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Rio de Janeiro, Brasil

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Aos meus pais, Danilo e Juliana.

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Resumo

Contruimos fluxos de curvatura média eternais do toro de Clifford em perturbações da esfera unitária stándard $\mathbf{S}^3$ in $\mathbb{R}^4$.

Palavras-chave: fluxos de curvatura média eternais, Toro de Clifford.

Abstract

We construct eternal mean curvature flows of tori in perturbations of the standard unit sphere $\mathbf{S}^3$ in $\mathbb{R}^4$.

Keywords: Eternal mean curvature curvature flow, Clifford Tori.

Contents

1 Introdução

Seja $\Sigma := \Sigma^m$ uma superfície compacta, orientada m-dimensional em $M := M^{m+1}$ uma variedade Riemanniana orientada $m+1$-dimensional. Definimos a variação de Σ do seguinte modo, para $t \in I \subset \mathbb{R}$ e $x \in \Sigma$, seja $e : \Sigma \times I \longrightarrow M$ uma função suave que toma valores em M^{m+1}, tal que $e_t(\cdot) := e(\cdot, t)$ é uma imersão e $\Sigma_t = \{e(x,t) \mid x \in \Sigma\}$ é uma família suave de superfícies em torno de Σ, e o vetor $\frac{\partial e(x,t)}{\partial t}$ é perpendicular a Σ_t, em $e(x,t)$. Uma familia a um parâmetro $e_t(.) = e(., t)$ de hipersuperficies é solução para o fluxo de curvatura média se satisfaz

$$\begin{cases} \frac{\partial e}{\partial t}(x,t) = -H(x,t)N(x,t) \\ e(x,0) = e(x) \end{cases},$$

onde $N_t(.) = N(., t)$ é o campo vetorial normal unitário sobre Σ_t e $H_t(.) = H(., t)$ é a curvatura média com relação a este normal. Em outras palavras, a função e é dita fluxo de curvatura média sempre que for solução da equação de fluxo de curvatura média

$$\left\langle \frac{\partial e_t}{\partial t}, N_t \right\rangle + H_t = 0. \tag{1.1}$$

O fluxo de curvatura média evolui hipersuperficies na direção normal com velocidade igual a curvatura média em cada ponto e é um fluxo do tipo gradiente para o funcional área [3], que se comporta como a equação do calor em um curto período de tempo.

Podemos classificar os fluxos de curvatura média a partir de seu intervalo de definição. Diremos que um fluxo de curvatura média é de tipo eterno quando $I = \mathbb{R}$, e se I não é limitado inferiormente ou superiormente, isto é, $I =]-\infty, b[$ e $I =]a, \infty[$ diremos que os fluxos são do tipo antigo e imortal, respectivamente.

Como os fluxos de curvatura média tendem a se tornar singulares, espera-se que cada uma dessas propriedades tenha implicações significativas para a geometria do fluxo. Destas propriedades, a de ser eterno é claramente a mais restritiva. Exemplos de fluxos eternos são muito poucos, assim resulta interessante descobrir novos exemplos para entender melhor sua estrutura.

Nossa construção de fluxos de curvatura média eternais envolve a construção de fluxos dados por subgrupos 1-parametro de grupos afins de $\mathbb{R}^{m+1}$.

Além disso, para ter uma ideia de quão restritiva é a condição de ser eterno, vejamos por exemplo que em [18], Hamilton demostrou

Theorem 1 (Hamilton-1995)**.** *Se F é uma solução eternal estritamente convexa ao fluxo de curvatura media e se a curvatura media de F atinge seu maximo valor em um ponto no space-time, então F é um soliton transladado.*

Da mesma forma, em [2], White conjeturou

Conjeture 1 (White 2003)**.** *Qualquer solução eternal convexa nonflat ao fluxo de curvatura média é um soliton trasladado.*

É útil pensar em curvatura média a partir das teorias de Morse.

De fato, assim como superfícies mínimas são pontos críticos da área funcional, os fluxos de curvatura media são fluxos gradientes desse funcional. Ideias da teoria de Morse e especialmente da teoria da homologia de Morse são particularmente utéis na construção de fluxos eternos.

A ideia chave é considerar fluxos de curvatura média eternais não como fluxos evoluindo no tempo, mas como soluções de operadores hipo-elípticos quase-lineares. Assim, adaptamos as construções de perturbações de superfí cies minimas no cenário elíptico para obter construções de perturbações de fluxos de curvatura média eterna.

Essa abordagem já permitiu construir fluxos de curvatura média eterna forçada por meio de técnicas de perturbações singulares (ver [7]). De fato, lembremos que a existência de famílias de hipersuperfícies com curvatura média constante concentrada em um ponto foi estudada por R. Ye [19]. Para qualquer ponto crítico não degenerado p da curvatura escalar S em (M^{m+1}, g), R. Ye demonstra a existência de uma família de hipersuperfícies submersas, com curvatura média constante, que estão concentradas em p quando a curvatura média tende ao infinito.

Adaptando este resultado ao cenário hipo-elíptico, G. Smith in [7] mostrou que se $\gamma : \mathbb{R} \longrightarrow M^{n+1}$ é um fluxo completo da função curvatura escalar S em M^{m+1} e que se γ é não degenerado com imagem compacta, então existe um fluxo de curvatura média forçado de esferas com o termo de forçado tendendo a infinito.

Este trabalho tem como objetivo principal a construção de fluxos de curvatura média eternos sem forçar. Para isto, usamos outra construção de superfícies mínimas. Nosso interesse será utilizar uma perturbação singular para construir fluxos de curvatura média eternos. Mais precisamente, vamos construir fluxos de toros em perturbações da esfera unitária $\mathbf{S}^3$. Para isto adaptamos o trabalho de White in [1] ao cenário hipo-elíptico. Ao longo deste trabalho, identificaremos $\mathbf{S}^3$ com a esfera unitária em $\mathbb{R}^4$; isto é,

$$\mathbf{S}^3 = \{x \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}.$$

Sejam $\mathbf{T}_0 = \mathbf{S}^1 \times \mathbf{S}^1$ o toro padrão e $(\mathbf{S}^3, g)$ a esfera unitária com métrica de curvatura constante em $\mathbb{R}^4$. Seja $C^\infty(\mathbf{S}^3)$ o espaço de funções suaves sobre $\mathbf{S}^3$ munido com a topologia da convergência suave e seja $\mathcal{U}$ um subconjunto de $C^\infty(\mathbf{S}^3)$, $\mathcal{U}$ é dito genérico sempre que contiver uma interseção contável de subconjuntos abertos e densos. Provamos que

Theorem 2. *Existe um subconjunto generico $\mathcal{U}$ de $C^\infty(\mathbf{S}^3)$ com a propriedade que, para todo $u \in \mathcal{U}$, existe $\epsilon > 0$ tal que, para todo $t \in]-\epsilon, \epsilon[$, existe um fluxo de curvatura média eternal $e : \mathbb{T} \times \mathbb{R} \to (\mathbf{S}^3, e^{2tu}g)$, isto é, se $(\mathbf{S}^3, g_1)$ uma 3-esfera com sua métrica de curvatura constante, para uma métrica $g = e^{2tu}g_1$ suficientemente perto de g_1, existe um fluxo de curvatura média eternal do toro em $(\mathbf{S}^3, g)$.*

A fim de mostrar nosso teorema principal, dividimos o nosso trabalho da seguinte maneira: No primeiro capítulo, vamos a construir uma métrica sobre $\mathbf{T}_0 \times \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[$ que será rotacionalmente simétrica, onde

$$\mathbf{T}_0 := \left\{(x, y) \in \mathbf{S}^3 \mid \|x\|^2 = \|y\|^2 = \frac{1}{\sqrt{2}}\right\}.$$

Fazemos isto para obter uma parametrização da esfera $\mathbf{S}^3$ que seja compatível com o toro de Clifford. Na continuação, estaremos interessados em estudar a geometria dos gráficos sobre $\mathbf{T}_0$, pois lembramos que na construção de White [1] utilizamos gráficos normais. Estes gráficos normais serão toros de Clifford que poderão ser escritos como gráficos sobre o toro inicial $\mathbf{T}_0$. Assim, uma vez feita a perturbação inicial faremos uma perturbação normal que vai nós permitir determinar a componente normal desta variação de famílias já que depois temos que compararlo com a curvatura média. Em seguida, realizaremos um análise de certas funções no espaço de toros de Clifford em $\mathbf{S}^3$ que será denotado por CL. Lembremos que o toro de Clifford padrão $\mathbf{T}_0$ em $\mathbf{S}^3$ é uma superfície mínima. Geralmente, os toros de Clifford em $\mathbf{S}^3$ são as imagens de $\mathbf{T}_0$ sob a ação do grupo ortogonal O(4). O espaço CL de toros de Clifford é uma variedade suave 4-dimensional difeomorfo a $\mathbb{RP}^2 \times \mathbb{RP}^2$.

Definimos o funcional linear $\mathrm{I} : C^\infty(\mathbf{S}^3) \longrightarrow C^\infty(\mathrm{CL})$ por

$$\mathrm{I}[u](T) := \int_T u \ \mathrm{dArea}_T,$$

onde para cada T, dArea_T denota a forma área. Estamos interessados nas propriedades analíticas das funções da forma $\mathrm{I}[u]$. Lembremos de que uma função suave é do tipo Morse sempre que todos os seus pontos críticos não são degenerados e é do tipo Morse-Smale sempre que além de ser do tipo Morse, toda variedade estável de seu fluxo gradiente é transversal a cada variedade instável desse fluxo.

Assim, no segundo capítulo daremos a prova do teorema principal. Este teorema é provado analogamente ao argumento de perturbação desenvolvido por White na seção 3 de [1]. Para isto, vamos a usar as parametrizações padrão do toro de Clifford introduzidas na seção 2.2 pois isto nos dará uma família canônica de parametrizações ao longo de uma dada família de superfícies. Embora isto seja um caso particular para o toro de Clifford na esfera $\mathbf{S}^3$, as técnicas que desenvolvemos são mais gerais.

2 Perturbations of the Clifford Torus

2.1 Notation for error terms

The formulae for the geometric quantities that we will study can become complicated. For this reason we condense notation as much as possible. In this section, we discuss the notational conventions we will use.

Let T be a Clifford torus. Given a function $f : \mathbb{R}^2 \times T \longrightarrow \mathbb{R}$ such that $f_{0,0} = 0$, we may write

$$f(s,t,\theta,\varphi) = f_{s,t}(\theta,\varphi) = sf^s(s,t,\theta,\varphi) + tf^t(s,t,\theta,\varphi)$$

where

$$f^s(s,t,\theta,\varphi) := \int_0^1 \frac{\partial f}{\partial s}(rs,t,\theta,\varphi)dr$$

and

$$f^t(s,t,\theta,\varphi) := \int_0^1 \frac{\partial f}{\partial t}(s,rt,\theta,\varphi)dr.$$

We use the expression $O^1_k(s,t)$ to denote any term, vanishing at $(0,0)$, which involves s,t,θ,φ and combinations of the functions f^s and f^t and their derivatives up to including order k. More precisely, the function F is of type $O^1_k(s,t)$ whenever

$$\begin{aligned} F_{s,t}(\theta,\varphi) =& sF^s\Big(s,t,\theta,\varphi,f^s(s,t,\theta,\varphi),Df^s(s,t,\theta,\varphi),\cdots,D^kf^s(s,t,\theta,\varphi),f^t(s,t,\theta,\varphi),\\ & Df^t(s,t,\theta,\varphi),\cdots,D^kf^t(s,t,\theta,\varphi)\Big) + tF^t\left(s,t,\theta,\varphi,f^s(s,t,\theta,\varphi),Df^s(s,t,\theta,\varphi),\right.\\ & \left.\cdots,D^kf^s(s,t,\theta,\varphi),f^t(s,t,\theta,\varphi),Df^t(s,t,\theta,\varphi),\cdots,D^kf^t(s,t,\theta,\varphi)\right) \end{aligned} \tag{2.1}$$

where F^s and F^t are smooth functions in their arguments. In the sequel, we will make particular use of $O^1_1(s,t)$.

Lemma 1. *If $F,G \in O^1_1(s,t)$ then the product $FG \in O^1_1(s,t)$*

Proof. Indeed, let

$$F(\theta,\varphi) = sF^s(s,t,\theta,\varphi,f^s,Df^s,f^t,Df^t) + tF^t(s,t,\theta,\varphi,f^s,Df^s,f^t,Df^t)$$
$$G(\theta,\varphi) = sG^s(s,t,\theta,\varphi,f^s,Df^s,f^t,Df^t) + tG^t(s,t,\theta,\varphi,f^s,Df^s,f^t,Df^t)$$

Thus

$$\begin{aligned} FG(\theta,\varphi) =& s^2F^s\left(s,t,\theta,\varphi,f^s,Df^s,f^t,Df^t\right)G^s\left(s,t,\theta,\varphi,f^s,Df^s,f^t,Df^t\right) + \\ & t^2F^t\left(s,t,\theta,\varphi,f^s,Df^s,f^t,Df^t\right)G^t\left(s,t,\theta,\varphi,f^s,Df^s,f^t,Df^t\right) + \cdots \end{aligned}$$

therefore $FG \in O^1_1$

□

Lemma 2. *If $F \in O_1^1(s,t)$ and G is a smooth function with $G(0) = 0$, then the composition $G \circ F \in O_1^1(s,t)$.*

Proof. Indeed,

$$F(\theta,\varphi) = sF^s\left(s,t,\theta,\varphi,f^s,Df^s,f^t,Df^t\right) + tF^t\left(s,t,\theta,\varphi,f^s,Df^s,f^t,Df^t\right)$$

However

$$G(x) = x\widetilde{G}(x)$$

where

$$\widetilde{G}(x) = \int_0^1 G'(rx)dr$$

thus

$$\begin{aligned} G\left(F(s,t,\theta,\varphi)\right) =& F(s,t,\theta,\varphi)\widetilde{G}\left(F(s,t,\theta,\varphi)\right) \\ =& sF^s(s,t,\theta,\varphi,f(s,t,\theta,\varphi),Df(s,t,\theta,\varphi))\widetilde{G}\left(sF^s(s,t,\theta,\varphi,f(s,t,\theta,\varphi),\right. \\ & \left. Df(s,t,\theta,\varphi)) + tF^t(s,t,\theta,\varphi,f(s,t,\theta,\varphi),Df(s,t,\theta,\varphi)\right) + \\ & tF^t(s,t,\theta,\varphi,f(s,t,\theta,\varphi),Df(s,t,\theta,\varphi))\widetilde{G}\left(sF^s(s,t,\theta,\varphi,f(s,t,\theta,\varphi),\right. \\ & \left. Df(s,t,\theta,\varphi)) + tF^t(s,t,\theta,\varphi,f(s,t,\theta,\varphi),Df(s,t,\theta,\varphi)\right). \end{aligned}$$

therefore $G \circ F \in O_1^1(s,t)$ □

We conclude by comparing this notation with the conventional notation. Recall that it is customary to write

$$f_{s,t} = O(s,t)$$

to show that there exist a constant $C > 0$ such that

$$|f_{s,t}| < C|s| + C|t|$$

for all s and for all t. Our terminology provides more informations. In particular, it allows us to study the continuity and smoothness properties of functionals between Banach spaces.

In a similar manner, we write $\Theta \in O_1^2(s,t)$ whenever

$$\begin{aligned} \Theta_{s,t}(\theta,\varphi) =& s^2\Theta^{ss}\left(s,t,\theta,\varphi,f^s(s,t,\theta,\varphi),Df^s(s,t,\theta,\varphi),f^t(s,t,\theta,\varphi),Df^t(s,t,\theta,\varphi)\right) \\ & + st\Theta^{st}\left(s,t,\theta,\varphi,f^s(s,t,\theta,\varphi),Df^s(s,t,\theta,\varphi),f^t(s,t,\theta,\varphi),Df^t(s,t,\theta,\varphi)\right) \\ & + t^2\Theta^{tt}\left(s,t,\theta,\varphi,f^s(s,t,\theta,\varphi),Df^s(s,t,\theta,\varphi),f^t(s,t,\theta,\varphi),Df^t(s,t,\theta,\varphi),\right) \end{aligned}$$

and so on.

2.2 Geometry of perturbed Clifford Tori

We derive expansions for the metric, the second fundamental form and the mean curvature of normal perturbations of the Clifford Torus. Let $\mathbf{S}^3$ be the 3-dimensional unit sphere in $\mathbb{R}^4$, that is

$$\mathbf{S}^3 := \{x \in \mathbb{R}^4 \mid \|x\|^2 = 1\}.$$

Let $\mathbf{T_0}$ denote the standard Clifford Torus in $\mathbf{S}^3$ given by

$$\mathbf{T_0} := \left\{(x, y) \in \mathbf{S}^3 \subset \mathbb{R}^2 \oplus \mathbb{R}^2 \mid \|x\|^2 = \|y\|^2 = \frac{1}{\sqrt{2}}\right\} \tag{2.2}$$

2.2.1 Fermi Coordinates

Fermi coordinates are coordinates that are adapted to the study of tubular neighborhoods of submanifolds and points. Indeed, let M be a Riemannian manifold. Let S be a compact embedded submanifold. Let TS and NS be its tangent and normal bundles respectively. For $\epsilon > 0$, let $N_\epsilon S$ be the open subset of NS consisting of vectors of length less than ϵ. Let $\operatorname{Exp} : TM \longrightarrow M$ be the exponential map of M. For sufficiently small ϵ the restriction of Exp to $N_\epsilon S$ defines a smooth diffeomorphism onto an open subset of M. This parametrization defines the Fermi coordinates of M about S. Observe that $\operatorname{Exp}(N_\epsilon S)$ is a tubular neighborhood of radius ϵ about S.

Recall that normal coordinates parameterize a neighborhood of a manifold in terms of geodesics emanating from a point. In a similar manner, Fermi coordinates parameterize a neighborhood of the manifold locally in terms of normal geodesics emanating from a submanifold.

The restriction of Exp to the zero section maps diffeomorphically onto S. Finally for any normal vector $\xi_x \in N_\epsilon S$, the curve $t \longrightarrow \operatorname{Exp}(t\xi_x)$ is a geodesic normal to S.

2.2.2 Mean Curvature Operator

We briefly recall the definition of the mean curvature of a hypersurface since this is one of the main objects of study of our work. Let S be a closed hypersurface that is embedded in a riemannian manifold $\mathbb{R}^{n+1}$. We denote the mean curvature and second fundamental form of S by H_S and II_S, respectively, and the outward-pointing unit normal vector field by N_S. Then, the linearization of the mean curvature operator on the space of normal graphs over S is usually referred to as the Jacobi operator or stability operator about S and is given by (c.f. [24])

$$J := -\Delta + \left(\|II_S\|^2 + \operatorname{Ric}(N_S, N_S)\right),$$

where Δ is the Laplace operator of S and Ric is the Ricci curvature tensor of M.
When $S = \mathbf{S}^3$, the linearized mean curvature reads

$$J = -\Delta + \left(\|II_S\|^2 + 2\right).$$

The extrinsic curvature of S is described by a symmetric two-tensor II_S, the second fundamental form of S that is defined by

$$II(\xi, \eta) = \langle D_\xi N_S, \eta \rangle .$$

The eigenvalues of II are referred to as the principal curvatures of S. Let λ_1, λ_2 be the principal curvatures of S. The sum of the principal curvatures is referred to as the mean curvature of S :

$$H = \lambda_1 + \lambda_2.$$

Geometrically, the mean curvature is the L^2 -gradient of the area functional; more precisely, given an immersion $\varphi : S \to \mathbb{R}^{n+1}$ of a smooth hypersurface in $\mathbb{R}^{n+1}$, if we consider a smooth one-parameter family of immersions $\varphi_t : S \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+1}$ with $t \in (-\varepsilon, \varepsilon)$ and $\varphi_0 = \varphi$, such that, outside of a compact set $K \subset S$, we have $\varphi_t(p) = \varphi(p)$ for every $t \in (-\varepsilon, \varepsilon)$, we have (c.f. [3])

Proposition 1. *The first variation of the Area functional depends only on the normal component of the infinitesimal generator* $X = \frac{\partial g_n}{\partial t}\Big|_{t-0}$ *of the variation* φ_l *precisely*

$$\frac{\partial}{\partial t} \operatorname{Area}(\varphi_t)\Big|_{t=0} = -\int_S H_S \langle X, N_S \rangle d\mu.$$

2.2.3 Families of embedded minimal tori in $\mathbf{S^3}$

In this section we are going to construct the family of Clifford tori in $\mathbf{S}^3$. Recall that $\mathbf{T_0}$ denote the standard Clifford torus in $\mathbf{S}^3$ and is naturally parameterized by $\mathbf{S}^1 \times \mathbf{S}^1$ as

$$\Phi_0(\theta, \varphi) = \frac{1}{\sqrt{2}} (\cos(\theta), \sin(\theta), \cos(\varphi), \sin(\varphi)) . \tag{2.3}$$

where $\mathbf{S}^1 = \{x \in \mathbb{R}^2 \mid \|x\|^2 = 1\}$.

The unit normal vector field over $\mathbf{T_0}$ is given by

$$N_0(\theta, \varphi) = \frac{1}{\sqrt{2}} (\cos(\theta), \sin(\theta), -\cos(\varphi), -\sin(\varphi)) . \tag{2.4}$$

Note that the principal curvatures of the Clifford Torus $\mathbf{T}_0$ are 1 and -1, so the mean curvature is zero.

Recall that the tangent bundle of $\mathbf{S}^3$ is

$$T\mathbf{S}^3 := \{(x, y) \in \mathbb{R}^4 \mid \|x\|^2 = 1, \ \langle y, x \rangle = 0\},$$

and the exponential map is given by

$$\operatorname{Exp}(x, y) = \cos(\|y\|)\, x + \sin(\|y\|) \frac{y}{\|y\|}.$$

The Fermi coordinates about $\mathbf{T_0}$ in $\mathbf{S}^3$

$$\Phi : \mathbf{S}^1 \times \mathbf{S}^1 \times \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\longrightarrow \mathbf{S}^3$$

are thus given by

$$\Phi(\theta, \varphi, r) := \cos(r)\Phi_0(\theta, \varphi) + \sin(r)N(\theta, \varphi). \tag{2.5}$$

Observe that this means that, for all (θ, φ, r), r is the distance from $\Phi(\theta, \varphi, r)$ to $\mathbf{T}_0$, and $\Phi(\theta, \varphi, 0)$ is the unique closest point in $\mathbf{T}_0$ to this point.

We want to determine a metric $\Phi^* g_1$, the pull-back through Φ of the spherical metric over $\mathbf{S}^3$. This will yield a parametrisation of the sphere that is compatible with the Clifford torus.
By (2.5) we have that the parametrisation is given, by

$$\Phi(\theta, \varphi, r) = \cos(r)(\underline{x}, \underline{y}) + \sin(r)(\underline{x}, -\underline{y}), \tag{2.6}$$

where

$$\begin{aligned} \underline{x} &= \frac{1}{\sqrt{2}} (\cos(\theta), \sin(\theta), 0, 0) \\ \underline{y} &= \frac{1}{\sqrt{2}} (0, 0, \cos(\varphi), \sin(\varphi)) . \end{aligned}$$

This expands to

$$\begin{aligned} \Phi(\theta, \varphi, r) = \frac{1}{\sqrt{2}} \Big[&\cos(r)\cos(\theta) + \sin(r)\cos(\theta), \cos(r)\sin(\theta) + \sin(r)\sin(\theta), \\ &\cos(r)\cos(\varphi) - \sin(r)\cos(\varphi), \cos(r)\sin(\varphi) - \sin(r)\sin(\varphi) \Big]. \end{aligned}$$

Recall that $\frac{1}{\sqrt{2}} = \sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$, so that

$$\frac{1}{\sqrt{2}} (\cos(r) + \sin(r)) = \cos(r)\sin\left(\frac{\pi}{4}\right) + \sin(r)\cos\left(\frac{\pi}{4}\right) = \sin\left(r + \frac{\pi}{4}\right).$$

It follows that

$$\begin{aligned} \Phi(\theta, \varphi, r) = \Bigg(&\sin\left(\frac{\pi}{4} + r\right) \cos(\theta), \sin\left(\frac{\pi}{4} + r\right) \sin(\theta), \\ &\sin\left(\frac{\pi}{4} - r\right) \cos(\varphi), \sin\left(\frac{\pi}{4} - r\right) \sin(\varphi) \Bigg). \end{aligned} \tag{2.7}$$

Note that when $r = 0$, $\Phi_0(\theta, \varphi) = \Phi(\theta, \varphi, 0)$ we have a parametrization of Clifford Torus.

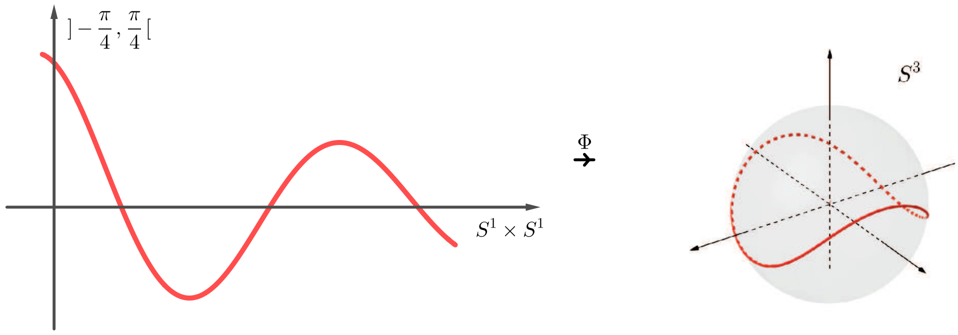

Figure 1 – The image of functions in $C^\infty\left(\mathbf{S}^1 \times \mathbf{S}^1, \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\right)$ by Φ are Clifford Tori in sphere $\mathbf{S}^3$.

Observe that

$$\begin{aligned}
\Phi_*\partial_\theta &= \left(-\sin\left(\frac{\pi}{4} + r\right)\sin(\theta), \sin\left(\frac{\pi}{4} + r\right)\cos(\theta)\sin(\theta), 0, 0\right) \\
\Phi_*\partial_\varphi &= \left(0, 0, -\sin\left(\frac{\pi}{4} - r\right)\sin(\varphi), \sin\left(\frac{\pi}{4} - r\right)\cos(\varphi)\sin(\theta)\right) \\
\Phi_*\partial_r &= \left(\cos\left(\frac{\pi}{4} + r\right)\cos(\theta), \cos\left(\frac{\pi}{4} + r\right)\sin(\theta),\right. \\
&\qquad \left. -\cos\left(\frac{\pi}{4} - r\right)\cos(\varphi), -\cos\left(\frac{\pi}{4} - r\right)\sin(\varphi)\right).
\end{aligned}$$

We readily determine

$$\langle \Phi_*\partial_\theta, \Phi_*\partial_\varphi\rangle = \langle \Phi_*\partial_\theta, \Phi_*\partial_r\rangle = \langle \Phi_*\partial_\varphi, \Phi_*\partial_r\rangle = 0,$$

and

$$\begin{aligned}
\|\partial_\theta\|^2 &= \sin^2\left(\tfrac{\pi}{4} + r\right) \\
\|\partial_\varphi\|^2 &= \sin^2\left(\tfrac{\pi}{4} - r\right) \\
\|\partial_r\|^2 &= 1
\end{aligned},$$

so that

$$\Phi^* g_1 = \sin^2\left(\frac{\pi}{4} + r\right) d\theta^2 + \sin^2\left(\frac{\pi}{4} - r\right) d\varphi^2 + dr^2. \tag{2.8}$$

In particular, this metric is orthogonal in the sense that $\partial_\theta, \partial_\varphi, \partial_r$ is an orthogonal (though not orthonormal) frame. We also see that the maximal domain over which Φ defines a diffeomorphism is $\mathbf{S}^1 \times \mathbf{S}^1 \times \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[$. The image of this domain is trivially a dense open subset of $\mathbf{S}^3$.

2.2.4 The Taylor Series of the Mean Curvature

In this section we determine the Taylor series of the mean curvature function. First, we want to determine the Levi-Civita covariant derivative of $\Phi^* g_1$ using only geometric considerations.

By definition, vertical lines are unit speed geodesics. Hence

$$\nabla_{\partial_r}\partial_r = 0.$$

Lemma 3. *Let M be a Riemannian manifold. Let $\phi : M \longrightarrow M$ be an isometry such that $\phi^2 = Id$. Suppose that $\Sigma = \{x \mid \phi(x) = x\}$ is a smooth hypersurface in M. Then Σ is totally geodesic.*

Proof. Let $N : \Sigma \longrightarrow TM$ be the unit normal vector field over Σ. We claim that $\forall\ x \in \Sigma$

$$D\phi(x) \cdot N(x) = -N(x).$$

Suppose the contrary. By hypotheses $\forall\ \xi \in T_x\Sigma$,

$$D\phi(x) \cdot \xi = \xi.$$

Since ϕ is an isometry, it follows that

$$D\phi(x) \cdot N(x) = \pm N(x).$$

Since $D\phi(x) \cdot N(x) \neq -N(x)$, it follows that

$$D\phi(x) \cdot N(x) = N(x).$$

Let $\gamma : \mathbb{R} \longrightarrow M$ be a geodesic such that $\gamma(0) = x$, $\partial_t\gamma(0) = N(x)$. Thus, $\phi \circ \gamma$ is a geodesic such that $(\phi \circ \gamma)(0) = 0$ and $\partial_t (\phi \circ \gamma)(0) = N(x)$. Thus

$$\gamma(t) = (\phi \circ \gamma)(t) \quad \forall t$$

so that $\gamma(t) \in \Sigma\ \ \forall t$. This is absurd. Hence

$$D\phi(x) \cdot N(x) = -N(x).$$

Now let ξ and η be vector fields tangent to Σ. For $x \in \Sigma$

$$\begin{aligned}
II(\xi(x), \eta(x)) &= g(\nabla_\xi \eta(x), N(x)) \\
&= -(\phi^* g)\Big((\phi^*\nabla)_{(\phi^*\xi)}(\phi^*\eta)(x), \phi^* N(x)\Big) \\
&= -g(\nabla_\xi \eta(x), -N(x)) \\
&= g(\nabla_\xi \eta(x), N(x)) \\
&= -II(\xi(x), \eta(x))
\end{aligned}$$

It follows that II vanishes, as desired.

□

For all φ_0,

$$\Phi^* g_1 = \sin^2\left(\frac{\pi}{4} + r\right) d\theta^2 + \sin^2\left(\frac{\pi}{4} - r\right) d\varphi^2 + dr^2.$$

is symmetric under the reflection

$$(\theta, \varphi, r) \mapsto (\theta, 2\varphi_0 - \varphi, r)\,. \tag{2.9}$$

The fixed surface of this reflection is

$$\Sigma_{\varphi_0} := \{(\theta, \varphi_0, r)\}$$

It follows that this surface is totally geodesic. Its unit normal vector field is

$$N = \frac{1}{\sin\left(\frac{\pi}{4} - r\right)} \partial_\varphi \tag{2.10}$$

thus

$$\nabla_{\partial_r} \frac{1}{\sin\left(\frac{\pi}{4} - r\right)} \partial_\varphi = 0$$

$$\frac{1}{\sin\left(\frac{\pi}{4} - r\right)} \nabla_{\partial_r} \partial_\varphi + \frac{1}{\sin\left(\frac{\pi}{4} - r\right)} \cdot \cot\left(\frac{\pi}{4} - r\right) \partial_\varphi = 0$$

hence

$$\nabla_{\partial_r} \partial_\varphi = -\cot\left(\frac{\pi}{4} - r\right) \partial_\varphi.$$

Similarly

$$\nabla_{\partial_r} \partial_\theta = \cot\left(\frac{\pi}{4} + r\right) \partial_\varphi.$$

Since $[\partial_r, \partial_\theta] = [\partial_r, \partial_\varphi] = 0$, we also have

$$\begin{aligned}
\nabla_{\partial_\theta} \partial_r &= \cot\left(\frac{\pi}{4} + r\right) \partial_\varphi \\
\nabla_{\partial_\varphi} \partial_r &= -\cot\left(\frac{\pi}{4} - r\right) \partial_\varphi.
\end{aligned}$$

Since the $\theta - r$ plane is a totally geodesic surface in $\mathbf{S}^3$, it is an equatorial sphere. By symmetry in θ, the curve $\{r = r_0\}$ in this plane has constant geodesic curvature and is therefore a circle. Since it has length $L = 2\pi \sin\left(\frac{\pi}{4} + r_0\right)$ its geodesic curvature is $K = \cot\left(\frac{\pi}{4} + r_0\right)$.

Its outward normal is ∂_r (as this is the direction of increasing length). Hence

$$\begin{aligned}
\nabla_{\partial_\theta} \partial_\theta &= -\|\partial_\theta\|^2 \cot\left(\frac{\pi}{4} + r_0\right) \partial_r \\
&= -\cos\left(\frac{\pi}{4} + r_0\right) \sin\left(\frac{\pi}{4} + r_0\right) \partial_r \\
&= -\frac{1}{2} \sin\left(\frac{\pi}{2} + 2r_0\right) \partial_r \\
&= -\frac{1}{2} \cos(2r) \partial_r,
\end{aligned}$$

likewise

$$\begin{aligned}\nabla_{\partial\varphi}\partial_\varphi &= \frac{1}{2}\sin\left(\frac{\pi}{2} - 2r_0\right)\partial_r\\ &= \frac{1}{2}\cos(2r)\partial_r\end{aligned}$$

from (2.10), we get that

$$\partial_\varphi = \sin\left(\frac{\pi}{4} - r\right)N$$

Thus

$$\begin{aligned}\nabla_{\partial_\theta}\partial_\varphi =&\sin\left(\frac{\pi}{4} - r\right)\nabla_{\partial_\theta}N\\ =&0.\end{aligned}$$

In summary, we have

$$\begin{aligned}&\nabla_{\partial_r}\partial_r = 0\\ &\nabla_{\partial_\theta}\partial_r = \nabla_{\partial_r}\partial_\theta = \cot\left(\frac{\pi}{4} + r\right)\partial_\theta = \frac{\cos(2r)}{1+\sin(2r)}\partial_\theta\\ &\nabla_{\partial_\varphi}\partial_r = \nabla_{\partial_r}\partial_\varphi = -\cot\left(\frac{\pi}{4} - r\right)\partial_\varphi = -\frac{\cos(2r)}{1-\sin(2r)}\partial_\varphi\\ &\nabla_{\partial_\theta}\partial_\theta = -\cos\left(\frac{\pi}{4} + r\right)\sin\left(\frac{\pi}{4} + r\right)\partial_r = -\frac{1}{2}\cos(2r)\partial_r\\ &\nabla_{\partial_\varphi}\partial_\varphi = -\cos\left(\frac{\pi}{4} - r\right)\sin\left(\frac{\pi}{4} - r\right)\partial_r = \frac{1}{2}\cos(2r)\partial_r\\ &\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = 0.\end{aligned}$$

Perturbations of $\mathbf{T}_0$ are given in the Fermi parametrization by functions

$$f : \mathbf{S}^1 \times \mathbf{S}^1 \longrightarrow \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[. \tag{2.11}$$

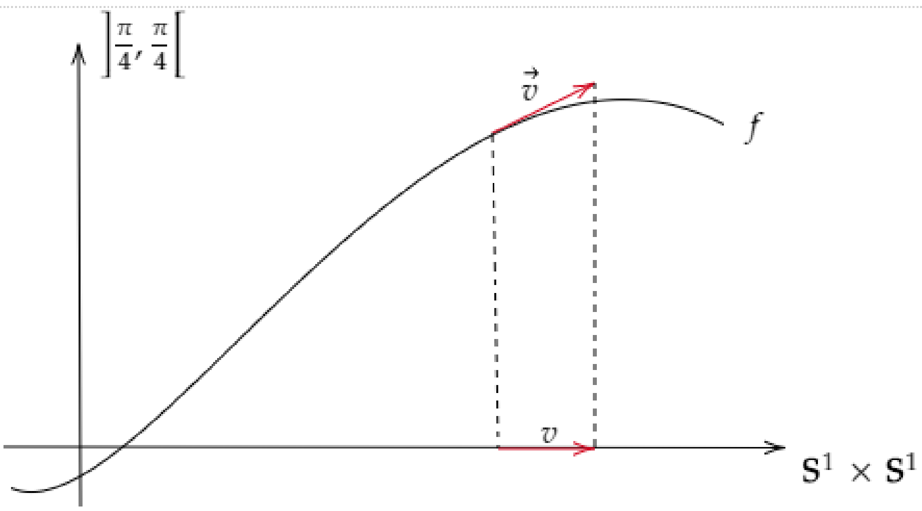

Figure 2 – The function f is a perturbation of $\mathbf{T}_0$

Given such a function f, denote:

$$\widehat{\partial}_\theta = \partial_\theta + f_\theta \partial_r$$
$$\widehat{\partial}_\varphi = \partial_\varphi + f_\varphi \partial_r$$

$$\widehat{N} = \left(-\sin^2\left(\frac{\pi}{4} - f\right) f_\theta, -\sin^2\left(\frac{\pi}{4} + f\right) f_\varphi, \sin^2\left(\frac{\pi}{4} - f\right)\sin^2\left(\frac{\pi}{4} + f\right)\right)$$

Observe that $\widehat{N}$ is the outward pointing normal over the graph of f. However $\widehat{N}$ does not have unit length. The unit normal vector field is

$$N := \frac{\widehat{N}}{\|\widehat{N}\|_{g_1}}$$

we have that

$$\begin{aligned} \sin^2\left(\frac{\pi}{4} - r\right) &= \frac{1}{2}\left[1 - \cos\left(\frac{\pi}{2} - 2r\right)\right] = \frac{1}{2}[1 - \sin(2r)] \\ \sin^2\left(\frac{\pi}{4} + r\right) &= \frac{1}{2}[1 + \sin(2r)] \\ \sin^2\left(\frac{\pi}{4} - r\right)\sin^2\left(\frac{\pi}{4} + r\right) &= \frac{1}{4}\left[1 - \sin^2(2r)\right] = \frac{1}{4}\cos^2(2r). \end{aligned}$$

Hence

$$\begin{aligned} \widehat{N} &= \left(-\frac{1}{2}(1 - \sin(2f))f_\theta, -\frac{1}{2}(1 + \sin(2f))f_\varphi, \frac{1}{4}\cos^2(2f)\right) \\ ||\widehat{N}||^2_{g_1} &= \frac{1}{8}\cos^2(2f)(1 - \sin(2f))f_\theta^2 + \frac{1}{8}\cos^2(2f)(1 + \sin(2f))f_\varphi^2 + \frac{1}{16}\cos^4(2f) \\ &= \frac{1}{4}\cos^2(2f)\left(\frac{1}{2}(1 - \sin(2f))f_\theta^2 + \frac{1}{2}(1 + \sin(2f))f_\varphi^2 + \frac{1}{4}\cos^2(2f)\right) \end{aligned}$$

Let Γ be the Christoffel symbol of the Levi Civita covariant derivative of g_1. That is,

$$\Gamma^k_{ij}(x)\partial_k = \nabla_{\partial_i}\partial_j - D_{\partial_i}\partial_j$$

where D denotes the canonical differentiation operator over $\mathbb{R}^4$. From the previous considerations, we have

$$\begin{aligned} \Gamma\left(\partial_r, \partial_r\right) &= \nabla_{\partial r}\partial_r - D_{\partial r}\partial_r = 0 \\ \Gamma\left(\partial_\theta, \partial_r\right) &= \nabla_{\partial_\theta}\partial_r - D_{\partial_\theta}\partial_r = \cot\left(\tfrac{\pi}{4} + r\right)\partial_\theta \\ \Gamma\left(\partial_\varphi, \partial_r\right) &= \nabla_{\partial_\varphi}\partial_r - D_{\partial_\varphi}\partial_r = -\cot\left(\tfrac{\pi}{4} - r\right)\partial_\varphi \\ \Gamma\left(\partial_\theta, \partial_\theta\right) &= \nabla_{\partial_\theta}\partial_\theta - D_{\partial_\theta}\partial_\theta = -\tfrac{1}{2}\cos(2r)\partial_r \\ \Gamma\left(\partial_\varphi, \partial_\varphi\right) &= \nabla_{\theta_\varphi}\partial_\varphi - D_{\partial_\varphi}\partial_\varphi = \tfrac{1}{2}\cos(2r)\partial_r \\ \Gamma\left(\partial_\theta, \partial_\varphi\right) &= \nabla_{\partial_\theta}\partial_\varphi - D_{\partial_\theta}\partial_\varphi = 0 \\ \Gamma\left(\partial_\varphi, \partial_\theta\right) &= \nabla\partial_\varphi\partial_\theta - D_{\partial_\varphi}\partial_\theta = 0 \end{aligned} \tag{2.12}$$

We now calculate the second fundamental form of the graphs of f.

$$D_{\widehat{\partial}_\theta}\widehat{N}=\left(\cos(2f)f_\theta^2+\frac{1}{2}(\sin(2f)-1)f_\theta,-\cos(2f)f_\theta f_\varphi-\frac{1}{2}(\sin(2f)+1)f_{\theta\varphi},-\cos(2f)\sin(2f)f_\theta\right)$$

$$D_{\widehat{\partial}_\varphi}\widehat{N}=\left(\cos(2f)f_\varphi f_\theta+\frac{1}{2}(\sin(2f)-1)f_{\varphi\theta},-\cos(2f)f_\varphi^2-\frac{1}{2}(\sin(2f)+1)f_{\varphi\varphi},-\cos(2f)\sin(2f)f_\varphi\right).$$

Then

$$\begin{aligned}\left\langle D_{\widehat{\partial}_\theta}\widehat{N},\widehat{\partial}_\theta\right\rangle_{g_1}=&\left\langle\left(\cos(2f)f_\theta^2+\frac{1}{2}(\sin(2f)-1)f_{\theta\theta},-\cos(2f)f_\theta f_\varphi-\frac{1}{2}(\sin(2f)+1)f_{\theta\varphi},\right.\right.\\&-\cos(2f)\sin(2f)f_\theta\right),(1,0,f_\theta)\rangle_{g_1}\\=&\sin^2\left(\frac{\pi}{4}+f\right)\cos(2f)f_\theta^2+\frac{1}{2}\sin^2\left(\frac{\pi}{4}+f\right)(\sin(2f)-1)f_{\theta\theta}-\cos(2f)\sin(2f)f_\theta^2\\=&\left(\frac{1}{2}(1+\sin(2f))\cos(2f)-\cos(2f)\sin(2f)\right)f_\theta^2+\frac{1}{4}\left(\sin^2(2f)-1\right)f_{\theta\theta}\\=&-\frac{1}{4}\cos^2(2f)f_{\theta\theta}+\frac{1}{2}(1-\sin(2f))\cos(2f)f_\theta^2.\end{aligned}$$

$$\begin{aligned}\left\langle D_{\widehat{\partial}_\theta}\widehat{N},\widehat{\partial}_\varphi\right\rangle_{g_1}=&\left\langle(\cos(2f)f_\theta^2+\frac{1}{2}(\sin(2f)-1)f_{\theta\theta},-\cos(2f)f_\theta f_\varphi-\frac{1}{2}(\sin(2f)+1)f_{\theta\varphi},\right.\\&-\cos(2f)\sin(2f)f_\theta,(0,1,f_\varphi)\rangle_{g_1}\\=&\frac{1}{2}\left(\sin((2f)-1)\cos(2f)f_\theta f_\varphi+\frac{1}{4}\left(\sin^2(2f)-1\right)f_{\theta\varphi}-\cos(2f)\sin(2f)f_\theta f_\varphi\right.\\=&\frac{1}{2}\sin(2f)\cos(2f)f_\theta f_\varphi-\frac{1}{2}\cos(2f)f_\theta f_\varphi+\frac{1}{4}\left(\sin^2(2f)-1\right)f_{\theta\varphi}-\cos(2f)\sin(2f)f_\theta f_\varphi\\=&-\frac{1}{2}\cos(2f)(\sin(2f)+1)f_\theta f_\varphi-\frac{1}{4}\cos^2(2f)f_{\theta\varphi}.\end{aligned}$$

Using (2.12) and the linearity of Cristhoffel symbols Γ, we obtain

$$\begin{aligned}\Gamma\left(\widehat{\partial}_\theta,\widehat{N}\right)=&\frac{1}{2}(\sin(2f)-1)f_\theta\Gamma\left(\partial_\theta,\partial_\theta\right)-\frac{1}{2}(\sin(2f)+1)f_\varphi\Gamma\left(\partial_\theta,\partial_\varphi\right)\\&+\frac{1}{4}\cos^2(2f)\Gamma\left(\partial_\theta,\partial_t\right)+\frac{1}{2}(\sin(2f)-1)f_\theta^2\Gamma\left(\partial_t,\partial_\theta\right)\\&-\frac{1}{2}(\sin(2f)+1)f_\theta f_\varphi\Gamma\left(\partial_t,\partial_\varphi\right)+\frac{1}{4}\left(\cos^2(2f)f_\theta\Gamma(\partial_t,\partial_t)\right.\\=&\left(\frac{1}{4}\cos^2(2f)+\frac{1}{2}(\sin(2f)-1)f_\theta^2\right)\cot\left(\frac{\pi}{4}+f\right)\partial_\theta\\&+\frac{1}{2}(\sin(2f)+1)f_\theta f_\varphi\cot\left(\frac{\pi}{4}-f\right)\partial_\varphi-\frac{1}{4}(\sin(2f)-1)f_\theta\cos(2t)\partial_t\end{aligned}$$

thus

$$\begin{aligned}\left\langle \Gamma\left(\widehat{\partial}_\theta, \widehat{N}\right), \widehat{\partial}_\theta \right\rangle =& \sin^2\left(\frac{\pi}{4}+f\right)\left(\frac{1}{4}\cos^2(2f)+\frac{1}{2}(\sin(2f)-1)f_\theta^2\right)\cot\left(\frac{\pi}{4}+f\right)\\ &-\frac{1}{4}(\sin(2f)-1)\cos(2f)f_\theta^2\\ =&\frac{1}{2}\left(\frac{1}{4}\cos^2(2f)+\frac{1}{2}(\sin(2f)-1)f_\theta^2\right)\cos(2f)-\frac{1}{4}(\sin(2f)-1)\cos(2f)f_\theta^2\\ =&\frac{1}{8}\cos^3(2f)+\frac{1}{4}(\sin(2f)-1)\cos(2f)f_\theta^2-\frac{1}{4}(\sin(2f)-1)\cos(2f)f_\theta^2\\ =&\frac{1}{8}\cos^3(2f)\end{aligned}$$

$$\begin{aligned}\left\langle \Gamma\left(\widehat{\partial}_\theta, \widehat{N}\right), \widehat{\partial}_\varphi \right\rangle =& \sin^2\left(\frac{\pi}{4}-f\right)\frac{1}{2}(\sin(2f)+1)f_\theta f_\varphi\cot\left(\frac{\pi}{4}-f\right)\\ &-\frac{1}{4}(\sin(2f)-1)\cos(2f)f_\theta f_\varphi\\ =&\frac{1}{2}(\sin(2f)+1)f_\theta f_\varphi\sin\left(\frac{\pi}{4}-f\right)\cos\left(\frac{\pi}{4}-f\right)-\frac{1}{4}(\sin(2f)-1)\cos(2f)f_\theta f_\varphi\\ =&\frac{1}{4}(\sin(2f)+1)\cos(2f)f_\theta f_\varphi-\frac{1}{4}(\sin(2f)-1)\cos(2f)f_\theta f_\varphi\\ =&\frac{1}{4}\cos(2f)f_\theta f_\varphi(\sin(2f)+1-\sin(2f)-1)\\ =&\frac{1}{2}\cos(2f)f_\theta f_\varphi.\end{aligned}$$

The second fundamental form is thus given by

$$\begin{aligned}II\left(\widehat{\partial}_\theta,\widehat{\partial}_\theta\right)=&\frac{1}{8}\cos^3(2f)-\frac{1}{4}\cos^2(2f)f_{\theta\theta}+\frac{1}{2}(1-\sin(2f))\cos(2f)f_\theta^2.\\ II\left(\widehat{\partial}_\theta,\widehat{\partial}_\varphi\right)=&-\frac{1}{2}\cos(2f)\sin(2f)f_\theta f_\varphi-\frac{1}{4}\cos^2(2f)f_{\theta\varphi}.\end{aligned}\tag{2.13}$$

and as there is a symmetry, such that

$$f\mapsto -f \quad \theta\mapsto\varphi \quad \varphi\mapsto\theta \quad II\mapsto -II$$

we have that

$$II\left(\widehat{\partial}_\varphi,\widehat{\partial}_\varphi\right)=-\frac{1}{8}\cos^3(2f)-\frac{1}{4}\cos^2(2f)f_{\varphi\varphi}-\frac{1}{2}(1+\sin(2f))\cos(2f)f_\varphi^2.\tag{2.14}$$

On the other hand, the first fundamental form is given by

$$\begin{aligned}I\left(\widehat{\partial_\theta},\widehat{\partial_\theta}\right)=&\frac{1}{2}(1+\sin(2f))+f_\theta^2.\\ I\left(\widehat{\partial_\theta},\widehat{\partial}_\varphi\right)=&f_\theta f_\varphi\\ I\left(\widehat{\partial}_\varphi,\widehat{\partial_\theta}\right)=&f_\theta f_\varphi\\ I\left(\widehat{\partial}_\varphi,\widehat{\partial}_\varphi\right)=&\frac{1}{2}(1-\sin(2f))+f_\varphi^2\end{aligned}\tag{2.15}$$

S

$$I = \begin{pmatrix} \frac{1}{2}(1+\sin(2f)) + f_\theta^2 & f_\theta f_\varphi \\ f_\theta f_\varphi & \frac{1}{2}(1-\sin(2f)) + f_\varphi^2 \end{pmatrix}$$

with $\det(I) = \frac{4}{\cos^2(2f)}\|\widehat{N}\|^2$, then

$$I^{-1} = \frac{\cos^2(2f)}{4\|\widehat{N}\|^2} \begin{pmatrix} \frac{1}{2}(1-\sin(2f)) + f_\varphi^2 & -f_\theta f_\varphi \\ -f_\theta f_\varphi & \frac{1}{2}(1+\sin(2f)) + f_\theta^2. \end{pmatrix}$$

From the Weingarten equations, the mean curvature H satisfies

$$H = \operatorname{trace}\left(I^{-1} II\right) = \frac{1}{||\widehat{N}||^2}\operatorname{trace}\left(I^{-1}\widehat{II}\right). \tag{2.16}$$

Using the notation given by (2.1) we have

$$\begin{aligned} \widehat{N} &= \left(-\frac{1}{2}f_\theta + O_1^2, -\frac{1}{2}f_\varphi + O_1^2, \frac{1}{4} + O_0^2\right) \\ &= \left(-\frac{1}{2}O_1^1 + O_1^2, -\frac{1}{2}O_1^1 + O_1^2, \frac{1}{4} + O_0^2\right) \\ \|\widehat{N}\|^2 &= \frac{1}{8}\left(1 + O_0^2\right)\left(\frac{1}{2} + O_1^2\right) = \frac{1}{16} + O_1^2. \end{aligned}$$

So

$$I = \begin{pmatrix} \frac{1}{2}(1+2f) + O_1^2 & O_1^2 \\ O_1^2 & \frac{1}{2}(1-2f) + O_1^2 \end{pmatrix},$$

then

$$I^{-1} = 2\begin{pmatrix} 1-2f+O_1^2 & O_1^2 \\ O_1^2 & 1+2f+O_1^2 \end{pmatrix},$$

and

$$\widehat{II} = \begin{pmatrix} \frac{1}{8} - \frac{1}{4}\left(1+O_0^2\right)f_{\theta\theta} + O_1^2 & -\frac{1}{4}\left(1+O_0^2\right)f_{\theta\varphi} + O_1^2 \\ -\frac{1}{4}\left(1+O_0^2\right)f_{\theta\varphi} + O_1^2 & -\frac{1}{8} - \frac{1}{4}\left(1+O_0^2\right)f_{\varphi\varphi} + O_1^2. \end{pmatrix}$$

On the other hand

$$\begin{aligned} \left(I^{-1}\widehat{II}\right)_{\theta\theta} &= 2(1-2f)\left(\frac{1}{8} - \frac{1}{4}f_{\theta\theta}\right) + O_1^2 + O_1^2\operatorname{Hess}(f) \\ &= \frac{1}{4} - \frac{f}{2} - \frac{1}{2}f_{\theta\theta} + f f_{\theta\theta} + O_1^2 + O_1^2\operatorname{Hess}(f) \end{aligned}$$

$$\begin{aligned} \left(I^{-1}\widehat{II}\right)_{\varphi\varphi} &= 2(1+2f)\left(-\frac{1}{8} - f_{\varphi\varphi}\right) + O_1^2 + O_1^2\operatorname{Hess}(f) \\ &= -\frac{1}{4} + \frac{f}{2} - \frac{1}{2}f_{\varphi\varphi} + f f_{\varphi\varphi} + O_1^2 + O_1^2\operatorname{Hess}(f) \end{aligned}$$

and as $II = \frac{1}{||\widehat{N}||}\widehat{II}$ we have

$$\begin{aligned} 4\left(I^{-1}II\right)_{\theta\theta} &= 4 - 8f - 8f_{\theta\theta} + 16 f f_{\theta\theta} + ... \\ 4\left(I^{-1}II\right)_{\varphi\varphi} &= -4 - 8f - 8f_{\varphi\varphi} + 16 f f_{\varphi\varphi} + ... \end{aligned}$$

Finally

$$H = -2(\Delta f + 2f) + 4f\,(f_{\theta\theta} - f_{\varphi\varphi}) + \operatorname{trace}\Big(O_1^2\,\mathrm{Hess}(f)\Big) + O_1^2. \tag{2.17}$$

Observe in particular that, when $f = 0$, $\widehat{N} = (0, 0, \frac{1}{4})$; $||\widehat{N}||^2 = \frac{1}{16}$ then

$$I^{-1} = \frac{1}{4}\begin{pmatrix} \dfrac{1}{2} & 0 \\ 0 & \dfrac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{8} \end{pmatrix}$$

and

$$II = \begin{pmatrix} \dfrac{1}{8} & 0 \\ 0 & -\dfrac{1}{8} \end{pmatrix}$$

and therefore $H = 0$.

On the other hand, note also that, all the previous terms vary when we vary f. Since f lives in a space of functions, we study the Frechet derivative. Let h be a function. Using the preceding relations, we determine partial derivations in the direction of h.

We have that $D_f||\widehat{N}||^2 = 0$, thus

$$D_f I^{-1}\Big|_{f=0} \cdot h = \begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}$$
$$D_f II\big|_{f=0} \cdot h = \begin{pmatrix} -h_{\theta\theta} & -h_{\theta\phi} \\ -h_{\phi\theta} & -h_{\phi\phi} \end{pmatrix}$$

Likewise

$$D_f H\big|_{f=0} \cdot h = -2\,(\Delta + 2)\,h$$

is the operator of the infinitesimal variation of the mean curvature, here Δ denotes the Laplace operator of the metric g_1. Indeed, this is the Jacobi operator of $\mathbf{T}_0$.

2.3 The differential geometries of $\mathcal{E}$ and CL

We consider the sets $\mathcal{E}$ and CL, spaces of embeddings that will be of interest to us, these sets will be defined as follows

$$\hat{\mathcal{E}} = \{e : \mathbf{S}^1 \times \mathbf{S}^1 \longrightarrow \mathbf{S}^3 \mid e \text{ is a smoothly embedded torus in } \mathbf{S}^3\}.$$

This space is furnished with the C^∞ topology. Let $\mathcal{D}$ denote the group of smooth diffeomorphisms of $\mathbf{S}^1 \times \mathbf{S}^1$. Observe that $\mathcal{D}$ acts on $\hat{\mathcal{E}}$ on the right by composition. Let $\mathcal{E} := \hat{\mathcal{E}}/\mathcal{D}$ denote the quotient space of this action furnished with the quotient topology, this space is a weakly smooth manifold which we call the space of unparametrised embeddings of $\mathbf{S}^1 \times \mathbf{S}^1$ in $\mathbf{S}^3$. Given an element e of $\hat{\mathcal{E}}$, we denote by $[e]$ its equivalence class in $\mathcal{E}$. Throughout the sequel, we will often identify the equivalence class $[e]$ with its representative element e.

Let $\mathcal{E}'$ denote the space of embedded tori in $\mathbf{S}^3$ furnished with the topology of smooth convergence. The map

$$\mathrm{Im} : \hat{\mathcal{E}} \to \mathcal{E}'$$

such that

$$e \mapsto \left\{e(\theta, \varphi) \mid (\theta, \varphi) \in \mathbf{S}^1\right\}$$

project to a homeomorphism from $\mathcal{E}$ to $\mathcal{E}'$. In what follows, $\mathcal{E}'$ will be identified with $\mathcal{E}$, that is, every element e of $\hat{\mathcal{E}}$ will be identified with its equivalence class $[e]$ in $\mathcal{E}$ and its image $\mathrm{Im}(e)$ in $\mathbf{S}^3$.

We define

$$\mathrm{CL} = \{T \mid T \text{ is Clifford torus in } \mathbf{S}^3\}$$

That is, CL is the orbit of $\mathbf{T}_0$ in $\mathcal{E}$ under the action of the orthogonal group O(4). Furthermore, in [10], White shows that this submanifold is diffeomorphic to $\mathbb{RP}^2 \times \mathbb{RP}^2$.

The Fermi parametrization of $\mathbf{S}^3$ defines a chart of $\mathcal{E}$ about $\mathbf{T}_0$. Indeed, define $\hat{e} : C^\infty\left(\mathbf{S}^1 \times \mathbf{S}^1, \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\right) \longrightarrow \hat{\mathcal{E}}$ by

$$\hat{e}[f](\theta, \varphi) := \Phi(\theta, \varphi, f(\theta, \varphi)) \tag{2.18}$$

and define $e : C^\infty\left(\mathbf{S}^1 \times \mathbf{S}^1, \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\right) \longrightarrow \mathcal{E}$ by

$$e := [\hat{e}] \tag{2.19}$$

In other words, $e[f]$ is the image under Φ of the graph of f. Finally, recall that the Jacobi operator of $\mathbf{T}_0$ is

$$\mathrm{J} := -2(\Delta + 2)$$

where Δ denotes the Laplace operator of $\mathbf{S}^1 \times \mathbf{S}^1$.

Lemma 4. *The tangent space at 0 of $e^{-1}(\mathrm{CL})$ is*

$$T_0 e^{-1}(\mathrm{CL}) = \mathrm{Ker}(\Delta + 2) \tag{2.20}$$

where Δ here denotes the standard Laplacian of $\mathbf{S}^1 \times \mathbf{S}^1$.

Proof. Choose $f \in T_0 e^{-1}(\mathrm{CL})$. Let $(f_t)_{t \in]-\epsilon, \epsilon[}$ be a family of functions such that $f_0 = f$ and $f_t \in e^{-1}(\mathrm{CL})$ for all t. Let $(H_t)_{t \in]-\epsilon, \epsilon[}$ be such that, for all t and for all (θ, φ), $H_t(\theta, \varphi)$ is the mean curvature of the embedding $\hat{e}\left[f_t\right]$ at the point (θ, φ). For all t, since $\hat{e}\left[f_t\right]$ is a Clifford torus, it is minimal, thus H_t vanishes for all f_t.

Thus, since J denotes the Jacobi operator of $\mathbf{T}_0$, by definition,

$$\mathrm{J}f = \left.\frac{\partial H_t}{\partial t}\right|_{t=0} = 0.$$

Since $\mathrm{J} = -2(\Delta + 2)$, it follows that $f \in \mathrm{Ker}(\Delta + 2)$. Since $f \in T_0 e^{-1}(\mathrm{CL})$ is arbitrary, it follows that

$$T_0 e^{-1}(\mathrm{CL}) \subseteq \mathrm{Ker}(\Delta + 2)$$

Finally, to obtain equality, notice that these two spaces are 4 dimensional. This completes the proof. □

Furnishing $\mathcal{E}$ with a smooth manifold structure is non-trivial. In fact, we would like to $\mathcal{E}$ to be a Banach manifold. However this is not the case, because the transition applications between two different charts are not differentiable. For this reason, we introduce the following terminology of **weakly smooth manifolds**, which allows us to endow $\mathcal{E}$ with all the structure required to develop our theory. We refer the reader to [Ros-S] for more details.

Let X be a smooth, compact, finite dimensional manifold. Given another smooth, finite dimensional manifold Y and a map, $\phi : Y \to C^\infty(X)$, we define $\tilde{\phi} : X \times Y \to \mathbb{R}$ such that $(x, y) \mapsto \phi(y)(x)$.

Definition 1. *ϕ is said to be strongly smooth whenever $\tilde{\phi}$ is smooth.*

Now consider another smooth, compact, finite dimensional manifold X'. Given an open subset $\Omega \subseteq C^\infty(X)$ and a continuous map $\Phi : \Omega \to C^\infty(X')$,

Definition 2. *Φ is said to be weakly smooth whenever the operation of composition by Φ sends strongly smooth maps continuously (in the C^∞_{loc} sense) into strongly smooth maps.*

Definition 3. *A weakly smooth manifold $\mathcal{M}$ modelled on $C^\infty(X)$, is now defined to be a Hausdorff topological space furnished with an atlas, all of whose charts are open subsets of $C^\infty(X)$, and all of whose transition maps are weakly smooth.*

Note that the function e defined by (2.19) is a weakly smooth diffeomorphism onto an open subset of $\mathcal{E}$. It thus defines a weakly smooth chart of $\mathcal{E}$ about $\mathbf{T}_0$. Finally, in this terminology, CL is a strongly smooth, 4 -dimensional embedded submanifold of $\mathcal{E}$ and observe that, the set $e^{-1}(\mathrm{CL})$ is a strongly smooth submanifold of $C^\infty\left(\mathbf{S}^1 \times \mathbf{S}^1, \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\right)$.

It follows that $\mathcal{E}$ naturally carries the structure of a weakly smooth manifold (see [4]). It therefore makes sense to talk about tangent vectors, tangent spaces, differentiability and so on.

2.3.1 The symmetric space structures of $\mathbf{S}^3$ and CL

In order to understand how the geometry of CL relates to that of $\mathbf{S}^3$, we review the symmetric space structures of these manifolds. Let O(4) denote the group of 4 -dimensional orthogonal matrices. Let $\Psi_{\mathbf{S}} : \mathrm{O}(4) \times \mathbf{S}^3 \longrightarrow \mathbf{S}^3$ and $\Psi_{\mathrm{CL}} : \mathrm{O}(4) \times \mathrm{CL} \to \mathrm{CL}$ denote the left action of O(4) on $\mathbf{S}^3$ and CL, respectively. That is, for all $M \in \mathrm{O}(4)$ and for all $x \in \mathbf{S}^3$ and $T \in \mathrm{CL}$

$$\Psi_{\mathbf{S}}(M, x) := Mx = \{Mx \mid x \in \mathbf{S}^3\} \text{ and}$$

$$\Psi_{\mathrm{CL}}(M, T) := MT = \{Mx \mid x \in T\}.$$

The subset of elements of the group that leaves a given element fixed plays an important role. The stabiliser of $\mathbf{T}_0$ in O(4) is

$$\mathrm{Stab}(\mathbf{T}_0) = \left\{ \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \mid M, N \in \mathrm{O}(2) \right\}$$

This subgroup naturally identifies with $\mathrm{O}(2) \times \mathrm{O}(2) = \mathrm{O}(2)^2$, and we write

$$\mathrm{Stab}\ (\mathbf{T}_0) = \mathrm{O}(2)^2$$

Thus, if $C : \mathrm{O}(4) \longrightarrow \mathrm{CL}$ is defined by

$$C(M) := M\mathbf{T}_0, \tag{2.21}$$

Then C descends to a smooth diffeomorphism from the homogeneous space $\mathrm{O}(4)/\mathrm{O}(2)^2$ into CL.In this manner, CL identifies with homogeneous space of O(4).

Let $\mathfrak{o}(4)$ denote the Lie algebra of O(4) which, we recall, identifies with the space of 4-dimensional antisymmetric matrices.

$$\mathfrak{o}(4) = \left\{ M \mid M + M^t = 0 \right\}$$

We furnish this Lie algebra with the positive-definite bilinear form

$$G(M, N) := -\pi^2 \operatorname{trace}(MN) \tag{2.22}$$

Define

$$\mathfrak{h} := \mathfrak{o}(2) \oplus \mathfrak{o}(2) \text{ and}$$
$$\mathfrak{k} := \mathfrak{h}^\perp$$

where the orthogonal complement is here taken with respect to G. We verify that

$$[\mathfrak{h}, \mathfrak{k}] \subseteq \mathfrak{k} \text{ and}$$
$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{h}$$

so that the decomposition

$$\mathfrak{o}(4) = \mathfrak{k} \oplus \mathfrak{h}$$

constitutes a polarisation of $\mathfrak{o}(2)$. It follows that CL is in fact a symmetric space (c.f. [22]). Since CL has a symmetric space structure we are going to study its Riemannian metric. For $A \in \mathrm{End}(\mathbb{R}^2)$, define $\xi_A \in \mathfrak{k}$ by

$$\xi_A := \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix}$$

The map $A \longrightarrow \xi_A$ trivially defines a linear isomorphism from $\mathrm{End}(\mathbb{R}^2)$ into $\mathfrak{k}$.

The metric over $\mathrm{End}\left(\mathbb{R}^2\right)$ is given by

$$\langle A, B\rangle := 2\pi^2 \,\mathrm{trace}\left(AB^t\right). \tag{2.23}$$

with these metrics, the above map becomes a linear isometry.

Let C be a funtion defined by (2.21), its derivative at Id defines a linear isomorphism from $\mathfrak{k}$ into the tangent space to CL at $\mathbf{T}_0$. By Lemma 4, this tangent space in turn identifies with $\mathrm{Ker}(\Delta+2) \subseteq C^\infty\left(\mathbf{T}_0,]-\frac{\pi}{4}, \frac{\pi}{4}[\right)$. For all $A \in \mathrm{End}\left(\mathbb{R}^2\right)$, let ϕ_A denote the image of ξ_A under this map. Following through these identifications, we readily obtain,

Lemma 5. *For all* $A \in \mathrm{End}(\mathbb{R}^2)$,

$$\phi_A(\theta, \varphi) = (\cos(\theta), \sin(\theta)) A \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix}. \tag{2.24}$$

Proof. Indeed, for sufficiently small $\epsilon > 0$, let $(\alpha_t)_{t\in]-\epsilon,\epsilon[} : \mathbf{S}^1 \times \mathbf{S}^1 \to \mathbf{S}^1 \times \mathbf{S}^1$ and $(h_t)_{t\in]-\epsilon,\epsilon[} : \mathbf{S}^1 \times \mathbf{S}^1 \to \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[$ be such that, for all (θ, φ)

$$\mathrm{Exp}(t\xi_A)\Phi_0(\theta, \varphi) = \Phi\left(\alpha_t(\theta, \varphi), h_t(\theta, \varphi)\right)$$

where Φ_0 is the canonical parametrization of the Clifford Torus $\mathbf{T}_0$.

For all t,

$$\mathrm{Exp}(t\xi_A)\,\mathrm{Im}\left(\Phi_0\right) = e\left[h_t \circ \alpha_t^{-1}\right]$$

it follows that

$$\phi_A = \frac{\partial}{\partial t} h_t \circ \alpha_t^{-1}\bigg|_{t=0}$$

since $\alpha_0 = \mathrm{Id}$ and $h_0 = 0$, the chain rule yields

$$\phi_A = \frac{\partial}{\partial t} h_t\bigg|_{t=0} = \left\langle \frac{\partial}{\partial t}\mathrm{Exp}(t\xi_A)\Phi_0(\theta, \varphi)\bigg|_{t=0}, N_0(\theta, \varphi)\right\rangle,$$

where N_0 is the unit normal vector field over Φ_0 given by (2.4). Thus

$$\phi_A = \langle \xi_A \Phi_0(\theta, \varphi), N_0(\theta, \varphi)\rangle$$

and the result follows, upon substituting (2.3), (2.4) into this relation.

□

By (2.24), for all $A, B \in \mathrm{End}\,\mathbb{R}^2$,

$$G(\xi_A, \xi_B) = \langle A, B\rangle = \int_{\mathbf{S}^1\times\mathbf{S}^1} \phi_A \phi_B \, d\theta d\varphi,$$

So that, $DC(\mathrm{Id})$ defines a linear isometry from $\mathfrak{k}$ to $T_{\mathbf{T}_0}\mathrm{CL}$, justifying the normalisations of (2.22) and (2.23).

Since B is bi-invariant under the action by conjugation of $\mathrm{O}(4)$ on $\mathfrak{o}(4)$, it extends by left and right translation to a unique Riemannian metric over $\mathrm{O}(4)$. Recall now that the tangent bundle of $\mathrm{O}(4)$ decomposes orthogonally with respect to this metric as

$$T\mathrm{O}(4) = \tau^L\mathfrak{h} \oplus \tau^L\mathfrak{k},$$

where

$$\tau^L\mathfrak{h} := \{(M, MA) | A \in \mathfrak{h}\}$$
$$\tau^L\mathfrak{k} := \{(M, MA) | A \in \mathfrak{k}\}$$

here denote the left translations of $\mathfrak{h}$ and $\mathfrak{k}$ respectively. Since the left action of O(4) on CL is also isometric, it follows that by the preceding paragraph C defines a riemannian submersion with kernel $\tau^L\mathfrak{h}$ such that C^*TCL is isometric to $\tau^L\mathfrak{k}$. In particular, C defines a smooth isometry from $\mathrm{O}(4)/\mathrm{O}(2)^2$ into CL.

2.3.2 Killing fields over CL.

For all $\xi \in \mathfrak{o}(4)$, let X_ξ denote the pull back through Φ of the vector field that it generates over $\mathbf{S}^3$

$$X_\xi = \Phi^* \left.\frac{\partial}{\partial t}(\mathrm{Exp}(t\xi_A)\cdot x)\right|_{t=0}$$

and let F_ξ denote its flow. Vector fields of this form are referred to as Killing fields. The space of Killings fields as a 6-dimensional vector subspace of $\Gamma\left(\mathbf{S}^1\times\mathbf{S}^1\times\left]-\frac{\pi}{4},\frac{\pi}{4}\right[\right)$.

Observe that, for all ξ and for all suitable (θ,φ,r) and t

$$\left(\Phi\circ F_{\xi,t}\right)(\theta,\varphi,r) = \left(\mathrm{Exp}\, t\xi\circ\Phi\right)(\theta,\varphi,r) \tag{2.25}$$

Observe that, when $\xi\in\mathfrak{h}$, X_ξ is tangent to the surface $\mathbf{S}^1\times\mathbf{S}^1\times\{0\}$. In particular, F_ξ preserves this surface. It follows that elements of $\mathfrak{h}$ yields Killing fields and flows which are trivial for own purposes. For this reason, we will only be interested in Killing fields and flows arising from elements of $\mathfrak{k}$. Thus for all $A\in\mathrm{End}(\mathbb{R}^2)$, denote

$$\begin{aligned} X_A :=& X_{\xi_A} \quad \text{and} \\ F_A :=& F_{\xi_A}. \end{aligned} \tag{2.26}$$

Lemma 6. *For all $A\in\mathrm{End}\left(\mathbb{R}^2\right)$*

$$X_A(\theta,\phi,r) = \cot\left(\frac{\pi}{4}+r\right)\frac{\partial\phi_A}{\partial\theta}(\theta,\varphi)\partial_\theta - \tan\left(\frac{\pi}{4}+r\right)\frac{\partial\phi_A}{\partial\varphi}(\theta,\varphi)\partial_\varphi + \phi_A(\theta,\varphi)\partial_r \tag{2.27}$$

Proof. Let g denote the pull-back through Φ of the constant curvature metric of $\mathbf{S}^3$. For all constant vectors ξ and η

$$\begin{aligned} \left(\mathcal{L}_{X_A}g\right)(\xi,\eta) &= \mathrm{D}_{X_A}(g(\xi,\eta)) - g\left(\left[X_A,\xi\right],\eta\right) - g\left(\xi,\left[X_A,\eta\right]\right) \\ &= \mathrm{D}_{X_A}(g(\xi,\eta)) + g\left(\mathrm{D}_\xi X_A,\eta\right) + g\left(\xi,\mathrm{D}_{X_A}\eta\right) \end{aligned}$$

However, since $F_{A,t}$ is a flow of isometries,

$$\mathcal{L}_{X_A}g = 0$$

By $g\left(\partial_r,\partial_r\right)=1$. Substituting $\xi=\eta=\partial_r$ therefore yields,

$$g\left(\mathrm{D}_{\partial_r}X_A,\partial_r\right) = -\mathrm{D}_{X_A}\left(g\left(\partial_r,\partial_r\right)\right) = 0$$

so that the ∂_r component of X_A is independent of r. However, when $r=0$, this component is equal to ϕ_A, so that

$$X_A(\theta,\varphi,r) = u(\theta,\varphi,r)\partial_\theta + v(\theta,\varphi,r)\partial_\varphi + \phi_A(\theta,\varphi)\partial_r$$

Substituting $\xi=\eta=\partial_\theta$ then yields

$$\phi_A\frac{\partial}{\partial r}\sin^2\left(\frac{\pi}{4}+r\right) + 2\sin^2\left(\frac{\pi}{4}+r\right)\frac{\partial u}{\partial\theta} = 0.$$

Since $\frac{\partial^2\phi_A}{\partial\theta^2} = -\phi_A$, this becomes

$$-\frac{\partial^2\phi_A}{\partial\theta^2}\frac{\partial}{\partial r}\sin^2(\frac{\pi}{4}+r) + 2\sin^2(\frac{\pi}{4}+r)\frac{\partial u}{\partial\theta} = 0$$

which is solved by

$$u(\theta,\varphi,r) = \cot(\frac{\pi}{4}+r)\frac{\partial\phi_A}{\partial\theta}(\theta,\varphi) + \tilde{u}(\varphi,r)$$

for some function $\tilde{u}$. In the same manner, we obtain

$$v(\theta,\varphi,r) = -\tan(\frac{\pi}{4}+r)\frac{\partial\phi_A}{\partial\varphi}(\theta,\varphi) + \tilde{v}(\theta,r)$$

for some function $\tilde{v}$Substituting $\xi = \partial_r$ and $\eta = \partial_\theta$ shows that $\tilde{u}$ is independent of r, whilst substituting $\xi = \partial_r$ and $\eta = \partial_\varphi$ shows that $\tilde{v}$ is also independent of r. Finally, substituting $\xi = \partial_\theta$ and $\eta = \partial_\varphi$ shows that

$$\frac{\partial\tilde{v}}{\partial\theta} + \frac{\partial\tilde{u}}{\partial\varphi} = 0,$$

so that, by the principle of separation of variables,

$$\frac{\partial\tilde{v}}{\partial\theta} = -\frac{\partial\tilde{u}}{\partial\varphi} = c,$$

for some constant c. Since $\mathbf{T}_0$ is compact, c vanishes so that both $\tilde{u}$ and $\tilde{v}$ are constant. Finally, by evaluating X_A at $r = 0$, we show that this constant vanishes, and this completes the proof.

□

2.3.3 Local Properties of the set of zeros of elements of $\mathrm{Ker}(\Delta + 2)$

In this section, we study the level sets (or sets of zeros) of elements of $\mathrm{Ker}(\Delta+2)$. The study of level sets of smooth functions is very delicate. Recall that, in a neighborhood of a regular point, the level set can be studied by the implicit function theorem. However, in a neighborhood of a singular point, the study is very delicate.
We shall provide a charming geometric description of Fermi coordinates about a Clifford torus.

For $A \in \mathrm{End}\,(\mathbb{R}^2)\setminus\{0\}$, Z_A will denote the set the points $(\theta,\varphi) \in T$ such that $\phi_A(\theta,\varphi) = 0$ that is,

$$Z_A := \phi_A^{-1}(\{0\}). \tag{2.28}$$

Observe that Z_A is a projective property of the matrix A in the sense that it unaffected by multiplication by non-zero scalar. Recall that the metric over $\mathrm{End}(\mathbb{R}^2)$, is given by

$$\langle A, B\rangle := 2\pi^2\,\mathrm{trace}(AB^t). \tag{2.29}$$

Consider the unit sphere in $\mathrm{End}(\mathbb{R}^2)$. This is the set given by

$$\Sigma := \{A \mid \|A\|^2 = 1\}. \tag{2.30}$$

Since Z_A is projective, it suffices to consider $A \in \Sigma$. Let

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

denote the standard convex structure over $\mathbb{R}^2$. Recall that, for all t,

$$\exp(tJ) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \tag{2.31}$$

Lemma 7. *For all $\theta, \varphi \in \mathbf{S}^1$ and for all s, t*

i) $\phi_A(\theta + s, \varphi + t) = \phi_{\mathrm{Exp}(-sJ)A\,\mathrm{Exp}(tJ)}$.

ii) $\partial_\theta \phi_A = -\phi_{JA}$.

iii) $\partial_\varphi \phi_A = \phi_{AJ}$.

iv) $\mathrm{Det}(A) = \phi_{AJ}\phi_{JA} - \phi_A \phi_{JAJ}$.
In particular $\phi_A(\theta + \pi, \varphi) = \phi_A(\theta, \varphi + \pi) = -\phi_A(\theta, \varphi)$.

Proof. *i*), *ii*), *iii*), *iv*) are obtained directly from the definition of ϕ_A. □

Finally, recall that, for all $A \in \mathrm{End}(\mathbb{R}^2)$,

$$\mathrm{Det}(A) = -\frac{1}{2}\,\mathrm{trace}(JAJA^t). \tag{2.32}$$

Denote:

$$\begin{aligned} \Sigma_0 :=& \{A \in \Sigma \mid \mathrm{Det}(A) = 0\} \text{ and} \\ \Sigma_{\pi/4} :=& \left\{ \frac{1}{\sqrt{2\pi}} A \mid A \in \mathrm{O}(2) \right\} \end{aligned} \tag{2.33}$$

Matrices in Σ_0 and $\Sigma_{\pi/4}$ will be said to be singular and special respectively, whilst all other matrices in Σ will be said to be generic. Suppose that $A \in \Sigma$ is singular. Upon composing with rotations, we may suppose that

$$A = \begin{pmatrix} \frac{1}{\sqrt{2\pi}} & 0 \\ 0 & 0 \end{pmatrix} \tag{2.34}$$

so that

$$\phi_A = \frac{1}{\sqrt{2\pi}} \cos(\theta)\cos(\varphi).$$

It follows that

$$Z_A = \left\{\theta = \frac{\pi}{2}\right\} \cup \left\{\theta = \frac{3\pi}{2}\right\} \cup \left\{\varphi = \frac{\pi}{2}\right\} \cup \left\{\varphi = \frac{3\pi}{2}\right\}. \tag{2.35}$$

In other words Z_A consists of the union of two horizontal and two vertical circles.

Suppose now that A is special. Again, upon composing with rotations, we may suppose that

$$A = \begin{pmatrix} \frac{1}{2\pi} & 0 \\ 0 & \frac{\epsilon}{2\pi} \end{pmatrix}$$

where $\epsilon = \mathrm{Sign}(\mathrm{Det}(A))$, so that

$$\phi_A = \frac{1}{2\pi}\cos(\theta)\cos(\varphi) + \frac{\epsilon}{2\pi}\sin(\theta)\sin(\varphi) = \frac{1}{2\pi}\cos(\theta - \epsilon\varphi)$$

It follows that

$$Z_A = \left\{\theta + \epsilon\varphi \in \pi\mathbb{Z} + \frac{\pi}{2}\right\}. \tag{2.36}$$

In other words Z_A consists of the union of two diagonal circles with constant gradient equal to ϵ. Furthermore, the sign of ϵ is equal to that of $\mathrm{Det}(A)$.

Finally, suppose that A is neither singular nor special, that its $A \in \Sigma \setminus (\Sigma_0 \cup \Sigma_{\frac{\pi}{4}})$. We first show that if $(\theta, \varphi) \in Z_A$, then $\partial_\theta \phi_A(\theta, \varphi) \neq 0$. Indeed, otherwise, by lemma 7, the vector $A(\cos(\varphi), \sin(\varphi))^t$ is orthogonal to both $(\cos(\theta), \sin(\theta))^t$ and $J(\cos(\theta), \sin(\theta))^t$, and therefore vanishes. This is absurd, since A is non-singular. Likewise, $\partial_\varphi \phi_A(\theta, \varphi) \neq 0$. It follows that Z_A is a union of smoothly embedded curves whose tangent lines are never horizontal nor vertical. Upon composing with rotations, we may now suppose that

$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

where both a and d are non-zero. It follows that

$$\phi_A(\theta, \varphi) = a\cos(\theta)\cos(\varphi) + b\sin(\theta)\cos(\varphi) + d\sin(\theta)\sin(\varphi).$$

Thus

$$\begin{aligned} \phi_A(\theta, \varphi) = 0 &\Leftrightarrow a\cos\theta\cos\varphi + b\sin\theta\cos\varphi + b\sin\theta\sin\varphi = 0 \\ &\Leftrightarrow a\cos\theta\cos\varphi = b\sin\theta\cos\varphi + d\sin\theta\sin\varphi \\ &\Leftrightarrow a\cot\theta = b + d\tan\varphi \\ &\Leftrightarrow a\tan(\theta - \frac{\pi}{2}) = b + d\tan(\varphi) \end{aligned} \tag{2.37}$$

Thus,we will consider a reparametrization

$$(\xi, \eta) := (\cot(\theta), \tan(\varphi)), \tag{2.38}$$

in this reparametrization we obtain

$$Z_A \cap \left( ]0, \pi[\times \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[\right) = \{(\xi, \eta) \mid a\eta + d\xi + b = 0\} \tag{2.39}$$

It follows by lemma 7 that, with the exception of 4 points Z_A coincides with the union of 4 translates of this set. In particular, Z_A is the union of two smooth, closed curves which, up to the transformation given by (2.38), are straight lines.

This representation can be improved slightly. Indeed, by lemma 7 together with standard formulae of curve theory (c.f. [3]), at any point (θ, φ) of Z_A, its geodesic curvature κ satisfies

$$\kappa = II^N \left(\frac{J\nabla\phi}{\|\nabla\phi\|}, \frac{J\nabla\phi}{\|\nabla\phi\|} \right)$$

thus

$$\begin{aligned} \|\nabla\phi\|^3 \kappa &= \|\phi\| II^N(J\nabla\phi, J\nabla\phi) \\ &= \mathrm{Hess}(\phi)(J\nabla\phi, J\nabla\phi) \\ &= \mathrm{Hess}\left(\begin{pmatrix} -\phi_{AJ} \\ \phi_{JA} \end{pmatrix}, \begin{pmatrix} -\phi_{AJ} \\ \phi_{JA} \end{pmatrix} \right) \\ &= -\phi_A \left(\phi_{AJ}^2 + \phi_{JA}^2 \right) - 2\phi_{JAJ}\phi_{AJ}\phi_{JA} \end{aligned}$$

Thus, by lemma 7 and (2.32), because we are along the level set $\phi_A = 0$

$$\|\nabla\phi\|^3 \kappa = -2\,\mathrm{Det}(A)\phi_{JAJ}.$$

Note that, the curvature vanishes when ϕ_{JAJ} vanishes. It is then straightforward to show that, since A is non-special, ϕ_{JAJ} vanishes at 4 evenly spaced points along each of the two components of Z_A. Furthermore, any two such points along the same component of Z_A are separated by an integer multiple of $\left(\frac{\pi}{2}, \frac{\epsilon\pi}{2}\right)$, where $\epsilon = \pm 1$ is equal to the sign of $\mathrm{Det}(A)$. These points are the inflection points of Z_A, these inflection points are precisely the points where the curvature is zero. Upon composing with rotations, we may suppose that one such point inflection point lies at $\left(\frac{\pi}{2}, 0\right)$, so that

$$A = \frac{1}{\sqrt{2}\pi}\begin{pmatrix} \cos(r) & 0 \\ 0 & -\sin(r) \end{pmatrix}$$

so that

$$Z_A \cap \left(\left]0,\pi\right[\times\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right) = \{(x,y) \mid y = \cot(r)x\}$$

In particular, the gradient of Z_A at $\left(\frac{\pi}{2}, 0\right)$ is equal to $\cot(r)$. Finally it is straightforward to show that if (θ, φ) and (θ', φ') are two consecutive inflexion points along the same component of Z_A, then their gradients γ and γ' satisfy $\gamma\gamma' = 1$.

2.3.3.1 Geometries of $\Sigma_0, \Sigma_{\frac{\pi}{4}}$ and Σ

Differentiating (2.32) we see that (JA, AJ) constitutes a basis of the tangent space to Σ_0 at a the point A. Likewise, the unit normal vector at this point is $-JAJ$. In particular,

$$-D_{JA}JAJ = AJ \text{ and}$$
$$-D_{AJ}JAJ = JA$$

The second fundamental form of Σ_0 with respect to this basis is therefore

$$\mathrm{II} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so that Σ_0 is minimal. Furthermore, by Lemma 7, the group $\mathrm{SO}(2) \times \mathrm{SO}(2)$ acts freely and transitively over Σ_0, so that this surface is a torus. Indeed, Σ_0 is a Clifford torus (c.f. [3]).

Given $A \in \Sigma_0$, the unit speed geodesic in Σ leaving A in the normal direction to Σ_0 is

$$\gamma(r) = \cos(r)A - \sin(r)JAJ$$

When A takes the form (2.34), this becomes

$$\gamma(r) = \frac{1}{\sqrt{2}\pi}\begin{pmatrix} \cos(r) & 0 \\ 0 & -\sin(r) \end{pmatrix}$$

It follows that matrices B lying at a distance θ from Σ_0 along Σ are precisely those for which Z_B has gradients $\tan(\theta)$ and $\cot(\theta)$ at its inflection points. In particular, along the set of special matrices is precisely the set of matrices lying at a distance $\pi/4$ from Σ_0. In other words, $\Sigma_{\pi/4}$ consists of the two circles lying at the two extremes of the Fermi parametrisation of Σ about Σ_0. The evolution of $Z_{\gamma(r)}$ as r varies over the interval $[0, \pi]$ is shown in Figure (3).

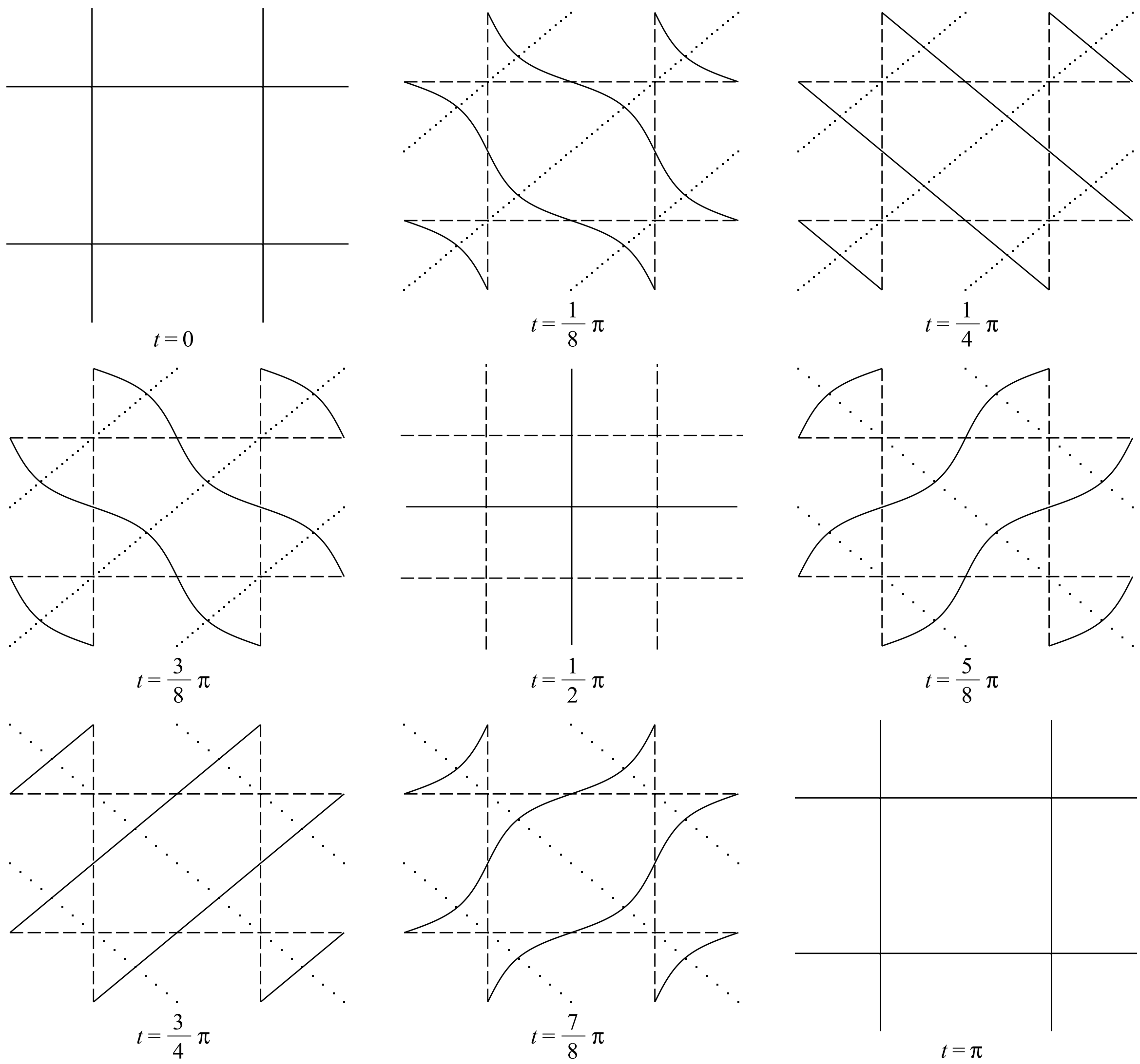

Figure 3 – The evolution of $Z_\gamma(r)$ as t varies over the interval $[0, \pi]$. The points of maximum curvature move along diagonal lines at non-constant speed.

2.4 The geometry of curves in CL

In this section, we derive an explicit formula for curves in CL which pass through $\mathbf{T}_0$ and which have prescribed first and second derivatives. Indeed, for $A, B \in \mathrm{End}\left(\mathbb{R}^2\right)$, define $\gamma : \mathbb{R} \to \mathrm{CL}$ such that

$$\begin{aligned}\gamma(0) &= \mathbf{T}_0\\ \dot\gamma(0) &= \phi_A \text{ and}\\ \left(\nabla_{\dot\gamma}\dot\gamma\right)(0) &= \phi_B\end{aligned}$$

where ∇ here denotes the Levi-Civita covariant derivative of CL. For sufficiently small ϵ, let $f :]-\epsilon, \epsilon[\to C^\infty\left(\mathbf{S}^1 \times \mathbf{S}^1,]-\frac{\pi}{4}, \frac{\pi}{4}[\right)$ be such that, for all t

$$e\left[f(t)\right] = \gamma(t).$$

Theorem 3. *For all* $A, B \in \mathrm{End}\left(\mathbb{R}^2\right)$ *and* f *defined as above, then*

$$f(t) = t\phi_A + \frac{1}{2}t^2\left(\phi_B - \left(\frac{\partial\phi_A}{\partial\theta}\right)^2 + \left(\frac{\partial\phi_A}{\partial\phi}\right)^2\right) + O\left(t^3\right).$$

In order to prove theorem 3, we first express f in terms of the flows of certain Killing fields. Indeed , by the classical theory of symmetric spaces, the exponential map of CL at $\mathbf{T}_0$ is

$$E : \mathrm{End}(\mathbb{R}^2) \longrightarrow \mathrm{CL}$$

such that $A \mapsto \mathrm{Exp}(\xi_A)\mathbf{T}_0$.
We may therefore suppose that

$$\gamma(t) := E\left(tA + \frac{t^2}{2}B\right)$$

As in the previous section, let X_A and X_B denote the vector fields defined over $\mathbf{S}^1 \times \mathbf{S}^1 \times]-\frac{\pi}{4}, \frac{\pi}{4}[$ by the pull-back through Φ of the infinitesimal flows defined over $\mathbf{S}^3$ by ξ_A and ξ_B respectively and let F_A and F_B denote their respective flows.

Recall that,for all (s, t),

$$\mathrm{Exp}(sA + tB) = \mathrm{Exp}(sA)\,\mathrm{Exp}(tB) + O(st)$$

so that, bearing in mind (2.25),

$$\begin{aligned}\gamma(t) &= \mathrm{Exp}\left(\frac{t^2}{2}B\right)\mathrm{Exp}(tA)\mathbf{T}_0 + O\left(t^3\right)\\ &= \left\{\left(\Phi \circ F_{B,\frac{t^2}{2}} \circ F_{A,t}\right)(\theta, \varphi, 0) + \mathrm{O}\left(t^3\right) \mid \theta, \varphi \in \mathbf{S}^1\right\}\end{aligned}$$

Now let $\alpha_{s,t}$ and $h_{s,t}$ be such that

$$F_{B,s}\left(F_{A,t}(\theta, \varphi, 0)\right) = \left(\alpha_{s,t}(\theta, \varphi), h_{s,t}(\theta, \varphi)\right).$$

Observe that $\alpha_{0,0} = \mathrm{Id}$, so that, for sufficiently small (s, t), $\alpha_{s,t}$ is a smooth diffeomorphism. For all such (s, t), let $\beta_{s,t}$ denote its inverse.

Lemma 8. *For all sufficiently small t*

$$f(t)(\theta,\varphi) = \left(h_{\frac{t2}{2},t} \circ \beta_{\frac{t2}{2},t}\right)(\theta,\varphi) + O\left(t^3\right)$$

Proof. Indeed, for all t

$$\begin{aligned}\gamma(t) &= \left\{\Phi\left(\alpha_{\frac{t2}{2},t}(\theta,\varphi), h_{\frac{t2}{2},t}(\theta,\varphi)\right) + O\left(t^3\right) \mid \theta,\varphi \in \mathbf{S}^1\right\} \\ &= \left\{\Phi\left(\theta,\varphi,\left(h_{\frac{t2}{2},t} \circ \beta_{\frac{t2}{2},t}\right)(\theta,\varphi)\right) + O\left(t^3\right) \mid \theta,\varphi \in \mathbf{S}^1\right\}\end{aligned}$$

However, by definition of $f(t)$ we have that

$$\gamma(t) := \left\{\Phi\left(\theta,\varphi,f(t)(\theta,\varphi)\right) + O\left(t^3\right) \mid \theta,\phi \in \mathbf{S}^1\right\}.$$

The result follows. ☐

Denote $\alpha_t := \alpha_{0,t}, h_t := h_{0,t}$ and $\beta_t := \beta_{0,t}$.

Lemma 9. *The first derivatives of h and β satisfy*

$$\begin{aligned}\left.\frac{\partial h_t}{\partial t}\right|_{t=0} &= \phi_A \text{ and} \\ \left.\frac{\partial \beta_t}{\partial t}\right|_{t=0} &= -\left(\frac{\partial \phi_A}{\partial \theta}\right)\partial_\theta + \left(\frac{\partial \phi_A}{\partial \varphi}\right)\partial_\varphi\end{aligned} \tag{2.40}$$

Proof. Indeed, by definition, for all t, and for all θ,φ

$$(\alpha_t(\theta,\phi), h_t(\theta,\phi)) = F_{A,t}(\theta,\phi,0)$$

Hence,

$$\left(\frac{\partial \alpha_t}{\partial t}, \frac{\partial h_t}{\partial_t}\right) = X_A\left(F_{A,t}(\cdot,0)\right) = X_A\left(\alpha_t, h_t\right)$$

since $\alpha_0 = \mathrm{Id}$ and $h_0 = 0$, this yields

$$\begin{aligned}\left.\frac{\partial h_t}{\partial t}\right|_{t=0} &= \phi_A \text{ and} \\ \left.\frac{\partial \alpha_t}{\partial t}\right|_{t=0} &= \left(\frac{\partial \phi_A}{\partial \theta}\right)\partial_\theta - \left(\frac{\partial \phi_A}{\partial \varphi}\right)\partial_\varphi\end{aligned}$$

By definition, for all t

$$\alpha_t \circ \beta_t = \mathrm{Id}$$

so that, by the chain rule,

$$\frac{\partial \alpha_t}{\partial t} \circ \beta_t + (\mathrm{D}\alpha_t \circ \beta_t)\frac{\partial \beta_t}{\partial t} = 0$$

since $\alpha_0 = \beta_0 = \mathrm{Id}$, this yields

$$\left.\frac{\partial \beta_t}{\partial t}\right|_{t=0} = -\left.\frac{\partial \alpha_t}{\partial t}\right|_{t=0} = -\left(\frac{\partial \phi_A}{\partial \theta}\right)\partial_\theta + \left(\frac{\partial \phi_A}{\partial \varphi}\right)\partial_\varphi$$

This completes the proof. ☐

Lemma 10. *The second derivative of h satisfies*

$$\left.\frac{\partial^2 h_t}{\partial t^2}\right|_{t=0} = \left(\frac{\partial \phi_A}{\partial \theta}\right)^2 - \left(\frac{\partial \phi_A}{\partial \varphi}\right)^2$$

Proof. As before

$$\left(\frac{\partial \alpha_t}{\partial t}, \frac{\partial h_t}{\partial t}\right) = X_A\left(\alpha_t, h_t\right)$$

It follows by (2.40) that

$$\frac{\partial h_t}{\partial t} = \phi_A \circ \alpha_t.$$

Differentiating a second time yields

$$\frac{\partial^2 h_t}{\partial t^2} = \left(\mathrm{D}\phi_A \circ \alpha_t\right) \frac{\partial \alpha_t}{\partial t}$$

so that, by Lemma 9 ,

$$\left.\frac{\partial^2 h_t}{\partial t^2}(x)\right|_{t=0} = \left(\frac{\partial \phi_A}{\partial \theta}\right)^2 - \left(\frac{\partial \phi_A}{\partial \varphi}\right)^2.$$

This completes the proof. □

We now prove Theorem 3

Proof of Theorem 3. Indeed, by Lemma 9

$$\left.\frac{\partial h_{s,t}}{\partial s}\right|_{s,t=0} = \phi_B \text{ and}$$
$$\left.\frac{\partial h_{s,t}}{\partial t}\right|_{s,t=0} = \phi_A.$$

Thus,

$$\begin{aligned}\frac{\partial f_t}{\partial t} =& t\left(\frac{\partial h}{\partial s}\right)_{\frac{t^2}{2},t} \circ \beta_{\frac{t^2}{2},t} + \left(\frac{\partial h}{\partial t}\right)_{\frac{t^2}{2},t} \circ \beta_{\frac{t^2}{2},t} \\ &+ \left(\mathrm{D}h_{\frac{t^2}{2},t} \circ \beta_{\frac{t^2}{2},t}\right)\left(t\left(\frac{\partial \beta}{\partial s}\right)_{\frac{t^2}{2},t} + \left(\frac{\partial \beta}{\partial t}\right)_{\frac{t^2}{2},t}\right) + O\left(t^2\right)\end{aligned}$$

since $h_{0,0} = 0$, and $\beta_{0,0} = 0$ evaluating at $t = 0$ yields

$$\left.\frac{\partial f_t}{\partial t}\right|_{t=0} = \left.\left(\frac{\partial h}{\partial t} h\right)_{s,t}\right|_{s,t=0} = \phi_A.$$

Differentiating a second time and evaluating at zero yields

$$\begin{aligned}\left.\frac{\partial^2 f_t}{\partial t^2}\right|_{t=0} =& \left.\left(\frac{\partial h}{\partial s}\right)_{s,t}\right|_{s,t=0} + \left.\left(\frac{\partial^2 h}{\partial t^2}\right)_{s,t}\right|_{s,t=0} \\ &+ 2\mathrm{D}\left.\left(\frac{\partial h}{\partial t}\right)_{s,t}\right|_{s,t=0}\left.\left(\frac{\partial \beta}{\partial t}\right)_{s,t}\right|_{s,t=0} \\ &+ \mathrm{D}^2 h_{s,t}\big|_{s,t=0}\left(\left.\left(\frac{\partial \beta}{\partial t}\right)_{s,t}\right|_{s,t=0}, \left.\left(\frac{\partial \beta}{\partial t}\right)_{s,t}\right|_{s,t=0}\right) \\ &+ \mathrm{D}h_{s,t}\big|_{s,t=0}\left.\left(\frac{\partial^2 \beta}{\partial t^2}\right)_{s,t}\right|_{s,t=0}\end{aligned}$$

since $h_{0,0} = 0$, the last two terms vanish. Thus, by Lemmas 9 and 10

$$\left.\frac{\partial^2 f_t}{\partial t^2}\right|_{t=0} = \phi_B - \left(\frac{\partial \phi_A}{\partial \theta}\right)^2 + \left(\frac{\partial \phi_A}{\partial \varphi}\right)^2$$

It follows by Taylor's theorem that

$$f_t = t\phi_A + \frac{1}{2}t^2 \left(\phi_B - \left(\frac{\partial \phi_A}{\partial \theta}\right)^2 + \left(\frac{\partial \phi_A}{\partial \varphi}\right)^2\right) + O\left(t^3\right)$$

This completes the proof. □

3 Proof of Main Theorem

Although conceptually straightforward, the application of perturbation theory to the construction of eternal mean curvature flows leads to cumbersome notation. This is mainly because it is rare to have a canonical parametrisation of any given embedded surface, and even rarer to have a canonical family of parametrisations along a given family of surfaces. In this chapter, we address this using the standard parametrisations of Clifford tori. Although this makes our presentation specific to the case of Clifford tori in $\mathbf{S}^3$, the techniques we develop are more general.

3.1 White's function construction

Let $\mathcal{G}^{1,\alpha}\left(\mathbf{S}^3\right)$ denote the Banach manifold of $C^{1,\alpha}$ Riemannian metrics over $\mathbf{S}^3$. Let $g_1 \in \mathcal{G}^{1,\alpha}\left(\mathbf{S}^3\right)$ denote the standard metric of constant unit curvature. Define

$$\mathrm{H} : \mathrm{O}(4) \times C^{2,\alpha}\left(\mathbf{S}^1 \times \mathbf{S}^1, \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\right) \times \mathcal{G}^{1,\alpha}\left(\mathbf{S}^3\right) \rightarrow C^{0,\alpha}\left(\mathbf{S}^1 \times \mathbf{S}^1\right)$$

such that, for all $M \in \mathrm{O}(4)$ for all $f \in C^{2,\alpha}\left(\mathbf{S}^1 \times \mathbf{S}^1, \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\right)$, for all $g \in \mathcal{G}^{1,\alpha}\left(\mathbf{S}^3\right)$ and for all $(\theta, \varphi) \in \mathbf{S}^1 \times \mathbf{S}^1$,

$$\mathrm{H}[M, f, g](\theta, \varphi) : \text{is the mean curvature of the embedding } Me[f] \text{ with respect to the metric } g \text{ at the point } (\theta, \varphi).$$

Defining H in this manner, we have that H is a smooth function between Banach manifolds. As we have shown in section 1.2, its partial derivative with respect the second component satisfies, for all $M \in \mathrm{O}(4)$

$$\mathrm{D}_2\mathrm{H}\left[M, 0, g_1\right] f = -2(\Delta + 2)f$$

Let K denote the kernel of $\Delta + 2$ in $L^2\left(\mathbf{S}^1 \times \mathbf{S}^1\right)$. Recall that K consists of all functions ϕ of the form

$$\phi = \phi_A,$$

where, for all $A \in \mathrm{End}\left(\mathbb{R}^2\right)$, ϕ_A is defined as in (2.24). Let $\mathrm{K}^\perp$ denote the orthogonal complement of K in $L^2\left(\mathbf{S}^1 \times \mathbf{S}^1\right)$. Let $\pi : L^2\left(\mathbf{S}^1 \times \mathbf{S}^1\right) \rightarrow \mathrm{K}$ and $\pi^\perp : L^2\left(\mathbf{S}^1 \times \mathbf{S}^1\right) \rightarrow \mathrm{K}^\perp$ denote the L^2-orthogonal projections of $L^2\left(\mathbf{S}^1 \times \mathbf{S}^1\right)$ onto K and $\mathrm{K}^\perp$, respectively. For all (k, α), let $\mathrm{K}^{\perp,k,\alpha}$ denote the intersection of $\mathrm{K}^\perp$ with $C^{k,\alpha}\left(\mathbf{S}^1 \times \mathbf{S}^1\right)$. For all (k, α), since K is a finite-dimensional space of smooth functions, $\pi^\perp$ maps $C^{k,\alpha}\left(\mathbf{S}^1 \times \mathbf{S}^1\right)$ continuously into $\mathrm{K}^{\perp,k,\alpha}$.

Theorem 4. *There exist a neighborhood Ω of g_1 in $\mathcal{G}^{k,\alpha}$ and a unique smooth funtion $\tilde{f} : \mathrm{O}(4) \times \Omega \longrightarrow \mathrm{K}^{\perp,2,\alpha}$ such that*

$$\tilde{f}\left[M, g_1\right] = 0,$$

and for all $M \in \mathrm{O}(4)$and $g \in \Omega$

$$\left(\pi^\perp \circ \mathrm{H}\right)\left[M, \tilde{f}[M, g], g\right] = 0$$

Proof. Consider the restriction of $\pi^\perp \circ \mathrm{H}$ to $\mathrm{O}(4) \times \mathrm{K}^\perp \times \mathcal{G}^{1,\alpha}$. By definition of the Jacobi operator, for all $M \in \mathrm{O}(4)$

$$\left(\pi^\perp \circ \mathrm{D}_2\mathrm{H}\right)\left[M, 0, g_1\right] \cdot f = \left(\pi^\perp \circ \mathrm{J}\right) \cdot f$$

Recall that, over $\mathbf{T}_0$

$$\mathrm{J} = -2(\Delta + 2)$$

In particular

$$\pi^{\perp} \circ \mathrm{J} = -2(\Delta + 2) \tag{3.1}$$

that is, for all $M \in \mathrm{O}(4)$, $\left(\pi^{\perp} \circ \mathrm{D}_2\mathrm{H}\right)[M, 0, g_1]$ defines a linear isomorphism from $\mathrm{K}^{\perp,2,\alpha}$ into $\mathrm{K}^{\perp,0,\alpha}$. It follows by the inverse function theorem that there exists a neighbourhood Ω of g_1 in $\mathcal{G}^{k,\alpha}$ having the property that there exists unique smooth function $\tilde{f} : \mathrm{O}(4) \times \Omega \to \mathrm{K}^{\perp,2,\alpha}$ such that, for all $M \in \mathrm{O}(4)$

$$\tilde{f}\,[M, g_1] = 0$$

and, for all $M \in \mathrm{O}(4)$ and for all $g \in \Omega$

$$\left(\pi^{\perp} \circ \mathrm{H}\right)[M, \tilde{f}[M, g], g] = 0.$$

□

Remark 1. *The function $\tilde{f}$ of the previous theorem will be called White's function.*

Define $\tilde{\mathrm{a}} : \mathrm{O}(4) \times \Omega \to \mathbb{R}$ such that, for all $M \in \mathrm{O}(4)$ and for all $g \in \Omega$,

$$\tilde{\mathrm{a}}[M, g] := \mathrm{Area}[Me[\tilde{f}[M, g]]], \tag{3.2}$$

that is, $\tilde{\mathrm{a}}[M, g]$ is the area of the embedding $Me[\tilde{f}[M, g]]$ with respect to the metric g.

Define $\tilde{\mathrm{h}} : \mathrm{O}(4) \times \Omega \to \mathrm{K}$ by

$$\tilde{\mathrm{h}}[M, g] := (\pi \circ \mathrm{H})\,[M, \tilde{f}[M, g], g] \tag{3.3}$$

By uniqueness, the functions $\tilde{f}$, $\tilde{\mathrm{h}}$ and $\tilde{a}$ are invariant under certain transformations. First, since $\mathrm{O}(4)$ is compact, we may suppose that Ω is invariant under the action of this group by pull-back on $\mathcal{G}^{1,\alpha}(\mathbf{S}^3)$. Now, for all $M, N \in \mathrm{O}(4)$, for all $f \in C^{2,\alpha}\left(\mathbf{S}^1 \times \mathbf{S}^1, \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\right)$ and for all $g \in \Omega$

$$\tilde{\mathrm{h}}\,[M, f, N^*g] = \tilde{\mathrm{h}}[NM, f, g]$$

so that, by uniqueness, for all $M, N \in \mathrm{O}(4)$ and for all $g \in \Omega$,

$$\begin{aligned} f\,[M, N^*g] =& f[NM, g], \\ \tilde{\mathrm{h}}\,[M, N^*g] =& \tilde{\mathrm{h}}[NM, g] \quad \text{and} \\ \tilde{\mathrm{a}}\,[M, N^*g] =& \tilde{\mathrm{a}}[NM, g]. \end{aligned} \tag{3.4}$$

Likewise, for all $M \in \mathrm{O}(4)$, for all $N \in \mathrm{O}(2)^2$, for all $f \in C^{2,\alpha}\left(\mathbf{S}^1 \times \mathbf{S}^1, \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\right)$ and for all $g \in \Omega$

$$\tilde{\mathrm{h}}[MN, f \circ N, g] = \tilde{\mathrm{h}}[M, f, g] \circ N$$

so that, by uniqueness again, for all $M \in \mathrm{O}(4)$, for all $N \in \mathrm{O}(2)^2$ and for all $g \in \Omega$

$$\begin{aligned} f[MN, g] &= f[M, g] \circ N \\ \tilde{\mathrm{h}}[MN, g] &= \tilde{\mathrm{h}}[M, g] \circ N \quad \text{and} \\ \tilde{\mathrm{a}}[MN, g] &= \tilde{\mathrm{a}}[M, g]. \end{aligned} \tag{3.5}$$

Recall that the functional $\mathrm{I} : C^\infty(\mathbf{S}^3) \to C^\infty(\mathrm{CL})$ is defined by

$$\mathrm{I}[u](T) := \int_T u \mathrm{dA}_T \tag{3.6}$$

where dA_T here denotes the area form of T.

Lemma 11. *If $h = 2ug_1$ for some $u \in C^\infty(\mathbf{S}^3)$ is a first order variation of a metric, then, for all $M \in \mathrm{O}(4)$*

$$\mathrm{D}_2\tilde{\mathrm{a}}\,[M, g_1]\,h = (\mathrm{I}[u] \circ C)(M). \tag{3.7}$$

Proof. By the left invariance given in (3.4) we may suppose that $M = \mathrm{Id}$. Let $(g_{1+s})_{s\in]-\epsilon,\epsilon[}$ be a smooth family of metrics about g_1 such that

$$\left.\frac{\partial g_{1+s}}{\partial s}\right|_{s=0} = h = 2ug_1.$$

For sufficiently small s, denote $e_s(\theta,\varphi) := \left(\theta, \varphi, \tilde{f}\,[\mathrm{Id}, g_{1+s}]\,(\theta,\varphi)\right)$. Note that we can represent every torus, close to the initial torus $\mathbf{T}_0$ as a graph over a torus. For sufficiently small s and t, let $a_{s,t}$ denote the area of the embedding e_s with respect to the metric $\Phi^* g_{1+t}$.

Note that, when $t = 0$, since $\mathbf{T}_0$ is minimal, the area form of e_s with respect to $\Phi^* g_1$ satisfies

$$\mathrm{d}A_{s,0} := \frac{1}{2} d\theta d\varphi + O\left(s^2\right).$$

Hence

$$a_{s,0} = \frac{1}{2}\int_{S^1\times S^1} d\theta d\varphi + O\left(s^2\right) = 2\pi^2 + O\left(s^2\right),$$

so that

$$\left.\frac{\partial a_{s,t}}{\partial s}\right|_{s,t=0} = 0.$$

On the other hand, when $s = 0$, since h is a first order variation of a metric, the area form of e_0 with respect to $\Phi^* g_{1+t}$ is

$$\mathrm{d}A_{0,t} := \frac{1}{2}\left(1 + 2ut\right) d\theta d\varphi + O(t^2).$$

Hence,

$$\begin{aligned} a_{0,t} &= \frac{1}{2}\int_{S^1\times S^1} \left(1 + 2ust\right) d\theta d\varphi + O(t^2) \\ &= 2\pi^2 + t\mathrm{I}[u] + O(t^2) \end{aligned}$$

so that

$$\left.\frac{\partial a_{s,t}}{\partial t}\right|_{s,t=0} = \mathrm{I}[u].$$

It follows by the chain rule that

$$\mathrm{D}_2\tilde{\mathrm{a}}\,[\mathrm{Id}, g_1]\,h = \left.\frac{\partial a_{s,s}}{\partial s}\right|_{s=0} = \left.\frac{\partial a_{s,t}}{\partial s}\right|_{s,t=0} + \left.\frac{\partial a_{s,t}}{\partial t}\right|_{s,t=0} = \mathrm{I}[u]\left(\mathbf{T}_0\right).$$

This completes the proof.

□

Lemma 12. *For all $M \in \mathrm{O}(4)$, for all $g \in \Omega$ and for all $A \in \mathrm{End}(\mathbb{R}^2)$*

$$\mathrm{D}_1\tilde{\mathrm{a}}[\mathrm{Id}, g]\xi_A = \frac{1}{2}\int_{\mathbf{S}^1\times\mathbf{S}^1} \phi_A \tilde{\mathrm{h}}[\mathrm{Id}, g] d\theta d\varphi + O\left(\|A\| \, \|g - g_1\|^2_{C^{1,\alpha}}\right) \tag{3.8}$$

Remark 2. *By the right-invariance given in (3.5), $\mathrm{D}_1\tilde{\mathrm{a}}[M, g]$ vanishes over $\left(\tau^L\mathfrak{h}\right)(M)$ so that (3.8) completely describes $\mathrm{D}_1\tilde{\mathrm{a}}[M, g]$.*

Remark 3. *By the left-invariance given in (3.4), for all $M \in \mathrm{O}(4)$, for all $g \in \Omega$ and for all $A \in \mathrm{End}(\mathbb{R}^2)$,*

$$\mathrm{D}_1\tilde{\mathrm{a}}[M, g]\left(\tau^L\xi_A\right)(M) = \frac{1}{2}\int_{\mathbf{S}^1\times\mathbf{S}^1} \phi_A \tilde{\mathrm{h}}[M, g] d\theta d\varphi + O\left(\|A\| \, \|g - g_1\|^2_{C^{1,\alpha}}\right)$$

Proof. First observe that, since $\tilde{\mathrm{h}}[\mathrm{Id}, g_1] = 0$,

$$\tilde{\mathrm{h}}[\mathrm{Id}, g] = O\left(\|g - g_1\|_{C^{1,\alpha}}\right) \tag{3.9}$$

Now, choose $A \in \mathrm{End}\left(\mathbb{R}^2\right)$. For all sufficiently small s and t, let $a_{s,t}$ denote the area of the embedding $\mathrm{Exp}\left(s\xi_A\right) e\left[\tilde{f}\left[\mathrm{Exp}\left(t\xi_A\right), g\right]\right]$ with respect to the metric g. Let X_A denote the pull-back through Φ of the Killing field generated over S^3 by ξ_A and let F_A denote its flow. For all sufficiently small s and t, denote

$$e_{s,t}(\theta, \varphi) := F_{A,s}\left(\theta, \varphi, \tilde{f}\left[\mathrm{Exp}\left(t\xi_A\right), g\right]\right)$$

then

$$\begin{aligned}\left(\Phi \circ e_{s,t}\right)(\theta, \varphi) &= \left(\Phi \circ F_{A,s}\right)\left(\theta, \varphi, \tilde{f}\left[\mathrm{Exp}\left(t\xi_A\right), g\right]\right)\\ &= \mathrm{Exp}\left(s\xi_A\right) e\left[\tilde{f}\left[\mathrm{Exp}\left(t\xi_A\right), g\right]\right](\theta, \varphi),\end{aligned}$$

it follows that $a_{s,t}$ is equal to the area of $e_{s,t}$ with respect to the metric Φ^*g. However, by definition of F_A

$$\left.\frac{\partial e_{s,t}}{\partial s}(\theta, \varphi)\right|_{s,t=0} = X_A(\theta, \varphi, \tilde{f}[\mathrm{Id}, g]).$$

The unit normal vector field over $e_{0,0}$ with respect to Φ^*g is

$$N := (0, 0, 1) + O\left(\|g - g_1\|_{C^{1,\alpha}}\right).$$

Furthermore,

$$\tilde{f}[\mathrm{Id}, g] = O\left(\|g - g_1\|_{C^{1,\alpha}}\right).$$

It follows (2.27) that

$$\left(\Phi^*g\right)\left(\left.\frac{\partial e_{s,t}}{\partial s}(\theta, \varphi)\right|_{s,t=0}, N(\theta, \varphi)\right) = \phi_A(\theta, \varphi) + O\left(\|A\| \, \|g - g_1\|_{C^{1,\alpha}}\right).$$

since the area form of $e_{0,0}$ with respect to Φ^*g is

$$\mathrm{d}A = \frac{1}{2} d\theta d\varphi + O\left(\|g - g_1\|_{C^{1,\alpha}}\right)$$

It follows by (3.9) and the first variation formula for area [24], that

$$\left.\frac{\partial a_{s,t}}{\partial s}\right|_{s,t=0} = \frac{1}{2}\int_{\mathbf{S}^1\times\mathbf{S}^1} \phi_A(\theta, \varphi)\tilde{\mathrm{h}}[\mathrm{Id}, g]\left(1 + O\left(\|A\| \, \|g - g_1\|^2_{C^{1,\alpha}}\right)\right) d\theta d\varphi$$

On the other hand, for all t

$$\tilde{f}\left[\operatorname{Exp}\left(t\xi_A\right), g_1\right] = 0,$$

so that

$$\frac{\partial}{\partial t}\tilde{f}\left[\operatorname{Exp}\left(t\xi_A\right), g\right] = O\left(\|A\| \left\|g - g_1\right\|_{C^{1,\alpha}}\right).$$

Thus, by (3.9) and the first variation formula for area again,

$$\begin{aligned}\left.\frac{\partial a_{s,t}}{\partial t}\right|_{s,t=0} &= \frac{1}{2}\int_{\mathbf{S}^1\times\mathbf{S}^1} \tilde{h}[\mathrm{Id}, g]\, O\left(\|A\| \left\|g - g_1\right\|_{C^{1,\alpha}}\right) d\theta d\varphi \\ &= O\left(\|A\| \left\|g - g_1\right\|^2_{C^{1,\alpha}}\right).\end{aligned}$$

It follows by the chain rule that

$$\begin{aligned}\mathrm{D}_1\tilde{\mathrm{a}}\left[\mathbf{T}_0, g\right]\xi_A &= \left.\frac{\partial a_{s,t}}{\partial s}\right|_{s,t=0} + \left.\frac{\partial a_{s,t}}{\partial t}\right|_{s,t=0} \\ &= \frac{1}{2}\int_{\mathbf{S}^1\times\mathbf{S}^1} \tilde{\mathrm{h}}[\mathrm{Id}, g]\left(\phi_A(\theta,\varphi) + O\left(\left\|g - g_1\right\|_{C^{1,\alpha}} \|A\|\right)\right) d\theta d\varphi\end{aligned}$$

Finally, since $\mathbf{T}_0$ is minimal, by (3.9) yields the desired result.
This completes the proof.

∎

Lemma 13. *For all $T \in \mathrm{CL}$, for all $g \in \Omega$ and for all $A, B \in \mathrm{End}\left(\mathbb{R}^2\right)$*

$$\operatorname{Hess}(\tilde{\mathrm{a}}[M, g])\left(\xi_A, \xi_B\right) = \frac{1}{2}\int_{\mathbf{S}^1\times\mathbf{S}^1} \phi_A \mathrm{D}_1\tilde{\mathrm{h}}[M, g]\left(\xi_B\right) d\theta d\varphi + O\left(\|A\|\|B\| \left\|g - g_1\right\|^2_{C^{1,\alpha}}\right)$$

Remark 4. *By the left-invariance given in (3.4), for all $M \in \mathrm{O}(4)$, for all $T \in \mathrm{CL}$, for all $g \in \Omega$, and for all $A, B \in \mathrm{End}(\mathbb{R}^2)$*

$$\begin{aligned}&\operatorname{Hess}(\tilde{\mathrm{a}}[M, g])\left(\left(\tau^L\xi_A\right)(M), \left(\tau^L\xi_B\right)(M)\right) \\ &\quad = \frac{1}{2}\int_{\mathbf{S}^1\times\mathbf{S}^1} \phi_A \mathrm{D}_1\tilde{\mathrm{h}}[M, g]\left(\tau^L\xi_B\right)(M) d\theta d\varphi + O\left(\|A\|\|B\| \left\|g - g_1\right\|^2_{C^{1,\alpha}}\right)\end{aligned}$$

We have chosen to state Lemma 13 in the above manner for ease of readability.

Proof. Choose $A, B \in \mathrm{End}(\mathbb{R}^2)$. Recall that

$$\operatorname{Exp}\left(s\xi_A + t\xi_B\right) = \operatorname{Exp}\left(t\xi_B\right)\operatorname{Exp}\left(s\xi_A\right)\operatorname{Exp}\left(\frac{1}{2}st\left[\xi_A, \xi_B\right]\right) + O\left(s^2 + t^2\right)$$

so that, since $\left[\xi_A, \xi_B\right] \in \mathfrak{h}$

$$\operatorname{Exp}\left(s\xi_A + t\xi_B\right) T_0 = \operatorname{Exp}\left(t\xi_B\right)\operatorname{Exp}\left(s\xi_A\right) T_0 + O\left(s^2 + t^2\right)$$

Thus, bearing in mind Lemma 12 and by the right-invariance given by (3.5), we have that

$$
\begin{aligned}
\mathrm{Hess}(\tilde{\mathrm{a}}[\mathrm{Id}, g])\,(\xi_A, \xi_B) &= \frac{\partial^2}{\partial s \partial t}\tilde{\mathrm{a}}\left[\mathcal{E}\left(s\xi_A + t\xi_B\right), g\right]\Big|_{s,t=0} \\
&= \frac{\partial^2}{\partial s \partial t}\tilde{\mathrm{a}}\left[\mathrm{Exp}\left(t\xi_B\right)\mathrm{Exp}\left(s\xi_A\right), g\right]\Big|_{s,t=0} \\
&= \frac{\partial^2}{\partial s \partial t}\tilde{\mathrm{a}}\left[\mathrm{Exp}\left(s\xi_A\right), \mathrm{Exp}\left(t\xi_B\right)^* g\right]\Big|_{s,t=0} \\
&= \frac{\partial}{\partial t}\frac{1}{2}\int_{\mathbf{S}^1\times\mathbf{S}^1}\phi_A(\theta,\varphi)\tilde{\mathrm{h}}\left[\mathrm{Id}, \mathrm{Exp}\left(t\xi_B\right)^* g\right] d\theta d\varphi\Big|_{t=0} + O\left(\|A\|\|B\|\,\|g-g_1\|^2_{C_{1,\alpha}}\right) \\
&= \frac{1}{2}\int_{\mathbf{S}^1\times\mathbf{S}^1}\phi_A(\theta,\varphi)\frac{\partial}{\partial t}\tilde{\mathrm{h}}\left[\mathrm{Exp}\left(t\xi_B\right), g\right]\Big|_{t=0} d\theta d\varphi + O\left(\|A\|\|B\|\,\|g-g_1\|^2_{C^{1,\alpha}}\right) \\
&= \frac{1}{2}\int_{\mathbf{S}^1\times\mathbf{S}^1}\phi_A(\theta,\varphi)\mathrm{D}_1\tilde{\mathrm{h}}[\mathrm{Id}, g]\xi_B d\theta d\varphi + \mathrm{O}\left(\|A\|\|B\|\,\|g-g_1\|^2_{C^{1,\alpha}}\right)
\end{aligned}
$$

The relation follows, and this completes the proof. □

3.2 The mean curvature flow operator

We now extend our framework to the time-dependent case which interests us. Thus, fix $u \in C^\infty(\mathbf{S}^3)$ and suppose that $\mathrm{I}[u]$ is of Morse Smale type. Let $\gamma : \mathbb{R} \longrightarrow \mathrm{CL}$ be a gradient complete flow of $\mathrm{I}[u]$, that is

$$\dot{\gamma}(t) = -\nabla \mathrm{I}[u](\gamma(t))$$

for all $t \in \mathbb{R}$. We have that for all t, $\gamma(t)$ represents a Clifford torus. Let $(g_{1+s})_{s\in]-\epsilon,+\epsilon[}$ be a smooth family of metrics in $\mathcal{G}^{1,\alpha}(\mathbf{S}^3)$ metrics that perturb g_1 such that

$$\frac{\partial g_{1+s}}{\partial s}\Big|_{s=0} = 2ug_1$$

Let Ω be as in the previous section. By invariance, $\tilde{\mathrm{a}}$ projects to a smooth function $\mathrm{a} : \mathrm{CL}\times\Omega \to \mathbb{R}$. For all s, define $a_{1+s} : \mathrm{CL} \to \mathbb{R}$ by

$$a_{1+s}[T] := \mathrm{a}\left[T, g_{1+s}\right]$$

Now, we are going to change the speed with which we walk along γ, that is, for all s, we consider $\gamma(st)$ as an approximate gradient flow of a_{1+s}. Indeed, by

$$a_{1+s}[T] = 2\pi^2 + s\mathrm{I}[u](T) + O\left(s^2\right)$$

the velocity of $\gamma(st)$ is at every point equal to minus the gradient of a_{1+s} up to an error of order 2 in s. In this section, we will show that, for sufficiently small s, $\gamma(st)$ perturbs in $\mathcal{E}$ to a complete gradient flow of the area functional of g_{1+s}, that is, an eternal mean curvature flow.

Recall that, γ can be represented as a curve within the Lie group $\mathrm{O}(4)$, that is, we may suppose that $\gamma(0) = \mathbf{T}_0$, let $M : \mathbb{R} \to \mathrm{O}(4)$ be the unique lift of γ such that $M(0) = \mathrm{Id}$, $\gamma(t) = M(t)\mathbf{T}_0$, and for all t, $M(t)^{-1}\dot{M}(t) \in \mathfrak{k}$. We verify that such a lift exist. Let $X := \nabla \mathrm{I}[u]$ be a vector field along of CL, recall that the map $\pi : \mathrm{O}(4) \longrightarrow \mathrm{CL}$ such that $C(M) = M\mathbf{T}_0$. Note that, for $M \in \mathrm{O}(4)$ we know that

$$T_M\mathrm{O}(4) = \tau^L\mathfrak{h} \oplus \tau^L\mathfrak{k}$$

For the other hand, we also know that $DC_M|_{\tau^L \mathfrak{k}}$ is a isomorphism, then there is $\tilde{X}(M) \in \tau^L \mathfrak{k}$ such that

$$DC_M \cdot \tilde{X}(M) = X(C(M)),$$

this defines a vector field $\tilde{X} : \mathrm{O}(4) \longrightarrow T\mathrm{O}(4)$. we can suppose applying isometries, that $\gamma(0) = \mathbf{T}_0$. Let M be a integral curve of $\tilde{X}$ such that $M_0 = \mathrm{Id}$. Then

$$\begin{aligned}\frac{\partial}{\partial t}(C \circ M) &= DC(M(t))\frac{\partial}{\partial t}M(t)\\ &= DC(M(t))\tilde{X}(M(t))\\ &= X(C(M(t)))\\ &= X((C \circ M)(t))\end{aligned}$$

Therefore $C \circ M$ is a integral curve of X, so by uniqueness $C \circ M = \gamma$. Note that γ leaves $\mathbf{T}_0$ and is also an integral curve. Thus we have a curve M which is a curve within the group.

3.2.0.1 Parabolic Operators

We first aim to determine formal solutions of the equation $\mathrm{MFC}(\eta, f, s) = 0$ for small values of s. To this end, we introduce the following functional analytic framework. For a finite-dimensional vector space E, and for $\alpha \in]0,1]$, define the Hölder seminorm of order α over $C^0(\mathbb{R}, E)$ by

$$[f]_\alpha := \sup_{0<|s-t|\leq 1} \frac{\|f(s) - f(t)\|}{|s-t|^\alpha} \tag{3.10}$$

For all k and for all $\alpha \in]0,1]$, define the *Hölder norm* of order (k, α) over $C^k(\mathbb{R}, E)$ by

$$\|f\|_{k,\alpha} := \sum_{i=0}^{k} \left\|\partial_t^i f\right\|_0 + \left[\partial_t^k f\right]_\alpha, \tag{3.11}$$

where $\|\cdot\|_0$ denotes the uniform norm. For all (k, α), define the *Hölder space* of order (k, α) by

$$C^{k,\alpha}(\mathbb{R}, E) := \left\{f \in C^k(\mathbb{R}, E) \mid \|f\|_{k,\alpha} < \infty\right\}.$$

Recall that $C^{k,\alpha}$ furnished with the norm $\|\cdot\|_{k,\alpha}$ constitutes a Banach space.

In the case where E is a vector space of infinite dimension, for all $\alpha \in]0,1]$ define the *Hölder seminorms* of order α over $C^0\left(\mathbb{R} \times \mathbf{S}^1 \times \mathbf{S}^1\right)$ by

$$\begin{aligned}[f]_{x,\alpha} &:= \sup_{t, x \neq y} \frac{|f(t,x) - f(t,y)|}{\|x-y\|^\alpha}\\ [f]_{t,\alpha} &:= \sup_{x, 0 \leq |t-s| \leq 1} \frac{|f(s,x) - f(t,x)|}{|s-t|^\alpha}\end{aligned} \tag{3.12}$$

For all $k \in \mathbb{N}$, let $C^k_{\mathrm{in}}\left(\mathbb{R} \times \mathbf{S}^1 \times \mathbf{S}^1, \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\right)$ be the set of all functions $f : \mathbb{R} \times \mathbf{S}^1 \times \mathbf{S}^1 \to \mathbb{R}$ which are continuously differentiable i times in the x direction and j times in the t direction for all $i + 2j \leq 2k$. For all $k \in \mathbb{N}$ and for all $\alpha \in]0, 1/2]$, define the anisotropic Hölder norm of order (k, α) over $C^k_{\mathrm{in}}\left(\mathbb{R} \times \mathbf{S}^1 \times \mathbf{S}^1, \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\right)$ by

$$\|f\|_{k,\alpha,\,\mathrm{in}} := \sum_{i+2j \leq 2k} \left\|D_x^i D_t^j f\right\|_0 + \sum_{i+2j=2k} \left[D_x^i D_t^j f\right]_{x,2\alpha} + \sum_{i+2j=2k} \left[D_x^i D_t^j f\right]_{t,\alpha} \tag{3.13}$$

For all k, α, define the anisotropic Hölder space of order (k, α) by

$$C^k_{\mathrm{in}}\left(\mathbb{R} \times \mathbf{S}^1 \times \mathbf{S}^1, \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\right) := \left\{f \in C^k_{\mathrm{in}}\left(\mathbb{R} \times \mathbf{S}^1 \times \mathbf{S}^1, \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\right) \mid \|f\|_{k,\alpha,\mathrm{in}} < \infty\right\}. \tag{3.14}$$

Now, for $\eta \in C^{1,\alpha}(\mathbb{R}, \mathfrak{k})$, for $f \in C^{1,\alpha}_{\text{in}}\left(\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbb{R}, \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\right)$ and for $s \in]-\epsilon, \epsilon[$, define $\mathrm{e}[\eta, f, s]$ such that, for all $(\theta, \varphi, t) \in \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbb{R}$

$$\mathrm{e}[\eta, f, s](\theta, \varphi, t) := \mathrm{Exp}(\eta(st))M(st)\Phi\left(\theta, \varphi, \tilde{f}\left[\mathrm{Exp}(\eta(st))M(st), g_{1+s}\right](\theta, \varphi) + sf(\theta, \varphi, t)\right).$$

For all such η, f and s, define $\mathrm{N}[\eta, f, s]$ such that, for all $(\theta, \varphi, t) \in \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbb{R}$, $N[\eta, f, s](\theta, \varphi, t)$ is the unit normal vector over the embedding $e[\eta, f, s](\cdot, \cdot, t)$ at the point (θ, φ).
Likewise, for all such η, f and s, define $\mathrm{H}[\eta, f, s]$ such that, for all $(\theta, \varphi, t) \in \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbb{R}$, $\mathrm{H}[\eta, f, s](\theta, \varphi, t)$ is the mean curvature of the embedding $e[\eta, f, s](\cdot, \cdot, t)$ with respect to the metric g_{1+s} at the point (θ, φ). The mean curvature flow operator

$$\mathrm{MCF} : C^{1,\alpha}(\mathbb{R}, \mathfrak{k}) \times C^{1,\alpha}_{\text{in}}\left(\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbb{R}, \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\right) \times]-\epsilon, \epsilon[\to C^{0,\alpha}_{\text{in}}\left(\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbb{R}\right)$$

is defined by

$$\mathrm{MCF}[\eta, f, s] := g_{1+s}\left(\frac{\partial}{\partial t}\mathrm{e}[\eta, f, s], \mathrm{N}[\eta, f, s]\right) + \mathrm{H}[\eta, f, s] \tag{3.15}$$

This operator defines a smooth function between Banach spaces. The function $\mathrm{MCF}[\eta, f, s]$ vanishes if and only of $\mathrm{e}[\eta, f, s]$ is an eternal mean curvature flow with respect to the metric g_{1+s}.

Define $\Psi : \mathfrak{k} \to K$ such that, for all $A \in \mathrm{End}(\mathbb{R}^2)$,

$$\Psi(\xi_A) = \phi_A.$$

Lemma 14. *For $\eta \in C^{1,\alpha}(\mathbb{R}, \mathfrak{k})$, for $f \in C^{1,\alpha}_{in}\left(\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbb{R}, \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\right)$ and for $s \in]-\epsilon, \epsilon[$, and for all (θ, φ, t)*

$$\begin{aligned} &g_{1+s}\Big(\frac{\partial}{\partial t}\mathrm{e}[\eta, f, s](\theta, \varphi, t), N[\eta, f, s](\theta, \varphi, t)\Big) \\ &\qquad = s\Psi\left(\dot{\eta}(t) + M^{-1}(t)\dot{M}(t)\right)(\theta, \varphi) + s\frac{\partial f}{\partial t}(\theta, \varphi, t) + O\left(s^2 + s\|\eta\|^2_{C^{1,\alpha}}\right) \end{aligned}$$

Proof. By time invariance of the mean curvature flow operator, we may suppose that $t = 0$. Let $A, B \in \mathrm{End}\left(\mathbb{R}^2\right)$ be such that

$$\begin{aligned} \dot{\eta}(0) &= \xi_A \quad \text{and} \\ \dot{M}(0) &= \xi_B \end{aligned}$$

Let X_A be the pull-backs through Φ of the Killing fields generated over $\mathbf{S}^3$ by ξ_A. Likewise, let X_B be the pull-backs through Φ of the Killing fields generated over $\mathbf{S}^3$ by ξ_B. Let F_A and F_B denote their respective flows. Define $\tilde{e}$ by

$$\tilde{e}(\theta, \varphi, t) = (F_{A,st} \circ F_{B,st})\left(\theta, \varphi, \tilde{f}\left[\mathrm{Exp}(\eta(st))M(st), g_{1+s}\right](\theta, \varphi) + sf(\theta, \varphi, t)\right)$$

First we show that

$$\mathrm{Exp}\left(st\xi_A\right) = \mathrm{Exp}\left(\eta(0)\right)^{-1}\mathrm{Exp}\left(\eta(st)\right) + O\left(s^2t^2 + \|\eta\|^2_{C^1}st\right) \tag{3.16}$$

Indeed, by Taylor's theorem and since η takes values in $\mathfrak{k}$, we have that

$$\eta(st) = \eta(0) + st\xi_A + O(s^2t^2)$$

then,

$$\begin{aligned}
\operatorname{Exp}(\eta(st)) &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\eta(0)+st\dot\eta(0)+O(s^2t^2)\right)^k\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}\eta(0)^{k-l}\left(st\dot\eta(0)+O(s^2t^2)\right)^l\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left[\eta(0)^k+\sum_{l=1}^{k}\binom{k}{l}\eta(0)^{k-l}\left(st\dot\eta(0)+O(s^2t^2)\right)^l\right]\\
&= \operatorname{Exp}(\eta(0))+st\dot\eta(0)+O(s^2t^2)+O\left(st|\eta(0)||\dot\eta(0)|\right)+O(s^2t^2)\\
&= \operatorname{Exp}(\eta(0))+st\dot\eta(0)+O\left(s^2t^2+\|\eta\|^2_{C^1}st\right)
\end{aligned}$$

Now, note that

$$\begin{aligned}
\operatorname{Exp}(\eta(0))\operatorname{Exp}(st\dot\eta(0)) &= \operatorname{Exp}(\eta(0))\left(\operatorname{Id}+st\dot\eta(0)+O\left(s^2t^2\right)\right)\\
&= \operatorname{Exp}(\eta(0))+\left(\operatorname{Id}+O\left(\|\eta\|^2_{C^1}\right)\right)st\dot\eta(0)+O\left(s^2t^2\right)\\
&= \operatorname{Exp}(\eta(0))+st\dot\eta(0)+O\left(s^2t^2+\|\eta\|^2_{C^1}st\right)
\end{aligned}$$

thus

$$\operatorname{Exp}(\eta(st))=\operatorname{Exp}(\eta(0))\operatorname{Exp}(st\dot\eta(0))+O\left(s^2t^2+\|\eta\|^2_{C^1}st\right)$$

so, we obtain the desired equality.
Then, by (3.16)

$$\begin{aligned}
(\Phi\circ\tilde e)(\theta,\varphi,t) &= \operatorname{Exp}\left(st\xi_A\right)\operatorname{Exp}\left(st\xi_B\right)\Phi\left(\theta,\varphi,\tilde f\left[\operatorname{Exp}(\eta(st))M(st),g_{1+s}\right](\theta,\varphi)+sf(\theta,\varphi,t)\right)\\
&= \operatorname{Exp}(\eta(0))^{-1}e[\eta,f,s](\theta,\varphi,t)+O\left(s^2t^2+st\|\eta\|^2_{C^{1,\alpha}}\right)
\end{aligned}$$

However,

$$\begin{aligned}
\frac{\partial\tilde e}{\partial t}(\theta,\varphi,t)\Big|_{t=0} &= sX_A\left(\theta,\varphi,\tilde f\left[\operatorname{Exp}(\eta(0)),g_{1+s}\right](\theta,\varphi)+sf(\theta,\varphi,0)\right)\\
&\quad+sX_B\left(\theta,\varphi,\tilde f\left[\operatorname{Exp}(\eta(0)),g_{1+s}\right](\theta,\varphi)+sf(\theta,\varphi,0)\right)\\
&\quad+\left(0,0,s\frac{\partial}{\partial t}\tilde f\left[\operatorname{Exp}(\eta(t))M(t),g_{1+s}\right](\theta,\varphi)\Big|_{t=0}\right)\\
&\quad+\left(0,0,s\frac{\partial f}{\partial t}(\theta,\varphi,0)\right)+O\left(s\|\eta\|^2_{C^{1,\alpha}}\right)
\end{aligned}$$

Recall that, for all M

$$\tilde f[M,g_1]=0$$

it follows that

$$\tilde f\left[\operatorname{Exp}(\eta(t))M(t),g_{1+s}\right]=O(s).$$

It follows by Lemma 6 that

$$\frac{\partial\tilde e}{\partial t}(\theta,\varphi,t)\Big|_{t=0}=\left(O(s),O(s),s\phi_A(\theta,\varphi)+s\phi_B(\theta,\varphi)+s\frac{\partial f}{\partial t}(\theta,\varphi,0)+O\left(s^2+s\|\eta\|^2_{C^{1,\alpha}}\right)\right)$$

Observe that $\tilde e(\cdot,\cdot,0)$ is the graph of a function of size $O(s)$, thus the unit normal vector field over $\tilde e(\cdot,\cdot,0)$ with respect to g_{1+s} is

$$\tilde N(\theta,\varphi)=(0,0,1)+O(s)$$

It follows that

$$
\begin{aligned}
g_{1+s}&\left(\frac{\partial}{\partial t}\mathrm{e}[\eta,f,s](\theta,\varphi,t),\ \tilde{N}[\eta,f,s](\theta,\varphi,t)\right)\\
&= (\mathrm{Exp}(\eta(0))\circ\Phi)^{*}\, g_{1+s}\left(\frac{\partial\tilde{e}}{\partial t}(\theta,\varphi,t)\bigg|_{t=0},\tilde{N}(\theta,\varphi)\right)\\
&= s\phi_A(\theta,\varphi)+s\phi_B(\theta,\varphi)+s\frac{\partial f}{\partial t}(\theta,\varphi,0)+O\left(s^2+s\|\eta\|^2_{C^{1,\alpha}}\right)\\
&= s\Psi\left(\dot{\eta}(0)+M^{-1}(0)\dot{M}(0)\right)(\theta,\varphi)+s\frac{\partial f}{\partial t}(\theta,\varphi)+O\left(s^2+s\|\eta\|^2_{C^{1,\alpha}}\right).
\end{aligned}
$$

The result follows.

□

Lemma 15. *For all η, for all f and for all s*

$$
\begin{aligned}
\mathrm{H}[\eta,f,s](\theta,\varphi,t) =& s\Psi M(t)^{-1}\left(\nabla\left(\mathrm{I}[u]\circ C\right)(M(t))+\mathrm{Hess}((\mathrm{I}[u]\circ C))(M(t))\tau^L\eta(t))(M(t))\right)\\
&-2s(\Delta+2)f(\theta,\varphi,t)+O\left(s^2+s\|\eta\|^2_{C^{1,\alpha}}\right)
\end{aligned}
\tag{3.17}
$$

Proof. By the invariance given in (3.4) we may suppose that $\gamma(0)=\mathbf{T}_0, M(0)=\mathrm{Id}$ and $t=0$. Denote

$$
e_0[\eta,f,s](\theta,\varphi) := \mathrm{Exp}(\eta(0))\Phi\left(\theta,\varphi,\tilde{\mathrm{f}}\left[\mathrm{Exp}(\eta(0)),g_{1+s}\right](\theta,\varphi)\right).
$$

By construction, the mean curvature of e_0 with respect to the metric g_{1+s} is equal to $\tilde{\mathrm{h}}\left[\mathrm{Exp}(\eta(0)),g_{1+s}\right]$. The Jacobi operator of $e_0[\eta,f,0]$ with respect to the metric g_1 is

$$
\mathrm{J}_0 := -2(\Delta+2)
$$

so that the Jacobi operator of $e_0[\eta,f,0]$ with respect to the metric g_{1+s} is

$$
\mathrm{J}_s := -2(\Delta+2)+O(s)
$$

The mean curvature of $\mathrm{e}[\eta,f,s]$ is thus

$$
\mathrm{H}[\eta,f,s](\theta,\varphi,0) = \tilde{\mathrm{h}}\left[\mathrm{Exp}(\eta(0)),g_{1+s}\right](\theta,\varphi)-2s(\Delta+2)f(\theta,\varphi,0)+O\left(s^2\right)
$$

The result now follows by Lemmas 12 and 13. □

Combining these relations yields

Lemma 16. *For all η and for all f*

$$
\begin{aligned}
\mathrm{MFC}[\eta,f,s] =& s\Psi\left(\frac{\partial\eta}{\partial t}(st)+M(st)^{-1}\,\mathrm{Hess}(\mathrm{I}[u]\circ C)(M(st))\tau^L\eta(st)(M(st))\right)\\
&+s\left(\frac{\partial f}{\partial t}-2(\Delta+2)f\right)+O\left(s^2+s\|n\|^2_{C^{1,\alpha}}\right).
\end{aligned}
\tag{3.18}
$$

Proof. It suffices to prove this relation at $t=0$. However, by hypothesis,

$$
\Psi(\dot{M}(0)) = \dot{\gamma}(0) = -\nabla\,\mathrm{I}[u]\left(\mathbf{T}_0\right) = -\Psi(\nabla\left(\mathrm{I}[u]\circ C\right)(\mathrm{Id}))
$$

The result now follows by Lemmas 14 and 15.

□

Theorem 5. *If* $\mathrm{I}[u]$ *is of Morse-Smale type then, upon reducing* ϵ *if necessary, there exist smooth functions* $\tilde{\eta} :]-\epsilon, \epsilon[\to C^{1,\alpha}(\mathbb{R}, \mathfrak{k})$ *and* $\tilde{f} :]-\epsilon, \epsilon[\to C^{1,\alpha}_{in}(\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbb{R},]-\frac{\pi}{4}, \frac{\pi}{4}[)$ *such that, for all* s*,*

$$\mathrm{MFC}[\tilde{\eta}(s), \tilde{f}(s), s] = 0$$

In particular, for all $s \in]-\epsilon, \epsilon[$, $\mathrm{e}[\tilde{\eta}(s), \tilde{f}(s), s]$ *is a complete mean curvature flow of tori with respect to the metric* g_{1+s}

Proof. Define $\mathrm{P} : C^{1,\alpha}(\mathbb{R}, \mathfrak{k}) \to C^{0,\alpha}(\mathbb{R}, \mathfrak{k})$ by

$$(\mathrm{P}\eta)(t) = \frac{\partial \eta}{\partial t}(t) + M(t)^{-1}\,\mathrm{Hess}(\mathrm{I}[u] \circ C)(M(t))\tau^L \eta(t)(M(t)).$$

Observe that this operator corresponds to the first summand in (3.18). Since $\mathrm{I}[u]$ is of Morse-Smale type, P is Fredholm and surjective , and thus has a left inverse L (see [12]).
Define the operator $Q : C^{1,\alpha}_{\mathrm{in}}(\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbb{R}) \to C^{0,\alpha}_{\mathrm{in}}(\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbb{R})$ by

$$Qf := \left(\frac{\partial f}{\partial t} - 2(\Delta + 2)f\right).$$

Observe that this operator corresponds to the second summand in (3.18) of MFC. Follows from the classical theory of parabolic operators that Q restricts to functions $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ such that, for all $t, f(\cdot, \cdot, t) \in \mathrm{K}^\perp$, defines a linear isomorphism from $C^{1,\alpha}_{\mathrm{in}}(\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbb{R})$ into $C^{0,\alpha}_{\mathrm{in}}(\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbb{R})$.
For all s, define $A_s : C^{0,\alpha}(\mathbb{R}, \mathfrak{k}) \to C^{0,\alpha}_{\mathrm{in}}(\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbb{R},]-\frac{\pi}{4}, \frac{\pi}{4}[)$ by

$$(A_s\eta)(t) := \Psi(\eta(st))$$

Observe that $\pi \circ A_s$ is a linear isomorphism and that

$$\begin{aligned} \|\pi \circ A_s\| &= O\left(\mathrm{Max}\left(1, s^\alpha\right)\right) \text{ and} \\ \left\|(\pi \circ A_s)^{-1}\right\| &= O\left(\mathrm{Max}\left(1, s^{-\alpha}\right)\right) \end{aligned}$$

It follows from the above that the operator

$$D_s(\eta, f) := A_s P\eta + Qf$$

defines a surjective, Fredholm map from $C^{1,\alpha}(\mathbb{R}, \mathfrak{k}) \oplus C^{1,\alpha}_{\mathrm{in}}(T \times \mathbb{R})$ into $C^{0,\alpha}_{\mathrm{in}}(T \times \mathbb{R})$, where $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ such that, for all $t, f(\cdot, \cdot, t) \in \mathrm{K}^\perp$, and thus has left inverse E_s that satisfies

$$\|E_s\| = O\left(\mathrm{Max}\left(1, s^{-\alpha}\right)\right)$$

Finally, by Lemma 16

$$\frac{1}{s}\mathrm{MCF}[\eta, f, s] = D_s(\eta, f) + \mathrm{O}\left(s + \|\eta\|^2_{C^{1,\alpha}}\right)$$

and the result now follows by the inverse function theorem. □

□

Appendix

APPENDIX A – Clifford Torus as graph

Initially, we obtained the perturbations of Clifford Torus as graphs (but these are not all perturbations) that are obtained locally. The other perturbations of the torus can be obtained by rotations of the sphere, but these rotations are also graphs.
We consider $\mathbf{T}_0 = \mathbf{S}^1 \times \mathbf{S}^1$, and $\phi : \mathbf{S}^1 \times \mathbf{S}^1 \longrightarrow \mathbf{S}^3$ the canonical embedding.
Suppose we have a sequence

$$\phi_m : \mathbf{S}^1 \times \mathbf{S}^1 \longrightarrow \mathbf{S}^3$$

of immersions in $\mathbf{S}^3$.
We say that $\phi_m \longrightarrow \phi$ if there is a sequence α_m of diffeomorphisms of $\mathbf{S}^1 \times \mathbf{S}^1$ such that $\phi_m \circ \alpha_m \longrightarrow \phi$ in C^∞ topology.

Let $\widetilde{\phi}_m = \phi_m \circ \alpha_m$ such that $\widetilde{\phi}_m \circ \alpha_m \longrightarrow \phi$ in C^0 topology. Recall that the C^0 topology is the compact-open topology, that is $\forall$ open subset U of $\mathbf{S}^3$ and $\forall$ compact subset K of $\mathbf{S}^1 \times \mathbf{S}^1$ such that $\phi(K) \subseteq U$, there is M such that for all $m \geq M$, $\widetilde{\phi}_m(K) \subseteq U$.
In particular, we can choose $K = \mathbf{S}^1 \times \mathbf{S}^1$ and $U = \mathrm{Im}(\phi)$, we know that for m large enough $\mathrm{Im}(\widetilde{\phi}_m) \subseteq \mathrm{Im}(\phi)$, so

$$\begin{aligned} \Phi^{-1} \circ \widetilde{\phi}_m &\longrightarrow \Phi^{-1} \circ \phi \\ (\theta, \phi) &\longrightarrow (\theta, \phi, 0) \end{aligned}$$

where Φ is the Fermi parametrization of $\mathbf{S}^3$. We can decompose this composition into two components, that is, $\Phi^{-1} \circ \widetilde{\phi}_m = (\beta_m, \mu_m)$ where $\beta_m : \mathbf{S}^1 \times \mathbf{S}^1 \longrightarrow \mathbf{S}^1 \times \mathbf{S}^1$ and $\mu_m : \mathbf{S}^1 \times \mathbf{S}^1 \longrightarrow \mathbb{R}$ are applications such that $\mu_m \longrightarrow 0$ and $\beta_m \longrightarrow \mathrm{Id}$ in the C^∞ topology. Note that Identity is a diffeomorphism and as diffeomorphisms are open apliccations within the space of smooth applications on $\mathbf{S}^1 \times \mathbf{S}^1$ itself, so for m sufficiently large β_m is invertible, then $\beta_m^{-1} \longrightarrow \mathrm{Id}$ and $\mu_m \circ \beta_m^{-1} \longrightarrow 0$ in the C^∞ topology. We define $f_m = \mu_m \circ \beta_m^{-1}$, so for m sufficiently large

$$\begin{aligned} \mathrm{Im}(\tilde{\Phi}_m) &= \Big\{ (\beta_m(\theta, \varphi), \mu_m(\theta, \varphi) \ \mid \theta, \varphi \in \mathbf{S}^1 \Big\} \\ &= \Big\{ \Big(\beta_m \left(\beta_m^{-1}(\theta, \varphi) \right), \mu_m \left(\beta_m^{-1}(\theta, \varphi) \right) \Big) \ \mid \theta, \varphi \in \mathbf{S}^1 \Big\} \\ &= \Big\{ (\theta, \varphi, f_m(\theta, \varphi)) \ \mid \theta, \varphi \in \mathbf{S}^1 \Big\} \\ &= \mathrm{Gr}(f_m). \end{aligned}$$

therefore, in the canonical topology of immersions we see that all small perturbation of ϕ is a graph.

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