UNIVERSIDADE FEDERAL DO RIO DE JANEIRO

PEDRO ROBERTO DE LIMA

STABILIZATION OF THERMOELASTIC BRESSE SYSTEMS

RIO DE JANEIRO
2019

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Tese de Doutorado apresentada ao Programa de Pós-graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio de Janeiro – UFRJ, como parte dos requisitos necessários para obtenção do título de Doutor em Ciências.

Orientador: Hugo Danilo Fernández Sare

Rio de Janeiro
2019

CIP - Catalogação na Publicação

L732s   Lima, Pedro Roberto de
   Stabilization of thermoelastic Bresse systems / Pedro Roberto de Lima. -- Rio de Janeiro, 2019.
   102 f.

   Orientador: Hugo Danilo Fernández Sare.
   Tese (doutorado) - Universidade Federal do Rio de Janeiro, Instituto de Matemática, Programa de Pós Graduação em Matemática, 2019.

   1. Sistema de Bresse. 2. Estabilidade. 3. Lei de Fourier. 4. Lei de Cattaneo. I. Fernández Sare, Hugo Danilo, orient. II. Título.

Elaborado pelo Sistema de Geração Automática da UFRJ com os dados fornecidos pelo(a) autor(a), sob a responsabilidade de Miguel Romeu Amorim Neto - CRB-7/6283.

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Aprovada em 22/08/2019

Hugo Danilo Fernández Sare
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Doutora – UEL

Reinhard Racke
Doutor – University of Konstanz

Dedico este trabalho...

... aos meus avós maternos,
Leovilce Baggio Ramim e Pedro Ramim.

... aos meus pais,
Maria Ines Ramim de Lima e Aparecido Flávio de Lima.

... às minhas irmãs,
Flávia Maria de Lima e Maria Antonia de Lima.

... à memória dos meus avós paternos,
Antônia Nardo de Lima e Jorge Roberto de Lima.

... à memória da minha bisavó paterna,
Maria Pereira Nardo.

AGRADECIMENTOS

Agradeço a Deus, por tudo. Agradeço aos meus pais, Flávio e Maria, e às minhas irmãs, Maria e Flávia, pelo apoio incondicional. Agradeço ao professor Hugo, meu orientador, por continuamente conceder atenção e dedicação à minha orientação, compartilhando sua experiência e seu conhecimento, tornando possível a realização deste trabalho. Agradeço aos professores do curso pelos conhecimentos compartilhados. Agradeço aos colegas de curso pelas ajudas. Agradeço aos membros da banca examinadora pela participação. Agradeço à professora Luci, orientadora de mestrado, pelo incentivo. Finalmente, mas não menos importante, agradeço à Capes pelo apoio financeiro.

Um dos maiores males que acompanham a busca do conhecimento, as pesquisas da ciência, é a disposição de exaltar o raciocínio humano acima de seu real valor e sua devida esfera. (E. G. White)

LIMA, Pedro Roberto de. **Stabilization of thermoelastic Bresse systems**. Rio de Janeiro, 2019. Tese (Doutorado em Matemática) - Instituto de Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro, 2019.

RESUMO

Nesta tese, estudamos sistemas de Bresse termoelásticos onde a condução do calor é modelada para ambas as leis: de Fourier e de Cattaneo. Inicialmente, estudamos o caso em que a temperatura age apenas na força axial. Para o caso de Fourier, provamos estabilidade exponencial se e somente se a condição de velocidades de onda iguais é satisfeita. Para o caso de Cattaneo, caracterizamos a estabilidade exponencial por meio de uma nova condição sobre os coeficientes do sistema. Também provamos, no caso geral, estabilidade polinomial de soluções. Depois, estudamos o caso em que a temperatura age não apenas na força axial mas também no momento fletor. Nesta situação, damos mais uma nova condição que caracteriza completamente a estabilidade exponencial do modelo e generaliza tanto a bem conhecida condição de velocidades de onda iguais quanto o número de estabilidade do sistema de Timoshenko.

Palavras-chave: Sistema de Bresse. Estabilidade. Lei de Fourier. Lei de Cattaneo.

LIMA, Pedro Roberto de. **Stabilization of thermoelastic Bresse systems**. Rio de Janeiro, 2019. Tese (Doutorado em Matemática) - Instituto de Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro, 2019.

ABSTRACT

In this thesis, we study thermoelastic Bresses systems where the heat conductions are modeled for both: Fourier's and Cattaneo's laws. Initially, we study the case in which the temperature acts only on the axial force. For the Fourier's case, we prove exponential stability of solutions if and only if the condition of equal wave speeds is satisfied. For the Cattaneo's case, we characterize the exponential stability by a new condition on the coefficients of the system. We also prove, in the general case, polynomial stability of solutions. Later, we study the case in which the temperature acts not only on the axial force but also on the bending moment. In this situation, we give another new condition which completely characterizes the exponential stability of the model and generalizes the well-known equal wave speed condition as well as the stability number of the Timoshenko system.

Keywords: Bresse system. Stability. Fourier's law. Cattaneo's law.

LIST OF SYMBOLS

ℓ	Length of the beam
W_1	Horizontal displacement
W_3	Vertical displacement
ϑ_2	Angle of rotation of the cross-section
h	Cross-sectional area
m_0	Density
G	Shear modulus
E	Young's modulus
κ_3	Initial curvature
I_{33}	Moment of inertia of the cross-section with respect to the vertical axis
u'	Derivative of u with respect to the position $x \in [0, \ell]$
$\dot{u}$	Derivative of u with respect to the time $t \geq 0$
φ	Vertical displacement ($= W_3$)
ψ	Angle of rotation of the cross-section ($= \vartheta_2$)
w	Horizontal displacement ($= W_1$)
l	Initial curvature (=κ_3)
ρ_1	Product $m_0 h$
ρ_2	Product $m_0 I_{33}$
k	Product Gh
b	Product EI_{33}
k_0	Product Eh
θ	Temperature deviation that acts on axial force
ϑ	Temperature deviation that acts on bending moment
c_v	Heat capacity
T_0	Reference temperature
κ	Heat conductivity
γ	Positive coupling constant
q	Heat flux
τ_0	Heat flux relaxation
k_1	Constant $\frac{1}{m_0 c_v}$
m	Constant $\frac{\gamma T_0}{m_0 c_v}$
τ	Constant $\frac{\tau_0}{\kappa}$
δ	Constant $\frac{1}{\kappa}$
χ_0	Constant $b - \frac{k\rho_2}{\rho_1}$
χ_1	Constant $k_0 - k$
ζ_τ	Constant $\frac{\tau\gamma m}{\left(\frac{k_1\rho_1}{k} - \tau\right)}$
χ	Constant $\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau\gamma m$

ξ	Constant $\left(\varsigma - \frac{k_1\rho_1}{k}\right)\left(b - \frac{k\rho_2}{\rho_1}\right) + \varsigma\gamma m$
η_0	Function $\sqrt{\kappa_1}\theta_x$
η_τ	Function $\sqrt{\delta k_1}q$ if $\tau > 0$
$\mathbb{C}$	Set of complex numbers
$\mathrm{Re}(z)$	Real part of a complex number z
$\mathrm{Im}(z)$	Imaginary part of a complex number z
$\lvert z\rvert$	Norm of a complex number z
$\bar{z}$	Conjugate of a complex number z
$\mathbb{R}$	Set of real numbers
$\mathbf{i}$	Imaginary unit
$\mathbf{i}\mathbb{R}$	Set of all complex numbers with zero real part
$\times$	Cartesian product
$\mathbb{R}^N$	Usual Euclidean space $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (N factors)
Ω	Open subset of $\mathbb{R}^N$
$\mathrm{med}\,\Omega$	Lebesgue measure of Ω
$\int_\Omega u(x)\ dx$	Lebesgue integral of u on $\Omega \subset \mathbb{R}^N$
$\int_a^b u(x)\ dx$	Lebesgue integral of u on $[a, b] \subset \mathbb{R}$
$C_0^\infty(\Omega)$	Usual space of the infinitely differentiable functions of compact support
$L^p(\Omega)$	Usual Lebesgue space
$W^{m,p}(\Omega)$	Usual Sobolev space
$W_0^{m,p}(\Omega)$	Usual Sobolev space (closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$)
$L^2(0,\ell)$	Usual Lebesgue space
$L_*^2(0,\ell)$	Space $\left\{u \in L^2(0,\ell) \;\middle|\; \int_0^\ell u(x)\ dx = 0\right\}$
$H^1(0,\ell)$	Usual Sobolev space ($= W^{1,2}(0,\ell)$)
$H_0^1(0,\ell)$	Usual Sobolev space ($= W_0^{1,2}(0,\ell)$)
$H_*^1(0,\ell)$	Space $H^1(0,\ell) \cap L_*^2(0,\ell)$
$H^2(0,\ell)$	Usual Sobolev space ($= W^{2,2}(0,\ell)$)
L^2	Space $L^2(0,\ell)$
L_*^2	Space $L_*^2(0,\ell)$
H^1	Space $H^1(0,\ell)$
H_0^1	Space $H_0^1(0,\ell)$
H_*^1	Space $H_*^1(0,\ell)$
H^2	Space $H^2(0,\ell)$
$\mathcal{H}_0$	Space $H_0^1 \times L^2 \times H_*^1 \times L_*^2 \times H_*^1 \times L_*^2 \times L^2$
$\mathcal{H}_\tau$	Space $H_0^1 \times L^2 \times H_*^1 \times L_*^2 \times H_*^1 \times L_*^2 \times L^2 \times L_*^2$ if $\tau > 0$
$\mathbb{H}$	Space $H_0^1 \times L^2 \times H_*^1 \times L_*^2 \times H_*^1 \times L_*^2 \times L^2 \times L_*^2 \times L^2 \times L_*^2$
$D(A)$	Domain of an operator A
$\rho(A)$	Resolvent set of an operator A

$\sigma(A)$	Spectrum of an operator A
$\sigma_p(A)$	Point spectrum of an operator A
$\|\cdot\|_{D(A)}$	Graph norm on $D(A)$
$\|\cdot\|_X$	Norm in a space X
$\|\cdot\|_\tau$	Norm in $\mathcal{H}_\tau$
$\lvert\cdot\rvert_\tau$	Usual norm in $\mathcal{H}_\tau$
$(X, \|\cdot\|)$	Space X equipped with a norm $\|\cdot\|$
$(\cdot,\cdot)_X$	Inner product on a space X
$(\cdot,\cdot)_\tau$	Inner product on $\mathcal{H}_\tau$
∇u	Gradient of $u \in W^{1,p}(\Omega)$
u_x	Weak derivative of u with respect to x
u_t	Derivative of u with respect to t
x_n	Term of a given sequence $(x_n)_{n\in\mathbb{N}}$
$x^{(n)}$	Term of a given sequence $(x^{(n)})_{n\in\mathbb{N}}$
$\mathcal{L}(X)$	Space of all bounded linear operators from a normed space X to X
I	Identity operator
$\{e^{tA}\}_{t\geq 0}$	Semigroup generated by an operator A
$\overset{n\to\infty}{\longrightarrow} 0$	Converges to zero as n goes to infinity
$\overset{c}{\hookrightarrow}$	Compact embedding
$\hookrightarrow$	Continuous embedding
$O(n^k)$	A sequence $(x_n)_{n\in\mathbb{N}}$ such that $\lvert x_n\rvert \leq Cn^k$ for all n and some $C > 0$

CONTENTS

1 INTRODUCTION

In this thesis we study asymptotic properties of thermoelastic systems modeling longitudinal, vertical and angular motions, the well known Bresse systems named in honor of the French engineer Jacques Antoine Charles Bresse (1822–1883).

Figure 1: Jacques Antoine Charles Bresse.

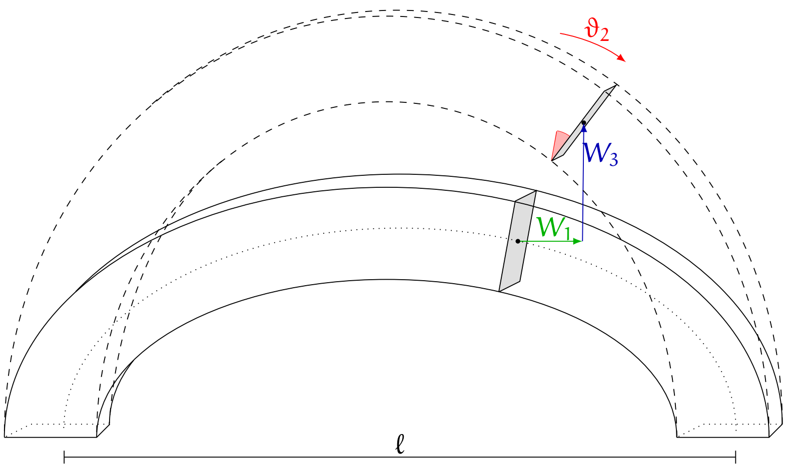
Source: Page 719 of [19].

These systems, following notations of [20], describe the behavior of a thin curved beam with length ℓ and have the form

$$\begin{aligned} m_0 h \ddot{W}_1 &= [Eh(W_1' - \kappa_3 W_3)]' - \kappa_3 Gh(\vartheta_2 + W_3' + \kappa_3 W_1) && \text{(longitudinal motion)} \\ m_0 h \ddot{W}_3 &= [Gh(\vartheta_2 + W_3' + \kappa_3 W_1)]' + \kappa_3 Eh[W_1' - \kappa_3 W_3] && \text{(vertical motion)} \\ m_0 I_{33} \ddot{\vartheta}_2 &= EI_{33}\vartheta_2'' - Gh(\vartheta_2 + W_3' + \kappa_3 W_1) && \text{(shear motion)} \end{aligned} \tag{1.1}$$

where W_1 is the horizontal displacement, W_3 is the vertical displacement, ϑ_2 is the angle of rotation of the cross-section, h is the cross-sectional area, m_0 is the density, G is the shear modulus, E is the Young's modulus, κ_3 is the initial curvature, I_{33} is the moment of inertia of the cross-section with respect to the vertical axis, the prime stands for the derivative with respect to the position $x \in [0, \ell]$ and the dot stands for the derivative with respect to the time $t \geq 0$ (see also [21]).

Figure 2: Displacement of a particle in the centerline of the beam due to deformation.



Source: Created by the author.

Using the notations

$$\varphi := W_3, \ \psi := \vartheta_2, \ w := W_1, \ l := \kappa_3, \ \rho_1 := m_0 h,$$
$$\rho_2 := m_0 I_{33}, \ k := Gh, \ b := EI_{33}, \ k_0 := Eh,$$

system (1.1) takes the form

$$\begin{aligned} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) = 0 \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) = 0, \end{aligned} \tag{1.2}$$

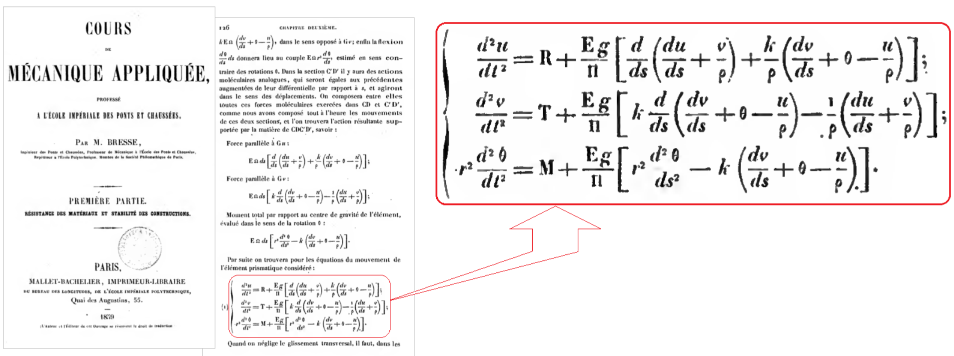
where $k(\varphi_x + \psi + lw)$ is the shear force, $b\psi_x$ is the bending moment, $k_0(w_x - l\varphi)$ is the axial force and all coefficients are positive constants. Note that, following the model described in [20, 21], the parameter k_0 can be completely determined by ρ_1, ρ_2 and b. More precisely:

$$k_0 = \frac{b\rho_1}{\rho_2}, \tag{1.3}$$

which means, from the physical point of view, that the shear and longitudinal motions have the same wave speeds. Of course, from the mathematical point of view, we can neglect condition (1.3) and study (1.2) without any restriction on k_0, which is the standard approach used by many authors. In our case, we will also work without any restriction on k_0, obtaining general results particularly valid in the case (1.3). However, we will keep in mind (1.3) and discuss its implications because, for the asymptotic stability of solutions, the relationship between the coefficients of the system will play a fundamental role. So, in order to distinguish both cases, we will refer to the general system (without any restriction on k_0) as *the mathematical system* and to the particular case with the restriction (1.3) as *the physical system*.

In this context, we will highlight briefly a few contributions concerning Bresse systems, which are the main references of this thesis. In 1859, the Bresse system was first derived in [4].

Figure 3: First publication of the Bresse system.



Source: Page 126 of [4].

In 1993-1994, in [20, 21], this model was derived again in a more general and modern approach which includes the Timoshenko system - a system that, roughly speaking, can be obtained from Bresse system by taking $w = 0$ and $l = 0$, i.e., neglecting longitudinal displacements and supposing zero initial curvature. To our best knowledge, the first work related to the well-posedness and stability of Bresse systems is given by [23], where the authors studied *the physical system*

$$\begin{aligned} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + l\gamma\theta = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\vartheta_x = 0 \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + \gamma\theta_x = 0 \\ \vartheta_t - \kappa_1\vartheta_{xx} + m\psi_{xt} = 0 \\ \theta_t - \kappa_1\theta_{xx} + m(w_{xt} - l\varphi_t) = 0, \end{aligned} \tag{1.4}$$

which contains two thermal dissipations θ and ϑ given by the Fourier's law, acting in axial force and bending moment, respectively. In the framework of semigroup theory, the authors proved existence and uniqueness of solution for two different boundary conditions: Dirichlet condition for all functions and the Dirichlet-Neumann condition

$$\varphi = \psi_x = w_x = \vartheta = \theta = 0 \quad \text{on} \quad \{0, \ell\} \times [0, \infty).$$

About stability, they proved that the solution is polynomially stable with rates of decay that depend on the boundary conditions. They also proved that for both boundary conditions, the solutions are exponentially stable provided that

$$\frac{\rho_1}{\rho_2} = \frac{k}{b}. \tag{1.5}$$

This condition means that the wave speeds of the vertical and longitudinal motions are equal. Additionally, for the Dirichlet-Neumann conditions, they proved that (1.5) is not only sufficient but also necessary for the exponential stability. The results of [23] have been extended to many other Bresse systems with thermal dampings as well as to Bresse systems with different kinds of dissipations, like frictional or memory dampings. One of them was given in [12], where the authors studied *the mathematical system*

$$\begin{aligned} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\vartheta_x = 0 \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) = 0 \\ \vartheta_t - \kappa_1\vartheta_{xx} + m\psi_{xt} = 0 \end{aligned} \tag{1.6}$$

which contains only one thermal dissipation, also given by the Fourier's law, acting in the bending moment. In this work, the authors proved existence, uniqueness and polynomial stability of solutions, also for two type of boundary conditions: Dirichlet condition for all functions and the Dirichlet-Neumann condition

$$\varphi = \psi_x = w_x = \vartheta = \theta = 0 \quad \text{on} \quad \{0, \ell\} \times [0, \infty). \tag{1.7}$$

In general, they obtained the same rate of polynomial decay for both boundary conditions which can be improved provided that $k = k_0$. Additionally they proved that, for both boundary conditions, the solutions are exponentially stable provided that

$$\frac{\rho_1}{\rho_2} = \frac{k}{b} \quad \text{and} \quad k = k_0. \tag{1.8}$$

This condition means that the wave speeds of the vertical, shear and longitudinal motions are equal. Finally, for the Dirichlet-Neumann conditions (1.7), they proved that the sufficient condition (1.8) is also necessary for the exponential stability. Note that, for *the physical system*, (1.8) reduces to (1.5).

The results in [12], concerning to boundary conditions (1.7), were extended in [8] to the case where the Fourier's law is replaced by the Gurtin-Pipkin thermal law. As a particular case, it was shown that *the mathematical system*

$$\begin{aligned} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) &= 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\vartheta_x &= 0 \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) &= 0 \\ \vartheta_t + k_1 p_x + m\psi_{xt} &= 0 \\ \tau p_t + \delta p + \vartheta_x &= 0 \end{aligned} \tag{1.9}$$

which contains a thermal dissipative mechanisms given by the Cattaneo's law acting only in the bending moment, is exponentially stable if and only if the equalities

$$\left(\frac{k_1\rho_1}{k} - \tau\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \tau\frac{\rho_1\gamma m}{bk} = 0 \quad \text{and} \quad k = k_0 \tag{1.10}$$

hold, which is the same result obtained in [17] for different boundary conditions. In particular, for $\tau > 0$, system (1.9) is not exponentially stable if $\frac{\rho_1}{\rho_2} = \frac{k}{b}$.

Note that, in none of the cases presented here, the dissipative mechanism is acting only in the axial force. Therefore, a natural question (to the best of our knowledge, still open) arises: can these results be extend to the Cattaneo-Fourier/Bresse system

$$\begin{aligned}
\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - l[k_0(w_x - l\varphi) - \gamma\theta] &= 0\\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) &= 0\\
\rho_1 w_{tt} - [k_0(w_x - l\varphi) - \gamma\theta]_x + kl(\varphi_x + \psi + lw) &= 0\\
m_0 c_v \theta_t = -q_x - \gamma T_0(w_x - l\varphi)_t & \qquad (1.11)\\
\tau_0 q_t = -(q + \kappa\theta_x), & \qquad (1.12)
\end{aligned}$$

which contains a thermal dissipative mechanisms acting only in the axial force? The aim of Chapter 3 is to give an answer to this question.

In (1.11)-(1.12), c_v is the heat capacity, T_0 is the reference temperature, κ is the heat conductivity, γ is a positive coupling constant, θ is the temperature deviation, q is the heat flux and τ_0 is a non negative constant standing for the heat flux relaxation. Note that if $\tau_0 = 0$, then (1.11)-(1.12) reduces to the classical Fourier's law

$$m_0 c_v \theta_t = \kappa\theta_{xx} - \gamma T_0(w_x - l\varphi)_t.$$

For simplicity, we will use the notations $k_1 := \frac{1}{m_0 c_v}$, $m := \frac{\gamma T_0}{m_0 c_v}$, $\tau := \frac{\tau_0}{\kappa}$ and $\delta := \frac{1}{\kappa}$. So, the problem under consideration takes the form

$$\begin{aligned}
\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + l\gamma\theta &= 0\\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) &= 0\\
\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + \gamma\theta_x &= 0\\
\theta_t + k_1 q_x + m(w_{xt} - l\varphi_t) &= 0\\
\tau q_t + \delta q + \theta_x &= 0
\end{aligned} \qquad (1.13)$$

where $\tau \geq 0$ and the other coefficients are positive constants. For both cases, $\tau > 0$ and $\tau = 0$, we consider boundary conditions

$$\varphi = \psi_x = w_x = \theta = 0 \quad \text{on} \quad \{0, \ell\} \times [0, \infty). \qquad (1.14)$$

The initial conditions, for the case $\tau > 0$, are given by

$$\left.\begin{array}{llll}
\varphi = \varphi_0, & \psi = \psi_0, & w = w_0, & \theta = \theta_0\\
\varphi_t = \varphi_1, & \psi_t = \psi_1, & w_t = w_1, & q = q_0
\end{array}\right\} \quad \text{on} \quad (0, \ell) \times \{0\}.$$

For the case $\tau = 0$, note that $q = -\kappa\theta_x$, then system (1.13) reduces to

$$\begin{aligned}
\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + l\gamma\theta = 0\\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) = 0\\
\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + \gamma\theta_x = 0\\
\theta_t - \kappa_1\theta_{xx} + m(w_{xt} - l\varphi_t) = 0
\end{aligned}$$

where $\kappa_1 = \frac{k_1}{\delta}$, with its corresponding initial conditions

$$\left.\begin{array}{llll}
\varphi = \varphi_0, & \psi = \psi_0, & w = w_0, & \theta = \theta_0\\
\varphi_t = \varphi_1, & \psi_t = \psi_1, & w_t = w_1 &
\end{array}\right\} \quad \text{on} \quad (0, \ell) \times \{0\}.$$

Under the above notations, the main contribution of Chapter 3 is to show that system (1.13)-(1.14), with its corresponding initial conditions, is exponentially stable if and only if the equalities

$$\frac{\rho_1}{\rho_2} = \frac{k}{b} \quad \text{and} \quad \underbrace{\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau\gamma m}_{:=\chi} = 0 \tag{1.15}$$

hold. In particular, for $\tau > 0$, system (1.13)-(1.14) is not exponentially if $k = k_0$. Note that (1.15) coincides with (1.10) if and only if $\tau = 0$, in which case they reduce to (1.8). Additionally, we obtain polynomial stability for the case where (1.15) is not satisfied. These stability results are interesting because, contrary to what happens with systems (1.4), (1.6) and (1.9), the Timoshenko version of (1.13) is conservative instead of dissipative.

Remark 1.1. From the physical point of view, in the derivation of the three-dimensional model for thin thermoelastic beams, the temperature difference ΔT at a point (x, y, z) of the beam is assumed to satisfy $\Delta T = \theta(x) + y\vartheta(x) + z\Theta(x)$, see [20, 21]. In our case, the model (1.13) is obtained under the assumption that $\Delta T = \theta(x)$. Physically, this means that the temperature difference is assumed to be constant on each cross-section of the beam (gray area in Figure 2), which is reasonable in the situation where y and z are very small.

Table 1 summarizes the results concerning to exponential stability discussed above, recovering the results for *the physical system* and comparing to Timoshenko versions in terms of the constants

$$\chi_0 := b - \frac{k\rho_2}{\rho_1}, \qquad \chi_1 := k_0 - k, \qquad \zeta_\tau := \frac{\tau\gamma m}{\left(\frac{k_1\rho_1}{k} - \tau\right)}.$$

Note that, for the case $\tau = 0$, we have $\zeta_0 = \frac{0}{\left(\frac{k_1\rho_1}{k}\right)} = 0$.

Table 1: Comparison between systems: known and new results on exponential stability.

Damping	Exponential decay		
	Mathematical system	**Physical system**	**Timoshenko version**
Fourier's law on bending moment [12, 28]	If and only if $\chi_0 = \zeta_0$ and $\chi_1 = \zeta_0$	If and only if $\chi_0 = \zeta_0$	If and only if $\chi_0 = \zeta_0$
Cattaneo's law on bending moment [8, 29]	If and only if $\chi_0 = \zeta_\tau$ and $\chi_1 = \zeta_0$	Does not decay (condition is never satisfied)	If and only if $\chi_0 = \zeta_\tau$
Fourier's law on axial force	If and only if $\chi_0 = \zeta_0$ and $\chi_1 = \zeta_0$	If and only if $\chi_0 = \zeta_0$	Does not decay (conservative system)
Cattaneo's law on axial force	If and only if $\chi_0 = \zeta_0$ and $\chi_1 = \zeta_\tau$	Does not decay (condition is never satisfied)	Does not decay (conservative system)

Taking into account the constants χ_0 and χ_1, we can reformulate condition (1.10) as

$$\chi_0\left(\tau - \frac{k_1\rho_1}{k}\right) + \tau\gamma m = 0 \quad \text{and} \quad \chi_1 = 0$$

and condition (1.15) as

$$\chi_1\left(\tau - \frac{k_1\rho_1}{k}\right) + \tau\gamma m = 0 \quad \text{and} \quad \chi_0 = 0,$$

which shows a similarity between the previous condition (1.10) existing in the literature and our new condition (1.15).

It is important to remark here that conditions (1.5), (1.10) and (1.15) are mathematically interesting but cannot be satisfied from the physical point of view. In fact, for conditions (1.5) and (1.15), this happens because the equality $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ is incompatible with the physical condition

$$\frac{\rho_1}{\rho_2} = 2(1+\nu)\frac{k}{b}, \tag{1.16}$$

which results from the relation $E = 2G(1+\nu)$ between the Young's modulus E, the shear modulus G and the Poisson's ratio ν. For conditions (1.10), this happens because, for *the physical system*, we have

$$k_0 - k = 0 \quad \Longleftrightarrow \quad \frac{\rho_1}{\rho_2} = \frac{k}{b}$$

and thus, in this case, we also obtain an incompatibility with (1.16).

Now, returning to the context of system (1.9), adding another thermal dissipation

θ acting in the axial force, the resulting system is exponentially stable even in the case where the second equality of (1.10) is dropped. In other words, and with a slightly different notation, if

$$\left(\varsigma - \frac{k_1\rho_1}{k}\right)\left(b - \frac{k\rho_2}{\rho_1}\right) + \varsigma\gamma m = 0, \tag{1.17}$$

then system

$$\begin{aligned} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - l[k_0(w_x - l\varphi) - \gamma\theta] &= 0 \\ \rho_2\psi_{tt} - [b\psi_x - \gamma\vartheta]_x + k(\varphi_x + \psi + lw) &= 0 \\ \rho_1 w_{tt} - [k_0(w_x - l\varphi) - \gamma\theta]_x + kl(\varphi_x + \psi + lw) &= 0 \\ \vartheta_t + k_1 p_x + m\psi_{xt} &= 0 \\ \varsigma p_t + \delta p + \vartheta_x &= 0 \\ \theta_t + k_1 q_x + m(w_{xt} - l\varphi_t) &= 0 \\ \tau q_t + \delta q + \theta_x &= 0 \end{aligned} \tag{1.18}$$

with boundary conditions

$$\varphi = \psi_x = w_x = \vartheta = \theta = 0 \quad \text{on} \quad \{0, \ell\} \times [0, \infty) \tag{1.19}$$

is exponentially stable. This result was proved in [13] and, to the best of our knowledge, it is the unique known result on exponential stability of (1.18)-(1.19) providing only a sufficient condition. In Chapter 4, we will prove a similar result but starting our analysis studying system (1.13) instead of (1.9). In fact, adding another thermal damping ϑ in system (1.13), now acting in the bending moment, we will prove that the resulting system is exponentially stable even in the case where the first equality of (1.15) is dropped. Note that in both cases, after adding the other damping, we obtain the same system (1.18). Therefore, our result says that if

$$\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau\gamma m = 0, \tag{1.20}$$

then system (1.18)-(1.19) is exponentially stable. This implies that the equalities (1.17) and (1.20) are two different sufficient conditions for the exponential stability of (1.18)-(1.19). So, combining these two results, we will prove that the sufficient and necessary condition for the exponential decay of system (1.18)-(1.19) is given by

$$\left[\left(\varsigma - \frac{k_1\rho_1}{k}\right)\left(b - \frac{k\rho_2}{\rho_1}\right) + \varsigma\gamma m\right]\left[\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau\gamma m\right] = 0. \tag{1.21}$$

Summarizing the previous discussion, the main contribution of Chapter 4 is to prove that the system (1.18)-(1.19) is exponentially stable if and only if equality (1.21) holds. Therefore, our results provide a complete characterization of the exponential stability of system (1.18)-(1.19) which, as far as we know, is an open problem in the literature. Observe that (1.21) generalizes the standard equal wave speed condition (1.5). Additionally, observe that the number

$$\left[\left(\varsigma-\frac{k_1\rho_1}{k}\right)\left(\rho_2-\frac{b\rho_1}{k}\right)-\frac{\varsigma\rho_1\gamma m}{k}\right]\left[\left(\tau-\frac{k_1\rho_1}{k}\right)\left(\rho_1-\frac{k_0\rho_1}{k}\right)-\frac{\tau\rho_1\gamma m}{k}\right],$$

obtained when we multiply (1.21) by $\left(-\frac{\rho_1}{k}\right)^2$, generalizes the stability number of the Timoshenko system with Cattaneo's law given in [29].

Remark 1.2. From the physical point of view, the relevance of our result is that, compared with system (1.4), system (1.18) considers thermal effects governed by the Cattaneo's law which removes the paradox of infinite propagation speed inherent in the Fourier's law. Additionally, compared with systems (1.9) and (1.13), system (1.18) is more realistic because it does not neglect the effects of the temperature in any direction. Furthermore, for *the physical system*, condition (1.21) reduces to

$$\left[\left(\varsigma-\frac{k_1\rho_1}{k}\right)\left(b-\frac{k\rho_2}{\rho_1}\right)+\varsigma\gamma m\right]\left[\left(\tau-\frac{k_1\rho_1}{k}\right)\left(\frac{b\rho_1}{\rho_2}-k\right)+\tau\gamma m\right]=0,$$

which is not incompatible with (1.16), contrary to what happens with conditions (1.5), (1.10) and (1.15).

Finally but not less important, we observe that there exists a quite large number of references in the literature studying Bresse systems. Until now, we have cited only a few of them which we believe are the most important to help the readers to understand our contributions. However, in order to give a most comprehensive survey on Bresse systems in the context of our work, we will cite some other papers about Bresse systems with thermal dampings on bounded domains. In [25] the authors studied a version of (1.4) with thermal effects of type III for θ and Fourier's law for ϑ. They proved that condition (1.5) is sufficient for the exponential stability and showed existence of global attractors. In [16] the authors studied a version of (1.4) with variable coefficients, boundary damping and heat flux given by Cattaneo's law. In [27] the authors studied existence and uniqueness of solutions for nonhomogeneous and nonlinear versions of (1.4). In [26] the authors extended the results of [12] to a version of (1.6) with locally distributed thermal damping. In [11] the authors extended the results of [12] to a version of (1.6) with nonlinear motion equation. In [30] the author extended the results of [12] to a version of (1.6) with heat conduction given by thermoelasticity of type III. In [15] the authors studied the system (1.9) with boundary damping. In [1] the authors studied a version of (1.9) with history. In [3], the authors studied a variant of (1.13) for different mixed boundary conditions. In such variant, looking to the precise Bresse model, the corresponding coupling terms are incomplete: the term θ in the first equation and the term φ_t in the coupling of the heat equation were omitted. In this case, the authors proved that condition (1.15) is no longer equivalent to the exponential decay and an extra restriction on the curvature is needed. Under the same condition on the curvature, polynomial stability was also obtained even in the case where (1.15) is not satisfied. Finally, in [14], the author obtained similar results for the same model with Cattaneo's law replaced by thermal effects of types I and III. In this case, condition (1.15) is replaced by the usual equal wave speeds condition, the same restriction on the curvature is needed.

The remainder of this thesis is organized as follows. In Chapter 3 we study the system (1.13)-(1.14): in Section 3.1 we formulate it, with its corresponding initial conditions, as an abstract Cauchy problem and prove existence and uniqueness of solutions; in Section 3.2 we prove exponential stability if and only if the condition (1.15) holds; in Section 3.3 we prove that, regardless of the relationship between the coefficients, the solution is always polynomially stable. In Chapter 4 we study the system (1.18)-(1.19): in Section 4.1 we formulate it, with its corresponding initial conditions, as an abstract Cauchy problem and prove existence and uniqueness of solution; in Section 4.2 we prove that the exponential decay of the system is characterized by the condition (1.21).

2 PRELIMINARY RESULTS

The aim of this chapter is to present the main tools which will by needed for the proofs presented in the next chapters. All results presented here are rather standard and, when a reference is not given, the result is very simple and a proof is presented only for convenience of the reader (not because it has any sort of originality).

2.1 Functional analysis

Let us recall that a space X equipped with a norm $\|\cdot\|$, represented by $(X, \|\cdot\|)$, is called a **Banach space** if it is complete with respect to $\|\cdot\|$. And two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ defined on the same space X are said to be **equivalent** if there exist two positive constants a and b such that

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1, \quad \forall\, x \in X.$$

Theorem 2.1 ([18], p. 106)**.** Let $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ be Banach spaces. If there exists a constant $c_1 > 0$ such that

$$\|x\|_1 \leq c_1\|x\|_2, \quad \forall\, x \in X$$

then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Let us recall also that a linear map $A : X \to Y$ is **compact** provided that $\overline{A(U)}$ is a compact subset of Y for all compact subset U of X. For the particular case in which X is identified with a subspace of Y, if the embedding $X \ni x \mapsto x \in Y$ is compact we say that X is **compactly embedded** in Y and write $X \overset{c}{\hookrightarrow} Y$.

Theorem 2.2 ([18], p. 231)**.** Let X and Y be two normed spaces and $B : X \to Y$ a linear map. Then, the following assertions are equivalent.

- B is compact.
- If $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence in X, then the sequence $\{Bx_n\}_{n\in\mathbb{N}}$ has a subsequence which converges in Y.

Now, let us recall that a linear operator with **compact resolvent** is a linear operator $A : D(A) \subset X \to X$ for which there exists $\lambda \in \rho(A)$ such that $(\lambda I - A)^{-1}$ is compact, where $\rho(A)$ is the **resolvent set** of A formed by all $\lambda \in \mathbb{C}$ such that the operator $(\lambda I - A)^{-1}$ exists, is bounded and has dense domain.

Theorem 2.3 ([9], p. 117)**.** Let $(X, \|\cdot\|_X)$ be a Banach space, $A : D(A) \subset X \to X$ a linear operator with nonempty resolvent and $\|\cdot\|_{D(A)}$ the **graph norm** defined on $D(A)$ by $\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X$. Then, the following assertions are equivalent.

- A has compact resolvent.
- $(D(A), \|\cdot\|_{D(A)}) \overset{c}{\hookrightarrow} (X, \|\cdot\|_X)$.

On the other hand, we recall that the **spectrum** of a linear operator $A : D(A) \subset X \to X$ is the set $\sigma(A) = \mathbb{C} \setminus \rho(A)$ and that the **point spectrum** of A is the subset $\sigma_p(A)$ of $\sigma(A)$ formed by all $\lambda \in \mathbb{C}$ such that $(\lambda I - A)$ is not invertible. The elements of $\sigma_p(A)$ are called **eigenvalues** of A.

Theorem 2.4 ([9], p. 248)**.** Let X be a Banach space and $A : D(A) \subset X \to X$ a linear operator. If A has compact resolvent, then $\sigma(A) = \sigma_p(A)$.

Finally, in order to finish this section with the the Lax-Milgram Theorem, we recall that given a Hilbert space H and a map $B : H \times H \to \mathbb{C}$ we say that B is:

- a **sesquilinear form** if

$$\left.\begin{aligned} &B(u+x, v) = B(u,v) + B(x,v) \text{ and } B(\alpha u, v) = \alpha B(u,v) \\ &B(u, v+x) = B(u,v) + B(u,x) \text{ and } B(u, \alpha v) = \overline{\alpha} B(u,v) \end{aligned}\right\} \quad \forall\, u, v, x \in H,\ \alpha \in \mathbb{C}.$$

- **continuous** if there exists a constant $C > 0$ such that

$$|B(u,v)| \leq C\|u\|\|v\|, \quad \forall\, u, v \in H.$$

- **coercive** if there exists a constant $M > 0$ such that

$$\mathrm{Re}(B(u,u)) \geq M\|u\|^2, \quad \forall\, u \in H.$$

Theorem 2.5 ([7], p. 376)**.** Let H be a Hilbert space on $\mathbb{C}$ and $B : H \times H \to \mathbb{K}$ a continuous coercive sesquilinear form. Then, given a bounded linear functional $f : H \to \mathbb{C}$, there exists a unique $z \in H$ such that

$$f(u) = B(u,z), \quad \forall\, u \in H.$$

2.2 Sobolev spaces

Let

$$L^p(\Omega), \quad W^{m,p}(\Omega) \quad \text{and} \quad W_0^{m,p}(\Omega) \tag{2.1}$$

be the usual Lebesgue and Sobolev spaces, defined as in [2]. Initially, let us recall that any element of $L^p(\Omega)$ can by approximated by elements of the space $C_0^\infty(\Omega)$, formed by the infinitely differentiable functions of compact support:

Theorem 2.6 ([2], p. 38)**.** Let Ω be an open subset of $\mathbb{R}^N$ and $1 \leq p < \infty$. Then, the space $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$.

Let us recall also that, for the space $W_0^{m,p}(\Omega)$, the Poincaré inequality is given as follows:

Theorem 2.7 ([2], p. 183)**.** Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^N$ be an open set of finite width (that is, an open set that lies between two parallel planes of dimension $N-1$). Then, there exists a constant $C > 0$ such that

$$\|u\|_{L^p} \leq C\|\nabla u\|_{L^p}, \quad \forall\, u \in W_0^{1,p}(\Omega).$$

And, for the space $W^{1,p}(\Omega)$, the Poincaré inequality takes the following form:

Theorem 2.8 ([10], p. 275)**.** Let $1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^N$ be an connected bounded open set of class C^1. Then, there exists a constant $C > 0$ such that

$$\left\| u - \frac{1}{\operatorname{med}\Omega}\int_\Omega u(x)\, dx \right\|_{L^p} \leq C\|\nabla u\|_{L^p}, \quad \forall\, u \in W^{1,p}(\Omega).$$

Let us recall finally that, under the notations $p_m^* := \frac{Np}{N-mp}$ and $W^{0,q} := L^q$, the Rellich-Kondrachov Theorem says the following:

Theorem 2.9 ([2], p. 168-169)**.** Let Ω be a bounded open subset of $\mathbb{R}^N$ with smooth boundary, m a positive integer, j a nonnegative integer and $1 \leq p < \infty$.

(a) If $mp < N$, then $W^{m+j,p}(\Omega) \overset{c}{\hookrightarrow} W^{j,q}(\Omega)$ for all $q \in [1, p_m^*)$.

(b) If $mp = N$, then $W^{m+j,p}(\Omega) \overset{c}{\hookrightarrow} W^{j,q}(\Omega)$ for all $q \in [1, \infty)$.

(c) If $mp > N$, then $W^{m+j,p}(\Omega) \overset{c}{\hookrightarrow} C^j(\overline{\Omega})$.

- In particular, $W^{m+j,p}(\Omega) \overset{c}{\hookrightarrow} W^{j,p}(\Omega)$ for all m, p and N.
- If we replace $W^{m+j,p}$ by $W_0^{m+j,p}$ in the previous embeddings, then the results are valid for a arbitrary open set Ω (not necessarily bounded with boundary not necessarily smooth).

In this thesis, we will use only the following particular cases of the spaces listed in (2.1), which are all Hilbert spaces:

- $L^2(0,\ell)$ equipped with the usual norm

$$\|u\|_{L^2} := \left(\int_0^\ell |u(x)|^2 \, dx\right)^{\frac{1}{2}}.$$

- $L^2_*(0,\ell) := \left\{u \in L^2(0,\ell) \;\middle|\; \int_0^\ell u \, dx = 0\right\}$ equipped with the norm

$$\|u\|_{L^2_*} := \|u\|_{L^2}$$

  induced from $L^2(0,\ell)$.

- $H^1(0,\ell) := W^{1,2}(0,\ell)$ equipped with the usual norm

$$\|u\|_{H^1} := \left(\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2\right)^{\frac{1}{2}}.$$

- $H_0^1(0,\ell) := W_0^{1,2}(0,\ell)$ equipped with the usual norm

$$\|u\|_{H_0^1} := \|u_x\|_{L^2}.$$

- $H_*^1(0,\ell) := H^1(0,\ell) \cap L_*^2(0,\ell)$ equipped with the norm

$$\|u\|_{H_*^1} := \|u_x\|_{L^2}.$$

- $H^2(0,\ell) := W^{2,2}(0,\ell)$ equipped with the usual norm

$$\|u\|_{H^2} := (\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2)^{\frac{1}{2}}.$$

We remark that in order to obtain estimates for expressions involving these norms, the Young's inequality

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}, \quad a, b \geq 0$$

will be tacitly used many times along the thesis.

Occasionally, we will omit the domain $(0,\ell)$ and simply represent the above spaces by

$$L^2, \quad L^2_*, \quad H^1, \quad H^1_0, \quad H^1_*, \quad H^2. \tag{2.2}$$

For these spaces, the Poincaré inequalities given in Theorem 2.7 and Theorem 2.8 are summarized as follows: there exists $C > 0$ such that

$$\|u\|_{L^2} \leq C\|u_x\|_{L^2}, \quad \forall\, u \in \left(H^1_0(0,\ell) \cup H^1_*(0,\ell)\right).$$

Furthermore, as consequence of Theorem 2.6, we have the following density results:

Theorem 2.10. The space $H^1_*(0,\ell)$ is dense in $L^2_*(0,\ell)$.

Proof. Take $f \in L^2_*(0,\ell)$. By Theorem 2.6, there exists a sequence $(f^{(n)})_{n\in\mathbb{N}}$ in $C^\infty_0(0,\ell)$ such that

$$\|f^{(n)} - f\|_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

Define $g^{(n)} = f^{(n)} - c_n$, where

$$c_n := \frac{1}{\ell}\int_0^\ell f^{(n)}\, dx.$$

Then, $g^{(n)} \in H^1_*(0,\ell)$. In addition, as $f \in L^2_*(0,\ell)$, we have

$$c_n = \frac{1}{\ell}\int_0^\ell (f^{(n)} - f)\, dx.$$

Therefore,

$$\begin{aligned}\|c_n\|_{L^2} = |c_n|\cdot\sqrt{\ell} = \left|\frac{1}{\ell}\int_0^\ell (f^{(n)} - f)\, dx\right|\cdot\sqrt{\ell} &\leq \frac{1}{\sqrt{\ell}}\int_0^\ell |f^{(n)} - f|\cdot 1\, dx \\ &\leq \frac{1}{\sqrt{\ell}}\|f^{(n)} - f\|_{L^2}\|1\|_{L^2} \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Thus,

$$\|g^{(n)} - f\|_{L^2} = \|f^{(n)} - c_n - f\|_{L^2} \leq \|f^{(n)} - f\|_{L^2} + \|c_n\|_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

This shows that there exists a sequence $(g^{(n)})_{n\in\mathbb{N}}$ in $H^1_*(0,\ell)$ that converges to f in $L^2_*(0,\ell)$. □

Theorem 2.11. The space $J := \{u \in H^1_*(0,\ell) \mid u_x \in H^1_0(0,\ell)\}$ is dense in $H^1_*(0,\ell)$.

Proof. Take $f \in H^1_*(0,\ell)$. By Theorem 2.6, there exists $(z^{(n)})_{n\in\mathbb{N}}$ in $C^\infty_0(0,\ell)$ such that

$$\|z^{(n)} - f_x\|_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

Define

$$g^{(n)}(x) = \int_0^x z^{(n)}(s)\ ds - \frac{1}{\ell}\int_0^\ell \int_0^y z^{(n)}(s)\ ds\ dy.$$

Then, $g^{(n)} \in J$. In addition,

$$\|g^{(n)} - f\|_{H^1_*} = \|g^{(n)}_x - f_x\|_{L^2} = \|z^{(n)} - f_x\|_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

This shows that there exists a sequence $(g^{(n)})_{n\in\mathbb{N}}$ in J that converges to f in $H^1_*(0,\ell)$. □

Now, in order to present compact embeddings for the spaces (2.2), we look at the one-dimensional version of Theorem 2.9:

Theorem 2.12 ([5], p. 217)**.** Let I be a bounded open interval, m a positive integer and $1 < p \le \infty$. Then,

$$W^{m,p}(I) \overset{c}{\hookrightarrow} C^{m-1}(\bar{I}).$$

Corollary 2.13. For the spaces (2.2), we have

(a) $H^1 \overset{c}{\hookrightarrow} L^2$.

(b) $H^1_* \overset{c}{\hookrightarrow} L^2_*$.

(c) $(H^2 \cap H^1_*) \overset{c}{\hookrightarrow} H^1_*$.

(d) $(H^2 \cap H^1_0) \overset{c}{\hookrightarrow} H^1_0$.

Proof.

(a) Taking $m = 1$ and $p = 2$ in Theorem 2.12, we conclude that $H^1(0,\ell) = W^{1,2}(0,\ell) \overset{c}{\hookrightarrow} C[0,\ell] \hookrightarrow L^2(0,\ell)$. Therefore, $H^1(0,\ell) \overset{c}{\hookrightarrow} L^2(0,\ell)$.

(b) Let $(u_n)_{n\in\mathbb{N}}$ be bounded in $H^1_*(0,\ell)$. Then, $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(0,\ell)$, which is compactly embedded in $L^2(0,\ell)$ by item (a). Thus, there exist a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and a function $u \in L^2(0,\ell)$ such that

$$u_{n_k} \overset{k\to\infty}{\longrightarrow} u \quad \text{in} \quad L^2(0,\ell).$$

Since $u_n \in L^2_*(0,\ell)$ for all $n \in \mathbb{N}$ and $L^2_*(0,\ell)$ is complete, it follows that $u \in L^2_*(0,\ell)$. Therefore,

$$u_{n_k} \overset{k\to\infty}{\longrightarrow} u \quad \text{in} \quad L^2_*(0,\ell),$$

which implies that $H^1_*(0,\ell) \overset{c}{\hookrightarrow} L^2_*(0,\ell)$ by Theorem 2.2.

(c) Let $(u_n)_{n\in\mathbb{N}}$ be bounded in $H^2(0,\ell)\cap H^1_*(0,\ell)$. Then, $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^2(0,\ell)$, which is compactly embedded in $H^1(0,\ell)$ by Theorem 2.12 (with $m=p=2$). Thus, there exist a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and a function $u\in H^1(0,\ell)$ such that

$$u_{n_k} \overset{k\to\infty}{\longrightarrow} u \quad \text{in} \quad H^1(0,\ell).$$

Since $u_n \in H^1_*(0,\ell)$ for all $n\in\mathbb{N}$ and $H^1_*(0,\ell)$ is complete, it follows that $u\in H^1_*(0,\ell)$. Therefore,

$$u_{n_k} \overset{k\to\infty}{\longrightarrow} u \quad \text{in} \quad H^1_*(0,\ell),$$

which implies that $[H^2(0,\ell)\cap H^1_*(0,\ell)] \overset{c}{\hookrightarrow} H^1_*(0,\ell)$ by Theorem 2.2.

(d) The proof is exactly as the proof of item (c), with H^1_* replaced by H^1_0. □

Finally, we finish this section with two simple existence lemmas.

Lemma 2.14. Let $l>0$, $f\in L^2(0,\ell)$ and $g\in L^2_*(0,\ell)$. Suppose that

$$l\ell \neq n\pi, \quad \forall\, n\in\mathbb{N}. \tag{2.3}$$

Then, there exist $\varphi\in H^1_0(0,\ell)$ and $w\in H^1_*(0,\ell)$ such that

$$\begin{cases} w_x - l\varphi = f, \\ \varphi_x + lw = g. \end{cases} \tag{2.4}$$

Proof. Define

$$\begin{aligned} \varphi(x) = c_1\cos(lx) - c_2\sin(lx) + \cos(lx)\int_0^x [g(s)\cos(ls)+f(s)\sin(ls)]\,ds \\ + \sin(lx)\int_0^x [g(s)\sin(ls)-f(s)\cos(ls)]\,ds \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} w(x) = c_1\sin(lx) + c_2\cos(lx) + \sin(lx)\int_0^x [g(s)\cos(ls)+f(s)\sin(ls)]\,ds \\ + \cos(lx)\int_0^x [f(s)\cos(ls)-g(s)\sin(ls)]\,ds. \end{aligned} \tag{2.6}$$

It is clear that $\varphi, w\in H^1(0,\ell)$. In addition, differentiating (2.5)-(2.6) and substituting into the equations, we see that φ and w satisfy the system (2.4). Taking $c_1=0$ and

$$c_2 = \frac{\cos(l\ell)}{\sin(l\ell)}\int_0^\ell [g(s)\cos(ls)+f(s)\sin(ls)]\,ds + \int_0^\ell [g(s)\sin(ls)-f(s)\cos(ls)]\,ds,$$

we conclude that $\varphi(0) = \varphi(\ell) = 0$. Therefore, $\varphi \in H_0^1(0,\ell)$ (note that c_2 is well defined because $\sin(l\ell) \neq 0$, which follows from assumption (2.3)). Finally, we conclude that $w \in H_*^1(0,\ell)$ because, integrating the second equation of (2.4) over $[0,\ell]$, we obtain

$$\int_0^\ell w(s)\, ds = 0.$$

□

Lemma 2.15. Let $l > 0$, $H \in L_*^2(0,\ell)$ and $F, G \in L^2(0,\ell)$. Suppose that condition (2.3) holds. Then, there exist $\varphi \in H_0^1(0,\ell)$ and $\psi, w \in H_*^1(0,\ell)$ such that

$$\begin{cases} \psi_x = F, \\ w_x - l\varphi = G, \\ \varphi_x + \psi + lw = H. \end{cases} \tag{2.7}$$

Proof. Define

$$\psi(x) = \int_0^x F(s)\, ds - \frac{1}{\ell}\int_0^\ell \left(\int_0^y F(s)\, ds \right) dy.$$

Then, $\psi \in H_*^1(0,\ell)$ and

$$\psi_x = F. \tag{2.8}$$

By Lemma 2.14, there exist $\varphi \in H_0^1(0,\ell)$ and $w \in H_*^1(0,\ell)$ such that

$$\begin{cases} w_x - l\varphi = G, \\ \varphi_x + lw = H - \psi. \end{cases} \tag{2.9}$$

Therefore, φ, ψ and w have the desired regularity and, by (2.8)-(2.9), satisfy (2.7). □

2.3 Semigroups of operators

Let us recall that a **semigroup** (of bounded linear operators) on a Banach space X is a family $\{S(t)\}_{t\geq 0}$ in $\mathcal{L}(X)$ which satisfies the following conditions:

- $S(0) = I$, where I is the identity operator from X to X.
- $S(s)S(t) = S(s+t)$, for all $s, t \geq 0$.

Let us recall also that a semigroup $\{S(t)\}_{t\geq 0}$ is called:

- **strongly continuous** (or C_0) if $\lim\limits_{t\to 0^+} \|S(t)x - x\|_X = 0$, for all $x \in X$.
- **bounded** if there exists a constant $M \geq 0$ such that $\|S(t)\|_{\mathcal{L}} \leq M$, for all $t \geq 0$.

- **of contractions** if $\|S(t)\|_{\mathcal{L}} \leq 1$, for all $t \geq 0$.

In addition, let us recal that the **infinitesimal generator** of a semigroup $\{S(t)\}_{t\geq 0}$ on X is the operator $A : D(A) \subset X \to X$ defined by

$$Ax = \lim_{t\to 0^+} \frac{S(t)x - x}{t}$$

with domain

$$D(A) = \left\{ x \in X \;\middle|\; \text{the limit } \lim_{t\to 0^+} \frac{S(t)x - x}{t} \text{ exists} \right\}.$$

Finally, let us recall the if $\{S(t)\}_{t\geq 0}$ is the semigroup whose infinitesimal generator is the operator A, then we write $S(t) = e^{tA}$.

For our purposes, the importance of the concept of C_0-semigroup is due to the following result.

Theorem 2.16 ([9], p. 145)**.** Let X be a Banach space and $A : D(A) \subset X \to X$ the infinitesimal generator of a C_0-semigroup on X. If $U_0 \in D(A)$, then the problem

$$\begin{cases} U_t = AU, \quad t > 0 \\ U(0) = U_0 \end{cases} \tag{2.10}$$

has a unique solution $U \in C^1\big([0, \infty); X\big)$, which is given by $U(t) = e^{tA}U_0$.

Therefore, in order to show that a problem of the form (2.10) is well-posed, it is enough to show that A is the infinitesimal generator of a C_0-semigroup on a suitable space. In this thesis, this goal will be attained as an application of the following well-known result.

Theorem 2.17 ([24], p. 3)**.** Let H be a Hilbert space and $A : D(A) \subset H \to H$ a linear operator. If $D(A)$ is dense in H, A is dissipative and $0 \in \rho(A)$, then A is the infinitesimal generator of a C_0-semigroup of contractions on H, where "dissipative" means that

$$\mathrm{Re}(Ax, x)_H \leq 0, \quad \forall\, x \in D(A).$$

Given that the solution of our problem has the form $U(t) = e^{tA}U_0$, as guaranteed by Theorem 2.16, the analysis of the asymptotic properties of the solution U reduces to the analysis of the asymptotic properties of the semigroup $\{e^{tA}\}_{t\geq 0}$. Therefore, in order to obtain exponential stability, we will use the Gearhart-Prüss Theorem:

Theorem 2.18 ([24], p. 4)**.** Let $\{e^{tA}\}_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space H. Then, $\{e^{tA}\}_{t\geq 0}$ is exponentially stable if and only if the following conditions are satisfied:

(a) $\mathbf{i}\mathbb{R} \subset \rho(A)$.

(b) $\limsup\limits_{|\beta|\to\infty} \|(\mathbf{i}\beta I - A)^{-1}\|_{\mathcal{L}(H)} < \infty$.

On the other hand, in order to obtain polynomial stability, we will use the Borichev-Tomilov Theorem:

Theorem 2.19 ([6], p. 459)**.** Let $\{e^{tA}\}_{t\geq 0}$ be a bounded semigroup on a Hilbert space H. Suppose that $\mathbf{i}\mathbb{R} \subset \rho(\mathcal{A})$. Then, fixed $\alpha > 0$, the following assertions are equivalent:

(a) There exist positive constants C and β_0 such that

$$\|(\mathbf{i}\beta I - \mathcal{A})^{-1})\|_{\mathcal{L}(H)} \leq C|\beta|^{\alpha}, \qquad \forall\, |\beta| \geq \beta_0.$$

(b) There exist positive constants C and t_0 such that

$$\|e^{t\mathcal{A}}\mathcal{A}^{-1}\|_{\mathcal{L}} \leq C\frac{1}{t^{1/\alpha}}, \qquad \forall\, t \geq t_0.$$

3 HEAT CONDUCTION ON AXIAL FORCE

In this chapter we prove that system (1.13)-(1.14), with its corresponding initial conditions, has a unique solution, which is exponentially stable if and only if condition (1.15) holds. We also prove polynomial stability in the case where the said condition is not satisfied.

3.1 Semigroup formulation and well-posedness

We will consider the problem in the framework of semigroup theory. Therefore, the aim of this section is to formulate it as an abstract Cauchy problem.

Initially, note that multiplying (formally) the equations of (1.13) by $\overline{\varphi}_t$, $\overline{\psi}_t$, $\overline{w}_t$, $\frac{\gamma}{m}\overline{\theta}$ and $\frac{\gamma k_1}{m}\overline{q}$, respectively, integrating over $[0,\ell]$ and taking the real part we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^\ell \big(\rho_1|\varphi_t|^2+\rho_2|\psi_t|^2+\rho_1|w_t|^2+b|\psi_x|^2+k|\varphi_x+\psi+lw|^2+k_0|w_x-l\varphi|^2\big)\,dx \\ +\frac{1}{2}\frac{d}{dt}\int_0^\ell\big(\tfrac{\gamma}{m}|\theta|^2+\tfrac{\gamma k_1\tau}{m}|q|^2\big)\,dx=-\frac{\delta\gamma k_1}{m}\int_0^\ell |q|^2\,dx,$$

which defines (formally) the energy of the system (1.13) by

$$E_\tau(t)=\rho_1\int_0^\ell|\varphi_t|^2\,dx+\rho_2\int_0^\ell|\psi_t|^2\,dx+\rho_1\int_0^\ell|w_t|^2\,dx+b\int_0^\ell|\psi_x|^2\,dx \\ +k\int_0^\ell|\varphi_x+\psi+lw|^2\,dx+k_0\int_0^\ell|w_x-l\varphi|^2\,dx+\frac{\gamma}{m}\int_0^\ell|\theta|^2\,dx+\frac{\gamma k_1\tau}{m}\int_0^\ell|q|^2\,dx.$$

Motivated by this calculation, using the notation presented in (2.2), we consider the phase space

$$\mathcal{H}_\tau=\begin{cases} H_0^1\times L^2\times H_*^1\times L_*^2\times H_*^1\times L_*^2\times L^2 & \text{for}\quad \tau=0\\ H_0^1\times L^2\times H_*^1\times L_*^2\times H_*^1\times L_*^2\times L^2\times L_*^2 & \text{for}\quad \tau>0\end{cases}$$

with inner product defined by

$$(\tilde{U},U)_\tau=\rho_1\int_0^\ell\tilde{\Phi}\overline{\Phi}\,dx+\rho_2\int_0^\ell\tilde{\Psi}\overline{\Psi}\,dx+\rho_1\int_0^\ell\tilde{W}\overline{W}\,dx+b\int_0^\ell\tilde{\psi}_x\overline{\psi_x}\,dx \\ +k\int_0^\ell(\tilde{\varphi}_x+\tilde{\psi}+l\tilde{w})\overline{(\varphi_x+\psi+lw)}\,dx+k_0\int_0^\ell(\tilde{w}_x-l\tilde{\varphi})\overline{(w_x-l\varphi)}\,dx \\ +\frac{\gamma}{m}\int_0^\ell\tilde{\theta}\overline{\theta}\,dx+\frac{\gamma k_1\tau}{m}\int_0^\ell\tilde{q}\overline{q}\,dx, \tag{3.1}$$

where U and $\tilde{U}$, are given by

$$U = \begin{cases} (\varphi, \Phi, \psi, \Psi, w, W, \theta) & \text{for} \quad \tau = 0 \\ (\varphi, \Phi, \psi, \Psi, w, W, \theta, q) & \text{for} \quad \tau > 0 \end{cases}$$

and

$$\tilde{U} = \begin{cases} (\tilde{\varphi}, \tilde{\Phi}, \tilde{\psi}, \tilde{\Psi}, \tilde{w}, \tilde{W}, \tilde{\theta}) & \text{for} \quad \tau = 0 \\ (\tilde{\varphi}, \tilde{\Phi}, \tilde{\psi}, \tilde{\Psi}, \tilde{w}, \tilde{W}, \tilde{\theta}, \tilde{q}) & \text{for} \quad \tau > 0. \end{cases}$$

Now, system (1.13)-(1.14) can be written as an evolution equation on $\mathcal{H}_\tau$ given by

$$\begin{cases} U_t = \mathcal{A}_\tau U, \quad t > 0 \\ U(0) = U_0, \end{cases} \tag{3.2}$$

where

$$U_0 = \begin{cases} (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, \theta_0) & \text{for} \quad \tau = 0 \\ (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, \theta_0, q_0) & \text{for} \quad \tau > 0, \end{cases}$$

and the operator $\mathcal{A}_\tau : D(\mathcal{A}_\tau) \subset \mathcal{H}_\tau \to \mathcal{H}_\tau$ is defined by

$$\mathcal{A}_\tau U = \begin{cases} \begin{pmatrix} \Phi \\ \frac{k}{\rho_1}(\varphi_x + \psi + lw)_x + \frac{k_0 l}{\rho_1}(w_x - l\varphi) - \frac{l\gamma}{\rho_1}\theta \\ \Psi \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi + lw) \\ W \\ \frac{k_0}{\rho_1}(w_x - l\varphi)_x - \frac{kl}{\rho_1}(\varphi_x + \psi + lw) - \frac{\gamma}{\rho_1}\theta_x \\ \kappa_1\theta_{xx} - m(W_x - l\Phi) \end{pmatrix} & \text{for} \quad \tau = 0 \\[2ex] \begin{pmatrix} \Phi \\ \frac{k}{\rho_1}(\varphi_x + \psi + lw)_x + \frac{k_0 l}{\rho_1}(w_x - l\varphi) - \frac{l\gamma}{\rho_1}\theta \\ \Psi \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi + lw) \\ W \\ \frac{k_0}{\rho_1}(w_x - l\varphi)_x - \frac{kl}{\rho_1}(\varphi_x + \psi + lw) - \frac{\gamma}{\rho_1}\theta_x \\ -k_1 q_x - m(W_x - l\Phi) \\ -\frac{\delta}{\tau}q - \frac{1}{\tau}\theta_x \end{pmatrix} & \text{for} \quad \tau > 0 \end{cases}$$

with domain

$$D(\mathcal{A}_\tau) = \begin{cases} \{U \in \mathcal{H}_0 \mid \varphi, \theta \in H^2, \;\; \Phi, \psi_x, w_x, \theta \in H_0^1, \;\; \Psi, W \in H^1\} & \text{for} \quad \tau = 0 \\ \{U \in \mathcal{H}_\tau \mid \varphi \in H^2, \;\; \Phi, \psi_x, w_x, \theta \in H_0^1, \;\; \Psi, W, q \in H^1\} & \text{for} \quad \tau > 0. \end{cases}$$

In this context, in view of Theorem 2.16, we will prove the well-posedness of system (1.13)-(1.14) by showing that $\mathcal{A}_\tau$ is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}_\tau$. First of all, we prove that $\mathcal{H}_\tau$ is indeed a Hilbert space.

Remark 3.1. As pointed out in [23] (p. 58), to ensure that the bilinear form $(\cdot,\cdot)_\tau$ given by (3.1) defines an inner product on $\mathcal{H}_\tau$, we have to assume that $l\ell$ is not a multiple of π, that is,

$$l\ell \neq n\pi, \quad \forall\, n \in \mathbb{N}. \tag{3.3}$$

Otherwise we can construct a vector $U \neq 0$ in $\mathcal{H}_\tau$ satisfying $(U,U)_\tau = 0$, for example, taking $\varphi(x) = \sin(lx)$, $w(x) = -\cos(lx)$ and the other components equal to zero. Therefore, here and thereafter, we take (3.3) as a hypothesis, otherwise we cannot speak of the “norm induced by (3.1)”.

Theorem 3.2. The space $\mathcal{H}_\tau$, equipped with the norm $\|\cdot\|_\tau$ induced by the inner product (3.1), is complete and thus it is a Hilbert space.

Proof. Suppose that $\tau > 0$. Then,

$$\begin{aligned} \|U\|_\tau^2 = \rho_1\|\Phi\|_{L^2}^2 + \rho_2\|\Psi\|_{L^2}^2 + \rho_1\|W\|_{L^2}^2 + b\|\psi_x\|_{L^2}^2 + k_0\|w_x - l\varphi\|_{L^2}^2 \\ + k\|\varphi_x + \psi + lw\|_{L^2}^2 + \frac{\gamma}{m}\|\theta\|_{L^2}^2 + \frac{\gamma k_1 \tau}{m}\|q\|_{L^2}^2. \end{aligned} \tag{3.4}$$

Therefore, taking a Cauchy sequence (U_n) in $\mathcal{H}_\tau$ and writing

$$U_n = (\varphi^{(n)}, \Phi^{(n)}, \psi^{(n)}, \Psi^{(n)}, w^{(n)}, W^{(n)}, \theta^{(n)}, q^{(n)}),$$

we conclude that

- $(\Phi^{(n)})$, $(\psi_x^{(n)})$, $(w_x^{(n)} - l\varphi^{(n)})$, $(\theta^{(n)})$ are Cauchy sequences in $(L^2(0,\ell), \|\cdot\|_{L^2})$.
- $(\Psi^{(n)})$, $(W^{(n)})$, $(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})$, $(q^{(n)})$ are Cauchy sequences in $(L^2_*(0,\ell), \|\cdot\|_{L^2})$.

Since $(L^2(0,\ell), \|\cdot\|_{L^2})$ and $(L^2_*(0,\ell), \|\cdot\|_{L^2})$ are complete spaces, it follows that there exist $\Phi, F, G, \theta \in L^2(0,\ell)$ and $\Psi, W, H, q \in L^2_*(0,\ell)$ such that

$$\begin{aligned} \|\Phi^{(n)} - \Phi\|_{L^2} &\overset{n\to\infty}{\longrightarrow} 0, \quad & \|\psi_x^{(n)} - F\|_{L^2} &\overset{n\to\infty}{\longrightarrow} 0, \\ \|w_x^{(n)} - l\varphi^{(n)} - G\|_{L^2} &\overset{n\to\infty}{\longrightarrow} 0, \quad & \|\theta^{(n)} - \theta\|_{L^2} &\overset{n\to\infty}{\longrightarrow} 0, \\ \|\Psi^{(n)} - \Psi\|_{L^2} &\overset{n\to\infty}{\longrightarrow} 0, \quad & \|W^{(n)} - W\|_{L^2} &\overset{n\to\infty}{\longrightarrow} 0, \\ \|\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)} - H\|_{L^2} &\overset{n\to\infty}{\longrightarrow} 0, \quad & \|q^{(n)} - q\|_{L^2} &\overset{n\to\infty}{\longrightarrow} 0. \end{aligned} \tag{3.5}$$

By Lemma 2.15, there exist $\varphi \in H^1_0(0,\ell)$ e $\psi, w \in H^1_*(0,\ell)$ such that

$$\begin{cases} \psi_x = F, \\ w_x - l\varphi = G, \\ \varphi_x + \psi + lw = H. \end{cases} \tag{3.6}$$

Taking $U := (\varphi, \Phi, \psi, \Psi, w, W, \theta, q)$, we conclude that $U \in \mathcal{H}_\tau$. In addition, substituting (3.6) into (3.5), we obtain

$$\|U_n - U\|^2_\tau \overset{n\to\infty}{\longrightarrow} 0.$$

Therefore, (U_n) converges in $\mathcal{H}_\tau$. This shows that $\mathcal{H}_\tau$ is complete.

The case $\tau = 0$ is similar (the only difference is that the vectors have seven components instead of eight). □

Remark 3.3. Let $|\cdot|_\tau$ be the usual norm defined in $\mathcal{H}_\tau$ by

$$|U|^2_\tau = \begin{cases} \|\varphi_x\|^2_{L^2} + \|\Phi\|^2_{L^2} + \|\psi_x\|^2_{L^2} + \|\Psi\|^2_{L^2} + \|w_x\|^2_{L^2} + \|W\|^2_{L^2} + \|\theta\|^2_{L^2}, & \tau = 0 \\ \|\varphi_x\|^2_{L^2} + \|\Phi\|^2_{L^2} + \|\psi_x\|^2_{L^2} + \|\Psi\|^2_{L^2} + \|w_x\|^2_{L^2} + \|W\|^2_{L^2} + \|\theta\|^2_{L^2} + \|q\|^2_{L^2}, & \tau > 0. \end{cases}$$

From (3.4), it is clear that there exists a constant $C > 0$ such that

$$\|U\|_\tau \le C|U|_\tau, \quad \forall\, U \in \mathcal{H}_\tau.$$

Therefore, in view of Theorem 3.2 and Theorem 2.1, we conclude that $\|\cdot\|_\tau$ and $|\cdot|\tau$ are equivalent. This equivalence will be tacitly used many times along the thesis.

Now, we are ready to state and prove the main result of this section.

Theorem 3.4 (Existence and uniqueness)**.** The operator $\mathcal{A}_\tau : D(\mathcal{A}_\tau) \subset \mathcal{H}_\tau \to \mathcal{H}_\tau$ is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}_\tau$. Therefore, for each initial data $U_0 \in D(\mathcal{A}_\tau)$, the problem (3.2) has a unique classical solution $U \in C^1([0,\infty); \mathcal{H}_\tau)$, which is given by $U(t) = e^{t\mathcal{A}_\tau}U_0$.

Proof. We will apply Theorem 2.17.

$D(\mathcal{A}_\tau)$ **is dense in** $\mathcal{H}_\tau$**.** Suppose that $\tau > 0$. Then,

$$\mathcal{H}_\tau = Y_1 \times Y_2 \times Y_3 \times Y_4 \times Y_5 \times Y_6 \times Y_7 \times Y_8,$$

$$D(\mathcal{A}_\tau) = S_1 \times S_2 \times S_3 \times S_4 \times S_5 \times S_6 \times S_7 \times S_8$$

and

$$|(y_1, y_2, ..., y_8)|_\tau^2 = \|y_1\|_{Y_1}^2 + \|y_2\|_{Y_2}^2 + \cdots + \|y_8\|_{Y_8}^2,$$

where

$$\begin{array}{llll} Y_1 = H_0^1, & Y_2 = Y_7 = L^2, & Y_3 = Y_5 = H_*^1, & Y_4 = Y_6 = Y_8 = L_*^2, \\ S_1 = H^2 \cap H_0^1, & S_2 = S_7 = H_0^1, & S_3 = S_5 = \{u \in H_*^1 \mid u_x \in H_0^1\}, & S_4 = S_6 = S_8 = H_*^1. \end{array}$$

Therefore, it is enough to show that S_i is a dense subspace of Y_i, for $i = 1, 2, 3, 4$:

- Since $C_0^\infty(0,\ell) \subset [H^2(0,\ell) \cap H_0^1(0,\ell)]$, S_1 is dense in Y_1 by the definition of $H_0^1(0,\ell)$.
- Since $C_0^\infty(0,\ell) \subset H_0^1(0,\ell)$, S_2 is dense in Y_2 by Theorem 2.6.
- S_3 is dense in Y_3 by Theorem 2.11.
- S_4 is dense in Y_4 by Theorem 2.10.

The case $\tau = 0$ is similar (the unique differences are that S_7 equals to S_1 instead of S_2 and there are only seven spaces instead of eight).

$\mathcal{A}_\tau$ is dissipative. A straightforward computation shows that

$$\mathrm{Re}(\mathcal{A}_\tau U, U)_\tau = -\frac{\gamma}{m}\|\eta_\tau\|_{L^2}^2, \quad \forall\, U \in D(\mathcal{A}_\tau) \tag{3.7}$$

where

$$\eta_\tau := \begin{cases} \sqrt{\kappa_1}\theta_x & \text{if} \quad \tau = 0 \\ \sqrt{\delta k_1} q & \text{if} \quad \tau > 0. \end{cases} \tag{3.8}$$

$0 \in \rho(\mathcal{A}_\tau)$. Suppose that $\tau > 0$. Let $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8) \in \mathcal{H}_\tau$. Then, taking

$$\Phi = f_1, \quad \Psi = f_3, \quad W = f_5, \tag{3.9}$$

we conclude that there exists a unique $q \in H_*^1(0,\ell)$ such that

$$-k_1 q_x = f_7 + mW_x - ml\Phi,$$

which is given by

$$\begin{aligned} q(x) = & -\frac{1}{k_1}\int_0^x \Big(f_7(y) + mW_x(y) - ml\Phi(y)\Big)dy \\ & + \frac{1}{k_1\ell}\int_0^\ell\int_0^x \Big(f_7(y) + mW_x(y) - ml\Phi(y)\Big)dy\ dx. \end{aligned} \tag{3.10}$$

Analogously, there exists a unique $\theta \in H_0^1(0,\ell)$ satisfying

$$-\theta_x = \delta q + \tau f_8,$$

which is given by

$$\theta(x) = -\int_0^x \Big(\delta q + \tau f_8\Big)\, dy. \tag{3.11}$$

Also, applying Theorem 2.5 with the sesquilinear form $B : (H_0^1 \times H_*^1 \times H_*^1)^2 \to \mathbb{C}$ defined by

$$\begin{aligned} B((\varphi^*, \psi^*, w^*), (\varphi, \psi, w)) = k \int_0^\ell (\varphi_x^* + \psi^* + l w^*)\overline{(\varphi_x + \psi + l w)}\, dx \\ + k_0 \int_0^\ell (w_x^* - l\varphi^*)\overline{(w_x - l\varphi)}\, dx + b \int_0^\ell \psi_x^* \overline{\psi_x}\, dx \end{aligned}$$

and the linear functional $f : H_0^1 \times H_*^1 \times H_*^1 \to \mathbb{C}$ defined by

$$f(\varphi^*, \psi^*, w^*) = -\int_0^\ell \overline{(\rho_1 f_2 + l\gamma\theta)}\varphi^*\, dx - \int_0^\ell \overline{\rho_2 f_4}\psi^*\, dx - \int_0^\ell \overline{(\rho_1 f_6 + \gamma\theta_x)} w^*\, dx$$

we conclude that the system

$$\begin{aligned} k(\varphi_x + \psi + l w)_x + k_0 l(w_x - l\varphi) &= \rho_1 f_2 + l\gamma\theta \\ b\psi_{xx} - k(\varphi_x + \psi + l w) &= \rho_2 f_4 \\ k_0(w_x - l\varphi)_x - kl(\varphi_x + \psi + l w) &= \rho_1 f_6 + \gamma\theta_x \end{aligned}$$

has a unique solution

$$(\varphi, \psi, w) \in [H^2(0,\ell) \cap H_0^1(0,\ell)] \times [H^2(0,\ell) \cap \times H_*^1(0,\ell)]^2 \quad \text{with} \quad \psi_x, w_x \in H_0^1(0,\ell)$$

(see Remark 3.5). This shows that there exists a unique $U \in D(\mathcal{A}_\tau)$ satisfying $\mathcal{A}_\tau U = F$ and thus $\mathcal{A}_\tau$ is bijective. Working with the components of $\mathcal{A}_\tau U = F$, we also conclude that $\|U\|_\tau \le C\|F\|_\tau$ which shows that $\mathcal{A}_\tau^{-1}$ is bounded. So, $0 \in \rho(\mathcal{A}_\tau)$.

For the case $\tau = 0$, we start with $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7) \in \mathcal{H}_0$. Then, taking Φ, Ψ and W as in (3.9), we conclude that there exists a unique $\theta \in H_0^1(0,\ell)$ satisfying

$$\kappa_1 \theta_{xx} = f_7 + mW_x - ml\Phi,$$

which is given by the expression (3.11) with $\tau = 0$ and q defined by (3.10). The rest of the argument is exactly the same. $\square$

Remark 3.5. Since B is continuous coercive and f is bounded, Theorem 2.5 implies that there exist a unique $(\varphi, \psi, w) \in (H_0^1 \times H_*^1 \times H_*^1)$ such that

$$B((\varphi^*, \psi^*, w^*), (\varphi, \psi, w)) = f(\varphi^*, \psi^*, w^*), \quad \forall\, (\varphi^*, \psi^*, w^*) \in (H_0^1 \times H_*^1 \times H_*^1). \tag{3.12}$$

In particular,

$$B((\varphi^*, 0, 0), (\varphi, \psi, w)) = f(\varphi^*, 0, 0), \quad \forall\, \varphi^* \in H_0^1 \tag{3.13}$$

$$B((0, \psi^*, 0), (\varphi, \psi, w)) = f(0, \psi^*, 0), \quad \forall\, \psi^* \in H_*^1 \tag{3.14}$$

$$B((0, 0, w^*), (\varphi, \psi, w)) = f(0, 0, w^*), \quad \forall\, w^* \in H_*^1. \tag{3.15}$$

Taking $\phi \in H_0^1$ and applying (3.14)-(3.15) with

$$\psi^* = w^* = \phi - \frac{1}{\ell}\int_0^\ell \phi\, dx$$

we conclude that

$$B((0, \phi, 0), (\varphi, \psi, w)) = f(0, \phi, 0), \quad \forall\, \phi \in H_0^1 \tag{3.16}$$

$$B((0, 0, \phi), (\varphi, \psi, w)) = f(0, 0, \phi), \quad \forall\, \phi \in H_0^1. \tag{3.17}$$

In view of the definition of weak derivative, equalities (3.13), (3.16) and (3.17) imply, respectively, that

$$(\varphi_x + \psi + lw) \in H^1, \quad \psi_x \in H^1 \quad \text{and} \quad (w_x - l\varphi) \in H^1 \tag{3.18}$$

with

$$(\varphi_x + \psi + lw)_x = \frac{1}{k}(\rho_1 f_2 + l\gamma\theta - k_0 l(w_x - l\varphi)) \tag{3.19}$$

$$\psi_{xx} = \frac{1}{b}(\rho_2 f_4 + k(\varphi_x + \psi + lw)) \tag{3.20}$$

$$(w_x - l\varphi)_x = \frac{1}{k_0}(\rho_1 f_6 + \gamma\theta_x + kl(\varphi_x + \psi + lw)) \tag{3.21}$$

From (3.18), we have $\varphi, \psi, w \in H^2$. In addition, integrating (3.14) by parts and using (3.20), we obtain

$$b\psi^*(\ell)\overline{\psi}_x(\ell) - b\psi^*(0)\overline{\psi}_x(0) = 0, \quad \forall\, \psi^* \in H_*^1.$$

We can choose ψ^* such that $\psi^*(\ell) = 1$ and $\psi^*(0) = 0$, which shows that $\psi_x(\ell) = 0$. On the other hand, we can choose ψ^* such that $\psi^*(\ell) = 0$ and $\psi^*(0) = 1$, which shows that $\psi_x(0) = 0$. Therefore, $\psi_x \in H_0^1$. Analogously, equations (3.15) and (3.21) imply that $w_x \in H_0^1$. This shows existence of a solution (φ, ψ, w) for the system (3.19)-(3.21) with the desired regularity. Since any solution with this regularity satisfies (3.12), the solution is unique.

3.2 Characterization of exponential stability

In this section we will prove that the solution of *the mathematical system* (1.13)-(1.14) is exponentially stable if and only if (1.15) holds. Initially, note that the dissipative condition (3.7) implies the following result.

Lemma 3.6. Assume that $U \in D(\mathcal{A}_\tau)$ satisfies the resolvent equation $\lambda U - \mathcal{A}_\tau U = F$ for some nonzero $\lambda \in i\mathbb{R}$ and some $F \in \mathcal{H}_\tau$. Then, η_τ given by (3.8) satisfies

$$\frac{\gamma}{m}\|\eta_\tau\|_{L^2}^2 = \mathrm{Re}(F, U)_\tau \leq \|F\|_\tau \|U\|_\tau.$$

Proof. Taking the inner product of $\lambda U - \mathcal{A}_\tau U = F$ with U, we get

$$\lambda\|U\|_\tau^2 - (\mathcal{A}_\tau U, U)_\tau = (F, U)_\tau.$$

Taking the real part, the desired result follows from (3.7). □

First, based in Theorem 2.18 and using the Lemma 3.6, we will prove that $i\mathbb{R} \subset \rho(\mathcal{A}_\tau)$. For this, we will need the following result.

Lemma 3.7. $(D(\mathcal{A}_\tau), \|\cdot\|_{D(\mathcal{A}_\tau)}) \overset{c}{\hookrightarrow} (\mathcal{H}_\tau, \|\cdot\|_\tau)$, where $\|\cdot\|_{D(\mathcal{A}_\tau)}$ is the graph norm

$$\|U\|_{D(\mathcal{A}_\tau)} = \|U\|_\tau + \|\mathcal{A}_\tau U\|_\tau.$$

Proof. In view of Remark 3.3, it is enough to show that $(D(\mathcal{A}_\tau), |\cdot|_{D(\mathcal{A}_\tau)}) \overset{c}{\hookrightarrow} (\mathcal{H}_\tau, |\cdot|_\tau)$, where

$$|U|_{D(\mathcal{A}_\tau)}^2 = |U|_\tau^2 + |\mathcal{A}_\tau U|_\tau^2.$$

Suppose that $\tau > 0$ and let $U_n = (\varphi^{(n)}, \Phi^{(n)}, \psi^{(n)}, \Psi^{(n)}, w^{(n)}, W^{(n)}, \theta^{(n)}, q^{(n)})$ be a sequence in $D(\mathcal{A}_\tau)$, bounded with respect to the norm $|\cdot|_{D(\mathcal{A}_\tau)}$. In view of Theorem 2.2, it is enough to show that $(U_n)_{n\in\mathbb{N}}$ has a subsequence which converges in $(\mathcal{H}_\tau, |\cdot|_\tau)$.

From the boundedness of $(U_n)_{n\in\mathbb{N}}$ with respect to $|\cdot|_{D(\mathcal{A}_\tau)}$, there exists a constant

$C > 0$ such that

$$
\begin{aligned}
|U_n|^2_{D(\mathcal{A}_\tau)} &= |U_n|^2_\tau + |\mathcal{A}_\tau U_n|^2_\tau \\
&= \|\varphi_x^{(n)}\|^2_{L^2} + \|\Phi^{(n)}\|^2_{L^2} + \|\psi_x^{(n)}\|^2_{L^2} + \|\Psi^{(n)}\|^2_{L^2} + \|w_x^{(n)}\|^2_{L^2} + \|W_x^{(n)}\|^2_{L^2} \\
&\quad + \|\theta^{(n)}\|^2_{L^2} + \|q^{(n)}\|_{L^2} + \|\Phi_x^{(n)}\|^2_{L^2} \\
&\quad + \frac{1}{{\rho_1}^2}\|k(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_x + k_0 l(w_x^{(n)} - l\varphi^{(n)}) - l\gamma\theta\|^2_{L^2} + \|\Psi_x^{(n)}\|^2_{L^2} \\
&\quad + \frac{1}{{\rho_2}^2}\|b\psi_{xx}^{(n)} - k(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})\|^2_{L^2} + \|W_x^{(n)}\|^2_{L^2} \\
&\quad + \frac{1}{{\rho_1}^2}\|k_0(w_x^{(n)} - l\varphi^{(n)})_x - kl(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}) - \gamma\theta_x^{(n)}\|^2_{L^2} \\
&\quad + \| - k_1 q_x^{(n)} - m(W_x^{(n)} - l\Phi^{(n)})\|^2_{L^2} + \frac{1}{\tau^2}\| - \delta q^{(n)} - \theta_x^{(n)}\|^2_{L^2} \\
&\le C
\end{aligned}
$$

for all $n \in \mathbb{N}$. In particular, the sequences $(\varphi_x^{(n)})_{n\in\mathbb{N}}$, $(\varphi^{(n)})_{n\in\mathbb{N}}$, $(\psi_x^{(n)})_{n\in\mathbb{N}}$, $(\psi^{(n)})_{n\in\mathbb{N}}$, $(w_x^{(n)})_{n\in\mathbb{N}}$, $(w^{(n)})_{n\in\mathbb{N}}$, $(\theta^{(n)})_{n\in\mathbb{N}}$, $(q^{(n)})_{n\in\mathbb{N}}$, $(W_x^{(n)})_{n\in\mathbb{N}}$ and $(\Phi^{(n)})_{n\in\mathbb{N}}$ are all bounded in $L^2(0,\ell)$. From this and from triangular inequality, we conclude that $(\theta_x^{(n)})_{n\in\mathbb{N}}$, $(\varphi_{xx}^{(n)})_{n\in\mathbb{N}}$, $(\psi_{xx}^{(n)})_{n\in\mathbb{N}}$, $(w_{xx}^{(n)})_{n\in\mathbb{N}}$ and $(q_x^{(n)})_{n\in\mathbb{N}}$ are also bounded in $L^2(0,\ell)$. Therefore, $(U_n)_{n\in\mathbb{N}}$ is bounded in the space

$$(H^2 \cap H^1_0) \times H^1 \times (H^2 \cap H^1_*) \times H^1_* \times (H^2 \cap H^1_*) \times H^1_* \times H^1 \times H^1_*$$

(equipped with the usual norm) which, by Corollary 2.13, is compactly embedded in

$$H^1_0 \times L^2 \times H^1_* \times L^2_* \times H^1_* \times L^2_* \times L^2 \times L^2_*$$

(equipped with the usual norm). As the last space (with its usual norm) is $(\mathcal{H}_\tau, |\cdot|_\tau)$, we conclude that there exists $U \in \mathcal{H}_\tau$ such that

$$|U_n - U|_\tau \overset{n\to\infty}{\longrightarrow} 0 \,.$$

□

Theorem 3.8. $i\mathbb{R} \subset \rho(\mathcal{A}_\tau)$.

Proof. By contradiction, supposing that the inclusion $i\mathbb{R} \subset \rho(\mathcal{A}_\tau)$ is not true, there exists $\lambda \in i\mathbb{R}$ such that $\lambda \in \sigma(\mathcal{A}_\tau)$, with $\lambda \neq 0$ because $0 \in \rho(\mathcal{A}_\tau)$. Now, by Lemma 3.7 and Theorem 2.3, A has compact resolvent. Thus, $\sigma(\mathcal{A}_\tau) = \sigma_p(\mathcal{A}_\tau)$ by Theorem 2.4. Therefore, λ is an eigenvalue of $\mathcal{A}_\tau$ which implies the existence of $U \neq 0$ in $D(\mathcal{A}_\tau)$ satisfying

$$\mathcal{A}_\tau U = \lambda U. \tag{3.22}$$

Note that for both cases, $\tau = 0$ and $\tau > 0$, the equality (3.22) is equivalent to

$$\lambda\varphi - \Phi = 0 \tag{3.23}$$

$$\rho_1\lambda\Phi - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + l\gamma\theta = 0 \tag{3.24}$$

$$\lambda\psi - \Psi = 0 \tag{3.25}$$

$$\rho_2\lambda\Psi - b\psi_{xx} + k(\varphi_x + \psi + lw) = 0$$

$$\lambda w - W = 0 \tag{3.26}$$

$$\rho_1\lambda W - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + \gamma\theta_x = 0 \tag{3.27}$$

$$\lambda\theta + k_1 q_x + mW_x - ml\Phi = 0 \tag{3.28}$$

$$\tau\lambda q + \delta q + \theta_x = 0. \tag{3.29}$$

It follows from Lemma 3.6 and (3.29) that $q = \theta_x = 0$ and thus $q_x = \theta = 0$. Then, substituting into (3.28) and using (3.23) and (3.26), we conclude that

$$w_x - l\varphi = \frac{1}{\lambda}(W_x - l\Phi) = 0. \tag{3.30}$$

These results, combined with equalities (3.23)-(3.24) and (3.26)-(3.27), yields

$$\rho_1\lambda^2\varphi - k(\varphi_x + \psi + lw)_x = 0 \tag{3.31}$$

$$\rho_1\lambda^2 w + kl(\varphi_x + \psi + lw) = 0. \tag{3.32}$$

Now, from (3.31)-(3.32) we get

$$l\varphi + w_x = 0. \tag{3.33}$$

Equalities (3.30) and (3.33) imply $w_x = 0$ and thus $w = 0$. Also, from (3.32)-(3.33) we obtain $\varphi = 0$ and $\psi = 0$. Finally, using (3.23), (3.25) and (3.26), we conclude that $U = 0$ which is a contradiction. □

Second, in order to verify the other condition of Theorem 2.18, we need some suitable estimates obtained from the resolvent equation

$$\lambda U - \mathcal{A}_\tau U = F \tag{3.34}$$

which, again for both cases ($\tau = 0$ and $\tau > 0$), is equivalent to

$$\lambda\varphi - \Phi = f_1 \tag{3.35}$$

$$\rho_1\lambda\Phi - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + l\gamma\theta = \rho_1 f_2 \tag{3.36}$$

$$\lambda\psi - \Psi = f_3 \tag{3.37}$$

$$\rho_2\lambda\Psi - b\psi_{xx} + k(\varphi_x + \psi + lw) = \rho_2 f_4 \tag{3.38}$$

$$\lambda w - W = f_5 \tag{3.39}$$

$$\rho_1\lambda W - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + \gamma\theta_x = \rho_1 f_6 \tag{3.40}$$

$$\lambda\theta + k_1 q_x + mW_x - ml\Phi = f_7 \tag{3.41}$$

$$\tau\lambda q + \delta q + \theta_x = \tau f_8. \tag{3.42}$$

Such estimates will be proved in the next lemmas, where $C > 0$ represents a generic constant whose value can change from line to line (or even within the same line).

Lemma 3.9. Assume that $U \in D(\mathcal{A}_\tau)$ satisfies (3.35)-(3.42) for some nonzero $\lambda \in i\mathbb{R}$ and some $F \in \mathcal{H}_\tau$. Then, there exists a constant $C > 0$ (independent of U, λ and F) such that

$$\|\Psi\|_{L^2}^2 \leq C\|\psi_x\|_{L^2}\|U\|_\tau + C\|U\|_\tau\|F\|_\tau.$$

Proof. From (3.37) and (3.38),

$$\begin{aligned}\|\Psi\|_{L^2}^2 &= \int_0^\ell (\lambda\psi - f_3)\overline{\Psi}\,dx = \frac{1}{\rho_2\overline{\lambda}}\int_0^\ell \lambda\psi\overline{(b\psi_{xx} - k(\varphi_x + \psi + lw) + \rho_2 f_4)}\,dx - \int_0^\ell f_3\overline{\Psi}\,dx\\ &= \frac{b}{\rho_2}\int_0^\ell \psi_x\overline{\psi_x}\,dx + \frac{k}{\rho_2}\int_0^\ell \psi\overline{(\varphi_x + \psi + lw)}\,dx - \int_0^\ell \psi\overline{f_4}\,dx - \int_0^\ell f_3\overline{\Psi}\,dx,\end{aligned}$$

which implies

$$\|\Psi\|_{L^2}^2 \leq C\|\psi_x\|_{L^2}\|U\|_\tau + C\|\psi\|_{L^2}\|U\|_\tau + C\|\psi\|_\tau\|F\|_\tau + C\|U\|_\tau\|F\|_\tau.$$

So, the desired result follows from Poincaré inequality applied to ψ. □

Lemma 3.10. Assume that $U \in D(\mathcal{A}_\tau)$ satisfies (3.35)-(3.42) for some $\lambda \in i\mathbb{R}$ satisfying $|\lambda| \geq 1$ and some $F \in \mathcal{H}_\tau$. Then, there exists a constant $C > 0$ (independent of U, λ and F) such that

$$\|\psi_x\|_{L^2}^2 \leq C\left(1+|\lambda|^2\left|\rho_1-\frac{k\rho_2}{b}\right|^2\right)\|\Phi\|_{L^2}^2 + C\|U\|_\tau\|F\|_\tau + C\|U\|_\tau\|\varphi_x+\psi+lw\|_{L^2}$$
$$+ C\|w_x - l\varphi\|_{L^2}\|U\|_\tau + C\|\theta\|_{L^2}\|U\|_\tau.$$

Proof. Multiplying (3.36) by $\overline{\psi}_x$ and integrating over $[0,\ell]$, we obtain

$$\begin{aligned} k\|\psi_x\|_{L^2}^2 = &\underbrace{\rho_1\lambda\int_0^\ell \Phi\overline{\psi}_x\,dx - k\int_0^\ell \varphi_{xx}\overline{\psi}_x\,dx}_{I_1} \\ &\underbrace{-\,kl\int_0^\ell w_x\overline{\psi}_x\,dx - k_0 l\int_0^\ell (w_x - l\varphi)\overline{\psi}_x\,dx}_{I_2} \\ &\underbrace{+\,l\gamma\int_0^\ell \theta\overline{\psi}_x\,dx - \rho_1\int_0^\ell f_2\overline{\psi}_x\,dx}_{I_3}. \end{aligned} \tag{3.43}$$

Now, in order to estimate I_1, note that (using (3.38))

$$\begin{aligned} I_1 &= \rho_1\lambda\int_0^\ell \Phi\overline{\psi}_x\,dx + \frac{k}{b}\int_0^\ell \varphi_x\overline{(\rho_2\lambda\Psi + k(\varphi_x+\psi+lw) - \rho_2 f_4)}\,dx \\ &= \rho_1\lambda\int_0^\ell \Phi\overline{\psi}_x\,dx + \frac{k\rho_2\lambda}{b}\int_0^\ell \varphi\overline{\Psi}_x\,dx + \underbrace{\frac{k^2}{b}\int_0^\ell \varphi_x\overline{(\varphi_x+\psi+lw)}\,dx - \frac{k\rho_2}{b}\int_0^\ell \varphi_x\overline{f_4}\,dx}_{I_4} \end{aligned}$$

where, by (3.35) and (3.37),

$$\begin{aligned} \frac{k\rho_2\lambda}{b}\int_0^\ell \varphi\overline{\Psi}_x\,dx &= \frac{k\rho_2}{b}\int_0^\ell (\Phi+f_1)\overline{\Psi}_x\,dx \\ &= -\frac{k\rho_2\lambda}{b}\int_0^\ell \Phi\overline{\psi}_x\,dx \underbrace{-\frac{k\rho_2}{b}\int_0^\ell \Phi\overline{f}_{3,x}\,dx - \frac{k\rho_2}{b}\int_0^\ell f_{1,x}\overline{\Psi}\,dx}_{I_5}. \end{aligned}$$

Consequently,

$$I_1 = \lambda\left(\rho_1 - \frac{k\rho_2}{b}\right)\int_0^\ell \Phi\overline{\psi}_x\,dx + I_4 + I_5.$$

On the other hand, for I_2, by (3.35) we have

$$\begin{aligned} I_2 &= -\,kl\int_0^\ell (w_x - l\varphi)\overline{\psi}_x\,dx - kl^2\int_0^\ell \varphi\overline{\psi}_x\,dx - k_0 l\int_0^\ell (w_x - l\varphi)\overline{\psi}_x\,dx \\ &= -\frac{kl^2}{\lambda}\int_0^\ell \Phi\overline{\psi}_x\,dx \underbrace{-(k+k_0)l\int_0^\ell (w_x - l\varphi)\overline{\psi}_x\,dx - \frac{kl^2}{\lambda}\int_0^\ell f_1\overline{\psi}_x\,dx.}_{I_6} \end{aligned}$$

Therefore, substituting the last two equalities for I_1 and I_2 into (3.43), we obtain

$$k\|\psi_x\|_{L^2}^2 = \left(-\frac{kl^2}{\lambda} + \lambda\left(\rho_1 - \frac{k\rho_2}{b}\right)\right)\int_0^\ell \Phi\overline{\psi}_x\, dx + I_3 + I_4 + I_5 + I_6. \tag{3.44}$$

Here, note that

$$|I_3| + |I_4| + |I_5| + |I_6| \le C\left(\|\theta\|_{L^2} + \|\varphi_x + \psi + lw\|_{L^2} + \|w_x - l\varphi\|_{L^2}\right)\|U\|_\tau + C\left(1 + \frac{1}{|\lambda|}\right)\|F\|_\tau\|U\|_\tau.$$

Finally, applying this inequality in (3.44) and recalling that $|\lambda| \ge 1$, we get the desired result. □

Lemma 3.11. Assume that $U \in D(\mathcal{A}_\tau)$ satisfies (3.35)-(3.42) for some nonzero $\lambda \in i\mathbb{R}$ and some $F \in \mathcal{H}_\tau$. Then, there exists a constant $C > 0$ (independent of U, λ and F) such that

$$\left|k\left(\tau - \frac{k_1\rho_1}{k}\right)\|\varphi_x + \psi + lw\|_{L^2}^2 + \frac{\rho_1}{k}\left(k\left(\tau - \frac{k_1\rho_1}{k}\right) + \chi\right)\|\Phi\|_{L^2}^2\right|$$
$$\le C|\chi\lambda|\int_0^\ell |w_x\overline{\Phi}|\, dx + C\|F\|_\tau\|U\|_\tau + C\|W\|_{L^2}\|U\|_\tau + C\|U\|_\tau\|\theta\|_{L^2} + C\|q\|_{L^2}\|U\|_\tau + C\|w_x - l\varphi\|_{L^2}^2 + C\|\theta\|_{L^2}^2$$

where χ is defined in (1.15).

Proof. Multiplying (3.40) by $\overline{(\varphi_x + \psi + lw)}$ and integrating over $[0, \ell]$, we get

$$kl\|\varphi_x + \psi + lw\|_{L^2}^2 = \underbrace{\rho_1\int_0^\ell f_6\overline{(\varphi_x + \psi + lw)}\, dx}_{I_1} \underbrace{- \rho_1\lambda\int_0^\ell W\overline{(\varphi_x + \psi + lw)}\, dx}_{I_2} \underbrace{+ k_0\int_0^\ell (w_x - l\varphi)_x\overline{(\varphi_x + \psi + lw)}\, dx}_{I_3} - \gamma\int_0^\ell \theta_x\overline{(\varphi_x + \psi + lw)}\, dx.$$

From (3.35), (3.37) and (3.39) note that

$$I_2 = \rho_1\int_0^\ell W\overline{(\Phi_x + \Psi + lW)}\, dx + \rho_1\int_0^\ell W\overline{(f_{1,x} + f_3 + lf_5)}\, dx = -\rho_1\lambda\int_0^\ell w_x\overline{\Phi}\, dx + I_4,$$

where

$$I_4 = \rho_1\int_0^\ell f_{5,x}\overline{\Phi}\, dx + \rho_1\int_0^\ell W\overline{\Psi}\, dx + \rho_1 l\int_0^\ell W\overline{W}\, dx + \rho_1\int_0^\ell W\overline{(f_{1,x} + f_3 + lf_5)}\, dx.$$

Also, from (3.35) and (3.36) we have

$$\begin{aligned} I_3 =& -\frac{k_0}{k}\int_0^\ell (w_x - l\varphi)\overline{(\rho_1\lambda\Phi - k_0 l(w_x - l\varphi) + l\gamma\theta - \rho_1 f_2)}\, dx \\ =& \frac{k_0\rho_1\lambda}{k}\int_0^\ell w_x\overline{\Phi}\, dx - \frac{k_0\rho_1 l}{k}\int_0^\ell |\Phi|^2\, dx + I_5, \end{aligned}$$

where

$$I_5 = -\frac{k_0\rho_1 l}{k}\int_0^\ell f_1\overline{\Phi}\, dx + \frac{k_0^2 l}{k}\int_0^\ell |w_x - l\varphi|^2\, dx - \frac{k_0 l\gamma}{k}\int_0^\ell (w_x - l\varphi)\overline{\theta}\, dx + \frac{k_0\rho_1}{k}\int_0^\ell (w_x - l\varphi)\overline{f_2}\, dx.$$

The above calculations show that

$$\begin{aligned} kl\|\varphi_x + \psi + lw\|_{L^2}^2 = I_1 + \lambda\left(\frac{k_0\rho_1}{k} - \rho_1\right)\int_0^\ell w_x\overline{\Phi}\, dx + I_4 - \frac{k_0\rho_1 l}{k}\int_0^\ell |\Phi|^2\, dx \\ + I_5 - \gamma\int_0^\ell \theta_x\overline{(\varphi_x + \psi + lw)}\, dx. \end{aligned} \tag{3.45}$$

On the other hand, multiplying (3.41) by $\frac{\tau\gamma\rho_1}{k}\overline{\Phi}$, integrating over $[0,\ell]$ and using (3.39), we obtain

$$\begin{aligned} \frac{\tau\gamma\rho_1\lambda}{k}\int_0^\ell \theta\overline{\Phi}\, dx + \frac{\tau\gamma\rho_1 k_1}{k}\int_0^\ell q_x\overline{\Phi}\, dx + \frac{\tau\gamma\rho_1 m\lambda}{k}\int_0^\ell w_x\overline{\Phi}\, dx - \frac{\tau\gamma\rho_1 ml}{k}\int_0^\ell |\Phi|^2\, dx \\ = \underbrace{\frac{\tau\gamma\rho_1}{k}\int_0^\ell f_7\overline{\Phi}\, dx - \frac{\tau\gamma\rho_1 m}{k}\int_0^\ell f_{5,x}\overline{\Phi}\, dx}_{J_1}. \end{aligned}$$

Also, multiplying $\overline{(3.36)}$ by $\frac{\tau\gamma}{k}\theta$ and integrating over $[0,\ell]$,

$$\begin{aligned} -\frac{\tau\gamma\rho_1\lambda}{k}\int_0^\ell \theta\overline{\Phi}\, dx + \tau\gamma\int_0^\ell \theta_x\overline{(\varphi_x + \psi + lw)}\, dx \\ = \underbrace{-\frac{\tau l\gamma^2}{k}\int_0^\ell |\theta|^2\, dx + \frac{\tau\gamma k_0 l}{k}\int_0^\ell \theta\overline{(w_x - l\varphi)}\, dx + \frac{\tau\gamma\rho_1}{k}\int_0^\ell \theta\overline{f_2}}_{J_2}, \end{aligned}$$

which implies, adding the last two equalities,

$$\begin{aligned} \frac{\tau\gamma\rho_1 ml}{k}\int_0^\ell |\Phi|^2\, dx = \frac{\tau\gamma\rho_1 k_1}{k}\int_0^\ell q_x\overline{\Phi}\, dx + \frac{\tau\gamma\rho_1 m\lambda}{k}\int_0^\ell w_x\overline{\Phi}\, dx \\ + \tau\gamma\int_0^\ell \theta_x\overline{(\varphi_x + \psi + lw)}\, dx - J_1 - J_2. \end{aligned} \tag{3.46}$$

Therefore, introducing the notation $\sigma_0 := \left(\tau - \frac{k_1\rho_1}{k}\right)$ and doing $\sigma_0\times$(3.45)+(3.46), we obtain

$$\sigma_0 kl\|\varphi_x+\psi+lw\|_{L^2}^2+\frac{\rho_1 l}{k}(\sigma_0 k+\chi)\|\Phi\|_{L^2}^2=\frac{\rho_1\chi}{k}\lambda\int_0^\ell w_x\overline{\Phi}\,dx+\sigma_0 I_1+\sigma_0 I_4+\sigma_0 I_5-J_1$$
$$-J_2+(\tau-\sigma_0)\gamma\int_0^\ell \theta_x\overline{(\varphi_x+\psi+lw)}\,dx$$
$$+\frac{\tau\gamma\rho_1 k_1}{k}\int_0^\ell q_x\overline{\Phi}\,dx$$

where, by (3.42) and (3.35),

$$\int_0^\ell \theta_x\overline{(\varphi_x+\psi+lw)}\,dx=\tau\int_0^\ell f_8\overline{\varphi_x}\,dx-\tau\lambda\int_0^\ell q\overline{\varphi_x}\,dx-\delta\int_0^\ell q\overline{\varphi_x}\,dx+\int_0^\ell \theta_x\overline{(\psi+lw)}\,dx$$
$$=-\tau\int_0^\ell q_x\overline{\Phi}\,dx+\underbrace{\tau\int_0^\ell q\overline{f_{1,x}}\,dx+\tau\int_0^\ell f_8\overline{\varphi_x}\,dx-\delta\int_0^\ell q\overline{\varphi_x}\,dx-\int_0^\ell \theta\overline{\psi}_x\,dx-l\int_0^\ell \theta\overline{w}_x\,dx}_{J_3},$$

which implies that

$$\sigma_0 kl\|\varphi_x+\psi+lw\|_{L^2}^2+\frac{\rho_1 l}{k}(\sigma_0 k+\chi)\|\Phi\|_{L^2}^2=\frac{\rho_1\chi}{k}\lambda\int_0^\ell w_x\overline{\Phi}\,dx+\sigma_0 I_1+\sigma_0 I_4+\sigma_0 I_5$$
$$-J_1-J_2+J_3$$
$$+\left(\sigma_0-\tau+\frac{\rho_1 k_1}{k}\right)\gamma\tau\int_0^\ell q_x\overline{\Phi}\,dx \tag{3.47}$$

where

$$|I_1|+|I_4|+|I_5|+|J_1|+|J_2|+|J_3|\le C\|F\|_\tau\|U\|_\tau+C\|W\|_{L^2}\|U\|_\tau+C\|q\|_{L^2}\|U\|_\tau$$
$$+C\|U\|_\tau\|\theta\|_{L^2}+C\|\theta\|_{L^2}^2+C\|w_x-l\varphi\|_{L^2}^2.$$

Finally, using this estimate into (3.47) and noting that $\left(\sigma_0-\tau+\frac{\rho_1 k_1}{k}\right)=0$, the desired result is obtained. □

Lemma 3.12. Assume that $U\in D(\mathcal{A}_\tau)$ satisfies (3.35)-(3.42) for some $\lambda\in i\mathbb{R}$ satisfying $|\lambda|\ge 1$ and some $F\in\mathcal{H}_\tau$. Then, there exists a constant $C>0$ (independent of U, λ and F) such that

$$\|W\|_{L^2}^2\le C\|w_x-l\varphi\|_{L^2}\|U\|_\tau+C\|\theta\|_{L^2}\|U\|_\tau+C\|\theta\|_{L^2}^2+C\|q\|_{L^2}\|U\|_\tau+C\|F\|_\tau\|U\|_\tau.$$

Proof. Let us define

$$p(x)=\int_0^x W(y)\,dy,\qquad g(x)=\int_0^x\int_0^y \theta(z)\,dz\,dy-\frac{x}{\ell}\int_0^\ell\int_0^y\theta(z)\,dz\,dy.$$

Note that $p_x = W$, $g_{xx} = \theta$, $p \in H^1_0(0,\ell)$ and $g_x \in H^1_*(0,\ell)$. Multiplying (3.41) by $\overline{p}$ and integrating over $[0,\ell]$ we get

$$m\|W\|^2_{L^2} = \lambda\int_0^\ell \theta\overline{p}\,dx - ml\int_0^\ell \Phi\overline{p}\,dx \underbrace{-\,k_1\int_0^\ell q\overline{W}\,dx - \int_0^\ell f_7\overline{p}\,dx}_{I_1}$$

where, from (3.40),

$$\begin{aligned}\lambda\int_0^\ell \theta\overline{p}\,dx =& -\lambda\int_0^\ell g_x\overline{W}\,dx\\ &= \int_0^\ell g_x\overline{f_6}\,dx + \frac{k_0}{\rho_1}\int_0^\ell \theta\overline{(w_x - l\varphi)}\,dx + \frac{kl}{\rho_1}\int_0^\ell g_x\overline{(\varphi_x + \psi + lw)}\,dx - \frac{\gamma}{\rho_1}\int_0^\ell |\theta|^2\,dx\\ &:= I_2,\end{aligned}$$

and, from (3.36),

$$\begin{aligned}-ml\int_0^\ell \Phi\overline{p}\,dx = \frac{mlk}{\rho_1\lambda}\int_0^\ell(\varphi_x + \psi + lw)\overline{W}\,dx\\ \underbrace{-\frac{ml}{\lambda}\int_0^\ell f_2\overline{p}\,dx - \frac{ml^2k_0}{\rho_1\lambda}\int_0^\ell (w_x - l\varphi)\overline{p}\,dx + \frac{ml^2\gamma}{\rho_1\lambda}\int_0^\ell \theta\overline{p}\,dx}_{I_3}.\end{aligned}$$

Consequently, the substitution in the first identity implies that

$$m\|W\|^2_{L^2} = \frac{mlk}{\rho_1\lambda}\int_0^\ell(\varphi_x + \psi + lw)\overline{W}\,dx + I_1 + I_2 + I_3. \tag{3.48}$$

On the other hand, multiplying (3.40) by $m\overline{W}$ and integrating over $[0,\ell]$ we get

$$\begin{aligned}m\|W\|^2_{L^2} =& -\frac{mkl}{\rho_1\lambda}\int_0^\ell(\varphi_x + \psi + lw)\overline{W}\,dx + \frac{mk_0}{\rho_1\lambda}\int_0^\ell (w_x - l\varphi)_x\overline{W}\,dx - \frac{m\gamma}{\rho_1\lambda}\int_0^\ell \theta_x\overline{W}\,dx\\ &\underbrace{+\,\frac{m}{\lambda}\int_0^\ell f_6\overline{W}\,dx}_{I_4}\end{aligned}$$

where, from (3.39) and (3.42),

$$\begin{aligned}\frac{mk_0}{\rho_1\lambda}\int_0^\ell (w_x - l\varphi)_x\overline{W}\,dx &= -\frac{mk_0}{\rho_1\lambda}\int_0^\ell (w_x - l\varphi)\overline{(\lambda w - f_5)_x}\,dx\\ &= \underbrace{\frac{mk_0}{\rho_1}\int_0^\ell (w_x - l\varphi)\overline{w}_x\,dx + \frac{mk_0}{\rho_1\lambda}\int_0^\ell (w_x - l\varphi)\overline{f}_{5,x}\,dx}_{I_5}\end{aligned}$$

and

$$-\frac{m\gamma}{\rho_1\lambda}\int_0^\ell \theta_x\overline{W}\,dx = \underbrace{-\frac{\tau m\gamma}{\rho_1\lambda}\int_0^\ell f_8\overline{W}\,dx + \frac{\tau m\gamma}{\rho_1}\int_0^\ell q\overline{W}\,dx + \frac{\delta m\gamma}{\rho_1\lambda}\int_0^\ell q\overline{W}\,dx}_{I_6}.$$

This shows that

$$m\|W\|_{L^2}^2 = -\frac{mkl}{\rho_1\lambda}\int_0^\ell (\varphi_x+\psi+lw)\overline{W}\,dx + I_4 + I_5 + I_6. \tag{3.49}$$

Finally, note that

$$\begin{aligned}\sum_{i=1}^{6}|I_i| \le{}& C\left(\|q\|_{L^2} + \frac{1}{|\lambda|}\|q\|_{L^2} + \|\theta\|_{L^2} + \|w_x - l\varphi\|_{L^2} + \|g_x\|_{L^2}\right)\|U\|_\tau \\ &+ C\|\theta\|_{L^2}^2 + C\left(\|p\|_{L^2} + \frac{1}{|\lambda|}\|p\|_{L^2} + \|g_x\|_{L^2}\right)\|F\|_\tau \\ &+ \frac{C}{|\lambda|}\left(\|w_x - l\varphi\|_{L^2} + \|\theta\|_{L^2}\right)\|p\|_{L^2} + C\left(1+\frac{1}{|\lambda|}\right)\|U\|_\tau\|F\|_\tau.\end{aligned} \tag{3.50}$$

Consequently, doing (3.48)+(3.49), using (3.50) and applying Poincaré inequality to p and g_x, recalling that $|\lambda| \ge 1$, the desired result follows without difficulties. □

Lemma 3.13. Assume that $U \in D(\mathcal{A}_\tau)$ satisfies (3.35)-(3.42) for some $\lambda \in i\mathbb{R}$ satisfying $|\lambda| \ge 1$ and some $F \in \mathcal{H}_\tau$. Then, there exists a constant $C > 0$ (independent of U, λ and F) such that

$$\|w_x - l\varphi\|_{L^2}^2 \le C\|\theta\|_{L^2}\|U\|_\tau + C\|q\|_{L^2}\|U\|_\tau + C\|q\|_{L^2}^2 + C\|F\|_\tau\|U\|_\tau.$$

Proof. From (3.35) and (3.39),

$$\begin{aligned}\|w_x - l\varphi\|_{L^2}^2 = \int_0^\ell (w_x - l\varphi)\overline{(w_x - l\varphi)}\,dx ={}& \frac{1}{\lambda}\int_0^\ell (W_x - l\Phi)\overline{(w_x - l\varphi)}\,dx \\ &+ \underbrace{\frac{1}{\lambda}\int_0^\ell (f_{5,x} - lf_1)\overline{(w_x - l\varphi)}\,dx}_{I_1}.\end{aligned}$$

Here, from (3.41) we have $W_x - l\Phi = \frac{1}{m}(f_7 - \lambda\theta - k_1 q_x)$ and thus

$$\begin{aligned}\frac{1}{\lambda}\int_0^\ell (W_x - l\Phi)\overline{(w_x - l\varphi)} ={}& \underbrace{\frac{1}{\lambda m}\int_0^\ell f_7\overline{(w_x - l\varphi)}\,dx - \frac{1}{m}\int_0^\ell \theta\overline{(w_x - l\varphi)}\,dx}_{I_2} \\ &- \frac{k_1}{\lambda m}\int_0^\ell q_x\overline{(w_x - l\varphi)}\,dx,\end{aligned}$$

which implies, substituting in the first equation, that

$$\|w_x - l\varphi\|_{L^2}^2 = -\frac{k_1}{\lambda m}\int_0^\ell q_x\overline{(w_x - l\varphi)}\,dx + I_1 + I_2. \tag{3.51}$$

Now, from (3.40), we have

$$\begin{aligned}
&-\frac{k_1}{\lambda m}\int_0^\ell q_x\overline{(w_x - l\varphi)}\,dx \\
&= \frac{k_1}{\lambda k_0 m}\int_0^\ell q\overline{(\rho_1\lambda W + kl(\varphi_x + \psi + lw) + \gamma\theta_x - \rho_1 f_6)}\,dx \\
&= \underbrace{-\frac{k_1\rho_1}{k_0 m}\int_0^\ell q\overline{W}\,dx + \frac{k_1 kl}{\lambda k_0 m}\int_0^\ell q\overline{(\varphi_x + \psi + lw)}\,dx - \frac{k_1\rho_1}{\lambda k_0 m}\int_0^\ell q\overline{f_6}\,dx}_{I_3} + \frac{k_1\gamma}{\lambda k_0 m}\int_0^\ell q\overline{\theta_x}\,dx,
\end{aligned}$$

where, from (3.42), the last term is estimated by

$$\frac{k_1\gamma}{\lambda k_0 m}\int_0^\ell q\overline{\theta_x}\,dx = \underbrace{\frac{\tau k_1\gamma}{\lambda k_0 m}\int_0^\ell q\overline{f_8}\,dx + \frac{\tau k_1\gamma}{k_0 m}\int_0^\ell |q|^2\,dx - \frac{\delta k_1\gamma}{\lambda k_0 m}\int_0^\ell |q|^2\,dx}_{I_4}.$$

So, substituting the above calculation in (3.51), we obtain

$$\|w_x - l\varphi\|_{L^2}^2 = I_1 + I_2 + I_3 + I_4.$$

Here, it is not difficult to verify that

$$\begin{aligned}
\sum_{i_i}^{4}|I_i| \le{}& \frac{C}{|\lambda|}\|F\|_\tau\|U\|_\tau + C\left(\|q\|_{L^2} + \frac{1}{|\lambda|}\|q\|_{L^2} + \|\theta\|_{L^2}\right)\|U\|_\tau \\
&+ \frac{C}{|\lambda|}\|U\|_\tau\|F\|_\tau + C\|q\|^2 + \frac{C}{|\lambda|}\|q\|^2,
\end{aligned}$$

which implies the desired result because $|\lambda| \ge 1$. □

Lemma 3.14. Assume that $U \in D(\mathcal{A}_\tau)$ satisfies (3.35)-(3.42) for some $\lambda \in i\mathbb{R}$ satisfying $|\lambda| \ge 1$ and some $F \in \mathcal{H}_\tau$. Then, there exists a constant $C > 0$ (independent of U, λ and F) such that

$$\|\theta\|_{L^2}^2 \le C\|q\|_{L^2}\|U\|_\tau + C\|F\|_\tau\|U\|_\tau.$$

Proof. Multiplying (3.41) by $\overline{\theta}$ and integrating over $[0,\ell]$,

$$\|\theta\|_{L^2}^2 = \underbrace{\frac{1}{\lambda}\int_0^\ell f_7\overline{\theta}\,dx}_{I_1} + \underbrace{\frac{k_1}{\lambda}\int_0^\ell q\overline{\theta}_x\,dx + \frac{m}{\lambda}\int_0^\ell W\overline{\theta}_x\,dx}_{I_2} + \frac{ml}{\lambda}\int_0^\ell \Phi\overline{\theta}\,dx. \tag{3.52}$$

Now, in order to estimate the last term of (3.52), we define

$$h(x) = \int_0^x \Phi(y)\,dy - \frac{1}{\ell}\int_0^\ell\int_0^z \Phi(y)\,dy\,dz.$$

Then, $h \in H_*^1(0,\ell)$ and $h_x = \Phi$. So, multiplying $\overline{(3.42)}$ by mlh and integrating over $(0,\ell)$, we obtain

$$\frac{ml}{\lambda}\int_0^\ell \overline{\theta}\Phi\,dx = \underbrace{\tau ml\int_0^\ell \overline{q}h\,dx + \frac{\delta ml}{\lambda}\int_0^\ell \overline{q}h\,dx - \frac{\tau ml}{\lambda}\int_0^\ell \overline{f_8}h\,dx}_{I_3}.$$

Substituting in (3.52), we get

$$\|\theta\|_{L^2}^2 = I_1 + I_2 + I_3.$$

Consequently, using (3.42), we deduce that

$$\begin{aligned}
I_2 &= \frac{k_1}{\lambda}\int_0^\ell q\overline{(\tau f_8 - \tau\lambda q - \delta q)}\,dx + \frac{m}{\lambda}\int_0^\ell W\overline{(\tau f_8 - \tau\lambda q - \delta q)}\,dx \\
&= \frac{\tau k_1}{\lambda}\int_0^\ell q\overline{f_8}\,dx + \tau k_1\int_0^\ell q\overline{q}\,dx - \frac{\delta k_1}{\lambda}\int_0^\ell q\overline{q}\,dx + \frac{\tau m}{\lambda}\int_0^\ell W\overline{f_8}\,dx + \tau m\int_0^\ell W\overline{q}\,dx \\
&\quad - \frac{\delta m}{\lambda}\int_0^\ell W\overline{q}\,dx,
\end{aligned}$$

which is used to prove that

$$\begin{aligned}
\|\theta\|_{L^2}^2 \le \sum_{i=1}^3 |I_i| \le \frac{C}{|\lambda|}\left(\|U\|_\tau + \|h\|_{L^2}\right)\|F\|_\tau + C\left(\tau + \frac{1}{|\lambda|}\right)\|q\|_{L^2}\|U\|_\tau \\
+ C\left(\tau + \frac{1}{|\lambda|}\right)\|q\|_{L^2}\|h\|_{L^2}.
\end{aligned}$$

Then, the desired result follows form Poincaré inequality applied to h, and from the fact that $|\lambda| \ge 1$. □

Now, we are ready to state and prove the main results of this section.

Theorem 3.15 (Exponential decay)**.** If

$$\frac{\rho_1}{\rho_2} = \frac{k}{b} \quad \text{and} \quad \underbrace{\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau\gamma m}_{\chi} = 0, \tag{3.53}$$

then the semigroup generated by $\mathcal{A}_\tau$ is exponentially stable.

Proof. In view of Theorem 3.8, it remains to show condition (b) of Theorem 2.18. To this purpose, taking $F \in \mathcal{H}_\tau$ and $\lambda \in i\mathbb{R}$ satisfying $|\lambda| \geq 1$, we define $U := (\lambda - \mathcal{A}_\tau)^{-1}F$. Then, U belongs to $D(\mathcal{A}_\tau)$ and satisfies (3.35)-(3.42).

Therefore, from Lemma 3.9, for $\varepsilon > 0$ arbitrary, we have

$$\rho_2\|\Psi\|_{L^2}^2 \leq C_\varepsilon\|\psi_x\|_{L^2}^2 + 2\varepsilon\|U\|_\tau^2 + C_\varepsilon\|F\|_\tau^2$$

and thus

$$\begin{aligned}(1-2\varepsilon)\|U\|_\tau^2 \leq \rho_1\|\Phi\|_{L^2}^2 + \rho_1\|W\|_{L^2}^2 + C_\varepsilon\|\psi_x\|_{L^2}^2 + k\|\varphi_x + \psi + lw\|_{L^2}^2 \\ + k_0\|w_x - l\varphi\|_{L^2}^2 + \frac{\gamma}{m}\|\theta\|_{L^2}^2 + \frac{\gamma k_1\tau}{m}\|q\|_{L^2}^2 + C_\varepsilon\|F\|_\tau^2.\end{aligned}$$

Similarly, using that $\frac{\rho_1}{\rho_2} = \frac{k}{b}$, Lemma 3.10 implies

$$\begin{aligned}C_\varepsilon\|\psi_x\|_{L^2}^2 \leq 5\varepsilon\|U\|_\tau^2 + C_\varepsilon\|F\|_\tau^2 + C_\varepsilon\|\varphi_x + \psi + lw\|_{L^2}^2 \\ + C_\varepsilon\|w_x - l\varphi\|_{L^2}^2 + C_\varepsilon\|\Phi\|_{L^2}^2 + C_\varepsilon\|F\|_\tau^2 + C_\varepsilon\|\theta\|_{L^2}^2\end{aligned}$$

and thus

$$\begin{aligned}(1-7\varepsilon)\|U\|_\tau^2 \leq C_\varepsilon\|\Phi\|_{L^2}^2 + \rho_1\|W\|_{L^2}^2 + C_\varepsilon\|\varphi_x + \psi + lw\|_{L^2}^2 \\ + C_\varepsilon\|w_x - l\varphi\|_{L^2}^2 + C_\varepsilon\|\theta\|_{L^2}^2 + \frac{\gamma k_1\tau}{m}\|q\|_{L^2}^2 + C_\varepsilon\|F\|_\tau^2.\end{aligned}$$

On the other hand, note that $\chi = 0$ implies $\left(\tau - \frac{k_1\rho_1}{k}\right) \neq 0$. Then, multiplying the estimate of Lemma 3.11 by $\left|\tau - \frac{k_1\rho_1}{k}\right|^{-1}$, we obtain

$$\begin{aligned}C_\varepsilon\|\varphi_x + \psi + lw\|_{L^2}^2 + C_\varepsilon\|\Phi\|_{L^2}^2 \leq C_\varepsilon\|F\|_\tau^2 + 4\varepsilon\|U\|_\tau^2 + C_\varepsilon\|W\|_{L^2}^2 + C_\varepsilon\|\theta\|_{L^2}^2 + C_\varepsilon\|q\|_{L^2}^2 \\ + C_\varepsilon\|w_x - l\varphi\|_{L^2}^2\end{aligned}$$

and thus

$$(1-11\varepsilon)\|U\|_\tau^2 \leq C_\varepsilon\|W\|_{L^2}^2 + C_\varepsilon\|w_x - l\varphi\|_{L^2}^2 + C_\varepsilon\|\theta\|_{L^2}^2 + C_\varepsilon\|q\|_{L^2}^2 + C_\varepsilon\|F\|_\tau^2.$$

Proceeding in a similar way with Lemmas 3.12, 3.13, 3.14 and 3.6, we get

$$(1-n\varepsilon)\|U\|_\tau^2 \leq C_\varepsilon \|F\|_\tau^2$$

for some suitable $n \in \mathbb{N}$. Therefore, taking $\varepsilon > 0$ small enough, we conclude that there exists a constant $C > 0$ (independent of λ and F) such that

$$\|(\lambda - \mathcal{A}_\tau)^{-1}F\|_\tau = \|U\|_\tau \leq C\|F\|_\tau, \quad \forall\, F \in \mathcal{H}_\tau,\ |\lambda| \geq 1.$$

This shows condition (b) of Theorem 2.18, which concludes the proof. □

Theorem 3.16 (Lack of exponential decay)**.** The converse of Theorem 3.15 is true. In other words: if

$$\frac{\rho_1}{\rho_2} \neq \frac{k}{b} \quad \text{or} \quad \underbrace{\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau\gamma m}_{\chi} \neq 0, \tag{3.54}$$

then the semigroup generated by $\mathcal{A}_\tau$ is not exponentially stable.

Proof. Assume (3.54). It is enough to show that there exist a sequence (β_n) of positive real numbers such that $\beta_n \overset{n\to\infty}{\longrightarrow} \infty$ and a bounded sequence (F_n) in $\mathcal{H}_\tau$ such that

$$\|(\mathbf{i}\beta_n I - \mathcal{A}_\tau)^{-1}F_n\|_\tau \overset{n\to\infty}{\longrightarrow} \infty \tag{3.55}$$

because, in this case, condition (b) of Theorem 2.18 fails.

In order to construct such sequences, let us write $c_n = \frac{n\pi}{\ell}$, take $\nu_1, \nu_2 \in \mathbb{R}$ (to be fixed later) and define

$$F_n = (0, \nu_1\rho_1^{-1}\sin(c_n x), 0, \nu_2\rho_2^{-1}\cos(c_n x), 0, ..., 0).$$

Then (F_n) is a bounded sequence in $\mathcal{H}_\tau$, with

$$\begin{aligned} \|(\mathbf{i}\beta_n I - \mathcal{A}_\tau)^{-1}F_n\|_\tau^2 = \rho_1\|\Phi^{(n)}\|_{L^2}^2 + \rho_2\|\Psi^{(n)}\|_{L^2}^2 + \rho_1\|W^{(n)}\|_{L^2}^2 + b\|\psi_x^{(n)}\|_{L^2}^2 \\ + k\|\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}\|_{L^2}^2 + k_0\|w_x^{(n)} - l\varphi^{(n)}\|_{L^2}^2 \\ + \frac{\gamma}{m}\|\theta^{(n)}\|_{L^2}^2 + \frac{\gamma k_1\tau}{m}\|q^{(n)}\|_{L^2}^2 \end{aligned} \tag{3.56}$$

where

$$U_n = \begin{cases} (\varphi^{(n)}, \Phi^{(n)}, \psi^{(n)}, \Psi^{(n)}, w^{(n)}, W^{(n)}, \theta^{(n)}), & \tau = 0 \\ (\varphi^{(n)}, \Phi^{(n)}, \psi^{(n)}, \Psi^{(n)}, w^{(n)}, W^{(n)}, \theta^{(n)}, q^{(n)}), & \tau > 0 \end{cases}$$

is the unique solution in $D(\mathcal{A}_\tau)$ of the resolvent equation

$$(\mathbf{i}\beta_n - \mathcal{A}_\tau)U_n = F_n.$$

In order to find (U_n) and (β_n), let us start by rewriting the resolvent equation in its components:

$$\begin{aligned}
\mathbf{i}\beta_n\varphi^{(n)} - \Phi^{(n)} &= 0\\
\rho_1\mathbf{i}\beta_n\Phi^{(n)} - k(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_x - k_0 l(w_x^{(n)} - l\varphi^{(n)}) + l\gamma\theta^{(n)} &= \nu_1\sin(c_n x)\\
\mathbf{i}\beta_n\psi^{(n)} - \Psi^{(n)} &= 0\\
\rho_2\mathbf{i}\beta_n\Psi^{(n)} - b\psi_{xx}^{(n)} + k(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}) &= \nu_2\cos(c_n x)\\
\mathbf{i}\beta_n w^{(n)} - W^{(n)} &= 0\\
\rho_1\mathbf{i}\beta_n W^{(n)} - k_0(w_x^{(n)} - l\varphi^{(n)})_x + kl(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}) + \gamma\theta_x^{(n)} &= 0\\
\mathbf{i}\beta_n\theta^{(n)} + k_1 q_x^{(n)} + mW_x^{(n)} - ml\Phi^{(n)} &= 0\\
\tau\mathbf{i}\beta_n q^{(n)} + \delta q^{(n)} + \theta_x^{(n)} &= 0.
\end{aligned}$$

Then,

$$\Phi^{(n)} = \mathbf{i}\beta_n\varphi^{(n)}, \qquad \Psi^{(n)} = \mathbf{i}\beta_n\psi^{(n)}, \qquad W^{(n)} = \mathbf{i}\beta_n w^{(n)}$$

which implies, substituting in the previous equations,

$$\begin{aligned}
\rho_1(\mathbf{i}\beta_n)^2\varphi^{(n)} - k(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_x - k_0 l(w_x^{(n)} - l\varphi^{(n)}) + l\gamma\theta^{(n)} &= \nu_1\sin(c_n x)\\
\rho_2(\mathbf{i}\beta_n)^2\psi^{(n)} - b\psi_{xx}^{(n)} + k(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}) &= \nu_2\cos(c_n x)\\
\rho_1(\mathbf{i}\beta_n)^2 w^{(n)} - k_0(w_x^{(n)} - l\varphi^{(n)})_x + kl(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}) + \gamma\theta_x^{(n)} &= 0\\
\mathbf{i}\beta_n\theta^{(n)} + k_1 q_x^{(n)} + m\mathbf{i}\beta_n w_x^{(n)} - ml\mathbf{i}\beta_n\varphi^{(n)} &= 0\\
\tau\mathbf{i}\beta_n q^{(n)} + \delta q^{(n)} + \theta_x^{(n)} &= 0.
\end{aligned}$$

Here, the last system can be solved by

$$\varphi^{(n)}(x) = A_n\sin(c_n x), \quad \psi^{(n)}(x) = B_n\cos(c_n x), \quad w^{(n)}(x) = C_n\cos(c_n x),$$
$$\theta^{(n)}(x) = D_n\sin(c_n x), \quad q^{(n)}(x) = E_n\cos(c_n x),$$

where $A_n, B_n, ..., E_n$ depend on (β_n) and will be determined in the sequel. In fact, substituting in the system, we conclude that the coefficients satisfy the linear system

$$\begin{aligned}\big(\rho_1(\mathbf{i}\beta_n)^2 + kc_n^2 + k_0 l^2\big)A_n + kc_nB_n + l(k+k_0)c_nC_n + l\gamma D_n &= \nu_1\\ kc_nA_n + \big(\rho_2(\mathbf{i}\beta_n)^2 + bc_n^2 + k\big)B_n + klC_n &= \nu_2\\ l(k+k_0)c_nA_n + klB_n + \big(\rho_1(\mathbf{i}\beta_n)^2 + k_0c_n^2 + kl^2\big)C_n + \gamma c_nD_n &= 0\\ -ml\mathbf{i}\beta_nA_n - m\mathbf{i}\beta_nc_nC_n + \mathbf{i}\beta_nD_n - k_1c_nE_n &= 0\\ (\tau\mathbf{i}\beta_n + \delta)E_n + c_nD_n &= 0\end{aligned}$$

which can be written as

$$E_n = -\frac{c_n}{\tau\mathbf{i}\beta_n + \delta}D_n, \qquad \underbrace{\begin{bmatrix} p_n^{(1)} & kc_n & l(k+k_0)c_n & l\gamma\\ kc_n & p_n^{(2)} & kl & 0\\ l(k+k_0)c_n & kl & p_n^{(3)} & \gamma c_n\\ -ml\mathbf{i}\beta_n & 0 & -m\mathbf{i}\beta_nc_n & p_n^{(4)}\end{bmatrix}}_{M_n}\begin{bmatrix}A_n\\ B_n\\ C_n\\ D_n\end{bmatrix} = \begin{bmatrix}\nu_1\\ \nu_2\\ 0\\ 0\end{bmatrix} \tag{3.57}$$

where

$$\begin{aligned} p_n^{(1)} &= \rho_1(\mathbf{i}\beta_n)^2 + kc_n^2 + k_0l^2, & p_n^{(2)} &= \rho_2(\mathbf{i}\beta_n)^2 + bc_n^2 + k,\\ p_n^{(3)} &= \rho_1(\mathbf{i}\beta_n)^2 + k_0c_n^2 + kl^2, & p_n^{(4)} &= \mathbf{i}\beta_n + \frac{k_1c_n^2}{\tau\mathbf{i}\beta_n + \delta}.\end{aligned} \tag{3.58}$$

Note that, solving (3.57), the unique solution $U_n \in D(\mathcal{A}_\tau)$ of the resolvent equation will be determined by

$$U_n = (A_n\sin(c_nx), \Phi^{(n)}, B_n\cos(c_nx), \Psi^{(n)}, C_n\cos(c_nx), W^{(n)}, D_n\sin(c_nx))$$

if $\tau = 0$, and

$$U_n = (A_n\sin(c_nx), \Phi^{(n)}, B_n\cos(c_nx), \Psi^{(n)}, C_n\cos(c_nx), W^{(n)}, D_n\sin(c_nx), E_n\cos(c_nx))$$

if $\tau > 0$, where

$$\Phi^{(n)} = \mathbf{i}\beta_nA_n\sin(c_nx), \qquad \Psi^{(n)} = \mathbf{i}\beta_nB_n\cos(c_nx), \qquad W^{(n)} = \mathbf{i}\beta_nC_n\cos(c_nx).$$

So, for the moment, let us assume that

$$\begin{aligned}\Delta_n :=\ &\det M_n\\ =\ &p_n^{(1)}p_n^{(2)}p_n^{(3)}p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2p_n^{(1)}p_n^{(2)} - k^2c_n^2p_n^{(3)}p_n^{(4)} - l^2(k+k_0)^2c_n^2p_n^{(2)}p_n^{(4)}\\ &- 2l^2(k+k_0)\gamma m(\mathbf{i}\beta_n)c_n^2p_n^{(2)} + l^2\gamma m(\mathbf{i}\beta_n)p_n^{(2)}p_n^{(3)} - k^2\gamma m(\mathbf{i}\beta_n)c_n^4\\ &- k^2l^2p_n^{(1)}p_n^{(4)} + 2k^2l^2(k+k_0)c_n^2p_n^{(4)} + 2k^2l^2\gamma m(\mathbf{i}\beta_n)c_n^2 - k^2l^4\gamma m(\mathbf{i}\beta_n)\end{aligned} \tag{3.59}$$

is nonzero. Then U_n takes the above specific form (in terms of sines and cosines) and thus, by (3.56),

$$\begin{aligned}\|(\mathbf{i}\beta_n - \mathcal{A})^{-1}F_n\|_{\tau}^2 &\geq \rho_1\|\Phi^{(n)}\|_{L^2}^2 + b\|\psi_x^{(n)}\|_{L^2}^2 \\ &= \rho_1\beta_n^2|A_n|^2\int_0^{\ell}|\sin(c_n x)|^2\,dx + \frac{bn^2\pi^2}{\ell^2}|B_n|^2\int_0^{\ell}|\sin(c_n x)|^2\,dx \\ &= \frac{\rho_1\ell}{2}\beta_n^2|A_n|^2 + \frac{b\pi^2}{2\ell}n^2|B_n|^2\end{aligned} \tag{3.60}$$

where A_n and B_n are given by the Cramer's formulas

$$A_n = \frac{\tilde{A}_n}{\Delta_n} \quad \text{and} \quad B_n = \frac{\tilde{B}_n}{\Delta_n}, \tag{3.61}$$

where

$$\begin{aligned}\tilde{A}_n &= \det\begin{bmatrix} \nu_1 & kc_n & l(k+k_0)c_n & l\gamma \\ \nu_2 & p_n^{(2)} & kl & 0 \\ 0 & kl & p_n^{(3)} & \gamma c_n \\ 0 & 0 & -mi\beta_n c_n & p_n^{(4)} \end{bmatrix}, \\ \tilde{B}_n &= \det\begin{bmatrix} p_n^{(1)} & \nu_1 & l(k+k_0)c_n & l\gamma \\ kc_n & \nu_2 & kl & 0 \\ l(k+k_0)c_n & 0 & p_n^{(3)} & \gamma c_n \\ -mli\beta_n & 0 & -mi\beta_n c_n & p_n^{(4)}. \end{bmatrix}.\end{aligned} \tag{3.62}$$

Consequently, in order to use formulas (3.61) let us start by proving that $\Delta_n = \det M_n \neq 0$ for all sufficiently large n, which will be consequence of a specific definition of (β_n). The following remark clarifies the situation.

Remark 3.17. Assuming that β_n has the form

$$\beta_n = \sqrt{\mu_1 c_n^2 + \mu_2 c_n + \mu_3} \tag{3.63}$$

with $\mu_1 > 0$ and $\mu_2, \mu_3 \geq 0$ and recalling that $c_n = \frac{n\pi}{\ell}$, we conclude that $p_n^{(1)}$, $p_n^{(2)}$ and $p_n^{(3)}$ are real-valued polynomials in the variable n of degree ≤ 2. In addition, it follows that

$$p_n^{(4)} = \frac{k_1\delta c_n^2}{\delta^2 + \tau^2\beta_n^2} + \mathbf{i}\beta_n\left(1 - \frac{k_1\tau c_n^2}{\delta^2 + \tau^2\beta_n^2}\right), \quad \forall\, n \in \mathbb{N}. \tag{3.64}$$

Using these facts, we conclude that the real and imaginary parts of Δ_n have the form

$$\mathrm{Re}(\Delta_n) = \underbrace{\frac{k_1\delta c_n^2}{\delta^2 + \tau^2\beta_n^2}}_{\neq 0} P_6(n), \qquad \mathrm{Im}(\Delta_n) = \underbrace{\frac{1}{\delta^2 + \tau^2\beta_n^2}}_{\neq 0} P_8(n)$$

where $P_k(n)$ is a polynomial in the variable n of degree $\leq k$. Therefore,

$$\begin{aligned}\Delta_n = 0 \quad &\Longleftrightarrow \quad \mathrm{Re}(\Delta_n) = 0 \text{ and } \mathrm{Im}(\Delta_n) = 0 \\ &\Longleftrightarrow \quad P_6(n) = 0 \text{ and } P_8(n) = 0.\end{aligned}$$

Thus, if $\Delta_n = 0$ for infinitely many values of n, then P_6 and P_8 are identically zero (because every nonzero polynomial has at most a finite number of roots). But, looking to the particular forms of $P_6(n)$ and $P_8(n)$ in each subcase below, we conclude that this is not the case. This shows that $\Delta_n = 0$ only for a finite number of values of n, i.e., $\Delta_n \neq 0$ for all n large enough.

Now, the specific choices of ν_1, ν_2 and (β_n) will be done in separated cases. We will use the notation $x_n = O(n^k)$ to indicate that there exists a positive constant C such that $|x_n| \leq Cn^k$ for all $n \in \mathbb{N}$.

• **Case 1: Fourier ($\tau = 0$).** In this case, condition (3.54) reduces to

$$\frac{\rho_1}{\rho_2} \neq \frac{k}{b} \quad \text{or} \quad k_0 \neq k$$

and can be separated in three subcases.

Subcase 1.1: $\frac{\rho_1}{\rho_2} \neq \frac{k}{b}$ and $k \neq k_0$. In this case, we define (β_n) by

$$\beta_n = \sqrt{\frac{1}{\rho_1}(kc_n^2 + k_0 l^2)}.$$

Then, remembering the definition of $c_n = \frac{n\pi}{\ell}$, we have $\beta_n = O(n)$ and, by (3.58),

$$\begin{aligned} p_n^{(1)} &= 0, \\ p_n^{(2)} &= \left(b - \frac{\rho_2 k}{\rho_1}\right) c_n^2 - \frac{\rho_2}{\rho_1} k_0 l^2 + k = O(n^2), \\ p_n^{(3)} &= (k_0 - k)c_n^2 + (k - k_0) l^2 = O(n^2), \\ p_n^{(4)} &= i\sqrt{\frac{1}{\rho_1}(kc_n^2 + k_0 l^2)} + \frac{k_1}{\delta} c_n^2 = O(n^2). \end{aligned}$$

So, substituting into (3.59), we conclude that

$$\begin{aligned}\Delta_n = & -k^2c_n^2p_n^{(3)}p_n^{(4)} - l^2(k+k_0)^2c_n^2p_n^{(2)}p_n^{(4)} - 2l^2(k+k_0)\gamma m(\mathbf{i}\beta_n)c_n^2p_n^{(2)} \\ & + l^2\gamma m(\mathbf{i}\beta_n)p_n^{(2)}p_n^{(3)} - k^2\gamma m(\mathbf{i}\beta_n)c_n^4 + 2k^2l^2(k+k_0)c_n^2p_n^{(4)} + 2k^2l^2\gamma m(\mathbf{i}\beta_n)c_n^2 \\ & - k^2l^4\gamma m(\mathbf{i}\beta_n) \\ = & -k^2c_n^2p_n^{(3)}p_n^{(4)} - l^2(k+k_0)^2c_n^2p_n^{(2)}p_n^{(4)} + O(n^5).\end{aligned}$$

Now, defining $\nu_1 = 1$ and $\nu_2 = 0$ we obtain from (3.62)

$$\tilde{A}_n = p_n^{(2)}p_n^{(3)}p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2p_n^{(2)} - k^2l^2p_n^{(4)} = p_n^{(2)}p_n^{(3)}p_n^{(4)} + O(n^5).$$

Here, using that

$$\frac{c_n^2}{n^2} \overset{n\to\infty}{\longrightarrow} \frac{\pi^2}{\ell^2}, \qquad \frac{p_n^{(2)}}{n^2} \overset{n\to\infty}{\longrightarrow} \left(b - \frac{\rho_2 k}{\rho_1}\right)\frac{\pi^2}{\ell^2}, \qquad \frac{p_n^{(3)}}{n^2} \overset{n\to\infty}{\longrightarrow} (k_0 - k)\frac{\pi^2}{\ell^2}, \qquad \frac{p_n^{(4)}}{n^2} \overset{n\to\infty}{\longrightarrow} \frac{k_1}{\delta}\frac{\pi^2}{\ell^2}$$

we deduce that

$$\frac{\tilde{A}_n}{n^6} = \frac{p_n^{(2)}}{n^2}\frac{p_n^{(3)}}{n^2}\frac{p_n^{(4)}}{n^2} + \frac{O(n^5)}{n^6} \overset{n\to\infty}{\longrightarrow} \left[\left(b - \frac{\rho_2 k}{\rho_1}\right)\frac{\pi^2}{\ell^2}\right]\left[(k_0 - k)\frac{\pi^2}{\ell^2}\right]\left[\frac{k_1}{\delta}\frac{\pi^2}{\ell^2}\right] := L_1$$

and, analogously,

$$\frac{\Delta_n}{n^6} \overset{n\to\infty}{\longrightarrow} -k^2\left[\frac{\pi^2}{\ell^2}\right]\left[(k_0 - k)\frac{\pi^2}{\ell^2}\right]\left[\frac{k_1}{\delta}\frac{\pi^2}{\ell^2}\right] - l^2(k+k_0)^2\left[\frac{\pi^2}{\ell^2}\right]\left[\left(b - \frac{\rho_2 k}{\rho_1}\right)\frac{\pi^2}{\ell^2}\right]\left[\frac{k_1}{\delta}\frac{\pi^2}{\ell^2}\right] := L_2.$$

Note that $L_1 \neq 0$, because $\left(b - \frac{\rho_2 k}{\rho_1}\right) \neq 0$ and $(k - k_0) \neq 0$. Then, using Remark 3.17 and writing

$$A_n = \frac{\tilde{A}_n}{\Delta_n} = \frac{\frac{\tilde{A}_n}{n^6}}{\frac{\Delta_n}{n^6}}$$

for all sufficiently large $n \in \mathbb{N}$, we have

$$|A_n| \overset{n\to\infty}{\longrightarrow} L = \begin{cases} \dfrac{|L_1|}{|L_2|} \neq 0 & \text{if} \quad L_2 \neq 0 \\ \infty & \text{if} \quad L_2 = 0. \end{cases}$$

Consequently, since $\beta_n \overset{n\to\infty}{\longrightarrow} \infty$, we obtain from (3.60)

$$\|(\mathbf{i}\beta_n - \mathcal{A})^{-1}F_n\|_\tau^2 \geq \frac{\rho_1\ell}{2}\beta_n^2|A_n|^2 \overset{n\to\infty}{\longrightarrow} \infty,$$

which implies (3.55).

Subcase 1.2: $\frac{\rho_1}{\rho_2} \neq \frac{k}{b}$ and $k = k_0$. In this case we define (β_n) by

$$\beta_n = \sqrt{\frac{1}{\rho_2}(bc_n^2 + k)}. \tag{3.65}$$

Then $\beta_n = O(n)$ and, using the equality $k = k_0$ in (3.58), we conclude that

$$\begin{aligned}
p_n^{(1)} &= \left(k - \frac{\rho_1 b}{\rho_2}\right) c_n^2 - \frac{\rho_1}{\rho_2} k + kl^2 = O(n^2), \\
p_n^{(2)} &= 0, \\
p_n^{(3)} &= p_n^{(1)}, \\
p_n^{(4)} &= \mathbf{i}\sqrt{\frac{1}{\rho_2}(bc_n^2 + k)} + \frac{k_1}{\delta} c_n^2 = O(n^2).
\end{aligned}$$

Then, it follows that the determinant (3.59) is simplified to

$$\begin{aligned}
\Delta_n = & -k^2 c_n^2 p_n^{(1)} p_n^{(4)} - k^2 \gamma m(\mathbf{i}\beta_n) c_n^4 - k^2 l^2 p_n^{(1)} p_n^{(4)} + 4k^3 l^2 c_n^2 p_n^{(4)} + 2k^2 l^2 \gamma m(\mathbf{i}\beta_n) c_n^2 \\
& - k^2 l^4 \gamma m(\mathbf{i}\beta_n) \\
= & -k^2 c_n^2 p_n^{(1)} p_n^{(4)} + O(n^5).
\end{aligned}$$

Now, defining $\nu_1 = 0$ and $\nu_2 = 1$ we obtain

$$\begin{aligned}
\tilde{B}_n &= [p_n^{(1)}]^2 p_n^{(4)} + \gamma m(\mathbf{i}\beta_n) c_n^2 p_n^{(1)} - 4l^2 k^2 c_n^2 p_n^{(4)} + l^2 \gamma m(\mathbf{i}\beta_n) p_n^{(1)} - 4l^2 k \gamma m(\mathbf{i}\beta_n) c_n^2 \\
&= [p_n^{(1)}]^2 p_n^{(4)} + O(n^5).
\end{aligned}$$

Analogously to the previous case,

$$\frac{c_n^2}{n^2} \overset{n\to\infty}{\longrightarrow} \frac{\pi^2}{\ell^2}, \qquad \frac{[p_n^{(1)}]^2}{n^4} \overset{n\to\infty}{\longrightarrow} \left(k - \frac{\rho_1 b}{\rho_2}\right) \frac{\pi^2}{\ell^2}, \qquad \frac{p_n^{(4)}}{n^2} \overset{n\to\infty}{\longrightarrow} \frac{k_1}{\delta} \frac{\pi^2}{\ell^2}$$

and thus

$$\frac{\tilde{B}_n}{n^6} = \frac{[p_n^{(1)}]^2}{n^4} \frac{p_n^{(4)}}{n^2} + \frac{O(n^5)}{n^6} \overset{n\to\infty}{\longrightarrow} \left[\left(k - \frac{\rho_1 b}{\rho_2}\right) \frac{\pi^2}{\ell^2}\right]^2 \left[\frac{k_1}{\delta} \frac{\pi^2}{\ell^2}\right] := L_1.$$

Also, we have

$$\frac{\Delta_n}{n^6} \overset{n\to\infty}{\longrightarrow} -k^2 \left[\frac{\pi^2}{\ell^2}\right] \left[\left(k - \frac{\rho_1 b}{\rho_2}\right) \frac{\pi^2}{\ell^2}\right] \left[\frac{k_1}{\delta} \frac{\pi^2}{\ell^2}\right] := L_2.$$

Using that $\left(k - \frac{\rho_1 b}{\rho_2}\right) \neq 0$, we conclude that $L_1 \neq 0$ and $L_2 \neq 0$. Therefore, from Remark

3.17,

$$|B_n| = \left|\frac{\tilde{B}_n}{\Delta_n}\right| = \left|\frac{\frac{\tilde{B}_n}{n^6}}{\frac{\Delta_n}{n^6}}\right| \stackrel{n\to\infty}{\longrightarrow} \frac{|L_1|}{|L_2|} \neq 0$$

which implies (3.55), because

$$\|(\mathbf{i}\beta_n - \mathcal{A})^{-1}F_n\|_\tau^2 \geq \frac{b\pi^2}{2\ell} n^2 |B_n|^2 \stackrel{n\to\infty}{\longrightarrow} \infty.$$

Subcase 1.3: $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ and $k \neq k_0$. Defining (β_n) by

$$\beta_n = \sqrt{\frac{k}{\rho_1}c_n^2 + \frac{k}{\sqrt{\rho_1\rho_2}}c_n}. \tag{3.66}$$

we have $\beta_n = O(n)$ and, using $\frac{\rho_1}{\rho_2} = \frac{k}{b}$, (3.58) implies that

$$\begin{aligned}
p_n^{(1)} &= -k\sqrt{\frac{\rho_1}{\rho_2}}c_n + k_0 l^2 = O(n),\\
p_n^{(2)} &= -k\sqrt{\frac{\rho_2}{\rho_1}}c_n + k = O(n),\\
p_n^{(3)} &= (k_0 - k)c_n^2 - k\sqrt{\frac{\rho_1}{\rho_2}}c_n + kl^2 = O(n^2),\\
p_n^{(4)} &= \mathbf{i}\sqrt{\frac{k}{\rho_1}c_n^2 + \frac{k}{\sqrt{\rho_1\rho_2}}c_n} + \frac{k_1}{\delta}c_n^2 = O(n^2).
\end{aligned}$$

Then, the determinant (3.59) can be written as

$$\begin{aligned}
\Delta_n = {} & p_n^{(1)}p_n^{(2)}p_n^{(3)}p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2 p_n^{(1)}p_n^{(2)} - k^2c_n^2p_n^{(3)}p_n^{(4)} - l^2(k+k_0)^2c_n^2p_n^{(2)}p_n^{(4)}\\
& - k^2\gamma m(\mathbf{i}\beta_n)c_n^4 + O(n^4).
\end{aligned}$$

Here, note that

$$\begin{aligned}
p_n^{(1)}p_n^{(2)}p_n^{(3)}p_n^{(4)} - k^2c_n^2p_n^{(3)}p_n^{(4)} &= (p_n^{(1)}p_n^{(2)} - k^2c_n^2)p_n^{(3)}p_n^{(4)}\\
&= -k\left(k_0l^2\sqrt{\frac{\rho_2}{\rho_1}} + k\sqrt{\frac{\rho_1}{\rho_2}}\right)c_np_n^{(3)}p_n^{(4)} + \underbrace{kk_0l^2p_n^{(3)}p_n^{(4)}}_{O(n^4)}
\end{aligned}$$

and

$$\gamma m(\mathbf{i}\beta_n)c_n^2 p_n^{(1)} p_n^{(2)} - k^2\gamma m(\mathbf{i}\beta_n)c_n^4 = (p_n^{(1)} p_n^{(2)} - k^2 c_n^2)\gamma m(\mathbf{i}\beta_n)c_n^2$$
$$= -k\left(\left(k_0 l^2\sqrt{\frac{\rho_2}{\rho_1}} + k\sqrt{\frac{\rho_1}{\rho_2}}\right)c_n - k_0 l^2\right)\gamma m(\mathbf{i}\beta_n)c_n^2$$
$$= O(n^4),$$

which simplify Δ_n to

$$\Delta_n = -k\left(k_0 l^2\sqrt{\frac{\rho_2}{\rho_1}} + k\sqrt{\frac{\rho_1}{\rho_2}}\right)c_n p_n^{(3)} p_n^{(4)} - l^2(k+k_0)^2 c_n^2 p_n^{(2)} p_n^{(4)} + O(n^4).$$

Now, defining $\nu_1 = 1$ and $\nu_2 = 0$ we obtain

$$\tilde{A}_n = p_n^{(2)} p_n^{(3)} p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2 p_n^{(2)} - k^2 l^2 p_n^{(4)} = p_n^{(2)} p_n^{(3)} p_n^{(4)} + O(n^4).$$

In this context,

$$\frac{c_n^2}{n^2} \overset{n\to\infty}{\longrightarrow} \frac{\pi^2}{\ell^2}, \qquad \frac{p_n^{(2)}}{n} \overset{n\to\infty}{\longrightarrow} -\frac{k\pi}{\ell}\sqrt{\frac{\rho_2}{\rho_1}}, \qquad \frac{p_n^{(3)}}{n^2} \overset{n\to\infty}{\longrightarrow} (k_0 - k)\frac{\pi^2}{\ell^2}, \qquad \frac{p_n^{(4)}}{n^2} \overset{n\to\infty}{\longrightarrow} \frac{k_1}{\delta}\frac{\pi^2}{\ell^2}$$

and thus

$$\frac{\tilde{A}_n}{n^5} = \frac{p_n^{(2)}}{n}\frac{p_n^{(3)}}{n^2}\frac{p_n^{(4)}}{n^2} + \frac{O(n^4)}{n^5} \overset{n\to\infty}{\longrightarrow} \left[-\frac{k\pi}{\ell}\sqrt{\frac{\rho_2}{\rho_1}}\right]\left[(k_0 - k)\frac{\pi^2}{\ell^2}\right]\left[\frac{k_1}{\delta}\frac{\pi^2}{\ell^2}\right] := L_1.$$

Analogously,

$$\frac{\Delta_n}{n^5} \overset{n\to\infty}{\longrightarrow} -k\left(k_0 l^2\sqrt{\frac{\rho_2}{\rho_1}} + k\sqrt{\frac{\rho_1}{\rho_2}}\right)\left[\frac{\pi}{\ell}\right]\left[(k_0 - k)\frac{\pi^2}{\ell^2}\right]\left[\frac{k_1}{\delta}\frac{\pi^2}{\ell^2}\right]$$
$$- l^2(k+k_0)^2\left[\frac{\pi^2}{\ell^2}\right]\left[-\frac{k\pi}{\ell}\sqrt{\frac{\rho_2}{\rho_1}}\right]\left[\frac{k_1}{\delta}\frac{\pi^2}{\ell^2}\right] := L_2.$$

Finally, using $(k_0 - k) \neq 0$, we conclude that $L_1 \neq 0$. Thus, from Remark 3.17,

$$|A_n| = \left|\frac{\tilde{A}_n}{\Delta_n}\right| = \left|\frac{\frac{\tilde{A}_n}{n^5}}{\frac{\Delta_n}{n^5}}\right| \overset{n\to\infty}{\longrightarrow} L = \begin{cases} \frac{|L_1|}{|L_2|} \neq 0 & \text{if} \quad L_2 \neq 0 \\ \infty & \text{if} \quad L_2 = 0, \end{cases}$$

which implies (3.55), because

$$\|(\mathbf{i}\beta_n - \mathcal{A})^{-1}F_n\|_\tau^2 \geq \frac{\rho_1\ell}{2}\beta_n^2|A_n|^2 \overset{n\to\infty}{\longrightarrow} \infty.$$

• **Case 2: Cattaneo ($\tau > 0$).** Here, we will consider two subcases.

Subcase 2.1: $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ and $\chi \neq 0$. In this subcase we use (β_n) defined by (3.66). Then $\beta_n = O(n)$ and, using $\frac{\rho_1}{\rho_2} = \frac{k}{b}$, from (3.58) we have

$$\begin{aligned}
p_n^{(1)} &= -k\sqrt{\frac{\rho_1}{\rho_2}}c_n + k_0 l^2 = O(n),\\
p_n^{(2)} &= -k\sqrt{\frac{\rho_2}{\rho_1}}c_n + k = O(n),\\
p_n^{(3)} &= (k_0 - k)c_n^2 - k\sqrt{\frac{\rho_1}{\rho_2}}c_n + kl^2 = O(n^2),\\
p_n^{(4)} &= \mathbf{i}\sqrt{\frac{k}{\rho_1}c_n^2 + \frac{k}{\sqrt{\rho_1\rho_2}}c_n} + \frac{k_1 c_n^2}{\tau\mathbf{i}\sqrt{\frac{k}{\rho_1}c_n^2 + \frac{k}{\sqrt{\rho_1\rho_2}}c_n} + \delta} = O(n).
\end{aligned}$$

It follows that the determinant (3.59) is written as

$$\begin{aligned}
\Delta_n = {} & p_n^{(1)}p_n^{(2)}p_n^{(3)}p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2 p_n^{(1)}p_n^{(2)} - k^2c_n^2p_n^{(3)}p_n^{(4)} - l^2(k+k_0)^2c_n^2p_n^{(2)}p_n^{(4)}\\
& - 2l^2(k+k_0)\gamma m(\mathbf{i}\beta_n)c_n^2p_n^{(2)} + l^2\gamma m(\mathbf{i}\beta_n)p_n^{(2)}p_n^{(3)} - k^2\gamma m(\mathbf{i}\beta_n)c_n^4 + O(n^3).
\end{aligned}$$

Here, note that

$$p_n^{(1)}p_n^{(2)}p_n^{(3)}p_n^{(4)} - k^2c_n^2p_n^{(3)}p_n^{(4)} = -k\left(k_0l^2\sqrt{\frac{\rho_2}{\rho_1}} + k\sqrt{\frac{\rho_1}{\rho_2}}\right)c_np_n^{(3)}p_n^{(4)} + \underbrace{kk_0l^2p_n^{(3)}p_n^{(4)}}_{=O(n^3)}$$

and

$$\begin{aligned}
\gamma m(\mathbf{i}\beta_n)c_n^2p_n^{(1)}p_n^{(2)} - k^2\gamma m(\mathbf{i}\beta_n)c_n^4 = {} & -k\left(k_0l^2\sqrt{\frac{\rho_2}{\rho_1}} + k\sqrt{\frac{\rho_1}{\rho_2}}\right)c_n\gamma m(\mathbf{i}\beta_n)c_n^2\\
& + \underbrace{kk_0l^2\gamma m(\mathbf{i}\beta_n)c_n^2}_{=O(n^3)},
\end{aligned}$$

which simplify Δ_n to

$$\begin{aligned}
\Delta_n = {} & -k\left(k_0l^2\sqrt{\frac{\rho_2}{\rho_1}} + k\sqrt{\frac{\rho_1}{\rho_2}}\right)c_n\left(p_n^{(3)}p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2\right) - l^2(k+k_0)^2c_n^2p_n^{(2)}p_n^{(4)}\\
& - 2l^2(k+k_0)\gamma m(\mathbf{i}\beta_n)c_n^2p_n^{(2)} + l^2\gamma m(\mathbf{i}\beta_n)p_n^{(2)}p_n^{(3)} + O(n^3).
\end{aligned}$$

Now, defining $\nu_1 = 1$ and $\nu_2 = 0$ we obtain

$$\tilde{A}_n = p_n^{(2)}p_n^{(3)}p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2p_n^{(2)} - k^2l^2p_n^{(4)} = p_n^{(2)}\left(p_n^{(3)}p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2\right) + O(n).$$

Then, using the convergences

$$\frac{c_n^2}{n^2} \overset{n\to\infty}{\longrightarrow} \frac{\pi^2}{\ell^2}, \qquad \frac{\beta_n}{n} \overset{n\to\infty}{\longrightarrow} \frac{\pi}{\ell}\sqrt{\frac{k}{\rho_1}}, \qquad \frac{p_n^{(2)}}{n} \overset{n\to\infty}{\longrightarrow} -\frac{k\pi}{\ell}\sqrt{\frac{\rho_2}{\rho_1}},$$

$$\frac{p_n^{(3)}}{n^2} \overset{n\to\infty}{\longrightarrow} (k_0-k)\frac{\pi^2}{\ell^2}, \qquad \frac{p_n^{(4)}}{n} \overset{n\to\infty}{\longrightarrow} \mathbf{i}\frac{\pi}{\ell}\sqrt{\frac{k}{\rho_1}}\left(1-\frac{k_1\rho_1}{\tau k}\right)$$

we obtain

$$\frac{\tilde{A}_n}{n^4} \overset{n\to\infty}{\longrightarrow} \left[-k\sqrt{\frac{\rho_2}{\rho_1}}\frac{\pi}{\ell}\right]\left(\left[(k_0-k)\frac{\pi^2}{\ell^2}\right]\left[\mathbf{i}\frac{\pi}{\ell}\sqrt{\frac{k}{\rho_1}}\left(1-\frac{k_1\rho_1}{\tau k}\right)\right]+\gamma m\mathbf{i}\left[\frac{\pi}{\ell}\sqrt{\frac{k}{\rho_1}}\right]\left[\frac{\pi^2}{\ell^2}\right]\right)$$
$$= -\mathbf{i}k\frac{\pi^4}{\ell^4}\frac{\sqrt{k\rho_2}}{\rho_1\tau}\chi := L_1.$$

Analogously, we can deduce that

$$\frac{\Delta_n}{n^4} \overset{n\to\infty}{\longrightarrow} L_2$$

for some constant $L_2 \in \mathbb{C}$. Finally, using $\chi \neq 0$, we conclude that $L_1 \neq 0$. Thus, from Remark 3.17,

$$|A_n| = \left|\frac{\tilde{A}_n}{\Delta_n}\right| = \left|\frac{\frac{\tilde{A}_n}{n^4}}{\frac{\Delta_n}{n^4}}\right| \overset{n\to\infty}{\longrightarrow} L = \begin{cases} \dfrac{|L_1|}{|L_2|} \neq 0 & \text{if} \quad L_2 \neq 0 \\ \infty & \text{if} \quad L_2 = 0, \end{cases}$$

which implies (3.55), because

$$\|(\mathbf{i}\beta_n - \mathcal{A})^{-1}F_n\|_\tau^2 \geq \frac{\rho_1\ell}{2}\beta_n^2|A_n|^2 \overset{n\to\infty}{\longrightarrow} \infty.$$

Subcase 2.2: $\frac{\rho_1}{\rho_2} \neq \frac{k}{b}$. Here we use (β_n) defined by (3.65). Then $\beta_n = O(n)$ and, by (3.58),

$$\begin{aligned}
p_n^{(1)} &= \left(k - \frac{\rho_1 b}{\rho_2}\right)c_n^2 - \frac{\rho_1}{\rho_2}k + k_0 l^2 = O(n^2),\\
p_n^{(2)} &= 0,\\
p_n^{(3)} &= \left(k_0 - \frac{\rho_1 b}{\rho_2}\right)c_n^2 - \frac{\rho_1}{\rho_2}k + kl^2 = O(n^2),\\
p_n^{(4)} &= \mathbf{i}\sqrt{\frac{1}{\rho_2}(bc_n^2+k)} + \frac{k_1 c_n^2}{\tau\mathbf{i}\sqrt{\frac{1}{\rho_2}(bc_n^2+k)}+\delta} = O(n).
\end{aligned}$$

It follows that the determinant (3.59) is simplified to

$$\begin{aligned}\Delta_n =& -k^2c_n^2p_n^{(3)}p_n^{(4)} - k^2\gamma m(\mathbf{i}\beta_n)c_n^4 - k^2l^2p_n^{(1)}p_n^{(4)} + 2k^2l^2(k+k_0)c_n^2p_n^{(4)}\\ &+ 2k^2l^2\gamma m(\mathbf{i}\beta_n)c_n^2 - k^2l^4\gamma m(\mathbf{i}\beta_n)\\ =& -k^2c_n^2\left(p_n^{(3)}p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2\right) + O(n^3).\end{aligned}$$

Now, defining $\nu_1 = 0$ and $\nu_2 = 1$ we obtain

$$\begin{aligned}\tilde{B}_n =& \, p_n^{(1)}p_n^{(3)}p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2p_n^{(1)} - l^2(k+k_0)^2c_n^2p_n^{(4)} + l^2\gamma m(\mathbf{i}\beta_n)p_n^{(3)}\\ &- 2l^2(k+k_0)\gamma m(\mathbf{i}\beta_n)c_n^2\\ =& \, p_n^{(1)}\left(p_n^{(3)}p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2\right) + O(n^3).\end{aligned}$$

So, using the convergences

$$\frac{c_n^2}{n^2} \overset{n\to\infty}{\longrightarrow} \frac{\pi^2}{\ell^2}, \qquad \frac{\beta_n}{n} \overset{n\to\infty}{\longrightarrow} \frac{\pi}{\ell}\sqrt{\frac{b}{\rho_2}}, \qquad \frac{p_n^{(1)}}{n^2} \overset{n\to\infty}{\longrightarrow} \left(k - \frac{\rho_1 b}{\rho_2}\right)\frac{\pi^2}{\ell^2},$$

$$\frac{p_n^{(3)}}{n^2} \overset{n\to\infty}{\longrightarrow} \left(k_0 - \frac{\rho_1 b}{\rho_2}\right)\frac{\pi^2}{\ell^2}, \qquad \frac{p_n^{(4)}}{n} \overset{n\to\infty}{\longrightarrow} \mathbf{i}\frac{\pi}{\ell}\sqrt{\frac{b}{\rho_2}}\left(1 - \frac{k_1\rho_2}{\tau b}\right)$$

we deduce that

$$\frac{\tilde{B}_n}{n^5} \overset{n\to\infty}{\longrightarrow} L_1,$$

where

$$\begin{aligned}L_1 :=& \left[\left(k - \frac{\rho_1 b}{\rho_2}\right)\frac{\pi^2}{\ell^2}\right]\\ &\times\left(\left[\left(k_0 - \frac{\rho_1 b}{\rho_2}\right)\frac{\pi^2}{\ell^2}\right]\left[\mathbf{i}\frac{\pi}{\ell}\sqrt{\frac{b}{\rho_2}}\left(1 - \frac{k_1\rho_2}{\tau b}\right)\right] + \gamma m\mathbf{i}\left[\frac{\pi}{\ell}\sqrt{\frac{b}{\rho_2}}\right]\left[\frac{\pi^2}{\ell^2}\right]\right),\end{aligned}$$

and (using $\frac{\rho_1}{\rho_2} \neq \frac{k}{b}$)

$$\begin{aligned}\frac{\Delta_n}{n^5} \overset{n\to\infty}{\longrightarrow} -k^2\left[\frac{\pi^2}{\ell^2}\right]&\left(\left[\left(k_0 - \frac{\rho_1 b}{\rho_2}\right)\frac{\pi^2}{\ell^2}\right]\left[\mathbf{i}\frac{\pi}{\ell}\sqrt{\frac{b}{\rho_2}}\left(1 - \frac{k_1\rho_2}{\tau b}\right)\right] + \gamma m\mathbf{i}\left[\frac{\pi}{\ell}\sqrt{\frac{b}{\rho_2}}\right]\left[\frac{\pi^2}{\ell^2}\right]\right)\\ &= -\frac{k^2}{\left(k - \frac{\rho_1 b}{\rho_2}\right)}L_1 := L_2\end{aligned}$$

Now, we consider two cases separately.

- $L_1 \neq 0$.

In this case we have $L_2 \neq 0$. Then, from Remark 3.17,

$$B_n = \frac{\tilde{B}_n}{\Delta_n} = \frac{\frac{\tilde{B}_n}{n^5}}{\frac{\Delta_n}{n^5}} \overset{n\to\infty}{\longrightarrow} \frac{L_1}{L_2} = -\frac{\left(k - \frac{\rho_1 b}{\rho_2}\right)}{k^2} \neq 0,$$

which implies (3.55), because

$$\|(\mathbf{i}\beta_n - \mathcal{A})^{-1}F_n\|_\tau^2 \geq \frac{b\pi^2}{2\ell} n^2 |B_n|^2 \overset{n\to\infty}{\longrightarrow} \infty.$$

- $L_1 = 0$.

In this case, $L_2 = 0$ and thus the previous argument does not work because we are led to the indetermination $\frac{0}{0}$. So, let us apply a different argument.

From Remark 3.17 (using that $\mathrm{Re}(p_n^{(3)} p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2) = \mathrm{Re}(p_n^{(4)})p_n^{(3)}$) we have

$$B_n = \frac{\tilde{B}_n}{\Delta_n} = \frac{p_n^{(1)}\left(p_n^{(3)}p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2\right) + O(n^3)}{-k^2 c_n^2 \left(p_n^{(3)}p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2\right) + O(n^3)} = \frac{\frac{p_n^{(1)}}{n^2} + \frac{O(n^3)}{n^2\left(p_n^{(3)}p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2\right)}}{-k^2 \frac{c_n^2}{n^2} + \frac{O(n^3)}{n^2\left(p_n^{(3)}p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2\right)}}$$

for all sufficiently large $n \in \mathbb{N}$. Then, assuming that

$$\frac{O(n^3)}{n^2\left(p_n^{(3)}p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2\right)} \overset{n\to\infty}{\longrightarrow} 0 \tag{3.67}$$

and using that $\frac{\rho_1}{\rho_2} \neq \frac{k}{b}$, we conclude

$$B_n \overset{n\to\infty}{\longrightarrow} \frac{\left(k - \frac{\rho_1 b}{\rho_2}\right)\frac{\pi^2}{\ell^2}}{-k^2 \frac{\pi^2}{\ell^2}} = -\frac{\left(k - \frac{\rho_1 b}{\rho_2}\right)}{k^2} \neq 0,$$

which implies (3.55), because

$$\|(\mathbf{i}\beta_n - \mathcal{A})^{-1}F_n\|_\tau^2 \geq \frac{b\pi^2}{2\ell} n^2 |B_n|^2 \overset{n\to\infty}{\longrightarrow} \infty.$$

So, in order to obtain our result, it is sufficient to show (3.67). In fact, note that L_1 can be rewritten as

$$\mathbf{i}\left(k - \frac{\rho_1 b}{\rho_2}\right)\frac{\pi^5}{\ell^5}\sqrt{\frac{b}{\rho_2}}\left(\left(k_0 - \frac{\rho_1 b}{\rho_2}\right)\left(1 - \frac{k_1\rho_2}{\tau b}\right) + \gamma m\right) = L_1 = 0,$$

which implies (using $\frac{\rho_1}{\rho_2} \neq \frac{k}{b}$) that

$$\left(k_0 - \frac{\rho_1 b}{\rho_2}\right)\left(1 - \frac{k_1\rho_2}{\tau b}\right) + \gamma m = 0.$$

Additionally, from the last equality we can deduce that $\left(k_0 - \frac{\rho_1 b}{\rho_2}\right) \neq 0$, because $\gamma m \neq 0$. Therefore

$$\begin{aligned}\frac{\mathrm{Re}\left((n^2\left(p_n^{(3)}p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2\right)\right)}{n^4} &= \frac{p_n^{(3)}}{n^2}\,\mathrm{Re}\left(p_n^{(4)}\right) = \frac{p_n^{(3)}}{n^2}\frac{\delta k_1 c_n^2}{\tau^2\beta_n^2 + \delta^2} \\ &\overset{n\to\infty}{\longrightarrow} \left[\left(k_0 - \frac{\rho_1 b}{\rho_2}\right)\frac{\pi^2}{\ell^2}\right]\left[\frac{\delta k_1\rho_2}{\tau^2 b}\right] \neq 0\end{aligned}$$

and thus

$$\begin{aligned}\left|\frac{O(n^3)}{n^2\left(p_n^{(3)}p_n^{(4)} + \gamma m(\mathbf{i}\beta_n)c_n^2\right)}\right|^2 &= \frac{\left|\frac{O(n^3)}{n^4}\right|^2}{\left[\frac{\mathrm{Re}\left(n^2\left(p_n^{(3)}p_n^{(4)}+\gamma m(\mathbf{i}\beta_n)c_n^2\right)\right)}{n^4}\right]^2 + \left[\frac{\mathrm{Im}\left(n^2\left(p_n^{(3)}p_n^{(4)}+\gamma m(\mathbf{i}\beta_n)c_n^2\right)\right)}{n^4}\right]^2} \\ &\leq \frac{\left|\frac{O(n^3)}{n^4}\right|^2}{\left[\frac{\mathrm{Re}\left(n^2\left(p_n^{(3)}p_n^{(4)}+\gamma m(\mathbf{i}\beta_n)c_n^2\right)\right)}{n^4}\right]^2} \overset{n\to\infty}{\longrightarrow} \frac{0}{\left[\left(k_0 - \frac{\rho_1 b}{\rho_2}\right)\frac{\pi^2}{\ell^2}\frac{\delta k_1\rho_2}{\tau^2 b}\right]^2} \\ &= 0,\end{aligned}$$

which proves (3.67). □

Remark 3.18. Theorems 3.15 and 3.16 show that the semigroup generated by $\mathcal{A}_\tau$ is exponentially stable if and only if condition (3.53) is satisfied. For *the physical system*, when (1.3) holds, we have

$$\begin{aligned}(3.53) \quad &\Longleftrightarrow \quad \frac{\rho_1}{\rho_2} = \frac{k}{b} \quad \text{and} \quad \left(\tau - \frac{k_1\rho_1}{k}\right)\left(\frac{b\rho_1}{\rho_2} - k\right) + \tau\gamma m = 0 \\ &\Longleftrightarrow \quad \frac{\rho_1}{\rho_2} = \frac{k}{b} \quad \text{and} \quad \tau = 0.\end{aligned}$$

Therefore, for the Fourier's case ($\tau = 0$), the semigroup associated with *the physical system* is exponentially stable provided that $\frac{\rho_1}{\rho_2} = \frac{k}{b}$, which is an unrealistic condition due to (1.16). And, for the Cattaneo's case ($\tau > 0$), the semigroup associated with *the physical system* will be always non-exponentially stable.

Remark 3.19. Note that the parameter τ affect the matrix M_n only in the expression of $p_n^{(4)}$, which plays an essential role in the calculations.

3.3 Polynomial Stability

In the previous section it was proved that, from the realistic point of view, *the physical system* is never exponentially stable, see Remark 3.18. Then, in order to complete our results, it is interesting to establish some rates of decay for both cases, Fourier and Cattaneo. In fact, the main result of this section is to show that *the mathematical system* decays polynomially to zero with rates of decay which can be improved provided that conditions of equal wave speeds are satisfied. The implications for *the physical system* are discussed in Remark 3.23.

Let us start by remembering the notations of the previous sections:

$$\chi_0 = b - \frac{k\rho_2}{\rho_1}, \qquad \chi_1 = k_0 - k, \qquad \sigma_0 = \left(\tau - \tfrac{k_1\rho_1}{k}\right), \qquad \chi = \chi_1\sigma_0 + \tau\gamma m.$$

Lemma 3.20. Assume that $U \in D(\mathcal{A}_\tau)$ satisfies (3.35)-(3.42) for some (nonzero) $\lambda \in i\mathbb{R}$ and $F \in \mathcal{H}_\tau$. Then, there exists a constant $C > 0$ (independent of U, λ and F) such that

$$\begin{aligned}\|\varphi_x + \psi + lw\|_{L^2}^2 + \|\Phi\|_{L^2}^2 \le{}& C(k_0-k)^2|\lambda|^2\|w_x - l\varphi\|_{L^2}^2 + C\tau|\lambda|^2\|q\|_{L^2}^2 + C\|F\|_\tau\|U\|_\tau \\ &+ C\|W\|_{L^2}\|U\|_\tau + C\|U\|_\tau\|\theta\|_{L^2} + C\|q\|_{L^2}\|U\|_\tau \\ &+ C\|w_x - l\varphi\|_{L^2}^2.\end{aligned}$$

Proof. As usual, we get our estimates from the resolvent system (3.35)-(3.42) and take advantage from some estimates proved in the previous section. From (3.35) we have

$$\begin{aligned}\int_0^\ell w_x\overline{\Phi}\,dx = \int_0^\ell (w_x - l\varphi)\overline{\Phi}\,dx + l\int_0^\ell \varphi\overline{\Phi}\,dx = \int_0^\ell (w_x - l\varphi)\overline{\Phi}\,dx + \frac{l}{\lambda}\int_0^\ell f_1\overline{\Phi}\,dx \\ + \frac{l}{\lambda}\int_0^\ell |\Phi|^2\,dx.\end{aligned}$$

Then, substituting into (3.45) we obtain

$$\begin{aligned}kl\|\varphi_x + \psi + lw\|_{L^2}^2 = \tilde{I}_1 + \left(\frac{k_0\rho_1}{k} - \rho_1\right)\lambda\int_0^\ell (w_x - l\varphi)\overline{\Phi}\,dx + I_4 - \rho_1 l\int_0^\ell |\Phi|^2\,dx \\ + I_5 - \gamma\int_0^\ell \theta_x\overline{(\varphi_x + \psi + lw)}\,dx.\end{aligned} \tag{3.68}$$

where

$$\tilde{I}_1 = I_1 + \left(\frac{k_0\rho_1}{k} - \rho_1\right) l\int_0^\ell f_1\overline{\Phi}\,dx$$

with I_1, I_4, I_5 defined in the proof of Lemma 3.11. Additionally, from (3.42) we have

$$\gamma \int_0^\ell \theta_x \overline{(\varphi_x + \psi + lw)}\, dx = \tau\gamma \int_0^\ell f_8 \overline{(\varphi_x + \psi + lw)}\, dx - \tau\gamma\lambda \int_0^\ell q \overline{(\varphi_x + \psi + lw)}\, dx - \delta\gamma \int_0^\ell q \overline{(\varphi_x + \psi + lw)}\, dx.$$

Finally, substituting into (3.68),

$$\begin{aligned} kl\|\varphi_x + \psi + lw\|_{L^2}^2 = \tilde{I}_1 + \frac{\rho_1}{k}(k_0 - k)\lambda \int_0^\ell (w_x - l\varphi)\overline{\Phi}\, dx + I_4 - \rho_1 l \int_0^\ell |\Phi|^2\, dx \\ + I_5 - \tau\gamma \int_0^\ell f_8 \overline{(\varphi_x + \psi + lw)}\, dx + \tau\gamma\lambda \int_0^\ell q \overline{(\varphi_x + \psi + lw)}\, dx \\ + \delta\gamma \int_0^\ell q \overline{(\varphi_x + \psi + lw)}\, dx \end{aligned}$$

which implies the desired result. □

Remark 3.21. In the formulation of Lemma 3.20, the dependence of τ in the second term on the right-hand side was written (explicitly) because it will be useful in the analysis of the cases $\tau = 0$ and $\tau > 0$ (see the arguments after (3.69) below). In the other lemmas of the previous section the constant C also depends on τ, but this dependence was not specified because it has the form $C = (1 + \tau)K$, where K does not depend on τ, and thus there is no difference between the cases $\tau > 0$ and $\tau = 0$.

Theorem 3.22 (Polynomial decay)**.** The C_0-semigroup generated by $\mathcal{A}_\tau$ is polynomially stable with the following rates of decay.

- For the Fourier's law:

1. If $\frac{\rho_1}{\rho_2} \neq \frac{k}{b}$, there exists a constant $C > 0$ (independent of U_0) such that

$$\|e^{t\mathcal{A}} U_0\|_\tau \leq \frac{C}{t^{1/16}} \|U_0\|_{D(\mathcal{A}_\tau)}, \quad \forall\, t \geq 0.$$

2. If $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ and $k \neq k_0$, there exists a constant $C > 0$ (independent of U_0) such that

$$\|e^{t\mathcal{A}} U_0\|_\tau \leq \frac{C}{t^{1/4}} \|U_0\|_{D(\mathcal{A}_\tau)}, \quad \forall\, t \geq 0.$$

- For the Cattaneo's law:

3. If $\frac{\rho_1}{\rho_2} \neq \frac{k}{b}$, there exists a constant $C > 0$ (independent of U_0) such that

$$\|e^{t\mathcal{A}}U_0\|_\tau \leq \frac{C}{t^{1/32}}\|U_0\|_{D(\mathcal{A}_\tau)}, \quad \forall\, t \geq 0.$$

4. If $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ and $k \neq k_0$, there exists a constant $C > 0$ (independent of U_0) such that

$$\|e^{t\mathcal{A}}U_0\|_\tau \leq \frac{C}{t^{1/8}}\|U_0\|_{D(\mathcal{A}_\tau)}, \quad \forall\, t \geq 0.$$

5. If $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ and $k = k_0$, there exists a constant $C > 0$ (independent of U_0) such that

$$\|e^{t\mathcal{A}}U_0\|_\tau \leq \frac{C}{t^{1/2}}\|U_0\|_{D(\mathcal{A}_\tau)}, \quad \forall\, t \geq 0.$$

Proof. Taking $F \in \mathcal{H}_\tau$ and $\lambda \in \mathbf{i}\mathbb{R}$ such that $|\lambda| \geq 1$, the vector $U := (\lambda I - \mathcal{A}_\tau)^{-1}F$ (which is well-defined by Theorem 3.8) satisfies the resolvent equation (3.34). Then, from Lemma 3.9,

$$\|\Psi\|_{L^2}^2 \leq C_\varepsilon\|\psi_x\|_{L^2}^2 + 2\varepsilon\|U\|_\tau^2 + C_\varepsilon\|F\|_\tau^2$$

and thus, using the expression (3.4),

$$\begin{aligned}(1-2\varepsilon)\|U\|_\tau^2 \leq{}& \rho_1\|\Phi\|_{L^2}^2 + \rho_1\|W\|_{L^2}^2 + C_\varepsilon\|\psi_x\|_{L^2}^2 + k\|\varphi_x + \psi + lw\|_{L^2}^2 \\ &+ k_0\|w_x - l\varphi\|_{L^2}^2 + \frac{\gamma}{m}\|\theta\|_{L^2}^2 + \frac{\gamma k_1 \tau}{m}\|q\|_{L^2}^2 + C_\varepsilon\|F\|_\tau^2.\end{aligned}$$

On the other hand, from Lemma 3.10,

$$\begin{aligned}C_\varepsilon\|\psi_x\|_{L^2}^2 \leq{}& C_\varepsilon(1+\chi_0^2|\lambda|^2)\|\Phi\|_{L^2}^2 + 5\varepsilon\|U\|_\tau^2 + C_\varepsilon\|F\|_\tau^2 + C_\varepsilon\|\varphi_x + \psi + lw\|_{L^2}^2 \\ &+ C_\varepsilon\|w_x - l\varphi\|_{L^2}^2 + C_\varepsilon\|\theta\|_{L^2}^2\end{aligned}$$

and thus

$$\begin{aligned}(1-7\varepsilon)\|U\|_\tau^2 \leq{}& C_\varepsilon C_\lambda\left(\|\Phi\|_{L^2}^2 + \|\varphi_x + \psi + lw\|_{L^2}^2\right) + \rho_1\|W\|_{L^2}^2 \\ &+ C_\varepsilon\|w_x - l\varphi\|_{L^2}^2 + C_\varepsilon\|\theta\|_{L^2}^2 + \frac{\gamma k_1 \tau}{m}\|q\|_{L^2}^2 + C_\varepsilon\|F\|_\tau^2\end{aligned}$$

where $C_\lambda := 1 + \chi_0^2|\lambda|^2$.

Additionally, from Lemma 3.20,

$$C_\varepsilon C_\lambda \left(\|\varphi_x+\psi+lw\|_{L^2}^2+\|\Phi\|_{L^2}^2\right) \leq C_\varepsilon C_\lambda(1+\chi_1^2|\lambda|^2)\|w_x-l\varphi\|_{L^2}^2+C_\varepsilon C_\lambda\tau|\lambda|^2\|q\|_{L^2}^2 + C_\varepsilon C_\lambda^2\|F\|_\tau^2+4\varepsilon\|U\|_\tau^2+C_\varepsilon C_\lambda^2\|W\|_{L^2}^2+C_\varepsilon C_\lambda^2\|\theta\|_{L^2}^2 + C_\varepsilon C_\lambda^2\|q\|_{L^2}^2$$

and thus

$$(1-11\varepsilon)\|U\|_\tau^2 \leq C_\varepsilon C_\lambda^2\|W\|_{L^2}^2+C_\varepsilon C_\lambda(1+\chi_1^2|\lambda|^2)\|w_x-l\varphi\|_{L^2}^2+C_\varepsilon C_\lambda^2\|\theta\|_{L^2}^2 + C_\varepsilon(C_\lambda^2+C_\lambda\tau|\lambda|^2)\|q\|_{L^2}^2+C_\varepsilon C_\lambda^2\|F\|_\tau^2.$$

Analogously, from Lemma 3.12,

$$C_\varepsilon C_\lambda^2\|W\|_{L^2}^2 \leq C_\varepsilon C_\lambda^4\|w_x-l\varphi\|_{L^2}^2+4\varepsilon\|U\|_\tau^2+C_\varepsilon C_\lambda^4\|\theta\|_{L^2}^2+C_\varepsilon C_\lambda^2\|\theta\|_{L^2}^2+C_\varepsilon C_\lambda^4\|q\|_{L^2}^2 + C_\varepsilon C_\lambda^4\|F\|_\tau^2$$

and thus, substituting in the previous estimate, we deduce that

$$(1-15\varepsilon)\|U\|_\tau^2 \leq C_\varepsilon\tilde{C}_\lambda\|w_x-l\varphi\|_{L^2}^2+C_\varepsilon C_\lambda^4\|\theta\|_{L^2}^2+C_\varepsilon(C_\lambda\tau|\lambda|^2+C_\lambda^4)\|q\|_{L^2}^2+C_\varepsilon C_\lambda^4\|F\|_\tau^2$$

where $\tilde{C}_\lambda := C_\lambda^4 + C_\lambda\chi_1^2|\lambda|^2$.

Similarly, from Lemma 3.13,

$$C_\varepsilon\tilde{C}_\lambda\|w_x-l\varphi\|_{L^2}^2 \leq C_\varepsilon\tilde{C}_\lambda^2\|\theta\|_{L^2}^2+3\varepsilon\|U\|_\tau^2+C_\varepsilon\tilde{C}_\lambda^2\|q\|_{L^2}^2+C_\varepsilon\tilde{C}_\lambda\|q\|_{L^2}^2+C_\varepsilon\tilde{C}_\lambda^2\|F\|_\tau^2$$

and thus

$$(1-18\varepsilon)\|U\|_\tau^2 \leq C_\varepsilon\tilde{C}_\lambda^2\|\theta\|_{L^2}^2+C_\varepsilon(C_\lambda\tau|\lambda|^2+\tilde{C}_\lambda^2)\|q\|_{L^2}^2+C_\varepsilon\tilde{C}_\lambda^2\|F\|_\tau^2. \tag{3.69}$$

At this point, we consider the Fourier and Cattaneo cases, separately.

- **Fourier ($\tau = 0$).** In this case, estimate (3.69) reduces to

$$(1-18\varepsilon)\|U\|_0^2 \leq C_\varepsilon\tilde{C}_\lambda^2\|\theta\|_{L^2}^2+C_\varepsilon\tilde{C}_\lambda^2\|q\|_{L^2}^2+C_\varepsilon\tilde{C}_\lambda^2\|F\|_0^2.$$

Now, from (3.42), we have $q=-\frac{1}{\delta}\theta_x$. Then, by Poincaré inequality we obtain

$$(1-18\varepsilon)\|U\|_0^2 \leq C_\varepsilon\tilde{C}_\lambda^2\|\theta_x\|_{L^2}^2+C_\varepsilon\tilde{C}_\lambda^2\|F\|_0^2.$$

Moreover, from Lemma 3.6,

$$C_\varepsilon \tilde{C}_\lambda^2 \|\theta_x\|_{L^2}^2 \le C_\varepsilon \tilde{C}_\lambda^4 \|F\|_0^2 + \varepsilon \|U\|_0^2$$

and thus

$$(1 - 19\varepsilon)\|U\|_0^2 \le C_\varepsilon \tilde{C}_\lambda^4 \|F\|_0^2$$

where

$$\begin{aligned}\tilde{C}_\lambda^4 = \left((1+\chi_0^2|\lambda|^2)^4 + (1+\chi_0^2|\lambda|^2)\chi_1^2|\lambda|^2\right)^4 &\le C\left(1 + \chi_1^8|\lambda|^8 + \chi_0^8\chi_1^8|\lambda|^{16} + \chi_0^{32}|\lambda|^{32}\right) \\ &\le \begin{cases} C|\lambda|^{32} & \text{if} \quad \chi_0 \neq 0, \\ C|\lambda|^8 & \text{if} \quad \chi_0 = 0 \text{ and } \chi_1 \neq 0. \end{cases}\end{aligned}$$

Therefore, taking $\varepsilon > 0$ small enough, we conclude that there exists a constant $C > 0$ (independent of λ and F) such that

$$\|(\lambda - \mathcal{A}_0)^{-1}F\|_0 = \|U\|_0 \le C|\lambda|^\alpha \|F\|_0,$$

where

$$\alpha = \begin{cases} 16 & \text{if} \quad \chi_0 \neq 0, \\ 4 & \text{if} \quad \chi_0 = 0 \text{ and } \chi_1 \neq 0. \end{cases}$$

Consequently, for the case $\chi_0 \neq 0$, the condition (a) of Theorem 2.19 holds with $\alpha = 16$ (and $\beta_0 = 1$). Hence there are constants $C, t_0 > 0$ such that

$$\|e^{t\mathcal{A}_0}\mathcal{A}_0^{-1}F\|_0 \le \frac{C}{t^{1/16}}\|F\|_0, \quad \forall\, t \ge t_0,\ F \in \mathcal{H}_0.$$

In particular,

$$\|e^{t\mathcal{A}_0}U_0\|_0 = \|e^{t\mathcal{A}_0}\mathcal{A}_0^{-1}\mathcal{A}_0U_0\|_0 \le \frac{C}{t^{1/16}}\|\mathcal{A}_0U_0\|_0 \le \frac{C}{t^{1/16}}\|U_0\|_{D(\mathcal{A}_0)}, \quad \forall\, t \ge t_0,\ U_0 \in D(\mathcal{A}_0)$$

which implies item 1. Analogously, for the case $\chi_0 = 0$ and $\chi_1 \neq 0$, there are constants $C, t_0 > 0$ such that

$$\|e^{t\mathcal{A}_0}\mathcal{A}_0^{-1}F\|_0 \le \frac{C}{t^{1/4}}\|F\|_0, \quad \forall\, t \ge t_0,\ F \in \mathcal{H}_0$$

and thus

$$\|e^{t\mathcal{A}_0}U_0\|_0 \le \frac{C}{t^{1/4}}\|U_0\|_{D(\mathcal{A}_0)}, \quad \forall\, t \ge t_0,\ U_0 \in D(\mathcal{A}_0)$$

which implies item 2.

- **Cattaneo ($\tau > 0$).** From Lemma 3.14,

$$C_\varepsilon \tilde{C}_\lambda^2 \|\theta\|_{L^2}^2 \leq C_\varepsilon \tilde{C}_\lambda^4 \|q\|_{L^2}^2 + 2\varepsilon \|U\|_\tau^2 + C_\varepsilon \tilde{C}_\lambda^4 \|F\|_\tau^2$$

and thus, returning to (3.69),

$$(1 - 20\varepsilon)\|U\|_\tau^2 \leq C_\varepsilon(\tilde{C}_\lambda^4 + C_\lambda \tau |\lambda|^2)\|q\|_{L^2}^2 + C_\varepsilon \tilde{C}_\lambda^4 \|F\|_\tau^2.$$

On the other hand, from Lemma 3.6,

$$C_\varepsilon(\tilde{C}_\lambda^4 + C_\lambda \tau |\lambda|^2)\|q\|_{L^2}^2 \leq C_\varepsilon(\tilde{C}_\lambda^4 + C_\lambda \tau |\lambda|^2)^2 \|F\|_\tau^2 + \varepsilon \|U\|_\tau^2$$

and thus

$$(1 - 21\varepsilon)\|U\|_\tau^2 \leq C_\varepsilon(\tilde{C}_\lambda^4 + C_\lambda \tau |\lambda|^2)^2 \|F\|_\tau^2$$

where

$$\begin{aligned}
(\tilde{C}_\lambda^4 + C_\lambda \tau|\lambda|^2)^2 &= \left(\left[(1+\chi_0^2|\lambda|^2)^4 + (1+\chi_0^2|\lambda|^2)\chi_1^2|\lambda|^2\right]^4 + (1+\chi_0^2|\lambda|^2)\tau|\lambda|^2\right)^2 \\
&\leq C\left(1 + \tau^2|\lambda|^4 + \chi_0^4\tau^2|\lambda|^8 + \chi_1^{16}|\lambda|^{16} + \chi_0^{32}\chi_1^{16}|\lambda|^{48} + \chi_0^{64}|\lambda|^{64}\right) \\
&\leq \begin{cases} C|\lambda|^{64} & \text{if} \quad \chi_0 \neq 0, \\ C|\lambda|^{16} & \text{if} \quad \chi_0 = 0 \text{ and } \chi_1 \neq 0, \\ C|\lambda|^4 & \text{if} \quad \chi_0 = \chi_1 = 0. \end{cases}
\end{aligned}$$

Therefore, taking $\varepsilon > 0$ small enough, we conclude that there exists a constant $C > 0$ (independent of λ and F) such that

$$\|(\lambda - \mathcal{A}_\tau)^{-1}F\|_\tau = \|U\|_\tau \leq C|\lambda|^\alpha \|F\|_\tau,$$

where

$$\alpha = \begin{cases} 32 & \text{if} \quad \chi_0 \neq 0, \\ 8 & \text{if} \quad \chi_0 = 0 \text{ and } \chi_1 \neq 0, \\ 2 & \text{if} \quad \chi_0 = \chi_1 = 0. \end{cases}$$

Analogously to the previous case, we obtain items 3, 4 and 5. □

Remark 3.23. In Theorem 3.22, the improvement of the polynomial rates of decay given by items 2 and 4 are interesting only from the mathematical point of view because those conditions cannot be fulfilled by *the physical system*. In fact, for *the physical* *system*, condition (1.3) implies that

$$\frac{\rho_1}{\rho_2} = \frac{k}{b} \qquad \Longleftrightarrow \qquad k_0 = k \qquad \Longleftrightarrow \qquad \frac{\rho_1}{\rho_2} = \frac{k}{b} \text{ and } k_0 = k.$$

Consequently, if $\frac{\rho_1}{\rho_2} = \frac{k}{b}$, then *the physical system* with Fourier's law decays exponentially (by Theorem 3.15), and *the physical system* with Cattaneo's law decays polynomially with rate of decay $t^{-1/2}$ (see item 5). However, since the condition $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ is not realistic, the best we obtained for the realistic *physical system* is polynomial decay with rate $t^{-1/16}$ in the Fourier case, and polynomial decay with rate $t^{-1/32}$ in the Cattaneo case. The optimality of all cases remains as an open problem.

4 HEAT CONDUCTION ON AXIAL FORCE AND BENDING MOMENT

In this chapter we prove that system (1.18)-(1.19), with its corresponding initial conditions, has a unique solution, which is exponentially stable if and only if condition (1.21) holds.

4.1 Semigroup formulation and well-posedness

We start by including our problem in the context of semigroups. For this, using the notations (2.2), we define the phase space

$$\mathbb{H} = H^1_0 \times L^2 \times H^1_* \times L^2_* \times H^1_* \times L^2_* \times L^2 \times L^2_* \times L^2 \times L^2_*$$

equipped with norm

$$\begin{aligned}\|U\|^2_{\mathbb{H}} = \rho_1\|\Phi\|^2_{L^2} + \rho_2\|\Psi\|^2_{L^2} + \rho_1\|W\|^2_{L^2} + b\|\psi_x\|^2_{L^2} + k\|\varphi_x + \psi + lw\|^2_{L^2} \\ + k_0\|w_x - l\varphi\|^2_{L^2} + \frac{\gamma}{m}\left(\|\vartheta\|^2_{L^2} + \|\theta\|^2_{L^2}\right) + \frac{\gamma k_1}{m}\left(\varsigma\|p\|^2_{L^2} + \tau\|q\|^2_{L^2}\right),\end{aligned} \tag{4.1}$$

where $U := (\varphi, \Phi, \psi, \Psi, w, W, \vartheta, p, \theta, q)$. As usual, under the assumption that $l\ell$ is not a multiple of π, the space $\mathbb{H}$ is a Hilbert space and $\|\cdot\|_{\mathbb{H}}$ is equivalent to the usual norm of $\mathbb{H}$ (the proofs follow as in Section 3.1).

Now, we define $\mathbb{A} : D(\mathbb{A}) \subset \mathbb{H} \to \mathbb{H}$ by

$$\mathbb{A}U = \begin{pmatrix} \Phi \\ \frac{k}{\rho_1}(\varphi_x + \psi + lw)_x + \frac{k_0 l}{\rho_1}(w_x - l\varphi) - \frac{l\gamma}{\rho_1}\theta \\ \Psi \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi + lw) - \frac{\gamma}{\rho_2}\vartheta_x \\ W \\ \frac{k_0}{\rho_1}(w_x - l\varphi)_x - \frac{kl}{\rho_1}(\varphi_x + \psi + lw) - \frac{\gamma}{\rho_1}\theta_x \\ -k_1 p_x - m\Psi_x \\ -\frac{\delta}{\varsigma}p - \frac{1}{\varsigma}\vartheta_x \\ -k_1 q_x - m(W_x - l\Phi) \\ -\frac{\delta}{\tau}q - \frac{1}{\tau}\theta_x \end{pmatrix}$$

with domain

$$D(\mathbb{A}) = \left\{U \in \mathbb{H} \mid \varphi \in H^2,\ \ \Phi, \psi_x, w_x, \vartheta, \theta \in H^1_0,\ \ \Psi, W, p, q \in H^1\right\}.$$

Remark 4.1. As in the proof of Theorem 3.7, we can show that if (U_n) is bounded in

$D(\mathbb{A})$, with respect to the graph norm $\|\cdot\|_{D(\mathbb{A})}$, then (U_n) is bounded in the space

$$(H^2 \cap H_0^1) \times H^1 \times (H^2 \cap H_*^1) \times H_*^1 \times (H^2 \cap H_*^1) \times H_*^1 \times [H^1 \times H_*^1]^2,$$

with respect to its usual norm, which is compactly embedded in $(\mathbb{H}, |\cdot|_{\mathbb{H}})$ by Corollary 2.13. Therefore, $(D(\mathbb{A}), \|\cdot\|_{D(\mathbb{A})})$ is compactly embedded in $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$, which implies that all elements of $\sigma(\mathbb{A})$ are eigenvalues of $\mathbb{A}$ (by Theorems 2.3 and 2.4).

Under this setting, problem (1.18)-(1.19) can be written as

$$\begin{cases} U_t = \mathbb{A}U, & t > 0 \\ U(0) = U_0 \end{cases} \tag{4.2}$$

where $U_0 := (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, \vartheta_0, p_0, \theta_0, q_0)$ and we can prove the main result of this section, which was already stated in [13].

Theorem 4.2 (Existence and uniqueness)**.** The operator $\mathbb{A}$ is the infinitesimal generator of a C_0-semigroups of contractions on $\mathbb{H}$ and thus, for each initial data $U_0 \in D(\mathbb{A})$, the problem (4.2) has a unique classical solution $U \in C^1([0, \infty); \mathbb{H})$, which is given by $U(t) = e^{t\mathbb{A}}U_0$.

Proof. We verify the hypotheses of Theorem 2.17. The final conclusion follows from Theorem 2.16.

- **$D(\mathbb{A})$ is dense in $\mathbb{H}$.**

We have

$$\mathbb{H} = Y_1 \times Y_2 \times Y_3 \times Y_4 \times Y_5 \times Y_6 \times Y_7 \times Y_8 \times Y_9 \times Y_{10}, \tag{4.3}$$

$$D(\mathbb{A}) = S_1 \times S_2 \times S_3 \times S_4 \times S_5 \times S_6 \times S_7 \times S_8 \times S_9 \times S_{10} \tag{4.4}$$

and

$$|(y_1, y_2, ..., y_{10})|_{\mathbb{H}}^2 = \|y_1\|_{Y_1}^2 + \|y_2\|_{Y_2}^2 + \cdots + \|y_8\|_{Y_{10}}^2, \tag{4.5}$$

where

$$S_i = \begin{cases} H^2 \cap H_0^1, & \text{if} \quad i = 1 \\ H_0^1, & \text{if} \quad i = 2, 7, 9 \\ \{u \in H_*^1 \mid u_x \in H_0^1\}, & \text{if} \quad i = 3, 5 \\ H_*^1, & \text{if} \quad i = 4, 6, 8, 10 \end{cases}$$

and

$$Y_i = \begin{cases} H_0^1, & \text{if} \quad i = 1 \\ L^2, & \text{if} \quad i = 2,7,9 \\ H_*^1, & \text{if} \quad i = 3,5 \\ L_*^2, & \text{if} \quad i = 4,6,8,10. \end{cases}$$

Due to the inclusion of $C_0^\infty(0,\ell)$ in $H^2(0,\ell) \cap H_0^1(0,\ell)$ and the density of $C_0^\infty(0,\ell)$ in $L^2(0,\ell)$, we show (as in the proof of Theorem 3.4) that S_i is a dense subspace of $(Y_i, \|\cdot\|_{Y_i})$, which implies the density of $D(\mathbb{A})$ in $\mathbb{H}$ by (4.3)-(4.5).

- **$\mathbb{A}$ is dissipative.**

A straightforward computation shows that

$$\Re(\mathbb{A}U, U)_{\mathbb{H}} = -\frac{\gamma\delta k_1}{m}(\|p\|_{L^2}^2 + \|q\|_{L^2}^2), \quad \forall\, U \in D(\mathbb{A}) \tag{4.6}$$

and this proves the dissipativity of $\mathbb{A}$.

- **$0 \in \rho(\mathbb{A})$.**

Suppose that $0 \notin \rho(\mathbb{A})$. Then, by Remark 4.1, 0 in an eigenvalue of $\mathbb{A}$. Therefore, there exist $U \neq 0$ in $D(\mathbb{A})$ satisfying the equation $\mathbb{A}U = 0$, which in terms of its components can be written as

$$\Phi = 0 \tag{4.7}$$
$$k(\varphi_x + \psi + lw)_x + k_0 l(w_x - l\varphi) - l\gamma\theta = 0 \tag{4.8}$$
$$\Psi = 0 \tag{4.9}$$
$$b\psi_{xx} - k(\varphi_x + \psi + lw) - \gamma\vartheta_x = 0 \tag{4.10}$$
$$W = 0 \tag{4.11}$$
$$k_0(w_x - l\varphi)_x - kl(\varphi_x + \psi + lw) - \gamma\theta_x = 0 \tag{4.12}$$
$$-k_1 p_x - m\Psi_x = 0 \tag{4.13}$$
$$-\delta p - \vartheta_x = 0 \tag{4.14}$$
$$-k_1 q_x - m(W_x - l\Phi) = 0 \tag{4.15}$$
$$-\delta q - \theta_x = 0. \tag{4.16}$$

From (4.7), (4.9) and (4.11), it follows that $\Phi = \Psi = W = 0$. Substituting this into (4.13) and (4.15), it follows that $p_x = q_x = 0$. Since $p, q \in H_*^1$, we conclude that $p = q = 0$. Substituting this into (4.14) and (4.16), it follows that $\vartheta_x = \theta_x = 0$. Since $\vartheta, \theta \in H_0^1$, we conclude that $\vartheta = \theta = 0$. Substituting this into (4.8), (4.10) and (4.12), it follows that

(φ, ψ, w) satisfies

$$B((\varphi^*, \psi^*, w^*), (\varphi, \psi, w)) = 0, \quad \forall\ (\varphi^*, \psi^*, w^*) \in H_0^1 \times H_*^1 \times H_*^1 \tag{4.17}$$

where B is defined by

$$\begin{aligned} B((\varphi^*, \psi^*, w^*), (\varphi, \psi, w)) = {} & k\int_0^\ell (\varphi_x^* + \psi^* + lw^*)\overline{(\varphi_x + \psi + lw)}\ dx \\ & + k_0\int_0^\ell (w_x^* - l\varphi^*)\overline{(w_x - l\varphi)}\ dx + b\int_0^\ell \psi_x^*\overline{\psi_x}\ dx. \end{aligned}$$

Since B is a continuous coercive sesquilinear form on $H_0^1 \times H_*^1 \times H_*^1$, it follows from Theorem 2.5 that (4.17) has a unique solution and thus $\psi = \psi = w = 0$. This shows that $U = 0$, which is a contradiction. Therefore, $0 \in \rho(\mathbb{A})$. $\square$

4.2 Characterization of exponential stability

In this section, we prove that condition (1.21) is sufficient and necessary for exponential stability of the semigroup generated by $\mathbb{A}$. We start by proving some lemmas about a sequence $(\beta_n)_{n\in\mathbb{N}}$ of real numbers and a sequence

$$U_n := (\varphi^{(n)}, \Phi^{(n)}, \psi^{(n)}, \Psi^{(n)}, w^{(n)}, W^{(n)}, \vartheta^{(n)}, p^{(n)}, \theta^{(n)}, q^{(n)}) \in D(\mathbb{A})$$

such that

$$\beta_n \overset{n\to\infty}{\longrightarrow} \infty, \quad (U_n)_{n\in\mathbb{N}} \text{ is bounded in } \mathbb{H} \quad \text{and} \quad \|(i\beta_n - \mathbb{A})U_n\|_{\mathbb{H}} \overset{n\to\infty}{\longrightarrow} 0. \tag{4.18}$$

Before to formulate the lemmas we observe that, using the equivalence between the norm $\|\cdot\|_{\mathbb{H}}$ given by (4.1) and the usual norm

$$\begin{aligned} |U|_{\mathbb{H}}^2 := {} & \|\varphi\|_{H^1}^2 + \|\Phi\|_{L^2}^2 + \|\psi\|_{H^1}^2 + \|\Psi\|_{L^2}^2 + \|w\|_{H^1}^2 + \|W\|_{L^2}^2 + \|\vartheta\|_{L^2}^2 + \|p\|_{L^2}^2 \\ & + \|\theta\|_{L^2}^2 + \|q\|_{L^2}^2, \end{aligned} \tag{4.19}$$

it follows from (4.18) that

$$i\beta_n\varphi^{(n)} - \Phi^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0 \tag{4.20}$$

$$i\beta_n\varphi_x^{(n)} - \Phi_x^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0 \tag{4.21}$$

$$\rho_1 i\beta_n\Phi^{(n)} - k(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_x - k_0 l(w_x^{(n)} - l\varphi^{(n)}) + l\gamma\theta^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0 \tag{4.22}$$

$$i\beta_n\psi^{(n)} - \Psi^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0 \tag{4.23}$$

$$i\beta_n\psi_x^{(n)} - \Psi_x^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0 \tag{4.24}$$

$$\rho_2 i\beta_n\Psi^{(n)} - b\psi_{xx}^{(n)} + k(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}) + \gamma\vartheta_x^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0 \tag{4.25}$$

$$i\beta_n w^{(n)} - W^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0 \tag{4.26}$$

$$i\beta_n w_x^{(n)} - W_x^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0 \tag{4.27}$$

$$\rho_1 i\beta_n W^{(n)} - k_0(w_x^{(n)} - l\varphi^{(n)})_x + kl(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}) + \gamma\theta_x^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0 \tag{4.28}$$

$$i\beta_n\vartheta^{(n)} + k_1 p_x^{(n)} + m\Psi_x^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0 \tag{4.29}$$

$$\varsigma i\beta_n p^{(n)} + \delta p^{(n)} + \vartheta_x^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0 \tag{4.30}$$

$$i\beta_n\theta^{(n)} + k_1 q_x^{(n)} + m(W_x^{(n)} - l\Phi^{(n)}) \stackrel{n\to\infty}{\longrightarrow} 0 \tag{4.31}$$

$$\tau i\beta_n q^{(n)} + \delta q^{(n)} + \theta_x^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0 \tag{4.32}$$

with all convergences in the sense of $L^2(0,\ell)$.

Lemma 4.3. Assume (4.18). Then, the following sequences are bounded in $L^2(0,\ell)$:

(a) $(\Phi^{(n)})_{n\in\mathbb{N}}$, $(\Psi^{(n)})_{n\in\mathbb{N}}$, $(W^{(n)})_{n\in\mathbb{N}}$, $(\psi_x^{(n)})_{n\in\mathbb{N}}$, $(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{n\in\mathbb{N}}$, $(w_x^{(n)} - l\varphi^{(n)})_{n\in\mathbb{N}}$, $(\vartheta^{(n)})_{n\in\mathbb{N}}$, $(\theta^{(n)})_{n\in\mathbb{N}}$, $(q^{(n)})_{n\in\mathbb{N}}$, $(p^{(n)})_{n\in\mathbb{N}}$.

(b) $(\varphi_x^{(n)})_{n\in\mathbb{N}}$, $(w_x^{(n)})_{n\in\mathbb{N}}$.

(c) $\left(\frac{\Phi_x^{(n)}}{i\beta_n}\right)_{n\in\mathbb{N}}$, $\left(\frac{\Psi_x^{(n)}}{i\beta_n}\right)_{n\in\mathbb{N}}$, $\left(\frac{W_x^{(n)}}{i\beta_n}\right)_{n\in\mathbb{N}}$.

(d) $\left(\frac{\psi_{xx}^{(n)}}{i\beta_n}\right)_{n\in\mathbb{N}}$, $\left(\frac{(w_x^{(n)} - l\varphi^{(n)})_x}{i\beta_n}\right)_{n\in\mathbb{N}}$.

Proof. Since $(U_n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{H}$, it follows from (4.1) that the following sequences are bounded in $L^2(0,\ell)$:

$$(\Phi^{(n)})_{n\in\mathbb{N}}, \quad (\Psi^{(n)})_{n\in\mathbb{N}}, \quad (W^{(n)})_{n\in\mathbb{N}}, \quad (\psi_x^{(n)})_{n\in\mathbb{N}}, \quad (\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{n\in\mathbb{N}},$$
$$(w_x^{(n)} - l\varphi^{(n)})_{n\in\mathbb{N}}, \quad (\vartheta^{(n)})_{n\in\mathbb{N}}, \quad (\theta^{(n)})_{n\in\mathbb{N}}, \quad (p^{(n)})_{n\in\mathbb{N}}, \quad (q^{(n)})_{n\in\mathbb{N}}.$$

Then, the sequences of (a) are bounded in $L^2(0,\ell)$. The boundedness of the sequences in (b) are a direct consequence of the equivalence of the norms (4.1) and (4.19). For (c), note that all sequences in (4.20)–(4.32) are bounded in $L^2(0,\ell)$. In particular, multiplying by the bounded sequence $\left(\frac{1}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}}$, the following sequences are also bounded:

$$\begin{aligned}
\left(\frac{\Phi_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}} &= (\varphi_x^{(n)})_{n\in\mathbb{N}} - \left(\varphi_x^{(n)} - \frac{\Phi_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}},\\
\left(\frac{\Psi_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}} &= (\psi_x^{(n)}) - \left(\psi_x^{(n)} - \frac{\Psi_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}},\\
\left(\frac{W_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}} &= (w_x^{(n)}) - \left(w_x^{(n)} - \frac{W_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}},
\end{aligned}$$

which proves (c). Additionally, we have the boundedness of the following sequences:

$$\begin{aligned}
\left(b\frac{\psi_{xx}^{(n)}}{\mathbf{i}\beta_n} - \gamma\frac{\vartheta_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}} &= \left(\rho_2\Psi^{(n)} + k\frac{(\varphi_x^{(n)}+\psi^{(n)}+lw^{(n)})}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}}\\
&\quad - \left(\rho_2\Psi^{(n)} - b\frac{\psi_{xx}^{(n)}}{\mathbf{i}\beta_n} + k\frac{(\varphi_x^{(n)}+\psi^{(n)}+lw^{(n)})}{\mathbf{i}\beta_n} + \gamma\frac{\vartheta_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}},\\
\left(k_0\frac{(w_x^{(n)}-l\varphi^{(n)})_x}{\mathbf{i}\beta_n} - \gamma\frac{\theta_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}} &= \left(\rho_1 W^{(n)} + kl\frac{(\varphi_x^{(n)}+\psi^{(n)}+lw^{(n)})}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}}\\
&\quad - \left(\rho_1 W^{(n)} - k_0\frac{(w_x^{(n)}-l\varphi^{(n)})_x}{\mathbf{i}\beta_n} + kl\frac{(\varphi_x^{(n)}+\psi^{(n)}+lw^{(n)})}{\mathbf{i}\beta_n} + \gamma\frac{\theta_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}},\\
\left(\frac{\vartheta_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}} &= \left(\varsigma p^{(n)} + \delta\frac{p^{(n)}}{\mathbf{i}\beta_n} + \frac{\vartheta_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}} - \left(\varsigma p^{(n)} + \delta\frac{p^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}},\\
\left(\frac{\theta_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}} &= \left(\tau q^{(n)} + \delta\frac{q^{(n)}}{\mathbf{i}\beta_n} + \frac{\theta_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}} - \left(\tau q^{(n)} + \delta\frac{q^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}}.
\end{aligned}$$

Then, the following sequences are also bounded in $L^2(0,\ell)$:

$$\begin{aligned}
\left(\frac{\psi_{xx}^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}} &= \frac{1}{b}\left(b\frac{\psi_{xx}^{(n)}}{\mathbf{i}\beta_n} - \gamma\frac{\vartheta_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}} + \frac{\gamma}{b}\left(\frac{\vartheta_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}},\\
\left(\frac{(w_x^{(n)}-l\varphi^{(n)})_x}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}} &= \frac{1}{k_0}\left(k_0\frac{(w_x^{(n)}-l\varphi^{(n)})_x}{\mathbf{i}\beta_n} - \gamma\frac{\theta_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}} + \frac{\gamma}{k_0}\left(\frac{\theta_x^{(n)}}{\mathbf{i}\beta_n}\right)_{n\in\mathbb{N}},
\end{aligned}$$

which completes the proof of (d). □

Lemma 4.4. Assume (4.18). Then, $q^{(n)} \overset{n\to\infty}{\longrightarrow} 0$ and $p^{(n)} \overset{n\to\infty}{\longrightarrow} 0$.

Proof. Let us write $G_n := \mathbf{i}\beta_n U_n - \mathbb{A}U_n$. Then, multiplying by U,

$$\mathbf{i}\beta_n\|U_n\|_{\mathbb{H}}^2 - (\mathbb{A}U_n, U_n)_{\mathbb{H}} = (G_n, U_n)_{\mathbb{H}}.$$

Now, taking the real part and recalling (4.18) together with equation (4.6), we conclude that

$$\|q^{(n)}\|_{L^2}^2 + \|p^{(n)}\|_{L^2}^2 \le \frac{m}{\gamma\delta k_1}\|G_n\|_{\mathbb{H}}\|U_n\|_{\mathbb{H}} = \frac{m}{\gamma\delta k_1}\|\mathbf{i}\beta_n U_n - \mathbb{A}U_n\|_{\mathbb{H}}\|U\|_{\mathbb{H}} \overset{n\to\infty}{\longrightarrow} 0$$

which proves the result. □

Lemma 4.5. Assume (4.18). Then,

(a) $\dfrac{1}{\mathbf{i}\beta_n}(W_x^{(n)}, \theta^{(n)})_{L^2} \stackrel{n\to\infty}{\longrightarrow} 0$, $\dfrac{1}{\mathbf{i}\beta_n}(q^{(n)}, \theta_x^{(n)})_{L^2} \stackrel{n\to\infty}{\longrightarrow} 0$.

(b) $\dfrac{1}{\mathbf{i}\beta_n}(\Psi_x^{(n)}, \vartheta^{(n)})_{L^2} \stackrel{n\to\infty}{\longrightarrow} 0$, $\dfrac{1}{\mathbf{i}\beta_n}(p^{(n)}, \vartheta_x^{(n)})_{L^2} \stackrel{n\to\infty}{\longrightarrow} 0$.

Proof. From (4.32) and Lemma 4.4 we deduce that $\frac{1}{\mathbf{i}\beta_n}\theta_x^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0$. Then, multiplying by $W^{(n)}$ and $q^{(n)}$ in $L^2(0,\ell)$, we obtain (a). Analogously, using equation (4.30) and Lemma 4.4 we have $\frac{1}{\mathbf{i}\beta_n}\vartheta_x^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0$. Then, multiplying by $\Psi^{(n)}$ and $p^{(n)}$ in $L^2(0,\ell)$, we deduce (b). □

Lemma 4.6. Assume (4.18). Then, $\theta^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0$ and $\vartheta^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0$.

Proof. From (4.31) we have

$$\theta^{(n)} + \frac{k_1}{\mathbf{i}\beta_n} q_x^{(n)} + \frac{m}{\mathbf{i}\beta_n} W_x^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0.$$

Then, multiplying by $\theta^{(n)}$ in $L^2(0,\ell)$,

$$\|\theta^{(n)}\|_{L^2}^2 - \frac{k_1}{\mathbf{i}\beta_n}(q^{(n)}, \theta_x^{(n)})_{L^2} + \frac{m}{\mathbf{i}\beta_n}(W_x^{(n)}, \theta^{(n)})_{L^2} \stackrel{n\to\infty}{\longrightarrow} 0,$$

which implies, using Lemma 4.5, that $\theta^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0$. Analogously, from (4.29) we have

$$\vartheta^{(n)} + \frac{k_1}{\mathbf{i}\beta_n} p_x^{(n)} + \frac{m}{\mathbf{i}\beta_n} \Psi_x^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0.$$

Then, multiplying by $\vartheta^{(n)}$ in $L^2(0,\ell)$ we obtain

$$\|\vartheta^{(n)}\|_{L^2}^2 - \frac{k_1}{\mathbf{i}\beta_n}(p^{(n)}, \vartheta_x^{(n)})_{L^2} + \frac{m}{\mathbf{i}\beta_n}(\Psi_x^{(n)}, \vartheta^{(n)})_{L^2} \stackrel{n\to\infty}{\longrightarrow} 0,$$

which implies, using Lemma 4.5, that $\vartheta^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0$. □

Lemma 4.7. Assume (4.18). Then, $w_x^{(n)} - l\varphi^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0$ and $\psi_x^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0$.

Proof. From (4.31), we deduce that

$$\theta^{(n)} + \frac{k_1}{\mathbf{i}\beta_n} q_x^{(n)} + \frac{m}{\mathbf{i}\beta_n}(W_x^{(n)} - l\Phi^{(n)}) \overset{n\to\infty}{\longrightarrow} 0.$$

On the other hand, from (4.20) and (4.27) we can obtain

$$\mathbf{i}\beta_n(w_x^{(n)} - l\varphi^{(n)}) - (W_x^{(n)} - l\Phi^{(n)}) \overset{n\to\infty}{\longrightarrow} 0. \tag{4.33}$$

Then, multiplying (4.33) by $\frac{m}{\mathbf{i}\beta_n}$ and adding to the first convergence, we obtain

$$\theta^{(n)} + \frac{k_1}{\mathbf{i}\beta_n} q_x^{(n)} + m(w_x^{(n)} - l\varphi^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

Finally, multiplying by the bounded sequence $(w_x^{(n)} - l\varphi^{(n)})$ in $L^2(0,\ell)$ we deduce that

$$(\theta^{(n)}, w_x^{(n)} - l\varphi^{(n)})_{L^2} + k_1\left(q^{(n)}, \frac{(w_x^{(n)} - l\varphi^{(n)})_x}{\mathbf{i}\beta_n}\right)_{L^2} + m\|w_x^{(n)} - l\varphi^{(n)}\|_{L^2}^2 \overset{n\to\infty}{\longrightarrow} 0.$$

Note that the first term goes to zero by Lemma 4.6, and the second also goes to zero by Lemmas 4.3 and 4.4. This implies that $w_x^{(n)} - l\varphi^{(n)} \overset{n\to\infty}{\longrightarrow} 0$. Analogously, from (4.29),

$$\vartheta^{(n)} + \frac{k_1}{\mathbf{i}\beta_n} p_x^{(n)} + \frac{m}{\mathbf{i}\beta_n}\Psi_x^{(n)} \overset{n\to\infty}{\longrightarrow} 0.$$

From (4.24), multiplying by $\frac{m}{\mathbf{i}\beta_n}$, we have

$$m\psi_x^{(n)} - \frac{m}{\mathbf{i}\beta_n}\Psi_x^{(n)} \overset{n\to\infty}{\longrightarrow} 0. \tag{4.34}$$

Then, adding the last convergences, we obtain

$$\vartheta^{(n)} + \frac{k_1}{\mathbf{i}\beta_n} p_x^{(n)} + m\psi_x^{(n)} \overset{n\to\infty}{\longrightarrow} 0,$$

which implies, multiplying by $\psi_x^{(n)}$ in $L^2(0,\ell)$, that

$$(\vartheta^{(n)}, \psi_x^{(n)})_{L^2} + k_1\left(p^{(n)}, \frac{\psi_{xx}^{(n)}}{\mathbf{i}\beta_n}\right)_{L^2} + m\|\psi_x^{(n)}\|_{L^2}^2 \overset{n\to\infty}{\longrightarrow} 0.$$

Again, the first term goes to zero by Lemma 4.6 and the second by Lemmas 4.3 and 4.4. This implies that $\psi_x^{(n)} \overset{n\to\infty}{\longrightarrow} 0$. □

Lemma 4.8. Assume (4.18). Then, $W^{(n)} \overset{n\to\infty}{\longrightarrow} 0$ and $\Psi^{(n)} \overset{n\to\infty}{\longrightarrow} 0$.

Proof. Multiplying (4.28) by $\frac{1}{\mathbf{i}\beta_n}$, which goes to zero, and using the boundedness of

$(\varphi_x + \psi + lw)$, which comes from Lemma 4.3, we deduce

$$W^{(n)} - \frac{k_0}{\rho_1 \mathbf{i}\beta_n}(w_x^{(n)} - l\varphi^{(n)})_x + \frac{\gamma}{\rho_1 \mathbf{i}\beta_n}\theta_x^{(n)} \overset{n\to\infty}{\longrightarrow} 0.$$

Then, multiplying by $W^{(n)}$ in $L^2(0,\ell)$,

$$\|W^{(n)}\|_{L^2}^2 - \frac{k_0}{\rho_1}\left(w_x^{(n)} - l\varphi^{(n)}, \frac{W_x^{(n)}}{\mathbf{i}\beta_n}\right)_{L^2} + \frac{\gamma}{\rho_1 \mathbf{i}\beta_n}(\theta_x^{(n)}, W^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

Note that the second term goes to zero by Lemmas 4.3 and 4.7. Also, the last term goes to zero by Lemmas 4.3 and 4.5. These convergences imply the convergence $W^{(n)} \overset{n\to\infty}{\longrightarrow} 0$. Analogously, multiplying (4.25) by $\frac{1}{\mathbf{i}\beta_n}$ and using the boundedness of $(\varphi_x + \psi + lw)$ again, we obtain

$$\Psi^{(n)} - \frac{b}{\rho_2 \mathbf{i}\beta_n}\psi_{xx}^{(n)} + \frac{\gamma}{\rho_2 \mathbf{i}\beta_n}\vartheta_x^{(n)} \overset{n\to\infty}{\longrightarrow} 0,$$

which implies, multiplying by $\Psi^{(n)}$ in $L^2(0,\ell)$,

$$\|\Psi^{(n)}\|_{L^2}^2 - \frac{b}{\rho_2}\left(\psi_x^{(n)}, \frac{\Psi_x}{\mathbf{i}\beta_n}\right)_{L^2} + \frac{\gamma}{\rho_2 \mathbf{i}\beta_n}(\vartheta_x^{(n)}, \Psi)_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

Again, the second term goes to zero by Lemmas 4.3 and 4.7. Also, the last term goes to zero by Lemmas 4.3 and 4.5. This implies the convergence $\Psi^{(n)} \overset{n\to\infty}{\longrightarrow} 0$. □

Lemma 4.9. Assume (4.18). Then,

(a) $\tau(q^{(n)}, \Phi_x^{(n)})_{L^2} - (\theta_x^{(n)}, \varphi_x^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$

(b) $\varsigma(p^{(n)}, \Phi_x^{(n)})_{L^2} - (\vartheta_x^{(n)}, \varphi_x^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$

Proof. From Lemmas 4.3 and 4.6,

$$(\theta_x^{(n)}, \psi^{(n)} + lw^{(n)})_{L^2} = -(\theta^{(n)}, \psi_x^{(n)})_{L^2} - l(\theta^{(n)}, w_x^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0, \tag{4.35}$$

which implies, multiplying (4.32) by the sequence $(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})$, bounded in $L^2(0,\ell)$, and using Lemma 4.4, that

$$\tau\mathbf{i}\beta_n(q^{(n)}, \varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{L^2} + (\theta_x^{(n)}, \varphi_x^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0. \tag{4.36}$$

On the other hand, from (4.21), (4.23) and (4.26) we obtain

$$\mathbf{i}\beta_n(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}) - (\Phi_x^{(n)} + \Psi^{(n)} + lW^{(n)}) \overset{n\to\infty}{\longrightarrow} 0, \tag{4.37}$$

which implies, multiplying by $q^{(n)}$ in $L^2(0,\ell)$ and using Lemmas 4.3 and 4.4,

$$-\mathbf{i}\beta_n(q^{(n)}, \varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{L^2} - (q^{(n)}, \Phi_x^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

Finally, multiplying by τ and adding to (4.36) we get the convergence (a). Analogously, from Lemmas 4.3 and 4.6,

$$(\vartheta_x^{(n)}, \psi^{(n)} + lw^{(n)})_{L^2} = -(\vartheta^{(n)}, \psi_x^{(n)})_{L^2} - l(\vartheta^{(n)}, w_x^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0, \tag{4.38}$$

which implies, multiplying (4.30) by $(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})$ in $L^2(0,\ell)$, that

$$\varsigma\mathbf{i}\beta_n(p^{(n)}, \varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{L^2} + (\vartheta_x^{(n)}, \varphi_x^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0. \tag{4.39}$$

Moreover, multiplying (4.37) by $p^{(n)}$ in $L^2(0,\ell)$ and using Lemmas 4.3 and 4.4,

$$-\mathbf{i}\beta_n(p^{(n)}, \varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{L^2} - (p^{(n)}, \Phi_x^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

Then, multiplying by ς and adding to (4.39) we obtain the convergence (b). □

Lemma 4.10. Assume (4.18) and condition (1.21). Then,

$$\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)} \overset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \Phi^{(n)} \overset{n\to\infty}{\longrightarrow} 0.$$

Proof. Note that condition (1.21) is equivalent to

$$\underbrace{\left(\varsigma - \frac{k_1\rho_1}{k}\right)\left(b - \frac{k\rho_2}{\rho_1}\right) + \varsigma\gamma m}_{:=\xi} = 0 \quad \text{or} \quad \underbrace{\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau\gamma m}_{:=\chi} = 0. \tag{4.40}$$

First, let us assume that $\chi = 0$. Then, multiplying (4.28) by $(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})$ in $L^2(0,\ell)$ and using (4.35),

$$\begin{aligned}\rho_1\mathbf{i}\beta_n(W^{(n)}, \varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{L^2} + k_0(w_x^{(n)} - l\varphi^{(n)}, (\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_x)_{L^2} \\ + kl\|\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}\|_{L^2} + \gamma(\theta_x^{(n)}, \varphi_x^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0.\end{aligned} \tag{4.41}$$

Moreover, multiplying (4.37) by $W^{(n)}$ in $L^2(0,\ell)$ and using Lemma 4.8, we deduce

$$-\mathbf{i}\beta_n(W^{(n)}, \varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{L^2} - (W^{(n)}, \Phi_x^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

Then, multiplying by ρ_1 and adding to (4.41),

$$-\rho_1(W^{(n)},\Phi_x^{(n)})_{L^2}+k_0(w_x^{(n)}-l\varphi^{(n)},(\varphi_x^{(n)}+\psi^{(n)}+lw^{(n)})_x)_{L^2} \\ +kl\|\varphi_x^{(n)}+\psi^{(n)}+lw^{(n)}\|_{L^2}+\gamma(\theta_x^{(n)},\varphi_x^{(n)})_{L^2}\overset{n\to\infty}{\longrightarrow}0. \tag{4.42}$$

On the other hand, from (4.22), Lemmas 4.6 and 4.7, we deduce that

$$\rho_1\mathbf{i}\beta_n\Phi^{(n)}-k(\varphi_x^{(n)}+\psi^{(n)}+lw^{(n)})_x\overset{n\to\infty}{\longrightarrow}0, \tag{4.43}$$

which implies, multiplying by $w_x^{(n)}-l\varphi^{(n)}$ in $L^2(0,\ell)$, that

$$-\rho_1\mathbf{i}\beta_n(w_x^{(n)}-l\varphi^{(n)},\Phi^{(n)})_{L^2}-k(w_x^{(n)}-l\varphi^{(n)},(\varphi_x^{(n)}+\psi^{(n)}+lw^{(n)})_x)_{L^2}\overset{n\to\infty}{\longrightarrow}0.$$

Also, multiplying (4.33) by $\rho_1\Phi^{(n)}$ in $L^2(0,\ell)$,

$$\rho_1\mathbf{i}\beta_n(w_x^{(n)}-l\varphi^{(n)},\Phi^{(n)})_{L^2}-\rho_1(W_x^{(n)}-l\Phi^{(n)},\Phi^{(n)})_{L^2}\overset{n\to\infty}{\longrightarrow}0,$$

which implies, adding the last two convergences, that

$$\rho_1(W^{(n)},\Phi_x^{(n)})_{L^2}+\rho_1 l\|\Phi^{(n)}\|_{L^2}^2-k(w_x^{(n)}-l\varphi^{(n)},(\varphi_x^{(n)}+\psi^{(n)}+lw^{(n)})_x)_{L^2}\overset{n\to\infty}{\longrightarrow}0.$$

Multiplying the last expression by $\frac{k_0}{k}$, adding to $\gamma\times$(a), where (a) is the first convergence of Lemma 4.9, and finally adding to (4.42), we obtain

$$\rho_1\left(\frac{k_0}{k}-1\right)(W^{(n)},\Phi_x^{(n)})_{L^2}+kl\|\varphi_x^{(n)}+\psi^{(n)}+lw^{(n)}\|_{L^2}+\frac{k_0\rho_1 l}{k}\|\Phi^{(n)}\|_{L^2}^2 \\ +\gamma\tau(q^{(n)},\Phi_x^{(n)})_{L^2}\overset{n\to\infty}{\longrightarrow}0. \tag{4.44}$$

On the other hand, multiplying (4.31) by $\Phi^{(n)}$ in $L^2(0,\ell)$,

$$\mathbf{i}\beta_n(\theta^{(n)},\Phi^{(n)})_{L^2}-k_1(q^{(n)},\Phi_x^{(n)})_{L^2}-m(W^{(n)},\Phi_x^{(n)})_{L^2}-ml\|\Phi^{(n)}\|_{L^2}^2\overset{n\to\infty}{\longrightarrow}0. \tag{4.45}$$

Also, multiplying (4.43) by $\theta^{(n)}$ in $L^2(0,\ell)$ and using (4.35), we obtain

$$-\rho_1\mathbf{i}\beta_n(\theta^{(n)},\Phi^{(n)})_{L^2}+k(\theta_x^{(n)},\varphi_x^{(n)})_{L^2}\overset{n\to\infty}{\longrightarrow}0.$$

Then, doing $k\times$(a), with (a) of Lemma 4.9, and adding to the last expression, we deduce that

$$-\rho_1\mathbf{i}\beta_n(\theta^{(n)},\Phi^{(n)})_{L^2}+\tau k(q^{(n)},\Phi_x^{(n)})_{L^2}\overset{n\to\infty}{\longrightarrow}0. \tag{4.46}$$

Now, in order to apply the same strategy of [8], let us define

$$\sigma_1 := \left(1 - \frac{k_1\rho_1}{\tau k}\right) \neq 0, \quad \text{because} \quad \chi = 0.$$

In addition, $\chi = 0$ implies $\frac{\rho_1}{k\sigma_1}\chi = 0$, that is,

$$\rho_1\left(\frac{k_0}{k} - 1\right) + \frac{\rho_1\gamma m}{k\sigma_1} = 0. \tag{4.47}$$

Multiplying (4.45) by $-\frac{\rho_1\gamma}{k\sigma_1}$, we have

$$-\frac{\rho_1\gamma}{k\sigma_1}\mathbf{i}\beta_n(\theta^{(n)},\Phi^{(n)})_{L^2}+\frac{\rho_1\gamma k_1}{k\sigma_1}(q^{(n)},\Phi_x^{(n)})_{L^2}+\frac{\rho_1\gamma m}{k\sigma_1}(W^{(n)},\Phi_x^{(n)})_{L^2}+\frac{\rho_1\gamma ml}{k\sigma_1}\|\Phi^{(n)}\|_{L^2}^2 \xrightarrow{n\to\infty} 0.$$

Multiplying (4.46) by $-\frac{\gamma}{k\sigma_1}$ and adding to the last expression,

$$\left(\frac{\rho_1\gamma k_1}{k\sigma_1} - \frac{\tau\gamma}{\sigma_1}\right)(q^{(n)},\Phi_x^{(n)})_{L^2} + \frac{\rho_1\gamma m}{k\sigma_1}(W^{(n)},\Phi_x^{(n)})_{L^2} + \frac{\rho_1\gamma ml}{k\sigma_1}\|\Phi^{(n)}\|_{L^2}^2 \xrightarrow{n\to\infty} 0,$$

which, added to (4.44) and using (4.47), yields

$$\begin{aligned} kl\|\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}\|_{L^2} + \left(\frac{\rho_1\gamma ml}{k\sigma_1} + \frac{k_0\rho_1 l}{k}\right)\|\Phi^{(n)}\|_{L^2}^2 \\ + \left(\frac{\rho_1\gamma k_1}{k\sigma_1} - \frac{\tau\gamma}{\sigma_1} + \gamma\tau\right)(q^{(n)},\Phi_x^{(n)})_{L^2} \xrightarrow{n\to\infty} 0 \end{aligned}$$

which implies the desired result because, again by (4.47),

$$\left(\frac{\rho_1\gamma ml}{k\sigma_1} + \frac{k_0\rho_1 l}{k}\right) = l\rho_1 > 0$$

and

$$\left(\frac{\rho_1\gamma k_1}{k\sigma_1} - \frac{\tau\gamma}{\sigma_1} + \gamma\tau\right) = \frac{\gamma}{k\sigma_1}(\rho_1 k_1 - k\tau + k\sigma_1\tau) = 0.$$

Second, let us assume that $\xi = 0$. Then, multiplying (4.25) by $(\varphi_x^{(n)}+\psi^{(n)}+lw^{(n)})$ in $L^2(0,\ell)$ and using (4.38),

$$\begin{aligned} \rho_2\mathbf{i}\beta_n(\Psi^{(n)},\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{L_2} + b(\psi_x^{(n)},(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_x)_{L_2} \\ +k\|\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}\|_{L_2}^2 + \gamma(\vartheta_x^{(n)},\varphi_x^{(n)})_{L_2} \xrightarrow{n\to\infty} 0. \end{aligned} \tag{4.48}$$

Multiplying (4.37) by $\Psi^{(n)}$ in $L^2(0,\ell)$ and using Lemma 4.8,

$$-\mathbf{i}\beta_n(\Psi^{(n)},\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{L^2} - (\Psi^{(n)},\Phi_x^{(n)})_{L^2} \xrightarrow{n\to\infty} 0,$$

which, multiplied by ρ_2 and added to (4.48), implies

$$\begin{aligned}-\rho_2(\Psi^{(n)}, \Phi_x^{(n)})_{L^2} + b(\psi_x^{(n)}, (\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_x)_{L^2}\\ +k\|\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}\|_{L^2} + \gamma(\vartheta_x^{(n)}, \varphi_x^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0.\end{aligned} \tag{4.49}$$

Also, multiplying (4.43) by $\psi_x^{(n)}$ in $L^2(0,\ell)$, we have

$$-\rho_1 \mathbf{i}\beta_n(\psi_x^{(n)}, \Phi^{(n)})_{L^2} - k(\psi_x^{(n)}, (\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_x)_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

Multiplying (4.34) by $\rho_1\Phi^{(n)}$ in $L^2(0,\ell)$

$$\rho_1 \mathbf{i}\beta_n(\psi_x^{(n)}, \Phi^{(n)})_{L^2} - \rho_1(\Psi_x^{(n)}, \Phi^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

Then, adding the last two convergences, we obtain

$$-\rho_1(\Psi_x^{(n)}, \Phi^{(n)})_{L^2} - k(\psi_x^{(n)}, (\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_x)_{L^2} \overset{n\to\infty}{\longrightarrow} 0,$$

which, multiplied by $\frac{b}{k}$, added to $\gamma\times$(b), where (b) is the second convergence of Lemma 4.9, and finally added to (4.49), implies that

$$\left(\tfrac{\rho_1 b}{k} - \rho_2\right)(\Psi^{(n)}, \Phi_x^{(n)})_{L^2} + k\|\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}\|_{L^2} + \gamma\varsigma(p^{(n)}, \Phi_x^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0. \tag{4.50}$$

On the other hand, multiplying (4.29) by $\Phi^{(n)}$ in $L^2(0,\ell)$,

$$\mathbf{i}\beta_n(\vartheta^{(n)}, \Phi^{(n)})_{L^2} - k_1(p^{(n)}, \Phi_x^{(n)})_{L^2} - m(\Psi^{(n)}, \Phi_x^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0. \tag{4.51}$$

Multiplying (4.43) by $\vartheta^{(n)}$ in $L^2(0,\ell)$ and using (4.38),

$$-\rho_1 \mathbf{i}\beta_n(\vartheta^{(n)}, \Phi^{(n)})_{L^2} + k(\vartheta_x^{(n)}, \varphi_x^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0,$$

which, added to $k \times (b)$, with (b) of Lemma 4.9, implies

$$-\rho_1 \mathbf{i}\beta_n(\vartheta^{(n)}, \Phi^{(n)})_{L^2} + \varsigma k(p^{(n)}, \Phi_x^{(n)})_{L^2} \overset{n\to\infty}{\longrightarrow} 0. \tag{4.52}$$

As before, let us define

$$\sigma_2 := \left(1 - \frac{k_1\rho_1}{\varsigma k}\right) \neq 0, \quad \text{because} \quad \xi = 0.$$

In addition, $\xi = 0$ implies $\dfrac{\rho_1}{k\sigma_2}\xi = 0$, that is,

$$\left(\frac{\rho_1 b}{k} - \rho_2\right) + \frac{\rho_1\gamma m}{k\sigma_2} = 0. \tag{4.53}$$

Multiplying (4.51) by $-\dfrac{\rho_1\gamma}{k\sigma_2}$, we have

$$-\frac{\rho_1\gamma}{k\sigma_2}\mathbf{i}\beta_n(\vartheta^{(n)},\Phi^{(n)})_{L^2}+\frac{\rho_1\gamma k_1}{k\sigma_2}(p^{(n)},\Phi_x^{(n)})_{L^2}+\frac{\rho_1\gamma m}{k\sigma_2}(\Psi^{(n)},\Phi_x^{(n)})_{L^2}\overset{n\to\infty}{\longrightarrow}0.$$

Multiplying (4.52) by $-\dfrac{\gamma}{k\sigma_2}$ and adding to the last expression,

$$\left(\frac{\rho_1\gamma k_1}{k\sigma_2}-\frac{\varsigma\gamma}{\sigma_2}\right)(p^{(n)},\Phi_x^{(n)})_{L^2}+\frac{\rho_1\gamma m}{k\sigma_2}(\Psi^{(n)},\Phi_x^{(n)})_{L^2}\overset{n\to\infty}{\longrightarrow}0,$$

which, added to (4.50) and using (4.53), implies

$$k\|\varphi_x^{(n)}+\psi^{(n)}+lw^{(n)}\|_{L^2}+\left(\frac{\rho_1\gamma k_1}{k\sigma_2}-\frac{\varsigma\gamma}{\sigma_2}+\gamma\varsigma\right)(p^{(n)},\Phi_x^{(n)})_{L^2}\overset{n\to\infty}{\longrightarrow}0,$$

which yields the first desired convergence because

$$\left(\frac{\rho_1\gamma k_1}{k\sigma_2}-\frac{\varsigma\gamma}{\sigma_2}+\gamma\varsigma\right)=\frac{\gamma}{k\sigma_2}\left(\rho_1k_1-k\varsigma+k\sigma_2\varsigma\right)=0.$$

Then, multiplying (4.22) by $\frac{1}{\mathbf{i}\beta_n}\Phi^{(n)}$ in $L^2(0,\ell)$, applying integration by parts and using the previous lemmas, we obtain the second desired convergence. □

Now, we are ready to prove the main results of this section.

Theorem 4.11 (Exponential decay)**.** Suppose that condition (1.21) is true, that is,

$$\Bigg[\underbrace{\left(\varsigma-\frac{k_1\rho_1}{k}\right)\left(b-\frac{k\rho_2}{\rho_1}\right)+\varsigma\gamma m}_{\xi}\Bigg]\Bigg[\underbrace{\left(\tau-\frac{k_1\rho_1}{k}\right)(k_0-k)+\tau\gamma m}_{\chi}\Bigg]=0. \qquad (4.54)$$

Then, the semigroup generated by $\mathbb{A}$ is exponentially stable.

Proof. As mentioned earlier, condition (4.54) is equivalent to (4.40). As seen in the Introduction, in [13] it was proved that $\xi=0$ implies exponential stability. So, it remains to show that the semigroup generated by $\mathbb{A}$ is exponentially stable provided that $\chi=0$. Nevertheless, as our argument is different, we give a complete proof. To this purpose, let us verify the conditions of Theorem 2.18. We emphasize, however, that the imaginary axis is always contained in $\rho(\mathbb{A})$, no matter if (4.54) is satisfied or not.

- $\mathbf{i}\mathbb{R}\subset\rho(\mathbb{A})$

Let us assume, by contradiction, that the inclusion is not valid. Then, there exists $\lambda\in\mathbf{i}\mathbb{R}$ such that $\lambda\in\sigma(\mathbb{A})$, with $\lambda\neq0$ because $0\in\rho(\mathbb{A})$ (as seen in the proof of

Theorem 4.2). By Remark 4.1, λ is an eigenvalue of $\mathbb{A}$. Therefore, there exists $U \neq 0$ in $D(\mathbb{A})$ satisfying the resolvent equation $\mathbb{A}U = \lambda U$, which in terms of its components can be written as

$$\lambda\varphi - \Phi = 0 \tag{4.55}$$
$$\rho_1\lambda\Phi - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + l\gamma\theta = 0 \tag{4.56}$$
$$\lambda\psi - \Psi = 0 \tag{4.57}$$
$$\rho_2\lambda\Psi - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\vartheta_x = 0 \tag{4.58}$$
$$\lambda w - W = 0 \tag{4.59}$$
$$\rho_1\lambda W - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + \gamma\theta_x = 0 \tag{4.60}$$
$$\lambda\vartheta + k_1 p_x + m\Psi_x = 0 \tag{4.61}$$
$$\varsigma\lambda p + \delta p + \vartheta_x = 0 \tag{4.62}$$
$$\lambda\theta + k_1 q_x + m(W_x - l\Phi) = 0 \tag{4.63}$$
$$\tau\lambda q + \delta q + \theta_x = 0. \tag{4.64}$$

Then, multiplying by $U \in D(\mathbb{A})$, it follows from the dissipative property (4.6) that $p = q = 0$. Now, substituting into (4.62), (4.64) we obtain

$$\vartheta_x = \theta_x = 0 \text{ in } L^2(0, L) \quad \Rightarrow \quad \vartheta = \theta = 0 \text{ in } L^2(0, L).$$

Therefore, by (4.61), (4.63), we have $\Psi_x = W_x - l\Phi = 0$. Then, from (4.55), (4.57) and (4.59) it follows that $\psi_x = w_x - l\varphi = 0$, which implies $\psi = \Psi = 0$. Finally, from (4.56), (4.58), (4.60) and applying all identities obtained, we conclude that

$$\rho_1\lambda\Phi - k(\varphi_x + lw)_x = 0, \quad k(\varphi_x + lw) = 0, \quad \rho_1\lambda W + kl(\varphi_x + lw) = 0$$

which implies, substituting the second equality in the others, that $\Phi = W = 0$ and thus $\varphi = w = 0$. This shows that $U = 0$, which is a contradiction. Therefore, the inclusion $\mathbf{i}\mathbb{R} \subset \rho(\mathbb{A})$ is valid.

- $\limsup\limits_{|\beta|\to\infty} \|(\mathbf{i}\beta I - \mathbb{A})^{-1}\|_{\mathcal{L}} < \infty.$

For this limit, it is sufficient to prove that there exist constants $C, \beta_0 > 0$ such that

$$\|(\mathbf{i}\beta I - \mathbb{A})^{-1})\|_{\mathcal{L}} \leq C, \qquad \forall\, \beta \geq \beta_0. \tag{4.65}$$

In fact, by contradiction, let us assume that (4.65) is not true. Then, given any $n \in \mathbb{N}$, it is not true that

$$\|(\mathbf{i}\beta - \mathbb{A})^{-1}\|_{\mathcal{L}} \leq n, \quad \forall\, \beta > n.$$

Consequently, there exists a sequence $\beta_n > n$ such that

$$\|(\mathbf{i}\beta_n - \mathbb{A})^{-1}\|_{\mathcal{L}} > n.$$

The last inequality implies that there exists a sequence $(F_n)_{n\in\mathbb{N}} \subset \mathbb{H}$ such that

$$\|(\mathbf{i}\beta_n - \mathbb{A})^{-1}F_n\|_{\mathbb{H}} > n\|F_n\|_{\mathbb{H}}.$$

Then, defining $U_n = \frac{(\mathbf{i}\beta_n - \mathbb{A})^{-1}F_n}{\|(\mathbf{i}\beta_n - \mathbb{A})^{-1}F_n\|_{\mathbb{H}}}$ we have

$$\|(\mathbf{i}\beta_n - \mathbb{A})U_n\|_{\mathbb{H}} = \frac{\|F_n\|_{\mathbb{H}}}{\|(\mathbf{i}\beta_n - \mathbb{A})^{-1}F_n\|_{\mathbb{H}}} < \frac{1}{n}.$$

The last inequality shows that, if (4.65) does not hold, then there exist a sequence $(\beta_n)_{n\in\mathbb{N}}$ of positive real numbers and a sequence $(U_n)_{n\in\mathbb{N}} \subset D(\mathbb{A})$ such that

$$\beta_n \overset{n\to\infty}{\longrightarrow} \infty, \qquad \|U_n\|_{\mathbb{H}} = 1, \qquad \|(\mathbf{i}\beta_n - \mathbb{A})U_n\|_{\mathbb{H}} \overset{n\to\infty}{\longrightarrow} 0. \tag{4.66}$$

Therefore, recalling assumption (4.54), we see that the hypothesis of Lemmas 4.3-4.10 are satisfied. Thus, from the convergences in Lemmas 4.4, 4.6, 4.7, 4.8 and 4.10, we conclude that $U_n \overset{n\to\infty}{\longrightarrow} 0$ in $\mathbb{H}$ which contradicts (4.66). Then, the second condition of Theorem 2.18 holds. □

Theorem 4.12 (Lack of exponential decay)**.** The converse of Theorem 4.11 is true. In other words: if

$$\Bigg[\underbrace{\left(\varsigma - \frac{k_1\rho_1}{k}\right)\left(b - \frac{k\rho_2}{\rho_1}\right) + \varsigma\gamma m}_{\xi}\Bigg]\Bigg[\underbrace{\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau\gamma m}_{\chi}\Bigg] \neq 0, \tag{4.67}$$

then the semigroup generated by $\mathbb{A}$ is not exponentially stable.

Proof. Assume (4.67), which is equivalent to

$$\xi \neq 0 \quad \text{and} \quad \chi \neq 0. \tag{4.68}$$

In view of Theorem 2.18, it is enough to show that there exist a sequence $(\beta_n)_{n\in\mathbb{N}}$ of positive real numbers such that $\beta_n \overset{n\to\infty}{\longrightarrow} \infty$ and a bounded sequence $(F_n)_{n\in\mathbb{N}}$ in $\mathbb{H}$ such that

$$\|(\mathbf{i}\beta_n I - \mathbb{A})^{-1}F_n\|_{\mathbb{H}} \overset{n\to\infty}{\longrightarrow} \infty. \tag{4.69}$$

Let us write $c_n = \frac{n\pi}{\ell}$ and define

$$\beta_n = \sqrt{\frac{1}{\rho_1}(kc_n^2 + k_0 l^2)}, \qquad F_n = (0, \rho_1^{-1}\sin(c_n x), 0, 0, 0, 0, 0).$$

Then, (F_n) is a bounded sequence in $\mathbb{H}$. In addition,

$$\begin{aligned} \|(\mathbf{i}\beta_n I - \mathbb{A})^{-1}F_n\|_{\mathbb{H}}^2 = \rho_1\|\Phi^{(n)}\|_{L^2}^2 + \rho_2\|\Psi^{(n)}\|_{L^2}^2 + \rho_1\|W^{(n)}\|_{L^2}^2 + b\|\psi_x^{(n)}\|_{L^2}^2 \\ + k\|\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}\|_{L^2}^2 + k_0\|w_x^{(n)} - l\varphi^{(n)}\|_{L^2}^2 \\ + \frac{\gamma}{m}\left(\|\vartheta^{(n)}\|_{L^2}^2 + \|\theta^{(n)}\|_{L^2}^2\right) + \frac{\gamma k_1}{m}\left(\varsigma\|p^{(n)}\|_{L^2}^2 + \tau\|q^{(n)}\|_{L^2}^2\right) \end{aligned} \tag{4.70}$$

where $(\varphi^{(n)}, \Phi^{(n)}, \psi^{(n)}, \Psi^{(n)}, w^{(n)}, W^{(n)}, \vartheta^{(n)}, p^{(n)}, \theta^{(n)}, q^{(n)}) := U_n$ is the unique solution in $D(\mathbb{A})$ of the resolvent equation

$$(\mathbf{i}\beta_n - \mathbb{A})U_n = F_n. \tag{4.71}$$

As before, by ansatz, we suppose that

$$\begin{aligned} &\varphi^{(n)}(x) = A_n \sin(c_n x), \quad \psi^{(n)}(x) = B_n \cos(c_n x), \quad w^{(n)}(x) = C_n \cos(c_n x), \\ &\vartheta^{(n)}(x) = \tilde{D}_n \sin(c_n x), \quad p^{(n)}(x) = \tilde{E}_n \cos(c_n x), \\ &\theta^{(n)}(x) = D_n \sin(c_n x), \quad q^{(n)}(x) = E_n \cos(c_n x). \end{aligned} \tag{4.72}$$

Then, substituting into the resolvent equation (4.71), we conclude that

$$U_n = (\varphi^{(n)}, \Phi^{(n)}, \psi^{(n)}, \Psi^{(n)}, w^{(n)}, W^{(n)}, \vartheta^{(n)}, p^{(n)}, \theta^{(n)}, q^{(n)})$$

given by (4.72) with

$$\Phi^{(n)} = \mathbf{i}\beta_n\varphi^{(n)}, \qquad \Psi^{(n)} = \mathbf{i}\beta_n\psi^{(n)}, \qquad W^{(n)} = \mathbf{i}\beta_n w^{(n)}$$

is the solution of (4.71) if and only if the coefficients A_n, B_n, C_n, $\tilde{D}_n$, $\tilde{E}_n$, D_n, and E_n satisfy the linear system

$$\begin{aligned} \left(\rho_1(\mathbf{i}\beta_n)^2 + kc_n^2 + k_0 l^2\right)A_n + kc_nB_n + l(k+k_0)c_nC_n + l\gamma D_n = 1 \\ kc_nA_n + \left(\rho_2(\mathbf{i}\beta_n)^2 + bc_n^2 + k\right)B_n + klC_n + \gamma c_n\tilde{D}_n = 0 \\ l(k+k_0)c_nA_n + klB_n + \left(\rho_1(\mathbf{i}\beta_n)^2 + k_0c_n^2 + kl^2\right)C_n + \gamma c_nD_n = 0 \\ -m\mathbf{i}\beta_nc_nB_n + \mathbf{i}\beta_n\tilde{D}_n - k_1c_n\tilde{E}_n = 0 \\ (\varsigma\mathbf{i}\beta_n + \delta)\tilde{E}_n + c_n\tilde{D}_n = 0 \\ -ml\mathbf{i}\beta_nA_n - m\mathbf{i}\beta_nc_nC_n + \mathbf{i}\beta_nD_n - k_1c_nE_n = 0 \\ (\tau\mathbf{i}\beta_n + \delta)E_n + c_nD_n = 0 \end{aligned} \tag{4.73}$$

which can be written as

$$\left.\begin{array}{l}\underbrace{\begin{bmatrix} p_n^{(1)} & kc_n & l(k+k_0)c_n & 0 & l\gamma \\ kc_n & p_n^{(2)} & kl & \gamma c_n & 0 \\ l(k+k_0)c_n & kl & p_n^{(3)} & 0 & \gamma c_n \\ 0 & -m\mathbf{i}\beta_n c_n & 0 & \tilde{p}_n^{(4)} & 0 \\ -ml\mathbf{i}\beta_n & 0 & -m\mathbf{i}\beta_n c_n & 0 & p_n^{(4)} \end{bmatrix}}_{\tilde{M}_n} \begin{bmatrix} A_n \\ B_n \\ C_n \\ \tilde{D}_n \\ D_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \tilde{E}_n = -\dfrac{c_n}{\varsigma\mathbf{i}\beta_n + \delta}\tilde{D}_n, \\ E_n = -\dfrac{c_n}{\tau\mathbf{i}\beta_n + \delta}D_n \end{array}\right\} \tag{4.74}$$

where

$$p_n^{(1)} = \rho_1(\mathbf{i}\beta_n)^2 + kc_n^2 + k_0 l^2, \qquad \tilde{p}_n^{(4)} = \mathbf{i}\beta_n + \frac{k_1 c_n^2}{\varsigma\mathbf{i}\beta_n + \delta},$$

$$p_n^{(2)} = \rho_2(\mathbf{i}\beta_n)^2 + bc_n^2 + k, \qquad p_n^{(4)} = \mathbf{i}\beta_n + \frac{k_1 c_n^2}{\tau\mathbf{i}\beta_n + \delta}.$$

$$p_n^{(3)} = \rho_1(\mathbf{i}\beta_n)^2 + k_0 c_n^2 + kl^2,$$

Now, using the the definition of β_n, we have

$$\begin{aligned} p_n^{(1)} &= 0, \\ p_n^{(2)} &= \left(b - \frac{\rho_2 k}{\rho_1}\right) c_n^2 - \frac{\rho_2}{\rho_1} k_0 l^2 + k = O(n^2), \\ p_n^{(3)} &= (k_0 - k)c_n^2 + (k - k_0)l^2 = O(n^2), \\ \tilde{p}_n^{(4)} &= \mathbf{i}\sqrt{\frac{1}{\rho_1}(kc_n^2 + k_0 l^2)} + \frac{k_1 c_n^2}{\varsigma\mathbf{i}\sqrt{\frac{1}{\rho_1}(kc_n^2 + k_0 l^2)} + \delta} = O(n), \\ p_n^{(4)} &= \mathbf{i}\sqrt{\frac{1}{\rho_1}(kc_n^2 + k_0 l^2)} + \frac{k_1 c_n^2}{\tau\mathbf{i}\sqrt{\frac{1}{\rho_1}(kc_n^2 + k_0 l^2)} + \delta} = O(n), \end{aligned}$$

which implies that

$$\begin{aligned} \tilde{\Delta}_n :=\ & \det(\tilde{M}_n) \\ =\ & \tilde{p}_n^{(4)}\Big[-k^2 c_n^2 p_n^{(3)} p_n^{(4)} - l^2(k+k_0)^2 c_n^2 p_n^{(2)} p_n^{(4)} - 2l^2(k+k_0)\gamma m(\mathbf{i}\beta_n) c_n^2 p_n^{(2)} \Big] \\ & + \tilde{p}_n^{(4)}\Big[l^2\gamma m(\mathbf{i}\beta_n) p_n^{(2)} p_n^{(3)} - k^2\gamma m(\mathbf{i}\beta_n) c_n^4 \Big] + O(n^5). \end{aligned}$$

Since $\tilde{\Delta}_n \neq 0$ (for all $n \in \mathbb{N}$ sufficiently large) we conclude that system (4.74) has a unique solution $[A_n, B_n, C_n, \tilde{D}_n, D_n]^T$ given by the Cramer's Rule. Consequently, system (4.73) has a unique solution $[A_n, B_n, C_n, \tilde{D}_n, \tilde{E}_n, D_n, E_n]^T$ which implies that the solution of the resolvent equation (4.71) is given by (4.72), for all $n \in \mathbb{N}$ sufficiently large. Therefore, we can estimate (4.70) by

$$\|(\mathbf{i}\beta_n - \mathbb{A})^{-1}F_n\|_{\mathbb{H}}^2 \geq \rho_1\|\Phi^{(n)}\|_{L^2}^2 = \rho_1\beta_n^2|A_n|^2\int_0^\ell |\sin(c_n x)|^2\,dx = \rho_1\frac{\ell}{2}\beta_n^2|A_n|^2, \tag{4.75}$$

where A_n is given by

$$A_n = \frac{\tilde{A}_n}{\tilde{\Delta}_n},$$

with $\tilde{A}_n$ defined as

$$\begin{aligned}\tilde{A}_n := \det\begin{bmatrix} 1 & kc_n & l(k+k_0)c_n & 0 & l\gamma \\ 0 & p_n^{(2)} & kl & \gamma c_n & 0 \\ 0 & kl & p_n^{(3)} & 0 & \gamma c_n \\ 0 & -m\mathbf{i}\beta_n c_n & 0 & \tilde{p}_n^{(4)} & 0 \\ 0 & 0 & -m\mathbf{i}\beta_n c_n & 0 & p_n^{(4)} \end{bmatrix} \\ = & -k^2l^2p_n^{(4)}\tilde{p}_n^{(4)} + \gamma^2m^2(\mathbf{i}\beta_n)^2c_n^4 + \gamma m\mathbf{i}\beta_n p_n^{(3)}p_n^{(4)}c_n^2 + \gamma m\mathbf{i}\beta_n p_n^{(2)}\tilde{p}_n^{(4)}c_n^2 \\ & + p_n^{(2)}p_n^{(3)}p_n^{(4)}\tilde{p}_n^{(4)} \\ = & \left(\gamma m\mathbf{i}\beta_n c_n^2 + p_n^{(3)}p_n^{(4)}\right)\left(\gamma m\mathbf{i}\beta_n c_n^2 + p_n^{(2)}\tilde{p}_n^{(4)}\right) + O(n^2).\end{aligned}$$

Here, using the convergences

$$\frac{p_n^{(2)}}{n^2} \overset{n\to\infty}{\longrightarrow} \left(b - \frac{\rho_2 k}{\rho_1}\right)\frac{\pi^2}{\ell^2}, \quad \frac{p_n^{(3)}}{n^2} \overset{n\to\infty}{\longrightarrow} (k_0 - k)\frac{\pi^2}{\ell^2}, \quad \frac{\beta_n}{n} \overset{n\to\infty}{\longrightarrow} \frac{\pi}{\ell}\sqrt{\frac{k}{\rho_1}}$$

$$\frac{\tilde{p}_n^{(4)}}{n} \overset{n\to\infty}{\longrightarrow} \mathbf{i}\frac{\pi}{\ell}\sqrt{\frac{k}{\rho_1}}\left(1 - \frac{k_1\rho_1}{\varsigma k}\right), \quad \frac{p_n^{(4)}}{n} \overset{n\to\infty}{\longrightarrow} \mathbf{i}\frac{\pi}{\ell}\sqrt{\frac{k}{\rho_1}}\left(1 - \frac{k_1\rho_1}{\tau k}\right)$$

we deduce

$$\begin{aligned}\frac{\tilde{A}_n}{n^6} \overset{n\to\infty}{\longrightarrow} & \left(\gamma m\mathbf{i}\left[\frac{\pi}{\ell}\sqrt{\frac{k}{\rho_1}}\right]\left[\frac{\pi^2}{\ell^2}\right] + \left[(k_0 - k)\frac{\pi^2}{\ell^2}\right]\left[\mathbf{i}\frac{\pi}{\ell}\sqrt{\frac{k}{\rho_1}}\left(1 - \frac{k_1\rho_1}{\tau k}\right)\right]\right) \\ & \times\left(\gamma m\mathbf{i}\left[\frac{\pi}{\ell}\sqrt{\frac{k}{\rho_1}}\right]\left[\frac{\pi^2}{\ell^2}\right] + \left[\left(b - \frac{\rho_2 k}{\rho_1}\right)\frac{\pi^2}{\ell^2}\right]\left[\mathbf{i}\frac{\pi}{\ell}\sqrt{\frac{k}{\rho_1}}\left(1 - \frac{k_1\rho_1}{\varsigma k}\right)\right]\right) := L_1.\end{aligned}$$

Analogously we can deduce that

$$\frac{\tilde{\Delta}_n}{n^6} \overset{n\to\infty}{\longrightarrow} L_2,$$

for some constant $L_2 \in \mathbb{C}$. Then, using (4.68), we conclude that

$$L_1 = -\frac{\pi^6}{\ell^6}\frac{k}{\rho_1}\frac{1}{\tau\varsigma}\chi\xi \neq 0$$

which implies

$$|A_n| = \left|\frac{\tilde{A}_n}{\Delta_n}\right| = \left|\frac{\frac{\tilde{A}_n}{n^6}}{\frac{\Delta_n}{n^6}}\right| \overset{n\to\infty}{\longrightarrow} L = \begin{cases} \dfrac{|L_1|}{|L_2|} \neq 0, & \text{if} \quad L_2 \neq 0 \\ \infty, & \text{if} \quad L_2 = 0. \end{cases}$$

Since $\beta_n \overset{n\to\infty}{\longrightarrow} \infty$, it follows from (4.75) that

$$\|(\mathbf{i}\beta_n - \mathbb{A})^{-1}F_n\|^2_{\mathbb{H}} \geq \rho_1\frac{\ell}{2}\beta_n^2|A_n|^2 \overset{n\to\infty}{\longrightarrow} \infty$$

which implies (4.69). □

4.3 Final Remarks

We have considered $\varsigma > 0, \tau > 0$. However, an analogous argument can be applied in general and the same results are valid for the other cases ($\varsigma > 0, \tau = 0$ or $\varsigma = 0, \tau > 0$ or $\varsigma = \tau = 0$), using also the same sequences $(F_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ used to prove the lack of exponential stability in Theorem 4.12. More precisely, under the boundary conditions (1.7):

- For the case $\varsigma > 0, \tau = 0$, *the mathematical system*

$$\begin{aligned} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - l[k_0(w_x - l\varphi) - \gamma\theta] &= 0 \\ \rho_2\psi_{tt} - [b\psi_x - \gamma\vartheta]_x + k(\varphi_x + \psi + lw) &= 0 \\ \rho_1 w_{tt} - [k_0(w_x - l\varphi) - \gamma\theta]_x + kl(\varphi_x + \psi + lw) &= 0 \\ \vartheta_t + k_1 p_x + m\psi_{xt} &= 0 \\ \varsigma p_t + \delta p + \vartheta_x &= 0 \\ \theta_t - \kappa_1\theta_{xx} + m(w_{xt} - l\varphi_t) &= 0 \end{aligned}$$

  is exponentially stable if and only if

$$\left[\left(\varsigma - \frac{k_1\rho_1}{k}\right)\left(b - \frac{k\rho_2}{\rho_1}\right) + \varsigma\gamma m\right](k_0 - k) = 0.$$

- For the case $\varsigma = 0, \tau > 0$, *the mathematical system*

$$
\begin{aligned}
\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - l[k_0(w_x - l\varphi) - \gamma\theta] = 0\\
\rho_2\psi_{tt} - [b\psi_x - \gamma\vartheta]_x + k(\varphi_x + \psi + lw) = 0\\
\rho_1 w_{tt} - [k_0(w_x - l\varphi) - \gamma\theta]_x + kl(\varphi_x + \psi + lw) = 0\\
\vartheta_t - \kappa_1\vartheta_{xx} + m\psi_{xt} = 0\\
\theta_t + k_1 q_x + m(w_{xt} - l\varphi_t) = 0\\
\tau q_t + \delta q + \theta_x = 0
\end{aligned}
$$

is exponentially stable if and only if

$$\left(b - \frac{k\rho_2}{\rho_1}\right)\left[\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau\gamma m\right] = 0.$$

- For the case $\varsigma = 0, \tau = 0$, *the mathematical system*

$$
\begin{aligned}
\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - l[k_0(w_x - l\varphi) - \gamma\theta] = 0\\
\rho_2\psi_{tt} - [b\psi_x - \gamma\vartheta]_x + k(\varphi_x + \psi + lw) = 0\\
\rho_1 w_{tt} - [k_0(w_x - l\varphi) - \gamma\theta]_x + kl(\varphi_x + \psi + lw) = 0\\
\vartheta_t - \kappa_1\vartheta_{xx} + m\psi_{xt} = 0\\
\theta_t - \kappa_1\theta_{xx} + m(w_{xt} - l\varphi_t) = 0
\end{aligned}
\tag{4.76}
$$

is exponentially stable if and only if

$$\left(b - \frac{k\rho_2}{\rho_1}\right)(k_0 - k) = 0. \tag{4.77}$$

In particular, if condition (1.3) holds, then equality (4.77) reduces to (1.5). As discussed in the Introduction, this result was obtained in [23], where the authors studied *the physical system* (4.76).

5 POSSIBLE FUTURE WORKS

Based on the existing literature until September 2019 and on the results proved in this thesis, we present in this section some possible future works. We use the same notations of the previous chapters.

5.1 Purely Dirichlet boundary conditions

From [12], it is known that system

$$\begin{aligned} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - lk_0(w_x - l\varphi) &= 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\vartheta_x &= 0 \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) &= 0 \\ \vartheta_t - k_1\theta_{xx} + m\psi_{xt} &= 0 \end{aligned}$$

with boundary conditions

$$\varphi = \psi = w = \vartheta = 0 \quad \text{on} \quad \{0, \ell\} \times [0, \infty)$$

is exponentially stable if

$$\frac{\rho_1}{\rho_2} = \frac{k}{b} \quad \text{and} \quad k = k_0.$$

Conjecture 5.1. System

$$\begin{aligned} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + l\gamma\theta &= 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) &= 0 \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + \gamma\theta_x &= 0 \\ \theta_t + k_1 q_x + m(w_{xt} - l\varphi_t) &= 0 \\ \tau q_t + \delta q + \theta_x &= 0 \end{aligned}$$

with boundary conditions

$$\varphi = \psi = w = \theta = 0 \quad \text{on} \quad \{0, \ell\} \times [0, \infty)$$

is exponentially stable if

$$\frac{\rho_1}{\rho_2} = \frac{k}{b} \quad \text{and} \quad \left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau\gamma m = 0.$$

From [23], it is known that system

$$
\begin{aligned}
\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - l\tfrac{b\rho_1}{\rho_2}(w_x - l\varphi) + l\gamma\theta = 0\\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\vartheta_x = 0\\
\rho_1 w_{tt} - \tfrac{b\rho_1}{\rho_2}(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + \gamma\theta_x = 0\\
\vartheta_t - \kappa_1\vartheta_{xx} + m\psi_{xt} = 0\\
\theta_t - \kappa_1\theta_{xx} + m(w_{xt} - l\varphi_t) = 0
\end{aligned}
$$

with boundary conditions

$$\varphi = \psi = w = \vartheta = 0 \quad \text{on} \quad \{0, \ell\} \times [0, \infty)$$

is exponentially stable if

$$\frac{\rho_1}{\rho_2} = \frac{k}{b}.$$

Conjecture 5.2. System

$$
\begin{aligned}
\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - lk_0(w_x - l\varphi) + l\gamma\theta = 0\\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\vartheta_x = 0\\
\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + \gamma\theta_x = 0\\
\vartheta_t + k_1 p_x + m\psi_{xt} = 0\\
\varsigma p_t + \delta p + \vartheta_x = 0\\
\theta_t + k_1 q_x + m(w_{xt} - l\varphi_t) = 0\\
\tau q_t + \delta q + \theta_x = 0
\end{aligned}
$$

with boundary conditions

$$\varphi = \psi = w = \vartheta = \theta = 0 \quad \text{on} \quad \{0, \ell\} \times [0, \infty)$$

is exponentially stable if

$$\left[\left(\varsigma - \frac{k_1\rho_1}{k}\right)\left(b - \frac{k\rho_2}{\rho_1}\right) + \varsigma\gamma m\right]\left[\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau\gamma m\right] = 0.$$

5.2 Heat flux given by Gurtin-Pipkin law

From [8], it is known that system

$$\begin{aligned}
\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - lk_0(w_x - l\varphi) &= 0\\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\vartheta_x &= 0\\
\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) &= 0\\
\vartheta_t - k_1\int_0^{\ell} g_1(s)\vartheta_{xx}(t-s)\ ds + m\psi_{xt} &= 0
\end{aligned}$$

with boundary conditions

$$\varphi = \psi_x = w_x = \vartheta = 0 \quad \text{on} \quad \{0,\ell\}\times[0,\infty)$$

is exponentially stable if and only if

$$\left(\frac{1}{g_1(0)k_1} - \frac{k_1\rho_1}{k}\right)\left(b - \frac{k\rho_2}{\rho_1}\right) + \frac{\gamma m}{g_1(0)k_1} = 0 \quad \text{and} \quad k = k_0.$$

Conjecture 5.3. System

$$\begin{aligned}
\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - lk_0(w_x - l\varphi) + l\gamma\theta &= 0\\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) &= 0\\
\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + \gamma\theta_x &= 0\\
\theta_t - k_1\int_0^{\infty} g_2(s)\theta_{xx}(t-s)\ ds + m(w_{xt} - l\varphi_t) &= 0
\end{aligned}$$

with boundary conditions

$$\varphi = \psi_x = w_x = \theta = 0 \quad \text{on} \quad \{0,\ell\}\times[0,\infty)$$

is exponentially stable if and only if

$$\frac{\rho_1}{\rho_2} = \frac{k}{b} \quad \text{and} \quad \left(\frac{1}{g_2(0)k_1} - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \frac{\gamma m}{g_2(0)k_1} = 0.$$

Conjecture 5.4. System

$$
\begin{aligned}
\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - lk_0(w_x - l\varphi) + l\gamma\theta = 0\\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\vartheta_x = 0\\
\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + \gamma\theta_x = 0\\
\vartheta_t - k_1\int_0^\ell g_1(s)\vartheta_{xx}(t-s)\ ds + m\psi_{xt} = 0\\
\theta_t - k_1\int_0^\infty g_2(s)\theta_{xx}(t-s)\ ds + m(w_{xt} - l\varphi_t) = 0
\end{aligned}
$$

with boundary conditions

$$\varphi = \psi_x = w_x = \vartheta = \theta = 0 \quad \text{on} \quad \{0,\ell\}\times[0,\infty)$$

is exponentially stable if and only if

$$\left[\left(\frac{1}{g_1(0)k_1} - \frac{k_1\rho_1}{k}\right)\left(b - \frac{k\rho_2}{\rho_1}\right) + \frac{\gamma m}{g_1(0)k_1}\right]\left[\left(\frac{1}{g_2(0)k_1} - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \frac{\gamma m}{g_2(0)k_1}\right] = 0.$$

5.3 Heat flux given by Coleman-Gurtin law

Also from [8], it is known that system

$$
\begin{aligned}
\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - lk_0(w_x - l\varphi) = 0\\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\vartheta_x = 0\\
\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) = 0\\
\vartheta_t - (1-\alpha)k_1\theta_{xx} - \alpha k_1\int_0^\ell g_1(s)\vartheta_{xx}(t-s)\ ds + m\psi_{xt} = 0
\end{aligned}
$$

with boundary conditions

$$\varphi = \psi_x = w_x = \vartheta = 0 \quad \text{on} \quad \{0,\ell\}\times[0,\infty)$$

is exponentially stable if and only if

$$\frac{\rho_1}{\rho_2} = \frac{k}{b} \quad \text{and} \quad k = k_0.$$

Conjecture 5.5. System

$$\begin{aligned}
\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - lk_0(w_x - l\varphi) + l\gamma\theta &= 0\\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) &= 0\\
\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + \gamma\theta_x &= 0\\
\theta_t - (1-\alpha)k_1\theta_{xx} - \alpha k_1\int_0^\infty g_2(s)\theta_{xx}(t-s)\, ds + m(w_{xt} - l\varphi_t) &= 0
\end{aligned}$$

with boundary conditions

$$\varphi = \psi_x = w_x = \theta = 0 \quad \text{on} \quad \{0,\ell\}\times[0,\infty)$$

is exponentially stable if and only if

$$\frac{\rho_1}{\rho_2} = \frac{k}{b} \quad \text{and} \quad k = k_0.$$

Conjecture 5.6. System

$$\begin{aligned}
\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - lk_0(w_x - l\varphi) + l\gamma\theta &= 0\\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\vartheta_x &= 0\\
\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + \gamma\theta_x &= 0\\
\vartheta_t - (1-\alpha)k_1\theta_{xx} - \alpha k_1\int_0^\ell g_1(s)\vartheta_{xx}(t-s)\, ds + m\psi_{xt} &= 0\\
\theta_t - (1-\alpha)k_1\theta_{xx} - \alpha k_1\int_0^\infty g_2(s)\theta_{xx}(t-s)\, ds + m(w_{xt} - l\varphi_t) &= 0
\end{aligned}$$

with boundary conditions

$$\varphi = \psi_x = w_x = \vartheta = \theta = 0 \quad \text{on} \quad \{0,\ell\}\times[0,\infty)$$

is exponentially stable if and only if

$$\left(\frac{\rho_1}{\rho_2} - \frac{k}{b}\right)(k - k_0) = 0.$$

6 CONCLUSION

In Chapter 3 of this thesis, which was recently published in [22], we dealt with the stability properties of a thermoelastic Bresse system where the dissipation is entirely contributed by the temperature and acts only on the axial force. The temperature evolution was given either by the Fourier's law or the Cattaneo's law of heat conduction. After proving the well-posedness of the model, we gave a necessary and sufficient condition for the exponential stability of the associated C_0-semigroup. In addition, we proved some polynomial decay estimates for the solutions, with a decay rate depending on some relations between the structural parameters of the problem. The only question left open is whether the obtained polynomial decay rates are optimal. The novelty with respect to the existing literature is that the coupling between the mechanical and the thermal part of the system takes place only through the axial force.

In Chapter 4, which is currently submitted for publication, the results of Chapter 3 concerning to the exponential stability were extended to the more realistic case where the dissipation acts not only in the axial force, but also in the bending moment. In this case, since the system have already been studied in the literature, the novelty of our contribution is the condition which completely characterizes the exponential stability.

All results in this thesis were obtained making use of the well-known Gearhart-Prüss and Borichev-Tomilov theorems. Although this mathematical methodology is rather standard, the complexity of the systems required very delicate and nontrivial computations. In addition, we emphasize that, in each chapter, the necessary and sufficient condition for the exponential stability obtained is different from the ones already present in the literature, reflecting the fact that each coupling mechanism considered has a nontrivial influence on the dynamics of the model.

In view of this, we believe that we have provided a quite complete picture for the stability of the said models, as illustrated in Appendix A, allowing us to enlarge the understanding on how the structural parameters affect the asymptotic behaviour of Bresse systems.

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APPENDIX A

Figure 4: How our results extends the known panorama on exponential stability of Bresse Systems. Continuous lines are known results and dashed lines are our contributions.

Two Fourier dampings in *the physical system* ([23]).
Necessary and **sufficient** condition for exponential stability:

$$b - \frac{k\rho_2}{\rho_1} = 0 \qquad (6.1)$$

The result remains valid if one damping is removed, and generalizes to *the mathematical system*:

One Fourier damping in bending moment ([12]). **Necessary** and **sufficient** condition:

$$b - \frac{k\rho_2}{\rho_1} = 0 \quad \text{and} \quad k_0 - k = 0$$

The result generalizes to Cattaneo's law, in which case the *first equality* of the condition is modified:

One Cattaneo damping in bending moment ([8]). **Necessary** and **sufficient** condition:

$$\left(\varsigma - \frac{k_1\rho_1}{k}\right)\left(b - \frac{k\rho_2}{\rho_1}\right) + \varsigma\gamma m = 0$$
and
$$k_0 - k = 0$$

The *second equality* of the condition can be dropped provided that a damping is introduced in axial force:

Two Cattaneo thermal dampings ([13]). **Sufficient** condition:

$$\left(\varsigma - \frac{k_1\rho_1}{k}\right)\left(b - \frac{k\rho_2}{\rho_1}\right) + \varsigma\gamma m = 0$$

One Fourier damping in axial force. **Necessary** and **sufficient** condition:

$$b - \frac{k\rho_2}{\rho_1} = 0 \quad \text{and} \quad k_0 - k = 0$$

The result generalizes to Cattaneo's law, in which case the *second equality* of the condition is modified:

One Cattaneo damping in axial force. **Necessary** and **sufficient** condition:

$$b - \frac{k\rho_2}{\rho_1} = 0$$
and
$$\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau\gamma m = 0$$

The *first equality* of the condition can be dropped provided that a damping is introduced in bending moment:

Two Cattaneo thermal dampings. **Sufficient** condition:

$$\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau\gamma m = 0$$

The system with two Cattaneo thermal dampings is exponentially stable provided that $\left[\left(\varsigma - \frac{k_1\rho_1}{k}\right)\left(b - \frac{k\rho_2}{\rho_1}\right) + \varsigma\gamma m\right]\left[\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau\gamma m\right] = 0$. The converse is true:

Two Cattaneo thermal dampings. **Necessary** and **sufficient** condition:

$$\left[\left(\varsigma - \frac{k_1\rho_1}{k}\right)\left(b - \frac{k\rho_2}{\rho_1}\right) + \varsigma\gamma m\right]\left[\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau\gamma m\right] = 0 \qquad (6.2)$$

For $k_0 = \frac{b\rho_1}{\rho_2}$ and $\varsigma = \tau = 0$, condition (6.2) reduces to (6.1)