UNIVERSIDADE FEDERAL DO RIO DE JANEIRO
CENTRO DE CIÊNCIAS MATEMÁTICAS E DA NATUREZA
INSTITUTO DE MATEMÁTICA

Generators for Residual Intersections.

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Rio de Janeiro
2019

Nome

por

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Tese de Doutorado apresentada ao Programa de Pós-graduação do Instituto de Matemática, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Doutor em Matemática.

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Agradecimentos

Agradeço aos meus pais, Jorge Marques da Costa Júnior e Rosana Bouça Marques da Costa que desde a minha infãncia investiram o que tinham e o que não tinham para que eu chegasse até este momento. Agradeço por sempre terem acreditado em mim e por me darem a total condição de seguir este caminho.

Agradeço à minha amada esposa Thais Rosa, por ser uma pessoa maravilhosa, que esteve ao meu lado em praticamente toda a minha trajetória acadêmica. Seu amor, carinho e apoio incondicionais foram fundamentais para ultrapassar todas as dificuldades que um doutorado impõe. Esta vitória também é sua. Eu te amo.

Agradeço aos meus avós maternos Lindoval e Marli, que tanto cuidaram de mim durante a infância e adolescência. Deixo também meu agradecimento a minha avó paterna Dirce e dedico esta tese a memória de meu avô Jorge.

Agradeço também aos meus sogros, Eduardo e Patrícia, por toda a torcida e apoio que sempre me deram.

Agradeço ao professor Sérgio Trindade que, durante minha época de Ensino Médio, foi o primeiro a me incentivar em relação a Matemática como profissão. Seu apoio foi fundamental para que eu fizesse a escolha por este caminho.

Agradeço a UFRJ pelos dez anos mais incríveis que eu poderia ter. Entre todas as alegrias e dificuldades, foi na UFRJ que eu me formei como profissional e como ser humano.

Agradeço a todos os professores com quem tive a honra de aprender matemática. Agradeço em especial ao meu grande amigo Jorge Barbosa, meu eterno monitor de Álgebra I, e aos professores Amílcar Pacheco, Luciane Quoos, Cecília Salgado e Alexander Arbieto.

Agradeço também aos meus amigos por todo o apoio e pelos grandes momentos. Agradeço especialmente ao meu grande amigo Renan, e os grandes parceiros de faculdade, Walter e Carol.

Por último, e extremamente importante, gostaria de deixar um agradecimento especial ao meu Professor e Orientador Hamid Hassanzadeh, que além de me ensinar uma quantidade enorme de matemática, sempre esteve ao meu lado quando as coisas não iam bem. Seu conhecimento, atenção, disponibilidade e companheirismo foram fundamentais para a realização deste trabalho. Foi uma honra dividir esta caminhada com alguém que, muito além de um excelente orientador, foi um grande amigo.

Agradeço às agências de fomento CAPES, CNPq e FAPERJ pelo auxílio financeiro que possibilitaram a realização deste trabalho.

Resumo

A primeira parte deste trabalho consiste em obter uma nova construção da família de complexos de Buchsbaum-Eisenbud (que contém como membros os complexos de Eagon-Northcott e de Buchsbaum-Rim) a partir de um estudo de sequências espectrais do tipo Koszul-Čech.

A segunda parte consiste em aplicar os resultados obtidos sobre as sequências espectrais de Koszul-Čech aos complexos de aproximação residuais para obter a estrutura das interseções residuais disfarçadas e atacamos uma conjectura que as relaciona com a interseção residual original . Como consequência, obtém-se resultados estruturais de interseções residuais no caso em que a mesma coincide com a sua interseção residual disfarçada.

Abstract

The first part of this thesis consists of obtaining a new construction of the Buchsbaum-Eisenbud family of complexes (which contain as members the Eagon-Northcott and the Buchsbaum-Rim complexes) from the study of spectral sequences of Koszul-Čech type.

The second part consists in applying the obtained results on Koszul-Čech spectral sequences to the residual approximation complexes to obtain the structure of the disguised residual intersections and to prove a conjecture relating them with the original residual intersection. As a consequence, several structural results about residual intersections are obtained in the case when it coincides with its disguised residual intersection.

Contents

Introduction

The theory of residual intersections, or residual schemes, has its roots in intersection theory, on the definition of the refined intersection products, which defines the refined intersection products (see [Fu, Corollary 9.2.3]). In the commutative algebra point of view the theory of residual intersections has its roots in the theory of Linkage. The very first aim is to classify varieties in the projective space, by saying that two subvarieties $X, Y \subset \mathbb{P}^n$ are in the same linkage class if there is a sequence of subvarieties $X = X_1, \dots, X_n = Y \subset \mathbb{P}^n$ such that $X_i \cup X_{i+1}$ is a complete intersection for each i. Algebraically we have:

Definition 0.0.1. *Let $(R, \mathfrak{m})$ be a local ring. We say that*

1. *Two ideals I, J are directly linked if there is a regular sequence $\alpha \subset I \cap J$ such that $I = (\alpha) : J$ and $J = (\alpha) : I$.*

2. *Two ideals I, J are linked if there is a sequence of ideals $I = I_1, \dots, I_n = J$ such that I_k and I_{k+1} are directly linked. In this case I, J are said to be in the same linkage class.*

The theory of linkage is very fruitful for the classification of curves in $\mathbb{P}^3$ (see [Eis,]). As Linkage is an equivalence relation, it is natural to ask what properties "pass through" a linkage. An example of such property is obtained by Peskine-Szpiro in [PS]:

Theorem 0.0.2. *If $(R, \mathfrak{m})$ is a Gorenstein ring and I, J are in the same linkage class, then R/I is Cohen-Macaulay if and only if R/J is.*

In the same paper the authors shows by an example that the theorem above is not true if we suppose that R is a Cohen-Macaulay ring. Huneke shows in [Hu] that if we suppose that not only R/I is Cohen-Macaulay but **all** of the Koszul homologies of a set of generators of I are Cohen-Macaulay. (we say that I is Strongly Cohen-Macaulay, or SCM, in this case), then for every ideal J in the same linkage class of I, R/J is Cohen-Macaulay. The notion of a residual intersection is a generalization of linkage by weakening the assumption that α is a regular sequence.

Definition 0.0.3. *Let R be a commutative ring, $I \subset R$ an ideal of height g and $s \geq g$. We say that an ideal J is an algebraic s-residual intersection if $J = (\mathfrak{a} : I)$ with $\mathfrak{a} = (a_1, \ldots, a_s) \subset I$ and ht(J)$\geq s$. Moreover, we say that*

1. *J is an arithmetic s-residual intersection if $\mu_{R_{\mathfrak{p}}}(\frac{I_{\mathfrak{p}}}{\mathfrak{a}_{\mathfrak{p}}}) \leq 1$ for all prime ideals $\mathfrak{p}$ with $\operatorname{ht} \mathfrak{p} = s$.*

2. *J is a geometric s-residual intersection if ht$(I + J) \geq s + 1$*

An important question in the theory of residual intersections is the following:

Question 0.0.4. *Let R be a Cohen-Macaulay Noetherian local ring. What conditions one must impose to an ideal I so that for every s-residual intersection of J of I, R/J is Cohen-Macaulay.*

In their seminal paper $[AN]$ Artin and Nagata introduce the following condition, in a attempt to ask the above question.

Definition 0.0.5. *An ideal I satisfies the G_s condition if for every prime ideal $\mathfrak{p}$ containing I with $\operatorname{ht} \mathfrak{p} \leq s-1$ we have $\mu_{R_{\mathfrak{p}}}(I_p) \leq \operatorname{ht} \mathfrak{p}$. We say that I satisfies G_∞ if I satisfies G_s for every s.*

In the same paper, the authors prove incorrectly that if R/I is Cohen-Macaulay ans satisfy G_s then R/J is Cohen-Macaulay for every s-residual intersection of I. Again in [Hu] Huneke proves that if I is SCM then for every *geometric* s-residual intersection J of I, R/J is Cohen-Macaulay. In [HVV] Herzog, Vasconcelos and Villarreal improves the result of Huneke for ideals satisfying the Sliding Depth condition, which we define below.

Definition 0.0.6. *Let $I = (f_1, \dots, f_r) = (\boldsymbol{f})$ be an ideal and k an integer. We say that*

1. *I satisfies the sliding depth Condition SD_k if*

$$\operatorname{depth}(H_i(\boldsymbol{f}; R)) \geq \min\{d - g, d - r + i + k\}$$

for all $i \geq 0$; also SD stands for SD_0.

2. *I satisfies the sliding depth condition on cycles SDC_k at level t if $\operatorname{depth}(Z_i(\boldsymbol{f}; R)) \geq \min\{d - r + i + k, d - g + 2, d\}$ for all $r - g - t \leq i \leq r - g$.*

It can be shown that these conditions do not depend on the choice of generators for I.

In another important work, [HU] Huneke and Ulrich proposed the following question.

Question 0.0.7. *Let R be a Cohen-Macaulay ring and I an ideal satisfying SD. Is every s-residual intersection of I Cohen-Macaulay?*

Affirmative answers to this question are obtained in a series of works. In [Ha] Hassanzadeh gives a positive answer for arithmetic residual intersections of ideals satisfying SD, and in [CNT] Chardin, Naéliton and Tran for algebraic residual intersections of ideals satisfying SD_1. Their techniques are based on what the authors call the residual approximation complexes. Roughly speaking, the residual approximation complexes of a pair of ideals $\mathfrak{a} \subset I$ are a family of complexes $\{{}_k\mathcal{Z}_\bullet^+\}$ of the same length such that

1. $H_0({}_0\mathcal{Z}_\bullet^+) = R/K$, where $K \subset R$ is an ideal.

2. $H_0({}_k\mathcal{Z}_\bullet^+) = \mathrm{Sym}^k(I/\mathfrak{a})$.

The ideal K above is an ideal that can be viewed as an approximation of the colon ideal $J = \mathfrak{a} : I$, and is called the *disguised residual intersection* in [HN]. It has the following properties:

- $K \subset J$.

- $K = J$ off $V(I)$.

- $\sqrt{K} = \sqrt{J}$.

In [Ha] Hassanzadeh shows that if R is Cohen-Macaulay, I satisfies SD and J is an s-residual intersection, then the complex ${}_0\mathcal{Z}_\bullet^+$ is acyclic and, moreover, R/K is a Cohen-Macaulay ring of dimension $d - s$. Furthermore, if J is an *arithmetic* residual intersection, then $K = J$ and hence R/J is Cohen-Macaulay. In [HN] the authors then propound the following conjecture.

Conjecture 0.0.8. *[HN, Conjecture 5.9] If R is a local Cohen-Macaulay ring and I an ideal satisfying SD and depth$(R/I) \geq d - s$. Then for any algebraic s-intersection $J = \mathfrak{a} : I$ we have $K = J$.*

In [CNT] the authors prove, by intricate use of dualities that $K = J$ when $\operatorname{ht} I = 2$ and I satisfies SD_1 and, by reduction to the height 2 case, that R/J is Cohen-Macaulay. This reduction does not show that $K = J$ for heights bigger than 2.

The ideal K has a quite esoteric nature, as it comes from a comparison map of a spectral sequence, and a description of the structure of this ideal is the main focus of this thesis. This is accomplished by showing that the residual approximation complex ${}_0\mathcal{Z}_\bullet^+$ is a subcomplex of a Eagon-Northcott complex. With this analysis a new ideal comes into play, the Koszul-Fitting ideal of the pair $\mathfrak{a} \subset I$, denoted by $\mathrm{Kitt}(\mathfrak{a}, I)$, defined as follows.

Let R be a commutative ring, $\mathfrak{a} = (a_1, \ldots, a_s) \subset I = (f_1, \ldots, f_r)$ two finitely generated ideals and $\Phi = [c_{ij}]$ a matrix such that $(a_1, \ldots, a_s) = (f_1, \ldots, f_r) \cdot \Phi$. Let $K_\bullet = R < e_1, \ldots, e_r; \partial(e_i) = f_i >$ be the Koszul complex on the generators of I with it's Differential Graded Algebra structure and consider two sub-algebras $\Gamma_\bullet, Z_\bullet$, where $\Gamma_\bullet$ is the sub-algebra generated by $\gamma_j = \sum_{i=1}^{r} c_{ij} e_i$, $1 \leq j \leq s$ and Z_γ is the sub-algebra of Koszul cycles. We then define

$$\mathrm{Kitt}(\mathfrak{a}, I) =< \Gamma_\bullet \cdot Z_\bullet >_r \subset K_r \simeq R$$

Our main results about the Kitt ideals are the following properties.

Theorem 0.0.9. *1.* $\mathrm{Kitt}(\mathfrak{a}, I)$ *depends only on the ideals* $\mathfrak{a}$ *and* I *and not on the generators or th representative matrix. (Propositions 4.2.5, 4.2.6 and 4.2.10).*

2. $\mathrm{Kitt}(\mathfrak{a}, I)$ *and the disguised residual intersection introduced above. (Theorems 4.2.3 and 4.2.4.)*

3. $\mathrm{Fitt}_0(I/\mathfrak{a}) \subseteq \mathrm{Kitt}(\mathfrak{a}, I) = \mathfrak{a}+ < \Gamma_\bullet \cdot \tilde{H}_\bullet >_r$ *where* $\tilde{H}_\bullet$ *is the sub-algebra of* $K_\bullet$ *generated by the representatives of Koszul homologies. (Proposition 4.2.13 and Theorem 4.2.15).*

4. If R *is a CM Noetherian ring,* $J = \mathfrak{a} :_R I$ *is an* s*-residual intersection and* I *satisfies* SD_1 *then* $\mathrm{Kitt}(\mathfrak{a}, I) = J$ *and hence* R/J *is Cohen-Macaulay. (Theorem 4.2.25).*

This ideal has a close relation with the DG-Algebra structure of the Koszul complex of I, as the previous results shows. Therefore the introduction of $\mathrm{Kitt}(\mathfrak{a}, I)$ is not only important as a tool to understand the Cohen-Macaulayness of residual intersections. Our applications of the Kitt ideals ranges from the existence of a generic s-residual intersection (Theorem 4.3.4) to nontrivial structural results in the case when $\mathrm{Kitt}(\mathfrak{a}, I) = \mathfrak{a} : I$ (Theorems 4.3.1,4.3.5, 4.3.6 and 4.3.7).

The thesis is organized as follows. In Chapter 1 we just set some notation and recall some basic constructions of commutative algebra as the symmetric and exterior algebras, the Koszul complexes, approximation complexes and the Čech complex.

Chapter 2 is devoted to spectral sequences and furnishes the language and the necessary tools for the understanding of the disguised residual intersection.

In Chapter 3 we have the first important result of this thesis. We recall the construction of the Buchsbaum-Eisenbud family of complexes and introduce a new family of complexes, called the Koszul-Čech family of complexes, constructed from the spectral sequences arising from tensoring a Koszul complex with a Čech complex. The main theorem of this section shows that these two families are isomorphic.

Chapter 4 applies all the machinery of the previous chapters on the theory of residual intersections. We give a brief review of the construction of the residual approximation complexes. Then we introduce the Kitt ideals and prove several of properties, and derive some applications to the theory of residual intersections.

Lastly, in Chapter 5 we make some remarks and state some open questions.

Chapter 1

Preliminaries

In this thesis R will always denote a commutative ring with 1 with no further hypothesis.

The aim of this chapter is to set up notations and recall some basic constructions from commutative algebra.

1.1 Sign functions and Matrices

In this section we define the sign function, state some lemmas that will be used in the sequel of the thesis and set notations about matrices and submatrices.

Definition 1.1.1. *Let $I = \{1, \dots, n\}$ be an ordered finite set, and let $J \subset I$ with $|J| = j$. We define*

$$\operatorname{sgn}(J \subset I)$$

as the sign of the permutation that puts the elements of J in the first j positions.

Example 1.1.2. *Consider $I = \{1, 2, 3, 4, 5\}$. Then $\operatorname{sgn}(\{2, 4\} \subset I) = -1$, because we need two transpositions to get $\{2, 4, 1, 3, 5\}$.*

We give some lemmas about the sign functions that we will use in some proofs in the next chapters. We advice the reader to skip them and come back when they are needed.

Lemma 1.1.3. *Let I be a finite ordered set and $i, j \in I$ two elements. Then the expressions*

$$\operatorname{sgn}(\{i\} \subset I).\operatorname{sgn}(\{j\} \subset I \setminus \{i\})$$

and

$$\operatorname{sgn}(\{j\} \subset I).\operatorname{sgn}(\{i\} \subset I \setminus \{j\})$$

have opposite signs.

Proof. Without loss of generality, we may suppose that $I = \{1, \ldots, n\}$ and, by applying a fixed permutation, that $i = 1$ and $j = 2$. The proof is now obvious. $\square$

Lemma 1.1.4. *Let I be a finite ordered set, $J = \{j_1, ..., j_p\} \subset I$. If $J_i = J \setminus \{j_i\}$, then*

$$\operatorname{sgn}(J_i \subset J)\operatorname{sgn}(\{j_i\} \subset J \setminus J_i) = (-1)^{p-i}\operatorname{sgn}(J \subset I).$$

Proof. To put J_i on the first positions, we multiply by $\operatorname{sgn}(J_i \subset J)$. Next, to put j_i right after J_i we multiply by $\operatorname{sgn}(\{j_i\} \subset J \setminus J_i)$. Finally we put j_i in its right position in J multiplying by $(-1)^{p-i}$. This proves the lemma. $\square$

Lemma 1.1.5. *Let I be a finite ordered set and $J \subset I$. Then the expression*

$$(\operatorname{sgn}(\{j\} \in J)\operatorname{sgn}(J \setminus \{j\} \subset I)\operatorname{sgn}(\{j\} \subset I \setminus (J \setminus \{j\}))$$

is the same for all $j \in J$.

Proof. Let $j_1 \neq j_2 \in J$. Without loss of generality, we may suppose that $I = \{1, \ldots, n\}$ and, applying a fixed permutation, that $J = \{1, \ldots, k\}, j_1 = 1$ and $j_2 = 2$. We have that

$$\operatorname{sgn}(\{1\} \subset J) = 1,$$

$$\mathrm{sgn}(J \setminus \{1\} \subset I) = (-1)^{k-1,}$$

$$\mathrm{sgn}(\{1\} \subset I \setminus (J \setminus \{1\})) = 1$$

and

$$\mathrm{sgn}(\{2\} \in J) = -1,$$

$$\mathrm{sgn}(J \setminus \{2\} \subset I) = (-1)^{k-1},$$

$$\mathrm{sgn}(\{2\} \subset I \setminus (J \setminus \{2\})) = 1$$

and both products are equal to $(-1)^k$ □

Lastly, we define some notations about matrices.

Definition 1.1.6. *We denote by* $\mathrm{id}_{r\times r}$ *the* $r \times r$ *identity matrix.*

Definition 1.1.7. *Let* $\Phi = [c_{ij}]$ *be a* $g \times f$ *matrix.*

1. *Let* $I \subset \{1, \ldots, g\}, J \subset \{1, \ldots, f\}$ *be two ordered subsets. Define*

$$\Phi_I^J$$

to be the submatrix with rows indexed by I *and columns indexed by* J*. If* $I = \{1, \ldots, g\}$ *we suppress the subscript and write*

$$\Phi^J.$$

We use an analogous notation if $J = \{1, \ldots, f\}$*.*

2. *Let* $I \subset \{1, \ldots, g\}, J_1, J_2 \subset \{1, \ldots, f\}$ *be three ordered subsets. Define*

$$\Phi_I^{J_1,J_2}$$

to be the submatrix with rows indexed by I*, the first columns indexed by* J_1 *and the last columns indexed by* J_2*.*

We again make the definition above more concrete by the means of an example.

Example 1.1.8. *Let* $\Phi = [a_{ij}]$ *be a* 3×4 *matrix. Let* $I = \{1, 3\}, J_1 = \{1, 4\}, J_2 = \{3\}, J_3 = \{3, 4\}$. *Then*

$$\Phi_I^{J_1} = \begin{bmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{bmatrix}, \Phi^{J_2} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}, \Phi^{J_2,J_1} = \begin{bmatrix} a_{13} & a_{11} & a_{14} \\ a_{23} & a_{21} & a_{24} \\ a_{33} & a_{31} & a_{33} \end{bmatrix}, \Phi^{J_3,J_2} = \begin{bmatrix} a_{13} & a_{14} & a_{13} \\ a_{23} & a_{24} & a_{23} \\ a_{33} & a_{34} & a_{33} \end{bmatrix}.$$

Definition 1.1.9. *Let* Φ *be a* $g \times f$ *matrix and* Ψ *a* $g \times f'$ *matrix. We denote by* $[\Phi|\Psi]$ *the* $g \times (f + f')$ *matrix whose first* f *columns are given by the columns of* Φ *and the last* f' *columns are given by the columns of* Ψ.

1.2 Exterior and Symmetric Algebras

In this section we recall the construction of some algebras from R-modules. We start by recalling the notion of the tensor algebra of a module. Let R be a ring and M be an R-module. The tensor algebra of M is the graded R-module

$$\bigotimes M = \bigoplus_{i=0}^{\infty} M^{\otimes i}$$

endowed with the product

$$(x_1 \otimes \cdots \otimes x_n) \odot (y_1 \otimes \cdots \otimes y_l) = x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_l.$$

The first important algebra that we will deal with is the exterior algebra, used to construct the Koszul complex. Let $\mathfrak{J}$ be the two-sided ideal generated by the elements of the form $x \otimes x$. This ideal is homogeneous, and the exterior algebra $\bigwedge M$ is the quotient

$$\bigwedge M = \bigotimes M/\mathfrak{J}. = \bigoplus \bigwedge^i M.$$

The projection of an element $x_1 \otimes \cdots \otimes x_n$ is denoted by $x_1 \wedge \cdots \wedge x_n$ and the product of two elements x, y is denoted by $x \wedge y$. This algebra is alternating:

$$x \wedge y = (-1)^{\deg(x)\deg(y)} y \wedge x, \text{ for } x, y \text{ homogeneous elements}$$
$$x \wedge x = 0 \text{ for any } x \in \bigwedge M$$

The second algebra is the symmetric algebra of the module M. Let $\mathfrak{A}$ be the two-sided ideal generated by the elements of the form $x \otimes y - y \otimes x$. This ideal is again homogeneous, and the symmetric algebra of M is the graded R-algebra

$$\operatorname{Sym}(M) = \bigotimes M/\mathfrak{A} = \bigoplus_{i=0}^{\infty} \operatorname{Sym}^i(M).$$

The projection of an element $x_1 \otimes \cdots \otimes x_n$ is denoted by $x_1 \cdots x_n$ and the product of two elements x, y is denoted by $x \cdot y$. This algebra is commutative.

The constructions of $\bigwedge M$ and $\operatorname{Sym}(M)$ are functorial: if $\psi : M \to N$ is an homomorphism of modules, then there are homomorphisms of algebras

$$\begin{array}{rcc} \bigwedge \psi : \bigwedge M & \to & \bigwedge N \\ m_1 \wedge \cdots \wedge m_n & \to & \psi(m_1) \wedge \cdots \wedge \psi(m_n) \end{array},$$

and

$$\begin{array}{rcc} \operatorname{Sym}(\psi) : \operatorname{Sym} M & \to & \operatorname{Sym} N \\ m_1 \cdots m_n & \to & \psi(m_1) \cdots \psi(m_n) \end{array}$$

Denote by $F(-)$ either of the functors $\bigwedge(-)$ or $\operatorname{Sym}(-)$. For any exact sequence

$$M \xrightarrow{\phi} N \xrightarrow{\psi} P \to 0$$

we have an exact sequence

$$F(N) \otimes M \xrightarrow{1 \otimes F(\phi)} F(N) \xrightarrow{F(\psi)} F(P) \to 0$$

This exact sequence allow us to describe the exterior and symmetric algebras for finitely generated R-modules.

1.2.1 Exterior Algebras

The exterior algebra of free modules are easy to explain. If F is a free R module with basis $e_1, ..., e_n$, we have:

Proposition 1.2.1. *For each i, $\bigwedge^i F$ is a free R module with basis*

$$e_I = e_{k_1} \wedge \cdots \wedge e_{k_i},$$

where I runs through all ordered subsets $\{k_1 < \cdots < k_i\} \subset \{1, \ldots, n\}$. In particular, $\bigwedge^i F$ has rank $\binom{n}{i}$.

The exterior algebra has a tight relation with determinants. Using the notation of 1.1.7 we have the following.

Proposition 1.2.2. *Let $v_j = \sum_{i=1}^{n} a_{ij} e_i \in \bigwedge^1 F$ for $1 \leq j \leq k$. If $\Phi = [a_{ij}]$, then*

$$v_1 \wedge \cdots \wedge v_k = \sum_{|I|=k} \det \Phi_I e_I$$

The last property of exterior algebras of free modules are self-dual.

Proposition 1.2.3. *Let F be a free module with basis $e_1, \ldots, e_n$. The natural multiplication map*

$$\bigwedge^i F \otimes_R \bigwedge^{r-i} F \to \bigwedge^r F \simeq R$$

is a perfect pairing and hence

$$\bigwedge^i F \simeq \mathrm{Hom}(\bigwedge^{r-i} F, R).$$

1.2.2 Symmetric Algebras

We begin by explaining the symmetric algebra of a free module.

Proposition 1.2.4. *Let F be a free R-module with basis $e_1, \dots, e_n$. Then the symmetric algebra* $\operatorname{Sym} F$ *is isomorphic to a polynomial ring $R[T_1, \dots, T_n]$, with each variable T_i corresponding to the basis element e_i.*

Now let M be a finitely generated module with a presentation

$$R^q \xrightarrow{\Phi=[a_{ij}]} R^p \to M \to 0.$$

Then $\operatorname{Sym}(M)$ has a presentation

$$R[T_1, ..., T_p] \otimes R^q \to R[T_1, ..., T_p] \to \operatorname{Sym}(M) \to 0.$$

If $f_1, \cdots, f_q$ is the basis of R^q, the image of the element $1 \otimes f_j$ through the left-hand map of the above exact sequence is $\sum_{i=1}^{p} a_{ij} T_i$. Therefore we have teh following.

Proposition 1.2.5. *If M is the cokernel of a linear map $R^q \xrightarrow{\Phi=[a_{ij}]} R^q$, then*

$$\operatorname{Sym}(M) \simeq R[T_1, ..., T_P]/(f_1, ..., f_q),$$

where $f_j = \sum_{i=1}^{p} a_{ij} T_i$.

1.3 Koszul Complex

One of the most important constructions in commutative algebra is the Koszul Complex. In this section we recall the construction and set notation for it.

Let R be a ring, $(x_1, \dots, x_n)$ a sequence of elements and let F be a free R module with basis $e_1, \dots, e_n$. Following the notation of Proposition 1.2.1 , we can define an homomorphism

$$d : \bigwedge F \to \bigwedge F$$

such that for each ordered subset $I \subset \{1, \dots, n\}$,

$$d(e_I) = \sum_{i \in I} \operatorname{sgn}(\{i\} \subset I) x_i e_{I \setminus \{i\}}.$$

It is easy to check, by Lemma 1.1.3, that $d^2 = 0$. Therefore we have a complex

$$0 \to \bigwedge^{n} F \xrightarrow{d} \bigwedge^{n-1} F \xrightarrow{d} \cdots \xrightarrow{d} \bigwedge^{1} F \xrightarrow{d} \bigwedge^{0} F = R \to 0$$

Definition 1.3.1. *The above complex is called the Koszul complex on the sequence $\boldsymbol{x} = (x_1, \cdots, x_r)$ with coefficients in R, and is denoted by $K_\bullet(\boldsymbol{x}; R)$. Moreover, if M is an R-module, then the Koszul complex on the sequence $\boldsymbol{x}$ with coefficients in M is the complex $K_\bullet(\boldsymbol{x}, M) = K_\bullet(\boldsymbol{x}, R) \otimes_R M$, and we denote by d_M the differential of this second complex.*

The following proposition shows that the Koszul complex has the structure of a differential graded algebra (DG-algebra) that comes from the algebra structure of $\bigwedge F$:

Theorem 1.3.2. *[BH, Proposition 1.6.2] Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a sequence of elements in R and M be an R-module. We have:*

1. *The Koszul complex $K_\bullet(\boldsymbol{x}, R)$ has a natural algebra structure coming from the algebra structure of $\bigwedge F$.*
2. *$K_\bullet(\boldsymbol{x}, M)$ is a $K_\bullet(\boldsymbol{x}, R)$-graded module in a natural way.*
3. *If $x \in K_\bullet(\boldsymbol{x}, R)$ and $y \in K_\bullet(\boldsymbol{x}, M)$, then*

$$d_M(x.y) = d(x).y + (-1)^{\deg x} x.d_M(y)$$

For any sequence $\mathbf{x} = (x_1, ..., x_n)$ in R and any R-module M set

$$Z_\bullet(\mathbf{x}; R) = \operatorname{Ker} d,\ Z_\bullet(\mathbf{x}; M) = \operatorname{Ker} d_M$$
$$B_\bullet(\mathbf{x}; R) = \operatorname{Im} d,\ B_\bullet(\mathbf{x}; M) = \operatorname{Im} d_M$$

Then it is not hard to see, using Theorem 1.3.2, (3) that

$$Z_\bullet(\mathbf{x};R)\cdot Z_\bullet(\mathbf{x};M)\subset Z_\bullet(\mathbf{x};M),$$
$$B_\bullet(\mathbf{x};R)\cdot Z_\bullet(\mathbf{x};M)\subset Z_\bullet(\mathbf{x};M) \text{ and}$$
$$B_\bullet(\mathbf{x};R)\cdot B_\bullet(\mathbf{x};M)\subset B_\bullet(\mathbf{x};M).$$

Therefore, we have

Theorem 1.3.3. *[BH, Proposition 1.6.4] For any sequence $\boldsymbol{x}=(x_1,\ldots,x_n)$ in R and any R-module M, we have:*

1. *$Z_\bullet(\boldsymbol{x};R)$ is a subalgebra of $K_\bullet(\boldsymbol{x};R)$, called the algebra of Koszul cycles of $\boldsymbol{x}$, and $Z_\bullet(\boldsymbol{x},M)$ is a $Z_\bullet(\boldsymbol{x};R)$-module in a natural way.*

2. *$B_\bullet(\boldsymbol{x};R)$ is an ideal of $Z_\bullet(\boldsymbol{x};R)$, called the ideal of Koszul boundaries of $\boldsymbol{x}$.*

3. *The homology $H_\bullet(\boldsymbol{x};R)=Z_\bullet(\boldsymbol{x};R)/B_\bullet(\boldsymbol{x};R)$ has an algebra structure induced by the one on $Z_\bullet(\boldsymbol{x};R)$, and $H_\bullet(\boldsymbol{x};M)$ has a natural structure of $H_\bullet(\boldsymbol{x};R)$-module.*

Because of the above theorem we say that the Koszul complex $K_\bullet(\mathbf{x};R)$ is a differential graded algebra or, more compactly, a DG-algebra.

1.4 Generalized Koszul Complex and Approximation Complexes

An important construction that is used in this thesis are the approximation complexes introduced by Herzog, Simis and Vasconcelos in [HSV], constructed from the generalized Koszul Complex that we explain below. In what follows, we denote by $S(G)$ the symmetric algebra of a module G and by $S_d(G)$ its degree d component.

Let $\varphi : F = R^f \to G = R^g$ be a linear map between two free R-modules, and let $e_1, \ldots, e_f$ be a basis for F. For each $s \in \mathbb{N}$ define a complex

$$\mathbb{K}_s(\varphi) := 0 \to \wedge^s F \to \wedge^{s-1} F \otimes S_1(G) \to \cdots \to S_s(G) \to 0$$

whose differential is given by

$$\partial \wedge^r F \otimes_R S_{r-s}(G) \to \wedge^{r-1} F \otimes_R S_{r-s+1}(G)$$

$$\partial(e_I \otimes w) = \sum_{i \in I} \operatorname{sgn}(\{i\} \subset I) e_{I \setminus i} \otimes \varphi(e_i) \cdot w,$$

where the last multiplication is the multiplication in $S(G)$, where $\varphi(e_i)$ is considered as an element of degree one on that algebra.

Definition 1.4.1. *The complex $\mathbb{K}(\varphi) = \bigoplus_{s=0}^{\infty} \mathbb{K}_s(\varphi)$ is called the generalized Koszul complex of the linear map φ.*

It is easy to describe this complex when we represent the map φ by matrices. Let $\Phi = [a_{ij}]$ be a matrix representing the map φ. Then the complex $\mathbb{K}(\varphi)$ is just the ordinary Koszul complex on the linear forms $\gamma_j = \sum_{i=1}^{g} a_{ij} T_i \in S(G) \simeq R[T_1, \cdots, T_g]$. With this description in mind we have the following theorem, whose proof is straightforward.

Proposition 1.4.2. *Let $\varphi : R^f \to R^g$ and $\psi : R^h \to R^g$ two linear maps. Then*

$$\mathbb{K}(\varphi \oplus \psi) \simeq \mathbb{K}(\varphi) \otimes \mathbb{K}(\psi).$$

The generalized Koszul complex is the basic ingredient of the approximation complexes. Let $\mathbf{x} = (x_1, \ldots, x_r)$ be a sequence of elements in R, $G = R^r$ and $\mathbb{K}$ the generalized Koszul complex of the identity mapping of G. We can make this complex a double complex introducing the differentials

$$\partial : \bigwedge^{r} G \otimes S_t(G) \to \bigwedge^{r-1} G \otimes S_t(G)$$

given by the Koszul complex on the sequence $\mathbf{x}$ with coefficients in $S(G)$ and the differential

$$\partial' : \bigwedge^{r} G \otimes S_t(G) \to \bigwedge^{r-1} G \otimes S_{t+1}(F)$$

given by the original differential of the generalized Koszul complex. It is not hard to show that these two differential skew-commute, that is,

$$\partial\partial' + \partial'\partial = 0.$$

We write $\mathcal{L}(\partial)$ and $\mathcal{L}(\partial')$ for the complex obtained by using one of the differentials. Let $\mathcal{Z}(\mathbf{x}; R)$ be the cycles, $\mathcal{B}(\mathbf{x}; R)$ the boundaries and $\mathcal{H}(\mathbf{x}, R)$ the homology of $\mathcal{L}(\partial)$. As $S(G)$ is faithfully flat it is easy to see that

$$\mathcal{Z}(\mathbf{x}; R) = Z(\mathbf{x}; R) \otimes S(G)$$

$$\mathcal{B}(\mathbf{x}; R) = B(\mathbf{x}; R) \otimes S(G).$$

$$\mathcal{H}(\mathbf{x}, R) = H(\mathbf{x}; R) \otimes S(G)$$

By the skew-commutativity of differentials, ∂' restricts to differential on $\mathcal{Z}(\mathbf{x}; R)$, $\mathcal{B}(\mathbf{x}; R)$ and $\mathcal{H}(\mathbf{x}, R)$.

Definition 1.4.3. *Let R be a commutative ring and $\boldsymbol{x} = (x_1, \cdots, x_r)$ a sequence of elements.*

1. *$(Z(\boldsymbol{x}; R), \partial')$ is called the $\mathcal{Z}$-complex of the sequence $\boldsymbol{x}$.*

2. *$(\mathcal{H}(\boldsymbol{x}; R), \partial')$ is called the $\mathcal{M}$-complex of the sequence $\boldsymbol{x}$.*

These complexes are candidates for a resolution of the modules

$$H_0(\mathcal{Z}(\mathbf{x}; R)) = \mathrm{Sym}(I)$$

$$H_0(\mathcal{M}(\mathbf{x}; R)) = \mathrm{Sym}(I/I^2)$$

and hence the name "approximation complexes". We shall use the $\mathcal{Z}$-complex in the sequel, and we usually write it as a graded S-complex, where $S = S(G)$:

$$0 \to Z_r(\mathbf{x}; R) \otimes S[-r] \to Z_{r-1}(\mathbf{x}; R) \otimes S[-r+1] \to \cdots \to Z_1(\mathbf{x}; R) \otimes S[-1] \to S \to 0$$

1.5 The Čech complex and the Local Cohomology Modules

Another important complex that we will use in the sequel of this thesis is the Čech complexes, whose homologies give the Local Cohomology modules.

Let R be a ring and $\mathbf{x} = (x_1, \cdots, x_n)$ be a sequence of elements in R. For every ordered subset $I \subset \{1, \cdots, n\}$, define

$$C^I(R) = R_{x_I},$$

where $x_I = \prod_{i \in I} x_i$ and R_x means the localization on the multiplicative set $\{1, x, x^2, \cdots\}$. For each $1 \leq i \leq n$ put

$$C^i(R) = \bigoplus_{|I|=i} C^I(R),$$

and define a map

$$\begin{array}{rcl} d^i : C^i(R) & \to & C^{i+1}(R) \\ m/(x_I)^n & \to & \sum_{i \notin I} \operatorname{sgn}(\{i\} \subset I \cup \{i\}) x_i^n . m/(x_{I \cup \{i\}})^n \end{array}$$

It is easy to see that $d^{i+1} \circ d^i = 0$. Therefore we have a complex

$$0 \to C^0(R) \xrightarrow{d^0} C^1(R) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^n(R) \to 0$$

Definition 1.5.1. *The complex defined above is called the Čech complex of the sequence $\boldsymbol{x}$ and is denoted by $\check{C}^{\bullet}_{\boldsymbol{x}}(R)$. More generally, if M is an R-module, then the Čech complex of $\boldsymbol{x}$ with coefficients in M is the module $\check{C}^{\bullet}_{\boldsymbol{x}}(M) = \check{C}^{\bullet}_{\boldsymbol{x}}(R) \otimes_R M$.*

It can be shown that the homologies of this complex depends only on the ideal generated by the sequence $\mathbf{x}$. Therefore the following definition makes sense.

Definition 1.5.2. *Let $\boldsymbol{x}$ be a sequence of elements of R and I the ideal they generate. The i-th cohomology module of $C^{\bullet}(M)$ is called the i-th local cohomology module of the module M with support on I, and is denoted by*

$$H^i_I(M).$$

We now state some important theorems on local cohomology modules.

Theorem 1.5.3. *[Grothendieck's Vanishing Theorem, [BS, Theorem 6.1.2]] Let R be a Noetherian ring. If M is an R-module of dimension d then*

$$H_I^i(M) = 0$$

for all ideals $I \subset R$ and $i \geq d$.

Theorem 1.5.4. *[Grothendieck's non-vanishing theorem, [BS, Theorem 6.1.4]] Let R be a local noetherian ring with maximal ideal $\mathfrak{m}$, and M a finitely generated R-module of dimension d. Then*

$$H_{\mathfrak{m}}^d(M) \neq 0$$

Theorem 1.5.5. *[BS, Theorem 6.2.7] If $I \subset R$ is an ideal and M is a finitely generated R-module, then*

$$\text{grade}(I, M) = \min\{i \mid H_I^i(M) \neq 0\}.$$

The last goal of this section is to describe the local cohomologies of a polynomial ring $S = R[T_1, \cdots, T_n]$ with respect to the ideal $\mathfrak{t} = (T_1, \cdots, T_n)$. By Theorems 1.5.3, 1.5.4 and 1.5.5 there is no cohomology besides the top local cohomology $H_{\mathfrak{t}}^n(S)$, and this module is easy to calculate: it is the cokernel of the map

$$\bigoplus_{|I|=n-1} S_{T_I} \xrightarrow{d^{n-1}} S_{T_1 \cdots T_g}$$

Let $m/(T_I)^n$ be a typical element of the source of d^{n-1}. If i is the unique index that is not in I, then we have $d^{n-1}(m) = x_i^n.m/(T_1 \cdots T_n)^k$. Therefore the image of the map d^{n-1} consists of elements of the form $m/(T_1 \cdots T_n)^k$ such that m is a polynomial of the form $x_i^k.m'$ for some polynomial m'. Moreover, it is easy to see that every element that satisfies this condition is in the image of the map d^{n-1}. Therefore, we have:

Theorem 1.5.6. *Let $S = R[T_1 \cdots T_n]$ and $\mathfrak{t} = (T_1, \cdots, T_n)$. We have:*

1. $H^i_{\mathfrak{t}}(S) = 0$ *for* $0 \leq i < n$

2. $H^n_{\mathfrak{t}}(S)$ *has an inverse polynomial structure: it is isomorphic to the S-module generated by monomials in $T_1, \cdots, T_n$ with strictly negative exponents with the following S-module structure:*

$$T_i \cdot (T_1^{\alpha_1} \cdots T_i^{\alpha_i} \cdots T_n^{\alpha_n}) = \begin{cases} T_1^{\alpha_1} \cdots T_i^{\alpha_i+1} \cdots T_n^{\alpha_n}, \ \textit{if } \alpha_i < -1 \\ 0, \ \textit{if } \alpha_i = -1 \end{cases}$$

Chapter 2

Spectral Sequences

The results that we develop in this thesis rely on constructions that involves the use of spectral sequences and maps involved in them, and this fact motivated the exposition of this topic in this chapter. Besides being present in almost every book on homological algebra, we give here a rather complete exposition giving special emphasis on the modules and maps that appear on each page of a spectral sequence. The chapter is organized as follows.

The first section is devoted to set the definitions of double and filtered complexes that we use in the sequel. In Section 2.2 we define the basic objects for the construction of spectral sequences: the exact and the derived couples. In this section we do not provide any proofs, as they are very messy, but give references on where these proofs can be found. We end the section defining what is a spectral sequence for a filtered complex.

Section 2.3 deals with the construction of the vertical and horizontal spectral sequences coming from the respective filtrations on the total complex. We describe completely the modules involved on each page of the respective spectral sequence, and fully describe the differentials appearing on such pages. The principal results in this section are Theorems 2.3.7, 2.3.8 and 2.3.9.

Finally, in Section 2.4 we discuss the key theorem on spectral sequences about convergence (see Theorem 2.4.1): if our double complex has finite diagonals, then the homology of the total complex can be , in some sense, recovered from the limit of the spectral sequence.

2.1 Setting the Stage

We begin the section with the definition of a bi-graded module and the first key definition: what is a double complex and a first quadrant double complex.

Definition 2.1.1. *A bi-graded module over a (ungraded) ring R is a module M with a direct sum decomposition $M = \bigoplus_{(p,q)\in\mathbb{Z}^2} M^{p,q}$. A homomorphism $f : M \to N$ is called a homogeneous homomorphism of bi-degree (a,b) if $f(M^{p,q}) \subset N^{p+a,q+b}$ for all $(p,q) \in \mathbb{Z}^2$.*

Definition 2.1.2. *A double complex is a triple (M, d, d'), where M is a bi-graded module and d and d' are homogeneous homomorphisms of a given bi-degree satisfying*

1. $d^2 = d'^2 = 0$.

2. $d \circ d' + d' \circ d = 0$*, that is, they skew-commute.*

A first quadrant bi-complex is a double complex $E = \bigoplus E^{p,q}$ such that $E^{p,q} = 0$ if $p < 0$ or $q < 0$ with differentials d_v of bi-degree $(0,1)$ and d_h, of bi-degree $(1,0)$.

Next we define the totalization of a double complex. The main goal of the theory of spectral sequences is to be able to calculate (or have a good approximation of) the homology of such complexes.

Definition 2.1.3. *Let* (E, d_v, d_h) *be a first quadrant double complex. The total complex of this bi-complex is the graded module*

$$\operatorname{Tot}(E^{\bullet,\bullet}) = \bigoplus_{k\in\mathbb{N}} \bigoplus_{p+q=k} E^{p,q}$$

with a degree one differential d *given by*

$$d(m) = (d_v(m), d_h(m)) \in E^{p+1,q} \oplus E^{p,q+1} \ \forall \ m \ \in \ E^{p,q}$$

Finally we define filtered complexes, one of the main ingredients of the construction of spectral sequences.

Definition 2.1.4. *Let* $\mathcal{C}^\bullet$ *be a complex. A filtration* $\{F^j\mathcal{C}^\bullet\}$ *of* $\mathcal{C}^\bullet$ *is a family of complexes* $F^j\mathcal{C}^\bullet$ *such that*

1. $F^{j+1}\mathcal{C}^i \subset F^j\mathcal{C}^i \subset \mathcal{C}^i$ *for all* $i, j \in \mathbb{Z}$.

2. *The differential of* $F^j\mathcal{C}^\bullet$ *is the restriction to* $F^j\mathcal{C}^\bullet$ *of the differential of* $\mathcal{C}^\bullet$.

A filtered complex is a complex with a given filtration.

2.2 Filtered Complexes and Exact Couples

Let $\mathcal{C}^\bullet$ be a complex with a given filtration $\{F^j\mathcal{C}^\bullet\}$. Then we can construct exact sequences

$$0 \to F^{p+1}\mathcal{C}^\bullet \xrightarrow{i} F^pC^\bullet \xrightarrow{\pi} F^pC^\bullet/F^{p+1}C^\bullet \to 0.$$

Summing over p we have the exact sequence

$$0 \to \bigoplus_p F^{p+1}\mathcal{C}^\bullet \xrightarrow{i} \bigoplus_p F^pC^\bullet \xrightarrow{\pi} \bigoplus_p F^pC^\bullet/F^{p+1}C^\bullet \to 0,$$

where i is induced by the inclusions $F^{p+1}\mathcal{C}\bullet \hookrightarrow F^p\mathcal{C}^\bullet$ and π is induced by the canonical projections $F^p\mathcal{C}^\bullet \to F^p\mathcal{C}^\bullet/F^{p+1}\mathcal{C}^\bullet$. Set $F = \bigoplus_p F^pC^\bullet$ and $E = \bigoplus_p F^pC^\bullet/F^{p+1}C^\bullet$.

Taking homology yields a long exact sequence, which can be pictured as a triangle

$$\begin{tikzcd}
H(F) \arrow[rr, "i^*"] & & H(F) \arrow[dl, "\pi^*"] \\
& H(E) \arrow[ul, "\delta"] &
\end{tikzcd}$$

The above structure appears repeatedly in the construction of spectral sequences that we want. We therefore give a general definition.

Definition 2.2.1. *An exact couple (or exact triangle) is a quintuple* $(M, N, \alpha, \beta, \gamma)$*, where*

1. M *and* N *are* R*-modules.*
2. $\alpha : M \to M$, $\beta : M \to N$, $\gamma : N \to M$ *are homomorphisms.*
3. *The triangle is exact in the following sense:* $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$, $\mathrm{Im}(\beta) = \mathrm{Ker}(\gamma)$ *and* $\mathrm{Im}(\gamma) = \mathrm{Ker}(\alpha)$.

We write an exact couple in the form of a triangle

$$\begin{tikzcd}
M \arrow[rr, "\alpha"] & & M \arrow[dl, "\beta"] \\
& N \arrow[ul, "\gamma"] &
\end{tikzcd}$$

Let $(M, N, \alpha, \beta, \gamma)$ be an exact couple. It's immediate to see that $d = \gamma \circ \beta$ satisfies $d^2 = 0$. The following proposition shows that we can, given an exact couple, construct a new one.

Proposition 2.2.2. *Let* $(M, N, \alpha, \beta, \gamma)$ *be an exact couple. Then*

$$d = \beta \circ \gamma : N \to N$$

satisfies $d^2 = 0$. *Moreover, the quintuple* $(M^{(1)}, N^{(1)}, \alpha^{(1)}, \beta^{(1)}, \gamma^{(1)})$ *is an exact couple, and is is called the first derived exact couple of the exact couple* $(M, N, \alpha, \beta, \gamma)$, *where*

1. $M^{(1)} = \alpha(M)$.

2. $N^{(1)} = H(N)$, *the homology of* (N, d).

3. $\alpha^{(1)}$ *is the restriction of* α *to* $M^{(1)}$.

4. $\beta^{(1)}(z)$ *is the class of* $\beta(\alpha^{-1}(z))$ *in* $N^{(1)}$, *where* $\alpha^{-1}(z)$ *is an element a with* $\alpha(a) = z$.

5. $\gamma^{(1)}$ *is the restriction of* γ *to* $\mathrm{Ker}(d)$.

Proof. See [Rot, Proposition 10.9]. □

The process of taking derived couples of an exact couple can be iterated:

Definition 2.2.3. *Let* $(M, N, \alpha, \beta, \gamma)$ *be an exact couple. The associated r-th derived couple, denoted by* $(M^{(r)}, N^{(r)}, \alpha^{(r)}, \beta^{(r)}, \gamma^{(r)})$, *is the first derived couple of the (r-1)-th derived couple* $(M^{(r-1)}, N^{(r-1)}, \alpha^{(r-1)}, \beta^{(r-1)}, \gamma^{(r-1)})$. *Moreover, the maps* $\alpha^{(r)}, \beta^{(r)}, \gamma^{(r)}$ *acts as follows:*

1. $\alpha^{(r)}$ *is just the restriction of* α *to* $\alpha^{(r)}(M)$

2. $\beta^{(r)}(z)$ *is the class of* $\beta(\alpha^{-r}(z))$ *in* $N^{(r)}$, *where* $\alpha^{-r}(z)$ *is an element a with* $\alpha^r(a) = z$

3. $\gamma^{(r)}$ *is the restriction of* γ *to* $\mathrm{Ker}(d^{(r-1)})$.

In each iteration of taking derived couples, we get a new module $N^{(n)}$ and a differential $d_n : N^{(n)} \to N^{(n)}$. Moreover we have that $N^{(n)} = H(N^{(n-1)})$. This is exactly the content of a spectral sequence, that we define generally below.

Definition 2.2.4. *A spectral sequence is a sequence* (E^n, d_n), *where the* E^n *are modules,* $d_n : E^n \to E^n$ *satisfies* $d_n^2 = 0$ *and* $E^n = H(E^{n-1})$.

2.3 The Spectral Sequences of a double complex

Let $E^{\bullet,\bullet}$ be a first quadrant double complex and $\mathrm{Tot}(E^{\bullet,\bullet})$ its totalization, as in Definition 2.1.3. There are two simple filtrations of the complex $\mathrm{Tot}(E^{\bullet,\bullet})$, the vertical and the horizontal one, that we define below.

Definition 2.3.1. *Let $E^{\bullet,\bullet}$ be a first quadrant double complex. We define the p-th vertical filtration of* $\mathrm{Tot}(E^{\bullet,\bullet})$ *as*

$$_{ver}\,\mathrm{Tot}(E^{\bullet,\bullet})^p = \bigoplus_k \bigoplus_{i\geq p} E^{i,k-i}$$

Similarly, we can define the p-th horizontal filtration as

$$_{hor}\,\mathrm{Tot}(E^{\bullet,\bullet})^p = \bigoplus_k \bigoplus_{i\geq p} E^{k-i,i}.$$

Both of these filtrations are filtrations of complexes as in Definition 2.1.4 and we denote by $(\mathrm{Tot}(E^{\bullet,\bullet})^p)^k$ *the k-th level of the complex* $(\mathrm{Tot}(E^{\bullet,\bullet})^p)$.

In the remaining of this chapter we will work out the spectral sequence of a first quadrant double complex $E^{\bullet,\bullet}$ arising from the vertical filtration of the totalization $\mathrm{Tot}(E^{\bullet,\bullet})$ and to clarify the notation, we omit the subscript *ver* from the filtration. As a first step, let

$$F = \bigoplus_p (\mathrm{Tot}(E^{\bullet,\bullet})^p)^\bullet$$

and

$$^0E = \bigoplus_p \frac{(\mathrm{Tot}(E^{\bullet,\bullet})^p)}{(\mathrm{Tot}(E^{\bullet,\bullet})^{p+1})}.$$

These modules are bi-graded, with grading given by bi-degrees (p, q) such that

$$F^{p,q} = (\mathrm{Tot}(E^{\bullet,\bullet})^p)^{p+q}$$

and

$$^0E^{p,q} = \bigoplus_p \frac{(\mathrm{Tot}(E^{\bullet,\bullet})^p)^{p+q}}{(\mathrm{Tot}(E^{\bullet,\bullet})^{p+1})^{p+q}} \simeq E^{p,q}$$

The differential d of the total complex induces differentials on E and F of bi-degree $(0,1)$. Moreover it is not hard to see that for any fixed p the differential induced in ${}^0E^{p,\bullet}$ is just the vertical differential d_v. As in the previous section, we have the exact sequence

$$0 \to \bigoplus_p \operatorname{Tot}(E^{\bullet,\bullet})^{p+1} \xrightarrow{i} \bigoplus_p \operatorname{Tot}(E^{\bullet,\bullet})^{p} \xrightarrow{\pi} \bigoplus_p \operatorname{Tot}(E^{\bullet,\bullet})^{p} / \operatorname{Tot}(E^{\bullet,\bullet})^{p+1} \to 0,$$

where the inclusion i is a map of bi-degree $(-1,1)$ and the projection map π has bi-degree $(0,0)$. As in the previous section, we can construct an exact couple $(H(F), H(E), i^*, \pi^*, \delta)$. Both $H(F)$ and $H(E)$ are again bi-graded by (p,q). Let $k = p+q$ be the total degree. We have

1. $H(F)^{p,q} = H^k(\operatorname{Tot}(E^{\bullet,\bullet})^p)$.

2. $H(E)^{p,q} = H^q(E^{p,\bullet})$.

3. The map i^* takes the cohomology class $(a_{p+1}, \ldots, a_k) \in H^k(\operatorname{Tot}(E^{\bullet,\bullet})^{p+1})$ to the cohomology class $(0, a_{p+1}, \ldots, a_k) \in H^k(\operatorname{Tot}(E^{\bullet,\bullet})^p)$ and has bi-degree $(-1,1)$.

4. The map π^* takes a cohomology class $(a_p, a_{p+1}, \ldots, a_k) \in H^k(\operatorname{Tot}(E^{\bullet,\bullet})^p)$ to the cohomology class of $a_p \in H^q(E^{p,\bullet})$. It has bi-degree (0,0).

5. The map δ resembles the connecting map of the snake lemma: takes an element $m \in H^q(E^{p,\bullet})$ to the element $(d_h(m), 0, \ldots, 0) \in H^{k+1}(\operatorname{Tot}(E^{\bullet,\bullet})^p)$ and has bi-degree $(0,1)$.

Therefore we can use the recipe in Proposition 2.2.2 to construct the first derived couple: put $F^{(1)} = H(F)$, ${}^1E = H(E)$ and $d_1 = \pi \circ \delta$ and iterate this process to get exact couples $(F^{(r)}, {}^rE, i^{(r)}, \pi^{(r)}, \delta^{(r)})$, and differentials $d_r = \pi^{(r)} \circ \delta^{(r)}$ on ${}^rE$.

Definition 2.3.2. *The spectral sequence* $({}^{r}E, d_r)$ *constructed above is the spectral sequence associated to the vertical filtration.*

Remark 2.3.3. The horizontal filtration gives another spectral sequence. Therefore, is common to refer to the vertical spectral sequence as ${}^{r}E_{vert}$ and, similarly, we use ${}^{r}E_{hor}$ for the horizontal one. As we have fixed the vertical filtration for the calculations, we do not use the subscript "vert" for the spectral sequences. All the results of this section can be translated easily to the case of horizontal spectral sequence.

For any bi-degree (p, q), define ${}^{0}Z^{p,q} = \mathrm{Ker}(d_0)^{p,q}$ and ${}^{0}B^{p,q} = \mathrm{Im}(d_0)^{p,q}$. Therefore, ${}^{1}E^{p,q} \simeq {}^{0}Z^{p,q}/ {}^{0}B^{p,q}$. Now every submodule of ${}^{1}E^{p,q}$ is of the form $M/ {}^{0}B^{p,q}$ for some submodule $M \supset {}^{0}B^{p,q}$. So there are submodules ${}^{1}Z^{p,q}$, ${}^{1}B^{p,q}$ such that ${}^{1}Z^{p,q}/ {}^{0}B^{p,q} \simeq \mathrm{Ker}(d_1)$ and ${}^{1}B^{p,q}/ {}^{0}B^{p,q} \simeq \mathrm{Im}(d_1)$. So we have a chain

$$ {}^{0}B^{p,q} \subset {}^{1}B^{p,q} \subset {}^{1}Z^{p,q} \subset {}^{0}Z^{p,q} $$

such that ${}^{1}Z^{p,q}/ {}^{1}B^{p,q} \simeq {}^{2}E^{p,q}$. Continuing this way, we reach at a chain of modules

$$ {}^{0}B^{p,q} \subset {}^{1}B^{p,q} \subset {}^{2}B^{p,q} \subset \cdots \subset {}^{2}Z^{p,q} \subset {}^{1}Z^{p,q} \subset {}^{0}Z^{p,q} $$

such that ${}^{r}Z^{p,q}/ {}^{r}B^{p,q} \simeq {}^{r+1}E^{p,q}$.

Definition 2.3.4. *The limit term of the vertical spectral sequence with bi-degree* (p, q) *is the module* ${}^{\infty}E^{p,q} = {}^{\infty}Z^{p,q}/{}^{\infty}B^{p,q}$, *where* ${}^{\infty}Z^{p,q} = \bigcap_{r=1}^{\infty} {}^{r}E^{p,q}$ *and* ${}^{\infty}B^{p,q} = \bigcup_{r=1}^{\infty} {}^{r}B^{p,q}$.

We are now ready to head towards the main goals of the section: describe completely the modules ${}^{r}Z^{p,q}$ and ${}^{r}B^{p,q}$ and describe completely the differentials d_r on each step of the iteration process. The following notion is key to this description.

Definition 2.3.5. *Let $m \in E^{p,q}$. We say that m is an r-liftable element, for some integer r, if $d_v(m) = 0$ and there is a sequence of elements $(m, a_1, \cdots, a_r)$ such that*

1. $d_h(m) = d_v(a_1)$
2. $d_h(a_i) = d_v(a_{i+1})$ *for every* $0 \leq i \leq r-1$.

The element a_r is called an r-th lift of m and the sequence $(m, a_1, \cdots, a_r)$ is called an r-lift sequence of m.

We remark that it is natural to consider this type of elements, since they are closely related to the cohomology classes of $\mathrm{Tot}(E^{\bullet,\bullet})$, as we indicate on the following remark.

Remark 2.3.6. Let $m \in E^{p,q}$, $k = p+q$ and $(m, a_1, \ldots, a_r)$ be an r-lift sequence with $d_h(a_r) = 0$. Then the element $(\mathbf{0}, m, -a_1, a_2, \ldots, (-1)^r a_r, \mathbf{0})$ gives a homology class in $H^k(\mathrm{Tot}(M^{\bullet,\bullet}))$. Conversely, if $(\mathbf{0}, m, a_1,$ and $\ldots, a_r, \mathbf{0})$ gives a homology class in $H^k(\mathrm{Tot}(M^{\bullet,\bullet}))$, then

$$d_h(m) = 0, d_v(a_1) = -d_h(m), d_v(a_{i+1}) = -d_h(a_i).$$

Therefore $(m, -a_1, a_2, \ldots, (-1)^r a_r)$ is an r-lift sequence for m.

We present now our first main theorem of this section. It shows that r-liftable elements belongs to ${}^rZ$ and tell us how to calculate d^{r+1} on such elements.

Theorem 2.3.7. *Let $m \in E^{p,q}$ be an r-liftable element, and $(m, a_1, a_2, ..., a_r)$ an r-lift sequence. Then $m \in {}^iZ^{p,q}$ for all $0 \leq i \leq r$ and $d^{r+1}(m)$ is the class of $d_h(a_r)$ in ${}^{r+1}E^{p+r+1,q-r}$.*

Proof. We need to follow the rules of definition of the spectral sequence by the exact couples. We've already seen that d^0 is given by d_v and that d^1 is induced by d_h in the vertical homologies.

To clarify the methods of the proof, we start with the case $r = 1$. Suppose that m is 1-liftable and let (m, a) be a 1-lift sequence. Considering the map δ of the derived couple (see the previous section)

$$\delta(m) = (d_h(m), 0, \ldots, 0) \in H^{k+1}(\mathrm{Tot}(E^{\bullet,\bullet})). \tag{2.1}$$

As (m, a) is a 1-lift sequence, we have $d_h(m) = -d_v(a)$. Moreover $(-d_v(a), 0, \ldots, 0)$ and $(0, d_h(a), \ldots, 0)$ are cohomologous in $H^{k+1}(\mathrm{Tot}(E^{\bullet,\bullet}))$. To calculate $d^1(m)$, we have to project $(0, d_h(a), \ldots, 0)$ onto the first coordinate. Therefore $m \in \ker d^1$. To calculate $d^2(m)$ we must calculate $\pi^{(1)}(0, d_h(a), \cdots, 0)$. By definition, we must take first the preimage by i^* once and then project onto the first coordinate. As

$$(i^*)^{-1}((0, d_h(a), \cdots, 0)) = (d_h(a), \cdots, 0), \tag{2.2}$$

$d^2(m)$ is the class of $d_h(a)$ in ${}^1E^{p+2,q-1}$.

Suppose now that r is arbitrary and m is r-liftable. Let $(m, a_1, \ldots, a_r)$ be an r-lift sequence. Again, applying δ to m, we have the same equation as in (2.1). Hence

$$(d_h(m), 0, \ldots, 0) \sim (-d_v(a_1), 0, \ldots, 0) \sim (0, d_h(a_1), 0, \ldots, 0) \sim (0, -d_v(a_2), 0, \ldots, 0) \sim$$

$$(0, 0, d_h(a_2), 0, \ldots, 0) \sim \cdots \sim (\overbrace{0, \ldots, 0}^{r}, d_h(a_r), 0, \ldots, 0), \tag{2.3}$$

where $\sim$ means cohomologous. To apply d^i, we must take $(i-1)$ times the preimage by i^* and then project onto the first coordinate. Thus, if $i \leq r$, then $m \in \ker d^i$. To calculate $d^{r+1}(m)$, we must take r times the preimage of m by i^*. As

$$(i^*)^r(\overbrace{0, \ldots, 0}^{r}, d_h(a_r), 0, \ldots, 0)) = (d_h(a_r), 0, \ldots, 0). \tag{2.4}$$

projecting on the first coordinate yields the result. $\square$

The following theorem shows that the converse of the above theorem also holds. Therefore we have described completely the modules ${}^rZ^{p,q}$ and the differentials d^r.

Theorem 2.3.8. *Let $m \in E^{p,q}$. If $m \in {}^rZ^{p,q}$, then m is r-liftable.*

Proof. For $r = 1$ the proof is easy: $d^1(m) = d_h(m) = 0 \in {}^1E^{p+1,q} = H^q(M^{p+1,\bullet})$ implies $d_h(m) = d_v(a)$ for some $a \in E^{p+1,q-1}$.

To illustrate the method of the proof, we do the case $r = 2$. Let $m \in {}^2Z^{p,q}$. In particular, $m \in {}^1Z^{p,q}$ which implies that m is 1-liftable. Let (m, a) be a 1 -lift sequence. As $d^2(m) \in {}^1B^{p+2,q-1}$, there is a' with $d_v(a') = 0$ such that $d_h(a) = d_h(a')$ in ${}^1E$. By the case $r = 1$, there is $a'' \in E^{p+2,q-2}$ with $d_v(a'') = d_h(a - a')$. Now, $(m, a - a', a'')$ is a 2-lift sequence for m.

For the general case, let $m \in {}^rZ^{p,q}$. Then, by induction, m is $(r-1)$-liftable. Let $(m, a_1, \cdots, a_{r-1})$ be an $(r-1)$-lift sequence. Then by Proposition 2.3.7, $d_h(a_{r-1}) = d^r(m) = 0$ in ${}^rE$. Therefore, there is $a_{r-1}^{(1)} \in {}^{r-1}Z^{p,q}$ such that $d_h(a_{r-1}) = d^{r-1}(a_1^{(1)})$ in ${}^{r-1}E$. Again by induction $a_1^{(1)}$ is $(r-2)$- liftable. Take an $(r-2)$-lift sequence

$$(0, a_1^{(1)}, a_2^{(1)}, \ldots, a_{r-1}^{(1)}) \tag{2.5}$$

such that $d_h(a_{r-1} - a_{r-1}^{(1)}) = 0$ in ${}^{r-1}E$. It is immediate to see that

$$(m, a_1 - a_1^{(1)}, a_2 - a_2^{(1)}, \ldots, a_{r-1} - a_{r-1}^{(1)}) \tag{2.6}$$

is an $(r-1)-$lift sequence. Now we may construct inductively, for each $1 \leq i \leq r-1$, $(r-1-i)$-lift sequences

$$(0, \ldots, 0, a_i^{(i)}, \ldots, a_{r-1}^{(i)}) \tag{2.7}$$

such that $d_h(a_{r-1} - a_1^{(i)} - \cdots - a_{r-1}^{(i)}) = 0$ in ${}^{r-i}E$. If $i = r-1$, we have that $d_h(a_{r-1} - \sum_{i=1}^{r-1} a_{r-1}^{(i)}) = 0$ in ${}^1E$, that is, there is a_r such that

$$d_h(a_{r-1} - \sum_{i=1}^{r-1} a_{r-1}^{(i)}) = d_v(a_r). \tag{2.8}$$

It follows that

$$(m, a_1 - a_1^{(1)}, a_2 - \sum_{i=1}^{2} a_2^{(i)}, \ldots, a_{r-1} - \sum_{i=1}^{r-1} a_{r-1}^{(i)}, a_r) \tag{2.9}$$

is an r-lift sequence. $\square$

The same method used in the proof of Theorem 2.3.7 gives us the characterization of the modules ${}^{r}B^{p,q}$.

Theorem 2.3.9. *Let $m \in M^{p,q}$. Then $m \in {}^{r}B^{p,q}$ if and only if there is an $r-1$-lift sequence $(a_1, ..., a_{r-1})$ and an element a_r such that $m = d_h(a_{r-1}) + d_v(a_r)$.*

Proof. As $m \in {}^{r}B^{p,q}$ there is an $(r-1)$-lift sequence $(b_1, \ldots, b_{r-1})$ such that $m = d_h(b_{r-1})$ in ${}^{r-1}E$. The proof now goes by the same lines as in the previous theorem. $\square$

2.4 Convergence

Recall that our main purpose on developing the machinery of spectral sequences is to calculate the homology of the total complex $\mathrm{Tot}(E^{\bullet,\bullet})$, and this is attained by the convergence theorem. Consider the total complex $\mathrm{Tot}(E^{\bullet,\bullet})$ of a first quadrant double complex with the vertical filtration as in Definition 2.3.1. The inclusions

$$i : \mathrm{Tot}(E^{\bullet,\bullet})^p \to \mathrm{Tot}(E^{\bullet,\bullet}) \tag{2.10}$$

induce maps on homology

$$i^* : H^k(\mathrm{Tot}(E^{\bullet,\bullet})^p) \to H^k(\mathrm{Tot}(E^{\bullet,\bullet})) \tag{2.11}$$

and these maps induce a filtration

$$H^k(\mathrm{Tot}(E^{\bullet,\bullet})) \supset i^*(H^k(\mathrm{Tot}(E^{\bullet,\bullet})^1)) \supset \cdots \supset i^*(H^k(\mathrm{Tot}(E^{\bullet,\bullet})^p)) \supset \cdots \tag{2.12}$$

of the homology of the totalization. The convergence theorem asserts that quotients of this filtration in fact the limit terms of the spectral sequence.

Theorem 2.4.1. *Let $E^{\bullet,\bullet}$ be a first quadrant double complex with the vertical filtration. Then the map*

$$\begin{array}{rcl}\varphi_{p,q} : i^*(H^k(\mathrm{Tot}(M^{\bullet,\bullet})^p))/i^*(H^k((\mathrm{Tot}(M^{\bullet,\bullet})^{p+1})\ ) & \to & {}^{\infty}E^{p,q} \\ (m, a_{p+1}, ..., a_k) & \to & \bar{m}\end{array},$$

where $q = k - p$ is well defined and is an isomorphism.

Proof. Remark 2.3.6 says that any cohomology class $(\mathbf{0}, m, a_1, ..., a_k, \mathbf{0})$ induces a lift sequence for m of any length that we want, and therefore $m \in {}^{\infty}Z^{p,q}$. As $E^{p,q}$ is a first quadrant bi-complex, for any lift sequence $(m, a_1, \cdots, a_r, \cdots)$ we have $a_i = 0$ for $i \gg 0$. On the other hand, Theorem 2.3.8 says that any element in ${}^{\infty}Z^{p,q}$ is r-liftable for every $r \in \mathbb{N}$. As any sufficiently large lift sequence of m must become zero, $m \in {}^{\infty}E^{p,q}$ induces a cohomology class in $i^*(\mathrm{Tot}(M^{\bullet,\bullet})^p)$. So if we prove that $\varphi_{p,q}$ is well defined, then it is certainly surjective. Consider first the map

$$\begin{array}{rcl}\Phi_{p,q} : i^*(H^k(\mathrm{Tot}(M^{\bullet,\bullet})^p)) & \to & {}^{\infty}E^{p,q} \\ (m, a_1, ..., a_k) & \to & \bar{m}\end{array}.$$

To show that $\Phi_{p,q}$ is well defined, let $(m, a_{p+1}, ..., a_k) = 0 \in i^*(H^k(\mathrm{Tot}(M^{\bullet,\bullet})^p))$. This means that there is $(b_1, ..., b_{k-1}) \in \mathrm{Tot}(M^{\bullet,\bullet})^{k-1}$ such that $d(b_0, ..., b_{k-1}) = (m, a_{p+1}, ..., a_k)$. We have

1. $d_v(b_0) = 0$.
2. $d_h(b_i) = -d_v(b_{i+1})$.
3. $m = d_h(b_{p-1}) + d_v(b_p)$.

Now, b_{p-1} is, up to a sign, a $(p-1)$-lift of b_0 and by Theorem 2.3.9 $m \in {}^{\infty}B^{p,q}$. This shows that $\Phi_{p,q}$ is well defined. It now remains to show that $\operatorname{Ker}\Phi = i^*(H^k(\mathrm{Tot}(M^{\bullet,\bullet})^{p+1}))$.

Suppose that $(m, a_1, ..., a_k) \in i^*(H^k(\mathrm{Tot}(M^{\bullet,\bullet})^p))$ with $\bar{m} = 0$ in ${}^{\infty}E^{p,q}$. By Theorem 2.3.9 there is an element b_0 and a lift sequence $(b_0, b_1, ..., b_{p-1})$ with $m = d_h(b_{p-1}) + d_v(a)$ for some a. Consider the element

$$\Gamma = ((-1)^r b_0, (-1)^{r-1} b_1, \cdots, b_r, a). \tag{2.13}$$

The cohomology class of $d(\Gamma) = (0, ..., m, d_h(a))$ is trivial. Therefore $(m, a_{p+1}, ..., a_k)$ is cohomologous to $(0, a_{p+1} - d_h(a), ..., a_k) \in i^*(H^k(\mathrm{Tot}(M^{\bullet,\bullet})^{p+1}))$. The reverse inclusion is obvious, and the theorem is proved $\square$

2.5 Comments and References

Our approach to spectral sequences follows closely the notation of [Eis, Appendix A 3.13] and in fact the work of this chapter is motivated by the phrase "and so on" of his explanation of the differentials d^r.

The degrees of the horizontal and vertical differentials of the double complex in definition 2.1.3 were chosen for the sake of simplicity. One needs only to adapt the notion of total complex and the statements of the results of Section 2.3. In this new setting they are still valid.

For the convergence theorem, the requirement of $E^{\bullet,\bullet}$ being a first quadrant double complex were again chosen for the sake of simplicity, as the proof of the convergence theorem only requires that, for each fixed total degree k, $E^{\bullet,\bullet}$ has finitely many nonzero components (we say that $E^{\bullet,\bullet}$ has finite diagonals in this case). Even if it is not the case, the spectral sequence can still provide some information about the quotients of the filtrations, as a careful analysis of the proof shows that the map $\Phi_{p,q}$ is always injective.

Other expositions and applications of spectral sequences can be found in the books [Rot], [W] and [Mc].

Chapter 3

Buchsbaum-Eisenbud complexes and the Koszul-Čech spectral sequence

In this chapter we present our first result in this thesis: we prove that the the Buchsbaum-Eisenbud family complexes defined in [Eis, A2.6] can be constructed via the spectral sequences of a Koszul-Čech double complex. The chapter is organized as follows.

In Section 3.1 we recall the construction of the Buchsbaum-Eisenbud family of complexes of a linear map $\Phi : R^f \to R^g$. The material of this section is essentially taken from [Eis].

Section 3.2 is about the Koszul-Čech spectral sequences. We use these spectral sequences to construct another family of complexes, that we call Koszul-Çech complexes associated to a linear map of free modules as in Section 3.1.

Finally, the work of the previous chapter in the maps appearing in the pages of spectral sequences rewards us with the proof that this new family of complexes is isomorphic to the Buchsbaum-Eisenbud family in Section 3.3.

3.1 The Buchsbaum-Eisenbud Family of Complexes

We begin this section with a notation.

Notation 3.1.1. *Let R be a ring and M an R-module. We denote by M^* the dual* $\operatorname{Hom}_R(M, R)$.

Proposition 3.1.2. *Let $G \simeq R^g$ be a free module. Then G^* acts on $\bigwedge G$ in the following way: if $\varphi \in G^*$ and $I \subset \{1, \dots, g\}$*

$$\varphi \cdot (e_I) = \sum_{j \in I} \operatorname{sgn}(\{j\} \subset I)\varphi(e_j)e_{I \setminus \{j\}}.$$

This action satisfies $\psi \cdot (\varphi \cdot (e_I)) = -\varphi \cdot (\psi \cdot (e_I))$, and therefore it extends to an action of $\bigwedge G^*$ on $\bigwedge G$ given by

$$(\varphi_1 \wedge \dots \wedge \varphi_n) \cdot e_I = \varphi_1 \cdots \varphi_n \cdot e_I.$$

Proof. For any $I \subset \{1, \dots, g\}$, we have

$$\psi \cdot \varphi \cdot e_I = \sum_{j \in I} \sum_{l \in I \setminus \{j\}} \operatorname{sgn}(\{j\} \subset I) \operatorname{sgn}(\{l\} \subset I \setminus \{j\}) \psi(e_l)\varphi(e_j)e_{I \setminus \{j,k\}}$$

Keeping the notation, we have

$$\varphi \cdot \psi \cdot e_I = \sum_{l \in I} \sum_{j \in I \setminus \{l\}} \operatorname{sgn}(\{l\} \subset I) \operatorname{sgn}(\{j\} \subset I \setminus \{l\}) \psi(e_l)\varphi(e_j)e_{I \setminus \{j,k\}}.$$

The signs on the expression are opposite by Lemma 1.1.3, and the proposition is proved. □

We now prepare the terrain for the construction of the Buchsbaum-Eisenbud complexes. Let $\varphi : F = R^f \to G = R^g$, $f \geq g$, be a linear map presented by a matrix $\Phi = [a_{ij}]$. Let $K = (\bigwedge F \otimes_R S, \partial)$, where $S = \operatorname{Sym}(G)$, be the generalized Koszul complex of the map φ (see Definition 1.4.1). Recall that this complex is just the Koszul complex on the linear forms $\gamma_j = \sum_{i=1}^{r} a_{ij}T_i$. As this complex is S-graded one can look at the degree d strand

$$K_{(d)}(\varphi) := 0 \to \bigwedge^{f} F \otimes_R S_{(d-f)} \to \cdots \to F \otimes_R S_{(d-1)} \to S_{(d)} \to 0. \tag{3.1}$$

The action of the Koszul differential can be reinterpreted in terms of the action described in Proposition 3.1.2: the module $\bigwedge G^*$ acts on $\bigwedge F$ via pullback by defining

$$\eta \cdot v = \varphi^*(\eta) \cdot v = (\eta \circ \varphi) \cdot v, \text{ if } \eta \in \bigwedge G^* \text{ and } v \in \bigwedge F. \tag{3.2}$$

Moreover, the module G acts naturally on $S = \mathrm{Sym}(G)$ via the multiplication of S. Therefore the module $G^* \otimes_R G$ acts on $\bigwedge F \otimes_R S$, and the action is given by

$$(\eta \otimes u) \cdot (v \otimes w) = \eta \cdot v \otimes uw. \tag{3.3}$$

for $\eta \in G^*, u \in G, v \in \bigwedge F$ and $w \in S$. Let now $T'_1, \dots, T'_g$ be a basis of G^* that is dual to the basis $T_1, \dots, T_g$ and consider the element $c = \sum_{i=1}^{g} T'_i \otimes T_i \in G^* \otimes G$. This element does not depend on the choice of a particular basis of G, as c is the pullback of 1 by the evaluation map $G \otimes_R G^* \to R$. This element acts on $\bigwedge F \otimes_R S$ as explained in (3.3), and the next proposition shows that c acts like the Koszul differential.

Proposition 3.1.3. *The Koszul differential ∂ in $\bigwedge F \otimes_R S$ is given by the multiplication by $c = \sum_{i=1}^{g} T'_i \otimes T_i$, via the action described in (3.3).*

Proof. The element $c \in G^* \otimes_R G$ pulls back to the element $c' = \sum_{i=1}^{g} T'_i \circ \varphi \otimes T_i$. Proposition 3.1.2 gives us

$$c \cdot (e_I \otimes f) = \sum_{i=1}^{g} (T'_i \circ \varphi(e_I) \otimes T_i f) = \sum_{i=1}^{g} (T'_i (\sum_{j \in I} (\mathrm{sgn}(\{j\} \subset I) \varphi(e_j) e_{I \setminus \{j\}}) \otimes T_i f) =$$

$$\sum_{i=1}^{g}(T_i'(\sum_{j\in I}(\mathrm{sgn}(\{j\}\subset I)(\sum_{k=1}^{g}c_{kj}T_k)e_{I\setminus\{j\}})\otimes T_if)=\sum_{i=1}^{g}(\sum_{j\in I}(\mathrm{sgn}(\{j\}\subset I)c_{ij}e_{I\setminus\{j\}})\otimes T_if)=$$

$$\sum_{j\in I}(e_{I\setminus\{j\}}\otimes\mathrm{sgn}(\{j\}\subset I)\sum_{i=1}^{g}c_{ij}T_if)=\sum_{j\in I}(e_{I\setminus\{j\}}\otimes\mathrm{sgn}(\{j\}\subset I)\gamma_jf)=\partial(e_I\otimes f).$$

□

Definition 3.1.4. *Let* $S = R[T_1, \ldots, T_g] = \bigoplus_{d\geq 0} S_{(d)}$. *Then the dual* $S^* = \mathrm{Hom}_R(S, R) = \bigoplus_{d\geq 0}(S_{(d)})^*$ *is a graded* S*-module. We write* $(T_1^{\alpha_1}...T_g^{\alpha_g})'$ *to be the basis element of* $(S_{(d)})^*$ *that is dual to* $(T_1^{\alpha_1}...T_g^{\alpha_g})$.

The S-module structure of S^* can be easily described in terms of the dual basis. Recall that or $f \in S,\ \psi \in S^*$, we have that

$$f\cdot\psi(x)=\psi(f\cdot x)$$

for all $x \in S$. Under this module structure, it is not hard to see that

$$T_i\cdot(T_1^{\alpha_1}\cdots T_g^{\alpha_g})'=(T_1^{\alpha_1}\cdots T_i^{\alpha_i-1}\cdots T_g^{\alpha_g})'$$

Consider now the dualization $K_{(d)}(\varphi)^* = \mathrm{Hom}(K_{(d)}(\varphi), R)$:

$$0\to(S_{(d)})^*\to\cdots\to\bigwedge^{i-1}F^*\otimes_R(S_{(d-i+1)})^*\to\bigwedge^{i}F^*\otimes_R(S_{(d-i)})^*\to\cdots\to\bigwedge^{f}F^*\otimes_R(S_{(d-f)})^*\to 0. \tag{3.4}$$

Using the duality property of the exterior powers (see Proposition 1.2.3), this complex is isomorphic to the complex

$$0\to\bigwedge^{f}F\otimes_R(S_{(d)}))^*\to\bigwedge^{f-1}F\otimes_R(S_{(d-1)})^*\to\cdots\to(S_{(d-f)})^*\to 0 \tag{3.5}$$

and it's easy to see that under this isomorphism, the complex (3.5) is a strand of the Koszul complex on the sequence $(T_1, \ldots, T_r)$ with coefficients in S^* and we have the following proposition, whose proof goes on the same lines as that of Proposition 3.1.3.

Proposition 3.1.5. *The S-module structure of S^* defines an action of $G^* \otimes_R G$ on $\bigwedge F \otimes_R S^*$, analogous to the action defined in (3.3), and the differential of the complex (3.5) is given by the multiplication by $c = \sum_{i=1}^{g} T_i' \otimes T_i$.*

After all this work we are ready to define the Buchsbaum-Eisenbud family of complexes. Let $d \leq f - g$. Then the complex $(K(\varphi)_{(d)})^*$ ends at $\bigwedge^{f-d} F \otimes_R (S_{(0)})^* \simeq \bigwedge^{f-d} F$, and the complex $K(\varphi)_{f-g-d}$ starts at $\bigwedge^{f-g-d} F \otimes_R S_0 \simeq \bigwedge^{f-g-d} F$. The goal now is to construct a connecting map

$$\varepsilon_{f-g-d} : \bigwedge^{f-d} F \to \bigwedge^{f-g-d} F$$

such that

$$0 \to \bigwedge^{f} F \otimes_R (S_{(d)})^* \to \cdots \bigwedge^{f-d} F \xrightarrow{\varepsilon_{f-g-d}} \bigwedge^{f-g-d} F \to \cdots \to S_{(d)} \to 0 \tag{3.6}$$

becomes a complex too.

To do this, recall that the pullback $\varphi^* : G^* \to F^*$ induces a map $\bigwedge^{g} \varphi^* : \bigwedge^{g} G^* \to \bigwedge^{g} F^*$. The element $\alpha = \bigwedge^{g} \varphi^*(T_1' \wedge \cdots \wedge T_g') \in \bigwedge F^*$ acts on $\bigwedge F$ by Proposition 3.1.2. Define

$$\begin{array}{rcl} \varepsilon_d : \bigwedge^{f-d} F & \to & \bigwedge^{f-d-g} F \\ v & \to & \alpha \cdot v \end{array} \tag{3.7}$$

The following proposition follows directly from Proposition 3.1.2 and the properties of exterior product given in Proposition 1.2.2. Recall the matrix notation in Definition 1.1.7.

Proposition 3.1.6. *The map $\varepsilon_{f-g-d} : \bigwedge^{f-d} F \to \bigwedge^{f-d-g} F$ takes the element e_I, $I \subset \{1, \ldots, f\}$, to the element*

$$\sum_{J \subset I, |J| = g} \det\Phi^J . \operatorname{sgn}(J \subset I) e_{I \setminus J}.$$

The following theorem can be proven directly (see [Eis, Theorem A2.10(a)]), but we give a different proof in Section 3.3.

Theorem 3.1.7. *The map ε given in Proposition 3.1.6 turns (3.6) into a complex.*

Definition 3.1.8. *We denote by $\mathcal{C}^d(\Phi)$ the complex obtained by joining $(K(\varphi)_{(f-g-d)})^*$ and $K(\varphi)_d$ via the map ε_d. This family of complexes is called the Buchsbaum-Eisenbud family of complexes. The complex $\mathcal{C}^0(\Phi)$ is called the Eagon-Northcott complex, and the complex $\mathcal{C}^1(\Phi)$ is the Buchsbaum-Rim complex.*

3.2 The Koszul-Čech spectral sequence

Recall the notation of the previous section: $\varphi : F = R^f \to R^g$ is a linear map presented by a matrix Φ and such that $f \geq g$. Let $S = R[T_1, \cdots, T_g]$, $\mathfrak{t} = (T_1, \cdots, T_g)$, $\underline{\gamma}$ be the sequence $(\gamma_1, \cdots, \gamma_s)$, where

$$\gamma_j = \sum_{i=1}^{g} a_{ij} T_i.$$

We consider the double complex $K_\bullet \otimes_S \check{C}^\bullet$, where $K_\bullet = K_\bullet(\underline{\gamma}; S)$ is the generalized Koszul complex of Φ (see Definition 1.4.1) and $\check{C}^\bullet = \check{C}^\bullet_{\mathfrak{t}}(S)$ is the Čech complex. We display this double complex as a third quadrant double complex as follows:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_f \otimes_R \check{C}^0_{\mathfrak{t}}(S) & \longrightarrow & \cdots & \longrightarrow & K_1 \otimes_R \check{C}^0_{\mathfrak{t}}(S) & \longrightarrow & K_0 \otimes_R \check{C}^0_{\mathfrak{t}}(S) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_f \otimes_R \check{C}^1_{\mathfrak{t}}(S) & \longrightarrow & \cdots & \longrightarrow & K_1 \otimes_R \check{C}^1_{\mathfrak{t}}(S) & \longrightarrow & K_0 \otimes_R \check{C}^1_{\mathfrak{t}}(S) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \cdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_f \otimes_R \check{C}^r_{\mathfrak{t}}(S) & \longrightarrow & \cdots & \longrightarrow & K_1 \otimes_R \check{C}^r_{\mathfrak{t}}(S) & \longrightarrow & K_0 \otimes_R \check{C}^r_{\mathfrak{t}}(S) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & \cdots & & 0 & & 0 & &
\end{array}$$

With the usual grading of S the differentials of this complex have degree zero. Therefore each strand of a given degree d defines a double complex over R. We start the construction of our new family of complexes by analyzing the vertical spectral sequences coming from each of these strands. By Theorem 1.5.6 $H^i_{\mathfrak{t}}(S) = 0$ unless $i = g$. Therefore the first page of the vertical spectral sequence ${}^1E^{p,q}_{ver}$ reads as

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\vdots & & \vdots & & \cdots & & \vdots & & \vdots & & \vdots \\
0 & \longrightarrow & H^g_{\mathfrak{t}}(K_f) & \longrightarrow & \cdots & \longrightarrow & H^g_{\mathfrak{t}}(K_1) & \longrightarrow & H^g_{\mathfrak{t}}(K_0) & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array} \tag{3.8}$$

As $K_i = S^{\binom{s}{i}}(-i)$ and $H^g_{\mathfrak{t}}(S)_{(d)} = 0$ for $d > -g$ we have for a fixed degree d a complex $({}^1E_{vert})^{\bullet,r}_{(d)}$ given by:

$$0 \to H^g_{\mathfrak{t}}(K_s)_{(d)} \to H^g_{\mathfrak{t}}(K_{s-1})_{(d)} \to \cdots \to H^g_{\mathfrak{t}}(K_{g+d+1})_{(d)} \xrightarrow{\psi_d} H^g_{\mathfrak{t}}(K_{g+d})_{(d)} \to 0 \tag{3.9}$$

Notice that $({}^2E^{d+g,g}_{vert})_{(d)} = ({}^\infty E^{d+g,g}_{vert})_{(d)} \simeq \mathrm{Coker}(\psi_d)$.

To construct the other part of the complex we analyze the horizontal spectral sequence. The first page ${}^1E^{p,q}_{hor}$ reads as

$$\begin{array}{cccc}
\check{C}^0_{\mathfrak{t}}(H_f(K_\bullet)) & \ldots & \check{C}^0_{\mathfrak{t}}(H_1(K_\bullet)) & \check{C}^0_{\mathfrak{t}}(H_0(K_\bullet)) \\
\downarrow & & \downarrow & \downarrow \\
\check{C}^1_{\mathfrak{t}}(H_f(K_\bullet)) & \ldots & \check{C}^1_{\mathfrak{t}}(H_1(K_\bullet)) & \check{C}^1_{\mathfrak{t}}(H_0(K_\bullet)) \\
\downarrow & & \downarrow & \downarrow \\
\vdots & \ldots & \vdots & \vdots \\
\downarrow & & \downarrow & \downarrow \\
\check{C}^r_{\mathfrak{t}}(H_f(K_\bullet)) & \ldots & \check{C}^r_{\mathfrak{t}}(H_1(K_\bullet)) & \check{C}^r_{\mathfrak{t}}(H_0(K_\bullet))
\end{array} \tag{3.10}$$

For the second page ${}^2E^{p,q}_{hor}$ we have

$$\begin{array}{cccc}
H^0_{\mathfrak{t}}(H_f(K_\bullet)) & \ldots & H^0_{\mathfrak{t}}(H_1(K_\bullet)) & H^0_{\mathfrak{t}}(H_0(K_\bullet)) \\
 & & & \\
H^1_{\mathfrak{t}}(H_f(K_\bullet)) & \ldots & H^1_{\mathfrak{t}}(H_1(K_\bullet)) & H^1_{\mathfrak{t}}(H_0(K_\bullet)) \\
 & & & \\
\vdots & \ddots & \vdots & \vdots \\
 & & & \\
H^r_{\mathfrak{t}}(H_f(K_\bullet)) & \ldots & H^r_{\mathfrak{t}}(H_1(K_\bullet)) & H^r_{\mathfrak{t}}(H_0(K_\bullet))
\end{array} \tag{3.11}$$

Again as $K_i = S^{\binom{s}{i}}(-i)$, we have that

$$0 \to (K_d)_{(d)} \xrightarrow{\mu_d} \cdots \to (K_0)_{(d)} \to 0; \tag{3.12}$$

and therefore $({}^2E^{d,0}_{hor})_{(d)} \simeq H^0_{\mathfrak{t}}(\mathrm{Ker}(\mu_d)) \subset \mathrm{Ker}(\mu_d)$. We now join together the complexes 3.9 and 3.12 By the convergence theorem (see Theorem 2.4.1) we have for the vertical spectral sequence that

$$({}^{\infty}E_{ver}^{-d-g,-g})_{(d)} \simeq H^d(\mathcal{D}_\bullet)_{(d)}. \tag{3.13}$$

Moreover, the horizontal spectral sequence gives us a filtration

$$H^d(\mathcal{D}_\bullet)_{(d)} = \mathcal{F}_{d,0} \supseteq \mathcal{F}_{d,1} \supseteq \cdots$$

such that

$$\frac{\mathcal{F}_{d,i}}{\mathcal{F}_{d,i+1}} \simeq {}^{\infty}(E_{hor}^{-d-i,-i})_{(d)}.$$

We then have a natural surjection $H^d(\mathcal{D}_\bullet)_{(d)} \to ({}^{\infty}E_{hor}^{d,0})_{(d)}$. We can define a map $\tau_d : H^g_{\mathfrak{t}}(K_{d+g})_{(d)} \to (K_d)_{(d)}$ as the composition

$$\begin{tikzcd}
H^g_{\mathfrak{t}}(K_{d+g})_{(d)} \arrow[r] & \mathrm{Coker}(\psi_d) \arrow[r] & H^d(\mathcal{D}_\bullet)_{(d)} \arrow[dll, two heads] \\
({}^{\infty}E_{hor}^{-d,0})_{(d)} \arrow[r, hook] & ({}^{1}E_{hor}^{-d,0})_{(d)} = \mathrm{Ker}(\mu_d) \arrow[r, hook] & (K_d)_{(d)}
\end{tikzcd} \tag{3.14}$$

We then define the complex $\mathcal{K}_d(\Phi)$ to joining

$$0 \to H^g_{\mathfrak{t}}(K_f)_{(d)} \to \ldots \xrightarrow{\psi_d} H^g_{\mathfrak{t}}(K_{d+g})_{(d)} \xrightarrow{\tau_d} (K_d)_{(d)} \xrightarrow{\mu_d} \cdots \to (K_0)_{(d)} \to 0. \tag{3.15}$$

Definition 3.2.1. *For each d, let $\mathcal{K}_d(\Phi)$ be the complex (3.15). We call this complex the Koszul-Čech degree d of degree d of the matrix Φ.*

3.3 The First Main Theorem.

This section is devoted to prove the following theorem.

Theorem 3.3.1. *Let $\Phi = (a_{ij})$ be a $g \times f$ matrix representing a linear map $\varphi : F = R^f \to G = R^g$, with $f \geq g$. Then the complexes $\mathcal{C}^d$ and $\mathcal{K}_d(\Phi)$ are isomorphic for $d \leq f - g$.*

The theorem is proved in a series of propositions. The complexes $\mathcal{C}^d$ and $\mathcal{K}_d(\Phi)$ are isomorphic on the right side of the maps τ_d and ε_d: they are both the Koszul complex on the forms $\sum_{i=1}^{g} a_{ij}T_i$. Our first step is to show that, at least componentwise, the left part of the complexes are isomorphic.

Proposition 3.3.2. *There is a perfect pairing $S_d \otimes_R H^g_{\mathfrak{t}}(S)_{(-d-g)} \to H^g_{\mathfrak{t}}(S)_{(-g)} \simeq R$ such that the isomorphism $(S_{(d)})^* \simeq H^g_{\mathfrak{t}}(S)_{(-d-g)}$ takes the element $(T_1^{\alpha_1} \dots T_g^{\alpha_g})'$ to the element $\frac{1}{T_1^{\alpha_1+1} \dots T_g^{\alpha_g+1}}$.*

Proof. Recall the inverse polynomial structure of $H^g_{\mathfrak{t}}(S)$ (see Theorem 1.5.6). $H^g_{\mathfrak{t}}(S)_{(r)}$ is a free R-module generated by the monomials $\frac{1}{T_1^{\alpha_1} \dots T_g^{\alpha_g}}$ with $\sum_{i=1}^{g} \alpha_i = r, \alpha_i \geq 1$ and we can define the pairing

$$S_{(d)} \otimes_R H^g_{\mathfrak{t}}(S)_{(-d-g)} \to H^g_{\mathfrak{t}}(S)_{-g}$$

given by the natural multiplication. It's easy to see that this is a perfect pairing and that $T_1^{\alpha_1} \dots T_g^{\alpha_g}$ is dual to $\frac{1}{T_1^{\alpha_1+1} \dots T_g^{\alpha_g+1}}$. $\square$

Therefore we have the isomorphisms

$$\mathcal{K}_d(\Phi))_{d+i} = H^g_{\mathfrak{t}}(K_{g+d+i-1})_{(d)} \simeq$$

$$H^g_{\mathfrak{t}}(\bigwedge^{d+g+i-1} F \otimes_R S(-g-d-i+1))_{(d)} \simeq$$

$$\bigwedge^{g+d+i-1} F \otimes_R H^g_{\mathfrak{t}}(S(-g-d-i+1))_{(d)} \simeq$$

$$\bigwedge^{g+d+i-1} F \otimes_R H^g_{\mathfrak{t}}(S)_{(-g-i+1)} \simeq \bigwedge^{g+d+i-1} F \otimes_R (S_{(i-1)})^* = (\mathcal{C}^d)_{d+i} \text{ (by Proposition 3.3.2).}$$

Therefore the left side of the complexes are isomorphic componentwise, and we need to see what happens to the differentials. The above isomorphism can be used to induce differentials in the left part of the complex $\mathcal{C}^d(\Phi)$ using the differentials of the complex $\mathcal{K}_d(\Phi)$. These differentials are not difficult to calculate: consider a typical basis element for $\mathcal{K}_d(\Phi))_i = \bigwedge^{g+i-1} F \otimes_R H^r_{\mathfrak{t}}(S)_{(d-i-g+1)}$ of the form

$$e_{j_1} \wedge \cdots \wedge e_{j_k} \otimes \frac{1}{T_1^{\alpha_1+1} ... T_g^{\alpha_g+1}}, \quad \sum_{i=1}^{g} \alpha_i = i - d - 1, \ \alpha_i \geq 0, \tag{3.16}$$

The differential of $\mathcal{K}_d(\Phi)$ is induced by the Koszul differential. Applying it to the basis element, we get

$$\sum_{l=1}^{k} [(-1)^l e_{j_1} \wedge \cdots \wedge \hat{e}_{j_l} \wedge \cdots \wedge e_{j_k} \otimes \sum_{m=1}^{g} \frac{a_{mj_k} T_m}{T_1^{\alpha_1+1} ... T_g^{\alpha_g+1}}] =$$

$$\sum_{l=1}^{k} [(-1)^l e_{j_1} \wedge \cdots \wedge \hat{e}_{j_l} \wedge \cdots \wedge e_{j_k} \otimes \sum_{m=1}^{g} \frac{a_{mj_k}}{T_1^{\alpha_1+1} ... T_m^{\alpha_m} ... T_g^{\alpha_g+1}}]. \tag{3.17}$$

Applying the duality of Proposition 3.3.2 we have the differential

$$\Delta(e_{j_1} \wedge \cdots \wedge e_{j_i} \otimes (T_1^{\alpha_1} ... T_g^{\alpha_g})') =$$

$$\sum_{l=1}^{k} [(-1)^l e_{j_1} \wedge \cdots \wedge \hat{e}_{j_l} \wedge \cdots \wedge e_{j_k} \otimes \sum_{m=1}^{g} a_{mj_k} (T_1^{\alpha_1} ... T_m^{\alpha_m - 1} ... T_g^{\alpha_g})'] \tag{3.18}$$

defined in the left part of $\mathcal{C}_d(\Phi)$.

Proposition 3.3.3. *Denote by δ the differential of the left part of $\mathcal{C}^d(\Phi)$ and let Δ be the differential defined in (3.18). Then $\delta = \Delta$.*

Proof. Proposition 3.1.5 says that δ is given by the multiplication by $c = \sum_{i=1}^{g} T_i' \otimes T^i \in G^* \otimes_R G$. Take a typical basis element of $\bigwedge^{g+i-1} F \otimes_R S^*_{i-d-1}$. We have:

$$\delta((e_{j_1} \wedge \cdots \wedge e_{j_k} \otimes (T_1^{\alpha_1} ... T_g^{\alpha_g})')) = c \cdot ((e_{j_1} \wedge \cdots \wedge e_{j_k} \otimes (T_1^{\alpha_1} ... T_g^{\alpha_g})')) =$$

$$\sum_{m=1}^{g} [(\sum_{l=1}^{k} (-1)^l T_m'(\varphi(e_{j_l})) e_{j_1} \wedge \cdots \wedge \hat{e}_{j_l} \wedge \cdots \wedge e_{j_k}) \otimes T_m (T_1^{\alpha_1} ... T_g^{\alpha_g})'] \tag{3.19}$$

Therefore,

$$\delta(e_{j_1} \wedge \cdots \wedge e_{j_k} \otimes (T_1^{\alpha_1}...T_g^{\alpha_g})') =$$

$$\sum_{l=1}^{k}[(-1)^l e_{j_1} \wedge \cdots \wedge \hat{e}_{j_l} \wedge \cdots \wedge e_{j_k} \otimes \sum_{m=1}^{g} a_{mj_k}(T_1^{\alpha_1}...T_m^{\alpha_m - 1}...T_g^{\alpha_g})'] \tag{3.20}$$

which matches with 3.18. $\square$

It remains to study the connection maps τ_d of the previous section and the maps ε_d of Proposition 3.1.6. The results of Chapter 2 allows us to give an explicit explanation of the map τ_d.

Let $z \in K_{g+d} \otimes \check{C}^g$ representing an element of $H^g_{\mathfrak{t}}(K_{r+d})_{(d)} \simeq \bigwedge^{r+d} F \otimes_R H^g_{\mathfrak{t}}(S)_{-g}$. By Theorem 2.3.8 this element is liftable through the double complex to an element z' in $(\bigwedge^d F \otimes \check{C}^0(S))_{(d)} \simeq (K_d)_{(d)}$. Then by the Convergence Theorem (see Theorem 2.4.1) $\tau_d(z) = z'$. The following theorem proves that the map τ_d is equal to ε_d up to a sign. Before stating the theorem, we introduce some notation.

Notation 3.3.4. *Let $T_1, \ldots, T_g$ be variables and $L \subset \{1, \ldots, g\}$. We define $T^L = \prod_{j \notin L} T_j$.*

We are now ready to state our main theorem for this section.

Theorem 3.3.5. *Let R be a commutative Noetherian ring, $\Phi = (a_{ij})$ be a $g \times f$ matrix in R with $f \geq g$. Let $S = R[T_1, \cdots, T_g]$ a polynomial extension. For $1 \leq j \leq f$ set $\gamma_j = \sum_{i=1}^{g} a_{ij} T_i$. Consider the double complex $K_\bullet \otimes \check{C}^\bullet = K_\bullet(\gamma_1, \ldots, \gamma_s) \otimes_S \check{C}^\bullet_{\mathfrak{t}}(S)$ and its horizontal and vertical spectral sequences. Then, for each $0 \leq d \leq f - g$ and $I \subset \{1, \cdots, f\}$ with $r + d$ elements, the element*

$$m_I = e_I \otimes \frac{1}{T_1 \ldots T_g} \in K_{r+d} \otimes \check{C}^g$$

is $g-$liftable. Moreover, the i-th lift of this element is given by

$$\sum_{L\subset\{1,\ldots,g\},J\subset I,|L|=|J|=i} e_{I\setminus J}\otimes \operatorname{sgn}(L\subset\{1,\ldots,g\})\operatorname{sgn}(J\subset I)\det(\Phi^J_L)\frac{1}{T^L}.$$

In particular,

$$\tau_d(m_I)=\sum_{J\subset I,|J|=g}\operatorname{sgn}(J\subset I)\det(\Phi^J)e_{I\setminus J}.$$

So that $\tau_d=\varepsilon_d$ up to a sign.

Proof. In the course of the proof we do all the liftings without concerning about the signs, for the sake of clarity. Finally, we stress that the signs involved depends only on g.

In $({}^{0}E_{ver}^{-d-g,-g})_{(d)}$ the differential $d^0=d_v$ is zero. As $\operatorname{end}(H^g_{\mathfrak{t}}(S))=-g$ $({}^{1}E_{ver}^{-d-g+1,-g})_{(d)}=0$ and $({}^{r}E_{ver}^{-d-g+r,-g-r+1})_{(d)}=0$ for all $r\geq 2$. Therefore all differentials d^r are zero in $({}^{r}E_{ver}^{-d-g,-g})_{(d)}$ and $({}^{0}E_{ver}^{-d-g,-g})_{(d)}=({}^{\infty}Z_{ver}^{-d-g,-g})_{(d)}$. Then by Theorem 2.3.8 m_I is g-liftable.

We prove the theorem by induction. Applying the Koszul differential on m_I, we get

$$\sum_{j\in I}e_{I\setminus\{j\}}\otimes\frac{\operatorname{sgn}(\{j\}\in I)\gamma_J}{T_1\cdots T_g}\tag{3.21}$$

For any $j\in I$, the corresponding summand is

$$\frac{\operatorname{sgn}(\{j\}\subset I)\sum_{l=1}^{g}a_{lj}T_l}{T_1\cdots T_g}\tag{3.22}$$

From (3.22) it's immediate to see that (3.21) is a Čech boundary, and the 1-lift is given by

$$e_{I\setminus\{j\}}\otimes\sum_{l\in\{1,\cdots,g\},j\in I}\operatorname{sgn}(\{l\}\subset\{1,\cdots,g\})\operatorname{sgn}(\{j\}\subset I)\det\Phi^{\{j\}}_{\{l\}}\frac{1}{T^{\{l\}}}.\tag{3.23}$$

Now, we proceed by induction. Let $L=\{l_1<\cdots<l_p\}\subset\{1,\ldots,g\}$ and $J=\{j_1<\cdots<j_p\}\subset I$. We want to calculate the coefficient of the p-th lift of m_I

with respect to the basis element $e_{I \setminus J} \otimes \frac{1}{T_L}$. We first notice that the components that lifts to $e_{I \setminus J} \otimes \frac{1}{T_L}$ are the components of the form $e_{I \setminus J_i} \otimes \frac{1}{T^{L'}}$, where $L' \subset L$ and $J_i = J \setminus \{j_i\}$. As this element is 1-liftable, we just need to analyze the components with a fixed L'. Choose $L' = \{l_1 < \cdots < l_{p-1}\}$. Looking only at these components, our induction hypothesis says that the $(p-1)$-lift of m_I is given by

$$\sum_{i=1}^{p} (e_{I \setminus J_i} \otimes \operatorname{sgn}(L' \subset \{1, \ldots, g\}) \operatorname{sgn}(J_i \subset I) \det \Phi_{J_i}^{L'} \frac{1}{T^{L'}}) \tag{3.24}$$

Applying the Koszul differential to (3.24), we get

$$e_{I \setminus J} \otimes \sum_{i=1}^{p} \operatorname{sgn}(L' \subset \{1, \ldots, g\}) \operatorname{sgn}(J_i \subset I \setminus J) \operatorname{sgn}(\{j_i\} \subset J_i) \Phi_{J_i}^{L'} \frac{\gamma_{j_i}}{T^{L'}} \tag{3.25}$$

By Lemma 1.1.4, $\operatorname{sgn}(J_i \subset J) \operatorname{sgn}(\{j_i\} \subset I \setminus J_i) = (-1)^{p-i} \operatorname{sgn}(J \subset I)$, we may rewrite (3.25) as

$$e_{I \setminus J} \otimes \operatorname{sgn}(L' \subset \{1, \ldots, g\}) \operatorname{sgn}(J \subset I) \sum_{i=1}^{p} \frac{(-1)^{p-i} \det \Phi_{L'}^{J_i} \gamma_{j_i}}{T^{L'}}, \tag{3.26}$$

and the coefficient of T_{l_p} in the sum $\sum_{i=1}^{p} \frac{(-1)^{p-i} \det \Phi_{L'}^{J_i} \gamma_{j_i}}{T^{L'}}$ is given by

$$\frac{(-1)^{p-i} \det \Phi_{L'}^{J_i} a_{l_p j_i} T_{l_p}}{T^{L'}} \tag{3.27}$$

On the other hand, Laplace's rule for the expansion of a determinant gives us

$$\det \Phi_L^J = \det \Phi_{L' \cup \{l_p\}}^J = \sum_{i=1}^{p} (-1)^{p+i} a_{a_p j_i} \det \Phi_{L'}^{J_i} \tag{3.28}$$

So we may rewrite (3.25) as

$$e_{I \setminus J} \otimes \operatorname{sgn}(L' \subset \{1, \ldots, g\}) \operatorname{sgn}(J \subset I) \sum_{i=1}^{p} \det \Phi_L^J \frac{T_l}{T^{L'}} \tag{3.29}$$

Therefore (3.25) is liftable by the Čech map, and the lifting is given by

$$e_{I \setminus J} \otimes \operatorname{sgn}(\{l_p\} \subset \{1, \cdots, g\} \setminus L') \operatorname{sgn}(\{L' \subset \{1, \cdots, g\}) \operatorname{sgn}(J \subset I) \frac{\det \Phi_L^J}{T^L} \tag{3.30}$$

As $\operatorname{sgn}(L' \subset \{1,\ldots,g\})\operatorname{sgn}(\{l_p\} \subset \{1,\cdots,g\} \setminus L') = \operatorname{sgn}(L \subset \{1,\ldots,g\})$, the coefficient of $e_{I\setminus J} \otimes \frac{1}{T_L}$ in the p-th lift of m_I is given by

$$\operatorname{sgn}(L \subset \{1,\ldots,g\})\operatorname{sgn}(J \subset I)\det \Phi_J^L, \tag{3.31}$$

as required.

To see that $\tau_d = \varepsilon_d$, we analyze the construction of τ_d given in (3.14). As

$$H_{\mathfrak{t}}^g(\bigwedge^{d+g}(S(-d-g))^f) \simeq \bigwedge^{d+g} F \otimes H_{\mathfrak{t}}^g(S)_{(-g)},$$

any element of $\operatorname{Coker}(\varphi_d)$ can be represented as a linear combination of some m_I's, and we calculate the image of this elements.

By theorem 2.4.1 the isomorphism $\operatorname{Coker}(\varphi_d) \simeq H^d(\mathcal{D}_\bullet)_{(d)}$ sends m_I to the cohomology class $(m_I, -a_1, ..., (-1)^g a_g) \in H^d(\mathcal{D}_\bullet)_{(d)}$, where $(m_I, a_1, \cdots, a_g)$ is a g-lift sequence for m_I.

Again by Theorem 2.4.1 the map $H^d(\mathcal{D}_\bullet)_{(d)} \to {}^\infty E^{-d,0}$ sends the cohomology class $(m_I, -a_1, ..., (-1)^g a_g)$ to the element $(-1)^g a_g$. As a_g is just the g-lift of m_I calculated above. Comparing with Definition 3.1.6 we see that $\tau_d = \varepsilon_d$ up to a sign. $\square$

The proof of Theorem 3.3.1 now follows from Proposition 3.3.3 and Theorem 3.3.5. Therefore the Koszul-Čech complexes $\mathcal{K}_d(\Phi)$ are isomorphic to the Buchsbaum-Eisenbud complexes $\mathcal{C}_\bullet^d(\Phi)$.

Chapter 4

Applications to Residual Intersections

In this chapter we apply the results of the previous chapters to the theory of residual intersections defined in the introduction.

In section 4.1 we recall the construction of the residual approximation complexes and give a more details of the methods of [Ha],[HN] and [CNT]. This construction leads us to the notion of disguised residual intersection of a pair of finitely generated ideals $\mathfrak{a} \subset I$.

Section 4.2 is devoted to the study of the structure of the disguised residual intersections, and is divided in two parts. First we show that the disguised residua intersection is in fact a invariant of the pair $\mathfrak{a} \subset I$. After it, we relate the structure of this ideal to the DG-Algebra structure of the Koszul complex of a set of generators of I. This leads to a proof of Conjecture 0.0.8 when I satisfies SD_1 (see Definition 0.0.6).

4.1 Residual Approximation Complexes

Let R be a commutative ring and let $\mathfrak{a} \subset I$ two finitely generated ideals of R. We recall here the construction of the residual approximation complexes following the constructions in [Ha], [HN] and [CNT].

As the first step, consider the $\mathcal{Z}$-approximation complex $\mathcal{Z}(\mathbf{f};R)$ (see Definition 1.4.3) on a set $(\mathbf{f})=(f_1,\dots,f_r)$ of generators of I:

$$0\to Z_r(\mathbf{f};R)\otimes S[-r]\to Z_{r-1}(\mathbf{f};R)\otimes S[-r+1]\to\cdots\to Z_1(\mathbf{f};R)\otimes S[-1]\to S\to 0,$$

where $S=R[T_1,\cdots,T_r]$.

Let now $(\mathbf{a})=(a_1,\cdots,a_s)$ a generating set for $\mathfrak{a}$ and $\Phi=[a_{ij}]$ a representation matrix, that is, a matrix such that $\mathbf{a}=\mathbf{f}\cdot\Phi$. Define $\gamma_j=\sum\limits_{i=1}^{r}a_{ij}T_i$, let $K(\underline{\gamma};S)$ be the generalized Koszul complex of the matrix Φ (see Definition 1.4.1) and construct the a new complex

$$\mathcal{D}_\bullet=\operatorname{Tot}(\mathcal{Z}(\mathbf{f};R)\otimes_S K(\underline{\gamma},S)).$$

Let $\mathfrak{t}=(T_1,\dots,T_r)$. We can apply the very same procedure of the construction of the Koszul-Čech complexes to the double complex

$$\mathcal{D}_\bullet\otimes\check{C}^\bullet_{\mathfrak{t}}(S)$$

we get new complexes ${}_d\mathcal{Z}^+_\bullet$ whose components are given by

$${}_d\mathcal{Z}^+_i=\bigoplus_{j+l=i}K_l(\underline{\gamma};S)_{(d)}\otimes_R Z_j(\mathbf{f};R)\quad\text{if}\quad i\le d$$
$${}_d\mathcal{Z}^+_i=\bigoplus_{j+l=d+r+i-1}(K_l(\underline{\gamma};S)\otimes_S H^r_{\mathfrak{t}}(S))_{(d)}\otimes_R Z_j(\mathbf{f};R)\quad\text{if}\quad i>d.$$

Definition 4.1.1. *The complex ${}_k\mathcal{Z}^+_\bullet$ constructed above is called the k-th residual approximation complex with respect to the generating sets $\boldsymbol{f}$ and $\boldsymbol{a}$ of I and $\mathfrak{a}$ and the matrix Φ.*

The methods involved in the mentioned papers are obtained by the analysis of the involved spectral sequences that does not lie on the choice of generators of I and $\mathfrak{a}$, neither on the choice of the matrix Φ.

The main advantage to have such approximation complexes is that they can give a lot of information about the module they resolve, if they are acyclic. A very useful criterion for acyclicity is the Lemme d'Aciclicité of Péskine and Szpiro, which we state below.

Lemma 4.1.2. *Let $(R, \mathfrak{m})$ be a local ring and*

$$0 \to C_r \to C_{r-1} \to \cdots C_1 \to C_0 \to 0$$

be a complex of finitely generated R-modules. Suppose that

1. $\operatorname{depth} H_i(C_\bullet) = 0$ *for* $i \geq 1$.
2. $\operatorname{depth} C_i \geq i$ *for all* i.

   Then the complex $C_\bullet$ is acyclic. Moreover, if

3. $\operatorname{depth} C_i \geq d + i$ *for all* i,

   then

$$\operatorname{depth} H_0(C_\bullet) \geq d.$$

Therefore if the modules of Koszul cycles $Z_\bullet(\mathbf{f}; R)$ has enough depth, then we can use the Lemma, and this is where the sliding depth conditions comes into play (see Definition 0.0.6). We state an acyclicity criterion given in [HN].

Theorem 4.1.3. *[HN, Theorem 2.6] Let R be a Cohen-Macaulay Noetherian local ring of dimension d, and let $I = (f_1, \ldots, f_r) = (\boldsymbol{f})$ be an ideal with $\operatorname{ht} I = g \geq 1$. Let $s \geq g$ and fix $0 \leq k \leq \max\{s, s - g + 2\}$. Suppose that one of the hypothesis below holds:*

1. $r + k \leq s$ *and I satisfies* SD
2. $r + k \geq s + 1$ *and I satisfies SDC_1 at level $s - g - k$ and depth $Z_i \geq d - s + k$ for $0 \leq i \leq k$*

3. $k \leq s - r + 2$ *and* I *is* SD

4. *depth* $H_i(f) \geq min\{d - s + k + 2, s - g\}$ *for* $0 \leq i \leq k - 1$ *and* I *is* SD.

5. *I is SCM.*

Then for any s-residual intersection $J = \mathfrak{a} : I$ *the complex* ${}_k\mathcal{Z}^+$ *is acyclic and* $H_0({}_k\mathcal{Z}^+)$ *is a Cohen-Macaulay module of dimension* $d - s$.

Consider the complex ${}_0\mathcal{Z}^+_\bullet$. The last map of this complex is given by

$$H^r_{\mathfrak{t}}(\mathcal{D}_r)_0 \to (D_0)_0 \simeq R$$

and $H_0({}_0\mathcal{Z}^+_\bullet) = R/K$. The following result, adapted from [H, Theorem 2.11] shows some properties of this ideal, as mentioned in the introduction.

Theorem 4.1.4. *Let* $I \subset R$ *be a finitely generated ideal. Then, if* $J = \mathfrak{a} : I$ *for any* $\mathfrak{a} \subset I$, *either* $J = R$ *or we have:*

1. $K \subset J$ *and* $V(K) = V(J)$

2. $J = K$ *off* $V(I)$.

   Moreover if the ring R *is Cohen-Macaulay,* I *has height* ≥ 1 *and satisfies* SD *and* J *is an s-residual intersection of* I, *then*

3. *K is a Cohen-Macaulay ideal of height s.*

4. *If J is an arithmetic residual intersection, then* $K = J$.

This new ideal K is equal to the original residual intersection $\mathfrak{a} : I$ in many cases. Hassanzadeh and Naéliton named K as the *disguised residual intersection* in [HN] and made the following conjecture.

The results above shows that the understanding of the disguised residual intersection is key to understand the Cohen-Macaulayness of the residual intersections.

As mentioned in the introduction, Chardin, Naéliton and Tran were able to prove in [CNT] the following result.

Theorem 4.1.5. *Let R be a Cohen-Macaulay ring and I is an ideal of height 2 satisfying* SD_1*. Then for every s-residual intersection $J = \mathfrak{a} : I$, we have $K = J$.*

The theorem above gives us that if I is a height 2 ideal satisfying SD_1 then every s-residual intersection of I is Cohen-Macaulay. The general case follows from this. If $\operatorname{ht} I < 2$, we can just add variables to the ideal I to increase the height. Suppose that $\operatorname{ht} I = g > 2$.

Notice first that $\operatorname{ht}\mathfrak{a} = \operatorname{ht} I$, as $\operatorname{ht}(\mathfrak{a} : I) \geq s \geq g$. Let $a_1, \ldots, a_s$ be a generating set of $\mathfrak{a}$. As R is Cohen-Macaulay, $\operatorname{grade}(\mathfrak{a}, R) = g$ and we can find a new generating set $a'_1, \cdots, a'_s$ such that $(\alpha) = (a'_1, \cdots, a'_{g-2})$ is an R-regular sequence (see [BH, Exercise 1.2.21]). The ideal $I/(\alpha)$ has height 2 in the ring $R/(\alpha)$ and still satisfies SD_1 (see [CNT, Proposition 4.1]). Therefore the ideal

$$J/(\alpha) = \mathfrak{a}/(\alpha) : I/(\alpha)$$

is an $s - g + 2$-residual intersection and therefore equal to the disguised residual intersection of $\mathfrak{a}/(\alpha) : I/(\alpha)$ and hence is Cohen-Macaulay. As

$$R/J \simeq \frac{R/(\alpha)}{J/(\alpha)},$$

we conclude that J is a Cohen-Macaulay ideal.

We could use a similar method as above to answer Conjecture 0.0.8 for ideals satisfying SD_1 of arbitrary height, but a more careful analysis of the spectral sequences comes into play. For instance, even the inclusion $\mathfrak{a} \subset I$ is not clear from the definition of K (this is obvious for the ideal $\mathfrak{a} : I$). Moreover, we don't know what happens to K if we change the set of generators of $\mathfrak{a}$ nor it's behavior under taking quotients. We deal with this issue in the next section, where we take out the disguise of the disguised residual intersection.

4.2 The structure of the Disguised Residual Intersections

As mentioned in the previous section to get more information about the disguised residual intersection we need a more careful approach to the construction of residual approximation complexes. We start the section with a more strict definition of the disguised residual intersection.

Definition 4.2.1. *Let R be a commutative ring, $I = (f_1, \cdots, f_r = (\boldsymbol{f})$, $\mathfrak{a} = (a_1, \cdots, a_s) = (\boldsymbol{a})$ two finitely generated ideals with $\mathfrak{a} \subset I$, Φ a matrix such that $\boldsymbol{a} = \boldsymbol{f} \cdot \Phi$ and ${}_0\mathcal{Z}^+_\bullet$ the zeroth residual approximation complexes with respect to $\boldsymbol{f}$, $\boldsymbol{a}$ and Φ (see Definition 4.1.1). We denote by*

$$K(\boldsymbol{a}, \boldsymbol{f}, \Phi)$$

the ideal given by the image of the map

$${}_0\mathcal{Z}^+_1 \to {}_0\mathcal{Z}^+_0 \simeq R$$

In order to get the structure of such ideals, we give a reinterpretation of the residual approximation complexes in terms of the Koszul-Čech complexes defined 3.2.1.

Recall from the last section that the residual approximation complexes where constructed from the spectral sequences that arises from the double complex

$$\mathcal{D}_\bullet \otimes \check{C}_t(S),$$

where $\mathcal{D}_\bullet = \mathrm{Tot}(K_\bullet(\underline{\gamma}; S) \otimes_S \mathcal{Z}_\bullet(\mathbf{f}; R))$.

The complex $\mathcal{Z}_\bullet(\mathbf{f}; R)$ is just the restriction to the Koszul cycles of I of the generalized Koszul complex $\mathbb{K}(\mathrm{id}_{r\times r})$. Hence the complex $\mathcal{D}_\bullet$ is just the restriction to the Koszul cycles of the generalized Koszul complex of the matrix

$$M = [\Phi|\mathrm{id}_{r\times r}]$$

and the complex ${}_0\mathcal{Z}_\bullet^+$ is a restriction of the Koszul-Čech complex of degree zero $\mathcal{K}_d(M)$ and this complex is nothing more than the Eagon-Northcott complex of the matrix M. Therefore is important to study the map ε_0 given in Proposition 3.1.6 in this situation.

For convenience, we recall and introduce some notation: $S = R[T_1, \ldots, T_r]$, $\gamma_j = \sum_{i=1}^{r} c_{ij} T_i$, where $\Phi = [c_{ij}]$. We write the generalized Koszul complex of M as the DG-Algebra

$$K_\bullet = S < e'_1, \ldots, e'_s, e_1, \ldots, e_r; \partial(e'_i) = \gamma_i, \partial(e_i) = T_i > .$$

Lemma 4.2.2. *Let $L_1 \subseteq \{1, \ldots, s\}$, $L_2 \subseteq \{1, \ldots, r\}$ subsets of L such that $|L_1| + |L_2| = r$. Then the image of the basis element $e'_{L_1} \otimes e_{L_2}$ via the connecting map in the Eagon-Northcott complex associated to the matrix M, defined above, is*

$$\operatorname{sgn}((L \setminus L_2) \subseteq L) \det \Phi^{L_1}_{L \setminus L_2}.$$

(The notation $\Phi^{L_1}_{L \setminus L_2}$ is already defined in Definition 1.1.7).

Proof. By Proposition 3.1.6, the image of the element $e'_{L_1} \otimes e_{L_2}$ is given by the determinant of the matrix $[\Phi^{L_1} | id^{L_2}_{(r \times r)}]$. Rearranging the rows, this determinant is equal to

$$\operatorname{sgn}(L \setminus L_2 \subseteq L) \det \begin{bmatrix} \Phi^{L_1}_{L \setminus L_2} & 0 \\ * & id_{|L_2| \times |L_2|} \end{bmatrix} = \operatorname{sgn}(L \setminus L_2 \subseteq L) \det \Phi^{L_1}_{L \setminus L_2} \tag{4.1}$$

□

Therefore we can describe the generators of the ideal $K(\mathbf{a}, \mathbf{f}, \Phi)$ by restricting the above map to the Koszul cycles of I. This is the content of the following theorem.

Proposition 4.2.3. *Let R be a commutative ring, $I = (f_1, \ldots, f_r) \subseteq R$ and $\mathfrak{a} = (a_1, \ldots, a_s) \subseteq I$ two finitely generated ideals, $\Phi = [c_{ij}]$ a matrix such that*

$\boldsymbol{a} = \boldsymbol{f} \cdot \Phi$, ε_0 *the connecting map in the Eagon-Northcott complex,* $L = \{1, \dots, r\}$. *Then the disguised residual intersection* $K = K(\boldsymbol{a}, \boldsymbol{f}, \Phi)$ *is generated by*

$$\{\varepsilon_0(e'_{L_1} \otimes z_j);\ r - s \leq j \leq r, L_1 \subset \{1, \dots, s\}, |L_1| = r - j, \quad \textit{and} \quad z_j \in Z_j(\boldsymbol{f}; R)\}.$$

More explicitly, for a cycle $z_j = \sum_{|L_2|=j} \alpha_{L_2} e_{L_2}$, *the element* $e'_{L_1} \otimes z_j$ *produces the generator*

$$\sum_{|L_2|=j} \alpha_{L_2}. \operatorname{sgn}(L \setminus L_2 \subseteq L) \det \Phi^{L_1}_{L \setminus L_2}.$$

Proof. By Definition 4.2.1, $K(\mathbf{a}, \mathbf{f}, \Phi)$ is the image of the map

$$\tau_0 : H^r_{\mathfrak{t}}(\mathcal{D}_r)_0 \to R$$

of the respective residual approximation complex.

Since

$$\mathcal{D}_r = \bigoplus_{i=0}^{r} K_i(\underline{\gamma}, S) \otimes_R Z_{r-i}(\mathbf{f}; R) \subseteq K_i(\underline{\gamma}, S) \otimes_R K_{r-i}(\mathbf{f}; R) \tag{4.2}$$

the map τ_0 is the restriction to $\mathcal{D}_r$ of the same named map in Theorem 3.3.5 which is equal to the connecting map in the Eagon-Northcott complex ε_0 up to a sign. The result then follows from Lemma 4.2.2. $\square$

The above Theorem can be explained by means of the DG-Algebra structure of the Koszul Complex of $\mathbf{f}$.

Theorem 4.2.4. *Let* R *be a commutative ring,* $I = (f_1, \dots, f_r) \subseteq R$ *and* $\mathfrak{a} = (a_1, \dots, a_s) \subseteq I$ *two finitely generated ideals and* $\Phi = [c_{ij}]$ *a matrix such that* $(\boldsymbol{a}) = (\boldsymbol{f}) \cdot \Phi$.

Consider the differential graded algebra $K_\bullet(\boldsymbol{f}; R) = R\langle e_1, \dots, e_r; \partial(e_i) = f_i \rangle$. *Let* $\zeta_i = \sum_{i=1}^{r} c_{ij} e_i, 1 \leq i \leq s$, $\Gamma_\bullet = R\langle \zeta_1, \cdots, \zeta_s \rangle$ *the sub-algebra generated by the* ζ*'s, and* $Z_\bullet = Z_\bullet(\boldsymbol{f}; R)$ *the sub-algebra of Koszul cycles. Then*

$$K(\boldsymbol{a}, \boldsymbol{f}, \Phi) = < \Gamma_\bullet \cdot Z_\bullet >_r .$$

Proof. Let $L_1 \subseteq \{1, \dots, s\}$ and $L_2 \subseteq \{1, \dots, r\}$ such that $|L_1| + |L_2| = r$. Then

$$\zeta_{L_1} \wedge e_{L_2} = \det[\Phi^{L_1} | id^{L_2}_{(r\times r)}] e_1 \wedge \cdots \wedge e_r = \operatorname{sgn}(L \setminus L_2 \subseteq L) \det \Phi^{L_1}_{L\setminus L_2} e_1 \wedge \cdots \wedge e_r$$

as calculated in the proof of Lemma 4.2.2. One can now follow the argument in Proposition 4.2.3.

We also notice that $< \Gamma_\bullet \cdot Z_\bullet >_r \subseteq K_r(\mathbf{f}; R) = \bigwedge^r R^r \simeq R$. Hence $< \Gamma_\bullet \cdot Z_\bullet >_r$ is isomorphic to an ideal of R. □

Given the above theorem we have two ways to follow: one is to study the dependence of $K(\mathbf{a}, \mathbf{f}, \Phi))$ on the choice of $\mathbf{a}, \mathbf{f}$ and Φ, and the other is to derive structural results about this ideal. We organize these two ways in the next two subsections.

4.2.1 The independence of the generating sets.

The independence of $K(\mathbf{a}, \mathbf{f}, \Phi)$ on the generating set is obtained in a series of propositions. The first one concerns the independence on the choice of the matrix Φ.

Proposition 4.2.5. *Let R be a commutative ring, $I = (f_1, \dots, f_r)$ and $\mathfrak{a} = (a_1, \dots, a_s) \subseteq I$ two finitely generated ideals and $\Phi = [c_{ij}]$ and $\widetilde{\Phi} = [\widetilde{c_{ij}}]$ be two matrices such that $\boldsymbol{a} = \boldsymbol{f} \cdot \Phi = \boldsymbol{f} \cdot \widetilde{\Phi}$. Then*

$$K(\boldsymbol{a}, \boldsymbol{f}, \Phi) = K(\boldsymbol{a}, \boldsymbol{f}, \widetilde{\Phi}).$$

.

Proof. Set $\zeta_j = \sum_{i=1}^r c_{ij} e_i$, $\widetilde{\zeta_j} = \sum_{i=1}^r \widetilde{c_{ij}} e_i$, $\Gamma_\bullet = R\langle \zeta_1, \cdots, \zeta_s \rangle$, $\widetilde{\Gamma}_\bullet = R\langle \widetilde{\zeta_1}, \cdots, \widetilde{\zeta_s} \rangle$, and $Z_\bullet$ the algebra of Koszul cycles of the sequence $\mathbf{f}$. By Theorem 4.2.4,

$$K(\mathbf{a}, \mathbf{f}, \Phi) = < \Gamma_\bullet \cdot Z_\bullet >_r \tag{4.3}$$

and

$$K(\mathbf{a}, \mathbf{f}, , \widetilde{\Phi}) =< \widetilde{\Gamma}_\bullet \cdot Z_\bullet >_r \tag{4.4}$$

.

As $\mathbf{f} \cdot (\Phi - \widetilde{\Phi}) = \mathbf{a} - \mathbf{a} = \mathbf{0}$ the columns of the matrix $\Phi - \widetilde{\Phi}$ are syzygies of the sequence $\mathbf{f}$. Therefore for all j we have that $\zeta_j = \widetilde{\zeta_j} + z_j$ for some $z_j \in Z_1$, hence

$$\widetilde{\Gamma}_i \subseteq \Gamma_i + \Gamma_{i-1} \cdot Z_1 + \cdots + \Gamma_1 \cdot Z_{i-1} + Z_i. \tag{4.5}$$

and, for $1 \leq i \leq s$,

$$\widetilde{\Gamma}_i \cdot Z_{r-i} \subseteq \Gamma_i \cdot Z_{r-i} + \Gamma_{i-1} \cdot Z_{r-i+1} + \cdots + \Gamma_1 \cdot Z_{r-1} + Z_r \tag{4.6}$$

This proves the inclusion

$$K(\mathbf{a}, \mathbf{f}, \Phi) \supseteq K(\mathbf{a}, \mathbf{f}, \widetilde{\Phi})$$

The opposite inclusion follows similarly. □

With the independence on the choice of the matrix Φ we are able to prove the independence of the choice of generators for $\mathfrak{a}$.

Proposition 4.2.6. *Let R be a commutative ring, $I = (f_1, \ldots, f_r) \subseteq R$ and $\mathfrak{a} \subseteq I$ two finitely generated ideals. Then the disguised residual intersection $K(\mathfrak{a}, I, \Phi)$ does not depend on the choice of generators of $\mathfrak{a}$.*

Proof. Let $(a_1, \ldots, a_s)$, $(a'_1, \ldots, a'_{s'})$ be two generating sets of the ideal $\mathfrak{a}$. There exists an $s \times s'$ matrix M, and an $s' \times s$ matrix M' such that

$$\mathbf{a} \cdot M = \mathbf{a}' \tag{4.7}$$

$$\mathbf{a}' \cdot M' = \mathbf{a} \tag{4.8}$$

Choosing Φ such that $\mathbf{a} = \mathbf{f} \cdot \Phi$. We have

$$\mathbf{a}' = \mathbf{f} \cdot \Phi \cdot M. \tag{4.9}$$

Let $\zeta_j, 1 \leq j \leq s$ be the ζ's associated to the matrix Φ and $\zeta'_j, 1 \leq j \leq s'$ be the ζ's associated to the matrix $\Phi \cdot M$. By elementary properties of the wedge product, any wedge product of the ζ'_j's is a linear combination of wedge products of the ζ_j's whose coefficients are some minors of the matrix M. Hence, by Theorem 4.2.4,

$$K(\mathbf{a}', f, \Phi \cdot M) \subseteq K(\mathbf{a}, \mathbf{f}, \Phi) \tag{4.10}$$

On the other hand, $\Phi \cdot M \cdot M'$ is a matrix such that $\mathbf{a} = \mathbf{f} \cdot \Phi \cdot M \cdot M'$. By the same argument as above, we have

$$K(\mathbf{a}, \mathbf{f}, \Phi \cdot M \cdot M') \subseteq K(\mathbf{a}, \mathbf{f}, \Phi \cdot M') \subseteq K(\mathbf{a}, \mathbf{f}, \Phi) \tag{4.11}$$

The result now follows from the independence from Proposition 4.2.5. $\square$

The last step is the dependency from the choice of generators of I. For that we need to study what happens to the Koszul cycles when we add a new element to the sequence $\mathbf{f}$. We need two lemmas.

Lemma 4.2.7. *Let R be a commutative ring, $I = (f_1, \ldots, f_r)$ an ideal, $1 \leq i \leq r+1$, $f_0 \in \operatorname{Ann} H_{i-1}(\boldsymbol{f}; R)$ and $K_\bullet(f_0, \boldsymbol{f}; R) = R\langle e_0, e_1, \ldots, e_r; \partial(e_i) = f_i\rangle$ the Koszul DG-Algebra. Then any cycle $z \in Z_i(f_0, \boldsymbol{f}; R)$ can be uniquely written in the form*

$$z = e_0 \wedge w + w'$$

where $w \in Z_{i-1}(\boldsymbol{f}; R)$, $w' \in K_i(f_0, \boldsymbol{f}; R)$ and $\partial(w') = -f_0 w$. Conversely, for any $w \in Z_{i-1}(\boldsymbol{f}; R)$ there exists $w' \in K_i(f_0, \boldsymbol{f}; R)$ such that $e_0 \wedge w + w' \in Z_i(f_0, \boldsymbol{f}; R)$.

Proof. Every element $z \in K_i(f_0, \mathbf{f}; R)$ can be uniquely written in the form $z = e_0 \wedge w + w'$ where $w \in K_{i-1}(\mathbf{f}; R)$ and $w' \in K_i(f_0, \mathbf{f}; R)$. If z is a cycle, then

$$0 = \partial(z) = f_0.w - e_0 \wedge \partial(w') + \partial(w). \tag{4.12}$$

Hence $\partial(w) = 0$ and $\partial(w') = -f_0 w$.

For the converse, suppose that $w \in Z_{i-1}(\mathbf{f}; R)$. Since $f_0 \in \operatorname{Ann} H_{i-1}(\mathbf{f}; R)$, $-f_0 w$ is a boundary, that is, there is $w' \in K_i(\mathbf{f}; R)$ with $\partial(w') = -f_0 w$. Taking $z = e_0 \wedge w + w' \in K_i(f_0, \mathbf{f}; R)$, we have

$$\partial(z) = \partial(e_0 \wedge w + w') = f_0 w + e_0 \wedge \partial(w) + \partial(w') = 0 \tag{4.13}$$

which proves the lemma. □

Lemma 4.2.8. *Let R be a commutative ring, $I = (f_1, \ldots, f_r)$ and $\mathfrak{a} = (a_1, \ldots, a_s) \subseteq I$ two finitely generated ideals, $f_0 \in \cap_{i=\max\{0,r-s\}}^{r} \operatorname{Ann} H_i(\boldsymbol{f}, R)$ and $K_\bullet(f_0, \boldsymbol{f}; R) = R\langle e_0, e_1, \ldots, e_r; \partial(e_i) = f_i\rangle$ the Koszul DG-Algebra. Let $\Phi = [c_{ij}]$ be a matrix such that $\boldsymbol{a} = \boldsymbol{f} \cdot \Phi$, then*

$$M = \begin{bmatrix} \boldsymbol{0} \\ \Phi \end{bmatrix}$$

satisfies $\boldsymbol{a} = (f_0, \boldsymbol{f}\,) \cdot M$, and

$$K(\boldsymbol{a}, \boldsymbol{f}, \Phi) = K(\boldsymbol{a}, (f_0, \boldsymbol{f}\,), M).$$

Proof. The assertion about the matrix is obvious. Let

$$\zeta_j = 0.e_0 + \sum_{i=1}^{r} c_{ij} e_i, \tag{4.14}$$

the ζ's corresponding to the representation matrix M. These elements can be viewed as the ζ's corresponding to the matrix Φ. By Theorem 4.2.4, we have $K(\mathbf{a}, (f_0, \mathbf{f}), M) =< \Gamma_\bullet{\cdot}Z_\bullet >_{r+1}$. Hence, to construct a generator of $K(\mathbf{a}, (f_0, \mathbf{f}), M)$, we take $z \in Z_j(f_0, \mathbf{f}; R)$, $r + 1 - s \le j \le r + 1$ and $L_1 \subseteq \{1, \ldots, s\}$ with $|L_1| = r + 1 - j$. By Lemma 4.2.7 $z = e_0 \wedge w + w'$, where $w \in Z_{j-1}(\mathbf{f}; R), w' \in K_j(\mathbf{f}; R)$. Therefore

$$\zeta_{L_1} \wedge z = \zeta_{L_1} \wedge e_0 \wedge w + \zeta_{L_1} \wedge w' \tag{4.15}$$

Since $\zeta_{L_1} \wedge w'$ is the wedge product of $r+1$ elements containing only $e_1, \dots, e_r$,

$$\zeta_{L_1} \wedge w' = 0. \tag{4.16}$$

The product $\zeta_{L_1} \wedge w$ is the product of a cycle of degree $j-1$ with $(r+1-j)$ ζ's. Hence it gives an element in $K(\mathbf{a}, \mathbf{f}, \Phi)$. Therefore

$$K(\mathbf{a}, (f_0, \mathbf{f}), M) \subseteq K(\mathbf{a}, \mathbf{f}, \Phi). \tag{4.17}$$

For the converse, let $w \in Z_j(\mathbf{f}; R)$. By Lemma 4.2.7, there exists $w' \in K_{j+1}(\mathbf{f}; R)$ such that

$$e_0 \wedge w + w' \in Z_{j+1}(a, (f_0, \mathbf{f}); R). \tag{4.18}$$

Let $L_1 \subseteq \{1, \dots, s\}$ with $|L_1| = r-j$. We have that $e_0 \wedge \zeta_{L_1} \wedge w = \zeta_{L_1} \wedge (e_0 \wedge w + w')$. This shows that $\zeta_{L_1} \wedge w \in K(\mathbf{a}, (f_0, \mathbf{f}), M)$ □

Remark 4.2.9. The hypothesis $f_0 \in \operatorname{Ann} H_{i-1}(\mathbf{f}; R)$ in Lemma 4.2.7 is used only to prove the "conversely" part of the theorem. Therefore the inclusion in (4.17) always holds.

We are now ready to prove the promised independence.

Proposition 4.2.10. *Let R be a commutative ring, $I \subseteq R$ and $\mathfrak{a} \subseteq I$ two finitely generated ideals. Then the disguised residual intersection does not depend on the choice of generators of I.*

Proof. Let $(f_1, \dots, f_r)$, $(f'_1, \dots, f'_t)$ be two sets of generators of I, $(a_1, \dots, a_s) = (\mathbf{a})$ a generating set for $\mathfrak{a}$ and Φ, Φ' matrices such that $\mathbf{a} = \mathbf{f} \cdot \Phi$ and $\mathbf{a} = \mathbf{f}' \cdot \Phi'$. By using repeatedly Lemma 4.2.8, we have that

$$M = \begin{bmatrix} \mathbf{0} \\ \Phi \end{bmatrix} \tag{4.19}$$

Satisfies $\mathbf{a} = (\mathbf{f'}, \mathbf{f}) \cdot M$, and $K(\mathbf{a}, \mathbf{f}, \Phi) = K(\mathbf{a}, (\mathbf{f'}, \mathbf{f}), M)$. By Proposition 4.2.5, $K(\mathbf{a}, (\mathbf{f'}, \mathbf{f}), M) = K(\mathbf{a}, (\mathbf{f'}, \mathbf{f}), M')$, where

$$M' = \begin{bmatrix} \Phi' \\ \mathbf{0} \end{bmatrix}. \tag{4.20}$$

Again, repeated applications of Lemma 4.2.8 gives us $K(\mathbf{a}, (\mathbf{f'}, \mathbf{f}), M') = K(\mathbf{a}, \mathbf{f'}, \Phi')$.

□

Now that we know the disguised residual intersection does not depend on any choice of generators for $\mathfrak{a}$ and I nor on the choice of a relating matrix Φ, we introduce the following notation.

Definition 4.2.11. *Let R be a commutative Noetherian ring and $\mathfrak{a} \subseteq I$ be two ideals. We denote the disguised residual intersection, $K(\boldsymbol{a}, \boldsymbol{f}, \Phi)$, defined in Definition 4.2.1 by* $\mathrm{Kitt}(\mathfrak{a}, I)$.

Lemmas 4.2.7 and 4.2.8 provides some unexpected results about the codimension of the colon ideals and at the same time on the structure of the common annihilators of Koszul homologies. Both of these topics were mentioned as desirable in the works [CHKV] and [U1].

Corollary 4.2.12. *Let R be a commutative ring, $I = (f_1, \ldots, f_r)$ and $\mathfrak{a} = (a_1, \ldots, a_s) \subseteq I$ two finitely generated ideals. Then $\mathrm{Kitt}(\mathfrak{a}, I) = \mathrm{Kitt}(\mathfrak{a}, I')$ for any ideal I' satisfying*

$$I \subseteq I' \subseteq \bigcap_{\max\{0, r-s\}}^{r} \operatorname{Ann} H_i(\boldsymbol{f}; R).$$

In particular, if $\mathfrak{a} : I$ is an s-residual intersection then so is $\mathfrak{a} : I'$.

Proof. Let $(f'_1, ..., f'_t)$ be a generating set for I'. The proof of Proposition 4.2.10 is applicable as it relies on of Lemma 4.2.8. Therefore $\mathrm{Kitt}(\mathfrak{a}, I) = \mathrm{Kitt}(\mathfrak{a}, I')$. The second part of the statement follows from Theorem 4.1.4, stating that of these ideals have the same radical. □

4.2.2 The Structure of Kitt ideals

Another consequence of Theorem 4.2.4 is that DG-Algebra structure of the Koszul cycles of $\mathbf{f}$ determines the structure of $\operatorname{Kitt}(\mathfrak{a}, I)$. Our first proposition deals with the sub-algebra generated by the cycles of degree one $< Z_1(\mathbf{f}; R) >$.

Proposition 4.2.13. *Let R be a commutative ring and keeping the same notation as in Theorem 4.2.4, we have*

$$\langle \Gamma_\bullet \cdot \langle Z_1(\boldsymbol{f}; R) \rangle \rangle_r = \operatorname{Fitt}_0(I/\mathfrak{a}).$$

In particular, if the algebra of Koszul cycles $Z_\bullet(\boldsymbol{f}; R)$ is generated by cycles of degree one, then $\operatorname{Kitt}(\mathfrak{a}, I) = \operatorname{Fitt}_0(I/\mathfrak{a})$.

Proof. Let Φ be an $r \times s$ matrix for which $\mathbf{a} = \mathbf{f} \cdot \Phi$ and $\Psi = [b_{ij}]$ be a syzygy matrix for the sequence $\mathbf{f}$ which has r rows. Then $Z_1 = Z_1(\mathbf{f}; R)$ is generated by the elements $z_i = \sum_{i=1}^{r} b_{ij} e_i$. Therefore, $\langle \Gamma_\bullet \cdot \langle Z_1(\mathbf{f}; R) \rangle \rangle_r$ is obtained by taking all the products of the form

$$\zeta_{L_1} \wedge z_{L_2}, |L_1| + |L_2| = r. \tag{4.21}$$

By elementary properties of the wedge product, a product as in (4.21) is an $r \times r$ minor of the matrix $[\Phi|\Psi]$. This matrix is the representation matrix of $I/\mathfrak{a}$ as it is obtained by taking the mapping-cone of the following diagram

$$\begin{tikzcd} R^t \arrow[r, "\Psi"] & R^r \arrow[r] & R \arrow[r] & R/I \arrow[r] & 0 \\ & R^s \arrow[u, "\Phi"] \arrow[r] & R \arrow[u, "="] \arrow[r] & R/\mathfrak{a} \arrow[u] \arrow[r] & 0. \end{tikzcd} \tag{4.22}$$

Thus $\langle \Gamma_\bullet \cdot \langle Z_1(\mathbf{f}; R) \rangle \rangle_r = \mathrm{I}_r([\Phi|\Psi]) = \operatorname{Fitt}_0(I/\mathfrak{a})$.

$\square$

We can also understand what happens if we turn to our attention to the ideal of Koszul boundaries. By doing this, we can conclude that the structure of $\operatorname{Kitt}(\mathfrak{a}, I)$

beyond $\mathfrak{a}$ is in fact encrypted on the structure of the Koszul homology algebra of I.

Lemma 4.2.14. *Let R be a commutative ring and keep the same notation as in Theorem 4.2.4. Let $B_\bullet(\boldsymbol{f}; R)$ be the ideal of Koszul boundaries. Then*

$$\langle \Gamma_\bullet \cdot B_\bullet \rangle_r = \mathfrak{a}.$$

Proof. In the Koszul complex $K_\bullet(\mathbf{f}; R)$, the module of boundaries of degree k is generated by elements of the form $\partial(e_{L_2})$ where $|L_2| = k+1$. For any $L_1 \subseteq \{1, \cdots, s\}$ with $|L_1| = r-k$ we have

$$\zeta_{L_1} \wedge \partial(e_{L_2}) = \zeta_{L_1} \wedge (\sum_{j\in L_2} \operatorname{sgn}(\{j\} \subseteq L_2) f_j e_{L_2 \setminus \{j\}}) = \sum_{j\in L_2} (\operatorname{sgn}(\{j\} \subseteq L_2) f_j \zeta_{L_1} \wedge e_{L_2\setminus\{j\}}). \tag{4.23}$$

According to Proposition 4.2.2, the above equation (4.23) can be written as

$$\sum_{j\in L_2} (\operatorname{sgn}(\{j\} \subseteq L_2) \operatorname{sgn}(L_2 \setminus \{j\} \subseteq L) \det \Phi^{I}_{L\setminus(L_2\setminus\{j\})} f_j. \tag{4.24}$$

If we rearrange every determinant in a way such that the j-th row becomes the first one (4.24) becomes

$$\sum_{j\in L_2} (\operatorname{sgn}(\{j\} \subseteq L_2) \operatorname{sgn}(L_2 \setminus \{j\} \subseteq L) \operatorname{sgn}(\{j\} \subseteq L \setminus (L_2 \setminus \{j\})) \det \Phi^{L_1}_{\{j\}, L\setminus L_2} f_j. \tag{4.25}$$

By Lemma 1.1.5 the expression $(\operatorname{sgn}(\{j\} \subseteq L_2) \operatorname{sgn}(L_2 \setminus \{j\} \subseteq L) \operatorname{sgn}(\{j\} \subseteq L \setminus (L_2 \setminus \{j\}))$ does not depend on $j \in L_2$. Thus we can ignore this product and consider

$$\zeta_{L_1} \wedge \partial(e_{L_2}) = \sum_{j\in L_2} \det \Phi^{L_1}_{\{j\}, L\setminus L_2} f_j. \tag{4.26}$$

If $j \notin L_2$ then $\det \Phi^{L_1}_{\{j\}, L\setminus L_2} = 0$, since $\Phi^{L_1}_{\{j\}, L\setminus L_2}$ has a repeated row. Therefore (4.26) is equal to

$$\sum_{j=1}^{r} \det \Phi^{L_1}_{\{j\}, L\setminus L_2} f_j. \tag{4.27}$$

Now, we expand every determinant in this sum over the first row of each matrix. Each summand has the form

$$\det \Phi^{L_1}_{\{j\},L\setminus L_2} f_j = \sum_{i\in L_1} \operatorname{sgn}(\{i\}\subset L_1) \det \Phi^{L_1\setminus\{i\}}_{L\setminus L_2} . c_{ji} f_j. \tag{4.28}$$

Summing over all j, we get

$$\zeta_{L_1} \wedge \partial(e_{L_2}) = \sum_{i\in L_1} \operatorname{sgn}(\{i\}\subset L_1) \det \Phi^{L_1\setminus\{i\}}_{L\setminus L_2} a_i. \tag{4.29}$$

This shows that $< \Gamma_\bullet \cdot B_\bullet >_r \subseteq \mathfrak{a}$.

As to the other inclusion, we consider the last boundary given by

$$\partial(e_1 \wedge \cdots \wedge e_r) = \sum_{i=1}^{r} (-1)^{i+1} f_i e_1 \wedge \dots \hat{e_i} \cdots \wedge e_r =: z \tag{4.30}$$

Then, for any $1 \leq j \leq s$, we have

$$\zeta_j \wedge z = \sum_{i=1}^{r} c_{ij} f_i = a_i. \tag{4.31}$$

□

As a corollary of the above lemma, we have the following structural theorem for $\mathrm{Kitt}(\mathfrak{a}, I)$

Theorem 4.2.15. *Let R be a commutative ring and keep the same notation as in Theorem 4.2.4, let $\tilde{H}_\bullet$ is the sub-algebra of $K_\bullet(\boldsymbol{f}; R)$ generated by the representatives of Koszul homologies. Then*

$$\mathrm{Kitt}(\mathfrak{a}, I) = \mathfrak{a} + \langle \Gamma_\bullet \cdot \tilde{H}_\bullet \rangle_r.$$

Proof. According to Theorem 4.2.4, $\mathrm{Kitt}(\mathfrak{a}, I) = \langle \Gamma_\bullet \cdot Z_\bullet \rangle_r$. Since $Z_\bullet = B_\bullet + \tilde{H}_\bullet$, we have $\mathrm{Kitt}(\mathfrak{a}, I) = \langle \Gamma_\bullet \cdot B_\bullet \rangle_r + \langle \Gamma_\bullet \cdot \tilde{H}_\bullet \rangle_r$. By Lemma 4.2.14, $\langle \Gamma_\bullet \cdot B_\bullet \rangle_r = \mathfrak{a}$ which yields the result. □

Corollary 4.2.16. *Let R be a commutative ring and $I = (f_1, \cdots, f_r)$ be a finitely generated ideal such that the Koszul homology algebra $H_\bullet(\boldsymbol{f}; R)$ is generated by elements of degree one. Then, for any $\mathfrak{a} = (a_1, \cdots, a_s) \subseteq I$, one has* $\mathrm{Kitt}(\mathfrak{a}, I) = \mathrm{Fitt}_0(I/\mathfrak{a}) + \mathfrak{a}$*. In particular, this is the case when $(f_1, \cdots, f_r)$ is an almost regular sequence (grade of I is $r-1$).*

If $(f_1, \cdots, f_r)$ is a regular sequence then $\mathrm{Kitt}(\mathfrak{a}, I) = I_r(\Phi) + \mathfrak{a}$ *where Φ is an $r \times s$ matrix satisfying $\boldsymbol{a} = \boldsymbol{f} \cdot \Phi$.*

Proof. Just notice that in the case of complete intersection $\tilde{H}_\bullet$ in Theorem 4.2.15 in concentrated in degree zero that is $\tilde{H}_\bullet = R$; so that

$$\langle \Gamma_\bullet \cdot \tilde{H}_\bullet \rangle_r = \langle \Gamma_\bullet \cdot R \rangle_r = I_r(\Phi)$$

□

We can also prove Theorem 4.1.4 (1) and (2) using the structural theorem for $\mathrm{Kitt}(\mathfrak{a}, I)$, instead of doing the analysis of the involved spectral sequences as in [Ha].

Proposition 4.2.17. *If $\mathfrak{a} \subset I$ are finitely generated ideals, then* $\mathrm{Kitt}(\mathfrak{a}, I) \subset \mathfrak{a} : I$

Proof. Let $b \in \mathrm{Kitt}(\mathfrak{a}, I)$ a generator obtained by $\gamma_I \wedge z \in (\Gamma \cdot Z)_r$ and $x \in I$. One then have

$$x \cdot be_1 \wedge \cdots \wedge e_r = x \cdot \gamma_I \wedge z = \gamma_I \wedge x \cdot z.$$

As $x \cdot z$ is a Koszul boundary $x \cdot b \in \mathfrak{a}$ by Lemma 4.2.14. □

Proposition 4.2.18. *Let R be a commutative ring and $\mathfrak{a}$ a finitely generated ideal. Then* $\mathrm{Kitt}(\mathfrak{a}, \mathfrak{a}) = R$.

Proof. Suppose that $\mathfrak{a} = (a_1, \ldots, a_s)$. Then we can use $\Phi = \mathrm{id}_{s \times s}$ as our required matrix. In this case, $\gamma_i = e_i$. As $1 \in Z_0(\mathbf{a}; R)$,

$$\gamma_1 \wedge \cdots \wedge \gamma_s = e_1 \wedge \cdots \wedge e_s$$

and $1 \in \operatorname{Kitt}(\mathfrak{a}, \mathfrak{a})$. $\square$

Corollary 4.2.19. *Let R be a commutative ring. If $\mathfrak{a} \subset I$ are two finitely generated ideals, then $\operatorname{Kitt}(\mathfrak{a}, I)$ and $J = \mathfrak{a} : I$ have the same radical.*

Proof. Both Kitt ideals and colon ideals localizes by the flatness of the localization. Therefore, if $\mathfrak{p} \not\supset J$ then $\mathfrak{a}_\mathfrak{p} = I_\mathfrak{p}$ and $\operatorname{Kitt}(;a_\mathfrak{p}, I_\mathfrak{p}) = R_\mathfrak{p}$. As $\operatorname{Kitt}(\mathfrak{a}, I) \subset J$ the corollary is proved. $\square$

Proposition 4.2.20. *Let R be a commutative ring. If $\mathfrak{a}$ is a finitely generated ideal then $\operatorname{Kitt}(\mathfrak{a}, (1)) = \mathfrak{a}$.*

Proof. The Koszul complex $K_\bullet((1); R)$ is exact. Therefore $Z_\bullet((1); R) = B_\bullet((1); R)$ and the Proposition follows from Lemma 4.2.14. $\square$

Corollary 4.2.21. *Let R be a commutative ring. If $\mathfrak{a} \subset I$ are two finitely generated ideals and $J = \mathfrak{a} : I$ then $\operatorname{Kitt}(\mathfrak{a}, I) = J$ off $V(I)$.*

Proof. Let $\mathfrak{p}$ be a prime ideal such that $\mathfrak{p} \not\supset I$. Then

$$J_\mathfrak{p} = \mathfrak{a}_\mathfrak{p} : I_\mathfrak{p} = \mathfrak{a}_\mathfrak{p} : R_\mathfrak{p} = \mathfrak{a}_\mathfrak{p}.$$

On the other hand,

$$\operatorname{Kitt}(\mathfrak{a}, I)_\mathfrak{p} = \operatorname{Kitt}(\mathfrak{a}_\mathfrak{p}, I_\mathfrak{p}) = \operatorname{Kitt}(\mathfrak{a}_\mathfrak{p}, R_\mathfrak{p}) = \mathfrak{a}_\mathfrak{p}$$

and the corollary is proved. $\square$

We can also use the above structural theorems to give affirmative answer to the Conjecture 0.0.8, as it may give nontrivial structural results on residual intersections. We first show using the Kitt ideals that a stronger version of Conjecture 0.0.8 if $I = (\mathfrak{a}, b)$ for some $b \in R$. This result was shown in [HN, Theorem 4.4] by a quite complicated analysis of the spectral sequence defining ${}_0\mathcal{Z}^+_\bullet$.

Proposition 4.2.22. *Let R be a commutative ring, For any finitely generated ideal $\mathfrak{a} \subset R$ and $b \in R$ we have*

$$\mathrm{Kitt}(\mathfrak{a}, (\mathfrak{a}, b)) = \mathfrak{a} : (\mathfrak{a}, b) = (\mathfrak{a} : b)$$

Proof. We need only to prove that $\mathrm{Kitt}(\mathfrak{a}, (\mathfrak{a}, b)) \supset (\mathfrak{a} : b)$. Take a generating set $\mathfrak{a} = (a_1, \cdots, a_r)$. The matrix

$$M = \begin{bmatrix} \mathrm{id}_{s\times s} \\ 0 \end{bmatrix}$$

satisfies $\mathbf{a} = (\mathbf{a}, b) \cdot M$.

Let $K_\bullet$ be the DG-Algebra $R < e_0, ..., e_s; \partial(e_i) = a_i, \partial(e_0) = b >$. By [HN, Lemma 5.2] we have exact sequences

$$0 \to Z_i(\mathbf{a}; R) \to Z_i(b, \mathbf{a}; R) \to (B_{i-1}(\mathbf{a}; R) :_{Z_{i-1}(\mathbf{a};R)} b) \to 0 \tag{4.32}$$

Where the first map is just the inclusion and the second map is the map $e_0 \wedge w + v \to w$.

Let $x \in (\mathfrak{a} : b) = (B_0(\mathbf{a}; R) :_{Z_0(\mathbf{a};R)} b)$. By the above exact sequence there is a cycle $w \in Z_1(\mathbf{a}; R)$ such that $a \cdot e_0 + v \in Z_1((b, \mathbf{a}); R)$. We have

$$(x \cdot e_0 + v) \cdot e_1 \wedge \cdots e_s = x \cdot e_0 \wedge \cdots \wedge e_s,$$

and by Theorem 4.2.4 $x \in \mathrm{Kitt}(\mathfrak{a}, (\mathfrak{a}, I))$. ☐

Recall that one of the obstructions for this is the independence on the generators of the ideal $\mathfrak{a}$ and this is addressed in Proposition 4.2.6. The second obstruction is the specialization. More precisely, we must prove that the construction of Kitt commutes with taking quotients by regular sequences, that is, if (α) is a regular sequence in the ideal $\mathfrak{a}$, then

$$\mathrm{Kitt}(\mathfrak{a}, I)/(\alpha) = \mathrm{Kitt}(\mathfrak{a}/(\alpha), I/(\alpha)).$$

Of course, as almost any proof involving regular sequences, the proof is inductive. We first recall a key lemma on Koszul homologies.

Lemma 4.2.23. *Let R be a commutative ring and $I = (f_1, \cdots, f_r)$ a finitely generated ideal. Let $f_0 \in I$ be a R-regular element and consider the Koszul complex $K_\bullet = R\langle e_0, \cdots, e_r : \partial(e_i) = f_i \rangle$. Then there is an isomorphism*

$$H_i(f_0, \boldsymbol{f}; R) \to H_i(\boldsymbol{f}; R/f_0)$$

given by the map

$$e_0 \wedge w + w' \to \tilde{w}'$$

where $w \in Z_{i-1}(\boldsymbol{f}; R)$, $w' \in K_i(f_0, \boldsymbol{f}; R)$ and $\partial(w') = -f_0 w$

Proof. The proof is essentially the one of Lemma 4.2.7; see also [BH, Proposition 1.6.12(c)]. □

We are now ready to state and prove the main theorem of this section, proving the specialization by regular sequences.

Theorem 4.2.24. *Let R be a commutative ring, $\mathfrak{a} \subseteq I$ two finitely generated ideals and $f_0 \in \mathfrak{a}$ an R-regular element. Then* $\mathrm{Kitt}(\mathfrak{a}, I)/(f_0) = \mathrm{Kitt}(\mathfrak{a}/(f_0), I/(f_0))$

Proof. First, we notice that $f_0 \in \mathrm{Kitt}(\mathfrak{a}, I)$ by Theorem 4.2.15. Also for an element $r \in R$, put $\widetilde{r}$ to denote the image of r via the projection homomorphism $R \to R/(f_0)$.

Fix generators $(f_1, \ldots, f_r)$ for I, $(a_1, \ldots, a_s)$ for $\mathfrak{a}$, a matrix $\Phi = [c_{ij}]$ such that $\mathbf{a} = \mathbf{f} \cdot \Phi$, and let $\zeta_j = \sum_{i=1}^r c_{ij} e_i \in K_1(f_1, \ldots, f_r; R)$. It's clear that $\widetilde{I} = (\widetilde{\mathbf{f}})$, $\widetilde{\mathfrak{a}} = (\widetilde{\mathbf{a}}$ and $\widetilde{\Phi}$ satisfies $\widetilde{\mathbf{a}} = \widetilde{\mathbf{f}} \cdot \widetilde{\Phi}$. Setting $\widetilde{\zeta_j} = \sum_{i=1}^r \widetilde{c_{ij}} e_i$ and $\widetilde{\Gamma}_\bullet = \frac{R}{(f_0)}[\widetilde{\zeta_1}, \ldots, \widetilde{\zeta_s}] \subseteq K_\bullet(\widetilde{\mathbf{f}}; R/(f_0))$, we have, by Theorem 4.2.4,

$$\mathrm{Kitt}(\frac{\mathfrak{a}}{(f_0)}, \frac{I}{(f_0)}) =< \widetilde{\Gamma}_\bullet \cdot Z_\bullet(\widetilde{\mathbf{f}}; R/(f_0)) >_r . \tag{4.33}$$

Let $z \in Z_j(\widetilde{\mathbf{f}}; R/(f_0))$, $0 \leq j \leq r$, and $L_1 \subseteq \{1, \ldots, s\}$ such that $|L_1| = r - j$. We need to prove that $\widetilde{\zeta}_{L_1} \wedge z$ is the specialization of some elements in $\operatorname{Kitt}(\mathfrak{a}, I)$. By Lemma 4.2.23, there is a cycle $c = e_0 \wedge w + w' \in Z_j(f_0, \mathbf{f}; R)$ such that $z = \widetilde{w'}$ in $H_j(\widetilde{\mathbf{f}}; R/(f_0))$.

According to Theorem 4.2.15, it suffices to prove that $\zeta_{L_1} \wedge w'$ is an element in $\operatorname{Kitt}(\mathfrak{a}, I)$. Since $f_0 \in \mathfrak{a}$ there exist $\alpha_i \in R$ such that

$$f_0 = \sum_{i=1}^{s} \alpha_i a_i. \tag{4.34}$$

Hence $e_0 - \sum_{i=1}^{s} \alpha_i \zeta_i \in Z_1(f_0, \mathbf{f}; R)$. Therefore, Theorem 4.2.4 implies that

$$\zeta_{L_1} \wedge (e_0 - \sum_{i=1}^{s} \alpha_i \zeta_i) \wedge c \in \operatorname{Kitt}(\mathfrak{a}, I).$$

On the other hand,

$$\zeta_{L_1} \wedge (e_0 - \sum_{i=1}^{s} \alpha_i \zeta_i) \wedge c = \zeta_{L_1} \wedge (-w' \wedge e_0 - \sum_{i=1}^{s} \alpha_i \zeta_i \wedge w \wedge e_0 - \sum_{i=1}^{s} \alpha_i \zeta_i \wedge w') \tag{4.35}$$

On the summands on the right side, we have

- $\zeta_{L_1} \wedge \sum_{i=1}^{s} \alpha_i \zeta_i \wedge w'$ which is zero, since it's a wedge product of $r+1$ elements involving only $e_1, \ldots, e_r$.

- $\zeta_{L_1} \wedge \sum_{i=1}^{s} \alpha_i \zeta_i \wedge w$ which gives us a generator of $\operatorname{Kitt}(\mathfrak{a}, I)$ by Theorem 4.2.4.

It then follows that $\zeta_{L_1} \wedge w'$ is an element in $\operatorname{Kitt}(\mathfrak{a}, I)$ as desired. $\square$

The ultimate consequence of this theorem is the proof of Conjecture 0.0.8 for ideals satisfying SD_1.

Theorem 4.2.25. *Let R be a Cohen-Macaulay ring and I be an ideal of height $g \geq 2$ which satisfies* SD_1 *condition. Then any algebraic s-residual intersection $J = \mathfrak{a} : I$ coincides with the disguised residual intersection* $\operatorname{Kitt}(\mathfrak{a}, I)$.

Proof. According to Theorem 4.1.4(i), $K \subseteq J$. So that to prove the equality we may assume without loss of generality that R is a complete Cohen-Macaulay local ring and hence possesses a canonical module.

By [BH, Exercise 1.2.21] and Proposition 4.2.6 we may suppose that $\mathfrak{a} = (\alpha_1, \cdots, \alpha_{g-2}, \ldots, a_s)$, where $\alpha = \alpha_1, \cdots, \alpha_{g-2}$ is a regular sequence. I/α still satisfies the SD_1 condition, e.g [CNT, Proposition 4.1]. So that [CNT, Theorem 4.5](Theorem4.1.4(4)) implies that

$$\mathrm{Kitt}(\frac{\mathfrak{a}}{\alpha}, \frac{I}{\alpha}) = (\frac{\mathfrak{a}}{\alpha} : \frac{I}{\alpha}) = \frac{J}{\alpha},$$

as $(\frac{\mathfrak{a}}{\alpha} : \frac{I}{\alpha})$ is a $s - g + 2$-residual intersection of the height 2 ideal $\frac{I}{\alpha}$. Now by Theorem 4.2.24, we have

$$\mathrm{Kitt}(\frac{\mathfrak{a}}{\alpha}, \frac{I}{\alpha}) = \frac{\mathrm{Kitt}(\mathfrak{a}, I)}{\alpha}$$

which proves the theorem. $\square$

4.3 Applications

The content of Theorem 4.2.25 is striking. Every structural result about $\mathrm{Kitt}(\mathfrak{a}, I)$ is in fact a structural result about the concrete residual intersection, as long as R is Cohen-Macaulay and I satisfies SD_1. For example, in [HN] Hassanzadeh and Naéliton studies the behavior of the Hilbert Function of the disguised residual intersections. Therefore their result is valid for residual intersections of SD_1 ideals:

Theorem 4.3.1. *Let R be a CM standard graded ring over an Artinian local ring R_0. Suppose that I satisfies SD_1 condition. Then for any s-residual intersection $J = (\mathfrak{a} : I)$, the Hilbert function of R/J depends only on the* ***degrees*** *of the generators of $\mathfrak{a}$ and the Koszul homologies of I.*

Proof. The fact has been already proved for disguised residual intersections in [HN, Proposition 3.1]. Due to Theorem 4.2.25 the disguised residual intersection is the same as the algebraic residual intersection for ideals with SD_1. □

A second application is the existence of the generic s-residual intersection. We recall a definition from [HU].

Definition 4.3.2. *Let R be a commutative ring and $I = (f_1, \cdots, f_r)$ a finitely generated ideal. Let $X = [X_{ij}]$ be an $r \times s$ matrix of variables, $S = R[X_{ij}]$ and $\boldsymbol{a} = \boldsymbol{f} \cdot X$. The generic s-residual intersection of I is the ideal*

$$\mathrm{RI}(s; I) = (\boldsymbol{a}) : I \cdot S$$

In [HU, Theorem 3.3] the authors prove that if R is a local Noetherian Cohen-Macaulay ring and I satisfies G_{s+1} then the generic s-residual intersection is a geometric s-residual intersection of the extension $I \cdot S$.

The G_{s+1} condition is equivalent to the existence of i-residual intersections of I for $\mathrm{ht}\, I \leq i \leq s$. Our aim is to show that only a s-residual intersection is needed.

We first begin with a lemma.

Theorem 4.3.3. *Let R be a commutative ring and $I = (f_1, \cdots, f_r)$ a finitely generated ideal. Let $X = [X_{ij}]$ be an $r \times s$ matrix of variables, $S = R[X_{ij}]$ and $\boldsymbol{a}' = \boldsymbol{f} \cdot X$. Let $\Phi = [c_{ij}]$ be a matrix with entries in R and $\boldsymbol{a} = \boldsymbol{f} \cdot \Phi$. Then the image of* $\mathrm{Kitt}((\boldsymbol{a}'), I \cdot S)$ *under the natural map*

$$S \to S/(X_{ij} - c_{ij}; 1 \leq i \leq r, 1 \leq j \leq j)$$

is $\mathrm{Kitt}((\boldsymbol{a}), I)$.

Proof. We have that

$$\mathrm{Kitt}((\mathbf{a}), I) =< \Gamma_\bullet \cdot Z_\bullet >,$$

where $Z_\bullet = Z_\bullet(\mathbf{f}; R)$ and $\Gamma_\bullet =< \sum_{i=1}^{r} c_{ij}e_i;\ 1 \leq j \leq s >$, and

$$\operatorname{Kitt}((\mathbf{a})', I \cdot S) =< \Gamma'_\bullet \cdot Z'_\bullet >_r,$$

where $Z'_\bullet = Z_\bullet(\mathbf{f}; S)$ and $\Gamma'_\bullet =< \sum_{i=1}^{r} X_{ij}e_i >$. Hence

$$\Gamma'_\bullet/\mathbf{c} \cdot \Gamma'_\bullet =< \sum_{i=1}^{r} c_{ij}e_i >= \Gamma_\bullet$$

and

$$Z'_i/\mathbf{c} \cdot Z'_i = Z_i(\mathbf{f}, R)[X_{ij}]/\mathbf{c} \cdot Z_i(\mathbf{f}, R)[X_{ij}] = Z_i(\mathbf{f}, R)$$

and these equalities proves the lemma. □

Theorem 4.3.4. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay Noetherian local ring of dimension d and $I \subset R$ an ideal that admits an s-residual intersection. Then the generic residual intersection* $\operatorname{RI}(s; I)$ *is an s-residual intersection of the extension $I \cdot S$.*

Proof. Let $(f_1, \ldots, f_r)$ be a set of generators, $(a_1, \ldots, a_s) : I$ an s-residual intersection and let $\Phi = [c_{ij}]$ be a matrix such that $\mathbf{a} = \mathbf{f} \cdot \Phi$.

Let $M = [X_{ij}]$ be a $r \times s$ matrix of indeterminates over R, $S = R[X_{ij}]$, $\mathbf{a}' = \mathbf{f} \cdot M$. and $\mathfrak{c} = (X_{ij} - c_{ij}; 1 \leq i \leq r, 1 \leq j \leq s)$. By Lemma 4.3.3 we have

$$\left(\frac{R[X_{ij}]}{\operatorname{Kitt}((\mathbf{a})', I \cdot S) + \mathfrak{c}}\right)_{\mathfrak{m}+\mathfrak{c}} \simeq R/\operatorname{Kitt}((\mathbf{a}), I).$$

As $R[X_{ij}]$ is Cohen-Macaulay,

$$\dim\left(\left(\frac{R[X_{ij}]}{\operatorname{Kitt}((\mathbf{a})', I \cdot S)}\right)_{\mathfrak{m}+\mathfrak{c}}\right) = d + rs - \operatorname{ht}\operatorname{Kitt}((\mathbf{a})', I \cdot S)$$

and therefore

$$\dim(\operatorname{Kitt}((\mathbf{a})', I \cdot S) + \mathfrak{c})_{\mathfrak{m}+\mathfrak{c}}) \geq d + r.s - \operatorname{ht}\operatorname{Kitt}((\mathbf{a})', I \cdot S) - r.s = d - \operatorname{ht}\operatorname{Kitt}((\mathbf{a})', I \cdot S).$$

Hence

$$d - \operatorname{ht}\operatorname{Kitt}((\mathbf{a}), I) = \dim(R/\operatorname{Kitt}((\mathbf{a}), I)) \leq d - \operatorname{ht}\operatorname{Kitt}((\mathbf{a})', I \cdot S),$$

which implies

$$\operatorname{ht}\operatorname{Kitt}((\mathbf{a})', I \cdot S) \geq \operatorname{ht}\operatorname{Kitt}((\mathbf{a}), I).$$

By Theorem 4.1.4 $\operatorname{ht}\operatorname{Kitt}((\mathbf{a}), I) = \operatorname{ht}((\mathbf{a}) : I)$ and the latter ideal is an s-residual intersection, we conclude that

$$\operatorname{ht}\operatorname{Kitt}((\mathbf{a}'), I \cdot S) \geq s$$

and again Theorem 4.1.4 gives us that $\operatorname{ht}\operatorname{RI}(s; I) \geq s$. □

Our third consequence of Theorem 4.2.25 is that, under it's hypothesis, we now know a set of generators of a residual intersection. Prior to this work, a set of generators is known in a few number of particular cases, for example:

1. If R is Gorenstein, I is perfect and $s = g$, [PS];
2. If R is a Gorenstein domain and I is complete intersection, [HU, Theorem 5.9(i)];
3. If R is Cohen-Macaulay, I is a complete intersection and J is a geometric residual intersection.
4. If R is Cohen-Macaulay and I is perfect ideal of height 2, [Hu],[KU] and [CEU, Theorem 1.1];
5. If R is Gorenstein and I is perfect Gorenstein ideal of height 3, [KU, Section 10].
6. If R is Gorenstein, I is Gorenstein licci, generically a complete intersection ideal and $s = g + 1$, [KMU, Corollary 2.18];

The cases 2 and 3 can be generalized for ideals satisfying SD_1 and R Cohen-Macaulay.

Theorem 4.3.5. *Let R be a Cohen-Macaulay ring and I be a complete intersection ideal generated by $\boldsymbol{f} = f_1, \cdots, f_r$. Suppose that $J = \mathfrak{a} : I$ is an algebraic s-residual intersection of I. Then $J = I_r(\Phi) + \mathfrak{a}$ where Φ is an $r \times s$ matrix satisfying $\boldsymbol{a} = \boldsymbol{f} \cdot \Phi$.*

Proof. Complete intersections are SD_1 obviously so that we have $J = \mathrm{Kitt}(\mathfrak{a}, I)$ by Theorem 4.2.25. The result now follows from Corollary 4.2.16. □

Prior to this work a description of the structure of residual intersections of almost complete intersections were not known. Using 4.2.16 we can give one for Cohen-Macaulay almost complete intersections.

Theorem 4.3.6. *Let R be a Cohen-Macaulay ring and I be an almost complete intersection ideal which is Cohen-Macaulay. Let $J = \mathfrak{a} : I$ be an algebraic s-residual intersection of I. Then $J = \mathrm{Fitt}_0(I/\mathfrak{a}) + \mathfrak{a}$.*

Proof. Since almost complete intersection CM ideals are SD_1, this is another consequence of Theorem 4.2.25 and Corollary 4.2.16. □

The DG-Algebra structure involved can give us non-trivial information as well. For example, we can give another proof of item 4 on the above list.

Theorem 4.3.7. *Let R be a Cohen-Macaulay ring and I be a perfect ideal of height 2. Let $J = \mathfrak{a} : I$ be an algebraic s-residual intersection of I. Then $J = \mathrm{Fitt}_0(I/\mathfrak{a})$.*

Proof. A result of Avramov-Herzog [AH, Proof of Theorem 2.1(e)] shows that for perfect ideals of height 2 the algebra of cycles of Koszul is generated in degree 1. So that the result follows from Theorem 4.2.25 and Proposition 4.2.13. □

We can also give a bound on the number of generators of the residual intersections.

Theorem 4.3.8. *Let R be a Cohen-Macaulay Noetherian local ring, $I \subset R$ an ideal of height g satisfying SD_1 and $J = \mathfrak{a} : I$ an s-residual intersection. If $(f_1, \ldots, f_r)$ is a set of generators of I then*

$$\mu(J) \leq s + \sum_{i=r-s}^{r-g} \binom{s}{r-i} \mu(H_i(\boldsymbol{f}; R)).$$

Proof. By [HN, Corollary 2.8a] $\mathfrak{a}$ is minimally generated by s elements.

The theorem then follows directly from Theorems 4.2.15 and 4.2.25. □

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We present a family of examples such that the above bound is sharp.

Example 4.3.9. *Let $I = (f_1, \ldots, f_4)$ a Cohen-Macaulay almost complete intersection, and suppose that $J = \mathfrak{a} : I$ is a 4-residual intersection. By Theorem 4.3.8 we have*

$$\mu(J) \leq 4 + \binom{4}{4}\mu(H_0(\boldsymbol{f}; R)) + \binom{4}{3}\mu(H_1(\boldsymbol{f}, R)) = 5 + 4\mu(H_1(\boldsymbol{f}, R)).$$

We can construct lots of ideals such that the above bound is sharp using the computer algebra system [Macaulay2] using the following commands:

- *loadPackage "RandomIdeals"*
- *R = ZZ_3[x_1..x_6] (creates the ring $R = \mathbb{Z}_3[x_1, \ldots, x_6]$).*
- *I = randomIdeal({2,2,2}, vars R) (create a random ideal generated by three quadratic forms).*
- *a = randomIdeal({3,3,3}, gens I) (creates an ideal generated by 3 forms of degree 3 in I)*
- *$L = a : I$*

Almost every time L is a 3*-residual intersection of* I*. Then L is a Cohen-Macaulay ideal of height 3 and has four generators, that is, is an almost complete intersection.*

- *b=randomIdeal(*$\{4,4,4,4\}$*, gens L)*
- $J = b : L$.

Almost every time J is a geometric 4-residual intersection of L, and the above bound holds. As L is linked to I one has $\mu(H_1(\boldsymbol{f}; R)) = \mu(I/a) = 3$*. Hence*

$$\mu(J) \leq 17,$$

where $(\boldsymbol{f})$ *are the generators of L.*

Another Macaulay2 computation shows that $\mu(J) = 17$ *and this gives the sharpness of the bound.*

Chapter 5

Questions and Remarks

In this chapter we make some remarks and propound further questions for future research.

With the description of the disguised residual intersection in hand, we were able to test in Macaulay2 if the Conjecture 0.0.8 was true or not. We tried to find counterexamples by testing cases were I does not satisfy any sliding depth condition, but the conjecture did hold in the tested cases. Therefore it is natural to state a more general conjecture.

Conjecture 5.0.1. *Let R be a Cohen-Macaulay ring. Then for any algebraic s-residual intersection $J = \mathfrak{a} : I$ we have*

$$J = \mathrm{Kitt}(\mathfrak{a}, I).$$

A proof of this conjecture is desirable as it gives us a explicit set of generators for a residual intersection.

The methods of the previous chapter shows us that if we are able to prove that, for a class of ideals, the conjecture is true for some height, then it is valid for all ideals on that class. The following example shows that the condition of being a residual intersection cannot be dropped.

Example 5.0.2. *Let $R = \mathbb{Z}_3[x, y, z, t]$, $I = (x^2, y^2, xy, xt-yz)$ and $\mathfrak{a} = (x^4, y^4, x^2y^2)$. Then $J = \mathfrak{a} : I$ has height 2 so that it is not a 3-residual intersection. One can check that the Koszul homology algebra of I is generated in degree 1; so that* $\mathrm{Kitt}(\mathfrak{a}, I) = \mathfrak{a} + \mathrm{Fitt}_0(I/a)$ *by Corollary 4.2.16. A Macaulay2 verification shows that* $\mathrm{Kitt}(\mathfrak{a}, I) \neq J$.

The verification of the validity of the above conjecture has some unexpected consequences. One of them is the following, about the common annihilator of Koszul homologies.

Proposition 5.0.3. *Let $I = (f_1, \cdots, f_r)$ and $\mathfrak{a} = (a_1, \cdots, a_r) \subset I$. If* $\mathrm{Kitt}(\mathfrak{a}, I) = \mathfrak{a} : I$ *then*

$$\mathfrak{a} : I = \mathfrak{a} : \bigcap_{i=r-s}^{r} \operatorname{Ann} H_i(\boldsymbol{f}; R)$$

.

Proof. We have by Corollary 4.2.12 that $\mathrm{Kitt}(\mathfrak{a}, I) = \mathrm{Kitt}(\mathfrak{a}, \bigcap_{i=r-s}^{r} \operatorname{Ann} H_i(\mathbf{f}; R))$. Therefore we have

$$\mathrm{Kitt}(\mathfrak{a}, I) = \mathrm{Kitt}(\mathfrak{a}, \bigcap_{i=r-s}^{r} \operatorname{Ann} H_i(\mathbf{f}; R)) \subset (\mathfrak{a} : \bigcap_{i=r-s}^{r} \operatorname{Ann} H_i(\mathbf{f}; R)) \subset (\mathfrak{a} : I) \qquad \square$$

The above proposition may give us a way to disprove Conjecture 5.0.1: find two ideals $\mathfrak{a} \subset I$ such that $\mathfrak{a} : I$ is an residual intersection and the equality on the above proposition is false.

The ideal $\mathrm{Kitt}(\mathfrak{a}, I)$ Shares some properties with the colon ideals $\mathfrak{a} : I$.

Proposition 5.0.4. *If $\mathfrak{a} \subset I_1 \subset I_2$ are finitely generated ideals then* $\mathrm{Kitt}(\mathfrak{a}, I_2) \subset \mathrm{Kitt}(\mathfrak{a}, I_1)$.

Proof. Remark 4.2.9 shows that the assertion holds if $I_2 = (I_1, b)$., and the general case follows by induction.

$\square$

Corollary 5.0.5. *For any two ideals I_1, I_2 containing $\mathfrak{a}$, we have* $\mathrm{Kitt}(\mathfrak{a}, I_1 + I_2) \subset \mathrm{Kitt}(\mathfrak{a}, I_1) \cap \mathrm{Kitt}(\mathfrak{a}, I_2)$

It would be nice to know conditions for the reverse inequality on the previous proposition is true. In fact, one always have

$$(\mathfrak{a}, (f_1, \ldots, f_r)) = \bigcap_{i=1}^{r} (\mathfrak{a} : \mathfrak{a}, f_i)) = \bigcap_{i=1}^{r} \mathrm{Kitt}(\mathfrak{a}, \mathfrak{a}, f_i) \supset \mathrm{Kitt}(\mathfrak{a}, I),$$

and the reverse inequality proves Conjecture 5.0.1. Of course Example 5.0.2 shows that this is not true in general, so that the residual intersection hypothesis is key here.

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