Symmetries of flows: The vector field and diffeomorphism centralizer

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Simetrias de fluxos: os centralizadores campo de vetor e difeomorfismo

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Esta tese é dedicada ao estudo de simetrias de fluxos. Há dois tipos de simetrias que estudaremos: o centralizador campo de vetores e o centralizador difeomorfismo. Nós estamos interessados em saber quando o centralizador de um campo de vetor é “trivial”, ou pequeno em algum sentido.

Para o centralizador campo de vetor, há dois tipos de resultados: um que se trata de achar condições dinâmicas gerais sobre o fluxo que implicam centralizador “trivial” e o outro úm resultado C^1-genérico. Introduzimos três tipos de “trivialidade” do centralizador, chamados colinear, quase-trivial e trivial. Em palavras, o centralizador de um campo é colinear quando qualquer outro campo que comuta com ele é colinear em todo ponto; é quase-trivial se qualquer outro campo que comuta com ele é dado por uma função vezes o campo original; é trivial se o centralizador coincide com constantes vezes o campo original. A seguir, resumimos alguns dos nossos resultados:

1. Um campo de vetor C^1-genérico tem centralizador quase-trivial. Além disso, se um campo de vetor C^1-genérico tem no máximo uma quantidade enumerável de classes de recorrência por cadeias então o seu centralizador é trivial.
2. Um fluxo transitivo e separador tem centralizador trivial.
3. Um campo de vetor com centralizador colinear cujas singularidades são hiperbólicas tem centralizador quase-trivial.
4. Um campo de vetor C^2 que preserva uma medida não-uniformemente hiperbólica com suporte total e com finitas singularidades tem centralizador trivial.

5. Em dimensão 3, um campo de vetor C^3 com um certo tipo de expansividade (chamado cinemático expansivo) e tal que todas as suas singularidades sejam hiperbólicas tem centralizador trivial.

Para o centralizador difeomorfismo, provamos que um campo de vetor C^1-genérico que possui no máximo finitos poços e fontes tem centralizador difeomorfismo quase-trivial. Além disso, se existirem no máximo uma quantidade enumerável de classes de recorrência por cadeias então o centralizador é trivial. Obtemos também um critério para que um difeomorfismo no centralizador seja uma reparametrização do fluxo.

Symmetries of flows: The vector field and diffeomorphism centralizer

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This thesis is dedicated to the study of symmetries of flows. There are two types of symmetries that will be considered: the vector field and the diffeomorphism centralizer. We are interested in knowing when the centralizer of a vector field is "trivial", or small in some sense.

For the vector field centralizer, there are two types of results: one is to find general dynamical conditions on the flow that imply "trivial" centralizer and the other is a C^1-generic result. We introduce three types of "triviality" of the centralizer, named collinear, quasi-trivial and trivial. In words, the centralizer of a vector field is collinear if any other vector field that commutes with it is collinear in every point; it is quasi-trivial if any other vector field that commutes with it is given by a function times the original one; it is trivial if the centralizer coincides with the set of constant times the vector field. In what follows, we summarize some of our results:

1. A C^1-generic vector field has quasi-trivial centralizer. Furthermore, if a C^1-generic vector field has at most countably many chain-recurrent classes then its centralizer is trivial.
2. A transitive separating flow has trivial centralizer.
3. A C^1 vector field with collinear centralizer such that all its singularities are hyperbolic has quasi-trivial centralizer.
4. A C^2 vector field that preserves a non-uniformly hyperbolic measure with full support and with finitely many singularities has trivial centralizer.

5. In dimension 3, a C^3 vector field with some type of expansiveness (called Kinematic expansiveness) and such that all its singularities are hyperbolic has trivial centralizer.

For the diffeomorphism centralizer, we prove that a C^1-generic vector field with at most finitely many sinks and sources has quasi-trivial diffeomorphism centralizer. Furthermore, if there are at most countably many chain-recurrent classes then the centralizer is trivial. We also obtain a criterion for a diffeomorphism in the centralizer to be a reparametrization of the flow.

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Chapter 1

Introduction

This thesis is dedicated to study symmetries of dynamical systems. Given an ODE (ordinary differential equation), one is interested in undestanding the asymptotic behavior of its solutions. In this generality, this is a hard problem. However, if one knows that this ODE has several symmetries, one might use them to obtain explicit solutions. Towards the end of the 19th century, this was done by Lie where he used symmetries of some differential equations to obtain its solutions. It was actually during this work that Lie developed the notion of Lie groups. The types of symmetries that we will be interested are the so called centralizers.

We focus on the study of different types of centralizers for flows. Since the problems we will study in this thesis are motivated by the study of centralizers for discrete time dynamical systems, in this introduction we will first discuss centralizers of diffeomorphisms.

1.1 Centralizers of diffeomorphisms

Let M be a compact riemannian manifold and for each $r \geq 1$ we consider $\mathrm{Diff}^r(M)$ to be the set of C^r-diffeomorphisms of M. For a given $f \in \mathrm{Diff}^r(M)$ and $s \in [1, r]$ we define its C^s-*centralizer* as

$$\mathfrak{C}^s(f) := \{g \in \mathrm{Diff}^s(M) : f \circ g = g \circ f\}.$$

In other words, it is the set of diffeomorphisms that commutes with f. Observe that some trivial solutions of the equation $f \circ g = g \circ f$ are given by $g = f^n$, for any $n \in \mathbb{Z}$. A natural question is to know when these are the only solutions of such equation. Whenever the centralizer is generated by f, we say that f has *trivial centralizer*. We remark that, whenever the centralizer of f is non-trivial, then f embeds into a non-trivial $\mathbb{Z}^2$-action.

Kopell in her Ph.D. thesis in 1970 proved that for $r \geq 2$ and when M is the circle S^1, there is an open and dense subset of C^r-diffeomorphisms with trivial centralizer (see [Kop70]). Motivated by Kopell's result, Smale asked the following question:

Question ([Sma91], [Sma98]). Is the set of C^r-diffeomorphisms with trivial centralizer a residual (or generic) subset? That is, does it contain a dense G_δ-subset of the space of C^r-diffeomorphisms? Is it open and dense?

This question remains open in this generality, but there are several partial answers. Let us mention some of these results. For high regularity, some type of hyperbolicity has played a key role in the proofs of triviality of the centralizer. For Axiom A systems in the C^∞-category, there are some very good answers given by Palis-Yoccoz in [PY89(1)], [PY89(2)] and Fisher in [Fi08].

In [PY89(1)], the authors proved that for a C^∞-open and dense subset of the Anosov diffeomorphisms on tori, the centralizer is trivial. For other results on the centralizer of Anosov diffeomorphisms on tori, we refer the reader to [Pl98], [Ro08], [BF14]. In [PY89(2)] and [Fi08], the authors obtain the triviality of the centralizer in a (C^∞) residual subset of Axiom A either with the strong transversality property (Palis-Yoccoz), or the no cycle condition (Fisher). For Axiom A systems that verifies either the strong transversality property or the no cycle condition, and that have a periodic sink or source, Palis-Yoccoz and Fisher can promote their result obtaining that triviality of the centralizer is open and dense, instead of just generic. We also refer the reader to [Ro93], [Fi09] and [RV18] for some related results in the hyperbolic setting.

In the partially hyperbolic setting, there are some results on centralizers by Burslem in [Bu04]. She proves that among the partially hyperbolic diffeomorphisms, the centralizer is discrete, i.e. it is a discrete group but not necessarily trivial, for a C^1-open and C^1-dense subset. For skew-products over Anosov and whose fibers are the circle, such that the center direction is tangent to these fibers, Burslem proved that the centralizer is trivial in a C^∞-residual subset. She also obtained a result for perturbations of the time t diffeomorphism of an Anosov flow.

Bonatti-Crovisier-Wilkinson gave a positive answer to Smale's question for the C^1-topology. They proved that a C^1-generic diffeomorphism has trivial C^1-centralizer (see [BCW09]). They also prove the same result restricted to the space of diffeomorphisms that preserves a volume form (see [BCW08]).

From the results of Bonatti-Crovisier-Wilkinson, a natural question is to know if for the C^1-topology the property of having trivial C^1-centralizer is open and dense. It turns out that the answer is no. This is given by Bonatti-Crovisier-Vago-Wilkinson ([BCVW08]), where they proved that any manifold admits a C^1-open set $\mathcal{U} \subset \mathrm{Diff}^1(M)$ such that there exists a subset $\mathcal{D} \subset \mathcal{U}$, which is C^1-dense in $\mathcal{U}$, with the property that any diffeomorphism $f \in \mathcal{D}$ has non-trivial centralizer.

A great portion of this thesis is dedicated to extend the result of Bonatti-Crovisier-Wilkinson in [BCW09] for flows. As we will see, there are some difficulties that arises when one studies the centralizer of flows, which do not appear for diffeomorphisms.

1.2 The vector field centralizer of flows

In this section we introduce one of the types of symmetries for flows that we will study. Since we will be restricted to the C^1-category, we can represent a flow by the vector field that generates it. Let $\mathfrak{X}^r(M)$ be the set of C^r-vector fields of M. Recall that a smooth manifold carries a Lie bracket operator $[.,.]$ that acts on $\mathfrak{X}^r(M) \times \mathfrak{X}^r(M)$. For

$X, Y \in \mathfrak{X}^r(M)$, it is defined by $[X, Y] = XY - YX$. If $[X, Y] = 0$, we say that X and Y commute.

Let $X \in \mathfrak{X}^r(M)$ and $1 \leq s \leq r$, we define the C^s*-vector field centralizer* of X by

$$\mathfrak{C}^s(X) := \{Y \in \mathfrak{X}^s(M) : [X, Y] = 0\}.$$

This is the set of vector fields that commutes with X. In this section, we will refer to the vector field centralizer of X as the centralizer of X. In the next section we will introduce a different type of centralizer.

Given X, the equation $[X, Y] = 0$ has some trivial solutions. Indeed, for any $c \in \mathbb{R}$, the vector field $Y = cX$ commutes with X. More generally, for any function $f : M \to \mathbb{R}$ such that $Xf = 0$, then the vector field $Y = fX$ also commutes with X. In what follows we will define different types of "triviality" for the centralizer of flows.

A C^r-vector field X has C^s*-collinear centralizer* if for any $Y \in \mathfrak{C}^s(X)$, for any point $x \in M$ the space generated by the vectors $X(x)$ and $Y(x)$ has dimension at most 1. This definition says that if Y commutes with X, then Y has the "same direction" of X.

Recall that two vector fields commute if and only if their flows commute, that is, for any $t_X, t_Y \in \mathbb{R}$ we have that $X_{t_X} \circ Y_{t_Y}(.) = Y_{t_Y} \circ X_{t_X}(.)$. Two commuting flows induce an $\mathbb{R}^2$-action. If X has collinear centralizer, then the flow generated by X does not embed into a non-trivial $\mathbb{R}^2$-action, that is, there are no orbits of the action with dimension 2.

A slightly stronger notion of triviality is the following: we say that X has C^s*-quasi-trivial centralizer* if for any $Y \in \mathfrak{C}^s(X)$ there is a continuous function $f : M \to \mathbb{R}$, which is differentiable along X-orbits, such that $Y = fX$.

At last, we say that X has C^s*-trivial centralizer* if the centralizer is given by the set $\{cX : c \in \mathbb{R}\}$. Observe that this is the smallest possible centralizer that a vector field may have. It is natural to ask in this context the following version of Smale's questions for the vector field centralizer.

Question. Is the set of C^r-vector fields with trivial (quasi-trivial or collinear) centralizer a residual (or generic) subset? Is it open and dense?

There are several works that study the different types of triviality of the vector field centralizer. In 1973, Kato-Morimoto proved that the centralizer of an Anosov flow is quasi-trivial (see [KM73]). The main feature used in their proof is a topological property called (Bowen-Walters) expansivity. We remark that there are several different notions of expansivity for flows.

A few years later in [Oka76], Oka extended Kato-Morimoto's result for (Bowen-Walters) expansive flows. This type of expansivity is somehow restrictive, since it implies that every singularity is an isolated point of the manifold.

In [Sad79], Sad in his Ph.D. thesis adapted the remarkable work of Palis-Yoccoz ([PY89(2)]) for flows. He proved that the triviality of the vector field centralizer holds for an C^∞-open and dense subset of C^∞-Axiom A vector fields that verify the strong transversality condition. The singularities of an axiom A flow are dynamically isolated, meaning that they are not contained in a non-trivial transitive set. For these type of flows the singularities do not give any trouble in the proofs of triviality of the centralizer.

Much more recently, in 2018, Bonomo-Rocha-Varandas ([BRV18]) studied the centralizer for Komuro expansive flows. We remark that Komuro expansivity allows the presence of singularities, this includes for instance Lorenz attractors. They prove the triviality of the centralizer of C^∞-Komuro expansive transitive flows whose singularities are hyperbolic and verify a non-ressonance condition.

Bonomo-Varandas proved in [BV19] that a C^1-generic divergence free vector field has trivial vector field centralizer (they also obtain a generic result for Hamiltonian flows in the same paper). In a different paper, [BV18], Bonomo-Varandas obtain that C^1-generic sectional axiom A vector fields have trivial vector field centralizer (see the introduction of [BV18] for the definition of sectional axiom A).

The goal of chapter 2 in this thesis, it is to study this type of centralizer. It contains a joint work of the author with Martin Leguil and Bruno Santiago [LOS18], where we obtain several restults for the centralizers of vector fields. In our work, there are two types of results: C^1-generic results and general results which study dynamical conditions on X that implies "triviality" of its centralizer. Let us make a few remarks comparing our results and previous results in the literature.

In the C^1-category, we prove that a very weak type of expansiveness called *separating* implies quasi-triviality of the centralizer. In particular, with the additional assumption of transitivity, in Theorem D, we obtain that a transitive separating vector field has trivial centralizer. We remark that the separating property is much weaker than Komuro expansiveness. This result generalizes to a much larger class of vector fields the results about centralizers of flows from [KM73], [Oka76] and [BRV18]. After our work, Bakker-Fisher-Hasselblatt in [BFH19] were able to prove a similar result in the C^0-category. However, their result uses a type of expansiveness stronger than separating, called kinematic expansiveness.

Our Theorem B states that for a C^1-generic vector field its centralizer is quasi-trivial. Moreover, if a C^1-generic vector field has at most countably many chain-recurrent classes then its centralizer is trivial. One of the corollaries of our result is the main result in [BV18], since sectional axiom A vector fields have finitely many chain-recurrent classes.

In what follows we will state our results for the vector field centralizer.

1.2.1 Quasi-trivial centralizers

We obtain some easy criteria that imply collinearity of the centralizer. A natural problem is to know when collinearity can be promoted to quasi-triviality. If Y commutes with X and Y is collinear to X, it is easy to see that there is a continuous function f, defined on regular (or non-singular) points such that $Y = fX$. The problem of going from collinearity to quasi-triviality is a problem of extending continuously the function f to the entire manifold. This is not always the case; indeed, in Section 2.3 we construct an example of a vector field with collinear centralizer which is not quasi-trivial.

Nevertheless, when all the singularities of a C^1 vector field are hyperbolic, collinearity can actually be promoted to quasi-triviality:

Theorem A. Let M be a compact manifold. If $X \in \mathfrak{X}^1(M)$ has collinear C^1-centralizer and all the singularities of X are hyperbolic, then X has quasi-trivial C^1-centralizer.

A significant part of chapter 2 is dedicated to the proof of the C^1-genericity of quasi-trivial centralizer. This is given in the following theorem:

Theorem B. Let M be a compact manifold. There exists a residual subset $\mathcal{R} \subset \mathfrak{X}^1(M)$ such that any $X \in \mathcal{R}$ has quasi-trivial C^1-centralizer. Furthermore, if X has at most countably many chain-recurrent classes, then the C^1-centralizer of X is trivial.

This result is a version for the vector field centralizer of Bonatti-Crovisier-Wilkinson's result [BCW09].

1.2.2 Trivial centralizers

Next we see in what conditions we can conclude the triviality of the centralizer. It is easy to construct examples of vector fields whose centralizer is quasi-trivial but not trivial. In Section 2.4 we explain how example 2.2.1 has quasi-trivial centralizer, but not trivial.

The problem of knowing if a vector field with quasi-trivial centralizer has trivial centralizer is reduced to the problem of knowing when an X-invariant function[1] is constant. This problem will be studied in Section 2.4.

Our first criterion to obtain triviality is based on the notion of *spectral decomposition*. We say that X admits a *countable spectral decomposition* if the non-wandering set, $\Omega(X)$, satisfies $\Omega(X) = \sqcup_{i\in\mathbb{N}}\Lambda_i$, where the sets Λ_i are pairwise disjoint, each of which is compact, X-invariant and transitive, i.e., contains a dense orbit.

Theorem C. Let M be a compact connected manifold and let $X \in \mathfrak{X}^1(M)$. Assume that all the singularities of X are hyperbolic, that X admits a countable spectral decomposition and that the C^1-centralizer of X is collinear. Then $\mathfrak{C}^1(X)$ is trivial.

With the assumption of a very weak type of expansivity, called *separating* (see definition 2.1.3), we can obtain the following result:

Theorem D. If X is a transitive, separating C^1-vector field, then X has trivial C^1-centralizer.

This generalizes to the C^1-category previous results of Oka [Oka76], Kato-Morimoto [KM73] and Bonomo-Rocha-Varandas [BRV18].

In higher regularity, Pesin's theory in the non-uniformly hyperbolic case and Sard's theorem give us two useful tools to verify triviality of the centralizer. Using Pesin's theory as a tool, we obtain the following result:

[1] A function is f is X-invariant if $Xf = 0$, which is equivalent to $f \circ X_t = f$, $\forall t \in \mathbb{R}$.

Theorem E. Let M be a compact manifold of dimension $d \geq 2$. Let $X \in \mathfrak{X}^2(M)$ be a vector field with finitely many singularities and let μ be a X-invariant probability measure such that $\text{supp}\mu = M$. If μ is non-uniformly hyperbolic[2] for X, then X has trivial C^1-centralizer.

Theorem E can be applied for non-uniformly hyperbolic geodesic flows, like the ones constructed by Donnay [Don88] and Burns-Gerber [BG89]. In particular, we obtain that non-uniformly hyperbolic geodesic flows have trivial centralizer.

In dimension three, under higher regularity assumptions, we are also able to obtain triviality, for a slightly stronger notion of expansiveness called *kinematic expansive*, which is stronger than separating (see definition 2.1.8).

Theorem F. Let M be a compact 3-manifold and consider $X \in \mathfrak{X}^3(M)$. If X is Kinematic expansive and all its singularities are hyperbolic, then its C^3-centralizer is trivial.

The technique we use in the above theorem, which relies on Sard's Theorem, also leads to a criteria to obtain triviality from a collinear centralizer of high regularity.

Theorem G. Let M be a compact, connected Riemannian manifold of dimension $d \geq 1$, and let $X \in \mathfrak{X}^d(M)$. Assume that every singularity and every periodic orbit of X is hyperbolic, that $\Omega(X) = \overline{\text{Per}(X)}$ and that the C^d-centralizer of X is collinear. Then X has trivial C^d-centralizer.

The criterion in Theorem G is not sufficient if we want to obtain a generic result, due to the lack of a general C^d-closing lemma. However, following the arguments of [Hur86, Man73], we can show that C^d-generically the triviality of the C^d-centralizer is equivalent to the collinearity of the C^d-centralizer.

Theorem H. Let M be a compact, connected Riemannian manifold of dimension $d \geq 1$. There exists a residual set $\mathcal{R}_T \subset \mathfrak{X}^d(M)$ such that for any $X \in \mathcal{R}_T$, the C^d-centralizer of X is collinear if and only if it is trivial.

1.3 The diffeomorphism centralizer

In this part we will introduce another type of symmetry that one can study for flows. Given a C^r-vector field X, we denote by X_t the flow on time t generated by X. For any $1 \leq s \leq r$, we define the *C^s-diffeomorphism centralizer of X* as

$$\mathfrak{C}^s_{\text{Diff}}(X) := \{f \in \text{Diff}^s(M) : f \circ X_t = X_t \circ f, \ \forall t \in \mathbb{R}\}. \tag{1.3.1}$$

This is the set of diffeomorphisms that commutes with the flow. The study of this type of centralizer is presented in chapter 3. We remark that this type of centralizer is less rigid

[2]See section 2.1.4 for the precise definition of non-uniform hyperbolicity.

than the vector field centralizer. In section 3.5 we present a simple example that justifies this claim.

Similarly to the vector field centralizer, there are different types of "triviality" that one may consider. We define two types. Given a C^r-vector field X, we say that it has *C^s-quasi-trivial centralizer* if for any $f \in \mathfrak{C}^s_{\mathrm{Diff}}(X)$, f is a reparametrization of the flow X_t, that is, there exists a continuous function $\tau : M \to \mathbb{R}$ such that $f(.) = X_{\tau(.)}(.)$. If X has quasi-trivial C^s-centralizer and for every element of the centralizer f, the function $\tau(.)$ is constant, then we say that the C^s-centralizer is *trivial*. In other words, X has trivial centralizer if the centralizer is the smallest possible one. For this type of centralizer, we are interested in the following version of Smale's question for flows:

Question. For a C^1-generic vector field X, is its diffeomorphism C^1-centralizer quasi-trivial? Is it trivial?

In [Mun17], the author proves that given a Morse function on a smooth manifold, for a generic riemannian metric the gradient flow has trivial centralizer. Another work related to this type of centralizer is the recent work of Bakker-Fisher-Hasselblatt [BFH19]. The authors prove the C^∞ genericity of trivial centralizer among Axiom A flows with no cycle. This is the analogous for flows of the previous result for diffeomorphism by Fisher [Fi08]. With some more dynamical assumptions they can also obtain C^1-open sets with trivial centralizer.

It is natural to study generic systems that present some form of "hyperbolicity". In this part we will focus on C^1-generic vector fields that have at most finitely many sinks or sources. In [ABC06], the authors proved that such systems have a weak form of hyperbolicity named dominated splitting (see Theorem 3.2.4). Our main result is the following:

Theorem I. There exists a C^1-residual subset $\mathcal{R} \subset \mathcal{X}^1(M)$ such that if $X \in \mathcal{R}$ has at most finitely many sinks or sources, then X has quasi-trivial diffeomorphism C^1-centralizer. Moreover, if in addition X has at most countably many chain-recurrent classes, then the C^1-centralizer of X is trivial.

We remark that the techniques used to prove C^∞-generic results on the centralizer of Axiom A vector fields (as used in [BFH19]), and the techniques that we use to prove Theorem I are completely different.

In [Pei60], Peixoto proved that a C^1-generic vector field on a compact surface is Morse-Smale. Recall that a vector field is Morse-Smale if the non-wandering set is the union of finitely many hyperbolic periodic orbits and hyperbolic singularities, and it verifies some transversality condition. In particular, the non-wandering set is finite. As a consequence of this result of Peixoto and Theorems B and C, we have the following corollary.

Corollary A. Let M be a compact connected surface. Then, there exists a residual subset $\mathcal{R}_\dagger \subset \mathfrak{X}^1(M)$ such that for any $X \in \mathcal{R}_\dagger$, the diffeomorphism C^1-centralizer of X is trivial.

A C^1-vector field X is *Axiom A* if the non-wandering set is hyperbolic and $\Omega(X) =$

$\overline{\mathrm{Per}(X)}$. It is well known that Axiom A vector fields admits a spectral decomposition, with finitely many basic pieces.

Corollary B. A C^1-generic Axiom A vector field has trivial diffeomorphism C^1-centralizer.

We remark that Corollary B actually holds for more a general type of hyperbolic system called *sectional Axiom A*. We refer the reader to [MM08] definition 2.14, for a precise definition.

Another corollary is for C^1-vector fields far from homoclinic tangencies in dimension three. Let us make it more precise. Recall that a vector field $X \in \mathfrak{X}^1(M)$ has a *homoclinic tangency* if there exists a hyperbolic non-singular closed orbit γ and a non-transverse intersection between $W^s(\gamma)$ and $W^u(\gamma)$. By the proof of Palis conjecture in dimension three given in [CY17], a C^1-generic $X \in \mathfrak{X}^1(M)$ which cannot be approximated by such vector fields admits a finite spectral decomposition, hence:

Corollary C. Let M be a compact connected 3-manifold. Then there exists a residual subset $\mathcal{R}_\ddagger \subset \mathfrak{X}^1(M)$ such that any vector field $X \in \mathcal{R}_\ddagger$ which cannot be approximated by vector fields exhibiting a homoclinic tangency has trivial diffeomorphism C^1-centralizer.

To prove Theorem I, a very important ingredient is given by the proposition below. It deals with the construction of the reparametrization of the flow, given a diffeomorphism in the centralizer that fixes orbits. We remark that this type of construction does not appear in the study of the centralizer for diffeomorphisms.

Proposition 3.0.2. Let $X \in \mathfrak{X}^1(M)$ be a C^1-vector field whose periodic orbits and singularities are all hyperbolic. Let $f \in \mathfrak{C}^1_{\mathrm{Diff}}(X)$ be an element of the centralizer with the following property: there exists a constant $T > 0$ such that for every $p \in M$, we have $f(p) \in X_{[-T,T]}(p)$, where $X_{[-T,T]}(p)$ is the piece of orbit of p from time $-T$ to T. Then there exists an X-invariant continuous function $\tau : M \to \mathbb{R}$ such that $f(.) = X_{\tau(.)}(.)$.

1.4 Thom's conjecture

We finish this introduction with a discussion in what is missing in Theorems B and I to obtain triviality of the centralizer.

Observe that in Theorem I, without the additional assumption of at most countably many chain recurrent classes, we do not get the triviality of the C^1-centralizer for a C^1-generic vector field that has at most finitely many sinks or sources. Similarly, for the vector field centralizer of a C^1-generic vector field, from Theorem B we cannot conclude the triviality of the centralizer without the assumption of at most countably many chain-recurrent classes. What is missing to obtain the triviality in Theorems B and I, is to prove that for a C^1-generic vector field every invariant continuous function is constant. This was conjectured (without indication on the regularities) by René Thom ([Thom]).

Conjecture ([Thom]). For a C^1-generic vector field, any C^1 (or C^0) invariant function of the manifold is constant.

Two results related to Thom's conjecture are given by Mañé in [Man73] and Cresson-Daniilidis-Shiota in [CDS08]. Mañé proves that C^r-generically, for any $1 \leq r \leq \infty$, any C^d-invariant function is constant, where d is the dimension of the manifold. His argument uses Sard's theorem, that explains why he needs C^d-functions. In [CDS08], the authors prove that C^1-generically any Lipschitz *definable* invariant function is constant. The notion of definable has to do with the so called o-minimal structures, which we do not define here. Their argument also passes through to obtain some generalization of Sard's theorem. We remark that neither of these works solve the conjecture as it is stated above, for general C^1 (or continuous) invariant functions.

1.5 Works contained in this thesis and other works

This thesis contains the following two works:

- *On the centralizer of vector fields: Criteria of triviality and genericity results.* Joint work with Martin Leguil and Bruno Santiago. Submited, preprint arXiv:1810.05085 (2018).
- *Symmetries of vector fields: The diffeomorphism centralizer.* Submited, preprint arXiv:1903.05883 (2019).

After starting my Ph.D. at UFRJ, the opportunity appeared to start a new Ph.D. at Université Paris-Sud (Orsay), under the supervision of Sylvain Crovisier. Since January 2017, I have been working on these two thesis, which are independent of each other. My thesis at Université Paris-Sud, which I will be defending in December 2019, is focused on problems in smooth ergodic theory (stable ergodicity, Lyapunov exponents, and SRB measures). I also mention the works finished (or almost finished) for my thesis in France:

- *On the stable ergodicity of Berger-Carrasco's example.* Ergodic Theory and Dynamical Systems, online version (2018).
- *On the stable ergodicity of diffeomorphisms with dominated splitting.* Nonlinearity, Vol.32, n.2, p.445–463 (2019).
- *On the genericity of positive exponents of conservative skew products with two-dimensional fibers.* Joint work with Mauricio Poletti. Submited, preprint arxiv:1809.03874 (2018).
- *A new example of robustly transitive diffeomorphism.* Joint work with Pablo Carrasco. Submited, preprint arxiv:1904.11788 (2019).
- *Open sets of partially hyperbolic skew products having an SRB measure.* In preparation.

1.6 Organization of this thesis

Chapter 2 contains the author's work with Martin Leguil and Bruno Santiago [LOS18]. This chapter deals with the vector field centralizer and it contains the proofs of Theorems A through H. Chapter 3 contains the author's work [Oba19]. This chapter contains the proof of Theorem I and it deals with the diffeomorphism centralizer. We remark that each chapter can be read separately, each containing its own introduction and preliminaries sections.

Chapter 2

The vector field centralizer

This chapter is dedicated to the study of the vector field centralizer.

2.1 Definitions and statement of the main results

In this part, we introduce some definitions and notations, and we summarize some of the results that we will show in the following.

2.1.1 General notions on vector fields

Let M be a smooth manifold of dimension $d \geq 1$, which we assume to be compact and boundaryless. For any $r \geq 1$, we denote by $\mathfrak{X}^r(M)$ the space of vector fields over M, endowed with the C^r topology. A property for vector fields in $\mathfrak{X}^r(M)$ is called *C^r-generic* if it is satisfied for any vector field in a residual set of $\mathfrak{X}^r(M)$. Recall that $\mathcal{R} \subset \mathfrak{X}^r(M)$ is *residual* if it contains a G_δ-dense subset of $\mathfrak{X}^r(M)$. In particular, it is dense in $\mathfrak{X}^r(M)$, by Baire's theorem.

In the following, given a vector field $X \in \mathfrak{X}^1(M)$, we denote by X_t the flow it generates. Recall that for any $Y \in \mathfrak{C}^1(X)$, and for any $s, t \in \mathbb{R}$, we have $Y_s \circ X_t = X_t \circ Y_s$. Differentiating this relation with respect to s at 0, we get

$$Y(X_t(x)) = DX_t(x) \cdot Y(x), \quad \forall x \in M. \tag{2.1.1}$$

We denote by $\mathrm{Zero}(X) := \{x \in M : X(x) = 0\}$ the set of *zeros*, or *singularities*, of the vector field X, and we set

$$M_X := M - \mathrm{Zero}(X). \tag{2.1.2}$$

For any $x \in M$ and any interval $I \subset \mathbb{R}$, we also let $X_I(x) := \{X_t(x) : t \in I\}$. In particular, we denote by $\mathrm{orb}^X(x) := X_{\mathbb{R}}(x)$ the orbit of the point x under X. Note that if $x \in M_X$, then $\mathrm{orb}^X(x) \subset M_X$, too.

Let $X \in \mathfrak{X}^1(M)$ be some C^1 vector field. The *non-wandering set* $\Omega(X)$ of X is defined as the set of all points $x \in M$ such that for any open neighbourhood $\mathcal{U}$ of x and for any $T > 0$, there exists a time $t > T$ such that $\mathcal{U} \cap X_t(\mathcal{U}) \neq \emptyset$.

Let us also recall another weaker notion of recurrence. Given two points $x, y \in M$, we write $x \prec_X y$ if for any $\varepsilon > 0$ and $T > 0$, there exists an (ε, T)-*pseudo orbit* connecting them, i.e., there exist $n \geq 2$, $t_1, t_2, \ldots, t_{n-1} \in [T, +\infty)$, and $x = x_1, x_2, \ldots, x_n = y \in M$, such that $d(X_{t_j}(x_j), x_{j+1}) < \varepsilon$, for $j \in \{1, \ldots, n-1\}$. The *chain recurrent set* $\mathcal{CR}(X) \subset M$ of X is defined as the set of all points $x \in M$ such that $x \prec_X x$. Restricted to $\mathcal{CR}(X)$, we consider the equivalence relation given by $x \sim_X y$ if and only if $x \prec_X y$ and $y \prec_X x$. An equivalence class under the relation $\sim_X$ is called a *chain recurrent class*: $x, y \in \mathcal{CR}(X)$ belong to the same chain recurrent class if $x \sim_X y$. In particular, chain recurrent classes define a partition of the chain recurrent set $\mathcal{CR}(X)$.

A point $x \in M$ is *periodic* if there exists $T > 0$ such that $X_T(x) = x$. The set of all periodic points is denoted by $\mathrm{Per}(X)$, observe that we are also including the singularities in this set.

An X-invariant compact set Λ is *hyperbolic* if there is a continuous decomposition of the tangent bundle over Λ, $T_\Lambda M = E^s \oplus \langle X \rangle \oplus E^u$ into DX_t-invariant sub-bundles that verifies the following property: there exists $T > 0$ such that for any $x \in \Lambda$, we have

$$\|DX_T(x)|_{E^s_x}\| < \frac{1}{2} \text{ and } \|DX_{-T}(x)|_{E^u_x}\| < \frac{1}{2}.$$

A periodic point $x \in \mathrm{Per}(X)$ is *hyperbolic* if $\mathrm{orb}^X(x)$ is a hyperbolic set. Let γ be a hyperbolic periodic orbit. We denote by $W^s(\gamma)$ the stable manifold of the periodic orbit γ, which is defined as the set of points $y \in M$ such that $d(X_t(y), \gamma) \to 0$ as $t \to +\infty$. We define in an analogous way the unstable manifold of γ. It is well known that the stable and unstable manifolds are C^1-immersed submanifolds. A hyperbolic periodic orbit is a *sink* if the unstable direction is trivial. It is a *source* if the stable direction is trivial. A hyperbolic periodic orbit is a *saddle* if it is neither a sink nor a source. For a hyperbolic periodic point p we defined its *index* by $\mathrm{ind}(p) := \dim(E^s)$.

2.1.2 Collinear centralizers

In this part, we consider a compact Riemannian manifold M, and we let $r, k \geq 1$ be positive integers. Given $x \in M$ and $u, v \in T_xM$ we denote by $\langle u, v \rangle$ the subspace spanned by u and v in T_xM.

Definition 2.1.1 (Collinear centralizer). We say that $X \in \mathfrak{X}^r(M)$ has a *collinear* C^k-*centralizer* if

$$\dim\langle X(x), Y(x) \rangle \leq 1,$$

for every $x \in M$ and every $Y \in \mathfrak{C}^k(X)$.

We have the following elementary result:

Lemma 2.1.2. *Let $X \in \mathfrak{X}^r(M)$ and assume that the vector field $Y \in \mathfrak{C}^k(M)$ satisfies $\dim\langle X(x), Y(x) \rangle \leq 1$, for every $x \in M$. Then, there exists a function $f \in C^s(M_X, \mathbb{R})$,* *with* $s := \min\{r,k\}$*, such that*

$$Y(x) = f(x)X(x), \quad \forall\, x \in M_X.$$

Moreover, the function f *is* X*-invariant, i.e.,*

$$f(X_t(x)) = f(x), \quad \forall\, x \in M_X,\ \forall\, t \in \mathbb{R}.$$

Proof. Let us denote by $(\cdot,\cdot)$ the scalar product associated to the Riemannian structure on M. For any $x \in M_X$ and for any $v \in T_xM$, we set $\pi_X(x,v) := \frac{(X(x),v)}{(X(x),X(x))}$. In particular, $\pi_X(x,v)X(x)$ is the orthogonal projection of the vector v on the direction spanned by $X(x)$. Let $Y \in \mathfrak{C}^k(M)$ be a vector field that satisfies $\dim\langle X(x), Y(x)\rangle \leq 1$. The function $f\colon M_X \to \mathbb{R}$, $x \mapsto \pi_X(x, Y(x))$ is of class C^s, with $s = \min\{r,k\}$. Moreover, by the collinearity of the vector fields X and Y, we have $Y = fX$.

By (2.1.1), it holds $Y(X_t(\cdot)) = DX_t \cdot Y(\cdot)$. Therefore, for any $x \in M_X$ and for any $t \in \mathbb{R}$, we have

$$f(X_t(x))X(X_t(x)) = DX_t(x)\cdot(f(x)X(x)) = f(x)DX_t(x)\cdot X(x) = f(x)X(X_t(x)),$$

where the last equality follows from (2.1.1), with Y in place of X. Since $X(X_t(x)) \neq 0$, we obtain $f(X_t(x)) = f(x)$, which concludes the proof. □

In this chapter, we obtain a few different criteria which ensure that the C^1-centralizer of a C^1 vector field is collinear. The following definition is a very weak form of expansiveness for flows.

Definition 2.1.3. A vector field $X \in \mathfrak{X}^1(M)$ is *separating* if there exists $\varepsilon > 0$ such that the following holds: if $d(X_t(x), X_t(y)) < \varepsilon$ for every $t \in \mathbb{R}$, then $y \in \mathrm{orb}^X(x)$.

In Section 2.2 we will elaborate on this property. In Section 2.2, we prove the following criterion for collinearity:

Proposition 2.1.4. *If* $X \in \mathfrak{X}^1(M)$ *is separating, then* X *has collinear* C^1*-centralizer.*

We remark that the separating property is not generic (see Appendix 2.6). So to obtain that the C^1-centralizer of a C^1-generic vector field is collinear we will need another criterion.

In Section 2.2, we define the notion of unbounded normal distortion (see Definition 2.2.3). This is an adaptation for flows of the definition of unbounded distortion used in [BCW09] to prove the triviality of the C^1-centralizer of a C^1-generic diffeomorphism. Using this property we obtain the following proposition.

Proposition 2.1.5. *Let* $X \in \mathfrak{X}^1(M)$*. Suppose that* X *verifies the following properties:*

- X *has unbounded normal distortion;*
- *every singularity and periodic orbit of* X *is hyperbolic;*
- $\mathcal{CR}(X) = \overline{\mathrm{Per}(X)}$.

Then X *has collinear* C^1*-centralizer.*

2.1.3 Quasi-trivial centralizers

Let M be a compact manifold.

Definition 2.1.6 (Quasi-trivial centralizer). Given two positive integers $1 \leq k \leq r$, we say that $X \in \mathfrak{X}^r(M)$ has a *quasi-trivial* C^k-centralizer if for every $Y \in \mathfrak{C}^k(X)$, there exists a C^1 function $f\colon M \to \mathbb{R}$ such that $X \cdot f \equiv 0$ and $Y(x) = f(x)X(x)$, for every $x \in M$.

In fact, by Lemma 2.1.2, if $X \in \mathfrak{X}^r(M)$ has a quasi-trivial C^k-centralizer, then for any $Y \in \mathfrak{C}^k(X)$, the function f in Definition 2.1.6 restricted to M_X is, in fact, of class C^k.

The difference between collinear and quasi-trivial centralizers is to know whether or not a C^k invariant function defined on M_X admits a C^1 extension to M. This is not always the case; indeed, in Section 2.3 we construct an example of a vector field with collinear centralizer which is not quasi-trivial.

Nevertheless, when all the singularities of a C^1 vector field are hyperbolic, collinearity can actually be improved to quasi-triviality:

Theorem A. *Let M be a compact manifold. If $X \in \mathfrak{X}^1(M)$ has collinear C^1-centralizer and all the singularities of X are hyperbolic, then X has quasi-trivial C^1-centralizer.*

A significant part of the present chapter is dedicated to the proof of the C^1-genericity of the unbounded normal distortion property (see Section 2.5). Since the other assumptions of Proposition 2.1.5 and Theorem A are already known to be C^1-generic, this allows us to conclude:

Theorem B. *Let M be a compact manifold. There exists a residual subset $\mathcal{R} \subset \mathfrak{X}^1(M)$ such that any $X \in \mathcal{R}$ has quasi-trivial C^1-centralizer. Furthermore, if X has at most countably many chain-recurrent classes, then the C^1-centralizer of X is trivial.*

2.1.4 Trivial centralizers

Let M be a compact manifold. Notice that for any $r \geq 1$ and $X \in \mathfrak{X}^r(M)$, we have that $cX \in \mathfrak{C}^k(X)$, for any $c \in \mathbb{R}$ and $1 \leq k \leq r$.

Definition 2.1.7 (Trivial centralizer). For any $1 \leq k \leq r$, we say that $X \in \mathfrak{X}^r(M)$ has a *trivial* C^k-centralizer if $\mathfrak{C}^k(X)$ is as small as it can be, i.e.,

$$\mathfrak{C}^k(X) = \{cX : c \in \mathbb{R}\}.$$

It is easy to construct examples of vector fields whose centralizer is quasi-trivial but not trivial. In Section 2.4 we explain how example 2.2.1 has quasi-trivial centralizer, but not trivial.

The problem of knowing if a quasi-trivial centralizer is trivial is reduced to the problem of knowing when a X-invariant function is constant. This problem will be studied in Section 2.4.

Our first criterion to obtain triviality is based on the notion of *spectral decomposition*. We say that X admits a *countable spectral decomposition* if the non-wandering set, $\Omega(X)$, satisfies $\Omega(X) = \sqcup_{i\in\mathbb{N}}\Lambda_i$, where the sets Λ_i are pairwise disjoint, each of which is compact, X-invariant and transitive, i.e., contains a dense orbit.

Theorem C. *Let M be a compact connected manifold and let $X \in \mathfrak{X}^1(M)$. Assume that all the singularities of X are hyperbolic, that X admits a countable spectral decomposition and that the C^1-centralizer of X is collinear. Then $\mathfrak{C}^1(X)$ is trivial.*

We also obtain the following theorem:

Theorem D. *If X is a transitive, separating C^1-vector field, then $\mathfrak{C}^1(X)$ is trivial.*

This generalizes to the C^1-category previous results of Oka [Oka76], Kato-Morimoto [KM73] and Bonomo-Rocha-Varandas [BRV18]. As a simple application of Theorem D, we can obtain the triviality of the centralizer of the following flow: starting with an irrational flow on $\mathbb{T}^2$, one may may multiply the vector field by a non negative function with only one zero on the torus. This flow is separating (see example 2.8 in [Art15]) and it verifies the conditions in Theorem D.

In higher regularity, Pesin theory in the non-uniformly hyperbolic case and Sard's theorem give us two useful tools to verify triviality of the centralizer. Consider a probability measure μ on M and $X \in \mathfrak{X}^1(M)$. We say that μ is X-invariant if for any measurable set $A \subset M$ and any $t \in \mathbb{R}$ we have $\mu(A) = \mu(X_t(A))$. By Oseledets theorem, for μ-almost every point x, there exist a number $1 \leq l(x) \leq d$ and $l(x)$-numbers $\lambda_1(x) < \ldots < \lambda_{l(x)}(x)$ with the following properties: there exist $l(x)$-subspaces $E_1(x), \ldots, E_{l(x)}(x)$ such that $T_xM = E_1(x) \oplus \cdots \oplus E_{l(x)}(x)$ and for each $i = 1, \ldots, l(x)$ and for any non zero vector $v \in E_i(x)$ we have

$$\lim_{t\to\pm\infty} \frac{\log \|DX_t(x) \cdot v\|}{t} = \lambda_i(x).$$

The numbers λ_i are called *Lyapunov exponents*. We say that μ is *non-uniformly hyperbolic* if for μ-almost every point all the Lyapunov exponents are non-zero except for the one in the direction generated by the vector field X. Using Pesin's theory, in the non-uniformly hyperbolic scenario, we obtain:

Theorem E. *Let M be a compact manifold of dimension $d \geq 2$. Let $X \in \mathfrak{X}^2(M)$ be a vector field with finitely many singularities and let μ be a X-invariant probability measure such that $\operatorname{supp}\mu = M$. If μ is non-uniformly hyperbolic for X, then X has trivial C^1-centralizer.*

Theorem E can be applied for non-uniformly hyperbolic geodesic flows, like the ones constructed by Donnay [Don88] and Burns-Gerber [BG89]. In particular, we obtain that non-uniformly hyperbolic geodesic flows have trivial centralizer.

In dimension three, under higher regularity assumptions, we are also able to obtain triviality, for a slightly stronger notion of expansiveness.

Definition 2.1.8. We say that $X \in \mathfrak{X}^1(M)$ is *Kinematic expansive* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in M$ satisfy $d(X_t(x), X_t(y)) < \delta$, for every $t \in \mathbb{R}$ then there exists $0 < |s| < \varepsilon$ such that $y = X_s(x)$.

The difference between the separating property and Kinematic expansiveness is that for the latter even points on the same orbit must eventually separate. In [Art16] it is described a vector field on the Möbius band which is separating but not Kinematic expansive.

Theorem F. *Let M be a compact 3-manifold and consider $X \in \mathfrak{X}^3(M)$. If X is Kinematic expansive and all its singularities are hyperbolic, then its C^3-centralizer is trivial.*

Remark 2.1.9. The Kinematic expansive condition does not imply that the system admits a countable spectral decomposition. Hence, we cannot use Theorem C to conclude Theorem F.

The technique we use in the above theorem, which relies on Sard's Theorem, also leads to a criteria to obtain triviality from a collinear centralizer of high regularity.

Theorem G. *Let M be a compact, connected Riemannian manifold of dimension $d \geq 1$, and let $X \in \mathfrak{X}^d(M)$. Assume that every singularity and periodic orbit of X is hyperbolic, that $\Omega(X) = \overline{\mathrm{Per}(X)}$ and that the C^d-centralizer of X is collinear. Then X has trivial C^d-centralizer.*

This criterion is not sufficient if we want to obtain a generic result, due to the lack of a general C^d-closing lemma. However, following the arguments of [Hur86, Man73], we can show that C^d-generically the triviality of the C^d-centralizer is equivalent to the collinearity of the C^d-centralizer.

Theorem H. *Let M be a compact, connected Riemannian manifold of dimension $d \geq 1$. There exists a residual set $\mathcal{R}_T \subset \mathfrak{X}^d(M)$ such that for any $X \in \mathcal{R}_T$, the C^d-centralizer $\mathfrak{C}^d(X)$ of X is collinear if and only if it is trivial.*

Organization of this chapter: The structure of this paper has two parts. The first part deals with general criteria for collinearity, quasi-triviality and triviality of the centralizer (Sections 2.2,2.3 and 2.4). The second part deals with our generic results (Section 2.5). Propositions 2.1.4 and 2.1.5 are proved in Section 2.2. In Section 2.3 we prove Theorem A. Theorems C, D, E, F, G and H are proved in Section 2.4. Finally in Section 2.5 we prove Theorem B.

2.2 Collinearity

In this section we obtain three criteria for collinear centralizer. The first criterion is given by Proposition 2.1.4, which is based on the notion of being *separating* (see Definition 2.1.3). There are several different notions of "expansiveness for flows. The property of being separating is a very weak form of expansiveness. Indeed, all the usual definitions for flows (Bowen-Walters expansive or Komuro expansive) imply that the flow is separating, see [Art16] for a discussion. Let us give an example of a separating flow.

Example 2.2.1. Fix two positive real numbers $0 < a < b$ and consider the annulus on $\mathbb{R}^2$ given by $A := \{(x, y) \in \mathbb{R}^2 : a \leq \|(x, y)\| \leq b\}$. Using polar coordinates (r, θ) on A, we consider the vector field $X(r,\theta) := \frac{\partial}{\partial\theta}$. Observe that every orbit of X is periodic with different period. It is easy to see that this flow is separating.

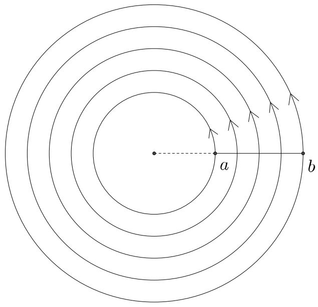

Figure 2.1: Example 2.2.1.

Proof of Proposition 2.1.4. Let $X \in \mathfrak{X}^1(M)$ be a separating vector field with separating constant $\varepsilon > 0$ and suppose that there exists $Y \in \mathfrak{C}^1(X)$ that is not collinear to X. Thus there is a point $x \in M$ such that $\dim\langle X(x), Y(x)\rangle = 2$.

Let (φ, U) be a small flow box for the flow X_t around x, that is, $\varphi\colon M \supset U \to W \subset \mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$ is a local chart such that $\varphi_* X = (1,0)$. In particular we have that for every point $p \in U$ there is a positive number $\rho(p) > 0$ such that

$$\varphi(X_{(-\rho(p),\rho(p))}(p)) \subset \varphi(p) + (-\rho(p), \rho(p)) \times \{0\}.$$

Fix $\delta > 0$ small enough such that for each $s \in (-\delta, \delta)$, we have $Y_s(x) \in U$, $d_{C^0}(Y_s, \mathrm{id}) < \varepsilon$, and $\dim\langle Y(Y_s(x)), X(Y_s(x))\rangle = 2$. Thus in the flow box, we have that $D\varphi(Y_s(x)) \cdot Y(Y_s(x)) = (Y_1(s), Y_2(s))$, with $Y_1(s) \in \mathbb{R}$ and $Y_2(s) \in \mathbb{R}^d - \{0\}$. In particular, for each $t \in \mathbb{R}$ such that $X_t(x) \in Y_{(-\delta,\delta)}(x)$, there exists an open interval $I_t := (t - \rho(X_t(x)), t + \rho(X_t(x))) \subset \mathbb{R}$ such that $\# X_{I_t}(x) \cap Y_{(-\delta,\delta)}(x) = 1$.

We conclude that the set $\mathrm{orb}^X(x) \cap Y_{(-\delta,\delta)}(x)$ is at most countable. Then, since $(-\delta, \delta)$ is uncountable, there is $s \in (-\delta, \delta)$ such that $Y_s(x) \notin \mathrm{orb}^X(x)$. Now, by commutativity, and by our choice of δ, we obtain

$$d(X_t(Y_s(x)), X_t(x)) = d(Y_s(X_t(x)), X_t(x)) < \varepsilon, \text{ for every } t \in \mathbb{R},$$

which is a contradiction. $\square$

The following theorem will be used to prove Theorem E. It also gives another criterion to obtain collinearity of the C^1-centralizer.

Proposition 2.2.2. *Let $X \in \mathfrak{X}^1(M)$. Suppose that X verifies the following condition: there exists a dense set $\mathcal{D} \subset M$ such that for any $x \in \mathcal{D}$ and any non-zero vector $v \in T_x M - \langle X(x)\rangle$, it holds*

$$\|DX_t(x) \cdot v\| \to +\infty, \text{ for } t \to +\infty \text{ or } t \to -\infty.$$

Then X has collinear centralizer.

Proof. Let $Y \in \mathfrak{C}^1(X)$. Then, by (2.1.1), for any $x \in M$, and $t \in \mathbb{R}$, it holds

$$Y(X_t(x)) = DX_t(x) \cdot Y(x).$$

Assume that $Y(x)$ is not collinear to $X(x)$. Since this is an open condition, we can take x belonging to the set $\mathcal{D}$. By compactness of M, we also have $\sup_{p \in M} \|Y(p)\| < +\infty$. However, by hypothesis,

$$\|DX_t(x) \cdot Y(x)\| \to +\infty, \text{ for } t \to +\infty \text{ or } t \to -\infty,$$

which is a contradiction. $\square$

Some examples of vector fields that verify the conditions of Proposition 2.2.2 are non-uniformly hyperbolic divergence-free vector fields and quasi-Anosov flows.

We remark that the conditions of collinearity in Propositions 2.1.4 and 2.2.2 are not generic (see Appendix 2.6). Therefore, to obtain that the C^1-centralizer of a C^1-generic vector field is collinear we will need another criterion, which is given by Proposition 2.1.5.

Let $X \in \mathfrak{X}^1(M)$ and let $M_X := M - \mathrm{Zero}(X)$ be as in (2.1.2). Over M_X we may consider the normal vector bundle N_X defined as $N_{X,p} := \langle X(p) \rangle^\perp$, for $p \in M_X$, where $\langle X(p) \rangle^\perp$ is the orthogonal complement of the direction $\langle X(p) \rangle$ inside T_pM. Let $\Pi^X : TM_X \to N_X$ be the orthogonal projection on N_X. On N_X we have a well defined flow, called the *linear Poincaré flow*, which is defined as follows: for any $p \in M_X$, any $v \in N_{X,p}$, and $t \in \mathbb{R}$, the image of v by the linear Poincaré flow is

$$P^X_{p,t}(v) := (\Pi^X_{X_t(p)} \circ DX_t(p)) \cdot v. \tag{2.2.1}$$

The key criterion to study the centralizer of C^1-generic vector fields is based on the following property.

Definition 2.2.3 (Unbounded normal distortion). Let $X \in \mathfrak{X}^1(M)$ be a C^1 vector field. We say that X verifies the *unbounded normal distortion* property if the following holds: there exists a dense subset $\mathcal{D} \subset M - \mathcal{CR}(X)$, such that for any $x \in \mathcal{D}$, $y \in M - \mathcal{CR}(X)$ such that $y \notin \mathrm{orb}^X(x)$ and $K \geq 1$, there is $n \in (0, +\infty)$, such that

$$|\log \det P^X_{x,n} - \log \det P^X_{y,n}| > K.$$

Proof of Proposition 2.1.5. Let $X \in \mathfrak{X}^1(M)$ be a vector field with the unbounded normal distortion property and let $\mathcal{D} \subset M - \mathcal{CR}(X)$ be the set given in Definition 2.2.3. Take $Y \in \mathfrak{C}^1(X)$. Assume by contradiction that Y is not collinear with X on $M - \mathcal{CR}(X)$. The set of points $x \in M$ such that $X(x)$ and $Y(x)$ are non-collinear is open, hence by density of the set $\mathcal{D}$, there exists a point $x \in \mathcal{D}$ such that $Y(x)$ and $X(x)$ are not collinear.

By the same argument as in the proof of Proposition 2.1.4, we can always find $s > 0$ arbitrarily close to 0 such that $Y_s(x) \notin \mathrm{orb}^X(x)$. Observe that for any $t \in \mathbb{R}$, it holds

$$|\det P^X_{Y_s(x),t}| = |\det \Pi^X_{X_t(Y_s(x))} \cdot \det DX_t(Y_s(x))|_{N_{X,Y_s(x)}}|.$$

Since X commutes with Y, we have that

$$DX_t(Y_s(x)) = DY_s(X_t(x)) \circ DX_t(x) \circ (DY_s(x))^{-1}. \tag{2.2.2}$$

Using the coordinates $N_X \oplus \langle X \rangle$ on TM_X, for each $s \in \mathbb{R}$, we obtain a linear map $L_{s,x} \colon N_{X,x} \to \langle X(x) \rangle$ such that

$$(DY_s(x))^{-1}(N_{X,Y_s(x)}) = \mathrm{graph}(L_{s,x}).$$

Furthermore, $\|L_{s,x}\|$ can be made arbitrarily small as $s \to 0$, since Y_s is C^1-close to the identity. Using the coordinates $N_{X,x} \oplus \langle X(x) \rangle$, any vector $v \in \mathrm{graph}(L_{s,x})$ can be written as $v = (v_N, L_{s,x}(v_N))$, where $v_N := \Pi^X_x(v)$. For any such vector v, for each $t \in \mathbb{R}$ and using the coordinates $N_{X,X_t(x)} \oplus \langle X(X_t(x)) \rangle$, we have

$$DX_t(x)v = \left(P^X_{x,t}(v_N), L_{s,x}(v_N)\frac{\|X(X_t(x))\|}{\|X(x)\|} + \left(DX_t(x)v_N, \frac{X(X_t(x))}{\|X(X_t(x))\|} \right) \right), \tag{2.2.3}$$

where $(\cdot,\cdot)$ inside the second coordinate of the right side of (2.2.3) denotes the scalar product given by the Riemannian structure.

On the other hand, for any vector $v_N \in N_{X,x}$ and any $t \in \mathbb{R}$, we have

$$DX_t(x)v_N = \left(P^X_{x,t}(v_N), \left(DX_t(x) \cdot v_N, \frac{X(X_t(x))}{\|X(X_t(x))\|} \right) \right). \tag{2.2.4}$$

Set $c := \|X(x)\| > 0$, and let $\tilde{c} \geq 1$ be a constant such that $\sup_{p \in M} \|X(p)\| < \tilde{c}$. For any vector $v_N \in N_{X,x}$, we obtain

$$|L_{s,x}(v_N)| \frac{\|X(X_t(x))\|}{\|X(x)\|} < \|L_{s,x}\| \cdot \|v_N\| \frac{\tilde{c}}{c},$$

which can be made arbitrarily close to 0 by taking s small enough. This holds for any $t \in \mathbb{R}$. Hence, comparing (2.2.3) and (2.2.4) we conclude that $DX_t(x)|_{\mathrm{graph(L_{s,x})}}$ is arbitrarily close to $DX_t(x)|_{N_{X,x}}$, for any $t \in \mathbb{R}$.

By (2.2.2), we obtain

$$\begin{aligned}
&\left|\det P^X_{Y_s(x),t}\right| = \\
&\left| \det \Pi^X_{Y_s(X_t(x))}|_{DY_s(X_t(x))DX_t(x)\cdot \mathrm{graph}(L_{s,x})}\right| \cdot \left|\det DY_s(X_t(x))|_{DX_t(x)\cdot \mathrm{graph}(L_{s,x})}\right| \cdot \\
&\cdot \left|\det DX_t(x)|_{\mathrm{graph}(L_{s,x})}\right| \cdot \left|(\det(DY_s(x))^{-1}|_{N_{X,Y_s(x)}}\right| =: A \cdot B \cdot C \cdot D.
\end{aligned}$$

Observe that

$$\left|\det P^X_{x,t}\right| = \left|\det \Pi^X_{X_t(p)}|_{DX_t(x)N_{X,x}}\right| . \left|\det DX_t(x)|_{N_{X,x}}\right| =: \mathrm{I} \cdot \mathrm{II}.$$

Notice that B and D are arbitrarily close to 1 if $s \in \mathbb{R}$ is small enough. By our previous discussion, for any $t \in \mathbb{R}$ the value of C is arbitrarily close to the value of II, for s sufficiently small.

Our previous discussion also implies that $DY_s(X_t(x))DX_t(x) \cdot \mathrm{graph}(L_{s,x})$ is close to $DX_t(x) \cdot N_{X,x}$, since $Y_s(X_t(x))$ is close to $X_t(x)$. Thus, the value of A can be made arbitrarily close to the value of I, for $s \in \mathbb{R}$ small enough. Hence, we can take s small such that $Y_s(x) \notin \mathrm{orb}^X(x)$ and

$$\frac{1}{2} < \frac{\left|\det P^X_{Y_s(x),t}\right|}{\left|\det P^X_{x,t}\right|} < 2, \text{ for any } t \in \mathbb{R}.$$

This is a contradiction with the unbounded normal distortion property. We conclude that any vector field $Y \in \mathfrak{C}^1(X)$ verifies that $Y|_{M-\mathcal{CR}(X)}$ is collinear to $X|_{M-\mathcal{CR}(X)}$.

Suppose that for some $x \in \mathcal{CR}(X)$ we have that $Y(x)$ is not collinear to $X(x)$. Since this is an open condition and the hyperbolic periodic points are dense in $\mathcal{CR}(X)$, we can suppose that x is a periodic point. By a calculation similar to the one made in the proof of Proposition 2.2.2 we would then have that $\|Y(X_t(x))\| \to +\infty$ for $t \to +\infty$ or $t \to -\infty$, which contradicts the fact that $\sup_{p\in M} \|Y(p)\| < +\infty$. Thus we have that Y is also collinear to X on $\mathcal{CR}(X)$. $\square$

2.3 Quasi-triviality

This section has two parts. In the first part we construct an example of a vector field whose centralizer is collinear but not quasi-trivial. In the second part we prove that under the condition that every singularity is hyperbolic we can improve the collinearity to quasi-triviality.

2.3.1 Collinear does not imply quasi-trivial

To obtain a quasi-trivial centralizer from a collinear centralizer is an issue of knowing whether an invariant function $f\colon M_X \to \mathbb{R}$ admits a C^1 extension to the set $Sing(X)$. The simple example below shows that this is not always possible. Indeed we construct an example of a vector field whose C^1-centralizer is collinear but not quasi-trivial.

Example 2.3.1. Let $V \in \mathfrak{X}^\infty(\mathbb{T}^2)$ generate an irrational flow. Fix a point $p \in \mathbb{T}^2$ and consider a function $\psi\colon \mathbb{T}^2 \to [0,1]$ such that $\psi(x) = 0 \iff x = p$. Let $Z \in \mathfrak{X}^\infty(\mathbb{T}^2)$ be defined by $Z = \psi V$. As it is described in example 2.8 in [Art16], Z is separating. Now, consider $f, g\colon [0,1) \to [1,+\infty)$ be given, respectively, by

$$f(t) = \frac{1}{1-t} \quad \text{and} \quad g(t) = \frac{1}{1-t^2}.$$

Observe that both functions diverge to $+\infty$ when $t \to 1$, but the function $\frac{f}{g} = 1+t$ extends smoothly to $[0,1]$. Consider $M = [0,1] \times \mathbb{T}^2$ and extend Z to M by $Z(t,x) = Z(x)$. Define the vector field $X(t,x) = \frac{1}{g(t)} Z(t,x)$. Notice that X is tangent to the fiber $\{t\} \times \mathbb{T}^2$, and the trajectories on each fiber are the same, but travelled with different speeds. Then, the proof of Proposition 2.1.4 shows that $X \in \mathfrak{X}^\infty(M)$ has collinear centralizer.

Nevertheless the vector field $Y = \frac{f}{g}Z$ is smooth and commutes with X. Indeed, both vector fields vanish at the fiber $\{1\} \times \mathbb{T}^2$. Moreover, both f and g are constant on each fiber and for $t < 1$ one has

$$Y(t,x) = f(t)X(t,x).$$

As X is tangent to each fiber $\{t\} \times \mathbb{T}^2$, we conclude that $[X,Y] = 0$. Since $f(t) \to \infty$ when $t \to 1$, this proves that X has not a quasi-trivial centralizer.

The above example has uncountably many singularities, and thus it is not separating. This raises the following question.

Question 1. Is there a separating vector field whose centralizer is not quasi-trivial?

We do not know what to expect as an answer to this question.

2.3.2 The case of hyperbolic zeros

The main result of this section is Theorem 2.3.3 below, in which we obtain the quasi-triviality from collinearity of $\mathfrak{C}^1(X)$ assuming only that all the singularities of X are hyperbolic.

Definition 2.3.2. A function $f\colon M \to \mathbb{R}$ is called a *first integral* of X if it is of class C^1 and satisfies $X \cdot f \equiv 0$. We denote by $\mathfrak{I}^1(X)$ the set of all such maps.

In our definition, first integrals are C^1 functions which are X-invariant. In particular, for any first integral f, we have $f \circ X_t = f$, for every $t \in \mathbb{R}$. In particular, for any $c \in \mathbb{R}$, the constant map $\underline{c}(x) := c$ is in $\mathfrak{I}^1(X)$, and then, we always have $\mathbb{R} \simeq \{\underline{c} : c \in \mathbb{R}\} \subset \mathfrak{I}^1(X)$. The following theorem is a reformulation in terms of $\mathfrak{I}^1(X)$ of Theorem A.

Theorem 2.3.3. *Let $X \in \mathfrak{X}^1(M)$. If X has collinear centralizer and all the singularities of X are hyperbolic, then X has quasi-trivial C^1-centralizer, in the sense of Definition 2.1.6. More precisely, we have*

$$\mathfrak{C}^1(X) = \{fX : f \in \mathfrak{I}^1(X)\}.$$

This theorem is an immediate consequence of Propositions 2.3.4, 2.3.5, and 2.3.6 below. We divide the proof into two subsections to emphasize that the technique to deal with singularities that are saddles is different from the technique to deal with sinks and sources. We also remark that Theorem 2.3.3 gives a significant improvement compared with previous works on centralizers of vector fields, since we only need C^1 regularity. The results that were known previously used Sternberg's linearisation results, which require higher regularity of the vector field and non-resonant conditions on the eigenvalues of the singularity, see for instance [BRV18].

2.3.3 When the singularity is of saddle type

Given any vector field $X \in \mathfrak{X}^1(M)$, and $Y \in \mathfrak{C}^1(X)$, by Lemma 2.1.2, we know that $Y|_{M_X} = fX|_{M_X}$, for some C^1, X-invariant function $f\colon M_X \to \mathbb{R}$. Assume that $\sigma \in$

$\mathrm{Zero}(X)$ is a saddle type singularity. In Propositions 2.3.4 and 2.3.5, we show that f can be extended to a C^1 function in a neighbourhood of σ.

Proposition 2.3.4. *Let $X \in \mathfrak{X}^1(M)$ and let $f\colon M_X \to \mathbb{R}$ be an X-invariant continuous function. If $\sigma \in \mathrm{Zero}(X)$ is a saddle type singularity, then f admits a continuous extension to σ.*

Proof. Recall that M has dimension $d \geq 0$. Fix a point $p \in W^s_{\mathrm{loc}}(\sigma)$. We claim that for any point $q \in W^u(\sigma)$ we have that $f(p) = f(q)$. By the X-invariance of f, it is enough to consider $q \in W^u_{\mathrm{loc}}(\sigma)$. Let $(D^s_n)_{n\in\mathbb{N}}$ be a sequence of discs of dimension $\mathrm{ind}(\sigma)$, centred on q, with radius $\frac{1}{n}$ and transverse to $W^u_{\mathrm{loc}}(\sigma)$. Similarly, consider a sequence $(D^u_n)_{n\in\mathbb{N}}$ of discs of dimension $d - \mathrm{ind}(\sigma)$, centred on p, with radius $\frac{1}{n}$, and transverse to $W^s_{\mathrm{loc}}(\sigma)$.

For each $n \in \mathbb{N}$, by the lambda-lemma (see [PM82] chapter 2.7) there exists $t_n > 0$ such that $X_{t_n}(D^u_n) \pitchfork D^s_n \neq \emptyset$. In particular, there exists a point $x_n \in D^u_n$ that verifies $X_{t_n}(x_n) \in D^s_n$. It is immediate that $x_n \to p$, as $n \to +\infty$. Since the function f is continuous on M_X, we have that $f(x_n) \to f(p)$. We also have that $X_{t_n}(x_n) \to q$ as $n \to +\infty$. Hence, $f(X_{t_n}(x_n)) \to f(q)$. By the X-invariance of f, we have

$$f(p) = \lim_{n\to+\infty} f(x_n) = \lim_{n\to+\infty} f(X_{t_n}(x_n)) = f(q).$$

Analogously, we can prove that for a fixed $q' \in W^u_{\mathrm{loc}}(\sigma)$ and for any $p' \in W^s(\sigma)$, it is verified $f(p') = f(q')$. We conclude that $f|_{W^s(\sigma)-\{\sigma\}} = f|_{W^u(\sigma)-\{\sigma\}} = c$, for some constant $c \in \mathbb{R}$. Set $f(\sigma) = c$. For any sequence of points $(x_n)_{n\in\mathbb{N}}$ converging to σ, by an argument similar to the above, using the invariance of f, we can conclude that this extension of f is continuous. $\square$

Proposition 2.3.5. *Let $X \in \mathfrak{X}^1(M)$ and let $f\colon M_X \to \mathbb{R}$ be an X-invariant function of class C^1. If $\sigma \in \mathrm{Zero}(X)$ is a saddle type singularity, then f can be extended to a C^1 function in a neighbourhood of σ, by setting $\nabla f(\sigma) := 0$.*

Proof. By Proposition 2.3.4, we already know that the function f admits a continuous extension to σ. We claim that $\lim_{x\to\sigma} \nabla f(x) = 0$. Let us fix $r > 0$ sufficiently small such that $B(\sigma, 2r) \cap \mathrm{Zero}(X) = \{\sigma\}$ and set $K^* := W^*_{\mathrm{loc}}(\sigma) \cap \partial B(\sigma, r)$, for $* \in \{s, u\}$. In the following, given any two points $p_s \in K^s$ and $q_u \in K^u$, we let $(D^u_n)_{n\in\mathbb{N}}$ be a sequence of discs of dimension $d - \mathrm{ind}(\sigma)$, centred on p_s, with radius $\frac{1}{n}$, transverse to $W^s_{\mathrm{loc}}(\sigma)$, and we let $(D^s_n)_{n\in\mathbb{N}}$ be a sequence of discs of dimension $\mathrm{ind}(\sigma)$, centred on q_u, with radius $\frac{1}{n}$, transverse to $W^u_{\mathrm{loc}}(\sigma)$.

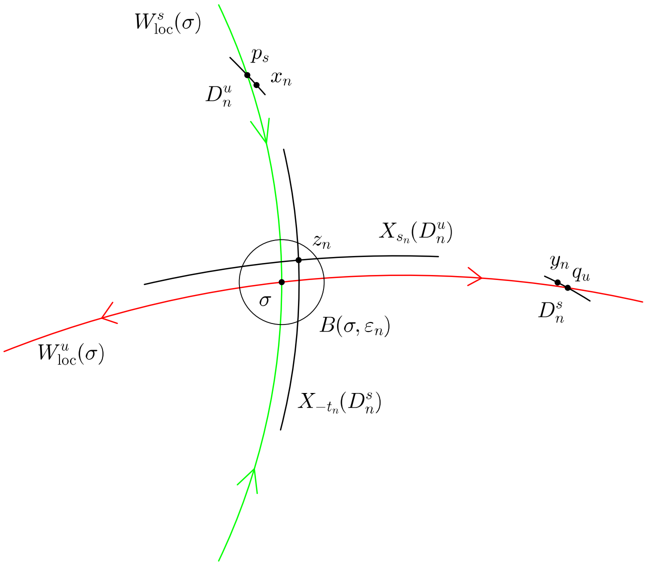

Figure 2.2: Proposition 2.3.5.

For any $n \geq 0$, by the lambda-lemma, there exists a sequence $(\varepsilon_n)_{n\geq 0} \in (\mathbb{R}^*_+)^{\mathbb{N}}$, with $\varepsilon_n \to 0$ as $n \to +\infty$, such that for any $z \in B(\sigma, \varepsilon_n)$, if $p_s \in K^s, q_u \in K^u$ are suitably chosen, and for D^u_n, D^s_n as defined previously, then there exist $x_n \in D^u_n$, $y_n \in D^s_n$, and $s_n, t_n > 0$, such that $z = X_{s_n}(x_n) = X_{-t_n}(y_n)$. Note that necessarily, $s_n, t_n \to +\infty$ as $n \to +\infty$. Fix $p_s \in K^s, q_u \in K^u$ and let $(z_n)_{n\geq 0}$ be a sequence of points such that $z_n = X_{s_n}(x_n) = X_{-t_n}(y_n) \in B(\sigma, \varepsilon_n)$, with $x_n \in D^u_n$, $y_n \in D^s_n$, $s_n, t_n > 0$, for all $n \geq 0$. It is immediate that $x_n \to p_s$ and $y_n \to q_u$, as $n \to +\infty$. Since the function f is C^1 on M_X, we deduce that

$$\lim_{n\to+\infty} \nabla f|_{D^u_n} = \nabla f(p_s), \qquad \lim_{n\to+\infty} \nabla f|_{D^s_n} = \nabla f(q_u). \tag{2.3.1}$$

We set $\mathbb{S}^u_n := \{v \in T_{x_n} D^u_n : \|v\| = 1\}$ and $\mathbb{S}^s_n := \{v \in T_{y_n} D^s_n : \|v\| = 1\}$. Let $v \in \mathbb{S}^u_n$. By the X-invariance, we have $f(z_n) = f(X_{s_n}(x_n)) = f(x_n)$. Differentiating the equation $f(.) = f \circ X_t(.)$ we obtain

$$(\nabla f(z_n), DX_{s_n}(x_n) \cdot v) = (\nabla f(X_{s_n}(x_n)), DX_{s_n}(x_n) \cdot v) = (\nabla f(x_n), v). \tag{2.3.2}$$

By the lambda-lemma, we know that $d_{C^1}(X_{s_n}(D^u_n), D^u) \to 0$, for some disc $D^u \subset W^u_{\mathrm{loc}}(\sigma)$. In particular, $\angle(DX_{s_n}(x_n) \cdot T_{x_n} D^u_n, E^u(\sigma)) \to 0$, and $\|DX_{s_n}(x_n) \cdot v\| \to +\infty$. By (2.3.1), by compactness of K^s, and since $\|v\| = 1$, the right hand side of (2.3.2) is uniformly bounded, independently of the choices of $p_s, q_u, (z_n)_n$ and n, thus,

$$\lim_{n\to+\infty} \left(\nabla f(z_n), \frac{DX_{s_n}(x_n) \cdot v}{\|DX_{s_n}(x_n) \cdot v\|}\right) = \lim_{n\to+\infty} \frac{(\nabla f(x_n), v)}{\|DX_{s_n}(x_n) \cdot v\|} = 0.$$

We deduce that $\lim_{n\to+\infty} \|\pi^u_n(\nabla f(z_n))\| = 0$, where $\pi^u_n \colon T_{z_n} M \to T_{z_n}(X_{s_n}(D^u_n))$ denotes the orthogonal projection onto $T_{z_n}(X_{s_n}(D^u_n))$. Arguing in the same way for $X_{-t_n}(D^s_n)$, we also have $\lim_{n\to+\infty} \|\pi_n^s(\nabla f(z_n))\| = 0$, where π_n^s is the orthogonal projection onto $T_{z_n}(X_{-t_n}(D_n^s))$. Since $T_{z_n}M = T_{z_n}(X_{s_n}(D_n^u)) \oplus T_{z_n}(X_{-t_n}(D_n^s))$, then for some sequence $(\delta_n)_{n\geq 0}$ going to 0, and for any $z \in B(\sigma, \varepsilon_n)$, we have

$$\|\nabla f(z)\| \leq \delta_n.$$

We conclude that ∇f can be extended by continuity to σ, by setting $\nabla f(\sigma) := 0$. In particular, the extension of f is C^1 in a neighbourhood of σ. $\square$

2.3.4 When the singularity is of the type sink or source

We now deal with hyperbolic singularities of the type sink or source.

Proposition 2.3.6. *Let $X, Y \in \mathfrak{X}^1(M)$ such that $[X,Y] = 0$ and $\dim\langle X(x), Y(x)\rangle \leq 1$, for every $x \in M$. Assume that $\sigma \in Sing(X)$ is a hyperbolic sink. Then, there exists $c \in \mathbb{R}$ such that $Y(x) = cX(x)$, for every $x \in W^s(\sigma)$.*

In the proof of Proposition 2.3.6 we shall use the following elementary lemma.

Lemma 2.3.7. *Let $(E, \|\cdot\|)$ be a finite-dimensional vector space endowed with a norm. Let Λ be an infinite set and assume that for each $\lambda \in \Lambda$, there exists a non-empty compact subset $K_\lambda \subset \mathbb{S} := \{v \in E : \|v\| = 1\}$ of the sphere of unit vectors in $(E, \|\cdot\|)$, such that*

$$\lambda' \neq \lambda \text{ in } \Lambda \quad \Longrightarrow \quad K_\lambda \cap K_{\lambda'} = \emptyset.$$

Suppose that $\dim E \geq 2$. Then, there exist a finite subset $\{\lambda, \lambda_1, \ldots, \lambda_k\} \subset \Lambda$ and vectors $\{u, u_1, \ldots, u_k\}$ such that

1. *$u \in K_\lambda$ and $u_\ell \in K_{\lambda_\ell}$, for each $\ell = 1, \ldots, k$;*
2. *u belongs to the subspace spanned by $\{u_1, \ldots, u_k\}$;*
3. *$\{u_1, \ldots, u_k\}$ is a linearly independent set.*

Proof. We begin with a simple observation that we will use repeatedly in this proof: for each $u \in \mathbb{S}$, $-u$ is the only other vector in $\mathbb{S}$ which is collinear with u.

Now, since Λ is infinite, we can pick a sequence $(\lambda_n)_{n\geq 0} \subset \Lambda$, whose terms are distinct. For each n, choose a vector $u_n \in K_{\lambda_n}$. Since $\dim E \geq 2$, and the sets K_λ are pairwise disjoint, by the simple observation above, we can assume without loss of generality that the set $\{u_1, u_2\}$ is linearly independent. Assume by contradiction that the conclusion does not hold. Then, we conclude by induction that for every n the set $\{u_1, u_2, u_3, \ldots, u_n\}$ must be linearly independent. But this is absurd as E is finite dimensional. $\square$

Proof of Proposition 2.3.6. By Lemma 2.1.2, for any $x \in M_X = M - Sing(X)$, we have $Y(x) = f(x)X(x)$, for some C^1 function $f\colon M_X \to \mathbb{R}$. Moreover, $f(X_t(x)) = f(x)$ for every $x \in M_X$ and $t \in \mathbb{R}$. Notice that, as σ is an isolated zero of X, we have $\sigma \in Sing(Y)$.

Take $\varepsilon > 0$ small so that $\overline{B}(\sigma, \varepsilon) \subset W^s(\sigma)$ and let $S := \partial B(\sigma, \varepsilon)$. In particular, notice that $x \in S$ implies $\lim_{t \to +\infty} X_t(x) = \sigma$. Also, for every $x \in W^s(\sigma)$, there exists $T \in \mathbb{R}$ such that $X_T(x) \in S$.

By the above remarks, the proof of the proposition is reduced to the proof of the following claim.

Claim 1. *$Df(p) = 0$ for every $p \in S$.*

We shall postpone the proof of Claim 1. Take a point $p \in S$ and consider the set

$$V(p) := \left\{ u \in T_\sigma M : \exists\, t_n \to \infty,\, u = \lim_{n \to \infty} \frac{X(X_{t_n}(p))}{\|X(X_{t_n}(p))\|} \right\}.$$

By compactness, $V(p)$ is non-empty, and every $u \in V(p)$ is a unit vector; in particular, $0 \notin V(p)$. The following claims are the key arguments for this proof.

Claim 2. *If $u \in V(p)$ then $DY(\sigma) \cdot u = f(p) DX(\sigma) \cdot u$.*

Proof. Fix some $t \in \mathbb{R}$. Since $Y(X_{t+s}(p)) = f(p) X(X_{t+s}(p))$ for every $s \in \mathbb{R}$, taking the derivative with respect to s on both sides we obtain

$$DY(X_t(p)) \cdot \left(\frac{X(X_t(p))}{\|X(X_t(p))\|} \right) = f(p) DX(X_t(p)) \cdot \left(\frac{X(X_t(p))}{\|X(X_t(p))\|} \right).$$

By using this formula with $t = t_n$ and letting $n \to \infty$ we conclude that $DY(\sigma) \cdot u = f(p) DX(\sigma) \cdot u$, proving the claim. $\square$

Claim 3. *If $p, q \in S$ and $V(p) \cap V(q) \neq \emptyset$ then $f(p) = f(q)$.*

Proof. Assume that there exists $u \in V(p) \cap V(q)$. Then, by Claim 2, one has

$$DY(\sigma) \cdot u = f(p) DX(\sigma) \cdot u = f(q) DX(\sigma) \cdot u.$$

As $DX(\sigma)$ is an invertible linear map (because all eigenvalues are negative) this implies that $(f(p) - f(q))u = 0$, and since $u \neq 0$, the claim is proved. $\square$

We are now in position to give the proof of Claim 1. Assume by contradiction that the claim is not true. Then, there exists $U \subset S$ and real numbers $a < b$ such that $f\colon U \to [a, b]$ is surjective.

Now, for every $t \in [a, b]$, we choose some point $p_t \in U \cap f^{-1}(t)$, and we consider the family of compact subsets $\{V(p_t)\}_{t \in [a,b]} \subset T_\sigma M$ of unit vectors. As $t \neq s$ implies $f(p_t) \neq f(p_s)$, one obtains from Claim 3 that the family $\{V(p_t)\}_{t \in [a,b]}$ satisfies all the assumptions of Lemma 2.3.7.

Thus, there exists a finite set $\{p, p_1, \ldots, p_k\} \subset U$ and vectors $u \in V(p)$, $u_\ell \in V(p_\ell)$, $\ell = 1, \ldots, k$, with $u \in \langle u_1, \ldots, u_k \rangle$ and $\{u_1, \ldots, u_k\}$ linearly independent, and such that $f(p_i) \neq f(p_j) \neq f(p)$, for every $i, j \in \{1, \ldots, k\}$.

Take $\alpha^1, \dots, \alpha^k \in \mathbb{R}$ such that $u = \sum_{\ell=1}^k \alpha^\ell u_\ell$. Using Claim 2 we can write

$$DY(\sigma) \cdot u = f(p) DX(\sigma) \cdot u = DX(\sigma) \cdot \left(\sum_{\ell=1}^k f(p) \alpha^\ell u_\ell \right).$$

Also

$$DY(\sigma) \cdot u_\ell = f(p_\ell) DX(\sigma) \cdot u_\ell, \ \forall \, \ell = 1, \dots, k,$$

which implies that

$$DY(\sigma) \cdot u = DX(\sigma) \cdot \left(\sum_{\ell=1}^k f(p_\ell) \alpha^\ell u_\ell \right).$$

Since $DX(\sigma)$ is invertible we must have $\sum_{\ell=1}^k f(p) \alpha^\ell u_\ell = \sum_{\ell=1}^k f(p_\ell) \alpha^\ell u_\ell$, and as $\{u_1, \dots, u_k\}$ is linearly independent, this gives

$$f(p) \alpha^\ell = f(p_\ell) \alpha^\ell, \text{ for every } \ell = 1, \dots, k.$$

Since $u \neq 0$ there exists some $\alpha^\ell \neq 0$. However, this implies that $f(p) = f(p_\ell)$, a contradiction. $\square$

We now give the proof of Theorem 2.3.3.

Proof of Theorem 2.3.3. Assume that $\mathfrak{C}^1(X)$ is collinear and that each singularity $\sigma \in \mathrm{Zero}(X)$ is hyperbolic. Let us consider $Y \in \mathfrak{C}^1(X)$. By Lemma 2.1.2, there exists a C^1 function $f \colon M_X \to \mathbb{R}$ which satisfies $X \cdot f \equiv 0$ on M_X and such that $Y(x) = f(x) X(x)$, for every $x \in M_X$. By assumption, the singularities of X are hyperbolic, hence they are isolated, and $Y(\sigma) = 0$, for all $\sigma \in \mathrm{Zero}(X)$. By Propositions 2.3.4, 2.3.5 and 2.3.6, we can extend f to a C^1 invariant function on M. We conclude that f is a first integral of X, and $Y = fX$.

Conversely, assume that $f \colon M \to \mathbb{R}$ is a first integral of X. We define a vector field $Y \in \mathfrak{X}^1(M)$ as $Y(x) := f(x) X(x)$, for every $x \in M$. Indeed, both f and X are of class C^1, thus Y is C^1 too. Moreover, we have $Y \in \mathfrak{C}^1(X)$, since

$$[X, Y] = (X \cdot f) X + f[X, X] = 0. \quad \square$$

2.4 The study of invariant functions and trivial centralizers

The main focus of this section is the study of invariant functions. An invariant function is also called a first integral of the system. There are several works that study the existence of non trivial (non constant) first integrals, see for instance [ABC16, FP15, FS04, Hur86, Man73, Pag11]. In this work we study dynamical conditions that imply the non-existence of first integrals.

First, it is easy to obtain examples of vector fields with quasi-trivial C^1-centralizer which is not trivial. Indeed consider the vector field in example 2.2.1. Since X is separating, it has collinear C^1-centralizer. This flow is non-singular, hence it has quasi-trivial C^1-centralizer. Now take any non-constant C^1-function f which is constant on each orbit, that is, a function which depends only on the coordinate r. The vector field $Y = fX$ belongs to the C^1-centralizer of X, therefore the centralizer of X is only quasi-trivial.

Let $X \in \mathfrak{X}^1(M)$. Recall that a compact set Λ is a basic piece for X if Λ is X-invariant and transitive, that is, it has a dense orbit. We say that X admits a countable spectral decomposition if $\Omega(X) = \sqcup_{i\in\mathbb{N}}\Lambda_i$, where the sets Λ_i are pairwise disjoint basic pieces.

Theorem 2.4.1. *Let $X \in \mathfrak{X}^1(M)$. If X admits a countable spectral decomposition then any continuous X-invariant function is constant.*

Proof. Let $f\colon M \to \mathbb{R}$ be a continuous X-invariant function. Suppose that f is not constant. Since M is connected, there exist two real numbers $a < b$ such that $f(M) = [a, b]$. It is easy to see that in each basic piece the function f is constant: this follows from the transitivity of each basic piece. For each $i \in \mathbb{N}$ define $c_i := f(\Lambda_i)$. Since X admits a countable spectral decomposition, the set $C := \{c_1, c_2, \dots\}$ is at most countable and in particular $[a, b] - C$ is non-empty. Take any value $c \in [a, b] - C$ and consider $\Lambda := f^{-1}(\{c\})$. Note that, since f is continuous, Λ is closed, and hence, compact.

The set Λ is compact and X-invariant. Hence, for any point $p \in \Lambda$ we must have $\omega(p) \subset \Lambda$, where $\omega(p)$ is the set of all accumulations points of the future orbit of p. By the countable spectral decomposition, $\omega(p)$ must be contained in some basic piece Λ_i, which implies that $\Lambda \cap \Lambda_i \neq \emptyset$. Since Λ is a level set of f, this implies that $c_i = f(\Lambda_i) = f(\Lambda) = c$ and this is a contradiction with our choice of c. $\square$

Theorem C follows easily from Theorems A and 2.4.1.

Proof of Theorem D. Let X be a vector field which is transitive and separating. Since X is separating, by proposition 2.1.4, we have that the C^1-centralizer of X is collinear. Let $Y \in \mathfrak{C}^1(X)$. By lemma 2.1.2, there exists a continuous function $f_Y : M_X \to \mathbb{R}$ such that $Y = f_Y X$ on M_X. By transitivity, one obtains that f_Y is constant to c_Y, for some $c_Y \in \mathbb{R}$, on M_X. Therefore, $Y = c_Y X$ on M. $\square$

2.4.1 First integrals and trivial C^1-centralizers

Let M be a compact connected manifold. Recall that for any $X \in \mathfrak{X}^1(M)$, we let $\mathfrak{I}^1(X) := \{f \in C^1(M, \mathbb{R}) : X \cdot f \equiv 0\}$ be the set of all C^1 functions which are invariant under X. As an easy consequence of Theorem A, we obtain the following lemma.

Lemma 2.4.2. *Let $X \in \mathfrak{X}^1(M)$. Assume that the singularities of X are hyperbolic and that the C^1-centralizer of X is collinear. Then X has trivial C^1-centralizer if and only if the set of first integrals of X is trivial, i.e., $\mathfrak{I}^1(X) \simeq \mathbb{R}$.*

As an immediate consequence of Theorem 2.4.1 and Lemma 2.4.2, we obtain:

Corollary 2.4.3. *Let $X \in \mathfrak{X}^1(M)$ be such that X admits a countable spectral decomposition and all its singularities are hyperbolic. If the C^1-centralizer of X is collinear, then it is trivial.*

The following lemma will be used several times in this section.

Lemma 2.4.4. *Let M be a compact manifold of dimension $d \geq 1$ and let $X \in \mathfrak{X}^1(M)$. Then, for any $f \in \mathfrak{I}^1(X)$ and for any hyperbolic singularity or hyperbolic periodic point $p \in \mathrm{Zero}(X) \cup \mathrm{Per}(X)$, it holds that $\nabla f(p) = 0$.*

Proof. Let $X \in \mathfrak{X}^1(X)$ be as above and let $f \in \mathfrak{I}^1(X)$. If $\sigma \in \mathrm{Zero}(X)$ is a hyperbolic singularity, then it follows from Propositions 2.3.5 and 2.3.6 that $\nabla f(\sigma) = 0$. Assume now that for some regular hyperbolic periodic point $p \in \mathrm{Per}(X)$, we have $\nabla f(p) \neq 0$. Then, we have the hyperbolic decomposition along its orbit given by

$$T_{\mathrm{orb}^X(p)} M = E^s \oplus \langle X \rangle \oplus E^u.$$

Note that $f|_{W^s(p)} = f|_{W^u(p)} = f(p)$: this follows easily from the X-invariance of f. Since $\nabla f(p) \neq 0$, by the local form of submersion, we have that $\Sigma := f^{-1}(\{f(p)\})$ is locally contained in a submanifold D of dimension $d-1$. In particular, $T_p D$ is a subspace of dimension $d-1$ contained in $T_p M$. However, our previous observation implies that $W^s_{\mathrm{loc}}(p) \subset \Sigma$ and $W^u_{\mathrm{loc}}(p) \subset \Sigma$. This implies that $E^s(p) \oplus \langle X(p) \rangle \oplus E^u(p) \subset T_p D$. By the hyperbolicity of p, we have that $T_p M = E^s(p) \oplus \langle X(p) \rangle \oplus E^u(p)$, but this is a contradiction with the fact that $T_p D$ has dimension $d-1$. □

By the Poincaré-Bendixson Theorem, for the two-sphere, any level set of an invariant function f has to contain a singularity or a periodic orbit, which forces f to be constant in the generic case where the latter are hyperbolic.

Proposition 2.4.5. *Let $M := \mathbb{S}^2$ be the two dimensional sphere, and let $X \in \mathfrak{X}^1(M)$ be such that every singularity and periodic orbit of X is hyperbolic. Then any continuous function that is invariant under the flow X is constant.*

Proof. Let $X \in \mathfrak{X}^1(M)$ be as above, and let $f\colon X \to \mathbb{R}$ be a continuous function which satisfies $f(X_t(x)) = f(x)$ for all $x \in M$ and $t \in \mathbb{R}$. Assume that f is non-constant. Then $f(M) = [a, b]$, with $a < b \in \mathbb{R}$. By assumption, each singularity of X is hyperbolic, hence there are finitely many of them. Let $c \in [a, b] - f(\mathrm{Zero}(X))$. For any $x \in f^{-1}(\{c\})$, it follows from Poincaré-Bendixson Theorem that $\omega(x)$ is a closed orbit formed by regular points, and by our assumption, $\omega(x)$ is hyperbolic. Moreover, $\omega(x) \subset f^{-1}(\{c\})$, since f is invariant under X. In particular, for each $c \in [a, b] - f(\mathrm{Zero}(X))$, the level set $f^{-1}(\{c\})$ contains a hyperbolic periodic orbit. This is a contradiction, since $[a, b] - f(\mathrm{Zero}(X))$ is uncountable, while there can be at most countably many hyperbolic periodic orbits. □

2.4.2 Some results in higher regularity

As we mentioned in Section 2.1, using Sard's theorem and Pesin theory we can obtain more information about the invariant functions.

Theorem 2.4.6. *Let M be a compact, connected Riemannian manifold of dimension $d \geq 1$ and let $X \in \mathfrak{X}^1(M)$. Suppose that X verifies the following conditions:*

- *every singularity and every periodic orbit of X is hyperbolic;*
- $\Omega(X) = \overline{\mathrm{Per}(X)}$.

Then any function $f \colon M_X \to \mathbb{R}$ which is X-invariant and such that $f|_{M_X}$ is of class C^d is constant.

Proof. Let $f \colon M_X \to \mathbb{R}$ be an X-invariant function such that $f|_{M_X}$ is of class C^d. By assumption, each singularity $\sigma \in \mathrm{Zero}(X)$ is hyperbolic, thus by Propositions 2.3.4 and 2.3.6, f admits a continuous extension to the whole manifold M. Suppose that f is not constant. Then, there exist two real numbers $a < b$ such that $f(M) = [a, b]$. All the singularities are hyperbolic, hence there are at most finitely many of them. In particular, $I \subset f(M) - f(\mathrm{Zero}(X))$ for some non-trivial open interval $I \subset \mathbb{R}$. Since $f|_{M_X}$ is of class C^d, then by Sard's theorem, there exists a set $R \subset I$ of full Lebesgue measure, such that each $c \in R$ is a regular value of f, that is, any $x \in f^{-1}(\{c\})$ verifies $\nabla f(x) \neq 0$.

Fix a value $c \in R - f(\mathrm{Zero}(X))$. By the same reason as in the proof of Theorem 2.4.1, we have that $f^{-1}(\{c\}) \cap \Omega(X) \neq 0$. The fact that c is a regular value implies that there exists $y \in \Omega(X) \cap M_X$ such that $\nabla f(y) \neq 0$, thus by the continuity of X and ∇f, there exists a neighbourhood $\mathcal{V} \subset M_X$ of y such that the gradient of f is non-zero at any $q \in \mathcal{V}$. Using the density of periodic points in the non-wandering set, we conclude that there exists a regular periodic point $p \in \mathrm{Per}(X) \cap \mathcal{V}$ such that $\nabla f(p) \neq 0$. By Lemma 2.4.4, we get a contradiction, since by assumption, the point p is hyperbolic. □

As a consequence of Theorem 2.4.6, we can prove Theorem G.

Proof of Theorem G. Let $X \in \mathfrak{X}^d(M)$ be as above and let $Y \in \mathfrak{C}^d(X)$. By the collinearity of $\mathfrak{C}^d(X)$, and since all the singularities of X are hyperbolic, Lemma 2.1.2 and Theorem 2.3.3 imply that $Y = fX$, where f is a X-invariant C^1 function such that $f|_{M_X}$ is of class C^d. We deduce from Theorem 2.4.6 that f is constant. Therefore, $\mathfrak{C}^d(X)$ is trivial. □

Using the ideas from [Man73], we are able to prove Theorem H.

Proof of Theorem H. By Kupka-Smale Theorem, there exists an open and dense subset $\mathcal{U}_{KS} \subset \mathfrak{X}^d(M)$ such that for any $X \in \mathcal{U}_{KS}$, any singularity of X is hyperbolic. Let $S(M)$ be the pseudometric space of subsets of M with the Hausdorff pseudometric. By [Tak71], there exists a residual subset $\mathcal{R}_d \subset \mathfrak{X}^d(M)$ such that the function $\Omega \colon \mathcal{R}_d \to S(M)$ which assigns to $X \in \mathcal{R}_d$ its non-wandering set is continuous. Let us define the residual set $\mathcal{R}_T := \mathcal{U}_{KS} \cap \mathcal{R}_d \subset \mathfrak{X}^d(M)$, and let $X \in \mathcal{R}_T$. Notice that X has finitely many singularities, since they are hyperbolic.

Suppose that X has collinear C^d-centralizer and let $Y \in \mathfrak{C}^d(X)$. By the collinearity, as a consequence of Lemma 2.1.2 and Theorem 2.3.3, we have $Y = fX$, for some X-invariant

C^1 function f such that $f|_{M_X}$ is of class C^d. Assume that f is non-constant. Then, as in the proof of Theorem 2.4.6, $f(M) - f(\text{Zero}(X))$ contains a non-trivial open interval $I \subset \mathbb{R}$. Consider a regular value $c \in I$ (this set is non-empty by Sard's theorem) and let $M_c = f^{-1}(\{c\})$. We now describe Mañé's argument from Theorem 1.2 in [Man73]. Let U be a small open neighbourhood of M_c. Since $\Omega(X) \cap U \neq \emptyset$, by the continuity of $\Omega(\cdot)$ at X, for any X' in a neighbourhood of X verifies $\Omega(X') \cap U \neq \emptyset$. Consider the gradient $\nabla f|_{M_c}$, since it is nonzero on M_c we can extend it to a vector field $V: U \to TU$ without singularities. We can take a C^1-vector field Z C^1-arbitrarily close to the zero vector field, with the following property: for any $x \in U$, $(Z(x), V(x)) > 0$. For the vector field $X' = X + Z$, it is easy to verify that $\Omega(X') \cap U = \emptyset$, a contradiction. We conclude that f is constant, and thus, $\mathfrak{C}^d(X)$ is trivial. □

Using Pesin's theory and ideas similar to the proof of Lemma 2.4.4, we can prove Theorem E.

Proof of Theorem E. Since the support of μ is the entire manifold, and by non-uniform hyperbolicity, we have that X verifies the conditions of Proposition 2.2.2, in particular, $\mathfrak{C}^1(X)$ is collinear. Let Y be a vector field in the C^1-centralizer of X. there exists a C^1-function $f: M_X \to \mathbb{R}$ such that $Y = fX$ on M_X.

Notice that M_X is a connected open and dense subset of M. If f were not constant, then there would exist a point $p \in M_X$ such that $\nabla f(p) \neq 0$. Since this condition is open we may take the point p to be a regular point of the measure μ. By Pesin's stable manifold theorem, there exists a C^1-stable manifold, $W^s_{\text{loc}}(p)$, which is tangent to $E^-(p) \oplus \langle X(p) \rangle$ on p. Similarly, there exists a C^1-unstable manifold which on p is tangent to $\langle X(p) \rangle \oplus E^+(p)$. The non-uniform hyperbolicity implies that $E^-(p) \oplus \langle X(p) \rangle \oplus E^+(p) = T_pM$.

Since p is a non-singular point, we have that $f|_{W^s_{\text{loc}}(p)} = f|_{W^u_{\text{loc}}(p)} = f(p)$. An argument similar to the one in the proof of Theorem 2.4.6 gives a contradiction and we conclude that $f|_{M_X}$ is constant. This implies that the centralizer of X is trivial. □

The C^3 centralizer of a C^3 kinematic expansive vector field

In this part we prove Theorem F. The proof is a combination of two results: Sard's Theorem and the proposition below.

Proposition 2.4.7. *Let $\mathbb{T}^2$ denote the two dimensional torus. If $X \in \mathfrak{X}^2(\mathbb{T}^2)$ and if Sing$(X) = \emptyset$ then X is not kinematic expansive.*

Proof. The argument follows closely some ideas in [Art16]. We present it here for the sake of completeness.

Assume by contradiction that there exists $X \in \mathfrak{X}^2(\mathbb{T}^2)$ a Kinematic expansive vector field. In particular it is separating. We fix $\varepsilon > 0$ to be a separation constant. Since X is C^2 we can apply Denjoy-Schwartz's Theorem [Sch63] and we have three possibilities for the dynamics:

1. each orbit is periodic and X is a suspension of the identity map $\mathrm{id}\colon \mathbb{S}^1 \to \mathbb{S}^1$;
2. there exist two distinct periodic orbits γ^s, γ^u and a non-periodic point x such that $\omega(x) = \gamma^s$ and $\alpha(x) = \gamma^u$;
3. X is a suspension of a C^3 diffeomorphism $f\colon \mathbb{S}^1 \to \mathbb{S}^1$, which is topologically conjugate to an irrational rotation.

We shall prove that each case leads us to a contradiction. In the first case, let $\tau\colon \mathbb{S}^1 \to (0, +\infty)$ be the first return time function. Then, $\tau(x)$ is the period of the orbit of x. As τ is a continuous function on the circle, there exists a maximum point x_0 and arbitrarily close to x_0 there are points x_1, x_2 such that $\tau(x_1) = \tau(x_2)$. This implies that one can choose those points so that

$$d(X_t(x_1), X_t(x_2)) \leq \varepsilon, \ \ \forall\, t \in \mathbb{R},$$

a contradiction.

Let us deal now with case (2). Fix an arbitrarily small number $\delta > 0$.

Take a small segment I transverse to X at a point $p \in \gamma^s$ and let $f\colon I \to I$ be the first return map, with $\tau\colon I \to (0, +\infty)$ the first return time function. There exists a time $T^s > 0$ such that $X_{T^s}(x) \in I$. Consider the fundamental domain $I_0^s = [f(x), x]$ for the dynamics of f and the sequence of image intervals $I_n^s = [f^{n+1}(x), f^n(x)]$, $n \geq 0$. Then, there exists $N^s > 0$ such that for $n \geq N^s$, it holds that $I_n^s \subset B(p, \delta)$. Pick $a, b \in I_0^s$ arbitrarily close.

Let $C > 0$ be the Lipschitz constant of τ. Then,

$$\left| \sum_{\ell=0}^{n} \tau(f^\ell(a)) - \sum_{\ell=0}^{n} \tau(f^\ell(b)) \right| \leq C \sum_{\ell=0}^{n} |f^\ell(a) - f^\ell(b)|.$$

The hight-hand side of above inequality is bounded by $\sum_n |I_n^s| = |I| < \infty$. Therefore, the left-hand side converges. Moreover, by continuity of f, if $d(a, b)$ is small enough then $\sum_{\ell=0}^{N^s} |f^\ell(a) - f^\ell(b)| < \delta$. Since $I_n^s \subset B(p, \delta)$ for every $n \geq N^s$, we have $\sum_{\ell=N^s}^{\infty} |f^\ell(a) - f^\ell(b)| < \delta$. We conclude that

$$\left| \sum_{\ell=0}^{\infty} \tau(f^\ell(a)) - \sum_{\ell=0}^{\infty} \tau(f^\ell(b)) \right| \leq 2C\delta.$$

Taking δ small enough, as the flow of X is the suspension of f with return time τ, we conclude that $d(X_t(a), X_t(b)) < \varepsilon$, for every $t \geq 0$.

Considering a small transverse segment to a point $q \in \gamma^u$ and arguing similarly with backwards iteration we obtain two arbitrarily close points a, b whose orbits are distinct and such that $d(X_t(a), X_t(b)) < \varepsilon$ for every $t \in \mathbb{R}$, a contradiction.

Finally, let us see that case (3) leads to a contradiction. This is essentially contained in the proof of Theorem 4.11 from [Art16] with a minor adaptation. We will sketch the main points of the proof. Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a C^3 diffeomorphism with irrational rotation number θ, and let $\tau\colon \mathbb{S}^1 \to (0,+\infty)$ be a C^1 function. It is well known that the Lebesgue measure is the only ergodic measure for an irrational rotation. Since f is C^3 by the usual Denjoy's theorem on the circle, f is conjugated with an irrational rotation, in particular, f has only one ergodic f-invariant probability measure μ.

Write $T := \int_{S^1} \tau(x)d\mu(x)$ and let $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ be the approximation of θ by rational numbers given by the continued fractions algorithm. From the corollary in [NT13], which is a version of Denjoy-Koksma inequality (Corollary C in [AK11]), we obtain the following

$$\lim_{n\to+\infty} \sup_{x\in S^1} \left| \sum_{l=0}^{q_n-1} \tau(f^l(x)) - Tq_n \right| = 0.$$

Following the same calculations in the proof of Theorem 4.11 from [Art16], for any $\epsilon > 0$ and for $n \in \mathbb{N}$ large enough, the points x and $f^{q_n}(x)$ are always ϵ-close for the future. One can argue similarly for f^{-1} and find points that are not separated for the past. Therefore, the flow cannot be Kinematic expansive. □

Remark 2.4.8. We do not know if there exists a separating suspension of an irrational rotation. The above proof shows that this is the only possibility for a separating non-singular vector field on $\mathbb{T}^2$.

Proof of Theorem F. Since all the singularities are hyperbolic, by Proposition 2.2.2 and Theorem 2.3.3, we have that $\mathfrak{C}^3(X)$ is quasi-trivial. Let $f\colon M \to \mathbb{R}$ be a C^1, X-invariant function such that $f|_{M_X}$ is C^3. We will prove that f is constant. Suppose not.

Since there are only finitely many singularities, then as in the proof of Theorem 2.4.6, if f were not constant, we would have $I \subset f(M) - f(\mathrm{Zero}(X))$, for some non-trivial open interval $I \subset \mathbb{R}$. By Sard's theorem, almost every value in I is a regular value.

Take a regular value $c \in I$. Hence, $S_c := f^{-1}(\{c\})$ is a compact surface that does not contain any singularity of X. Furthermore, since f is X-invariant, we have that $X|_{S_c}$ is a C^3 non-singular vector field on S_c. Up to considering a double orientation covering, this implies that S_c is a torus, since it is the only orientable closed surface that admits a non-singular vector field.

Notice that $X|_{S_c}$ induces a Kinematic expansive flow. However this contradicts Proposition 2.4.7. We conclude that f is constant, and this implies that the C^3-centralizer of X is trivial. □

In the higher dimensional case, and at a point of continuity of $\Omega(\cdot)$, we also have:

Proposition 2.4.9. *Let M be a compact manifold of dimension $d \geq 1$. Assume that $X \in \mathfrak{X}^d(M)$ is separating, that all its singularities are hyperbolic, and that X is a point of continuity of the map $\Omega(\cdot)$. Then the C^d-centralizer of X is trivial.*

Remark 2.4.10. As noted in the proof of Theorem H, the last two assumptions are satisfied by a residual subset of vector fields in $\mathfrak{X}^d(M)$.

Proof of Proposition 2.4.9. Since X is separating and its singularitis are hyperbolic, it follows from Proposition 2.1.4 and Theorem A that its C^1-centralizer is quasi-trivial. Take any vector field Y in the C^d-centralizer of X. By the quasi-triviality, and by Lemma 2.1.2, there exists a C^1 function $f\colon M \to \mathbb{R}$ such that $f|_{M_X}$ is of class C^d and $Y = fX$. If f is not constant, then as in the proof of Theorem H, by continuity of $\Omega(\cdot)$ at X, and by considering a regular value $c \in f(M) - f(\mathrm{Zero}(X))$ of $f|_{M_X}$, we reach a contradiction. We conclude that the C^d-centralizer is trivial. $\square$

2.5 The generic case

Our goal in this section is to prove the following theorem:

Theorem B. There exists a residual subset $\mathcal{R} \subset \mathfrak{X}^1(M)$ such that if $X \in \mathcal{R}$ then X has quasi-trivial C^1-centralizer. Furthermore, if X has at most countably many chain recurrent classes then its C^1-centralizer is trivial.

To prove this theorem, we will use a few generic results. In the following statement we summarize all the results we will need.

Theorem 2.5.1 ([BC04], [Cro06] and [PR83]). *There exists a residual subset $\mathcal{R}_*$ such that if $X \in \mathcal{R}_*$, then the following properties are verified:*

1. $\overline{\mathrm{Per}(X)} = \Omega(X) = \mathcal{CR}(X)$;
2. *every periodic orbit, and every singularity, is hyperbolic;*
3. *if C is a chain recurrent class, then there exists a sequence of periodic orbits $(\gamma_n)_{n\in\mathbb{N}}$ such that $\gamma_n \to C$ in the Hausdorff topology.*

We first prove that C^1-generically the centralizer is collinear. This proof is an adaptation for flows of Theorem *A* in [BCW09]. Once we have collinearity, using the criterion for quasi-triviality given by Theorem 2.3.3, we conclude that quasi-triviality of the C^1-centralizer is a C^1-generic property. At the end of this section we will show that for a C^1-generic vector field X that has at most countably many chain recurrent classes has trivial C^1-centralizer.

Idea of the proof of collinearity– In [BCW09] the authors prove that a version of the unbounded normal distortion holds C^1-generically for diffeomorphisms. To prove this, the key perturbative result is a perturbation made on a linear cocycle over $\mathbb{Z}$. To reduce the proof to this linear cocycle scenario, after several reductions, they introduce some change of coordinates that linearizes the dynamics around the orbit of a point for a finite time. Using the compactness of the manifold, they get uniform estimates on the C^1-norm of these changes of coordinates.

Our strategy is to reduce our problem to a perturbation of a linear cocycle over $\mathbb{Z}$. In order to do that, we study the Poincaré maps between a sequence of transverse sections. Since we are dealing with wandering points, this can be defined for a sequence of times arbitrarily large. Using these Poincaré maps we also introduce some change of coordinates to linearize the dynamics given by these maps for a finite time. However, the space where this can be defined is no longer compact, since the Poincaré map is only defined over non-singular points. Nevertheless, we can obtain uniform estimates for the C^1-norm of these change of coordinates.

We also need to prove that any perturbation of a Poincaré map, that verifies some conditions, can be realized as the Poincaré map of a perturbed vector field. All of these perturbations have to be done with precise control on the estimates that appear. These two ingredients are given in Lemma 2.5.8. Once we have that, we can adapt the proof of Bonatti-Crovisier-Wilkinson in [BCW09] and obtain that the unbounded normal distortion property is C^1-generic.

2.5.1 Unbounded normal distortion is C^1-generic

The goal of this section is to prove the following theorem:

Theorem 2.5.2. *There exists a residual subset of $\mathcal{R} \subset \mathfrak{X}^1(M)$ such that if $X \in \mathcal{R}$ then X has unbounded normal distortion.*

Linearizing coordinates

Let $X \in \mathfrak{X}^1(M)$, and as before, set $M_X := M - Sing(X)$. For $p \in M_X$ and $t \in \mathbb{R}$, for any two submanifolds Σ_1 and Σ_2 which are transverse to the orbit segment $O := X_{[0,t]}(p)$, each of which intersects O only at one point, we define the *Poincaré map* between these two transverse sections as follows: let $p_1 := O \cap \Sigma_1$ and $p_2 := O \cap \Sigma_2$. If a point $q \in \Sigma_1$ is sufficiently close to p_1, then $X_{[-t,2t]}(q)$ intersects Σ_2 at a unique point $\mathcal{P}^X_{\Sigma_1,\Sigma_2}(q)$. The map $q \mapsto \mathcal{P}^X_{\Sigma_1,\Sigma_2}(q)$ is called the Poincaré map between Σ_1 and Σ_2.

This map is a C^1-diffeomorphism between a neighbourhood of p_1 and its image. It also holds that for any vector field $Y \in \mathfrak{X}^1(M)$ sufficiently C^1-close to X, the Poincaré map $\mathcal{P}^Y_{\Sigma_1,\Sigma_2}$ for Y is well defined in some neighbourhood of p_1 in Σ_1.

Let $R > 0$ be smaller than the radius of injectivity of M. Using the exponential map, for each $p \in M_X$ and $r \in (0, R)$, we define the submanifold $\mathcal{N}_{X,p}(r) = \exp_p(N_{X,p}(r))$, where $N_{X,p}(r)$ is the ball of center 0 and radius r contained in $N_{X,p}$.

Remark 2.5.3. Considering R to be small enough, for each $p \in M_X$ and for each $q \in \mathcal{N}_{X,p}(R)$ we have that the C^1-norm of $\Pi^X_q|_{T_q\mathcal{N}_{X,p}(R)}$ is close to 1.

It is known that for each $t \in \mathbb{R}$, there exists a constant $\beta_t = \beta(X, t) > 0$ such that for any point $p \in M_X$, the Poincaré map is a C^1 diffeomorphism from $\mathcal{N}_{X,p}(\beta_t\|X(p)\|)$ to its image inside $\mathcal{N}_{X,X_t(p)}(R)$. We denote this map by $\mathcal{P}^X_{p,t}$ and we write $\beta := \beta_1$. For a fixed $\delta > 0$, we can choose β sufficiently small such that for any $p \in M_X$ and any

$q \in \mathcal{N}_{X,p}(\beta\|X(p)\|)$, it holds

$$\|D\mathcal{P}^X_{p,1}(q) - D\mathcal{P}^X_{p,1}(p)\| < \delta, \tag{2.5.1}$$

we refer the reader to Section 2.2 in [GY18] for more details. By our choices of transversals, we remark that $D\mathcal{P}^X_{p,1}(p) = P^X_{p,1}$.

Definition 2.5.4. For any $C > 1$ we say that a vector field $X \in \mathfrak{X}^1(M)$ is *bounded by* C if it holds

- $\sup\limits_{x\in M} |X(p)| < C$;
- $\sup_{p\in M} \|DX(p)\| < C$;
- $C^{-1} < \inf\limits_{x\in M} \inf\limits_{t\in[-1,1]} \|(DX_t(x))^{-1}\|^{-1} \leq \sup\limits_{x\in M} \sup\limits_{t\in[-1,1]} \|DX_t(x)\| < C$;
- $C^{-1} < \inf\limits_{p\in M_X} \inf\limits_{t\in[-1,1]} \|(P^X_{p,t})^{-1}\|^{-1} \leq \sup\limits_{p\in M_X} \sup\limits_{t\in[-1,1]} \|P^X_{p,t}\| < C$;
- there exists $\beta > 0$ small, such that

$$C^{-1} < \|(D\mathcal{P}^X_{p,1}(q))^{-1}\|^{-1} \leq \|D\mathcal{P}^X_{p,1}(q)\| < C, \text{ for any } q \in \mathcal{N}_{X,p}(\beta\|X(p)\|).$$

By (2.5.1), for any vector field $X \in \mathfrak{X}^1(M)$, there is a constant $C > 1$ such that X is bounded by C.

Let $X \in \mathfrak{X}^1(M)$ be a vector field bounded by $C > 1$. Using the exponential map, for $p \in M_X$, we consider the *lifted Poincaré map*

$$\widetilde{\mathcal{P}}^X_{p,1} = \exp^{-1}_{X_1(p)} \circ \mathcal{P}^X_{p,1} \circ \exp_p,$$

which goes from $N_{X,p}(\beta\|X(p)\|)$ to $N_{X,X_1(p)}(R)$. Observe that

$$\|X(X_1(p))\| > C^{-1}\|X(p)\|. \tag{2.5.2}$$

By (2.5.2) and the last item in Definition 2.5.4, for any $n \in \mathbb{N}$, the map $\mathcal{P}^X_{p,n}$ is well defined on $\mathcal{N}_{X,p}\big(\frac{\beta}{C^n}\|X(p)\|\big)$, while the lifted map $\widetilde{\mathcal{P}}^X_{p,n}$ is well defined on $V^X_{p,n} := N_p\big(\frac{\beta}{C^n}\|X(p)\|\big)$.

For each $n \in \mathbb{N}$ and $p \in M_X$, we define the change of coordinates $\psi_{p,n} = P^X_{X_{-n}(p),n} \circ (\widetilde{\mathcal{P}}^X_{X_{-n}(p),n})^{-1}$, which is a C^1 diffeomorphism from $\widetilde{\mathcal{P}}^X_{p,n}(V^X_{X_{-n}(p),n})$ to $P^X_{X_{-n}(p),n}(V^X_{X_{-n}(p),n}) \subset N_{X,p}$. Observe that $\psi_{p,0} = \mathrm{id}$. The sequence $(\psi_{X_j(p),j})_{j\in\mathbb{N}}$ verifies the following equality:

$$\psi_{X_n(p),n} \circ \widetilde{\mathcal{P}}^X_{p,n} = P^X_{p,n} \circ \psi_{p,0},$$

which holds on $V^X_{p,n}$. In other words, this change of coordinates linearizes the dynamics of $\widetilde{\mathcal{P}}^X_{p,n}$.

For all $y \in \mathcal{N}_{X,p}(\frac{\beta}{C^n}\|X(p)\|)$, we define the *hitting time* $\tau^X_{p,n}(y)$ as the first positive time where the trajectory starting at y hits the transverse section $\mathcal{N}_{X,X_n(p)}(R)$, that is,

$$\tau^X_{p,n}(y) := \min\{t \geq 0 : X_t(y) \in \mathcal{N}_{X,X_n(p)}(R)\}.$$

Notation. Let $p \in M_X$ and $n \in \mathbb{N}$. Suppose that for $Y \in \mathfrak{X}^1(M)$ the submanifolds $\mathcal{N}_{X,p}\big(\frac{\beta}{C^n}\|X(p)\|\big)$ and $\mathcal{N}_{X,X_n(p)}(R)$ are transverse to Y, and that the Poincaré map for Y between these transverse sections is well defined on $\mathcal{N}_{X,p}\big(\frac{\beta}{C^n}\|X(p)\|\big)$. Then we denote this Poincaré map for Y by $\mathcal{P}^Y_{X,p,n}$. Accordingly, we denote its lift by $\widetilde{\mathcal{P}}^Y_{X,p,n}$ and its hitting time by $\tau^Y_{X,p,n}$. We also extend those notations for non-integer times: given an integer $n \geq 1$ and $t \in [n-1, n]$, we let $\mathcal{P}^Y_{X,p,t}$ be the Poincaré map between the transversals $\mathcal{N}_{X,p}\big(\frac{\beta}{C^n}\|X(p)\|\big)$ and $\mathcal{N}_{X,X_t(p)}(R)$.

In the next definition we introduce the type of perturbations of the Poincaré map that we will consider in the sequel.

Definition 2.5.5. For each $\delta > 0$ and given an open set $U \subset \mathcal{N}_{X,p}(\beta\|X(p)\|)$, a C^1 map $g\colon \mathcal{N}_{X,p}(\beta\|X(p)\|) \to \mathcal{N}_{X,X_1(p)}(R)$ is called a *δ-perturbation of $\mathcal{P}^X_{p,1}$ with support in U* if the following holds:

- $d_{C^1}(\mathcal{P}^X_{p,1}, g) < \delta$;
- the image of g coincides with the image of $\mathcal{P}^X_{p,1}$;
- the map g is a C^1 diffeomorphism into its image;
- the support of $(\mathcal{P}^X_{p,1})^{-1} \circ g$ is contained in U.

For any $n \in \mathbb{N}$ and any $U \subset \mathcal{N}_{X,p}\big(\frac{\beta}{C^n}\|X(p)\|\big)$, we define

$$\mathcal{I}^X(p, U, n) := \{(y, t) : y \in U,\ t \in [0, \tau^X_{p,n}(y)]\}, \tag{2.5.3}$$

and we let $\mathcal{U}^X(p, U, n)$ be the image of $\mathcal{I}^X(p, U, n)$ under the map $(y, t) \mapsto X_t(y)$:

$$\mathcal{U}^X(p, U, n) := \bigcup_{y \in U} \bigcup_{t \in [0, \tau^X_{p,n}(y)]} X_t(y). \tag{2.5.4}$$

Remark 2.5.6. For a vector field $X \in \mathfrak{X}^1(M)$ we can fix a constant $\alpha = \alpha(X)$ small enough such that for any $t \in [-\alpha, \alpha]$ and $p \in M_X$, it holds that $|\det P^X_{p,t} - 1| < \frac{\log 2}{2}$.

Remark 2.5.7. Let $\alpha > 0$ be as in Remark 2.5.6. Then for $\beta > 0$ sufficiently small, for any $p \in M_X$ and $q \in \mathcal{N}_{X,p}\big(\frac{\beta}{C^n}\|X(p)\|\big)$, it holds that $\tau^X_{p,n}(q) \in [n - \alpha, n + \alpha]$. From now on we will always assume that β verifies this condition for this choice of α.

A realization lemma

We state and prove below a lemma that allows us to realise a non-linear perturbation of the linear Poincaré flow as the lifted Poincaré map of a vector field nearby.

Lemma 2.5.8. *For any $C, \varepsilon > 0$, there exists $\delta = \delta(C, \varepsilon) > 0$ that verifies the following. For any vector field $X \in \mathfrak{X}^1(M)$ that is bounded by C, any $0 < \delta_1 < \delta$ and any integer $n \in \mathbb{N}$, there is $\rho = \rho(X, \varepsilon, \delta_1) > 0$ with the following property.*

For any $p \in M_X$ and $U \subset \mathcal{N}_{X,p}(\rho\|X(p)\|)$ such that the map $(y, t) \mapsto X_t(y)$ restricted to the set $\mathcal{I}^X(p, U, n)$ is injective, then the following holds:

1. *Set $\widetilde{U} := \exp_p^{-1}(U)$. Then for every $i \in \{0, \dots, n\}$, the map $\Psi_{X_i(p),i} := \psi_{X_i(p),i} \circ \exp_{X_i(p)}^{-1}$ induces a C^1 diffeomorphism from $\mathcal{P}^X_{p,i}(U)$ onto $P^X_{p,i}(\widetilde{U})$ such that*

$$\max\{\|D\Psi_{X_i(p),i}\|, \|D\Psi^{-1}_{X_i(p),i}\|, |\det D\Psi_{X_i(p),i}|, |\det D\Psi^{-1}_{X_i(p),i}|\} < 2. \tag{2.5.5}$$

2. *For $i \in \{1, \dots, n\}$, let $\tilde{g}_i \colon N_{X,X_{i-1}(p)} \to N_{X,X_i(p)}$ be any C^1 diffeomorphism such that the support of $(P^X_{X_i(p),1})^{-1} \circ \tilde{g}_i$ is contained in $P^X_{p,i-1}(\widetilde{U})$, and which satisfies $d_{C^1}(\tilde{g}_i, P^X_{X_{i-1}(p),1}) < \delta_1$. Let g_i be the map defined as follows:*
   - *$g_i(y) := \mathcal{P}^X_{X_{i-1}(p),1}(y)$, if $y \notin \mathcal{P}^X_{p,i-1}(U)$;*
   - *$g_i(y) := \Psi^{-1}_{X_i(p),i} \circ \tilde{g}_i \circ \Psi_{X_{i-1}(p),i-1}(y)$, if $y \in \mathcal{P}^X_{p,i-1}(U)$.*

   Then the map g_i is a δ-perturbation of $\mathcal{P}^X_{X_{i-1}(p),1}$ with support in $\mathcal{P}^X_{p,i-1}(U)$.

3. *There exists $Y \in \mathfrak{X}^1(M)$ such that $d_{C^1}(X, Y) < \varepsilon$, and the Poincaré map $\mathcal{P}^Y_{X,X_i(p),1}$ for the vector field Y between $\mathcal{N}_{X_{i-1}(p)}(\rho\|X(p)\|)$ and $\mathcal{N}_{X_i(p)}(R)$ is well defined and is given by g_i, for each $i \in \{1, \dots, n\}$. Moreover, the support of $X - Y$ is contained in $\mathcal{U}^X(p, U, n)$ and the image of $\tau^Y_{X,p,n}$ coincides with the image of $\tau^X_{p,n}$. In particular, it is contained in $[n - \alpha, n + \alpha]$.*

Before proving this lemma, let us say a few words on items 2 and 3 in the statement. Item 2 states that we can obtain perturbations of the Poincaré map by perturbing its lift, with precise estimates on the size of each of these perturbations we consider. Observe that this only gives C^1 diffeomorphisms between certain transverse sections. Item 3 states that any such perturbation can be realized as the Poincaré map of a vector field C^1-close to X, with precise estimates on its distance to X. Furthermore, the hitting time is "the same" as the hitting time of X, in the sense that they have the same image as a function of the transverse section to $\mathbb{R}$. These two properties will be crucial in our proof, because it will allow us to reduce the proof of the theorem to the discrete case, after several adaptations.

Proof. We will obtain δ later, as consequence of a finite number of inequalities. In the following, we always assume that $0 < \rho \leq \frac{\beta}{C^n}$. By the previous discussion, this ensures that $\mathcal{P}^X_{p,n}$ is well defined on $\mathcal{N}_{X,p}\big(\rho\|X(p)\|\big)$, for all $p \in M_X$.

Point (1) follows from the following facts:

- It holds

$$C^{-n} < \inf_{p \in M_X,\ t \in [-n,n]} \|(P^X_{p,t})^{-1}\|^{-1} \leq \sup_{p \in M_X,\ t \in [-n,n]} \|P^X_{p,t}\| < C^n, \tag{2.5.6}$$

and by (2.5.1), we have similar estimates for the Poincaré maps $\mathcal{P}^X_{p,t}$, uniformly in $p \in M_X$ and $t \in [-n, n]$.

- By choosing $\rho > 0$ sufficiently small, the set

$$\bigcup_{y \in N_p(\rho\|X(p)\|)} \bigcup_{t\in[0,n]} P^X_{p,t}(y)$$

can be made arbitrarily close to the 0 section, uniformly in $p \in M_X$. Similarly, the set $\mathcal{U}^X(p, \mathcal{N}_{X,p}(\rho\|X(p)\|), n)$ defined in (2.5.4) can be made arbitrarily close to the orbit segment $\{X_t(p) : 0 \leq t \leq n\}$, uniformly in $p \in M_X$.

- The map $D\exp_p^{-1}$ is uniformly close to the identity in a neighbourhood of p.

- Since the vector field X is of class C^1 and by choosing $\rho > 0$ sufficiently small, the maps $\psi_{X_i(p),i}$ used to linearize the dynamics can be made uniformly C^1-close to the identity for $i \in \{0, \ldots, n\}$ and $p \in M_X$. Therefore, the map $\Psi_{X_i(p),i}$ can be made arbitrarily C^1 close to $\exp^{-1}_{X_i(p)}$.

In particular, we obtain a uniform control of $\Psi_{X_i(p),i}$ for $p \in M_X$ and $i \in \{0, \ldots, n\}$ even though the space M_X is not compact.

By Definition 2.5.5, the proof of (2) follows easily from the first point. Indeed, given $i \in \{1, \ldots, n\}$ and $p \in M_X$, we use the maps $\Psi_{X_{i-1}(p),i-1}$ and $\Psi_{X_i(p),i}$ to conjugate $\mathcal{P}^X_{X_{i-1}(p),1}$ to the linear Poincaré map $P^X_{X_{i-1}(p),1}$. By the previous discussion, for $\rho > 0$ small enough, the maps $\Psi_{X_{i-1}(p),i-1}$ and $\Psi_{X_i(p),i}$ are arbitrarily C^1-close to $\exp^{-1}_{X_{i-1}(p)}$ and $\exp^{-1}_{X_i(p)}$ respectively. The estimate on the C^1 distance between g_i and $\mathcal{P}^X_{X_{i-1}(p),1}$ follows, since we assume $d_{C^1}(\tilde{g}_i, P^X_{X_{i-1}(p),1}) < \delta_1$, and $\delta_1 < \delta$.

The proof of point (3) follows from arguments similar to those presented in Pugh-Robinson [PR83] (see in particular Lemma 6.5 in that paper).

More precisely, let $i \in \{1, \ldots, n\}$, and let $\tilde{g}_i \colon N_{X_{i-1}(p)} \to N_{X_i(p)}$ be a C^1 diffeomorphism satisfying the assumptions of point (2). We pull back $\tilde{g}_i$ to a C^1 diffeomorphism $\hat{g}_i \colon N_{X_{i-1}(p)} \to N_{X_{i-1}(p)}$ by letting $\hat{g}_i := P^X_{X_i(p),-1} \circ \tilde{g}_i$. By assumption, the support of $\hat{g}_i$ is contained in $P^X_{p,i-1}(\widetilde{U})$, with $\widetilde{U} := \exp_p^{-1}(U)$ and $U \subset \mathcal{N}_{X,p}(\rho\|X(p)\|)$, hence by (2.5.6), we get

$$d_{C^0}(\hat{g}_i, \mathrm{id}) \leq 2C\rho \max_{p\in M} \|X(p)\|. \tag{2.5.7}$$

Then for all $t \in [i-1, i]$, we define a map $\tilde{g}_t \colon N_{X_{i-1}(p)} \to N_{X_t(p)}$ as $\tilde{g}_t := P^X_{X_{i-1}(p),t-i+1} \circ \hat{g}_i$. By the above estimate, and by (2.5.6), we deduce that

$$d_{C^0}(\tilde{g}_t, P^X_{X_{i-1}(p),t-i+1}) \leq 2C^2\rho \max_{p\in M} \|X(p)\|, \qquad \forall t \in [i-1, i]. \tag{2.5.8}$$

Moreover, for any $t \in [i-1, i]$, we have $D\tilde{g}_t = P^X_{X_{i-1}(p),t-i+1} \cdot D\hat{g}_i = P^X_{X_{i-1}(p),t-i+1} \circ P^X_{X_i(p),-1} \cdot D\tilde{g}_i$. Since $d_{C^1}(\tilde{g}_i, P^X_{X_{i-1}(p),1}) < \delta_1$, we obtain

$$d_{C^1}(\tilde{g}_t, P^X_{X_{i-1}(p),t-i+1}) \leq C^2\delta_1, \qquad \forall t \in [i-1, i]. \tag{2.5.9}$$

Let us fix a C^∞ bump function $\chi\colon \mathbb{R} \to [0,1]$ which is 0 near 0 and 1 near 1. Fix $i \in \{1,\dots,n\}$ and set $\chi_{i-1}(\cdot) := \chi(\cdot - i + 1)$. For $k \in \{0,\dots,n\}$, we also let $\mathcal{N}_{p,k} := \mathcal{N}_{X,X_k(p)}\big(\frac{\beta}{C^{n-k}}\|X(p)\|\big)$. Then for any $t \in [i-1,i]$, we let $h_t^{(i)}\colon \mathcal{N}_{p,i-1} \to \mathcal{N}_{p,i}$ be the map defined as

- $h_t^{(i)}(y) := \mathcal{P}^X_{X_{i-1}(p),t-i+1}(y)$, if $y \notin \mathcal{P}_p^{i-1}(U)$;
- $h_t^{(i)}(y) := \Psi^{-1}_{X_t(p),t} \circ \Big(\chi_{i-1}(t)\tilde{g}_t + (1-\chi_{i-1}(t))P^X_{X_{i-1}(p),t-i+1}\Big) \circ \Psi_{X_{i-1}(p),i-1}(y)$, if $y \in \mathcal{P}^X_{p,i-1}(U)$,

where we have extended the previous notation by setting

$$\Psi_{X_t(p),t} := P^X_{p,t} \circ \widetilde{\mathcal{P}}^X_{X_t(p),-t} \circ \exp^{-1}_{X_t(p)}.$$

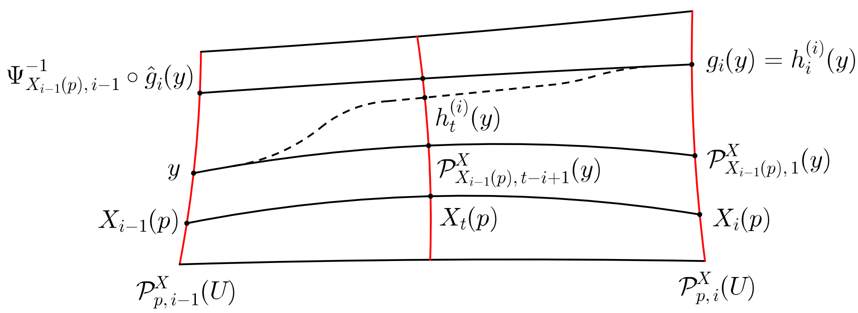

Figure 2.3: Interpolation between the initial Poincaré map and g_i.

In particular, we note that for $t = i-1$, we have $h_t^{(i)} = h_{i-1}^{(i)} = \mathrm{id}$, while for $t = i$, $h_t^{(i)} = h_i^{(i)}$ coincides with the map g_i defined in item (2).

By (2.5.8), for all $t \in [i-1,i]$, we have

$$d_{C^0}\Big(\chi_{i-1}(t)\tilde{g}_t + (1-\chi_{i-1}(t))P^X_{X_{i-1}(p),t-i+1}, P^X_{X_{i-1}(p),t-i+1}\Big) \leq 2C^2\rho \max_{p\in M}\|X(p)\|.$$

Since $\mathcal{P}^X_{X_{i-1}(p),t-i+1} = \Psi^{-1}_{X_t(p),t} \circ P^X_{X_{i-1}(p),t-i+1} \circ \Psi_{X_{i-1}(p),i-1}$, by the definition of $h_t^{(i)}$ and by (2.5.5), we can thus make the C^0 distance between $h_t^{(i)}$ and $\mathcal{P}^X_{X_{i-1}(p),t-i+1}$ arbitrarily small, provided that $\rho > 0$ is taken small enough.

For any $t \in [i-1,i]$, we have

$$\begin{aligned} &D\Big(\chi_{i-1}(t)\tilde{g}_t + (1-\chi_{i-1}(t))P^X_{X_{i-1}(p),t-i+1}\Big) \\ &= DP^X_{X_{i-1}(p),t-i+1} + \chi_{i-1}(t)\big(D\tilde{g}_t - P^X_{X_{i-1}(p),t-i+1}\big). \end{aligned}$$

By (2.5.6) and (2.5.9), we thus get

$$d_{C^1}(h_t^{(i)}, \mathcal{P}^X_{X_{i-1}(p),t-i+1}) \le 4C^2\delta_1. \tag{2.5.10}$$

For any $t \in [i-1, i]$, we also have:

$$\begin{aligned}
&\partial_t \left(\chi_{i-1}(t)\tilde{g}_t + (1-\chi_{i-1}(t))P^X_{X_{i-1}(p),t-i+1}\right) - \partial_t P^X_{X_{i-1}(p),t-i+1} \\
&= \chi'_{i-1}(t)\big(\tilde{g}_t - P^X_{X_{i-1}(p),t-i+1}\big) + \chi_{i-1}(t)\partial_t\big(\tilde{g}_t - P^X_{X_{i-1}(p),t-i+1}\big) \\
&= \chi'_{i-1}(t)P^X_{X_{i-1}(p),t-i+1} \circ \big(\hat{g}_i - \mathrm{id}\big) + \chi_{i-1}(t)\partial_t P^X_{X_{i-1}(p),t-i+1} \circ \big(\hat{g}_i - \mathrm{id}\big).
\end{aligned}$$

By (2.5.5), (2.5.6) and (2.5.7), we deduce that

$$\begin{aligned}
&\max_{t\in[i-1,i]} \max_{y\in U} |\partial_t \mathcal{P}^X_{X_{i-1}(p),t-i+1}(y) - \partial_t h_t^{(i)}(y)| \\
&\le 8C \max\left(C, \sup_{t\in[0,1]} \|\partial_t P^X_{X_{i-1}(p),t}\|\right) \|\chi\|_{C^1} \rho \max_{p\in M} \|X(p)\|.
\end{aligned} \tag{2.5.11}$$

Recall that for $k \in \{0, \dots, n\}$, we denote $\mathcal{N}_{p,k} := \mathcal{N}_{X,X_k(p)}\big(\frac{\beta}{C^{n-k}}\|X(p)\|\big)$. As in (2.5.3), given a set $V \subset \mathcal{N}_{p,0}$, we set

$$\mathcal{I}^X(p, V, n) := \left\{(y,t) : y \in V,\ t \in [0, \tau^X_{p,n}(y)]\right\}.$$

Let us assume that $U \subset \mathcal{N}_{X,p}(\rho\|X(p)\|)$ is such that the map $(y,t) \mapsto X_t(y)$ is injective on the set $\mathcal{I}^X(p,U,n)$. For $\rho > 0$ small, the hitting time function $\tau^X_{p,n}$ is uniformly close to n on $\mathcal{N}_{X,p}(\rho\|X(p)\|)$, and the C^1 distance between the maps $(y,t) \mapsto \mathcal{P}^X_{p,t}(y)$ and $(y,t) \mapsto X_t(y)$ restricted to $\mathcal{I}^X(p, \mathcal{N}_{X,p}(\rho\|X(p)\|), n)$ is small. Given $i \in \{1, \dots, n\}$, let us consider the map $h^{(i)} \colon (y,t) \mapsto h_t^{(i)}(y)$ defined on $\mathcal{N}_{p,i-1} \times [i-1, i]$ as above. By (2.5.10) and (2.5.11), and since $0 < \delta_1 < \delta$, the maps $\mathcal{N}_{p,i-1} \times [i-1,i] \ni (y,t) \mapsto \mathcal{P}^X_{X_{i-1}(p),t-i+1}(y)$ and $h^{(i)}$ can be made arbitrarily C^1-close by taking $\delta > 0$ small enough. For $\delta > 0$ sufficiently small, we deduce that the map $h^{(i)}$ is locally injective on the interior of $\mathcal{P}^X_{p,i-1}(U) \times [i-1,i]$. Besides, as we have seen, $h^{(i)}_{i-1}|_{\mathcal{N}_{p,i-1}} = \mathrm{id}|_{\mathcal{N}_{p,i-1}}$, while $h^{(i)}_i|_{\mathcal{N}_{p,i-1}} = g_i|_{\mathcal{N}_{p,i-1}}$ is a C^1 diffeomorphism.

Now, we define a map H on $\mathcal{N}_{p,0} \times [0,n]$ by setting

$$\begin{aligned}
&H(y,t) := h_t^{(i)} \circ g_{i-1} \circ g_{i-2} \circ \cdots \circ g_1(y), \\
&\forall y \in \mathcal{N}_{p,0},\ t \in [i-1,i],\ i \in \{1, \dots, n\}.
\end{aligned} \tag{2.5.12}$$

By what precedes, the map H is locally injective on the interior of the set $U \times [0,n]$. Moreover, $\partial\,(U \times [0,n]) = (U \times \{0\}) \cup (U \times \{n\}) \cup (\partial U \times [0,n])$. On the one hand, we have $H(\cdot, 0)|_U = \mathrm{id}|_U$, and by construction, the map $H(\cdot, n)|_U$ coincides with $g_n \circ g_{n-1} \circ \cdots \circ g_1|_U$, hence it is a C^1 diffeomorphism from U to $\mathcal{P}^X_{p,n}(U) \subset \mathcal{N}_{p,n}$. On the other hand, by point (2), each diffeomorphism g_i is a δ_2-perturbation of $\mathcal{P}^X_{X_{i-1}(p),1}$ with support in $\mathcal{P}^X_{p,i-1}(U)$. Therefore the restriction of H to the set $\partial U \times [0,n]$ coincides with the restriction of the map $(y,t) \mapsto \mathcal{P}^X_{p,t}(y)$. In particular, we deduce that the restriction $H|_{\partial(U\times[0,n])}$ of H to the boundary of $U \times [0,n]$ is injective. From Lemma 6.5 in Pugh-Robinson

[PR83], we conclude that H embeds $U\times[0,n]$ into the set $\mathcal{U}^X(p,U,n)$ introduced in (2.5.4).

In the same way as before, for any $y\in\mathcal{N}_{X,p}(\rho\|X(p)\|)$ and $t\in[0,n]$, we set

$$\tau^X_{p,t}(y):=\min\{s\geq 0: X_s(y)\in\mathcal{N}_{X,X_t(p)}(R)\}.$$

By definition, $\mathcal{P}^X_{p,t}(y)=X_{\tau^X_{p,t}(y)}(y)$, for any $(y,t)\in\mathcal{N}_{X,p}(\rho\|X(p)\|)\times[0,n]$, thus

$$X(\mathcal{P}^X_{p,t}(y))=(\partial_t\tau^X_{p,t}(y))^{-1}\partial_t\mathcal{P}^X_{p,t}(y). \tag{2.5.13}$$

Moreover, $\tau^X_{p,\cdot}(p)=\mathrm{id}$, and the map $(y,t)\mapsto\tau^X_{p,t}(y)$ is C^1 on $\mathcal{N}_{X,p}(\rho\|X(p)\|)\times[0,n]$, hence for $\rho>0$ sufficiently small, we have

$$\frac{1}{2}<|\partial_t\tau^X_{p,t}(y)|<2,\qquad \forall p\in M_X,\ y\in\mathcal{N}_{X,p}(\rho\|X(p)\|),\ t\in[0,n]. \tag{2.5.14}$$

As we have noted above, on the complement of $U\times[0,n]$, the maps H and $(y,t)\mapsto\mathcal{P}^X_{p,t}(y)$ coincide. We thus define a vector field $Y\in\mathfrak{X}^1(M)$ on M by setting

- $Y(q):=X(q)$, if $q\in M-\mathcal{U}^X(p,U,n)$;
- $Y(q):=(\partial_t|_{t=t_0}\tau^X_{p,t}(y))^{-1}\partial_t|_{t=t_0}H(y_0,t)$, if $q\in\mathcal{U}^X(p,U,n)$, where $(y_0,t_0):=H^{-1}(q)\in U\times[0,n]$.

For each $i\in\{1,\dots,n\}$, by the definition of H in (2.5.12) and since $h^{(i)}_i=g_i$, it follows that the Poincaré map $\mathcal{P}^Y_{X,X_{i-1}(p),1}$ for the vector field Y between $\mathcal{N}_{p,i-1}$ and $\mathcal{N}_{p,i}$ is given by g_i. By definition, the support of $X-Y$ is contained in $\mathcal{U}^X(p,U,n)$. Moreover, given any point $q=\mathcal{P}^X_{p,t}(y)=H(y',t)\in\mathcal{U}^X(p,U,n)$, say $(y,t)\in U\times[i-1,i]$, letting $z:=\mathcal{P}^X_{p,i-1}(y)$ and $z':=g_{i-1}\circ g_{i-2}\circ\cdots\circ g_1(y')$, we obtain

$$\begin{aligned}
\mathcal{P}^X_{p,t}(y)&=\mathcal{P}^X_{X_{i-1}(p),t-i+1}(z)\\
&=\Psi^{-1}_{X_t(p),t}\circ P^X_{X_{i-1}(p),t-i+1}\circ\Psi_{X_{i-1}(p),i-1}(z);\\
H(y',t)&=h^{(i)}_t(z')\\
&=\Psi^{-1}_{X_t(p),t}\circ\Big(\chi_{i-1}(t)\tilde{g}_t+(1-\chi_{i-1}(t))P^X_{X_{i-1}(p),t-i+1}\Big)\circ\Psi_{X_{i-1}(p),i-1}(z')\\
&=\Psi^{-1}_{X_t(p),t}\circ P^X_{X_{i-1}(p),t-i+1}\circ(\chi_{i-1}(t)(\hat{g}_i-\mathrm{id})+\mathrm{id})\circ\Psi_{X_{i-1}(p),i-1}(z').
\end{aligned}$$

Set

$$w:=\Psi_{X_{i-1}(p),i-1}(z)=(\chi_{i-1}(t)(\hat{g}_i-\mathrm{id})+\mathrm{id})\circ\Psi_{X_{i-1}(p),i-1}(z').$$

We deduce that

$$\begin{aligned}
\partial_t\mathcal{P}^X_{p,t}(y)&=\partial_t\left(\Psi^{-1}_{X_t(p),t}\circ P^X_{X_{i-1}(p),t-i+1}\right)(w),\\
\partial_t H(y',t)&=\partial_t\left(\Psi^{-1}_{X_t(p),t}\circ P^X_{X_{i-1}(p),t-i+1}\right)(w)+D_w\left(\Psi^{-1}_{X_t(p),t}\circ P^X_{X_{i-1}(p),t-i+1}\right)\\
&\qquad\cdot\partial_t\left((\chi_{i-1}(t)(\hat{g}_i-\mathrm{id})+\mathrm{id})\circ\Psi_{X_{i-1}(p),i-1}(z')\right)\\
&=\partial_t\mathcal{P}^X_{p,t}(y)+\chi'_{i-1}(t)D_{\Psi_{X_t(p),t}(q)}\Psi^{-1}_{X_t(p),t}\circ P^X_{X_{i-1}(p),t-i+1}\circ\\
&\qquad\circ(\hat{g}_i-\mathrm{id})\left(\Psi_{X_{i-1}(p),i-1}(z')\right),
\end{aligned}$$

and

$$Y(q) - X(q) = \frac{\chi'_{i-1}(t)}{\partial_t \tau^X_{p,t}(y)} D_{\Psi_{X_t(p),t}(q)} \Psi^{-1}_{X_t(p),t} \circ P^X_{X_{i-1}(p),t-i+1} \circ (\hat{g}_i - \mathrm{id}) \left(\Psi_{X_{i-1}(p),i-1}(z')\right),$$

where the last equality follows from (2.5.13) and the definition of Y. In particular, the difference between the vector fields X and Y is essentially controlled by the C^0 distance between $\hat{g}_i$ and id. More precisely, by (2.5.5), (2.5.6), (2.5.7), and (2.5.14), we deduce that

$$|X(q) - Y(q)| \leq 8\|\chi\|_{C^1} C^2 \rho \max_{p\in M} \|X(p)\|,$$

and we argue similarly for the derivatives. Therefore, by taking ρ sufficiently small, we can ensure that $d_{C^1}(X, Y) < \varepsilon$, which concludes the proof of point (3), and then, of Lemma 2.5.8. $\square$

Producing unbounded normal distortion by perturbation

We are now in position to prove the main perturbation result (Proposition 2.5.11 below) that will allow us to obtain unbounded normal distortion generically. The key tool behind this is a perturbation result for linear cocycles taken from [BCW09].

Proposition 2.5.9. *For any $d \geq 2$, $C > 1$, $K, \varepsilon > 0$, let $\delta = \delta(C, \varepsilon)$ be the constant given by Lemma 2.5.8. There exists $n_0 = n_0(d, C, K, \varepsilon) \in \mathbb{N}$ with the following property.*

For any d-dimensional manifold M, any vector field $X \in \mathfrak{X}^1(M)$ which is bounded by C, there exists $\rho_0 = \rho_0(d, C, K, \varepsilon) > 0$ such that for any $\eta > 0$, any compact set $\Delta \subset M_X$ and $x, p \in M_X$ satisfying:

(a) there exists an open set U inside $\mathcal{N}_{X,p}(\rho_0\|X(p)\|)$, such that $\Delta \subset U$;

(b) the map $(y, t) \mapsto X_t(y)$ is injective on $\mathcal{I}^X(p, U, n_0)$ (see (2.5.3));

(c) $\mathrm{orb}^X(x) \cap U = \emptyset$,

there exists a vector field $Y \in \mathfrak{X}^1(M)$ such that

1. *the support of $X - Y$ is contained in $\mathcal{U}^X(p, U, n_0)$ (see (2.5.4));*
2. *$d_{C^1}(X, Y) < \varepsilon$;*
3. *for any $i \in \{0, \ldots, n_0 - 1\}$, it is verified $d_{C^1}(\mathcal{P}^X_{X_i(p),1}, \mathcal{P}^Y_{X,X_i(p),1}) < \delta$, where $\mathcal{P}^Y_{X,X_i(p),1}$ is the Poincaré map between $\mathcal{N}_{X,X_i(p)}(\beta\|X(X_i(p))\|)$ and $\mathcal{N}_{X,X_{i+1}(p)}(R)$;*
4. *$d_{C^0}(\mathcal{P}^X_{X_i(p),t}, \mathcal{P}^Y_{X,X_i(p),t}) < \eta$, for all $t \in [0, 1]$;*
5. *for all $y \in \Delta$, there exists an integer $n \in \{1, \ldots, n_0\}$ such that*

$$|\log\det P^Y_{x,n} - \log\det P^Y_{y,n}| > K.$$

Proposition 2.5.9 is the analogous for flows of Proposition 8 in [BCW09]. Using Lemma 2.5.8, we will reduce the proof of this proposition to a discrete scenario where we can apply the following proposition from [BCW09].

Proposition 2.5.10 (Proposition 9 in [BCW09]). *For any $d \geq 1$, and any $C, K, \varepsilon > 0$, there exists $n_1 = n_1(d, C, K, \varepsilon) \geq 1$ with the following property.*

Consider any sequence $(A_i) \in \mathrm{GL}(d, \mathbb{R})$ with $\|A_i\|, \|A_i^{-1}\| < C$ and the associated cocycle $\tilde{f}$ on $\mathbb{Z} \times \mathbb{R}^d$ defined by $\tilde{f}(i, v) := (i+1, A_i v)$. Then, for any open set $U \subset \mathbb{R}^d$, for any compact set $\Delta \subset U$ and any $\eta > 0$, there exists a diffeomorphism $\tilde{g}$ of $\mathbb{Z} \times \mathbb{R}^d$ such that:

- $d_{C^1}(\tilde{f}, \tilde{g}) < \varepsilon$*;*
- $d_{C^0}(\tilde{f}, \tilde{g}) < \eta$*;*
- $\tilde{f} = \tilde{g}$ *on the complement of* $\bigcup_{i=0}^{2n_1-1} \tilde{f}^i(\{0\} \times U)$*;*
- *for all* $y \in \{0\} \times \Delta$*, there exists* $n \in \{1, \dots, n_1\}$ *such that*

$$|\log\det D\tilde{f}^n(y) - \log\det D\tilde{g}^n(y)| > K.$$

Proof of Proposition 2.5.9 from Proposition 2.5.10. Fix any $\delta_1 \in (0, \delta)$ and $K_0 > 2K + 10\log 2$. Let $n_1 = n_1(d-1, C, K_0, \delta_1)$ be the constant given by Proposition 2.5.10 for $d-1$, C, K_0, ε and let $n_0 = 2n_1$. Let $X \in \mathfrak{X}^1(M)$ be a vector field bounded by C and let $\rho > 0$ be the constant given by Lemma 2.5.8 for C, ε, δ_1, n_0 and X. Fix $\rho_0 \in (0, \frac{\rho}{C^{n_0}})$.

Let $\Delta \subset M_X$, $x, p \in M_X$ and $\eta > 0$ be such that conditions (a), (b) and (c) in Proposition (2.5.9) are verified. Let $U \subset \mathcal{N}_{X,p}(\rho_0\|X(x)\|)$ be the open set given by condition (a). Consider $O_{X_1}(p) = \{\dots, X_{-1}(p), p, X_1(p), \dots\}$ and observe that this set is naturally identified with $\mathbb{Z}$. We consider the normal bundle, with respect to X, over $O_{X_1}(p)$ and the linear cocycle defined as follows: for $i \in \mathbb{Z}$ and $v \in N_{X, X_i(p)}$ set $\tilde{f}(i, v) := (i+1, P^X_{X_i(p),1} v)$.

Recall that $\widetilde{U} = \exp_p^{-1}(U)$. By item (1) in Lemma 2.5.8, for any $i \in \{0, \dots, n_0\}$, we obtain C^1 diffeomorphisms $\Psi_i := \Psi_{X_i(p),i} \colon \mathcal{P}^X_{p,i}(U) \to P^X_{p,i}(\widetilde{U})$, such that for any $q \in \mathcal{P}^X_{p,i}(U)$ it holds that

$$P^X_{X_i(p),1}(\Psi_i(q)) = \Psi_{i+1}(\mathcal{P}^X_{X_i(p),1}(q)). \tag{2.5.15}$$

Write $\Psi \colon \bigcup_{i=0}^{n_0} \mathcal{P}^X_{p,i}(U) \to \bigcup_{i=0}^{n_0} P^X_{p,i}(\widetilde{U})$ as the C^1 diffeomorphism which is equal to Ψ_i on $\mathcal{P}^X_{p,i}$.

For the cocycle $\tilde{f}$, we apply Proposition 2.5.10 and obtain a δ_1-perturbation $\tilde{g}$ of $\tilde{f}$ supported on $\bigcup_{i=0}^{n_0-1} \tilde{f}^i(\{0\} \times \widetilde{U})$, such that for every $q \in \Psi_0(\Delta)$, it holds:

- $d_{C^0}(\tilde{f}, \tilde{g}) < \frac{\eta}{2}$;
- $\tilde{f} = \tilde{g}$ on the complement of $\bigcup_{i=0}^{n_0-1} \tilde{f}^i(\{0\} \times \widetilde{U})$;

- for every $q \in \Psi_0(\Delta)$, there exists $n \in \{1, \dots, n_0\}$ such that
$$|\log \det D\tilde{f}^n(q) - \log \det D\tilde{g}^n(q)| > K_0.$$

For each $i \in \{0, \dots, n_0 - 1\}$, let $\tilde{g}_i := g|_{\{i\} \times N_{X, X_i(p)}}$ and observe that $d_{C^1}(\tilde{g}_i, P^X_{X_i(p),1}) < \delta_1$. By item (2) of Lemma 2.5.8, we obtain a δ-pertubation g_i of $\mathcal{P}^X_{X_i(p),1}$. By (2.5.5) and (2.5.15), we have
$$d_{C^0}(g_i, \mathcal{P}^X_{X_i(p),1}) < 2d_{C^0}(\tilde{g}_i, P^X_{X_i(p),1}) < \eta.$$
Moreover, by the estimates in (2.5.5), we conclude that for any $q \in \Delta$, there exists $n \in \{1, \dots, n_0 - 1\}$ such that
$$|\log \det D\mathcal{P}^X_{p,n}(q) - \log \det D(g^n)(q)| > K_0 - 4\log 2, \tag{2.5.16}$$
where $g^n(q) := g_n \circ \cdots \circ g_1(q)$.

Recall that for $n \in \{0, \dots, n_0 - 1\}$, the maps $\mathcal{P}^X_{p,n}$ and $P^X_{p,n}$ are conjugated on Δ by Ψ. By (2.5.5), we obtain that for any $q \in \Delta$, it holds
$$|\log \det D\mathcal{P}^X_{p,n}(q) - \log \det P^X_{p,n}| \leq 2\log 2. \tag{2.5.17}$$
Suppose there exists $n \in \{0, \dots, n_0 - 1\}$ such that $|\log \det P^n_p - \log \det P^n_x| > K + 3\log 2$. By (2.5.17) and Remark 2.5.3, for any $q \in \Delta$ it holds that
$$\left|\log \det P^X_{q,\tau^X_{p,n}(q)} - \log \det P^X_{x,n}\right| > K + \log 2.$$
By Remark 2.5.6 and item 3 of Lemma 2.5.8, we conclude that
$$|\log \det P^X_{q,n} - \log \det P^X_{x,n}| > K.$$
In this case we do not make any perturbation. Suppose that for every $n \in \{0, \dots, n_0 - 1\}$ and every $q \in \Delta$ we have
$$|\log \det P^X_{q,n} - \log \det P^X_{x,n}| \leq K + 3\log 2.$$
Consider the maps $g_1, \dots, g_{n_0}$ as it was explained above (obtained using Proposition 2.5.10). Applying Lemma 2.5.8, we obtain a C^1 vector field Y that verifies the following properties:

- $d_{C^1}(X, Y) < \varepsilon$;
- the support of $X - Y$ is contained in $\mathcal{U}^X(p, U, n_0)$;
- for each $i \in \{1, \dots, n_0\}$, we have that $\mathcal{P}^Y_{X, X_i(p),1} = g_i$.

By (2.5.16) and (2.5.17), we conclude that for each $q \in \Delta$, there exists $n \in \{1, \dots, n_0\}$ such that

$$
\begin{aligned}
\left|\log\left(\frac{\det P^Y_{x,n}}{\det P^Y_{q,n}}\right)\right| &\geq \left|\log\left(\frac{\det D\mathcal{P}^X_{p,n}(q)}{\det P^Y_{q,n}}\right)\right| - \left|\log\left(\frac{\det P^X_{x,n}}{\det D\mathcal{P}^X_{p,n}(q)}\right)\right| \\
&\geq \left|\log\left(\frac{\det D\mathcal{P}^X_{p,n}(q)}{\det Dg^n(q)}\right)\right| - \left|\log\left(\frac{\det Dg^n(q)}{\det P^Y_{q,n}}\right)\right| - \\
&\quad - \left|\log\left(\frac{\det P^X_{x,n}}{\det P^X_{q,n}}\right)\right| - \left|\log\left(\frac{\det P^X_{q,n}}{\det D\mathcal{P}^X_{p,n}(q)}\right)\right| \\
&> K_0 - \log 2 - K - 4\log 2 - \log 2 > K.
\end{aligned}
$$

This concludes the proof of Proposition 2.5.9. $\square$

The following proposition is the version for flows of Proposition 7 in [BCW09].

Proposition 2.5.11. *Consider a vector field $X \in \mathfrak{X}^1(M)$, a compact set $D \subset M_X$, an open set $O \subset M_X$ and a point $x \in M_X$ satisfying:*

- *for any $y \in O$, any $t \geq 0$, we have $X_t(y) \in O$ and $X_1(\overline{O}) \subset O$;*
- *$D \subset O - X_1(\overline{O})$;*
- *$\mathrm{orb}^X(x) \cap D = \emptyset$.*

Then for any $K, \varepsilon > 0$, there exists a vector field $Y \in \mathfrak{X}^1(M)$ with $d_{C^1}(X, Y) < \varepsilon$ which satisfies the following property: for all $y \in D$, there exists $n \geq 1$ such that

$$|\log \det P^Y_{x,n} - \log \det P^Y_{y,n}| > K.$$

Moreover, the support of $X - Y$ is contained in the complement of the chain recurrent set of X.

Proof. Let X, D, O, x be as in the statement of Proposition 2.5.11. Let $C > 1$ be chosen such that the vector field X is bounded by C, and let $n_0 = n_0(d, C, 3K, \varepsilon)$, $\rho_0 = \rho_0(d, C, K, \varepsilon)$ be chosen according to Proposition 2.5.9. We set $N := 2^d n_0$. Without loss of generality, we also assume that K satisfies $K > 2d\log(2C) > 0$.

We fix a finite cover $\mathcal{F} = \{D_1, \dots, D_\ell\}$ of D by compact sets satisfying:

1. $D \subset \bigcup_{j=1}^{\ell} \mathrm{int}(D_j) \subset O - X_1(\overline{O})$;
2. for each $j \in \{1, \dots, \ell\}$, there exists a real number $\tau_j \in (0, 1)$, a point $p_j \in O - X_1(\overline{O})$, an open set $U_j \subset \mathcal{N}_{X,p_j}(\rho_0|X(p_j)|)$, and a compact set $\Delta_j \subset U_j$, such that the following properties hold:

(a) we have

$$D_j = \{X_t(y) : y \in \Delta_j,\ t \in [0, \tau_j]\}, \tag{2.5.18}$$

and

$$\mathrm{int}(D_j) \subset \{X_t(y) : y \in U_j,\ t \in (0, \tau_j)\} \subset O - X_1(\overline{O});$$

(b) for each $t \in [0, N]$, we have $\mathcal{P}^X_{p_j,t}(U_j) \subset \mathcal{N}_{X,X_t(p_j)}(\rho_0|X_t(p_j)|)$;

(c) for each $t \in [0, N-1]$, for each $t' \in [0, 1]$, and for each $y_1, y_2 \in \mathcal{P}^X_{p_j,t}(\Delta_j)$, it holds

$$d(\mathcal{P}^X_{X_t(p_j),t'}(y_1), \mathcal{P}^X_{X_t(p_j),t'}(y_2)) \leq 2Cd(y_1, y_2); \tag{2.5.19}$$

3. $\mathrm{orb}^X(x) \cap \bigcup_{j=1}^{\ell} U_j = \emptyset$;

4. for each $j \in \{1, \ldots, \ell\}$, the map $(y, t) \mapsto X_t(y)$ is injective restricted to the set $U_j \times [0, 1]$, and thus, it is also injective on the whole set $\mathcal{I}^X(p_j, U_j, N)$;[1]

5. there exists a partition $\{1, \ldots, \ell\} = J_0 \sqcup \cdots \sqcup J_{2^d-1}$ such that for each $k \in \{0, \ldots, 2^d - 1\}$, and for each $j_1 \neq j_2 \in J_k$, we have

$$\mathcal{U}^X(p_{j_1}, U_{j_1}, 1)\ \cap\ \mathcal{U}^X(p_{j_2}, U_{j_2}, 1) = \emptyset.$$

One can obtain $\mathcal{F}$ by tiling the compact set D by arbitrarily small cubes as in (2.5.18), i.e., obtained by flowing small transversals Δ_j under X, for $j = 1, \ldots, \ell$. Besides, since we assume that $D \subset M_X$, properties (1)-(4) are satisfied provided that D_j, U_j and Δ_j are chosen sufficiently small, for all $j \in \{1, \ldots, \ell\}$. In particular, (2.5.19) is true provided that D_j and Δ_j are chosen small enough, for all $j \in \{1, \ldots, \ell\}$, since X is bounded by C. Moreover, item (5) holds true provided that the diameter of the sets $U_1, \ldots, U_\ell$ is small enough, since M has dimension d.

For each $j \in \{1, \ldots, \ell\}$, and for each $i, m \geq 0$, we set

$$\mathcal{V}^X_j(i, m) := \mathrm{int}(\mathcal{U}^X(X_i(p_j), \mathcal{P}^X_{p_j,i}(U_j), m)).$$

Each set $\mathcal{V}^X_j(i, m)$ is open: it is the interior of the "tube" obtained by flowing points in the transversal $\mathcal{P}^X_{p_j,i}(U_j)$ under X until they hit the section $\mathcal{P}^X_{p_j,i+m}(U_j)$. We have the following properties:

- for each $j \in \{1, \ldots, \ell\}$, the sets $\mathcal{V}^X_j(0, 1), \mathcal{V}^X_j(1, 1), \ldots, \mathcal{V}^X_j(N-1, 1)$ are pairwise disjoint;
- for each $j \in \{1, \ldots, \ell\}$, the orbit $\mathrm{orb}^X(x)$ is disjoint from $\mathcal{U}^X(p_j, U_j, N)$;
- for each $(k_1, j_1) \neq (k_2, j_2)$ with $k_1, k_2 \in \{0, \ldots, 2^d - 1\}$ and $j_1 \in J_{k_1}$, $j_2 \in J_{k_2}$, we have

$$\mathcal{V}^X_{j_1}(n_0 k_1, n_0) \cap \mathcal{V}^X_{j_2}(n_0 k_2, n_0) = \emptyset. \tag{2.5.20}$$

[1] Indeed, for $t > 1$, we have $X_t(U_j) = X_1(X_{t-1}(U_j))$, and $X_{t-1}(U_j) \subset O - X_1(\overline{O})$.

Indeed, the first item is a consequence of point (4) above, the second one follows from point (3) above, and the third one is a consequence of points (4) and (5) above.

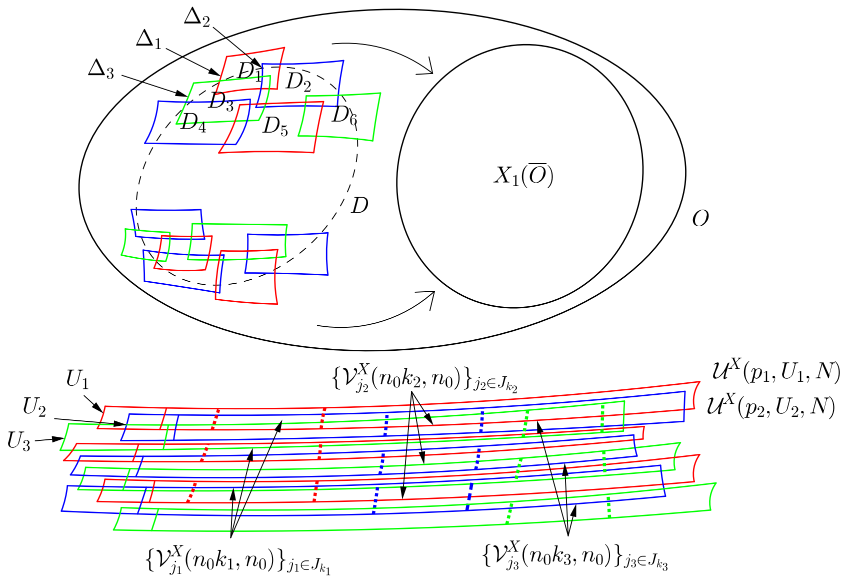

Figure 2.4: Selection of the perturbation times for the different tiles.

Claim 4. *There exists $\lambda > 0$ such that for each $y \in \bigcup_{j=1}^{\ell} D_j$, there exist $j \in \{1, \ldots, \ell\}$, $z \in \Delta_j$ and $u \in [0,1]$ such that $y = X_u(z)$, and $\mathcal{N}_{X,z}(2\lambda) \subset \Delta_j$.*

Proof. Let $\lambda_1 > 0$ be a Lebesgue number of the cover $\mathcal{F}$. We choose $\lambda_2 > 0$ such that $\mathcal{N}_{X,y}(\lambda_2) \subset B(y, \lambda_1)$, for any $y \in \bigcup_{j=1}^{\ell} D_j$, and we take $\lambda > 0$ such that $\mathcal{P}^X_{z,u}(\mathcal{N}_{X,z}(2\lambda)) \subset \mathcal{N}_{X,X_u(z)}(\lambda_2)$ for any $z \in \bigcup_{j=1}^{\ell} \Delta_j$ and $u \in [0,1]$. The existence of $\lambda > 0$ follows from the compactness of $\bigcup_{j=1}^{\ell} \Delta_j$ and from the fact that X is bounded $C > 0$. By the definition of λ_1 and $D_1, \ldots, D_\ell$, for each $y \in \bigcup_{j=1}^{\ell} D_j$, there exist $j \in \{1, \ldots, \ell\}$, $z \in \Delta_j$, and $u \in [0,1]$ such that $y = X_u(z)$, and $B(y, \lambda_1) \subset D_j$. By the definition of λ_2, we also have $\mathcal{N}_{X,y}(\lambda_2) \subset B(y, \lambda_1)$. Then, by the definition of λ and D_j, and since $y = X_u(z) \in \mathcal{N}_{X,y}(\lambda_2) \subset D_j$, we deduce that $\mathcal{N}_{X,z}(2\lambda) \subset (\mathcal{P}^X_{z,u})^{-1}(\mathcal{N}_{X,y}(\lambda_2)) \subset \Delta_j$. $\square$

For any $\eta > 0$, we define a sequence $(a_\eta(m))_{m\geq 0}$ inductively as follows:

$$a_\eta(0) := 0; \qquad a_\eta(m+1) := 2Ca_\eta(m) + \eta.$$

Note that $\lim_{\eta\to 0} a_\eta(N) = 0$. In the following, we fix $\eta_0 > 0$ small enough that

$$a_{\eta_0}(N) < (2C)^{-N}\lambda, \qquad \eta_0 < \frac{\lambda}{2}.$$

For each $k \in \{0, \ldots, 2^d - 1\}$ and $j \in J_k$, the set $\mathcal{P}^X_{p_j, n_0k}(\Delta_j)$ and the point $X_{n_0k}(x)$ satisfy the hypotheses of Proposition 2.5.9. We obtain a vector field $\widetilde{Y} \in \mathfrak{X}^1(M)$ such that the support of $X - \widetilde{Y}$ is contained in $\overline{\mathcal{V}_j^X(n_0k, n_0)}$. Moreover, for distinct choices of (k, j), (2.5.20) guarantees that the associated perturbations will be disjointly supported. Hence, applying Proposition 2.5.9 over all pairs (k, j) with $k \in \{0, \dots, 2^d - 1\}$ and $j \in J_k$, we obtain a vector field $Y \in \mathfrak{X}^1(M)$ with the following properties:

- the support of $X - Y$ is contained in

$$\bigcup_{k=0}^{2^d-1} \bigcup_{j \in J_k} \overline{\mathcal{V}_j^X(n_0k, n_0)} \subset \bigcup_{j=1}^{\ell} \mathcal{U}^X(p_j, U_j, N);$$

- $d_{C^1}(X, Y) < \varepsilon$;
- $d_{C^1}(\mathcal{P}^X_{X_i(p_j),1}, \mathcal{P}^Y_{X,X_i(p_j),1}) < \delta(\varepsilon)$, for all $i \in \{0, \dots, N\}$ and $j \in \{1, \dots, \ell\}$;
- $d_{C^0}(\mathcal{P}^X_{X_i(p_j),1}, \mathcal{P}^Y_{X,X_i(p_j),1}) < \eta_0$, for all $i \in \{0, \dots, N\}$ and $j \in \{1, \dots, \ell\}$;
- for each $k \in \{0, \dots, 2^d - 1\}$ and for each $z \in \bigcup_{j \in J_k} \Delta_j$, there exists an integer $n \in \{1, \dots, n_0\}$ such that:

$$\left| \log \det P^Y_{X_{n_0k}(x),n} - \log \det P^Y_{\mathcal{P}^X_{p_j,n_0k}(z),n} \right| > 3K.$$

Claim 5. *For each $y \in \bigcup_{j=1}^{\ell} D_j$, there exist $k \in \{0, \dots, 2^d - 1\}$, $j \in J_k$, and $t \in [0, 2]$, such that $y = Y_t(w)$, with $w \in \Delta_j$ and $\mathcal{P}^Y_{X,p_j,n_0k}(w) \in \mathcal{P}^X_{p_j,n_0k}(\Delta_j)$.*

Proof. Let $y \in \bigcup_{j=1}^{\ell} D_j$. By Claim 4, there exist $j \in \{1, \dots, \ell\}$, $z \in \Delta_j$ and $u \in [0, \frac{3}{2}]$ such that $y = \mathcal{P}^X_{p_j,u}(z)$, and $\mathcal{N}_{X,z}(2\lambda) \subset \Delta_j$. We have $d_{C^0}((\mathcal{P}^X_{p_j,u})^{-1}, (\mathcal{P}^Y_{X,p_j,u})^{-1}) < \eta_0 < \frac{\lambda}{2}$, hence $y = \mathcal{P}^Y_{X,p_j,u}(w) = Y_t(w)$, for some $t \in [0, 2]$, and $w \in \Delta_j$ satisfying $\mathcal{N}_{X,w}(\lambda) \subset \Delta_j$. Moreover, X is bounded by C, hence $\mathcal{N}_{X,\mathcal{P}^i_{X,p_j}(w)}((2C)^{-i}\lambda) \subset \mathcal{P}^X_{p_j,i}(\Delta_j)$, for all $i \in \{0, \dots, N-1\}$. For any $i \in \{0, \dots, N-1\}$, by (2.5.19), and by the fact that $d_{C^0}(\mathcal{P}^X_{X_i(p_j),1}, \mathcal{P}^Y_{X,X_i(p_j),1}) < \eta_0$, we have the estimate

$$\begin{aligned} d(\mathcal{P}^X_{p_j,i+1}(w), \mathcal{P}^Y_{X,p_j,i+1}(w)) &\leq d(\mathcal{P}^X_{X_i(p_j),1} \circ \mathcal{P}^X_{p_j,i}(w), \mathcal{P}^X_{X_i(p_j),1} \circ \mathcal{P}^Y_{X,p_j,i}(w)) \\ &\quad + d(\mathcal{P}^X_{X_i(p_j),1} \circ \mathcal{P}^Y_{X,p_j,i}(w), \mathcal{P}^Y_{X,X_i(p_j),1} \circ \mathcal{P}^Y_{X,p_j,i}(w)) \\ &\leq 2C d(\mathcal{P}^X_{p_j,i}(w), \mathcal{P}^Y_{X,p_j,i}(w)) + \eta_0. \end{aligned}$$

Thus, for any $i \in \{0, \dots, N-1\}$, we obtain

$$d(\mathcal{P}^X_{p_j,i}(w), \mathcal{P}^Y_{X,p_j,i}(w)) \leq a_{\eta_0}(i) < (2C)^{-N}\lambda.$$

Let $k \in \{0, \dots, 2^d - 1\}$ be such that $j \in J_k$. We conclude that $\mathcal{P}^Y_{X,p_j,n_0k}(w) \in \mathcal{N}_{X,\mathcal{P}^Y_{X,p_j,n_0k}(w)}((2C)^{-n_0k}\lambda) \subset \mathcal{P}^X_{p_j,n_0k}(\Delta_j)$, where $Y_t(w) = y$. ∎

We deduce that for each $y \in D \subset \cup_{j=1}^{\ell} D_j$, there exist $k \in \{0, \dots, 2^d - 1\}$, $j \in J_k$, $w \in \Delta_j$, $t \in [0, 2]$, such that $y = Y_t(w)$, and there exists $n \in \{1, \dots, n_0\}$, such that

$$|\log \det P^Y_{X_{n_0 k}(x),n} - \log \det P^Y_{\mathcal{P}^Y_{X,p_j,n_0 k}(w),n}| > 3K.$$

Since the vector field $X - Y$ has support in $\bigcup_{j=1}^{\ell} \mathcal{U}^X(p_j, U_j, N)$, which is disjoint from the orbit $\mathrm{orb}^X(x)$, we have $X_{n_0 k}(x) = Y_{n_0 k}(x)$. Moreover, there exists $t' \in [n_0 k - 2, n_0 k + 2]$ such that $\mathcal{P}^Y_{X,p_j,n_0 k}(w) = Y_{t'}(y)$. We thus have

$$|\log \det P^Y_{Y_{n_0 k}(x),n} - \log \det P^Y_{Y_{t'}(y),n}| > 3K.$$

We have $P^Y_{Y_{t'}(y),n} = P^Y_{Y_{n_0 k}(y),n} \circ P^Y_{Y_{t'}(y),n_0 k - t'}$, with $n_0 k - t' \in [-2, 2]$. Recall that $K > 0$ was chosen such that $K > 2d \log(2C)$. Since Y is close to X, we can assume that Y is bounded by $2C$. We thus get

$$\begin{aligned}
&|\log \det P^Y_{Y_{n_0 k}(x),n} - \log \det P^Y_{Y_{n_0 k}(y),n}| \\
\geq\ &|\log \det P^Y_{Y_{n_0 k}(x),n} - \log \det P^Y_{Y_{t'}(y),n}| - \max_{u' \in [-2,2]} \max_{y' \in M_X} |\log \det P^Y_{y',u'}| \\
>\ &3K - 2d \log(2C) > 2K.
\end{aligned}$$

Besides, $P^Y_{z,n+n_0 k} = P^Y_{Y_{n_0 k}(z),n} \circ P^Y_{z,n_0 k}$, hence of the following two cases holds:

- $|\log \det P^Y_{x,n_0 k} - \log \det P^Y_{y,n_0 k}| > K$;
- $|\log \det P^Y_{x,n+n_0 k} - \log \det P^Y_{y,n+n_0 k}| > K$.

In either case, $|\log \det P^Y_{x,n'} - \log \det P^Y_{y,n'}| > K$, for some $n' \in \{1, \dots, N\}$, as required.

By construction, the support of $X - Y$ is contained in at most N iterates of $O - X_1(\overline{O})$ for some trapping region O, and thus, the iterates of $O - X_1(\overline{O})$ for X and Y coincide. This implies that the vector fields X and Y have the same chain recurrent set, and they coincide on this set, which concludes the proof. $\square$

Proof of Theorem 2.5.2

Let $\mathcal{F}$ be a countable and dense subset of M, and let $\mathcal{K} = \{D_n\}_{n \in \mathbb{N}}$ be a countable collection of compact sets D_n, that verifies the following conditions:

- $\mathrm{diam} D_n \to 0$, as $n \to +\infty$;
- for any $n_0 \geq 1$, it holds $\displaystyle\bigcup_{n \geq n_0} D_n = M$.

For each $D \in \mathcal{K}$ we define the following set

$$\mathcal{O}_D = \{X \in \mathfrak{X}^1(M) : \exists \text{ open set } U,\ X_1(\overline{U}) \subset U \text{ and } D \subset (U - X_1(\overline{U}))\}.$$

It is easy to see that $\mathcal{O}_D$ is open. For any point $x \in \mathcal{F}$ we define

$$\mathcal{U}_{x,D} = \{X \in \mathcal{O}_D : x \notin \mathrm{Zero}(X) \text{ and } \mathrm{orb}^X(x) \cap D = \emptyset\}.$$

This set is not open. The next lemma gives a criterion for a vector field X to be in its interior.

Lemma 2.5.12. *Let $X \in \mathcal{U}_{x,D}$ and let $U \subset M$ be an open subset such that $X_1(\overline{U}) \subset U$ and $D \subset (U - X_1(\overline{U}))$. Assume that $\mathrm{orb}^X(x) \cap (U - X_1(\overline{U})) \neq \emptyset$. Then X belongs to the interior of $\mathcal{U}_{x,D}$, in particular, for any $Y \in \mathfrak{X}^1(M)$ sufficiently close to X it holds that $\mathrm{orb}^Y(x) \cap D = \emptyset$.*

Proof. Observe that the conditions $X_1(\overline{U}) \subset U$ and $D \subset (U - X_1(\overline{U}))$ are open. If $\mathrm{orb}^X(x) \cap U \neq \emptyset$, we can fix $t_1, t_2 \in \mathbb{R}$ such that $\big(\mathrm{orb}^X(x) \cap U - X_1(\overline{U})\big) \subset X_{[t_1,t_2]}(x)$. We can also assume that this property is open, that is, for any C^1 vector field Y sufficiently close to X, it holds

$$\mathrm{orb}^Y(x) \cap (U - Y_1(\overline{U})) \subset Y_{[t_1,t_2]}(x).$$

Since D and $X_{[t_1,t_2]}(x)$ are compact and disjoint, the distance between them is strictly positive. This implies that for any Y sufficiently C^1-close to X it holds that $Y_{[t_1,t_2]}(x)$ does not intersect D. Since for any Y close to X, $U - Y_1(\overline{U})$ is a fundamental domain for the attracting region Y, we conclude that $\mathrm{orb}^Y(x) \cap D = \emptyset$. In particular, X belongs to the interior of $\mathcal{U}_{x,D}$. □

The proof of the following lemma is the same as Lemma 15 in [BCW09].

Lemma 2.5.13. *The set $\mathrm{Int}(\mathcal{U}_{x,D}) \cup \mathrm{Int}(\mathcal{O}_D - \mathcal{U}_{x,D})$ is open and dense inside $\mathcal{O}_D$.*

First, observe that if $X \in \mathcal{U}_{x,D}$ then $D \cup \{x\}$ do not have any singularity of X. In particular, the linear Poincaré flow is well defined for any point $y \in D \cup \{x\}$. For $x \in \mathcal{F}$, $D \in \mathcal{K}$ and any $K \in \mathbb{N}$, we define:

$$\mathcal{V}_{x,D,K} := \left\{X \in \mathrm{Int}(\mathcal{U}_{x,D}) : \forall y \in D,\ \exists n \geq 1,\ |\log\det P^X_{y,n} - \log\det P^X_{x,n}| > K\right\}.$$

Using the fact that D is compact, it is easy to see that $\mathcal{V}_{x,D,K}$ is open inside $\mathrm{Int}(\mathcal{U}_{x,D})$. Proposition 2.5.11 implies that $\mathcal{V}_{x,D,K}$ is dense in $\mathrm{Int}(\mathcal{U}_{x,D})$. Therefore, the set

$$\mathcal{W}_{x,D,K} = \mathcal{V}_{x,D,K} \cup \mathrm{Int}(\mathcal{O}_D - \mathcal{U}_{x,D}) \cup \mathrm{Int}(\mathfrak{X}^1(M) - \mathcal{O}_D)$$

is open and dense in $\mathfrak{X}^1(M)$. Define the set

$$\mathcal{R}_0 = \bigcap_{x \in \mathcal{F}, D \in \mathcal{K}, K \in \mathbb{N}} \mathcal{W}_{x,D,K}.$$

By Baire's theorem, this set is residual in $\mathfrak{X}^1(M)$. Let $\mathcal{R} = \mathcal{R}_0 \cap \mathcal{R}_*$, where $\mathcal{R}_*$ is the residual subset given by Theorem 2.5.1.

Let $X \in \mathcal{R}$. Consider $x \in \mathcal{F} - \mathrm{Zero}(X)$ and $y \in M - \mathrm{CR}(X)$ such that $y \notin \mathrm{orb}^X(x)$. Since $y \notin \mathrm{CR}(X)$, by Conley's theory there exists an open set $U \subset M$ such that $X_1(\overline{U}) \subset$

U and $y \in (U - X_1(\overline{U}))$ (see for instance chapter 4 in [AN07]). Observe also that $\mathrm{orb}^X(x) \cap (U - X_1(\overline{U}))$ is either empty or a compact orbit segment. Take $D \in \mathcal{K}$ a compact set that contains y. If its diameter is sufficiently small, we have that $D \subset (U - X_1(\overline{U})$ and $\mathrm{orb}^X(x) \cap D = \emptyset$.

Hence $X \in \mathcal{U}_{x,D}$. Since $X \in \mathcal{R}_0$ and by the definition of $\mathcal{R}_0$, for every $K \in \mathbb{N}$, it holds that $X \in \mathcal{W}_{x,D,K}$. By the definition of $\mathcal{W}_{x,D,K}$ and since $X \in \mathcal{U}_{x,D}$, we have that $X \in \mathcal{V}_{x,D,K}$. Therefore, for any $K \in \mathbb{N}$, there exists $n \geq 1$ such that

$$|\log \det P^X_{x,n} - \log \det P^X_{y,n}| > K.$$

We conclude that X verifies the unbounded normal distortion property. $\square$

2.5.2 Collinearity

Once we have established Theorem 2.5.2, by combining Proposition 2.1.5 and some known generic results one obtains the collinearity of the centralizer of a C^1-generic vector field.

Theorem 2.5.14. *There exists a residual subset of $\mathcal{R} \subset \mathfrak{X}^1(M)$ such that if $X \in \mathcal{R}$ then the C^1-centralizer of X is collinear.*

Proof. The result follows directly from Proposition 2.1.5, and Theorems 2.5.1 and 2.5.2. $\square$

2.5.3 Quasi-triviality

By Theorem 2.5.1, we have that C^1-generically all the singularities are hyperbolic. As a consequence of Theorem 2.3.3, since C^1-generically the C^1-centralizer is collinear and all the singularities are hyperbolic, we conclude that C^1-generically the C^1-centralizer is quasi-trivial. More precisely, we have

Theorem 2.5.15. *Let M be a compact manifold. there exists a residual subset $\mathcal{R}_1 \subset \mathfrak{X}^1(M)$ such that if $X \in \mathcal{R}_1$, then any singularity and periodic orbit of X is hyperbolic, $\overline{\mathrm{Per}(X)} = \Omega(X) = \mathcal{CR}(X)$, and*

$$\mathfrak{C}^1(X) = \{fX : f \in \mathfrak{I}^1(X)\}, \text{ where } \mathfrak{I}^1(X) = \{f \in C^1(M, \mathbb{R}),\ X \cdot f \equiv 0\}.$$

Proof. By Theorem 2.5.14, there exists a residual subset $\mathcal{R} \subset \mathfrak{X}^1(M)$ whose elements have collinear C^1-centralizer. Moreover, by Theorem 2.5.1, there exists a residual subset $\mathcal{R}_* \subset \mathfrak{X}^1(M)$ such that for any $X \in \mathcal{R}_*$, any singularity and periodic orbit of X is hyperbolic, and $\overline{\mathrm{Per}(X)} = \Omega(X) = \mathcal{CR}(X)$. Then, $\mathcal{R}_1 := \mathcal{R} \cap \mathcal{R}_*$ is residual, and any $X \in \mathcal{R}_1$ satisfies the hypotheses of Theorem 2.3.3, which concludes. $\square$

2.5.4 Triviality

C^1-generic triviality for systems with a countable number of chain recurrent classes

We can now conclude the proof of Theorem B. To prove that we need the following lemma.

Lemma 2.5.16. *there exists a residual subset $\mathcal{R}_{\mathcal{CR}} \subset \mathfrak{X}^1(M)$ such that if $X \in \mathcal{R}_{\mathcal{CR}}$ and $f \in C^0(M)$ is an X-invariant function, then f is constant on chain-recurrent classes.*

Proof. By Theorem 1 in [Cro06], there exists a residual subset $\mathcal{R}_{\mathcal{CR}} \subset \mathfrak{X}^1(M)$ that verifies the following: if $X \in \mathcal{R}_{\mathcal{CR}}$ and $C \subset \mathcal{CR}(X)$ is a chain-recurrent class, then there exists a sequence of periodic orbits $(O(p_n))_{n\in\mathbb{N}}$ that converges to C in the Hausdorff topology.

By this property, for any two points $x, y \in C$, there exist two sequences of points $(q_n)_{n\in\mathbb{N}}$ and $(q'_n)_{n\in\mathbb{N}}$, with $q_n, q'_n \in O(p_n)$, such that $q_n \to x$ and $q'_n \to y$ as $n \to +\infty$. Let f be a continuous function which is X-invariant. By continuity,

$$\lim_{n\to+\infty} f(q_n) = f(x) \text{ and } \lim_{n\to+\infty} f(q'_n) = f(y).$$

However, since f is X-invariant and by our choice of q_n and q'_n, we have that $f(p_n) = f(q_n) = f(q'_n)$, which implies that $f(x) = f(y)$. □

Proof of Theorem B. Take $\mathcal{R} := \mathcal{R}_1 \cap \mathcal{R}_{\mathcal{CR}}$, where $\mathcal{R}_1$ is the residual subset given by Theorem 2.5.15. Using the conclusion of Lemma 2.5.16 and arguments analogous to the proof of Theorem 2.4.1 we can easily obtain the conclusion of Theorem B. □

2.6 Appendix: The separating property is not generic

In this section we prove that the separating property is not generic. Let M be a compact, connected Riemannian manifold. Take any Morse function $f \in C^2(M, \mathbb{R})$ and let $X := \nabla f$ be the gradient vector field which is C^1. It holds that X has two hyperbolic singularities, σ_s and σ_u with the following properties:

- σ_s is a hyperbolic sink and σ_u is a hyperbolic source;
- $W^s(\sigma_s) \cap W^u(\sigma_u) \neq \emptyset$;
- for any C^1 vector field Y which is sufficiently C^1-close to X, then $W^s(\sigma_s(Y)) \cap W^u(\sigma_u(Y)) \neq \emptyset$, where $\sigma_*(Y)$ is the continuation of σ_* for the vector field Y, for $* = s, u$.

We claim that X is C^1-robustly not separating. Let U be a compact ball inside $(W^s(\sigma_s) \cap W^u(\sigma_u)) - \{\sigma_s, \sigma_u\}$. Since compact parts of stable and unstable manifolds vary continuously with the vector field, it holds for any Y sufficiently C^1-close to X it holds that $U \subset (W^s(\sigma_s(Y)) \cap W^u(\sigma_u(Y))) - \{\sigma_s, B_u\}$.

Take any $\varepsilon > 0$ and consider the the balls $B(\sigma_s, \frac{\varepsilon}{2})$ and $B(\sigma_u, \frac{\varepsilon}{2})$. Since U is compact, there exists $T_X = T(\varepsilon) > 0$ such that any point $x \in U$ verifies

$$X_{-t}(x) \in B\left(\sigma_u, \frac{\varepsilon}{2}\right) \text{ and } X_t(x) \in B\left(\sigma_s, \frac{\varepsilon}{2}\right), \text{ for all } t \geq T. \tag{2.6.1}$$

Notice that for any two points $x, y \in B(\sigma_s, \frac{\varepsilon}{2})$ it holds that $d(X_t(x), X_t(y)) < \varepsilon$, for all $t \geq 0$. Similar statement is true for points in $B(\sigma_u, \frac{\varepsilon}{2})$ and the backward orbit.

Since T that verifies (2.6.1) is fixed, there exists $\delta > 0$ such that for any $x \in U$ and any $y \in B(x, \delta) \subset U$, it holds that

$$d(X_t(x), X_t(y)) < \varepsilon, \text{ for any } t \in \mathbb{R}.$$

In particular X is not separating. Also, observe that this holds for any Y sufficiently C^1-close to X. Thus we conclude that X is C^1-robustly not separating.

Remark 2.6.1. It is easy to see that the same type of example proves that the hypothesis of Proposition 2.2.2 is not generic. We conclude that the hypotheses of Propositions 2.1.4 and 2.2.2 are not generic.

Chapter 3

The diffeomorphism centralizer

3.1 Introduction

In this chapter we study a different type of symmetry for flows called the diffeormosphism centralizer. For any $1 \leq s \leq r$, we define the C^s*-diffeomorphism centralizer of* X as

$$\mathfrak{C}^s_{\mathrm{Diff}}(X) := \{f \in \mathrm{Diff}^s(M) : f \circ X_t = X_t \circ f, \ \forall t \in \mathbb{R}\}. \tag{3.1.1}$$

This is the set of diffeomorphisms that commute with the flow. Throughout this chapter we will refer to the C^1-diffeomorphism centralizer of a vector field X as the C^1-centralizer of X. We remark that this type of centralizer is less rigid than the vector field centralizer, see section 3.5 for a discussion on that.

We define two types of "triviality" for this centralizer. Given a C^r-vector field X, we say that it has *quasi-trivial* C^s*-centralizer* if for any $f \in \mathcal{Z}^s_{\mathrm{Diff}}(X)$, f is a reparametrization of the flow X_t, that is, there exists a continuous function $\tau : M \to \mathbb{R}$ such that $f(.) = X_{\tau(.)}(.)$. If X has quasi-trivial C^s-centralizer and for every element of the centralizer f, the function $\tau(.)$ is constant, then we say that the C^s-centralizer is *trivial*.

Observe that the set $\{X_t(.) : t \in \mathbb{R}\}$ is always contained in the centralizer of X. So a vector field has trivial centralizer if the centralizer is the smallest one possible.

It is natural to study generic systems that present some form of "hyperbolicity". In this chapter we will focus on C^1-generic vector fields that have at most finitely many sinks or sources. In [ABC06], the authors proved that such systems have a weak form of hyperbolicity named dominated splitting (see theorem 3.2.4). Our main result is the following:

Theorem I. *There exists a* C^1*-residual subset* $\mathcal{R} \subset \mathcal{X}^1(M)$ *such that if* $X \in \mathcal{R}$ *has at most finitely many sinks or sources, then* X *has quasi-trivial* C^1*-centralizer. Moreover, if in addition* X *has at most countably many chain-recurrent classes, then* X *has trivial* C^1*-centralizer.*

We state some consequences of this theorem. In [Pei60], Peixoto proved that a C^1-generic vector field on a compact surface is Morse-Smale. Recall that a vector field is

Morse-Smale if the non-wandering set is the union of finitely many hyperbolic periodic orbits and hyperbolic singularities, and it verifies some transversality condition. In particular, the non-wandering set is finite. As a consequence of Theorem I and the result of Peixoto, we have the following corollary.

Corollary A. *Let M be a compact connected surface. Then, there exists a residual subset $\mathcal{R}_\dagger \subset \mathfrak{X}^1(M)$ such that for any $X \in \mathcal{R}_\dagger$, the C^1-centralizer of X is trivial.*

A C^1-vector field X is *Axiom A* if the non-wandering set is hyperbolic and the periodic points are dense in the non-wandering set. It is well known that Axiom A vector fields have finitely many chain-recurrent classes.

Corollary B. *A C^1-generic Axiom A vector field has trivial C^1-centralizer.*

Remark 3.1.1. Corollary B actually holds for more a general type of hyperbolic system called sectional Axiom A.

Another corollary is for C^1-vector fields far from homoclinic tangencies in dimension three (see [CY17] for precise definitions). By the proof of the Palis conjecture in dimension three given in [CY17], a C^1-generic $X \in \mathfrak{X}^1(M)$ which cannot be approximated by such vector fields is singular axiom A (or sectional axiom A), in particular, it has a finite number of chain-recurrent classes. Hence:

Corollary C. *Let M be a compact connected 3-manifold. Then there exists a residual subset $\mathcal{R}_\ddagger \subset \mathfrak{X}^1(M)$ such that any vector field $X \in \mathcal{R}_\ddagger$ which cannot be approximated by vector fields exhibiting a homoclinic tangency has trivial C^1-centralizer.*

To prove Theorem I, we will need the proposition below. This proposition deals with the construction of the reparametrization of the flow, given a diffeomorphism in the centralizer that fixes orbits.

Proposition 3.1.2. *Let $X \in \mathfrak{X}^1(M)$ be a C^1-vector field whose periodic orbits and singularities are all hyperbolic. Let $f \in \mathfrak{C}^1_{\mathrm{Diff}}(X)$ be an element of the centralizer with the following property: there exists a constant $T > 0$ such that for every $p \in M$, we have $f(p) \in X_{[-T,T]}(p)$, where $X_{[-T,T]}(p)$ is the piece of orbit of p from time $-T$ to T. Then there exists an X-invariant continuous function $\tau : M \to \mathbb{R}$ such that $f(.) = X_{\tau(.)}(.)$.*

Observe that in Theorem I, without the additional assumption of at most countably many chain recurrent classes, we do not get the triviality of the C^1-centralizer for a C^1-generic vector field that has at most finitely many sinks or sources. What is missing to obtain the triviality of the C^1-centralizer in this case it is to prove that for a C^1-generic vector field every invariant continuous function is constant. This was conjectured (without specific requirement on the regularities) by René Thom ([Thom]).

Conjecture 1 ([Thom]). For a C^1-generic vector field, any C^1 (or C^0) invariant function of the manifold is constant.

Also, after the previous chapter, to conclude that a C^1-generic vector field has trivial C^1-vector field centralizer it is equivalent to proving Thom's conjecture.

Our approach to prove Theorem I is an adaptation for flows of the approach used by Bonatti-Crovisier-Wilkinson in [BCW09]. We organize this paper as follows. In section 3.2 we will introduce some basic notions and notations of vector fields that we will use, we will also recall the main tools from C^1-generic dynamics that will be used. The proof of proposition 3.1.2 is given in section 3.3. The proof of Theorem I is given in section 3.4. We conclude this paper with one example that justifies our claim that the diffeomorphism centralizer is less rigid than the vector field centralizer, in section 3.5.

3.2 Preliminaries

In this section we introduce the notations we will use throughout this article and state some preliminary results on C^1-generic dynamics that will be used in our proofs.

3.2.1 General notions on vector fields

Let M be a smooth manifold of dimension $d \geq 1$, which we assume to be compact and boundaryless. For any $r \geq 1$, we denote by $\mathfrak{X}^r(M)$ the space of vector fields over M, endowed with the C^r topology. A property $\mathcal{P}$ for vector fields in $\mathfrak{X}^r(M)$ is called C^r-generic if it is satisfied for any vector field in a *residual set* of $\mathfrak{X}^r(M)$. Recall that $\mathcal{R} \subset \mathfrak{X}^r(M)$ is *residual* if it contains a dense G_δ-subset of $\mathfrak{X}^r(M)$.

In the following, given a vector field $X \in \mathfrak{X}^1(M)$, we denote by $\mathrm{Sing}(X) := \{x \in M : X(x) = 0\}$ the set of *singularities* (or *zeros*) of X. The set of (non-singular) periodic points will be denoted by $\mathrm{Per}(X)$, and we set $\mathrm{Crit}(X) = \mathrm{Per}(X) \cup \mathrm{Sing}(X)$.

For any $x \in M$ and any interval $I \subset \mathbb{R}$, we also let $X_I(x) := \{X_t(x) : t \in I\}$. In particular, we denote by $orb(x) := X_{\mathbb{R}}(x)$ the orbit of the point x under X.

Let $X \in \mathfrak{X}^1(M)$ be some C^1 vector field. The *non-wandering set* $\Omega(X)$ of X is defined as the set of all points $x \in M$ such that for any open neighbourhood U of x and for any $T > 0$, there exists a time $t > T$ such that $U \cap X_t(U) \neq \emptyset$.

Let us also recall another weaker notion of recurrence. Given two points $x, y \in M$, we write $x \prec_X y$ if for any $\varepsilon > 0$ and $T > 0$, there exists an (ε, T)-*pseudo orbit* connecting them, i.e., there exist $n \geq 2$, $t_1, t_2, \ldots, t_{n-1} \in [T, +\infty)$, and $x = x_1, x_2, \ldots, x_n = y \in M$, such that $d(X_{t_j}(x_j), x_{j+1}) < \varepsilon$, for $j \in \{1, \ldots, n-1\}$. The *chain recurrent set* $\mathcal{CR}(X) \subset M$ of X is defined as the set of all points $x \in M$ such that $x \prec_X x$. Restricted to $\mathcal{CR}(X)$, we consider the equivalence relation given by $x \sim_X y$ if and only if $x \prec_X y$ and $y \prec_X x$. An equivalence class under the relation $\sim_X$ is called a *chain recurrent class*: $x, y \in \mathcal{CR}(X)$ belong to the same chain recurrent class if $x \sim_X y$. In particular, chain recurrent classes define a partition of the chain recurrent set $\mathcal{CR}(X)$.

An X-invariant compact set Λ is *hyperbolic* if there is a continuous decomposition of the tangent bundle over Λ, $T_\Lambda M = E^s \oplus \langle X \rangle \oplus E^u$ into DX_t-invariant subbundles that verifies the following property: there exists $T > 0$ such that for any $x \in \Lambda$, we have

$$\|DX_T(x)|_{E^s_x}\| < \frac{1}{2} \text{ and } \|DX_{-T}(x)|_{E^u_x}\| < \frac{1}{2}.$$

A periodic point $x \in \mathrm{Per}(X)$ is hyperbolic if $orb(x)$ is a hyperbolic set. Let $p \in \mathrm{Per}(X)$ be a hyperbolic periodic point and γ_p be its orbit. We define its strong stable manifold as

$$W^{ss}(p) := \{x \in M : d(X_t(x), X_t(p)) \xrightarrow{t \to +\infty} 0\}. \tag{3.2.1}$$

The stable manifold theorem states that $W^{ss}(p)$ is an immersed submanifold of dimension $\dim(E^s(p))$ tangent to $E^s(p)$ at p. We define the *stable manifold* of the orbit of p as

$$W^s(p) = \bigcup_{q \in \gamma_p} W^{ss}(q).$$

A hyperbolic periodic orbit is a sink if the unstable direction is trivial. It is a source if the stable direction is trivial. A hyperbolic periodic orbit is a saddle if it is neither a sink nor a source.

3.2.2 C^1-generic dynamics, the unbounded normal distortion and large normal derivative properties

In this part we will present the main tools from C^1-generic dynamics that we will need to prove Theorem I. Let us first fix some notation.

For a given vector field $X \in \mathfrak{X}^1(M)$, we define the non-singular set as $M_X := M - Sing(X)$. Observe that for a fixed riemannian metric, for any point $p \in M_X$, it is well defined the subspace orthogonal to the vector field direction, $N^X(p) = \langle X(p) \rangle^{\perp}$. This define the normal bundle

$$N^X = \bigsqcup_{p \in M_X} N^X(p)$$

over M_X. Let $\Pi_X : TM_X \to N^X$ be the orthogonal projection on N^X. Whenever it is clear that we have fixed a vector field X, we will denote the normal bundle and othogonal projection by N and Π.

On N^X we have a well defined flow, called the *linear Poincaré flow* defined as follows: for any point $p \in M_X$, vector $v \in N^X(p)$ and time $t \in \mathbb{R}$, the image of v by the linear Poincaré flow is

$$P_t(p).v = \Pi_X(X_t(p)) \circ DX_t(p)v.$$

Next, we define two notions that will be crucial in our proof.

Definition 3.2.1 (Unbounded normal distortion (UND)). Let $X \in \mathfrak{X}^1(M)$ be a C^1-vector field. We say that X verifies the *unbounded normal distortion* property if the following holds: there exists a dense subset $\mathcal{D} \subset M - \mathcal{CR}(X)$, such that for any $K \geq 1$, $x \in \mathcal{D}$ and $y \in M - \Omega(X)$ verifying $y \notin orb(x)$, there is $n \in (0, +\infty)$, such that

$$|\log \det P_n(x) - \log \det P_n(y)| > K.$$

Definition 3.2.2 (Unbounded normal distortion on stable manifolds (UND^s)). Let $X \in \mathfrak{X}^1(M)$ be a C^1-vector field. We say that X verifies the *unbounded normal distortion on stable manifolds* property if the following holds: for any $p \in Crit(X)$, there exists a dense subset $\mathcal{D}_p^s \subset W^s(p)$, such that for any $K \geq 1$, $x \in \mathcal{D}_p^s$ and $y \in W^s(p)$ verifying $y \notin orb(x)$, there is $n \in (0, +\infty)$, such that

$$|\log \det \left(P_n(x)|_{T_x W^s(p)}\right) - \log \det \left(P_n(y)|_{T_y W^s(p)}\right)| > K.$$

Given a vector field X that has the UND^s property, we define

$$\mathcal{D}^s = \bigcup_{p \in \mathrm{Per}(X)} D_p^s.$$

In [BCW09], the authors introduce these notions for diffeomorphisms and they use it as the main ingredient to obtain triviality of the centralizer in an open and dense subset of the manifold. They prove that these properties actually hold C^1-generically. For vector fields, the C^1-genericity of the UND property was proved in [LOS18] (theorem 6.3), and the C^1-genericity of the UND^s property was proved in [BV18](theorem 3.2). We summarize it in the following theorem:

Theorem 3.2.3 ([LOS18] and [BV18])**.** *There exists a residual subset $\mathcal{R}_1 \subset \mathfrak{X}^1(M)$ such that any vector field $X \in \mathcal{R}_1$ verifies the* UND *and* UND^s *properties.*

Given a non-singular invariant set $\Lambda \subset M_X$, we say that it admits a dominated splitting for the linear Poincaré flow if there exists a P_t-invariant, non-trivial decomposition of the normal bundle $N_\Lambda^X = E \oplus F$ and a constant $T > 0$ such that for any $x \in \Lambda$

$$\|P_T(x)|_E\|.\|(P_T(x)|_F)^{-1}\| < \frac{1}{2}.$$

In [ABC06], the authors proved that C^1-generically for a diffeomorphism far from the existence of infinitely many periodic sinks or sources (Newhouse phenomenon), one can obtain that the non-wandering set is decomposed into the disjoint union of finitely many periodic sinks or sources and invariant sets each of which admits a dominated decomposition. The key ingredient in their proof is a result of Bonatti-Gourmelon-Vivier (see corollary 2.19 in [BGV06]), which is a generalization of a previous theorem in [BDP03]. In [BGV06], the authors also prove a version of corollary 2.19 for flows, given by corollary 2.22. Using this, it is easy to adapt the proof of Abdenur-Bonatti-Crovisier to obtain the following statement:

Theorem 3.2.4 ([ABC06])**.** *There exists a residual subset $\mathcal{R}_2 \subset \mathfrak{X}^1(M)$ such that for any $X \in \mathcal{R}_2$, either (1) or (2) holds:*

1. *the non-wandering set admits a decomposition*

$$\Omega(X) = \mathrm{Sink}(X) \sqcup \mathrm{Source}(X) \sqcup \Lambda_1 \sqcup \cdots \sqcup \Lambda_{k_X}, \tag{3.2.2}$$

such that $\mathrm{Sink}(X)$ *is the set of periodic sinks of X, the set* $\mathrm{Source}(X)$ *is the set of periodic sources of X, and each $\Lambda_i - Sing(X)$ admits a dominated splitting for the linear Poincaré flow;*

2. *there are infinitely many periodic sinks or sources.*

In [BCW09], the authors also introduce the notion of large derivative for diffeomorphisms (see section 2.3 in [BCW09]). This is the key property to pass from triviality of the centralizer in an open and dense subset of M to triviality in the entire manifold. For vector fields we introduce the following similar definition:

Definition 3.2.5 (Large normal derivative (LND)). A vector field $X \in \mathfrak{X}^1(M)$ satisfies the LND property if for any $K > 0$, there exists $T = T(K) > 0$ such that for any $p \in M_X$ and $t > T$, there exists $s \in \mathbb{R}$ that verifies:

$$\max\{\|P_t(X_s(p))\|, \|P_{-t}(X_{s+T}(p))\|\} > K.$$

If the chain-recurrent set admits a decomposition as in (3.2.2), then the LND property holds. This was remarked for diffeomorphisms by Bonatti-Crovisier-Wilkinson (see remark 8 of Appendix A in [BCW09]). The same holds in our context for flows, we make it precise in the following corollary:

Corollary 3.2.6. *Let $\mathcal{R}_2 \subset \mathfrak{X}^1(M)$ be the residual subset from theorem 3.2.4. If $X \in \mathcal{R}_2$ and X does not have infinitely many sinks or sources, then X has the LND property.*

In the next statement, we summarize some other C^1-generic properties that we will use.

Theorem 3.2.7. *There exists $\mathcal{R}_3 \subset \mathfrak{X}^1(M)$ a residual subset such that any $X \in \mathcal{R}_3$ verifies:*

- *every periodic orbit, and singularity, is hyperbolic (Kupka-Smale);*
- *two distinct periodic orbits have different periods;*
- *any connected component O of the interior of $\Omega(X)$ is contained in the closure of the stable manifold of a periodic point (Bonatti-Crovisier, [BC04]).*

3.3 Proof of Proposition 3.1.2

Throughout this section we fix a vector field $X \in \mathfrak{X}^1(M)$ whose periodic orbits and singularities are all hyperbolic. We also fix $f \in \mathfrak{C}^1_{\mathrm{Diff}}(X)$ a C^1-diffeomorphism in the centralizer of X that verifies the conditions of proposition 3.1.2. The goal of this section is to construct an X-invariant continuous function $\tau : M \to \mathbb{R}$ such that $f(.) = X_{\tau(.)}(.)$.

3.3.1 Non-critical points

In this subsection we construct the function τ for non-critical points. This is given in the following lemma:

Lemma 3.3.1. *There exists an X-invariant continuous function $\tau_1 : M - Crit(X) \to \mathbb{R}$ such that $f|_{M-Crit(X)}(.) = X_{\tau_1(.)}(.)$.*

Proof. Let $p \in M - Crit(X)$. Since p is a non-critical point and f fixes its orbit, there is an unique $T_p \in \mathbb{R}$ such that $f(p) = X_{T_p}(p)$. We claim that for any $q \in orb(p)$ we have $f(q) = X_{T_p}(q)$.

Indeed, let $q \in orb(p)$ and let $s \in \mathbb{R}$ be such that $q = X_s(p)$. Hence

$$f(q) = f(X_s(p)) = X_s(f(p)) = X_s(X_{T_p}(p)) = X_{T_p}(X_s(p)) = X_{T_p}(q).$$

Define $\tau_1(p) = T_p$ for any $p \in M - Crit(X)$.

Claim 6. *The function τ_1 is continuous on $M - Crit(X)$.*

Proof. Let $T > 0$ be the constant that appears in the hypothesis of proposition 3.1.2, that is, for any $q \in M$, $f(q) \in X_{[-T,T]}(q)$. Fix $p \in M - Crit(X)$, we will prove that τ_1 is continuous on p.

For each $\delta > 0$, define $\mathcal{N}(p, \delta) := \exp_p(N(p, \delta))$, where $\exp_p$ is the exponential map on p and $N(p, \delta)$ is the ball of radius δ inside $N(p) \subset T_pM$. For δ small enough, the following map is a C^1-diffeomorphism

$$\begin{array}{rcl} \Psi : (-T-1, T+1) \times \mathcal{N}(p, \delta) & \longrightarrow & M \\ (t, x) & \mapsto & X_t(x). \end{array}$$

Let $V = Im(\Psi)$. The pair (Ψ^{-1}, V) is a flow box around the piece of orbit $X_{(-T-1,T+1)}(p)$. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of points contained in $\mathcal{N}(p, \delta) \cap M - Crit(X)$, which converges to p. Since $f(p_n) \in X_{[-T,T]}(p_n)$, we have that $\Psi^{-1}(f(p_n)) = (\tau_1(p_n), p_n)$. By the continuity of f, we obtain that $\lim_{n\to+\infty} \tau_1(p_n) = \tau_1(p)$ and τ_1 is continuous on $M - Crit(X)$. ☐

This claim concludes the proof of lemma 3.3.1. ☐

3.3.2 Periodic points

In this subsection we prove that there exists a continuous extension of the function τ_1 constructed in the previous section to the periodic points. We prove the following lemma:

Lemma 3.3.2. *There exists an X-invariant continuous function $\tau_2 : M - Sing(X) \to \mathbb{R}$ that verifies the following:*

- $f|_{M-Sing(X)}(.) = X_{\tau_2(.)}(.)$;
- $\tau_2|_{M-Crit(X)} = \tau_1$.

Proof. Let $p \in Per(X)$ be a hyperbolic periodic point and let $\pi(p)$ be the period of p. Notice that the equation $X_s(p) = f(p)$ has infinitely many solutions for $s \in \mathbb{R}$, thus the same strategy that we used for non-periodic points does not apply in this case. However, we will prove that the function τ_1 is constant on $W^s(p)$ and $W^u(p)$. This will allow us to extend it to the periodic orbit.

Claim 7. *The function τ_1 is constant on $W^s(p) - orb(p)$ and $W^u(p) - orb(p)$.*

Proof. Recall that

$$W^s(p) = \bigcup_{t \in [0, \pi(p)]} W^{ss}(X_t(p))$$

and that the strong stable manifolds forms a foliation of the stable manifold, in particular, if $t, s \in [0, \pi(p))$ are such that $t \neq s$ then $W^{ss}(X_t(p)) \cap W^{ss}(X_s(p)) = \emptyset$.

Observe that $W^s(p) - orb(p) \subset M - Crit(X)$, hence, the function τ_1 is well defined on $W^s(p) - orb(p)$. Since the function τ_1 is also X-invariant, it is enough to prove that τ_1 is constant along $W^{ss}(p) - \{p\}$.

Since f is a C^1-diffeomorphism that commutes with the flow, it is well known that $f(W^{ss}(p)) = W^{ss}(f(p))$. If τ_1 was not constant along $W^{ss}(p)$ there would be two points $x, y \in W^{ss}(p)$ such that $0 < |\tau_1(x) - \tau_1(y)| < \pi(p)$. Hence,

$$f(x) = X_{\tau_1(x)}(x) \in W^{ss}(X_{\tau_1(x)}(p)) \text{ and } f(y) = X_{\tau_1(y)}(y) \in W^{ss}(X_{\tau_1(y)}(p)).$$

However, $W^{ss}(X_{\tau_1(x)}(x)) \cap W^{ss}(X_{\tau_1(y)}(y)) = \emptyset$. This implies that $f(W^{ss}(p))$ is not contained in any strong stable manifold, which is a contradiction with the fact that $f(W^{ss}(p)) = W^{ss}(f(p))$. This proves that f is constant on each connected component of $W^s(p) - orb(p)$. Observe that $W^s(p) - orb(p)$ has at most two connected components.

Since f fixes the orbit of p, it induces a C^1-diffeomorphism f^s on $W^s(p)$. If τ_1 was not constant on $W^s(p)$, it would take two different values T_1 on the connected component O_1 and T_2 on the connected component O_2. From the above calculation, there exists $k \in \mathbb{Z} - \{0\}$ such that $T_2 = T_1 + k\pi(p)$.

On $TW^s(p)$ we consider the normal bundle $N_s = \langle X \rangle^\perp$, for the riemannian metric induced by the metric of the manifold on $W^s(p)$. Let $\Pi^s : TW^s(p) \to N_s$ be the orthogonal projection on N_s and let $P_t^s(.)$ be the linear Poincaré flow restricted to $W^s(p)$.

For any $q_1 \in O_1$ and $q_2 \in O_2$, we have the following formulas:

$$\Pi^s(f^s(q_1))Df^s(q_1)\Pi^s(q_1) = P_{T_1}^s(q_1) \text{ and } \Pi^s(f^s(q_2))Df^s(q_2)\Pi^s(q_2) = P_{T_2}^s(q_2).$$

Take q_1^n a sequence in O_1 converging to p and q_2^n a sequence in O_2 converging to p. Since f^s is C^1, we would have that

$$\begin{aligned} \lim_{n\to+\infty} \Pi^s(f^s(q_1^n))Df^s(q_1^n)\Pi^s(q_1^n) &= \\ \lim_{n\to+\infty} \Pi^s(f^s(q_2^n))Df^s(q_2^n)\Pi^s(q_2^n) &= \Pi^s(f^s(p))Df^s(p)\Pi^s(p). \end{aligned}$$

However, since p is a hyperbolic periodic point and $T_2 \neq T_1$, we have

$$\|P_{T_1}^s(p)\| \neq \|P_{T_2}^s(p)\|.$$

This is a contradiction and τ_1 is constant on $W^s(p)$. $\square$

Claim 7 implies that if p is a periodic sink or source, then the function τ_2 has a continuous extension to the orbit of p.

Let us assume that p is a hyperbolic saddle. We remark that the following claim is independent of claim 7.

Claim 8. *There is a constant $c_p \in \mathbb{R}$ such that $\tau_1|_{(W^s(p)-orb(p))} = \tau_1|_{(W^u(p)-orb(p))} = c_p$.*

Proof. The proof of this fact is the same as the proof of Proposition 4.4 in [LOS18]. For the sake of completeness we will repeat it here.

Fix a point $p_s \in W^s(p) - orb(p)$. We will prove that for any point $p_u \in W^u(p) - orb(p)$, we have $\tau_1(p_u) = \tau_1(p_s)$. By the X-invariance of τ_1, it is enough to consider $p_u \in W^u_{\text{loc}}(\sigma)$. Let $(D^s_n)_{n\in\mathbb{N}}$ be a sequence of discs transverse to $W^u_{loc}(p)$ and with radius $\frac{1}{n}$. By the lambda-lemma (see [PM82] chapter 2.7), for each $n \in \mathbb{N}$ and for any disc D^u transverse to $W^s_{loc}(p)$, there exists $t_n > 0$ such that $X_{t_n}(D^u) \pitchfork D^s_n \neq \emptyset$. Since this holds for any disc D^u and there are only countably many periodic orbits, for each $n \in \mathbb{N}$ we can find a disc D^u_n centered in p_s with radius smaller than $\frac{1}{n}$ and a point $q_n \in (X_{t_n}(D^u) \pitchfork D^s_n)$ which is non-periodic.

It is immediate that $q_n \to p_s$, as $n \to +\infty$. Since the function τ_1 is continuous on $M_X - \text{Crit}(X)$, we have that $\tau_1(q_n) \to \tau_1(p_s)$. We also have that $X_{t_n}(q_n) \to p_u$ as $n \to +\infty$. Hence, $\tau_1(X_{t_n}(q_n)) \to \tau_1(p_u)$. By the X-invariance of τ_1, we obtain

$$\tau_1(p_s) = \lim_{n\to+\infty} \tau_1(q_n) = \lim_{n\to+\infty} \tau_1(X_{t_n}(q_n)) = \tau_1(p_u).$$

This implies that for any $p_u \in W^u(p) - orb(p)$, we have $\tau_1(p_s) = \tau_1(p_u)$. Analogously, we can prove that for a fixed $p'_u \in W^u_{\text{loc}}(p) - orb(p)$ and for any $p'_s \in W^s_{\text{loc}}(p) - orb(p)$, it is verified $\tau_1(p'_s) = \tau_1(p'_u)$. We conclude that $\tau_1|_{W^s(p)-orb(p)} = \tau_1|_{W^u(p)-orb(p)} = c_p$, for some constant $c_p \in \mathbb{R}$. $\square$

From this claim, we can define an extension of τ_1 to the set of periodic points by setting $\tau_2|_{orb(p)} := c_p$, for $p \in Per(X)$. Let us prove that τ_2 is continuous on $M - Sing(X)$.

Fix $p \in Per(X)$. Since τ_2 is constant on $W^s(p)$, in the case that p is a sink, it is immediate that τ_2 is continuous on p. Similarly, we conclude continuity of τ_2 on p in the case that p is a source. Suppose that p is a saddle and let $(p_n)_{n\in\mathbb{N}} \subset M - Sing(X)$ be a sequence converging to p.

Suppose first that the sequence $(p_n)_{n\in\mathbb{N}}$ is formed by non-periodic points, hence, by the continuity of τ_1 and using the same argument as in the proof of claim 8, we can conclude that $\lim\limits_{n\to+\infty} \tau_2(p_n) = \tau_2(p)$.

In the case that $(p_n)_{n\in\mathbb{N}}$ is formed by periodic points, we may choose a sequence of points $(q_n)_{n\in\mathbb{N}} \subset M - Crit(X)$ such that each $n \in \mathbb{N}$, the point q_n is contained in the stable manifold of p_n and $d(q_n, p_n) < \frac{1}{n}$. Observe that $\lim_{n\to+\infty} q_n = p$. By claim 8, we have that $\tau_2(q_n) = \tau_2(p_n)$, hence $\lim\limits_{n\to+\infty} \tau_2(p_n) = \lim\limits_{n\to+\infty} \tau_2(q_n) = \tau_2(p)$. We conclude that τ_2 is continuous. $\square$

3.3.3 Singularities

We conclude the proof of proposition 3.1.2, by extending continuously the function τ_2 from lemma 3.3.2 to the singularities. This extension will give us the function τ that we wanted. We separate the proof into the case when the singularity is a saddle and when the singularity is a sink or source. This is given by the two lemmas below.

Lemma 3.3.3. *Let $\sigma \in Sing(X)$ be a hyperbolic singularity which is a saddle. Then the function τ_2 can be extended continuously to σ.*

Proof. The proof of this lemma is the same as the proof of claim 8 (see also Proposition 4.4 in [LOS18]). □

Lemma 3.3.4. *Let $\sigma \in Sing(X)$ be a singularity which is a sink. Then the function τ_2 is constant on $W^s(\sigma)$, in particular, it can be extended continuously to σ. A similar statement holds if σ is a source.*

Proof. Suppose that τ_2 is not constant on $W^s(\sigma)$, then there exists an open set $U \subset W^s(\sigma)$ such that $\tau_2(U) = (a, b)$, where $a \neq b$. Suppose also that $(a, b) \subset (0, +\infty)$. For each $t \in (a, b)$ we fix a point $x_t \in U$ such that $\tau_2(x_t) = t$. Thus, $f(x_t) = X_t(x_t)$. By the X-invariance of the function τ_2, we obtain that $f^n(x_t) = X_{nt}(x_t)$.

Let $\lambda_1, \cdots, \lambda_l \in \mathbb{C}$ be the eigenvalues of $DX(\sigma)$. For each $j \in \{1, \cdots, l\}$, consider the number $c_j = \mathfrak{R}(\lambda_j)$, where $\mathfrak{R}(.)$ is the real part of a number. Since σ is a hyperbolic sink for X, we have that $c_j < 0$, for each $j = 1, \cdots, l$. For each $t \in (a, b)$ the value

$$h_t := \limsup_{n \to +\infty} \frac{1}{n} \log d(\sigma, f^n(x_t)) = \limsup_{n \to +\infty} \frac{1}{n} \log d(\sigma, X_{nt}(x_t)) \tag{3.3.1}$$

belongs to the set $C_t := \{tc_1, \cdots, tc_l\}$.

Since σ is a fixed point of the C^1-diffeomorphisms f, we have that $Df(\sigma)$ has at most d different eigenvalues $\tilde{\lambda}_1, \cdots, \tilde{\lambda}_k \in \mathbb{C}$, where $1 \leq k \leq d$. For each $i \in \{1, \cdots, k\}$, let $a_i = \log |\tilde{\lambda}_i|$ and let $A = \{a_1, \cdots, a_k\}$. We remark that if $q \in M$ is a point such that $\lim_{n \to +\infty} f^n(q) = p$, then

$$\limsup_{n \to +\infty} \frac{1}{n} \log d(f^n(q), p) \in A. \tag{3.3.2}$$

Notice that the sets C_t varies continuously with $t \in (a, b)$. Observe also that the set (a, b) is uncountable and the set A is finite. Therefore, there exists $t_0 \in (a, b)$ such that $C_{t_0} \cap A = \emptyset$. By (3.3.1) and (3.3.2) this is a contradiction, since

$$h_{t_0} \in A \text{ and } h_{t_0} \in C_{t_0}.$$

We conclude that τ_2 is constant on $W^s(\sigma)$. If $(a, b) \subset (-\infty, 0)$ we can repeat this argument for f^{-1} and we would obtain the same conclusion. This implies that the function τ_2 can be continuously extended to a function τ defined on σ. □

3.4 Proof of Theorem I

In this section we prove Theorem I. Let $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ be the residual subsets given by theorems 3.2.3, 3.2.4 and 3.2.7, respectively. Consider the residual set $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3$. We claim that $\mathcal{R}$ is the residual subset that verifies the conditions of Theorem I.

From now on we fix $X \in \mathcal{R}$ such that X does not have infinitely many sinks or sources, and we fix $f \in \mathfrak{C}^1_{\mathrm{Diff}}(X)$. Since X is fixed, we will denote Π_X, and N^X, by Π, and N.

We want to prove that f verifies the conditions of 3.1.2. Let $\mathcal{D}$ and $\mathcal{D}^s$ be the subsets given by the UND and UND^s properties.

Lemma 3.4.1. *For each $x \in \mathcal{D} \cup \mathcal{D}^s$, we have that $f(orb(x)) = orb(x)$.*

Proof. Fix $x \in \mathcal{D}$. Since f commutes with the flow X_t, it is easy to see that $M - \Omega(X)$ is an f-invariant set.

Notice that $DX_t(f(x)) = Df(X_t(x))DX_t(x)Df^{-1}(f(x))$. Since the vector field direction, $\langle X\rangle$, is invariant by Df, Df^{-1} and DX_t, we have that

$$
\begin{aligned}
&|\det P_t(f(x))| = |\det \Pi(X_t(f(x)))DX_t(f(x))|_{N(f(x))}| = \\
&|\det \Pi(X_t(f(x)))Df(X_t(x))DX_t(x)Df^{-1}(f(x))|_{N(f(x))}| = \\
&|\det\left(\Pi(X_t(f(x)))Df(X_t(x))\right) \circ \left(\Pi(X_t(x))DX_t(x)\right) \circ \left(\Pi(x)Df^{-1}(f(x))\right)|_{N(f(x))}| = \\
&|\det \Pi(X_t(f(x)))Df(X_t(x))|_{N(X_t(x))}|.|\det \Pi(X_t(x))DX_t(x)|_{N(x)}|. \\
&.|\det \Pi(x)Df^{-1}(f(x))|_{N(f(x))}| = A_t.|\det P_t(x)|.C_t.
\end{aligned}
$$

Since f is a C^1-diffeomorphism and its derivative preserves the vector field direction, there exists a constant $\tilde{K} > 1$ such that

$$\tilde{K}^{-1} \leq \min\{A_t, C_t\} \leq \max\{A_t, C_t\} \leq \tilde{K}, \forall t \in \mathbb{R}.$$

Therefore, for every $t \in \mathbb{R}$ we have that

$$|\log|\det P_t(x)| - \log|\det P_t(f(x))|| \leq 2\log\tilde{K}. \tag{3.4.1}$$

Take $K > 2\log\tilde{K}$. If $f(x)$ did not belong to the orbit of x, by the UND property there would be $T \in \mathbb{R}$ such that

$$|\log\det P_T(x) - \log\det P_T(f(x))| > K.$$

This is a contradiction with (3.4.1). Hence, f fixes the orbit of x.

Let $x \in \mathcal{D}^s$. Since f commutes with the flow, it takes periodic orbits into periodic orbits of the same period. For a generic vector field, any two distinct periodic orbits have different periods. We conclude that f fixes each periodic orbit. Since f also takes stable manifolds into stable manifolds, we obtain that f fixes stable manifolds. Therefore, we can apply the same calculations made before, restricting the jacobian to the stable manifolds and this will imply that f fixes the orbit of x. The result then follows. □

This lemma states that the UND and UND^s properties provide some type of "local" triviality of the centralizer, meaning that f fixes a dense set of orbits. Now we want to be able to extend this to the entire manifold. For that, we will use the LND property. We will need the following lemma to work with points in $M - \mathcal{CR}(X)$:

Lemma 3.4.2. *There exists an open set $V \subset M - \mathcal{CR}(X)$, which is dense in $M - \mathcal{CR}(X)$, and a C^1-function $\tau : V \to \mathbb{R}$ such that $f(.) = X_{\tau(.)}(.)$ on V.*

Proof. For each $x \in \mathcal{D}$, there is an unique number $T_x \in \mathbb{R}$ such that $f(x) = X_{T_x}(x)$. Let $p \in \mathcal{D}$. By Conley's theory, there is an open set U such that $X_1(\overline{U}) \subset U$ and $p \in U - X_1(\overline{U})$ (see chapter 4 in [AN07]). Since f fixes the orbit of p, there is an unique $n \in \mathbb{Z}$ such that $f(p) \in X_{n-1}(U) - X_n(\overline{U})$.

Recall that for each $\delta > 0$, we defined $\mathcal{N}(p, \delta) := \exp_p(N(p, \delta))$. Consider the map $\Psi(t, x) = X_t(x)$ defined on $(-2|T_p|, 2|T_p|) \times \mathcal{N}(p, \delta)$. For δ small enough Ψ is a C^1-diffeomorphism and $f(\mathcal{N}(p, \delta)) \subset X_{n-1}(U) - X_n(\overline{U})$. This implies that for each $q \in \mathcal{N}(p, \delta) \cap \mathcal{D}$, we have that $f(q) \subset \Psi((-2|T_p|, 2|T_p|), q)$. Let $V_p = Im(\Psi)$.

Since $\mathcal{D}$ is dense, for each point $z \in \mathcal{N}(p, \delta)$, we can take a sequence $(z_n)_{n \in \mathbb{N}}$ contained in $\mathcal{N}(p, \delta) \cap \mathcal{D}$ and converging to z. Let $I_p = [-2|T_p|, 2|T_p|]$, By the continuity of the flow, we have

$$X_{I_p}(z_n) \xrightarrow{n \to +\infty} X_{I_p}(z).$$

Since $f(z_n) \in X_{I_p}(z_n)$ and $\lim_{n \to +\infty} f(z_n) = f(z)$, we conclude that $f(z) \in X_{I_p}(z)$. In particular f fixes the orbit of z. Therefore, for each $z \in \mathcal{N}(p, \delta)$, there is a number $T_z \in \mathbb{R}$ such that $f(z) = X_{T_z}(z)$. For each $x \in V_p$, consider the function $\hat{\tau}(x) = T_x$. Since f and Ψ^{-1} are C^1, we have that $\Psi^{-1} \circ f$ is also C^1. For any $z \in \mathcal{N}(p, \delta)$, we have $\Psi^{-1}(f(z)) = (\hat{\tau}(z), z)$. From this formula, since $\hat{\tau}$ is constant along orbits and $\Psi^{-1} \circ f$ is C^1, we conclude that $\hat{\tau}$ is C^1 on V_p.

Take $V = \bigcup_{p \in \mathcal{D}} V_p$. Observe that $\hat{\tau}$ is uniquely defined on V (since f fixes orbits of V and its points are non critical) and it is a C^1-function such that $f(.) = X_{\hat{\tau}(.)}(.)$ on V. ☐

To deal with points in $int(\Omega(X))$ we need the following lemma:

Lemma 3.4.3. *For each hyperbolic periodic point $p \in \mathrm{Per}(X)$, there exists a number $T_p \in \mathbb{R}$ such that for any $q \in W^s(p)$ it is verified that $f(q) = X_{T_p}(q)$.*

Proof. Let $p \in \mathrm{Per}(X)$ be a hyperbolic periodic point of X. Since any two different periodic orbits have different periods (theorem 3.2.7), f fixes the orbit of p. Since the stable manifold $W^s(p)$ is a C^1-immersed submanifold, and f fixes $W^s(p)$, we have that f induces a C^1-diffeomorphim f^s on $W^s(p)$, for the intrinsic topology. By lemma 3.4.1, f fixes the orbits of the points in $\mathcal{D}_p^s$.

The points in a stable manifold are non-recurrent for the intrinsic topology, hence, by an argument similar to the one in the proof of lemma 3.4.2, we obtain an open set V_p^s which is dense in $W^s(p)$, and a C^1-function $\tau^s : V_p^s \to \mathbb{R}$ such that $f^s(.) = X_{\tau^s(.)}(.)$.

We claim that $\tau^s(.)$ is constant on V_p^s. The proof of this fact is essentially contained in the proof of claim 7 in lemma 3.3.2. We sketch the proof of this fact here, for more details on the arguments see the proof of claim 7 in lemma 3.3.1.

First we prove that τ^s is constant on each connected component of V_p^s. If it was not the case, we could find two points $x, y \in W^{ss}(p) - \{p\}$ such that $0 < |\tau^s(x) - \tau^s(y)| < \pi(p)$, where $\pi(p)$ is the period of p. This implies that $f(x) \in X_{\tau^s(x)}(W^{ss}(p))$ and $f(y) \in X_{\tau^s(y)}(W^{ss}(p))$. From this, one can deduce that $W^{ss}(f(x)) \cap W^{ss}(f(y)) = \emptyset$, which is a contradiction with the fact that $f(W^{ss}(p)) = W^{ss}(f(p))$. This implies that τ^s is constant in each connected component of V_p^s.

Recall that on $TW^s(p)$ we defined the normal bundle $N_s = \langle X \rangle^{\perp}$. Let $\Pi^s : TW^s(p) \to N_s$ be the orthogonal projection on N_s and let $P_t^s(.)$ be the linear Poincaré flow restricted to $W^s(p)$.

Suppose that τ^s does not take the same value in every connected component of V_p^s. Let V_1 and V_2 be two connected components of V_p^s such that the numbers $T_1 := \tau^s|_{V_1}$ and $T_2 := \tau^s|_{V_2}$ are not equal. We may choose q_1^n a sequence in V_1 converging to p and q_2^n a sequence in V_2 converging to p. Since f^s is C^1, we would have that

$$\begin{aligned}\lim_{n\to+\infty} \Pi^s(f^s(q_1^n))Df^s(q_1^n)\Pi^s(q_1^n) &= \\ \lim_{n\to+\infty} \Pi^s(f^s(q_2^n))Df^s(q_2^n)\Pi^s(q_2^n) &= \Pi^s(f^s(p))Df^s(p)\Pi^s(p).\end{aligned}$$

However, by the hyperbolicity of p and since $T_2 \neq T_1$, we have $\|P_{T_1}^s(p)\| \neq \|P_{T_2}^s(p)\|$. This is a contradiction and τ^s is constant on V_p^s. Hence, there exists $T_p \in \mathbb{R}$ such that $\tau^s(q) = T_p$ for every $q \in V_p^s$. This easily implies that $f(q) = X_{T_p}(q)$, for any $q \in W^s(p)$. □

By theorem 3.2.7, in each connected component O of $int(\Omega(X))$, there exists a periodic point p whose stable manifold is dense in O. From lemma 3.4.3, there exists a number T_p such that $f(q) = X_{T_p}(q)$, for any $q \in W^s(p)$. This implies that $f(q) = X_{T_p}(q)$ for any $q \in O$.

Consider the open and dense set $W = V \cup int(\Omega(X))$. From the discussion above, there is a C^1-function $\hat{\tau} : W \to \mathbb{R}$ such that $f(.) = X_{\hat{\tau}(.)}(.)$ on W.

Lemma 3.4.4. *There exists a constant $T > 0$ such that $|\hat{\tau}(x)| \leq T$ for any $x \in W$.*

Proof. For $x \in W$ and any vector $v \in T_xM$, the following formula holds:

$$Df(x)v = DX_{\hat{\tau}(x)}(x)v + X(X_{\hat{\tau}(x)}(x))D\hat{\tau}(x)v \tag{3.4.2}$$

From this formula, one can see that $\Pi(f(x)) \circ Df(x)|_{N(x)} = P_{\hat{\tau}(x)}(x)$. Take $K > \|f\|_{C^1}$ and let $T = T(K) > 0$ be the uniform time given by the LND property.

If $\hat{\tau}$ was not uniformly bounded, there would be a point $x \in W$ such that $|\hat{\tau}(x)| > T$. By the LND property, there exists a point $y \in orb(x)$ such that

$$\max\{\|P_{\hat{\tau}(x)}(y)\|, \|P_{-\hat{\tau}(x)}(X_{\hat{\tau}(x)}(y))\|\} > K.$$

Then,

$$\max\{\|Df(y)\|, \|Df^{-1}(f(y))\|\} \geq \max\{\|P_{\hat{\tau}(x)}(y)\|, \|P_{-\hat{\tau}(x)}(X_{\hat{\tau}(x)}(y))\|\} > K.$$

This is a contradiction since $K > \|f\|_{C^1}$. Therefore, $\hat{\tau}$ is uniformly bounded on W. $\square$

Let $I = [-T, T]$. From lemma 3.4.4, we obtain that for any point $x \in W$, it is verified that $f(x) \in X_I(x)$. Since W is dense, for any point $z \in M$, there is a sequence $(y_n)_{n\in\mathbb{N}}$ contained in W such that $\lim_{n\to+\infty} y_n = y$. Since $X_I(y_n) \xrightarrow{n\to+\infty} X_I(y)$, and by the continuity of f, we conclude that $f(y) \in X_I(y)$. In particular, f fixes every orbit of X and $f(p) \in X_{[-T,T]}(p)$, for any $p \in M$.

By proposition 3.1.2, there is a continuous function $\tau : M \to \mathbb{R}$ such that $f(.) = X_{\tau(.)}(.)$. This proves that the centralizer is quasi-trivial.

In the case that X has at most countably many chain-recurrent classes, since the function τ is an X-invariant continuous function, using the same arguments as in section 6.4 of [LOS18], we conclude that τ is a constant function. In particular, the centralizer is trivial.

3.5 A remark on the centralizers

As we mentioned before, the diffeomorphism centralizer is less rigid than the vector field centralizer, mentioned in the introduction. In this section we give one example that justifies it.

By the work of [KM73] the vector field centralizer of an Anosov flow is trivial. However, it is easy to construct an Anosov flow and a diffeomorphism that commutes with the flow and which does not fixes the orbits of the flow. For example, one can take two hyperbolic matrices $A, B \in SL(n, \mathbb{Z})$ that commute and such that the group they generate is isomorphic to $\mathbb{Z}^2$; that is, $A^l B^k = Id$ if and only if $l = k = 0$. For example, in dimension 3 one may consider the matrices

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

These matrices induce Anosov diffeomorphisms on the torus $\mathbb{T}^3$. Consider the suspension flow of A, this gives an Anosov flow X_t^A. Using B one can easily construct a diffeomorphism f_B that commutes with the the flow X_t^A and that does not fixes orbit. In particular, the centralizer of X_t^A is not trivial.

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