |e^{-i\,\left(|\xi|^{2s}+\varepsilon^a\,|\xi|^2\right)t}\widehat{u}(\xi)|^2\,d\xi = \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2\,d\xi. \end{aligned}$$

Therefore, the family $(U_\varepsilon(t))_{t\in\mathbb{R}}$ is a group of isometries in $L^2(\mathbb{R}^n)$.

One remarks that, $H^s(\mathbb{R}^n)$, $(s>0)$, is invariant by the isometry group $(U_\varepsilon(t))_{t\in\mathbb{R}}$. For each $u \in H^s(\mathbb{R}^n)$, we have

$$\begin{aligned} \|U_\varepsilon(t)u\|^2_{H^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (1+|\xi|^2)^s\, |\widehat{U_\varepsilon(t)}u(\xi)|^2\, d\xi \\ &= \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2\, d\xi = \|u\|^2_{H^s(\mathbb{R}^n)}. \end{aligned}$$

Thus $U_\varepsilon(t)(H^s(\mathbb{R}^n))$ is a closed subspace in $H^s(\mathbb{R}^n)$ and, we have

$$H^s(\mathbb{R}^n) = U_\varepsilon(t)(H^s(\mathbb{R}^n)) \oplus (U_\varepsilon(t)(H^s(\mathbb{R}^n)))^\perp.$$

Moreover,

$$\begin{aligned} (U_\varepsilon(t)u, w)_{H^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (1+|\xi|^2)^s\, \widehat{U_\varepsilon(t)}u(\xi)\, \overline{\widehat{w}}(\xi)\, d\xi \\ &= \int_{\mathbb{R}^n} (1+|\xi|^2)^s e^{-i\left(|\xi|^{2s}+\varepsilon^a\,|\xi|^2\right)t}\, \widehat{u}(\xi)\, \overline{\widehat{w}}(\xi)\, d\xi \\ &= \int_{\mathbb{R}^n} (1+|\xi|^2)^s\, \widehat{u}(\xi)\, \overline{\widehat{U_\varepsilon(-t)}w(\xi)}\, d\xi = (u, U_\varepsilon(-t)w)_{H^s(\mathbb{R}^n)} \end{aligned}$$

and also an isometry, it follows that $(U_\varepsilon(t)(H^s(\mathbb{R}^n)))^\perp = \{0\}$.

• **Semigroups of contractions for the heat equation**.

For each $\varepsilon > 0$, we consider the following Cauchy problem for $v(t,x) \in \mathbb{R}$, driven by the linear Heat equation

$$\begin{cases} \partial_t v - \varepsilon^b \Delta v = 0, \ \ x \in \mathbb{R}^n, \ \ t>0, \\ v(0,x) = v_0(x), \ \ x \in \mathbb{R}^n, \end{cases} \tag{2.11}$$

where $b \in \mathbb{R}$ is a fixed parameter chosen a posteriori. Again, applying the Fourier transform in the spatial variable, we obtain

$$\begin{cases} \partial_t \widehat{v}(t,\xi) + \varepsilon^b\, |\xi|^2\, \widehat{v}(t,\xi) = 0, \ \ \xi \in \mathbb{R}^n, \ \ t>0, \\ \widehat{v}(0,\xi) = \widehat{v_0}(\xi), \ \ \xi \in \mathbb{R}^n, \end{cases}$$

which solution is given by $\widehat{v}(t,\xi) = e^{-\varepsilon^b\,|\xi|^2 t}\, \widehat{v_0}(\xi)$. Consequently,

$$v(t,x) = \mathcal{F}^{-1}\Big\{e^{-\,\varepsilon^b\,|\xi|^2 t}\, \mathcal{F}v_0(\xi)\Big\}(x)$$

solves the Cauchy problem (2.11), and it is well known that, for $v_0 \in L^2(\mathbb{R}^n)$, $(\mathcal{F}v_0 \in L^2(\mathbb{R}^n))$, it follows that $e^{-\varepsilon^b\,|\xi|^2 t}\, \mathcal{F}v_0(\xi) \in L^2(\mathbb{R}^n)$.

Similarly, we define for each $t>0$ the operator

$$v \mapsto W_\varepsilon(t)v = \mathcal{F}^{-1} e^{-\varepsilon^b\,|\xi|^2 t}\, \mathcal{F}v. \tag{2.12}$$

The operator $W_\varepsilon(t)$ is bounded in $L^2(\mathbb{R}^n)$, in fact the family $\{W_\varepsilon(t)v\}_{t>0}$ is a semi-group of contractions. Indeed, for any $t > 0$, $\|W_\varepsilon(t)v_0\|_{L^2(\mathbb{R})} \leq \|v_0\|_{L^2(\mathbb{R})}$, for any $v \in L^2(\mathbb{R}^n)$. Also in $H^1(\mathbb{R}^n)$, that is

$$\begin{aligned} \|W_\varepsilon(t)v_0\|^2_{H^1(\mathbb{R})} &= \int_{\mathbb{R}} (1 + |\xi|^2) \left|\widehat{W_\varepsilon(t)v_0}(\xi)\right|^2 d\xi, \\ &= \int_{\mathbb{R}} (1 + |\xi|^2) \left|e^{-\varepsilon^b|\xi|^2 t}\, \widehat{v_0}(\xi)\right|^2 d\xi \leq \|v_0\|^2_{H^1(\mathbb{R})}. \end{aligned}$$

One recalls that, the Heat kernel has a regularity effect. Indeed, a refined estimate is given by the following

Lemma 2.11. *For any $v \in L^2(\mathbb{R})$, there exists a constant $C > 0$ independent of t and v, such that for any $t > 0$*

$$\|\partial_x W_\varepsilon(t)v\|_{L^2(\mathbb{R})} \leq \frac{C}{\sqrt{t}}\|v\|_{L^2(\mathbb{R})}. \tag{2.13}$$

Proof. From (2.12), we have

$$W_\varepsilon(t)v = \mathcal{F}^{-1} e^{-\varepsilon^b\, |\xi|^2 t}\, \mathcal{F}v = \frac{1}{\sqrt{4\pi\varepsilon^b t}} e^{-\frac{x^2}{4\varepsilon^b t}} * v,$$

then

$$\partial_x W_\varepsilon(t)v = \frac{1}{\sqrt{4\pi\varepsilon^b t}} \frac{-2x}{4\varepsilon^b t} e^{-\frac{x^2}{4\varepsilon^b t}} * v.$$

Applying Young's inequality, it follows that

$$\begin{aligned} \|\partial_x W_\varepsilon(t)v\|_{L^2(\mathbb{R})} &\leq \frac{2}{4\varepsilon^b t}\, \Big\|\frac{x}{\sqrt{4\pi\varepsilon^b t}}\, e^{-\frac{x^2}{4\varepsilon^b t}}\Big\|_{L^1(\mathbb{R})} \|v\|_{L^2(\mathbb{R})} \\ &= \frac{1}{\sqrt{\pi\varepsilon^b}}\, \frac{1}{t^{1/2}}\, \|v\|_{L^2(\mathbb{R})}. \end{aligned}$$

□

Therefore, for any $v \in L^2(\mathbb{R}^n)$ we have

$$\|W_\varepsilon(t)v\|_{H^1(\mathbb{R}^n)} \leq (1 + \frac{C}{\sqrt{t}})\|v\|_{L^2(\mathbb{R})}. \tag{2.14}$$

2.4 Generalized grownwall lemma

In this section, we consider a nonlinear generalisations of Gronwall's inequality, that will be used in the proof of the second estimate.

Theorem 2.1. *Let $\eta(t)$ be a nonnegative function that satisfies the integral inequality*

$$\eta(t) \leq C + \int_{t_0}^{t} \Big(a(\tau)\,\eta(\tau) + b(\tau)\,\eta^{\sigma}(\tau) \Big)\, d\tau, \quad C \geq 0, \sigma \geq 0 \tag{2.15}$$

where $a(t)$ and $b(t)$ are continuous nonnegative functions for $t \geq t_0$. Then

1. For $0 \leq \sigma < 1$,

$$\begin{aligned} \eta(t) \leq \Big\{ & C^{1-\sigma} \exp\Big[(1-\sigma) \int_{t_0}^{t} a(\tau)\, d\tau\Big] \\ & + (1-\sigma) \int_{t_0}^{t} b(\tau) \exp\Big[(1-\sigma) \int_{\tau}^{t} a(r)\, dr\Big] d\tau \Big\}^{\frac{1}{1-\sigma}}. \end{aligned} \tag{2.16}$$

2. For $\sigma = 1$,

$$\eta(t) \leq C \exp\Big\{ \int_{t_0}^{t} [a(\tau) + b(\tau)]\, d\tau \Big\}. \tag{2.17}$$

3. For $\sigma > 1$, with the additional hypothesis

$$C < \Big\{ \exp\Big[(1-\sigma) \int_{t_0}^{t_0+h} a(\tau)\, d\tau\Big] \Big\}^{\frac{1}{\sigma-1}} \Big\{ (\sigma-1) \int_{t_0}^{t_0+h} b(\tau)\, d\tau \Big\}^{-\frac{1}{\sigma-1}}, \tag{2.18}$$

we also get for $t_0 \leq t \leq t_0 + h$, for $h > 0$

$$\eta(t) \leq C \Big\{ \exp\Big[(1-\sigma)\int_{t_0}^{t} a(\tau)\, d\tau\Big] - C^{-1}(\sigma-1)\int_{t_0}^{t} b(\tau) \exp\Big[(1-\sigma)\int_{\tau}^{t} a(r)\, dr\Big] d\tau \Big\}^{\frac{1}{\sigma-1}}. \tag{2.19}$$

Proof. To Prove the Theorem see Sever Silvestru Dragomir [15] Theorem 21, page 11. □

2.5 Aubin-Lions's Theorem

In this section, we recall the basic definitions to enunciate Aubin-Lions Theorem, following Málek, Necas, Rokyta and Ruzicka [27] Chapter 1.

Let X, Y be two Banach spaces equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$.

Definition 2.1. *Let X and Y be Banach spaces, $X \subset Y$. We say that X is (continuously) imbedded into Y, written*

$$X \hookrightarrow Y,$$

if only if there exists $c > 0$ such that $\|x\|_Y \leq c\|x\|_X$, for all $x \in X$.

Definition 2.2. *Let X and Y be Banach spaces, $X \subset Y$. We say that X is compactly imbedded into Y, written*

$$X \hookrightarrow\hookrightarrow Y,$$

provided

(1) $X \hookrightarrow Y$.

(2) *The identity map $I : X \to Y$ is compact, i.e. $I(B)$ is compact in Y for every bounded subset B of X.*

Let X be a Banach space and $T > 0$. The space $L^p((0,T);X)$, $1 \leq p \leq \infty$ we denote the space of all measurable functions $u : (0,T) \to X$ for which the norm

$$\|u\|_{L^p((0,T),X)} = \left\{ \int_0^T \|u(t)\|_X^p dt \right\}^{1/p}, \quad p < \infty,$$

or

$$\|u\|_{L^\infty((0,T),X)} = \operatorname{ess} \sup_{t\in(0,T)} \|u(t)\|_X, \quad p = \infty,$$

respectively, is finite. That space are called Bochner spaces. Now we enunciate the Aubin-Lions's Theorem.

Theorem 2.2 (Aubin-Lions)**.** *Let $1 < \alpha, \beta < \infty$. Let X be a Banach space, and let X_0, X_1 be separable and reflexive Banach spaces. Provided that $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$ we have*

$$\left\{ v \in L^\alpha((0,T);X_0); \frac{dv}{dt} \in L^\beta((0,T);X_1) \right\} \hookrightarrow\hookrightarrow L^\alpha((0,T);X).$$

Proof. To Prove the Theorem See J. Málek, J. Necas, M. Rokita and M. Ruzicka [27] page 36. □

Chapter 3

Existence of Weak Solutions

The main issue of this section is to show the solvability of the Cauchy problem (1.1), that is, we prove Theorem 1.1 (Main Theorem). To this end, we perturbe the equations in the system (1.1), and the new insight is to add Laplacian terms in both equations with different velocities of perturbation.

3.1 Perturbed system

Specifically, let $a, b > 0$ fixed parameters and for each $\varepsilon \in (0,1)$, we consider the following system posed in $(0,T) \times \mathbb{R}$,

$$\begin{cases} i\,\partial_t u^\varepsilon - (-\Delta)^s u^\varepsilon + \varepsilon^a \Delta u^\varepsilon = \alpha\, v^\varepsilon\, u^\varepsilon + |u^\varepsilon|^2 u^\varepsilon, \\ \partial_t v^\varepsilon - \varepsilon^b\, \Delta v^\varepsilon = \beta\,(-\Delta)^{s/2}|u^\varepsilon|^2 - (-\Delta)^{s/2} g(v^\varepsilon), \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), \;\; v^\varepsilon(0,x) = v_0^\varepsilon(x), \end{cases} \tag{3.1}$$

where $T > 0$ is a real number, and the pair $(u_0^\varepsilon, v_0^\varepsilon) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ is an approaching sequence converging strongly to (u_0, v_0) in $H^s(\mathbb{R}) \times H^{s/2}(\mathbb{R})$. First, we show (local in time) existence and uniqueness of mild solution to (3.1). Then, we derive a priori important estimates, which allowed to extend the local in time solution. Moreover, we stress that these a priori estimates will be also important to pass to the limit as $\varepsilon \to 0$.

3.2 Important inequalities

The next two auxiliary results will be used broadly in this thesis.

Proposition 3.1. *(Chain Rule) Let $f \in H^s(\mathbb{R}^n)$, $0 < s < 1$, $F \in C^1(\mathbb{C})$ with $\|F'\|_{L^\infty(\mathbb{R})} \leq M$ for some $M > 0$. Then*

$$\|(-\Delta)^{s/2} F(f)\|_{L^2(\mathbb{R}^n)} \leq \|F'\|_{L^\infty(\mathbb{R})}\, \|(-\Delta)^{s/2} f\|_{L^2(\mathbb{R}^n)}. \tag{3.2}$$

Proof. The proof follows directly from (2.2) □

Proposition 3.2. *Let $f \in H^s(\mathbb{R})$. Then,*

$$\|f\|_{L^\infty(\mathbb{R})} \leq \Big(\frac{2}{\sqrt{\pi}} + \frac{2}{\sqrt{\pi(2s-1)}}\Big) \|f\|_{L^2(\mathbb{R})}^{1-\frac{1}{2s}} \, \|(-\Delta)^{s/2} f\|_{L^2(\mathbb{R})}^{\frac{1}{2s}}, \quad \textit{for each } s > \frac{1}{2}, \tag{3.3}$$

$$\|(-\Delta)^{s/2}|f|^2\|_{L^2(\mathbb{R})} \leq 2\, \|f\|_{L^\infty(\mathbb{R})} \, \|(-\Delta)^{s/2} f\|_{L^2(\mathbb{R})}, \quad \textit{for } \frac{1}{2} < s < 1. \tag{3.4}$$

Proof. 1. First, since $s > 1/2$, it follows from the well-known Embedding Theorem that, H^s is an algebra of functions. Moreover, a function $f \in H^s(\mathbb{R})$ may be represented by a continuous function which vanishes at infinity. Let us show (3.3), hence applying the inverse Fourier transform, we have for each $x \in \mathbb{R}$

$$\begin{aligned} |f(x)| &= \Big|\frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{ix\xi} \widehat{f}(\xi)\, d\xi\Big| \leq \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} |\widehat{f}(\xi)|\, d\xi \\ &= \frac{1}{(2\pi)^{1/2}} \Big(\int_{|\xi| \leq R} |\widehat{f}(\xi)|\, d\xi + \int_{|\xi| \geq R} \frac{|\xi|^s}{|\xi|^s} |\widehat{f}(\xi)|\, d\xi \Big), \end{aligned}$$

where $R > 0$ is any fixed real number. Then, applying the Cauchy-Schwartz inequality

$$\begin{aligned} |f(x)| &\leq \frac{1}{(2\pi)^{1/2}} \Big(\int_{|\xi| \leq R} 1\, d\xi \Big)^{1/2} \Big(\int_{|\xi| \leq R} |\widehat{f}(\xi)|^2\, d\xi \Big)^{1/2} \\ &+ \frac{1}{(2\pi)^{1/2}} \Big(\int_{|\xi| \geq R} \frac{1}{|\xi|^{2s}}\, d\xi \Big)^{1/2} \Big(\int_{|\xi| \geq R} |\xi|^{2s}\, |\widehat{f}(\xi)|^2\, d\xi \Big)^{1/2} \\ &\leq \frac{1}{(2\pi)^{1/2}} \Big(\sqrt{2}\, R^{1/2}\, \|f\|_{L^2(\mathbb{R})} + \sqrt{\frac{2}{2s-1}}\, R^{\frac{1}{2}-s}\, \|(-\Delta)^{s/2} f\|_{L^2(\mathbb{R})} \Big) \\ &\leq \frac{1}{\sqrt{\pi}} \Big(1 + \frac{1}{\sqrt{2s-1}} \Big) \Big(R^{1/2} \|f\|_{L^2(\mathbb{R})} + R^{\frac{1}{2}-s} \|(-\Delta)^{s/2} f\|_{L^2(\mathbb{R})} \Big). \end{aligned} \tag{3.5}$$

Conveniently, we consider $R = \|f\|_{L^2(\mathbb{R})}^{-\frac{1}{s}} \, \|(-\Delta)^{s/2} f\|_{L^2(\mathbb{R})}^{\frac{1}{s}}$ in (3.5) to obtain

$$|f(x)| \leq \Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}} \Big) \Big(\|f\|_{L^2(\mathbb{R})}^{1-\frac{1}{2s}} \|(-\Delta)^{s/2} f\|_{L^2(\mathbb{R})}^{\frac{1}{2s}} + \|f\|_{L^2(\mathbb{R})}^{1-\frac{1}{2s}} \|(-\Delta)^{s/2} f\|_{L^2(\mathbb{R})}^{\frac{1}{2s}} \Big).$$

2. Now, we prove (3.4). Again, from (2.5) and the definition of the Fractional Laplacian, we obtain

$$\|(-\Delta)^{s/2}|f|^2\|_{L^2(\mathbb{R})}^2 = \frac{C_{n,s}}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{||f|^2(x) - |f|^2(y)|^2}{|x-y|^{1+2s}}\, dxdy$$

$$\leq C_{n,s}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)\,(\overline{f}(x)-\overline{f}(y))|^2}{|x-y|^{1+2s}}\,dxdy$$

$$+\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\overline{f}(y)\,(f(x)-f(y))|^2}{|x-y|^{1+2s}}\,dxdy\Big).$$

Therefore, it follows that

$$\|(-\Delta)^{s/2}|f|^2\|_{L^2(\mathbb{R})} \leq 2\,\|f\|_{L^\infty(\mathbb{R})}\,\|(-\Delta)^{s/2}f\|_{L^2(\mathbb{R})}.$$

□

Proposition 3.3. *Let $v \in H^s(\mathbb{R}^n)$, $0 < s < 1$, $G \in C^1(\mathbb{R})$ with $M \geq G'(\cdot) \geq m > 0$. Then*

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2}G(v)\,v\,dx \geq m\,\|(-\Delta)^{s/4}v\|^2_{L^2(\mathbb{R}^n)}.$$

Proof. It is sufficient to consider $v \in C_c^\infty(\mathbb{R}^n)$. Therefore, we have

$$\int_{\mathbb{R}^n} (-\Delta)(-\Delta)^{\frac{s}{2}-1}G(v)(x)\,v(x)\,dx$$

$$= \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}-1}\nabla G(v)(x)\,\nabla v(x)\,dx$$

$$= \int_{\mathbb{R}^n} \nabla G(v)(x)\,(-\Delta)^{\frac{s}{2}-1}\nabla v(x)\,dx$$

$$= C_{n,s}\iint_{\mathbb{R}^{2n}} \nabla G(v)(x)\,\frac{1}{|x-y|^{n-2+2(s/2)}}\,\nabla v\,dydx$$

$$= \frac{C^2_{n,s}}{C}\iint_{\mathbb{R}^{2n}} (G(v(x))-G(v(y)))\,\frac{1}{|x-y|^{n+s}}\,(v(x)-v(y))\,dxdy,$$

where we have integrated by parts, and used (2.7), (2.8). Hence applying the intermediate value theorem, we obtain

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2}G(v)(x)\,v(x)\,dx$$

$$\geq \frac{m\,C^2_{n,s}}{C}\iint_{\mathbb{R}^{2n}} (v(x)-v(y))\,\frac{1}{|x-y|^{n+s}}\,(v(x)-v(y))\,dxdy$$

$$= m\,C_{n,s}\iint_{\mathbb{R}^{2n}} \nabla v(x)\frac{1}{|x-y|^{n-2+s}}\nabla v(y)\,dxdy,$$

$$= m\,C_{n,s}\int_{\mathbb{R}^n} \nabla v(x)\Big(\int_{\mathbb{R}^n} \frac{\nabla v(y)}{|x-y|^{n-2(1-s/2)}}\,dy\Big)\,dx,$$

$$= m\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{4}-\frac{1}{2}}\nabla v(x)(-\Delta)^{\frac{s}{4}-\frac{1}{2}}\nabla v(x)\,dx,$$

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2} G(v)(x)\, v(x)\, dx \geq m \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{4}} v(x)|^2\, dx.$$

□

3.3 Existence and uniqueness of the Perturbed system

The following definition tells us in which sense the pair $(u^\varepsilon, v^\varepsilon)$ is a solution of the Cauchy problem (3.1).

Definition 3.1. *The pair* $(u^\varepsilon, v^\varepsilon) \in C([0,T]; H^1(\mathbb{R})) \times C([0,T]; H^1(\mathbb{R}))$ *is called a mild solution of* (3.1) *if satisfies the following integral equations*

$$\begin{cases} u^\varepsilon(t) = U_\varepsilon(t)\, u_0^\varepsilon - i \displaystyle\int_0^t U_\varepsilon(t-t')\Big(\alpha\, v^\varepsilon(t')\, u^\varepsilon(t') + \ |u^\varepsilon(t')|^2\, u^\varepsilon(t')\Big)\, dt', \\ v^\varepsilon(t) = W_\varepsilon(t)\, v_0^\varepsilon + \displaystyle\int_0^t W_\varepsilon(t-t')\Big(\beta\, (-\Delta)^{s/2} |u^\varepsilon(t')|^2 - (-\Delta)^{s/2} g(v^\varepsilon)\Big)\, dt', \end{cases} \tag{3.6}$$

where $U_\varepsilon(t)$, $W_\varepsilon(t)$ *are given respectively by* (2.10) *and* (2.12).

We are going to apply the Banach Fixed Point Theorem to show the local-in-time existence of solutions as defined above. To begin, we consider the following lemma (we put $\varepsilon = 1$ for simplicity).

Lemma 3.1. *Let* $\frac{1}{2} < s < 1$, $g \in C^1(\mathbb{R})$, *satisfying* $0 < m \leq g'(x) \leq M < \infty$, $g(0) = 0$. *For* $T > 0$, *let* $(\tilde{u}, \tilde{v}) \in C([0,T]; H^1(\mathbb{R})) \times C([0,T]; H^1(\mathbb{R}))$, *then for each* $(u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ *the Cauchy problem (decoupled system)*

$$\begin{cases} \partial_t u + i\, (-\Delta)^s u - i\, \Delta u = -i\, \alpha\, \tilde{v}\, \tilde{u} - i\ |\tilde{u}|^2\, \tilde{u}, & x \in \mathbb{R}\ \ t > 0, \\ \qquad \partial_t v - \Delta v = \beta\, (-\Delta)^{s/2} |\tilde{u}|^2 - (-\Delta)^{s/2} g(\tilde{v}), & x \in \mathbb{R}\ \ t > 0, \\ u(0,x) = u_0(x),\ \ v(0,x) = v_0(x), \end{cases} \tag{3.7}$$

admits a unique mild solution $(u, v) \in C([0,T]; H^1(\mathbb{R})) \times C([0,T]; H^1(\mathbb{R}))$.

Proof. First, we define for each $t \in (0, T)$

$$F(t) := -\, i\, \alpha\, \tilde{v}(t)\, \tilde{u}(t) - i\ |\tilde{u}|^2(t)\, \tilde{u}(t),$$

$$G(t) := \beta(-\Delta)^{s/2}(|\tilde{u}|^2)(t) - (-\Delta)^{s/2} g(\tilde{v})(t).$$

Claim 1: The complex valued function $F \in C([0,T]; L^2(\mathbb{R}))$.

Proof of Claim: Indeed, for each $t \in [0, T]$

$$|\tilde{u}|^2(t)\, \tilde{u}(t) \in H^1(\mathbb{R}), \quad \text{and} \quad \tilde{u}(t)\, \tilde{v}(t) \in H^1(\mathbb{R}).$$

Then, for h sufficiently small

$$\begin{aligned} F(t+h) - F(t) &= i\,\alpha\,\Big(\tilde{v}(t)\,\tilde{u}(t) - \tilde{v}(t+h)\,\tilde{u}(t+h)\Big) \\ &\quad + i\,\Big(|\tilde{u}|^2(t)\,\tilde{u}(t) - |\tilde{u}|^2(t+h)\,\tilde{u}(t+h)\Big) \\ &= i\,\alpha\,I_1 + i\,I_2, \end{aligned}$$

with obvious notation. A simple algebraic computation shows that

$$\lim_{h\to 0} \|I_1\|_{L^2(\mathbb{R})} = 0, \quad \text{and} \quad \lim_{h\to 0} \|I_2\|_{L^2(\mathbb{R})} = 0,$$

from which the claim is proved.

Claim 2: The real value function $G \in C([0,T]; L^2(\mathbb{R}))$.

Proof of Claim: We observe that $(-\Delta)^{s/2}(|\tilde{u}|^2)(t) \in L^2(\mathbb{R})$, for each $t \in (0,T)$. Also from the assumptions for the function g, that is $g \in C^1(\mathbb{R})$, $g(0) = 0$ and $|g'(v)| \leq M$, $(\forall v \in \mathbb{R})$, it follows that $(-\Delta)^{s/2} g(\tilde{v})(t) \in L^2(\mathbb{R})$. Now, for h sufficiently small, we have

$$\begin{aligned} G(t+h) - G(t) &= \beta\,\Big((-\Delta)^{s/2}(|\tilde{u}|^2)(t+h) - (-\Delta)^{s/2}(|\tilde{u}|^2)(t)\Big) \\ &\quad - \Big((-\Delta)^{s/2} g(\tilde{v})(t+h) - (-\Delta)^{s/2} g(\tilde{v})(t)\Big) \\ &= \beta\,J_1 - J_2, \end{aligned}$$

with obvious notation. Then, from (2.6) and the embedding theorem, see Lemma 2.4

$$\|J_1\|^2_{L^2(\mathbb{R})} \leq \|\,|\tilde{u}|^2(t+h) - |\tilde{u}|^2(t)\|^2_{H^s(\mathbb{R})} \leq \|\,|\tilde{u}|^2(t+h) - |\tilde{u}|^2(t)\|^2_{H^1(\mathbb{R})}.$$

Analogously, we have

$$\begin{aligned} \|J_2\|^2_{L^2(\mathbb{R})} &\leq \|g(\tilde{v})(t+h) - g(\tilde{v})(t)\|^2_{H^s(\mathbb{R})} \leq \|g(\tilde{v})(t+h) - g(\tilde{v})(t)\|^2_{H^1(\mathbb{R})} \\ &= \|g(\tilde{v})(t+h) - g(\tilde{v})(t)\|^2_{L^2(\mathbb{R})} + \|\partial_x g(\tilde{v})(t+h) - \partial_x g(\tilde{v})(t)\|^2_{L^2(\mathbb{R})} \\ &\leq M^2\Big(\int_{\mathbb{R}} |\tilde{v}(t+h,x) - \tilde{v}(t,x)|^2\,dx + \int_{\mathbb{R}} \big|\partial_x \tilde{v}(t+h,x) - \partial_x \tilde{v}(t,x)\big|^2\,dx\Big) \\ &\leq 2M^2\,\|\tilde{v}(t+h) - \tilde{v}(t)\|^2_{H^1(\mathbb{R})}. \end{aligned}$$

Then, passing to the limit as $h \to 0$, the claim is proved.

Finally, since $F, G \in C([0,T]; L^2(\mathbb{R}))$ applying Lemma 4.15 and Corollary 4.12 in [9], there exists a unique solution $(u,v) \in C([0,T]; H^1(\mathbb{R})) \times C([0,T]; H^1(\mathbb{R}))$ given by

$$\begin{aligned} u(t) &= U(t)\,u_0 - i\int_0^t U(t-t')\,(\alpha\,\widetilde{v}(t')\,\tilde{u}(t') + \,|\tilde{u}(t')|^2\,\tilde{u}(t'))\,dt', \\ v(t) &= W(t)\,v_0 + \beta\int_0^t W(t-t')\,(\beta\,(-\Delta)^{s/2}|\tilde{u}(t')|^2 - (-\Delta)^{s/2} g(\tilde{v})(t'))\,dt', \end{aligned} \tag{3.8}$$

where $U(t) \equiv U_{\varepsilon=1}(t)$, $W(t) \equiv W_{\varepsilon=1}(t)$ are given respectively by (2.10), and (2.12). □

Proposition 3.4. *Let* $\frac{1}{2} < s < 1$, $g \in C^1(\mathbb{R})$, $0 < m \leq g'(\cdot) \leq M < \infty$, $g(0) = 0$. *Then, for any* $(u_0^\varepsilon, v_0^\varepsilon) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$, *there exists* $T > 0$ *such that, the Cauchy problem* (3.1) *has a unique mild solution.*

Proof. 1. Hereupon, we denote by X_T the Banach space $C([0,T]; H^1(\mathbb{R}))$, where $T > 0$ is chosen a posteriori. For $R > 2\max\{\|u_0^\varepsilon\|_{H^1(\mathbb{R})}, \|v_0^\varepsilon\|_{H^1(\mathbb{R})}\}$, we define

$$B_R^T := \{f \in X_T : \|f\|_{L^\infty(0,T;H^1(\mathbb{R}))} \leq R\},$$

and the mapping $\Phi : B_R^T \times B_R^T \to X_T \times X_T$, $(\tilde{u}, \tilde{v}) \mapsto (u^\varepsilon, v^\varepsilon) \equiv \Phi(\tilde{u}, \tilde{v})$, where $(u^\varepsilon, v^\varepsilon)$ is the unique mild solution of the Cauchy problem (3.7) (for each $\varepsilon > 0$ fixed). Then, from (3.8) we have for any $t \in [0, T]$

$$\Phi_1(\tilde{u}, \tilde{v}) \equiv u^\varepsilon(t) = U_\varepsilon(t)u_0^\varepsilon - i \int_0^t U_\varepsilon(t - t') \, (\alpha \, \tilde{v}(t') \, \tilde{u}(t') + \, |\tilde{u}(t')|^2 \, \tilde{u}(t')) \, dt',$$

$$\Phi_2(\tilde{u}, \tilde{v}) \equiv v^\varepsilon(t) = W_\varepsilon(t)v_0^\varepsilon + \int_0^t W_\varepsilon(t - t') \, (\beta \, (-\Delta)^{s/2} |\tilde{u}(t')|^2 - (-\Delta)^{s/2} g(\tilde{v})(t')) \, dt'.$$

2. First, we show that $(\Phi_1(\tilde{u}, \tilde{v}), \Phi_2(\tilde{u}, \tilde{v})) \in B_R^T \times B_R^T$. Indeed, since for each $t \in [0, T]$, $\|U_\varepsilon(t)u_0^\varepsilon\|_{H^1(\mathbb{R})} = \|u_0^\varepsilon\|_{H^1(\mathbb{R})}$, then

$$\|U_\varepsilon(t)u_0^\varepsilon\|_{L^\infty(0,T;H^1(\mathbb{R}))} = \|u_0^\varepsilon\|_{H^1(\mathbb{R})}.$$

Moreover, we have

$$\begin{aligned}
&\| \int_0^t U_\varepsilon(t - t') \, (\alpha \, \tilde{v}(t') \, \tilde{u}(t') + \, |\tilde{u}(t')|^2 \, \tilde{u}(t')) \, dt' \|_{H^1(\mathbb{R})} \\
&\quad \leq \int_0^t \|(\alpha \, \tilde{v}(t') \, \tilde{u}(t') + \, |\tilde{u}(t')|^2 \, \tilde{u}(t'))\|_{H^1(\mathbb{R})} \, dt' \\
&\quad \leq \int_0^t C \, (|\alpha| \, \|\tilde{v}(t')\|_{H^1(\mathbb{R})} \, \|\tilde{u}(t')\|_{H^1(\mathbb{R})} + \, \||\tilde{u}(t')|^2\|_{H^1(\mathbb{R})} \, \|\tilde{u}(t')\|_{H^1(\mathbb{R})}) \, dt' \\
&\quad = C \, |\alpha| \int_0^t \|\tilde{v}(t')\|_{H^1(\mathbb{R})} \, \|\tilde{u}(t')\|_{H^1(\mathbb{R})} \, dt' + C \, \int_0^t \|\tilde{u}(t')\|^3_{H^1(\mathbb{R})} \, dt' \\
&\quad \leq 2\max\{|\alpha|, R\} \, C \, R^2 \, T,
\end{aligned}$$

where we have used (2.3). Consequently, for T satisfying

$$T < \frac{1}{4\max\{|\alpha|, R\} \, C \, R}, \tag{3.9}$$

$$\|\Phi_1(\tilde{u},\tilde{v})\|_{L^\infty(0,T;H^1(\mathbb{R}))} \leq \|u_0^\varepsilon\|_{H^1(\mathbb{R})} + 2\max\{|\alpha|, R\}\, C\, R^2\, T$$
$$< \frac{R}{2} + \frac{R}{2} = R.$$

Similarly, we estimate $\|\Phi_2(\tilde{u},\tilde{v})\|_{L^\infty(0,T;H^1(\mathbb{R}))}$. Applying (2.14), it follows that

$$\begin{aligned}
&\| \int_0^t W_\varepsilon(t-t')\,(\beta\,(-\Delta)^{s/2}(|\tilde{u}(t')|^2) - (-\Delta)^{s/2} g(\tilde{v})(t'))\, dt'\|_{H^1(\mathbb{R})} \\
&\quad \leq \int_0^t (1 + \frac{C}{(t-t')^{1/2}}) \|\beta\,(-\Delta)^{s/2}(|\tilde{u}(t')|^2) - (-\Delta)^{s/2} g(\tilde{v})(t')\|_{L^2(\mathbb{R})}\, dt' \\
&\quad \leq \int_0^t |\beta|\,(1 + \frac{C}{(t-t')^{1/2}})\, \|(-\Delta)^{s/2}(|\tilde{u}(t')|^2)\|_{L^2(\mathbb{R})}\, dt' \\
&\quad + \int_0^t (1 + \frac{C}{(t-t')^{1/2}})\, \|(-\Delta)^{s/2} g(\tilde{v})(t')\|_{L^2(\mathbb{R})}\, dt' = I_1 + I_2,
\end{aligned}$$

with obvious notation. To follow, we have

$$\begin{aligned}
I_1 &\leq \frac{1}{m_s}\int_0^t |\beta|\,(1 + \frac{C}{(t-t')^{1/2}})\, \|\tilde{u}(t')\|^2_{H^s(\mathbb{R})}\, dt' \\
&\leq \frac{|\beta|}{m_s}\int_0^t (1 + \frac{C}{(t-t')^{1/2}})\, \|\tilde{u}(t')\|^2_{H^1(\mathbb{R})}\, dt' < \frac{|\beta|}{m_s} R^2\,(T + 2C\sqrt{T}),
\end{aligned}$$

and

$$\begin{aligned}
I_2 &\leq \frac{1}{m_s}\int_0^t (1 + \frac{C}{(t-t')^{1/2}})\, \|g(\tilde{v})(t')\|_{H^s(\mathbb{R})}\, dt' \\
&\leq \frac{1}{m_s}\int_0^t (1 + \frac{C}{(t-t')^{1/2}})\, (\|g(\tilde{v})(t')\|^2_{L^2(\mathbb{R})} + \|\partial_x g(\tilde{v})(t')\|^2_{L^2(\mathbb{R})})^{1/2}\, dt' \\
&\leq \frac{M}{m_s}\int_0^t (1 + \frac{C}{(t-t')^{1/2}})\, \|\tilde{v}(t')\|_{H^1(\mathbb{R})}\, dt' < \frac{M}{m_s} R\,(T + 2C\sqrt{T}).
\end{aligned}$$

Consequently, for T satisfying

$$T < \min\{\frac{m_s}{8\max\{|\beta|R, M\}}, \frac{(m_s)^2}{256C^2(\max\{|\beta|R, M\})^2}\}, \tag{3.10}$$

$$\|\Phi_2(\tilde{u},\tilde{v})\|_{L^\infty(0,T;H^1(\mathbb{R}))} \leq \|v_0^\varepsilon\|_{H^1(\mathbb{R})} + \frac{R}{m_s}(|\beta|R + M)(T + 2C\sqrt{T})$$
$$< \frac{R}{2} + \frac{R}{2} = R.$$

3. Now, we show that Φ is a contraction on $B_R^T \times B_R^T$. Let $(\tilde{u}_i, \tilde{v}_i) \in B_R^T \times B_R^T$,

($i = 1, 2$), then we have

$$
\begin{aligned}
\|\Phi_1(\tilde{u}_1, \tilde{v}_1) &- \Phi_1(\tilde{u}_2, \tilde{v}_2)\|_{H^1(\mathbb{R})} \\
&\le |\alpha| \int_0^t \|U_\varepsilon(t-t')\,(\tilde{v}_2(t')\,\tilde{u}_2(t') - \tilde{v}_1(t')\,\tilde{u}_1(t'))\|_{H^1(\mathbb{R})}\,dt' \\
&+ \int_0^t \|U_\varepsilon(t-t')(|\tilde{u}_2(t')|^2\,\tilde{u}_2(t') - |\tilde{u}_1(t')|^2\,\tilde{u}_1(t'))\|_{H^1(\mathbb{R})}\,dt' \\
&\le |\alpha| \int_0^t \|\tilde{v}_2(t')\,\tilde{u}_2(t') - \tilde{v}_1(t')\,\tilde{u}_1(t')\|_{H^1(\mathbb{R})}\,dt' \\
&+ \int_0^t \|\,|\tilde{u}_2(t')|^2\,\tilde{u}_2(t') - |\tilde{u}_1(t')|^2\,\tilde{u}_1(t')\|_{H^1(\mathbb{R})}\,dt' = |\alpha|\,J_1 + J_2.
\end{aligned}
\tag{3.11}
$$

Applying (2.3) we obtain

$$
\begin{aligned}
|\alpha| J_1 \le\; & C\,|\alpha| \int_0^t \|\tilde{v}_2(t')\|_{H^1(\mathbb{R})}\,\|\tilde{u}_2(t') - \tilde{u}_1(t')\|_{H^1(\mathbb{R})}\,dt' \\
&+\; C\,|\alpha| \int_0^t \|\tilde{u}_1(t')\|_{H^1(\mathbb{R})}\,\|\tilde{v}_2(t') - \tilde{v}_1(t')\|_{H^1(\mathbb{R})}\,dt' \\
&\le C\,|\alpha|\,R\,T\,\big(\|\tilde{u}_2 - \tilde{u}_1\|_{L^\infty(0,T;H^1(\mathbb{R}))} + \|\tilde{v}_2 - \tilde{v}_1\|_{L^\infty(0,T;H^1(\mathbb{R}))}\big).
\end{aligned}
\tag{3.12}
$$

Similarly, we also have

$$
\begin{aligned}
J_2 \le C & \int_0^t \big(\|\tilde{u}_2(t')\|^2_{H^1(\mathbb{R})}\,\|\tilde{u}_2(t') - \tilde{u}_1(t')\|_{H^1(\mathbb{R})} \\
&+ \|\tilde{u}_1(t')\|_{H^1(\mathbb{R})}\,\|\,|\tilde{u}_2(t')|^2 - |\tilde{u}_1(t')|^2\|_{H^1(\mathbb{R})}\big)\,dt' \\
\le 3C\,R^2 & \int_0^t \|\tilde{u}_2(t') - \tilde{u}_1(t')\|_{H^1(\mathbb{R})}\,dt' \le 3C\,R^2\,T\,\|\tilde{u}_2 - \tilde{u}_1\|_{L^\infty(0,T;H^1(\mathbb{R}))}.
\end{aligned}
\tag{3.13}
$$

Therefore, from (3.11)–(3.13), it follows that

$$
\begin{aligned}
\|\Phi_1(\tilde{u}_1,\tilde{v}_1) &- \Phi_1(\tilde{u}_2, \tilde{v}_2)\|_{H^1(\mathbb{R})} \\
&\le 2C\,R\,\max\{|\alpha|, 3R\}\,T\,\big(\|\tilde{u}_1 - \tilde{u}_2\|_{L^\infty(0,T;H^1(\mathbb{R}))} + \|\tilde{v}_1 - \tilde{v}_2\|_{L^\infty(0,T;H^1(\mathbb{R}))}\big).
\end{aligned}
$$

To this end, we have

$$
\begin{aligned}
\|\Phi_2(\tilde{u}_1, \tilde{v}_1) &- \Phi_2(\tilde{u}_2, \tilde{v}_2)\|_{H^1(\mathbb{R})} \\
&\le \int_0^t |\beta|\,\|W_\varepsilon(t-t')((-\Delta)^{s/2}|\tilde{u}_1(t')|^2 - (-\Delta)^{s/2}|\tilde{u}_2(t')|^2)\|_{H^1(\mathbb{R})}\,dt' \\
&+ \int_0^t \|W_\varepsilon(t-t')((-\Delta)^{s/2}g(\tilde{v}_2)(t') - (-\Delta)^{s/2}g(\tilde{v}_1)(t'))\|_{H^1(\mathbb{R})}\,dt'
\end{aligned}
$$

$$
\begin{aligned}
&\|\Phi_2(\tilde u_1,\tilde v_1)-\Phi_2(\tilde u_2,\tilde v_2)\|_{H^1(\mathbb{R})}\\
&\qquad\le \int_0^t |\beta|\,(1+\frac{C}{(t-t')^{1/2}})\,\|(-\Delta)^{s/2}(|\tilde u_1(t')|^2-|\tilde u_2(t')|^2)\|_{L^2(\mathbb{R})}\,dt'\\
&\qquad+\int_0^t (1+\frac{C}{(t-t')^{1/2}})\,\|(-\Delta)^{s/2}(g(\tilde v_2)(t')-g(\tilde v_1)(t'))\|_{L^2(\mathbb{R})}\,dt'\\
&\qquad= K_1+K_2,
\end{aligned}
\tag{3.14}
$$

where we have used (2.14), and obvious notation. Applying (2.6), we obtain

$$
\begin{aligned}
K_1 &\le \frac{|\beta|}{m_s}\int_0^t (1+\frac{C}{(t-t')^{1/2}})\,\|\,|\tilde u_1(t')|^2-\tilde u_2(t')|^2\|_{H^s(\mathbb{R})}\,dt'\\
&\le \frac{|\beta|}{m_s}\int_0^t (1+\frac{C}{(t-t')^{1/2}})\Big(\|\tilde u_1(t')\|_{H^1(\mathbb{R})}\,\|\tilde u_1(t')-\tilde u_2(t')\|_{H^1(\mathbb{R})}\\
&\qquad\qquad+\|\tilde u_2(t')\|_{H^1(\mathbb{R})}\,\|\tilde u_1(t')-\tilde u_2(t')\|_{H^1(\mathbb{R})}\Big)\,dt'\\
&\le 2R\,\frac{|\beta|}{m_s}\,(T+2C\sqrt{T})\,\|\tilde u_1-\tilde u_2\|_{L^\infty(0,T;H^1(\mathbb{R}))},
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
K_2 &\le \frac{1}{m_s}\int_0^t (1+\frac{C}{(t-t')^{1/2}})\,\|g(\tilde v_2)(t')-g(\tilde v_1)(t')\|_{H^s(\mathbb{R})}\,dt'\\
&\le \frac{1}{m_s}\int_0^t (1+\frac{C}{(t-t')^{1/2}})\Big(\|g(\tilde v_2)(t')-g(\tilde v_1)(t')\|^2_{L^2(\mathbb{R})}\\
&\qquad\qquad+\|\partial_x g(\tilde v_2)(t')-\partial_x g(\tilde v_1)(t')\|^2_{L^2(\mathbb{R})}\Big)^{1/2}\,dt'\\
&\le \frac{M}{m_s}\,(T+2C\sqrt{T})\,\|\tilde v_1-\tilde v_2\|_{L^\infty(0,T;H^1(\mathbb{R}))}.
\end{aligned}
\tag{3.16}
$$

Consequently, from (3.14)–(3.16) we obtain

$$
\begin{aligned}
&\|\Phi_2(\tilde u_1,\tilde v_1)-\Phi_2(\tilde u_2,\tilde v_2)\|_{H^1(\mathbb{R})}\\
&\qquad\le \frac{2\max\{R\,|\beta|,M\}}{m_s}\,(T+2C\sqrt{T})\,\big(\|\tilde u_1-\tilde u_2\|_{L^\infty(0,T;H^1(\mathbb{R}))}+\|\tilde v_1-\tilde v_2\|_{L^\infty(0,T;H^1(\mathbb{R}))}\big).
\end{aligned}
$$

4. Finally, from items 2 and 3 there exists a $T>0$, sufficiently small, such that $\Phi : B_R^T\times B_R^T\to B_R^T\times B_R^T$ is a (strict) contraction. Hence we can apply the Banach Fixed Point Theorem and obtain a unique (local in time) solution $(u^\varepsilon,v^\varepsilon)$ of the Cauchy problem (3.1). □

3.4 A priori Estimates

For each $\varepsilon > 0$, let $(u^\varepsilon, v^\varepsilon)$ be the unique solution for the Cauchy problem (3.1), and recall that, the sequences $\{u_0^\varepsilon\}$ and $\{v_0^\varepsilon\}$ are uniformly bounded in $H^1(\mathbb{R})$ with respect to $\varepsilon > 0$. Fix $T > 0$ (arbitrary).

Lemma 3.2 (First estimate). *Let $\frac{1}{2} < s < 1$. Then, for each $t \in (0, T)$:*

$$\frac{d}{dt} \int_{\mathbb{R}} |u^\varepsilon(t,x)|^2 \, dx = 0, \tag{3.17}$$

$$\begin{aligned} \frac{d}{dt}\Big(\int_{\mathbb{R}} |(-\Delta)^{s/2} u^\varepsilon(t,x)|^2 \, dx + \varepsilon^a \int_{\mathbb{R}} |\partial_x u^\varepsilon(t,x)|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}} |u^\varepsilon(t,x)|^4 \, dx \\ + \alpha \int_{\mathbb{R}} v^\varepsilon(t,x) \, |u^\varepsilon(t,x)|^2 \, dx \Big) = \alpha \beta \int_{\mathbb{R}} (-\Delta)^{s/2}(|u^\varepsilon(t,x)|^2) \, |u^\varepsilon(t,x)|^2 \, dx \\ - \alpha \int_{\mathbb{R}} |u^\varepsilon(t,x)|^2 \, (-\Delta)^{s/2} g(v^\varepsilon(t,x)) \, dx - \alpha \, \varepsilon^b \int_{\mathbb{R}} \partial_x |u^\varepsilon(t,x)|^2 \, \partial_x v^\varepsilon(t,x) \, dx, \end{aligned} \tag{3.18}$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} |v^\varepsilon(t,x)|^2 \, dx + \int_{\mathbb{R}} (-\Delta)^{s/2} g(v^\varepsilon)(t,x) \, v^\varepsilon(t,x) \, dx + \varepsilon^b \int_{\mathbb{R}} |\partial_x v^\varepsilon(t,x)|^2 \, dx \\ = \beta \int_{\mathbb{R}} (-\Delta)^{s/2}(|u^\varepsilon(t,x)|^2) \, v^\varepsilon(t,x) \, dx. \end{aligned} \tag{3.19}$$

Proof. 1. First, by approximating the initial data in $H^1(\mathbb{R})$ by functions in $C_c^\infty(\mathbb{R})$, and a standard limit argument, we can assume that $(u^\varepsilon, v^\varepsilon)$ satisfies the Cauchy problem (3.1), (at least almost everywhere), and we are allowed to make the computations below. Indeed, since $H^s(\mathbb{R})$ is an algebra for any $s > 1/2$, we may follow the same strategy developed in the previous section, and for $0 < T' < T$, we obtain $(u^\varepsilon, v^\varepsilon) \in \big(C([0,T']; H^k(\mathbb{R})) \cap C^1([0,T']; H^{k-2}(\mathbb{R}))\big)^2$, for each integer $k > 2$.

2. To follow, multiplying equation $(3.1)_1$ by $\overline{u^\varepsilon}(t,x)$ and integrating in $\mathbb{R}$, we have

$$\begin{aligned} i \int_{\mathbb{R}} \partial_t u^\varepsilon(t,x) \, \overline{u^\varepsilon}(t,x) \, dx - \int_{\mathbb{R}} |(-\Delta)^{s/2} u^\varepsilon(t,x)|^2 \, dx - \varepsilon^a \int_{\mathbb{R}} |\partial_x u^\varepsilon(t,x)|^2 \, dx \\ = \alpha \int_{\mathbb{R}} v^\varepsilon(t,x) \, |u^\varepsilon(t,x)|^2 \, dx + \int_{\mathbb{R}} |u^\varepsilon(t,x)|^4 \, dx. \end{aligned}$$

Therefore, taking the imaginary part of the above equation, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} |u^\varepsilon(t,x)|^2 \, dx = \operatorname{Re} \int_{\mathbb{R}} \partial_t u^\varepsilon(t,x) \, \overline{u^\varepsilon}(t,x) \, dx = 0.$$

3. Now, let us multiply equation $(3.1)_1$ by $\partial_t \overline{u^\varepsilon}(t,x)$, and integrate in $\mathbb{R}$ to obtain

$$
\begin{aligned}
& i \int_{\mathbb{R}} \partial_t u^\varepsilon(t,x)\, \partial_t \overline{u^\varepsilon}(t,x)\, dx - \int_{\mathbb{R}} (-\Delta)^s u^\varepsilon(t,x)\, \partial_t \overline{u^\varepsilon}(t,x)\, dx \\
& \quad + \varepsilon^a \int_{\mathbb{R}} \Delta u^\varepsilon(t,x)\, \partial_t \overline{u^\varepsilon}(t,x)\, dx \\
& \quad = \alpha \int_{\mathbb{R}} v^\varepsilon(t,x)\, u^\varepsilon(t,x)\, \partial_t \overline{u^\varepsilon}(t,x)\, dx + \int_{\mathbb{R}} |u^\varepsilon(t,x)|^2\, u^\varepsilon(t,x)\, \partial_t \overline{u^\varepsilon}(t,x)\, dx.
\end{aligned}
$$

Then, writing $u^\varepsilon = u_1^\varepsilon + i u_2^\varepsilon$ and integrating by parts, it follows that

$$
\begin{aligned}
& i \int_{\mathbb{R}} |\partial_t u^\varepsilon(t,x)|^2\, dx - \int_{\mathbb{R}} (-\Delta)^{s/2} u^\varepsilon(t,x)\, \partial_t \overline{(-\Delta)^{s/2} u^\varepsilon}(t,x)\, dx \\
& \quad - \varepsilon^a \int_{\mathbb{R}} \partial_x u^\varepsilon(t,x)\, \partial_t \overline{\partial_x u^\varepsilon}(t,x)\, dx \\
& = \alpha \int_{\mathbb{R}} v^\varepsilon(t,x)\, (u_1^\varepsilon(t,x) \partial_t u_1^\varepsilon(t,x) + u_2^\varepsilon(t,x) \partial_t u_2^\varepsilon(t,x)) dx \\
& \quad + i\, \alpha \int_{\mathbb{R}} v^\varepsilon(t,x)\, (u_2^\varepsilon(t,x) \partial_t u_1^\varepsilon(t,x) - u_1^\varepsilon(t,x) \partial_t u_2^\varepsilon(t,x))\, dx \\
& \quad + \frac{1}{2} \int_{\mathbb{R}} (u^\varepsilon(t,x))^2\, \partial_t (\overline{(u^\varepsilon)^2}(t,x))\, dx.
\end{aligned}
$$

Taking the real part we have

$$
\begin{aligned}
& \frac{d}{dt} \Big[\int_{\mathbb{R}} |(-\Delta)^{s/2} u^\varepsilon(t,x)|^2\, dx + \varepsilon^a \int_{\mathbb{R}} |\partial_x u^\varepsilon(t,x)|^2\, dx + \frac{1}{2} \int_{\mathbb{R}} |u^\varepsilon(t,x)|^4\, dx \\
& + \alpha \int_{\mathbb{R}} v^\varepsilon(t,x)\, |u^\varepsilon(t,x)|^2\, dx \Big] = \alpha \int_{\mathbb{R}} |u^\varepsilon(t,x)|^2\, \partial_t v^\varepsilon(t,x)\, dx.
\end{aligned}
\tag{3.20}
$$

The right-hand side of the above equation is computed by multiplying $(3.1)_2$ by $\alpha |u^\varepsilon(t,x)|^2$ and integrating in $\mathbb{R}$, that is to say

$$
\begin{aligned}
& \alpha \int_{\mathbb{R}} |u^\varepsilon(t,x)|^2\, \partial_t v^\varepsilon(t,x)\, dx = \alpha\, \beta \int_{\mathbb{R}} (-\Delta)^{s/2} (|u^\varepsilon|^2)(t,x)\, |u^\varepsilon(t,x)|^2\, dx \\
& - \alpha \int_{\mathbb{R}} |u^\varepsilon(t,x)|^2\, (-\Delta)^{s/2} g(v^\varepsilon)(t,x)\, dx - \alpha\, \varepsilon^b \int_{\mathbb{R}} \partial_x |u^\varepsilon(t,x)|^2\, \partial_x v^\varepsilon(t,x)\, dx,
\end{aligned}
$$

and replacing it in (3.20), we obtain

$$
\begin{aligned}
& \frac{d}{dt} \Big[\int_{\mathbb{R}} |(-\Delta)^{s/2} u^\varepsilon(t,x)|^2\, dx + \varepsilon^a \int_{\mathbb{R}} |\partial_x u^\varepsilon(t,x)|^2\, dx + \frac{1}{2} \int_{\mathbb{R}} |u^\varepsilon(t,x)|^4\, dx \\
& + \alpha \int_{\mathbb{R}} v^\varepsilon(t,x)\, |u^\varepsilon(t,x)|^2\, dx \Big] = \alpha\, \beta \int_{\mathbb{R}} (-\Delta)^{s/2} (|u^\varepsilon|^2)(t,x)\, |u^\varepsilon(t,x)|^2\, dx \\
& - \alpha \int_{\mathbb{R}} |u^\varepsilon(t,x)|^2\, (-\Delta)^{s/2} g(v^\varepsilon)(t,x)\, dx - \alpha\, \varepsilon^b \int_{\mathbb{R}} \partial_x |u^\varepsilon|^2(t,x)\, \partial_x v^\varepsilon(t,x)\, dx.
\end{aligned}
$$

4. Finally, equation (3.19) follows directly by multiplying $(3.1)_2$ by $v^\varepsilon(t,x)$ and integrating in $\mathbb{R}$. Indeed, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} |v^\varepsilon(t,x)|^2\,dx + \int_{\mathbb{R}} (-\Delta)^{s/2} g(v^\varepsilon)(t,x)\,v^\varepsilon(t,x)\,dx + \varepsilon^b \int_{\mathbb{R}} |\partial_x v^\varepsilon(t,x)|^2\,dx$$
$$= \beta \int_{\mathbb{R}} (-\Delta)^{s/2}(|u^\varepsilon|^2)(t,x)\,v^\varepsilon(t,x)\,dx.$$

□

Now we pass to the second estimate.

Theorem 3.1 (Second estimate)**.** *Let $a,b > 0$ fixed parameters with $3a \le 2b$, $\frac{1}{2} < s < 1$, and $g \in C^1(\mathbb{R})$ satisfying*

$$0 < m \le g'(\cdot) \le M.$$

Then, there exist $\alpha_0 > 0$ and $E_0 > 0$, such that, for each $t \in (0,T)$

$$\int_{\mathbb{R}} |(-\Delta)^{s/2} u^\varepsilon(t,x)|^2\,dx + \varepsilon^a \int_{\mathbb{R}} |\partial_x u^\varepsilon(t,x)|^2\,dx + \frac{1}{4}\int_{\mathbb{R}} |u^\varepsilon(t,x)|^4\,dx \le h(t) \quad (3.21)$$

and

$$\int_{\mathbb{R}} |v^\varepsilon(t,x)|^2\,dx + m\int_0^t \|(-\Delta)^{s/4} v^\varepsilon(\tau)\|^2_{L^2(\mathbb{R})}\,d\tau + 2\,\varepsilon^b \int_0^t \|\partial_x v^\varepsilon(\tau)\|^2_{L^2(\mathbb{R})}\,d\tau$$
$$\le \|v_0^\varepsilon\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{m}\Big(\frac{4}{\sqrt{\pi}} + \frac{4}{\sqrt{\pi(2s-1)}}\Big)^2 \|u_0^\varepsilon\|^{2-\frac{1}{s}}_{L^2(\mathbb{R})} \int_0^t \|(-\Delta)^{s/2} u^\varepsilon(\tau)\|^{2+\frac{1}{s}}_{L^2(\mathbb{R})}\,d\tau, \quad (3.22)$$

for $|\alpha| \le \alpha_0$ or $\|u_0\|_{L^2(\mathbb{R})} \le E_0$, where h is a continuous positive function (independent of ε).

Proof. 1. First, from Proposition 3.3

$$\int_{\mathbb{R}} (-\Delta)^{s/2} g(v^\varepsilon)(t,x)\,v^\varepsilon(t,x)\,dx \ge m\,\|(-\Delta)^{s/4} v^\varepsilon(t)\|^2_{L^2(\mathbb{R})}.$$

From the above inequality and equation (3.19), it follows that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} |v^\varepsilon(t,x)|^2\,dx + m\int_{\mathbb{R}} |(-\Delta)^{s/4} v^\varepsilon(t,x)|^2\,dx + \varepsilon^b \int_{\mathbb{R}} |\partial_x v^\varepsilon(t,x)|^2\,dx$$
$$\le \beta \int_{\mathbb{R}} (-\Delta)^{s/4} |u^\varepsilon(t,x)|^2\,(-\Delta)^{s/4} v^\varepsilon(t,x)\,dx$$
$$\le \frac{\beta^2}{2m}\int_{\mathbb{R}} |(-\Delta)^{s/4}|u^\varepsilon|^2(t,x)|^2\,dx + \frac{m}{2}\int_{\mathbb{R}} |(-\Delta)^{s/4} v^\varepsilon(t,x)|^2\,dx,$$

where we have used Young's inequality. Then, integrating from 0 to $t > 0$,

$$\int_{\mathbb{R}} |v^\varepsilon(t,x)|^2 \, dx + m \int_0^t \|(-\Delta)^{s/4} v^\varepsilon(\tau)\|^2_{L^2(\mathbb{R})} \, d\tau + 2\, \varepsilon^b \int_0^t \|\partial_x v^\varepsilon(\tau)\|^2_{L^2(\mathbb{R})} \, d\tau$$
$$\leq \int_{\mathbb{R}} |v_0^\varepsilon(x)|^2 \, dx + \frac{\beta^2}{m} \int_0^t \|(-\Delta)^{s/4} |u^\varepsilon|^2(\tau)\|^2_{L^2(\mathbb{R})} \, d\tau. \tag{3.23}$$

2. Now, applying Proposition 3.2 and equation (3.17), we have

$$\|(-\Delta)^{s/2} |u^\varepsilon|^2(t)\|_{L^2(\mathbb{R})} \leq 2 \|u^\varepsilon(t)\|_{L^\infty(\mathbb{R})} \, \|(-\Delta)^{s/2} u^\varepsilon(t)\|_{L^2(\mathbb{R})}$$
$$\leq \Big(\frac{4}{\sqrt{\pi}} + \frac{4}{\sqrt{\pi(2s-1)}} \Big) \|u_0^\varepsilon\|^{1-\frac{1}{2s}}_{L^2(\mathbb{R})} \, \|(-\Delta)^{s/2} u^\varepsilon(t)\|^{1+\frac{1}{2s}}_{L^2(\mathbb{R})}.$$

Then, replacing in (3.23) we have

$$\int_{\mathbb{R}} |v^\varepsilon(t,x)|^2 \, dx + m \int_0^t \|(-\Delta)^{s/4} v^\varepsilon(\tau)\|^2_{L^2(\mathbb{R})} \, d\tau + 2\, \varepsilon^b \int_0^t \|\partial_x v^\varepsilon(\tau)\|^2_{L^2(\mathbb{R})} \, d\tau$$
$$\leq \|v_0^\varepsilon\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{m} \Big(\frac{4}{\sqrt{\pi}} + \frac{4}{\sqrt{\pi(2s-1)}} \Big)^2 \|u_0^\varepsilon\|^{2-\frac{1}{s}}_{L^2(\mathbb{R})} \int_0^t \|(-\Delta)^{s/2} u^\varepsilon(\tau)\|^{2+\frac{1}{s}}_{L^2(\mathbb{R})} \, d\tau. \tag{3.24}$$

From equations (3.18)

$$\frac{d}{dt} \Big[\int_{\mathbb{R}} |(-\Delta)^{s/2} u^\varepsilon(t,x)|^2 \, dx + \varepsilon^a \int_{\mathbb{R}} |\partial_x u^\varepsilon(t,x)|^2 \, dx$$
$$+ \frac{1}{2} \int_{\mathbb{R}} |u^\varepsilon(t,x)|^4 \, dx + \alpha \int_{\mathbb{R}} v^\varepsilon(t,x) \, |u^\varepsilon(t,x)|^2 \, dx \Big]$$
$$\leq |\alpha| \int_{\mathbb{R}} \Big| (-\Delta)^{s/4} |u^\varepsilon|^2(t,x) \, (-\Delta)^{s/4} g(v^\varepsilon)(t,x) \Big| \, dx$$
$$+ |\alpha| \, |\beta| \int_{\mathbb{R}} \big| (-\Delta)^{s/2} (|u^\varepsilon|^2)(t,x) \big| \, |u^\varepsilon(t,x)|^2 \, dx + |\alpha| \, \varepsilon^b \int_{\mathbb{R}} \Big| \partial_x |u^\varepsilon(t,x)|^2 \, \partial_x v^\varepsilon(t,x) \Big| \, dx$$
$$=: |\alpha| E + |\alpha| \, |\beta| F + |\alpha| \, \varepsilon^b G, \tag{3.25}$$

with obvious notation. Again, from Proposition 3.2 and equation (3.17), we may write:

(i) $E \leq \|(-\Delta)^{s/4} g(v^\varepsilon)(t)\|_{L^2(\mathbb{R})} \, \|(-\Delta)^{s/4} |u^\varepsilon|^2(t)\|_{L^2(\mathbb{R})}$

$$\leq \|(-\Delta)^{s/4} g(v^\varepsilon)(t)\|_{L^2(\mathbb{R})} \, \|(-\Delta)^{s/2} |u^\varepsilon|^2(t)\|_{L^2(\mathbb{R})}$$
$$\leq \|g'\|_{L^\infty(\mathbb{R})} \, \|(-\Delta)^{s/4} v^\varepsilon(t)\|_{L^2(\mathbb{R})} \, 2\, \|u^\varepsilon(t)\|_{L^\infty(\mathbb{R})} \, \|(-\Delta)^{s/2} u^\varepsilon(t)\|_{L^2(\mathbb{R})}$$
$$\leq \Big(\frac{4}{\sqrt{\pi}} + \frac{4}{\sqrt{\pi(2s-1)}} \Big) \|g'\|_{L^\infty(\mathbb{R})} \, \|u_0^\varepsilon\|^{1-\frac{1}{2s}}_{L^2(\mathbb{R})} \, \|(-\Delta)^{s/4} v^\varepsilon(t)\|_{L^2(\mathbb{R})} \, \|(-\Delta)^{s/2} u^\varepsilon(t)\|^{1+\frac{1}{2s}}_{L^2(\mathbb{R})}.$$

$$(ii)\ F \leq \|u^\varepsilon(t)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |u^\varepsilon(t,x)|\, |(-\Delta)^{s/2}(|u^\varepsilon|^2)(t,x)|\, dx$$

$$\leq \|u^\varepsilon(t)\|_{L^\infty(\mathbb{R})}\, \|u^\varepsilon(t)\|_{L^2(\mathbb{R}))}\, \|(-\Delta)^{s/2}|u^\varepsilon|^2(t)\|_{L^2(\mathbb{R}))}$$

$$\leq \Big(\frac{2}{\sqrt{\pi}} + \frac{2}{\sqrt{\pi(2s-1)}}\Big)\, \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{1-\frac{1}{2s}}\, \|(-\Delta)^{s/2} u^\varepsilon(t)\|_{L^2(\mathbb{R})}^{\frac{1}{2s}}\, \|u_0^\varepsilon\|_{L^2(\mathbb{R})}\, 2\, \|u^\varepsilon(t)\|_{L^\infty(\mathbb{R})}\, \|(-\Delta)^{s/2} u^\varepsilon(t)\|_{L^2(\mathbb{R})}$$

$$\leq 8\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2\, \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{3-\frac{1}{s}}\, \|(-\Delta)^{s/2} u^\varepsilon(t)\|_{L^2(\mathbb{R})}^{1+\frac{1}{s}}.$$

$$(iii)\ G \leq \int_{\mathbb{R}} \Big(|\partial_x u^\varepsilon(t,x)|\, |\overline{u^\varepsilon}(t,x)| + |u^\varepsilon(t,x)|\, |\partial_x \overline{u^\varepsilon}(t,x)|\Big)\, |\partial_x v^\varepsilon(t,x)|\, dx$$

$$\leq 2\|u^\varepsilon(t)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\partial_x u^\varepsilon(t,x)|\, |\partial_x v^\varepsilon(t,x)|\, dx$$

$$\leq 2\|u^\varepsilon(t)\|_{L^\infty(\mathbb{R})}\, \|\partial_x v^\varepsilon(t)\|_{L^2(\mathbb{R})}\, \|\partial_x u^\varepsilon(t)\|_{L^2(\mathbb{R})}$$

$$\leq \frac{8}{\sqrt{\pi}}\, \|u^\varepsilon(t)\|_{L^2(\mathbb{R})}^{1/2}\, \|\partial_x u^\varepsilon(t)\|_{L^2(\mathbb{R})}^{1/2}\, \|\partial_x v^\varepsilon(t)\|_{L^2(\mathbb{R})}\, \|\partial_x u^\varepsilon(t)\|_{L^2(\mathbb{R})}$$

$$\leq \frac{8}{\sqrt{\pi}}\, \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{1/2}\, \|\partial_x v^\varepsilon(t)\|_{L^2(\mathbb{R})}\, \|\partial_x u^\varepsilon(t)\|_{L^2(\mathbb{R})}^{3/2}.$$

Replacing in equation (3.25)

$$\frac{d}{dt}\Big[\int_{\mathbb{R}} |(-\Delta)^{s/2} u^\varepsilon(t,x)|^2\, dx + \varepsilon^a \int_{\mathbb{R}} |\partial_x u^\varepsilon(t,x)|^2\, dx$$

$$+ \frac{1}{2}\int_{\mathbb{R}} |u^\varepsilon(t,x)|^4\, dx + \alpha \int_{\mathbb{R}} v^\varepsilon(t,x)\, |u^\varepsilon(t,x)|^2\, dx\Big]$$

$$\leq |\alpha| \Big(\frac{4}{\sqrt{\pi}} + \frac{4}{\sqrt{\pi(2s-1)}}\Big)\, \|g'\|_{L^\infty(\mathbb{R})}\, \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{1-\frac{1}{2s}}\, \|(-\Delta)^{s/4} v^\varepsilon(t)\|_{L^2(\mathbb{R})} \|(-\Delta)^{s/2} u^\varepsilon(t)\|_{L^2(\mathbb{R})}^{1+\frac{1}{2s}}$$

$$+ 8|\alpha|\, |\beta| \Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2\, \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{3-\frac{1}{s}}\, \|(-\Delta)^{s/2} u^\varepsilon(t)\|_{L^2(\mathbb{R})}^{1+\frac{1}{s}}$$

$$+ \frac{8}{\sqrt{\pi}} |\alpha|\, \varepsilon^b\, \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{1/2}\, \|\partial_x v^\varepsilon(t)\|_{L^2(\mathbb{R})}\, \|\partial_x u^\varepsilon(t)\|_{L^2(\mathbb{R})}^{3/2}$$

and integrating from 0 to $t > 0$

$$\int_{\mathbb{R}} |(-\Delta)^{s/2} u^\varepsilon(t,x)|^2\, dx + \varepsilon^a \int_{\mathbb{R}} |\partial_x u^\varepsilon(t,x)|^2\, dx + \frac{1}{2}\int_{\mathbb{R}} |u^\varepsilon(t,x)|^4\, dx$$

$$\leq \|(-\Delta)^{s/2} u_0^\varepsilon\|_{L^2(\mathbb{R})}^2 + \varepsilon^a\, \|\partial_x u_0^\varepsilon\|_{L^2(\mathbb{R})}^2 + \frac{1}{2}\|u_0^\varepsilon\|_{L^4(\mathbb{R})}^4 + \int_{\mathbb{R}} |v_0^\varepsilon(x)|\, |u_0^\varepsilon(x)|^2\, dx$$

$$+ |\alpha| \Big(\frac{4}{\sqrt{\pi}} + \frac{4}{\sqrt{\pi(2s-1)}}\Big)\, \|g'\|_{L^\infty(\mathbb{R})}\, \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{1-\frac{1}{2s}} \int_0^t \|(-\Delta)^{s/4} v^\varepsilon(\tau)\|_{L^2(\mathbb{R})}\, \|(-\Delta)^{s/2} u^\varepsilon(\tau)\|_{L^2(\mathbb{R})}^{1+\frac{1}{2s}}\, d\tau$$

$$+ 8|\alpha|\,|\beta|\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2 \,\|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{3-\frac{1}{s}} \int_0^t \|(-\Delta)^{s/2}u^\varepsilon(\tau)\|_{L^2(\mathbb{R})}^{1+\frac{1}{s}}\,d\tau$$

$$+ \frac{8}{\sqrt{\pi}}|\alpha|\,\varepsilon^b\,\|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{1/2} \int_0^t \|\partial_x v^\varepsilon(\tau)\|_{L^2(\mathbb{R})}\,\|\partial_x u^\varepsilon(\tau)\|_{L^2(\mathbb{R})}^{3/2}\,d\tau$$

$$+ |\alpha| \int_{\mathbb{R}} |v^\varepsilon(t,x)|\,|u^\varepsilon(t,x)|^2\,dx.$$

Then, we have

$$\int_{\mathbb{R}} |(-\Delta)^{s/2}u^\varepsilon(t,x)|^2\,dx + \varepsilon^a \int_{\mathbb{R}} |\partial_x u^\varepsilon(t,x)|^2\,dx + \frac{1}{2}\int_{\mathbb{R}} |u^\varepsilon(t,x)|^4\,dx$$

$$\le \|(-\Delta)^{s/2}u_0^\varepsilon\|_{L^2(\mathbb{R})}^2 + \varepsilon^a\,\|\partial_x u_0^\varepsilon\|_{L^2(\mathbb{R})}^2 + \frac{1}{2}\|u_0^\varepsilon\|_{L^4(\mathbb{R})}^4 + \|u_0^\varepsilon\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |v_0^\varepsilon(x)|\,|u_0^\varepsilon(x)|\,dx$$

$$+ |\alpha|\Big(\frac{4}{\sqrt{\pi}} + \frac{4}{\sqrt{\pi(2s-1)}}\Big)\|g'\|_{L^\infty(\mathbb{R})}\,\|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{1-\frac{1}{2s}} \int_0^t \|(-\Delta)^{s/4}v^\varepsilon(\tau)\|_{L^2(\mathbb{R})}\,\|(-\Delta)^{s/2}u^\varepsilon(\tau)\|_{L^2(\mathbb{R})}^{1+\frac{1}{2s}}\,d\tau$$

$$+ 8|\alpha|\,|\beta|\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2 \,\|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{3-\frac{1}{s}} \int_0^t \|(-\Delta)^{s/2}u^\varepsilon(\tau)\|_{L^2(\mathbb{R})}^{1+\frac{1}{s}}\,d\tau$$

$$+ \frac{8}{\sqrt{\pi}}|\alpha|\,\varepsilon^b\,\|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{1/2} \int_0^t \|\partial_x v^\varepsilon(\tau)\|_{L^2(\mathbb{R})}\,\|\partial_x u^\varepsilon(\tau)\|_{L^2(\mathbb{R})}^{3/2}\,d\tau$$

$$+ \int_{\mathbb{R}} (\sqrt{2}\,|\alpha|\,|v^\varepsilon(t,x)|)\,\Big(\frac{|u^\varepsilon(t,x)|^2}{\sqrt{2}}\Big)\,dx,$$

from which follows that

$$\begin{aligned}
&\int_{\mathbb{R}} |(-\Delta)^{s/2}u^\varepsilon(t,x)|^2\,dx + \varepsilon^a \int_{\mathbb{R}} |\partial_x u^\varepsilon(t,x)|^2\,dx + \frac{1}{4}\int_{\mathbb{R}} |u^\varepsilon(t,x)|^4\,dx\\
&\quad\le \|(-\Delta)^{s/2}u_0^\varepsilon\|_{L^2(\mathbb{R})}^2 + \varepsilon^a\,\|\partial_x u_0^\varepsilon\|_{L^2(\mathbb{R})}^2 + \frac{1}{2}\|u_0^\varepsilon\|_{L^4(\mathbb{R})}^4 + \|u_0^\varepsilon\|_{L^\infty(\mathbb{R})}\|v_0^\varepsilon\|_{L^2(\mathbb{R})}\|u_0^\varepsilon\|_{L^2(\mathbb{R})}\\
&\quad+ |\alpha|\Big(\frac{4}{\sqrt{\pi}} + \frac{4}{\sqrt{\pi(2s-1)}}\Big)\|g'\|_{L^\infty(\mathbb{R})}\,\|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{1-\frac{1}{2s}} \int_0^t \|(-\Delta)^{s/4}v^\varepsilon(\tau)\|_{L^2(\mathbb{R})}\,\|(-\Delta)^{s/2}u^\varepsilon(\tau)\|_{L^2(\mathbb{R})}^{1+\frac{1}{2s}}\,d\tau\\
&\quad+ 8|\alpha|\,|\beta|\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2 \,\|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{3-\frac{1}{s}} \int_0^t \|(-\Delta)^{s/2}u^\varepsilon(\tau)\|_{L^2(\mathbb{R})}^{1+\frac{1}{s}}\,d\tau\\
&\quad+ \frac{8}{\sqrt{\pi}}|\alpha|\,\varepsilon^b\|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{1/2} \int_0^t \|\partial_x v^\varepsilon(\tau)\|_{L^2(\mathbb{R})}\,\|\partial_x u^\varepsilon(\tau)\|_{L^2(\mathbb{R})}^{3/2}\,d\tau,\\
&\quad+ |\alpha|^2 \int_{\mathbb{R}} |v^\varepsilon(t,x)|^2\,dx.
\end{aligned} \tag{3.26}$$

3. Now, replacing (3.24) in (3.26), we have

$$\int_{\mathbb{R}} |(-\Delta)^{s/2} u^\varepsilon(t,x)|^2\, dx + \varepsilon^a \int_{\mathbb{R}} |\partial_x u^\varepsilon(t,x)|^2\, dx + \frac{1}{4}\int_{\mathbb{R}} |u^\varepsilon(t,x)|^4\, dx$$

$$\leq \|(-\Delta)^{s/2} u_0^\varepsilon\|^2_{L^2(\mathbb{R})} + \varepsilon^a\, \|\partial_x u_0^\varepsilon\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|u_0^\varepsilon\|^4_{L^4(\mathbb{R})} + \|u_0^\varepsilon\|_{L^\infty(\mathbb{R})}\|v_0^\varepsilon\|_{L^2(\mathbb{R})}\|u_0^\varepsilon\|_{L^2(\mathbb{R})}$$

$$+ |\alpha|\Big(\frac{4}{\sqrt{\pi}} + \frac{4}{\sqrt{\pi(2s-1)}}\Big)\, \|g'\|_{L^\infty(\mathbb{R})}\, \|u_0^\varepsilon\|^{1-\frac{1}{2s}}_{L^2(\mathbb{R})} \int_0^t \|(-\Delta)^{s/4} v^\varepsilon(\tau)\|_{L^2(\mathbb{R})}\, \|(-\Delta)^{s/2} u^\varepsilon(\tau)\|^{1+\frac{1}{2s}}_{L^2(\mathbb{R})}\, d\tau$$

$$+ 8|\alpha|\, |\beta|\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2\, \|u_0^\varepsilon\|^{3-\frac{1}{s}}_{L^2(\mathbb{R})} \int_0^t \|(-\Delta)^{s/2} u^\varepsilon(\tau)\|^{1+\frac{1}{s}}_{L^2(\mathbb{R})}\, d\tau$$

$$+ \frac{8}{\sqrt{\pi}}|\alpha|\, \varepsilon^b \|u_0^\varepsilon\|^{1/2}_{L^2(\mathbb{R})} \int_0^t \|\partial_x v^\varepsilon(\tau)\|_{L^2(\mathbb{R})}\, \|\partial_x u^\varepsilon(\tau)\|^{3/2}_{L^2(\mathbb{R})}\, d\tau,$$

$$+ |\alpha|^2 \|v_0^\varepsilon\|^2_{L^2(\mathbb{R})} + \frac{16\, |\alpha|^2\beta^2}{m}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2 \|u_0^\varepsilon\|^{2-\frac{1}{s}}_{L^2(\mathbb{R})} \int_0^t \|(-\Delta)^{s/2} u^\varepsilon(\tau)\|^{2+\frac{1}{s}}_{L^2(\mathbb{R})}\, d\tau$$

or conveniently we write

$$\begin{aligned}
1 + \int_{\mathbb{R}} |(-\Delta)^{s/2} u^\varepsilon(t,x)|^2\, dx + \varepsilon^a \int_{\mathbb{R}} |\partial_x u^\varepsilon(t,x)|^2\, dx + \frac{1}{4}\int_{\mathbb{R}} |u^\varepsilon(t,x)|^4\, dx \leq \theta(t) := 1 \\
+ \|(-\Delta)^{s/2} u_0^\varepsilon\|^2_{L^2(\mathbb{R})} + \varepsilon^a\, \|\partial_x u_0^\varepsilon\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|u_0^\varepsilon\|^4_{L^4(\mathbb{R})} \\
+ \|u_0^\varepsilon\|_{L^\infty(\mathbb{R})}\|v_0^\varepsilon\|_{L^2(\mathbb{R})}\|u_0^\varepsilon\|_{L^2(\mathbb{R})} + |\alpha|^2\|v_0^\varepsilon\|^2_{L^2(\mathbb{R})} \\
+ |\alpha|\Big(\frac{4}{\sqrt{\pi}} + \frac{4}{\sqrt{\pi(2s-1)}}\Big)\, \|g'\|_{L^\infty(\mathbb{R})}\, \|u_0^\varepsilon\|^{1-\frac{1}{2s}}_{L^2(\mathbb{R})} \int_0^t \|(-\Delta)^{s/4} v^\varepsilon(\tau)\|_{L^2(\mathbb{R})}\, \|(-\Delta)^{s/2} u^\varepsilon(\tau)\|^{1+\frac{1}{2s}}_{L^2(\mathbb{R})}\, d\tau \\
+ 8|\alpha|\, |\beta|\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2\, \|u_0^\varepsilon\|^{3-\frac{1}{s}}_{L^2(\mathbb{R})} \int_0^t \|(-\Delta)^{s/2} u^\varepsilon(\tau)\|^{1+\frac{1}{s}}_{L^2(\mathbb{R})}\, d\tau \\
+ \frac{8}{\sqrt{\pi}}|\alpha|\, \varepsilon^b \|u_0^\varepsilon\|^{1/2}_{L^2(\mathbb{R})} \int_0^t \|\partial_x v^\varepsilon(\tau)\|_{L^2(\mathbb{R})}\, \|\partial_x u^\varepsilon(\tau)\|^{3/2}_{L^2(\mathbb{R})}\, d\tau, \\
+ \frac{16|\alpha|^2\beta^2}{m}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2 \|u_0^\varepsilon\|^{2-\frac{1}{s}}_{L^2(\mathbb{R})} \int_0^t \|(-\Delta)^{s/2} u^\varepsilon(\tau)\|^{2+\frac{1}{s}}_{L^2(\mathbb{R})}\, d\tau.
\end{aligned} \tag{3.27}$$

From the above definition, we have

$$\begin{aligned}
\theta'(t) \leq |\alpha|\Big(\frac{4}{\sqrt{\pi}} + \frac{4}{\sqrt{\pi(2s-1)}}\Big)\, \|g'\|_{L^\infty(\mathbb{R})}\, \|u_0^\varepsilon\|^{1-\frac{1}{2s}}_{L^2(\mathbb{R})}\, \|(-\Delta)^{s/4} v^\varepsilon(t)\,\|_{L^2(\mathbb{R})}\, \theta(t)^{\frac{1}{2}+\frac{1}{4s}} \\
+ 8|\alpha|\, |\beta|\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2\, \|u_0^\varepsilon\|^{3-\frac{1}{s}}_{L^2(\mathbb{R})}\, \theta(t)^{\frac{1}{2}+\frac{1}{2s}} \\
+ \frac{8|\alpha|\, \varepsilon^b}{\sqrt{\pi}\varepsilon^{3a/4}}\, \|u_0^\varepsilon\|^{1/2}_{L^2(\mathbb{R})}\, \|\partial_x v^\varepsilon(t)\|_{L^2(\mathbb{R})}\, \theta(t)^{3/4} + \frac{16|\alpha|^2\beta^2}{m}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2 \|u_0^\varepsilon\|^{2-\frac{1}{s}}_{L^2(\mathbb{R})}\, \theta(t)^{1+\frac{1}{2s}},
\end{aligned}$$

where we have used (3.27). Since $1/2 < s < 1$, then

$$\frac{3}{4} < \frac{1}{2} + \frac{1}{4s} < 1, \quad 1 < \frac{1}{2} + \frac{1}{2s} < \frac{3}{2},$$

and consequently dividing the above inequality by $\theta(t)^{\frac{1}{2}+\frac{1}{4s}}$, we obtain

$$\frac{1}{\frac{1}{2}-\frac{1}{4s}}\,[\theta(t)^{\frac{1}{2}-\frac{1}{4s}}]' e^t \le |\alpha|\Big(\frac{4}{\sqrt{\pi}} + \frac{4}{\sqrt{\pi(2s-1)}}\Big)\, \|g'\|_{L^\infty(\mathbb{R})}\, \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{1-\frac{1}{2s}}\, \|(-\Delta)^{s/4} v^\varepsilon(t)\|_{L^2(\mathbb{R})}\, e^t$$

$$+ 8|\alpha|\,|\beta|\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2 \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{3-\frac{1}{s}}\, \theta(t)^{\frac{1}{4s}}\, e^t$$

$$+ \frac{8}{\sqrt{\pi}}|\alpha|\,\varepsilon^b\,\varepsilon^{-3a/4}\, \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{1/2}\, \|\partial_x v^\varepsilon(t)\|_{L^2(\mathbb{R})}\, e^t$$

$$+ \frac{16|\alpha|^2\beta^2}{m}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2 \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{2-\frac{1}{s}}\, \theta(t)^{\frac{1}{2}+\frac{1}{4s}}\, e^t,$$

where we have multiplied the inequality by e^t. Then, integrating from 0 to $t > 0$

$$\int_0^t [\theta(\tau)^{\frac{1}{2}-\frac{1}{4s}}]'\, e^\tau\, d\tau$$

$$\le \frac{|\alpha|(4s-2)}{8s}\Big(\frac{4}{\sqrt{\pi}} + \frac{4}{\sqrt{\pi(2s-1)}}\Big)\, \|g'\|_{L^\infty(\mathbb{R})}\, \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{1-\frac{1}{2s}} \int_0^t \|(-\Delta)^{s/4} v^\varepsilon(\tau)\|_{L^2(\mathbb{R})}\, e^\tau d\,\tau$$

$$+ \frac{8|\alpha|\,|\beta|(4s-2)}{8s}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2 \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{3-\frac{1}{s}} \int_0^t \theta(\tau)^{\frac{1}{4s}}\, e^\tau\, d\tau$$

$$+ \frac{8|\alpha|\,\varepsilon^b\,\varepsilon^{-3a/4}(4s-2)}{8\sqrt{\pi}s}\, \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{1/2} \int_0^t \|\partial_x v^\varepsilon(\tau)\|_{L^2(\mathbb{R})}\, e^\tau\, d\tau$$

$$+ \frac{16|\alpha|^2\beta^2(4s-2)}{8ms}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2 \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{2-\frac{1}{s}} \int_0^t \theta(\tau)^{\frac{1}{2}+\frac{1}{4s}}\, e^\tau\, d\tau$$

and integrating by parts in the left hand side

$$
\begin{aligned}
&\theta(t)^{\frac{1}{2}-\frac{1}{4s}}\ e^t \leq \theta(0)^{\frac{1}{2}-\frac{1}{4s}} + \int_0^t \theta(\tau)^{\frac{1}{2}-\frac{1}{4s}}\ e^\tau\ d\tau \\
&\quad + \frac{|\alpha|(2s-1)}{\sqrt{2}s}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big) \|g'\|_{L^\infty(\mathbb{R})}\ \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{1-\frac{1}{2s}}\ (e^{2t}-1)^{1/2} \\
&\quad \times \Big(\frac{\|v_0^\varepsilon\|_{L^2(\mathbb{R})}^2}{m} + \frac{16\beta^2}{m^2}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2 \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{2-\frac{1}{s}} \int_0^t \|(-\Delta)^{s/2}u^\varepsilon(\tau)\|_{L^2(\mathbb{R})}^{2+\frac{1}{s}}\ d\tau\Big)^{1/2} \\
&\quad + \frac{|\alpha|\ |\beta|(2s-1)}{s}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2 \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{3-\frac{1}{s}} \int_0^t \theta(\tau)^{\frac{1}{4s}}\ e^\tau\ d\tau \\
&\quad + \frac{2|\alpha|\ \varepsilon^b\ \varepsilon^{-3a/4}(2s-1)}{\sqrt{2\pi}s}\ \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{1/2}(e^{2t}-1)^{1/2} \\
&\quad \times \Big(\frac{\|v_0^\varepsilon\|_{L^2(\mathbb{R})}^2}{2\varepsilon^b} + \frac{8\beta^2}{\varepsilon^b\ m}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2 \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{2-\frac{1}{s}} \int_0^t \|(-\Delta)^{s/2}u^\varepsilon(\tau)\|_{L^2(\mathbb{R})}^{2+\frac{1}{s}}\ d\tau\Big)^{1/2} \\
&\quad + \frac{16|\alpha|^2\beta^2(2s-1)}{ms}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2 \|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{2-\frac{1}{s}} \int_0^t \theta(\tau)^{\frac{1}{2}+\frac{1}{4s}}\ e^\tau\ d\tau,
\end{aligned} \tag{3.28}
$$

where we have used Holder's inequality and equation (3.24) two times.

4. The goal now is to apply the Generalized Grownwall Lemma (Section 2.4). We observe that

$$
\begin{aligned}
\theta(0) = \Big[&1 + \|(-\Delta)^{s/2}u_0^\varepsilon\|_{L^2(\mathbb{R})}^2 + \varepsilon^a\ \|\partial_x u_0^\varepsilon\|_{L^2(\mathbb{R})}^2 + \frac{1}{2}\|u_0^\varepsilon\|_{L^4(\mathbb{R})}^4 \\
&+ \|u_0^\varepsilon\|_{L^\infty(\mathbb{R})}\|v_0^\varepsilon\|_{L^2(\mathbb{R})}\|u_0^\varepsilon\|_{L^2(\mathbb{R})} + |\alpha|^2\|v_0^\varepsilon\|_{L^2(\mathbb{R})}^2\Big],
\end{aligned}
$$

hence from that and taking the square in equation (3.28), we have

$$
\begin{aligned}
\theta(t)^{1-\frac{1}{2s}}\ e^{2t} \leq{}& 2^6\Big[1 + \|(-\Delta)^{s/2}u_0^\varepsilon\|^2_{L^2(\mathbb{R})} + \varepsilon^a\ \|\partial_x u_0^\varepsilon\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|u_0^\varepsilon\|^4_{L^4(\mathbb{R})} \\
&+ \|u_0^\varepsilon\|_{L^\infty(\mathbb{R})}\|v_0^\varepsilon\|_{L^2(\mathbb{R})}\|u_0^\varepsilon\|_{L^2(\mathbb{R})} + |\alpha|^2\|v_0^\varepsilon\|^2_{L^2(\mathbb{R})}\Big]^{1-\frac{1}{2s}} \\
&+ 2^6 t^2\Big(\fint_0^t \theta(\tau)^{\frac{1}{2}-\frac{1}{4s}}\ e^\tau\ d\tau\Big)^2 + \frac{2^5|\alpha|^2\ (2s-1)^2}{s^2}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2\ \|g'\|^2_{L^\infty(\mathbb{R})}\ \|u_0^\varepsilon\|^{2-\frac{1}{s}}_{L^2(\mathbb{R})}\ e^{2t} \\
&\times\Big(\frac{\|v_0^\varepsilon\|^2_{L^2(\mathbb{R})}}{m} + \frac{16\beta^2}{m^2}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2\ \|u_0^\varepsilon\|^{2-\frac{1}{s}}_{L^2(\mathbb{R})}\int_0^t \theta(\tau)^{1+\frac{1}{2s}}\ d\tau\Big) \\
&+ \frac{2^6|\alpha|^2\ |\beta|^2(2s-1)^2 t^2}{s^2}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^4\ \|u_0^\varepsilon\|^{6-\frac{2}{s}}_{L^2(\mathbb{R})}\Big(\fint_0^t \theta(\tau)^{\frac{1}{4s}}\ e^\tau\ d\tau\Big)^2 \\
&+ \frac{2^7|\alpha|^2\ \varepsilon^{2b}\ \varepsilon^{-3a/2}\ (2s-1)^2}{\pi s^2}\ \|u_0^\varepsilon\|_{L^2(\mathbb{R})}\ e^{2t} \\
&\times\Big(\frac{\|v_0^\varepsilon\|^2_{L^2(\mathbb{R})}}{2\varepsilon^b} + \frac{8\beta^2}{m\varepsilon^b}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2\ \|u_0^\varepsilon\|^{2-\frac{1}{s}}_{L^2(\mathbb{R})}\int_0^t \theta(\tau)^{1+\frac{1}{2s}}\ d\tau\Big) \\
&+ \frac{2^{14}|\alpha|^4\ \beta^4(2s-1)^2}{m^2 s^2}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^4\|u_0^\varepsilon\|^{4-\frac{2}{s}}_{L^2(\mathbb{R})}\ t^2\Big(\fint_0^t \theta(\tau)^{\frac{1}{2}+\frac{1}{4s}}\ e^\tau\ d\tau\Big)^2.
\end{aligned}
$$

Then, we apply Jesen's inequality to obtain

$$
\begin{aligned}
\theta(t)^{1-\frac{1}{2s}}\ e^{2t} \leq 2^6\Big[& 1+\|(-\Delta)^{s/2}u_0^\varepsilon\|^2_{L^2(\mathbb{R})}+\varepsilon^a\ \|\partial_x u_0^\varepsilon\|^2_{L^2(\mathbb{R})}+\frac{1}{2}\|u_0^\varepsilon\|^4_{L^4(\mathbb{R})} \\
& +\|u_0^\varepsilon\|_{L^\infty(\mathbb{R})}\|v_0^\varepsilon\|_{L^2(\mathbb{R})}\|u_0^\varepsilon\|_{L^2(\mathbb{R})}+|\alpha|^2\|v_0^\varepsilon\|^2_{L^2(\mathbb{R})}\Big]^{1-\frac{1}{2s}} \\
& +2^6T\int_0^t\theta(\tau)^{1-\frac{1}{2s}}\ e^{2\tau}\ d\tau+\frac{2^5|\alpha|^2\ (2s-1)^2}{s^2}\Big(\frac{1}{\sqrt{\pi}}+\frac{1}{\sqrt{\pi(2s-1)}}\Big)^2\ \|g'\|^2_{L^\infty(\mathbb{R})}\ \|u_0^\varepsilon\|^{2-\frac{1}{s}}_{L^2(\mathbb{R})}\ e^{2T} \\
& \times\Big(\frac{\|v_0^\varepsilon\|^2_{L^2(\mathbb{R})}}{m}+\frac{16\beta^2}{m^2}\Big(\frac{1}{\sqrt{\pi}}+\frac{1}{\sqrt{\pi(2s-1)}}\Big)^2\ \|u_0^\varepsilon\|^{2-\frac{1}{s}}_{L^2(\mathbb{R})}\int_0^t\theta(\tau)^{1+\frac{1}{2s}}\ d\tau\Big) \\
& +\frac{2^6|\alpha|^2\ |\beta|^2(2s-1)^2T}{s^2}\Big(\frac{1}{\sqrt{\pi}}+\frac{1}{\sqrt{\pi(2s-1)}}\Big)^4\ \|u_0^\varepsilon\|^{6-\frac{2}{s}}_{L^2(\mathbb{R})}\int_0^t\theta(\tau)^{\frac{1}{2s}}\ e^{2\tau}\ d\tau \\
& +\frac{2^7|\alpha|^2\ \varepsilon^{2b}\ \varepsilon^{-3a/2}\ (2s-1)^2}{\pi s^2}\ \|u_0^\varepsilon\|_{L^2(\mathbb{R})}\ e^{2T} \\
& \times\Big(\frac{\|v_0^\varepsilon\|^2_{L^2(\mathbb{R})}}{2\varepsilon^b}+\frac{8\beta^2}{m\varepsilon^b}\Big(\frac{1}{\sqrt{\pi}}+\frac{1}{\sqrt{\pi(2s-1)}}\Big)^2\ \|u_0^\varepsilon\|^{2-\frac{1}{s}}_{L^2(\mathbb{R})}\int_0^t\theta(\tau)^{1+\frac{1}{2s}}\ d\tau\Big) \\
& +\frac{2^{14}|\alpha|^4\ \beta^4(2s-1)^2}{m^2s^2}\Big(\frac{1}{\sqrt{\pi}}+\frac{1}{\sqrt{\pi(2s-1)}}\Big)^4\|u_0^\varepsilon\|^{4-\frac{2}{s}}_{L^2(\mathbb{R})}\ T\int_0^t\theta(\tau)^{1+\frac{1}{2s}}\ e^{2\tau}\ d\tau.
\end{aligned}
$$

Moreover, after an algebraic manipulation and using that $e^t > 1$ for any $t > 0$, we may write

$$
\begin{aligned}
\theta(t)^{1-\frac{1}{2s}}\ e^{2t} \leq C + C_1\int_0^t\theta(\tau)^{1-\frac{1}{2s}}\ e^{2\tau}\ d\tau & \\
+ C_2\int_0^t\theta(\tau)^{\frac{1}{2s}}\ e^{2\tau}\ d\tau + C_3\int_0^t\theta(\tau)^{1+\frac{1}{2s}}e^{2\tau}\ d\tau, &
\end{aligned} \tag{3.29}
$$

where

$$
\begin{aligned}
C &:= 2^6\Big[1 + \|(-\Delta)^{s/2}u_0^\varepsilon\|_{L^2(\mathbb{R})}^2 + \|\partial_x u_0^\varepsilon\|_{L^2(\mathbb{R})}^2 + \frac{1}{2}\|u_0^\varepsilon\|_{L^4(\mathbb{R})}^4 \\
&\quad + \|u_0^\varepsilon\|_{L^\infty(\mathbb{R})}\|v_0^\varepsilon\|_{L^2(\mathbb{R})}\|u_0^\varepsilon\|_{L^2(\mathbb{R})} + |\alpha|^2\|v_0^\varepsilon\|_{L^2(\mathbb{R})}^2\Big]^{1-\frac{1}{2s}} \\
&\quad + \frac{2^5|\alpha|^2\,(2s-1)^2}{ms^2}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2\,\|g'\|_{L^\infty(\mathbb{R})}^2\,\|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{2-\frac{1}{s}}\|v_0^\varepsilon\|_{L^2(\mathbb{R})}^2\,e^{2T} \\
&\quad + \frac{2^6|\alpha|^2\,\varepsilon^b\,\varepsilon^{-3a/2}\,(2s-1)^2}{\pi s^2}\,\|u_0^\varepsilon\|_{L^2(\mathbb{R})}\|v_0^\varepsilon\|_{L^2(\mathbb{R})}^2\,e^{2T}, \\
C_1 &:= 2^6 T, \\
C_2 &:= \frac{2^6|\alpha|^2\,|\beta|^2(2s-1)^2 T}{s^2}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^4\,\|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{6-\frac{2}{s}}, \\
C_3 &:= \frac{2^9|\alpha|^2\,\beta^2}{m^2 s^2}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^4\,\|g'\|_{L^\infty(\mathbb{R})}^2\,\|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{4-\frac{2}{s}}\,e^{2T} \\
&\quad + \frac{2^{10}|\alpha|^2\,\beta^2\,\varepsilon^b\,\varepsilon^{-3a/2}\,(2s-1)^2}{m\pi s^2}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2\,\|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{3-\frac{1}{s}}\,e^{2T} \\
&\quad + \frac{2^{14}|\alpha|^4\,\beta^4(2s-1)^2}{m^2 s^2}\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^4\|u_0^\varepsilon\|_{L^2(\mathbb{R})}^{4-\frac{2}{s}}\,T.
\end{aligned}
$$

Therefore, as $3a \leq 2b$ the above positive constants C, C_1, C_2 and C_3 are independent of $\varepsilon > 0$. Now, since

$$
(1-\frac{1}{2s})(\frac{2s+1}{2s-1}) = 1 + \frac{1}{2s}, \quad \text{and} \quad (1-\frac{1}{2s})(\frac{1}{2s-1}) = \frac{1}{2s},
$$

then we have from (3.29)

$$
\begin{aligned}
\theta(t)^{1-\frac{1}{2s}}\,e^{2t} &\leq C + C_1\int_0^t \theta(\tau)^{1-\frac{1}{2s}}\,e^{2\tau}\,d\tau \\
&\quad + C_2\int_0^t (\theta(\tau)^{1-\frac{1}{2s}})^{\frac{1}{2s-1}}\,e^{2\tau}\,d\tau + C_3\int_0^t (\theta(\tau)^{1-\frac{1}{2s}})^{\frac{2s+1}{2s-1}} e^{2\tau}\,d\tau.
\end{aligned}
$$

For each $1/2 < s < 1$ we have

$$
1 < \frac{1}{2s-1} < \frac{2s+1}{2s-1}
$$

therefore from the above inequality we may write

$$
\begin{aligned}
\theta(t)^{1-\frac{1}{2s}}\,e^{2t} &\leq C + C_1\int_0^t \theta(\tau)^{1-\frac{1}{2s}}\,e^{2\tau}\,d\tau \\
&\quad + C_2\int_0^t (\theta(\tau)^{1-\frac{1}{2s}}\,e^{2\tau})^{\frac{2s+1}{2s-1}}\,d\tau + C_3\int_0^t (\theta(\tau)^{1-\frac{1}{2s}} e^{2\tau})^{\frac{2s+1}{2s-1}}\,d\tau
\end{aligned}
$$

or defining $\eta(t) := \theta(t)^{1-\frac{1}{2s}}\ e^{2t}$

$$\eta(t) \leq C + \int_0^t \Big[C_1 \eta(\tau) + \Big(C_2 + C_3\Big)(\eta(\tau))^{\frac{2s+1}{2s-1}} \Big]\, d\tau. \tag{3.30}$$

Therefore, applying the Generalized Grownwall Lemma, more precisely (2.19) with $\alpha = \frac{2s+1}{2s-1} > 1$, we must have for each $s \in (1/2, 1)$

$$\begin{aligned} C &< \Big\{ \exp\Big[\Big(1 - \frac{2s+1}{2s-1}\Big) \int_0^T C_1\, d\tau\Big]\Big\}^{\frac{1}{\frac{2s+1}{2s-1}-1}} \Big\{\Big(\frac{2s+1}{2s-1} - 1\Big) \int_0^T \Big(C_2 + C_3\Big)\, d\tau\Big\}^{-\frac{1}{\frac{2s+1}{2s-1}-1}} \\ &= \exp\big[-C_1 T\big] \Big\{ \frac{2}{2s-1} \Big(C_2 + C_3\Big)\, T\Big\}^{\frac{1-2s}{2}} \\ &= \frac{(2s-1)^{\frac{2s-1}{2}}\ \exp\big[-C_1 T\big]}{\Big\{2\Big(C_2 + C_3\Big)\, T\Big\}^{\frac{2s-1}{2}}} \end{aligned}$$

or equivalently

$$C\, \Big(C_2 + C_3\Big)^{\frac{2s-1}{2}} \exp[64T^2]\, T^{\frac{2s-1}{2}} \leq \Big(\frac{2s-1}{2}\Big)^{\frac{2s-1}{2}}. \tag{3.31}$$

One remarks that

$$\lim_{s \to \frac{1}{2}} \Big(\frac{2s-1}{2}\Big)^{\frac{2s-1}{2}} = 1.$$

Hence for any $s \in (1/2, 1)$ fixed, there exists $\alpha_0 > 0$ and $E_0 > 0$, such that condition (3.31) is satisfied when $\|u_0\|_{L^2(\mathbb{R})} \leq E_0$, or $|\alpha| \leq \alpha_0$. In fact, if there is no coupling, that is $\alpha = 0$ ($C_2 = C_3 = 0$), then condition (3.31) is trivially satisfied. Consequently, we have

$$\begin{aligned} \eta(t) \leq C\, &\Big\{ \exp\Big[\Big(1 - \frac{2s+1}{2s-1}\Big) \int_0^t C_1\, d\tau\Big] \\ &- C^{-1} \Big(\frac{2s+1}{2s-1} - 1\Big) \int_0^t \Big(C_2 + C_3\Big) \exp\Big[\Big(1 - \frac{2s+1}{2s-1}\Big) \int_\tau^t C_1\, dr\Big]\, d\tau\Big\}^{\frac{1}{\frac{2s+1}{2s-1}-1}} \\ &= C\, \Big\{ \exp\Big[\frac{2C_1}{1-2s} t\Big] - C^{-1} \frac{2}{2s-1} \Big(C_2 + C_3\Big) \int_0^t \exp\big[\frac{2C_1}{1-2s}(t-\tau)\big]\, d\tau\Big\}^{\frac{2s-1}{2}} \\ &= C\, \Big\{ \exp\Big[\frac{2C_1}{1-2s} t\Big] - \frac{C^{-1}\Big(C_2 + C_3\Big)}{C_1} \Big(1 - \exp\Big[\frac{2C_1}{1-2s} t\Big]\Big)\Big\}^{\frac{2s-1}{2}}, \end{aligned}$$

from which follows the proof of the theorem. □

3.5 Limit transition

The aim of this section is to pass to the limit in (3.1), which is to say, as the parameter $\varepsilon > 0$ goes to zero. More precisely, from the equivalence of mild solutions and weak solutions, we obtain a weak formulation from (3.6), then we pass $\varepsilon \to 0$, and obtain a solution of the Cauchy problem (1.1) in the sense of Definition 1.1.

First, we have the following

Lemma 3.3. *Let $\alpha_0 > 0, E_0 > 0$ be given by Theorem 3.1, such that, $|\alpha| \leq \alpha_0$ or $\|u_0\|_{L^2(\mathbb{R})} \leq E_0$. Then, the unique mild solution $(u^\varepsilon, v^\varepsilon)$ of* (3.1) *satisfies, for any $T > 0$*

$$
\begin{aligned}
& i\int_0^T \int_{\mathbb{R}} \Big(u^\varepsilon(t,x)\, \partial_t\overline{\varphi}(t,x) + (-\Delta)^{s/2}u^\varepsilon(t,x)\, (-\Delta)^{s/2}\overline{\varphi}(t,x)\Big)dxdt + i\int_{\mathbb{R}} u_0^\varepsilon(x)\, \overline{\varphi}(0,x)dx \\
& - \varepsilon^a \int_0^T \int_{\mathbb{R}} u^\varepsilon(t,x)\, \Delta\overline{\varphi}\, dxdt + \alpha \int_0^T \int_{\mathbb{R}} v^\varepsilon(t,x)\, u^\varepsilon(t,x)\, \overline{\varphi}(t,x)\, dxdt \\
& \quad + \int_0^T \int_{\mathbb{R}} |u^\varepsilon(t,x)|^2\, u^\varepsilon(t,x)\, \overline{\varphi}(t,x)\, dxdt = 0,
\end{aligned} \tag{3.32}
$$

$$
\begin{aligned}
& \int_0^T \int_{\mathbb{R}} v^\varepsilon(t,x)\, \partial_t\psi(t,x) - g(v^\varepsilon(t,x))\, (-\Delta)^{s/2}\psi(t,x)\, dxdt + \int_{\mathbb{R}} v_0^\varepsilon(x)\, \psi(0,x)\, dx \\
& \quad + \varepsilon^b \int_0^T \int_{\mathbb{R}} v^\varepsilon(t,x)\, \Delta\psi(t,x)\, dxdt + \beta \int_0^T \int_{\mathbb{R}} |u^\varepsilon|^2(t,x)\, (-\Delta)^{s/2}\psi(t,x)\, dtdx = 0
\end{aligned} \tag{3.33}
$$

for each test functions $\varphi, \psi \in C_c^\infty((-\infty, T) \times \mathbb{R})$, with φ being complex-valued and ψ real-valued.

Moreover, there exists a positive constant C independent of $\varepsilon > 0$, such that

$$
\int_0^T \|\partial_t u^\varepsilon(t)\|_{H^{-1}(\mathbb{R})} dt \leq C, \qquad \int_0^T \|\partial_t v^\varepsilon(t)\|_{H^{-1}(\mathbb{R})} dt \leq C. \tag{3.34}
$$

Proof. Equations (3.32), (3.33) are obtained from (3.6), that is, applying the equivalence between mild solutions and weak solutions, (see Ball [2], p. 371), which are obtained via functional analysis arguments. Similarly, the inequalities in equation (3.34) are obtained from the weak formulation, i.e. equations (3.32) and (3.33), applying standard functional analysis results, the uniform boundedness of u_0^ε, v_0^ε, and also the uniform estimates from Lemma 3.2 and Theorem 3.1. this is

As $u^\varepsilon(t) \in H^1(\mathbb{R})$ then $\Delta u^\varepsilon(t), (-\Delta)^s u^\varepsilon(t) \in H^{-1}(\mathbb{R})$ and

$$
\begin{aligned}
\langle (-\Delta)^s u^\varepsilon(t), w \rangle_{H^{-1}(\mathbb{R}) \times H^1(\mathbb{R})} = & \int_{\mathbb{R}} (-\Delta)^{s/2} u^\varepsilon(t,x)\, (-\Delta)^{s/2} w(x)\, dx \\
& \int_{\mathbb{R}} |(-\Delta)^{s/2} u^\varepsilon(t,x)\, (-\Delta)^{s/2} w(x)| dx
\end{aligned}
$$

$$
\begin{aligned}
&\le \|(-\Delta)^{s/2}u^{\varepsilon}(t)\|_{L^2(\mathbb{R})}\ \|(-\Delta)^{s/2}w\|_{L^2(\mathbb{R})}\\
&\le \|u^{\varepsilon}(t)\|_{H^s(\mathbb{R})}\ \|w\|_{H^s(\mathbb{R})}\\
&\le \|u^{\varepsilon}(t)\|_{H^s(\mathbb{R})}\ \|w\|_{H^1(\mathbb{R})}\\
&\le \|u^{\varepsilon}(t)\|_{H^s(\mathbb{R})}
\end{aligned}
$$

for all $w \in H^1(\mathbb{R})$ such that $\|w\|_{H^1(\mathbb{R})} \le 1$. From the estimates obtained in Theorem 3.1, we have

$$\left\{(-\Delta)^s u^{\varepsilon}\right\}_{\varepsilon} \text{ bounded in } L^{\infty}(0,T;H^{-1}(\mathbb{R})).$$

$$
\begin{aligned}
\langle \varepsilon^a\ \Delta u^{\varepsilon}(t), w\rangle_{H^{-1}(\mathbb{R})\times H^1(\mathbb{R})} &= \varepsilon^a \int_{\mathbb{R}} \partial_x u^{\varepsilon}(t,x)\ \partial_x w(x)\ dx\\
&\le |\varepsilon^a| \int_{\mathbb{R}} |\partial_x u^{\varepsilon}(t,x)\ \partial_x w(x)|\ dx\\
&\le \varepsilon^a \|\partial_x u^{\varepsilon}(t)\|_{L^2(\mathbb{R})}\ \|\partial_x w\|_{L^2(\mathbb{R})}\\
&\le \varepsilon^a\ \|\partial_x u^{\varepsilon}(t)\|_{L^2(\mathbb{R})}\ \|w\|_{H^1(\mathbb{R})}\\
&\le \varepsilon^a\ \|\partial_x u^{\varepsilon}(t)\|_{L^2(\mathbb{R})}
\end{aligned}
$$

for all $w \in H^1(\mathbb{R})$ such that $\|w\|_{H^1(\mathbb{R})} \le 1$. From the estimates obtained in Theorem 3.1, we have

$$\left\{\varepsilon^a \Delta u^{\varepsilon}\right\}_{\varepsilon} \text{ bounded in } L^{\infty}(0,T;H^{-1}(\mathbb{R})).$$

Then from the fractional schrodinger equation (in a distributional sense) we have

$$\left\{\partial_t u^{\varepsilon}\right\}_{\varepsilon} \text{ bounded in } L^{\infty}(0,T;H^{-1}(\mathbb{R})).$$

also, as $v^{\varepsilon}(t) \in H^1(\mathbb{R})$ then $\Delta v^{\varepsilon}(t) \in H^{-1}(\mathbb{R})$

$$
\begin{aligned}
\langle \varepsilon^b \Delta v^{\varepsilon}(t), w\rangle_{H^{-1}(\mathbb{R})\times H^1(\mathbb{R})} &= \varepsilon^b \int_{\mathbb{R}} \partial_x v^{\varepsilon}(t,x)\ \partial_x w(x)\ dx\\
&\le \varepsilon^b \int_{\mathbb{R}} |\partial_x v^{\varepsilon}(t,x)\ \partial_x w(x)|\ dx\\
&\le \varepsilon^b \|\partial_x v^{\varepsilon}(t)\|_{L^2(\mathbb{R})}\ \|\partial_x w\|_{L^2(\mathbb{R})}\\
&\le \varepsilon^b\ \|\partial_x v^{\varepsilon}(t)\|_{L^2(\mathbb{R})}\ \|w\|_{H^1(\mathbb{R})}\\
&\le \varepsilon^b\ \|\partial_x v^{\varepsilon}(t)\|_{L^2(\mathbb{R})}
\end{aligned}
$$

for all $w \in H^1(\mathbb{R})$ such that $\|w\|_{H^1(\mathbb{R})} \leq 1$. From the estimates obtained in Theorem 3.1, we have

$$\left\{\varepsilon^b \Delta v^\varepsilon\right\}_\varepsilon \text{ bounded in } L^2(0,T;H^{-1}(\mathbb{R})).$$

Then from the fractional porous medium equation (in a distributional sense) we have

$$\left\{\partial_t v^\varepsilon\right\}_\varepsilon \text{ bounded in } L^2(0,T;H^{-1}(\mathbb{R})).$$

inequalities (3.34) is demonstrated.

□

Now, we are ready to show the main result of this thesis:

Proof Main Theorem. 1. Under the conditions of Lemma 3.3, for each $\varepsilon > 0$, let $(u^\varepsilon, v^\varepsilon) \in C([0,T);H^1(\mathbb{R})) \times C([0,T);H^1(\mathbb{R}))$ be the unique mild solution of (3.1), satisfying (3.6) for any $T > 0$. Then, the pair$(u^\varepsilon(t,x), v^\varepsilon(t,x))$ satisfies the equations (3.32) and (3.33).

2. Now, to obtain (1.9), (1.10) we pass to the limit respectively in (3.32) and (3.33) as $\varepsilon \to 0^+$. Therefore, we need to show compactness of the sequences $\{u^\varepsilon\}_{\varepsilon>0}$, and $\{v^\varepsilon\}_{\varepsilon>0}$. From (3.17), (3.21), it follows that $\{u^\varepsilon\}_{\varepsilon>0}$ is (uniformly) bounded in $L^\infty(0,T;H^s(\mathbb{R}))$, hence it is possible to select a subsequence, still denoted by $\{u^\varepsilon\}_{\varepsilon>0}$, which converges weakly-$\star$ to u in $L^\infty(0,T;H^s(\mathbb{R}))$. Similarly, from (3.22) it follows that $\{v^\varepsilon\}_{\varepsilon>0}$ is (uniformly) bounded in $L^2(0,T;H^{s/2}(\mathbb{R}))$, hence it is also possible to select a subsequence, still denoted by $\{v^\varepsilon\}_{\varepsilon>0}$, which converges weakly to v in $L^2(0,T;H^{s/2}(\mathbb{R}))$. Due to a standard diagonalization procedure, these two weak convergences are enough to pass to the limit as $\varepsilon \to 0$ in the linear terms of the equations (3.32) and (3.33).

Applying the Lemma 2.5, for any compact set $K \subset \mathbb{R}$, the embedding of $H^s(K)$ in $L^2(K)$ is compact. Therefore, since the sequence $\{\partial_t u^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^2(0,T;H^{-1}(\mathbb{R}))$, we apply the Aubin-Lions's Theorem 2.2 and obtain (along a suitable subsequence) that u^ε converges strongly to u in $L^2(0,T;L^2(K))$, and thus, $u^\varepsilon(t,x) \to u(t,x)$ as $\varepsilon \to 0$ almost everywhere in $(0,T) \times \mathbb{R}$. Analogously, we obtain that $v^\varepsilon(t,x) \to v(t,x)$ as $\varepsilon \to 0$ almost everywhere in $(0,T) \times \mathbb{R}$. Hence from these two a.e. convergences, we apply the Dominated Convergence Theorem to pass to the limit as $\varepsilon \to 0$ in the nonlinear terms of the equations (3.32) and (3.33).

3. Finally, we obtain the solvability of the Cauchy problem (1.1) applying the Definition 1.1, which finish the proof. □

Chapter 4

Statement of the degenerate case

In this section we consider the initial value problem (1.2), that is to say, given an initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times L^2(\mathbb{R})$, we seek for a pair $(u(t,x), v(t,x))$ which satisfies the following system

$$\begin{cases} i\,\partial_t u - (-\Delta)^s u = \alpha\, v\, u + |u|^2 u, \\ \qquad\qquad \partial_t v = \beta\,(-\Delta)^{s/2}|u|^2, \end{cases}$$

posed in $(0,T)\times\mathbb{R}$, for any $T>0$, with parameter $\frac{1}{2} < s < 1$.

The definition bellow give us in which sense a pair $(u(t,x), v(t,x))$ is a weak solution to the Cauchy problem (1.2); one remarks the spatial regularity of $v(t,x)$.

Definition 4.1. *Given a initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times L^2(\mathbb{R})$, ($\frac{1}{2} < s < 1$), a pair*

$$(u,v) \in L^\infty(0,T; H^s(\mathbb{R}) \times L^2(0,T; L^2(\mathbb{R}))$$

is called a weak solution of the Cauchy problem (1.2)*, when it satisfies:*

$$\begin{aligned} i\int_0^T\!\!\int_{\mathbb{R}} \Big(u(t,x)\,\partial_t\overline{\varphi}(t,x) + (-\Delta)^{s/2}u(t,x)\,(-\Delta)^{s/2}\overline{\varphi}(t,x)\Big)dxdt + i\int_{\mathbb{R}} u_0(x)\,\overline{\varphi}(0,x)dx \\ + \alpha\int_0^T\!\!\int_{\mathbb{R}} v(t,x)\,u(t,x)\,\overline{\varphi}(t,x)dxdt + \int_0^T\!\!\int_{\mathbb{R}} |u(t,x)|^2\,u(t,x)\,\overline{\varphi}(t,x)dxdt = 0, \end{aligned} \tag{4.1}$$

$$\begin{aligned} \int_0^T\!\!\int_{\mathbb{R}} v(t,x)\,\partial_t\psi(t,x)dxdt + \int_{\mathbb{R}} v_0(x)\,\psi(0,x)dx \\ + \beta\int_0^T\!\!\int_{\mathbb{R}} |u|^2(t,x)\,(-\Delta)^{s/2}\psi(t,x)dxdt = 0, \end{aligned} \tag{4.2}$$

for each test function $\varphi, \psi \in C_c^\infty((-\infty,T)\times\mathbb{R})$, with φ being complex-valued and ψ real-valued.

One attempts to show the solvability of the Cauchy problem (1.2) is to follow Tsutsumi, Hatano [32]. More precisely, integrate with respect to t the second equation in (1.2) to get $v(t, x)$, and hence replace it into the first equation in (1.2), that is, obtaining an integro-differential equation for $u(t, x)$. We do not apply this strategy, and instead of that we perturb the equation for $v(t, x)$, conveniently, and make use of the results established before in Chapter 3.

4.1 Perturbed system

For each $\varepsilon \in (0, 1)$, we consider from (1.2) the following perturbed system

$$\begin{cases} i\,\partial_t u^\varepsilon - (-\Delta)^s u^\varepsilon = \alpha\, v^\varepsilon\, u^\varepsilon + |u^\varepsilon|^2 u^\varepsilon, \\ \partial_t v^\varepsilon + \varepsilon\,(-\Delta)^{s/2} v^\varepsilon = \beta\,(-\Delta)^{s/2} \big|u^\varepsilon\big|^2, \\ u^\varepsilon(0, x) = u_0^\varepsilon(x),\;\; v^\varepsilon(0, x) = v_0^\varepsilon(x), \end{cases} \tag{4.3}$$

where $(u_0^\varepsilon, v_0^\varepsilon) \in H^s(\mathbb{R}) \times H^{s/2}(\mathbb{R})$ is an approaching sequence converging strongly to $(u_0, v_0) \in H^s(\mathbb{R}) \times L^2(\mathbb{R})$.

The existence of weak solutions for (4.3) is given by Theorem 1.1 (Main Theorem). Therefore, it remains to obtain compactness for the family $\{(u^\varepsilon, v^\varepsilon)\}_{\varepsilon>0}$, and thus pass to the limit as $\varepsilon \to 0$ in (4.3) to get a weak solution to the Cauchy problem (1.2). To this end, we need as before a priori estimates (uniformly w.r.t. $\varepsilon > 0$). The proof of this result is similar to the one exposed to establish Lemma 3.2 and Theorem 3.1, hence we here describe (formally) the required estimates.

• **A priori estimates**

Let $(u^\varepsilon, v^\varepsilon)$ be a weak solution for the Cauchy problem (4.3).

Lemma 4.1 (First estimate). *Let $\frac{1}{2} < s < 1$. Then, for each $t \in (0, T)$*

$$\frac{d}{dt} \int_{\mathbb{R}} |u^\varepsilon(t, x)|^2\, dx = 0, \tag{4.4}$$

$$\frac{d}{dt}\Big(\int_{\mathbb{R}} |(-\Delta)^{s/2} u^\varepsilon(t, x)|^2\, dx + \frac{1}{2} \int_{\mathbb{R}} |u^\varepsilon(t, x)|^4 dx + \alpha \int_{\mathbb{R}} v^\varepsilon(t, x) |u^\varepsilon(t, x)|^2\, dx\Big)$$
$$= \alpha\beta \int_{\mathbb{R}} (-\Delta)^{s/2}\, |u^\varepsilon(t, x)|^2 |u^\varepsilon(t, x)|^2 dx - \alpha\,\varepsilon \int_{\mathbb{R}} |u^\varepsilon(t, x)|^2\, (-\Delta)^{s/2} v^\varepsilon(t, x) dx, \tag{4.5}$$

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}} |v^\varepsilon(t, x)|^2 dx + \varepsilon \int_{\mathbb{R}} |(-\Delta)^{s/4} v^\varepsilon(t, x)|^2\, dx = \beta \int_{\mathbb{R}} (-\Delta)^{s/2} |u^\varepsilon(t, x)|^2\, v^\varepsilon(t, x) dx. \tag{4.6}$$

Proof. 1. First, we multiply equation $(4.3)_1$ by $\overline{u^\varepsilon}(t, x)$ and integrate in $\mathbb{R}$ to obtain

$$i \int_{\mathbb{R}} \partial_t u^\varepsilon(t, x)\, \overline{u^\varepsilon}(t, x) dx - \int_{\mathbb{R}} |(-\Delta)^{s/2} u^\varepsilon(t, x)|^2 dx$$
$$= \alpha \int_{\mathbb{R}} v^\varepsilon(t, x) |u^\varepsilon(t, x)|^2 dx + \int_{\mathbb{R}} |u^\varepsilon(t, x)|^4\, dx.$$

Then, taking the imaginary part of the above equation

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} |u^\varepsilon(t,x)|^2 \, dx = \text{Re} \int_{\mathbb{R}} \partial_t u^\varepsilon(t,x) \, \overline{u^\varepsilon}(t,x) \, dx = 0.$$

2. Now, let us multiply equation $(4.3)_1$ by $\partial_t \overline{u^\varepsilon}(t,x)$ and integrate in $\mathbb{R}$, it follows that

$$\begin{aligned} & i \int_{\mathbb{R}} \partial_t u^\varepsilon(t,x) \, \partial_t \overline{u^\varepsilon}(t,x) \, dx - \int_{\mathbb{R}} (-\Delta)^s u^\varepsilon(t,x) \, \partial_t \overline{u^\varepsilon}(t,x) \, dx \\ & = \alpha \int_{\mathbb{R}} v^\varepsilon(t,x) \, u^\varepsilon(t,x) \, \partial_t \overline{u^\varepsilon}(t,x) \, dx + \int_{\mathbb{R}} \left| u^\varepsilon(t,x) \right|^2 u^\varepsilon(t,x) \, \partial_t \overline{u^\varepsilon}(t,x) \, dx. \end{aligned} \tag{4.7}$$

Then, writing $u^\varepsilon(t,x) = u_1^\varepsilon(t,x) + i u_2^\varepsilon(t,x)$, integrating by parts, and taking the real part we have

$$\begin{aligned} & \frac{d}{dt}\Big[\int_{\mathbb{R}} |(-\Delta)^{s/2} u^\varepsilon(t,x)|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}} |u^\varepsilon(t,x)|^4 \, dx + \alpha \int_{\mathbb{R}} v^\varepsilon(t,x) |u^\varepsilon(t,x)|^2 \, dx \Big] \\ & = \alpha \int_{\mathbb{R}} |u^\varepsilon(t,x)|^2 \, \partial_t v^\varepsilon(t,x) dx. \end{aligned} \tag{4.8}$$

Again, the right hand side of the above equation is computed by multiplying $(4.3)_2$ by $\alpha |u^\varepsilon(t,x)|^2$ and integrating in $\mathbb{R}$

$$\begin{aligned} & \alpha \int_{\mathbb{R}} |u^\varepsilon(t,x)|^2 \, \partial_t v^\varepsilon(t,x) \, dx = \alpha \, \beta \int_{\mathbb{R}} (-\Delta)^{s/2} (|u^\varepsilon|^2)(t,x) \, |u^\varepsilon(t,x)|^2 dx \\ & - \alpha \, \varepsilon \int_{\mathbb{R}} \left| u^\varepsilon(t,x) \right|^2 (-\Delta)^{s/2} v^\varepsilon(t,x) dx \end{aligned}$$

and inserting it in (4.8), we have

$$\begin{aligned} & \frac{d}{dt}\Big[\int_{\mathbb{R}} |(-\Delta)^{s/2} u^\varepsilon(t,x)|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}} |u^\varepsilon(t,x)|^4 \, dx \\ & + \alpha \int_{\mathbb{R}} v^\varepsilon(t,x) \, |u^\varepsilon(t,x)|^2 \, dx \Big] = \alpha \, \beta \int_{\mathbb{R}} (-\Delta)^{s/2} (|u^\varepsilon|^2)(t,x) \, |u^\varepsilon(t,x)|^2 \, dx \\ & - \alpha \, \varepsilon \int_{\mathbb{R}} |u^\varepsilon(t,x)|^2 \, (-\Delta)^{s/2} v^\varepsilon(t,x) \, dx. \end{aligned}$$

3. Finally, multiplying $(4.3)_2$ by $v^\varepsilon(t,x)$, and integrating in $\mathbb{R}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} |v^\varepsilon(t,x)|^2 dx + \varepsilon \int_{\mathbb{R}} |(-\Delta)^{s/4} v^\varepsilon(t,x)|^2 dx = \beta \int_{\mathbb{R}} (-\Delta)^{s/2} (|u^\varepsilon|^2)(t,x) \, v^\varepsilon(t,x) dx.$$

□

Proposition 4.1 (Second estimate)**.** *Let $\frac{1}{2} < s < 1$. Then, there exists $s_0 > 1/2$, such that, for any $s \in (s_0, 1)$ fixed, there exists $\alpha_0 > 0$ and $E_0 > 0$, such that, for each $t \in (0, T)$*

$$\int_{\mathbb{R}} |(-\Delta)^{s/2} u^{\varepsilon}(t,x)|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^4 \, dx \le h(t) \tag{4.9}$$

and

$$\begin{aligned} &\int_{\mathbb{R}} |v^{\varepsilon}(t,x)|^2 dx \le e^T \|v_0^{\varepsilon}\|_{L^2(\mathbb{R})} \\ &+ 16\, |\alpha|\, (1 + |\beta| + \beta^2 + |\alpha|\, \beta^2 e^T)\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2 \|u_0^{\varepsilon}\|_{L^2(\mathbb{R})}^{2-\frac{1}{s}} \int_0^t h(\tau)^{1+\frac{1}{2s}} \, d\tau, \end{aligned} \tag{4.10}$$

for $|\alpha| \le \alpha_0$ or $\|u_0\|_{L^2(\mathbb{R})} \le E_0$, where h is a continuous positive function (independent of ε).

Proof. 1. First, from equation (4.6) and applying Young's inequality, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}} |v^{\varepsilon}(t,x)|^2 \, dx &+ \varepsilon \int_{\mathbb{R}} |(-\Delta)^{s/4} v^{\varepsilon}(t,x)|^2 dx \\ &= \beta \int_{\mathbb{R}} (-\Delta)^{s/4} (|u^{\varepsilon}(t,x)|^2) \, (-\Delta)^{s/4} v^{\varepsilon}(t,x) \, dx \\ &\le \frac{\beta^2}{2\varepsilon} \int_{\mathbb{R}} \Big|(-\Delta)^{s/4} |u^{\varepsilon}(t,x)|^2\Big|^2 \, dx + \frac{\varepsilon}{2} \int_{\mathbb{R}} |(-\Delta)^{s/4} v^{\varepsilon}(t,x)|^2 dx. \end{aligned}$$

Then, integrating from 0 to $t > 0$,

$$\begin{aligned} \int_{\mathbb{R}} |v^{\varepsilon}(t,x)|^2 \, dx &+ \varepsilon \int_0^t \int_{\mathbb{R}} |(-\Delta)^{s/4} v^{\varepsilon}(\tau,x)|^2 \, dx d\tau \\ &\le \|v_0^{\varepsilon}\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{\varepsilon} \int_0^t \|(-\Delta)^{s/2} |u^{\varepsilon}(\tau)|^2\|_{L^2(\mathbb{R})}^2 \, d\tau. \end{aligned} \tag{4.11}$$

Similarly, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}} |v^{\varepsilon}(t,x)|^2 \, dx &\le \int_{\mathbb{R}} |\beta| \, |(-\Delta)^{s/2} |u^{\varepsilon}(t,x)|^2| \, |v^{\varepsilon}(t,x)| \, dx \\ &\le \frac{\beta^2}{2} \int_{\mathbb{R}} |(-\Delta)^{s/2} |u^{\varepsilon}(t,x)|^2|^2 dx + \frac{1}{2} \int_{\mathbb{R}} |v^{\varepsilon}(t,x)|^2 dx, \end{aligned}$$

and applying Gronwall's Lemma

$$\int_{\mathbb{R}} |v^{\varepsilon}(t,x)|^2 \, dx \le e^T \|v_0^{\varepsilon}\|_{L^2(\mathbb{R})}^2 + \beta^2 e^T \int_0^t \|(-\Delta)^{s/2} |u^{\varepsilon}(\tau)|^2\|_{L^2(\mathbb{R})}^2 d\tau. \tag{4.12}$$

2. Now, from equation (4.5) and applying Young's inequality, we have

$$\frac{d}{dt}\Big(\int_{\mathbb{R}}|(-\Delta)^{s/2}u^\varepsilon(t,x)|^2\,dx+\frac{1}{2}\int_{\mathbb{R}}|u^\varepsilon(t,x)|^4\,dx+\alpha\int_{\mathbb{R}}v^\varepsilon(t,x)\,|u^\varepsilon(t,x)|^2\,dx\Big)$$
$$\leq|\alpha||\beta|\,\|(-\Delta)^{s/2}|u^\varepsilon(t)|^2\|^2_{L^2(\mathbb{R})}+\frac{|\alpha|}{2}\|(-\Delta)^{s/2}|u^\varepsilon(t)|^2\|^2_{L^2(\mathbb{R})}$$
$$+\frac{|\alpha|\varepsilon^2}{2}\|(-\Delta)^{s/4}v^\varepsilon(t)\|^2_{L^2(\mathbb{R})},$$

where we have used the Embedding Theorem. Then, integrating from 0 to $t>0$

$$\int_{\mathbb{R}}|(-\Delta)^{s/2}u^\varepsilon(t,x)|^2\,dx+\frac{1}{2}\int_{\mathbb{R}}|u^\varepsilon(t,x)|^4\,dx+\alpha\int_{\mathbb{R}}v^\varepsilon(t,x)\,|u^\varepsilon(t,x)|^2\,dx$$
$$\leq C_0+|\alpha||\beta|\int_0^t\|(-\Delta)^{s/2}|u^\varepsilon(\tau)|^2\|^2_{L^2(\mathbb{R})}d\tau+\frac{|\alpha|}{2}\int_0^t\|(-\Delta)^{s/2}|u^\varepsilon(\tau)|^2\|^2_{L^2(\mathbb{R})}d\tau$$
$$+\frac{|\alpha|\varepsilon^2}{2}\int_0^t\|(-\Delta)^{s/4}v^\varepsilon(\tau)\|^2_{L^2(\mathbb{R})}d\tau,$$

where

$$C_0=\|(-\Delta)^{s/2}u_0^\varepsilon\|^2_{L^2(\mathbb{R})}+\|u_0^\varepsilon\|^4_{L^4(\mathbb{R})}+\frac{\alpha^2}{2}\|v_0^\varepsilon\|^2_{L^2(\mathbb{R})}.$$

Hence from estimate (4.11), we obtain

$$\int_{\mathbb{R}}|(-\Delta)^{s/2}u^\varepsilon(t,x)|^2\,dx+\frac{1}{2}\int_{\mathbb{R}}|u^\varepsilon(t,x)|^4\,dx+\alpha\int_{\mathbb{R}}v^\varepsilon(t,x)\,|u^\varepsilon(t,x)|^2\,dx$$
$$\leq C_0+|\alpha|\,\|v_0^\varepsilon\|^2_{L^2(\mathbb{R})}+|\alpha|(1+|\beta|+\beta^2)\int_0^t\|(-\Delta)^{s/2}|u^\varepsilon(\tau)|^2\|^2_{L^2(\mathbb{R})}d\tau.$$

Therefore, it follows that

$$\begin{aligned}\int_{\mathbb{R}}|(-\Delta)^{s/2}u^\varepsilon(t,x)|^2\,dx+\frac{1}{4}\int_{\mathbb{R}}|u^\varepsilon(t,x)|^4\,dx\leq C_0+|\alpha|\,\|v_0^\varepsilon\|^2_{L^2(\mathbb{R})}\\+|\alpha|(1+|\beta|+\beta^2)\int_0^t\|(-\Delta)^{s/2}|u^\varepsilon(\tau)|^2\|^2_{L^2(\mathbb{R})}d\tau+\alpha^2\int_{\mathbb{R}}|v^\varepsilon(t,x)|^2\,dx.\end{aligned}\tag{4.13}$$

3. To follow, we apply Proposition 3.2 and equation (4.4) to obtain

$$\begin{aligned}\|(-\Delta)^{s/2}|u^\varepsilon|^2(t)\|_{L^2(\mathbb{R})}&\leq 2\|u^\varepsilon(t)\|_{L^\infty(\mathbb{R})}\,\|(-\Delta)^{s/2}u^\varepsilon(t)\|_{L^2(\mathbb{R})}\\&\leq\Big(\frac{4}{\sqrt{\pi}}+\frac{4}{\sqrt{\pi(2s-1)}}\Big)\|u_0^\varepsilon\|^{1-\frac{1}{2s}}_{L^2(\mathbb{R})}\,\|(-\Delta)^{s/2}u^\varepsilon(t)\|^{1+\frac{1}{2s}}_{L^2(\mathbb{R})}.\end{aligned}\tag{4.14}$$

Replacing (4.12) and (4.14) in (4.13), we have

$$\int_{\mathbb{R}}|(-\Delta)^{s/2}u^\varepsilon(t,x)|^2\,dx+\frac{1}{4}\int_{\mathbb{R}}|u^\varepsilon(t,x)|^4\,dx\leq C_0+|\alpha|(1+|\alpha|e^T)\|v_0^\varepsilon\|^2_{L^2(\mathbb{R})}$$
$$+16\,|\alpha|\,(1+|\beta|+\beta^2+|\alpha|\,\beta^2e^T)\Big(\frac{1}{\sqrt{\pi}}+\frac{1}{\sqrt{\pi(2s-1)}}\Big)^2\|u_0^\varepsilon\|^{2-\frac{1}{s}}_{L^2(\mathbb{R})}\int_0^t\|(-\Delta)^{s/2}u^\varepsilon(\tau)\|^{2+\frac{1}{s}}_{L^2(\mathbb{R})}d\tau,$$

or conveniently we write

$$
\begin{aligned}
1+&\int_{\mathbb{R}} |(-\Delta)^{s/2} u^{\varepsilon}(t,x)|^2 \, dx + \frac{1}{4}\int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^4 \, dx \\
&\leq \theta(t) := 1 + C_0 + |\alpha|(1+|\alpha| e^T) \|v_0^{\varepsilon}\|^2_{L^2(\mathbb{R})} \\
&+ 16\,|\alpha|\,(1+|\beta| + \beta^2 + |\alpha|\,\beta^2 e^T)\Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2 \|u_0^{\varepsilon}\|^{2-\frac{1}{s}}_{L^2(\mathbb{R})} \int_0^t \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|^{2+\frac{1}{s}}_{L^2(\mathbb{R})} d\tau \\
&= C_1 + C_2 \int_0^t \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|^{2+\frac{1}{s}}_{L^2(\mathbb{R})} d\tau,
\end{aligned}
$$

with obvious notation. So, we may write

$$
\theta(t) \leq C_1 + C_2 \int_0^t \theta(\tau)^{1+\frac{1}{2s}} \, d\tau.
$$

Finally, we apply the Generalized Grownwall Lemma, more precisely (2.19) with $\sigma = 1 + \frac{1}{2s} > 1$, hence we must have for each $s \in (1/2, 1)$

$$
C_1 < \Big\{ \frac{16\,|\alpha|\,(1+|\beta| + \beta^2 + |\alpha|\,\beta^2 e^T)}{2s} \Big(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi(2s-1)}}\Big)^2 \|u_0^{\varepsilon}\|^{2-\frac{1}{s}}_{L^2(\mathbb{R})} T \Big\}^{-2s}
$$

or equivalently

$$
C_1\,|\alpha|^{2s}\Big(1+|\beta|+\beta^2+|\alpha|\,\beta^2 e^T\Big)^{2s} \|u_0^{\varepsilon}\|^{4s-2}_{L^2(\mathbb{R})}\, T^{2s} < \Big\{ \frac{s\pi^2(2s-1)}{8(\sqrt{\pi} + \sqrt{\pi(2s-1)})^2} \Big\}^{2s}. \quad (4.15)
$$

Therefore, there exists $s_0 > 1/2$, such that, for any $s \in (s_0, 1)$ fixed, there exists $\alpha_0 > 0$ and $E_0 > 0$, such that condition (4.15) is satisfied when $\|u_0\|_{L^2(\mathbb{R})} \leq E_0$, or $|\alpha| \leq \alpha_0$. Again, if there is no coupling, that is $\alpha = 0$, then condition (4.15) is trivially satisfied. Then, we have

$$
\theta(t) \leq C_1 \Big(1 - \frac{C_2}{C_1}\frac{t}{2s}\Big)^{2s},
$$

from which follows the proof of the proposition. □

4.2 Existence of weak solutions

The aim of this section is to prove the following

Theorem 4.1. *Let* $(u_0, v_0) \in H^s(\mathbb{R}) \times L^2(\mathbb{R})$, $(\frac{1}{2} < s < 1)$. *Then, there exists* $s_0 > 1/2$, *such that, for any* $s \in (s_0, 1)$ *fixed, there exists* $\alpha_0 > 0$ *and* $E_0 > 0$, *such that, if* $|\alpha| \leq \alpha_0$ *or* $\|u_0\|_{L^2(\mathbb{R})} \leq E_0$, *then there exists a weak solution*

$$
(u, v) \in L^{\infty}(0, T; H^s(\mathbb{R})) \times L^2(0, T; L^2(\mathbb{R}))
$$

of the Cauchy problem (1.2).

Proof. 1. First, under the conditions of Theorem 1.1, for each $\varepsilon > 0$ fixed, let $(u^\varepsilon, v^\varepsilon) \in L^\infty(0,T; H^s(\mathbb{R})) \times L^2(0,T; H^{s/2}(\mathbb{R}))$ be the weak solution of the Cauchy problem (4.3), that is, the pair $(u^\varepsilon, v^\varepsilon)$ satisfies for any $T > 0$,

$$
\begin{aligned}
i\int_0^T \int_{\mathbb{R}} \Big(u^\varepsilon(t,x)\, \partial_t \overline{\varphi}(t,x) + (-\Delta)^{s/2} u^\varepsilon(t,x)\, (-\Delta)^{s/2}\overline{\varphi}(t,x)\Big) dx dt + i\int_{\mathbb{R}} u_0^\varepsilon(x)\, \overline{\varphi}(0,x) dx \\
+ \alpha \int_0^T \int_{\mathbb{R}} v^\varepsilon(t,x)\, u^\varepsilon(t,x)\, \overline{\varphi}(t,x) dx dt + \int_0^T \int_{\mathbb{R}} |u^\varepsilon(t,x)|^2\, u^\varepsilon(t,x)\, \overline{\varphi}(t,x) dx dt = 0,
\end{aligned}
\tag{4.16}
$$

$$
\begin{aligned}
\int_0^T \int_{\mathbb{R}} v^\varepsilon(t,x)\, \partial_t \psi(t,x) - \varepsilon\, v^\varepsilon(t,x)\, (-\Delta)^{s/2} \psi(t,x) dx dt + \int_{\mathbb{R}} v_0^\varepsilon(x)\, \psi(0,x) dx \\
+ \beta \int_0^T \int_{\mathbb{R}} |u^\varepsilon|^2(t,x)\, (-\Delta)^{s/2} \psi(t,x) dx dt = 0,
\end{aligned}
\tag{4.17}
$$

for each test function $\varphi, \psi \in C_c^\infty((-\infty, T) \times \mathbb{R})$, with φ being complex-valued and ψ real-valued.

2. Now, to obtain (4.1), (4.2) we pass to the limit respectively in (4.16) and (4.17) as $\varepsilon \to 0^+$. Again, it is needed to show compactness of the sequences $\{u^\varepsilon\}_{\varepsilon>0}$, and $\{v^\varepsilon\}_{\varepsilon>0}$. From (4.4), (4.9), it follows that $\{u^\varepsilon\}_{\varepsilon>0}$ is (uniformly) bounded in $L^\infty(0,T; H^s(\mathbb{R}))$, hence we select a subsequence, still denoted by $\{u^\varepsilon\}_{\varepsilon>0}$, which converges weakly-$\star$ to u in $L^\infty(0,T; H^s(\mathbb{R}))$. Analogously, from (4.10) we have that $\{v^\varepsilon\}_{\varepsilon>0}$ is (uniformly) bounded in $L^2(0,T; L^2(\mathbb{R}))$, hence it is also possible to select a subsequence, still denoted by $\{v^\varepsilon\}_{\varepsilon>0}$, which converges weakly to v in $L^2(0,T; L^2(\mathbb{R}))$. To pass to the limit as $\varepsilon \to 0$ in all terms of (4.16) and (4.17), it is enough to show strong convergence just to $\{u^\varepsilon\}_{\varepsilon>0}$ sequence.

From (4.16) and the a priori estimates, we obtain that, the sequence $\{u^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^2(0,T; H^{-1}(\mathbb{R}))$. Therefore, we apply the Aubin-Lions's Theorem and obtain (along a suitable subsequence) that u^ε converges strongly to u in $L^2(0,T; L^2(K))$, for any compact set $K \subset \mathbb{R}$. Moreover, $u^\varepsilon(t,x) \to u(t,x)$ as $\varepsilon \to 0$ almost everywhere in $(0,T) \times \mathbb{R}$. Then, we are ready to obtain equations (4.1) and (4.2).

3. Finally, we get the solvability of the Cauchy problem (1.2) applying the Definition 4.1, where the conditions on the statement of Theorem 4.1 comes from the conditions under Theorem 1.1, and also Proposition 4.1. □

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