On the arithmetic of Elliptic Surfaces in Positive Characteristic

Vinicius Martins Teodosio Rocha

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Tese de Doutorado apresentada ao programa de Pós-Graduação em Matemática da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Doutor em Matemática.

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Abstract

In this thesis we study the behavior of the rank of the fibers of elliptic surfaces in positive characteristic, compared to the rank of the generic fiber. In particular, we prove that the rank of the fibers is strictly greater than the generic rank for infinitely many fibers under the hypothesis that the surface is unirational, generalizing a result of Salgado in characteristic 0. We then proceed to analyze some examples where the distinctions between the zero and positive characteristic plays a role.

Keywords: *Elliptic fibrations, K3 surfaces, Rational surfaces, Rank variation.*

Introduction

Arithmetic geometry deals with number theoretic problems by the means of geometric tools. One of its main subareas is the study of rational points on algebraic varieties[1]. Let us start with a curve C and a global field K. If C is a curve of genus 0, then it is a classical result that if C contains a K-rational point, i.e., a point with coordinates in K, then it contains infinitely many points. These are the rational curves, i.e., lines and conics, and their set of K-rational points is well understood. Now, if we consider a curve E of genus 1, then E may have no K-rational points, a finite number of them or infinitely many. If it contains at least one rational point, the set of these points can be equipped with a group structure defined by the tangent-chord law, such curves are called elliptic curves. A classical theorem of Mordell says that such group, denoted by $E(K)$, is a finitely generated abelian group for $K = \mathbb{Q}$. This theorem was later generalized to the case where K is a number field by André Weil, and when K is a function field by Lang and Néron. By the structure theorem for abelian groups, i.e., finitely generated $\mathbb{Z}$-modules, the free part of the

[1]Here, rational points means K-rational, for a given global or finite field K, i.e., points with all coordinates in the field K.

finitely generated group $E(K)$ is a free $\mathbb{Z}$-module of finite rank, which from now on we refer to as the rank of the elliptic curve.

The rank of elliptic curves over global fields is one of the most studied subjects in number theory. In this thesis we consider the problem of the variation of the ranks of elliptic curves over global fields of positive characteristic in an algebraic family. In particular, we are interested in the relation between these ranks and the rank of the generic fiber of such a family. Let us start to fix some notations : our base field is a global field K. Recall that K is either a number field or the function field of a smooth projective geometrically connected curve C over a finite field $\mathbb{F}_q$, where q is a power of a prime p, and we will deal mostly with the latter.

The main objects of this work are families of elliptic curves: given a 'parameter' space, in our case another variety V/K, we can consider families of elliptic curves parametrized by V, i.e. $\{E_v : v \in V\}$. We focus our attention on the case where V is a curve X/K, so our one-dimensional family of elliptic curves is parametrized by the points in the curve X. We can associate two objects to such family, the most natural one being an elliptic surface $\mathscr{E} \to X$, whose fibers are naturally identified with the curves E_v on the family we started with. The other important object is an elliptic curve E over the function field of X, say $\mathcal{K} = K(X)$, which arises as the generic fiber of the fibration $\mathscr{E} \to X$. Note that in the characteristic zero case, $\mathcal{K}$ is a function field of transcendence degree one over a number field and in the positive characteristic case $\mathcal{K}$ is a function field of transcendence degree two over a finite field. In both cases, there exists a generalization of the Mordell-Weil theorem, the Lang-

Néron theorem, which asserts that the group of $\mathcal{K}$-rational points on E is finitely generated, so we have a rank associated with the generic fiber of the family $\{E_v\}_{v \in X}$, which we denote by r_η.

A theorem of Silverman (See [41]) implies that the inequality

$$r_\eta \leq r_b \tag{1}$$

holds for all, except finitely many, $b \in X(K)$, where $r_b := \operatorname{rank} E_b(K)$.

Even though Silverman's Theorem does not distinguish between the cases of characteristics zero and $p > 0$, it cannot be applied to some settings that may occur in characteristic $p > 0$. In particular, it cannot be applied as it is proved when it is stated for a family of abelian varieties over function fields such that no smooth model is known for such family. The first contribution of this thesis is to state and prove Silverman's Theorem to this specific setting in prime characteristic.

Questions arise regarding the inequality (1). The first one is when is it a strict inequality? Examples are known where the inequality is strict for all but finitely many points $b \in X(K)$, for K a number field, but these examples are isotrivial, i.e. have constant j-invariant. In [6], the authors describe (conditional to the parity conjecture) a family of non-isotrivial elliptic curves satisfying this property in characteristic p and provide an heuristic argument to indicate that this behavior cannot happen in characteristic zero. We can, though, ask a modified version of this question, namely, are there elliptic surfaces satisfying the strict inequality in (1) for an infinite number of fibers? In [35], Salgado proves that this always happens in the number field setting for elliptic surfaces that are unirational over the base field. Salgado's proof however does not hold as it is in prime characteristic since an important part of the proof is to find bounds for the extensions generated by the division points of rational points, who behave differently in positive characteristic. The second main contribution of this thesis is to adapt Salgado's main results for the positive characteristic setting.

This adaptation together with recent results by Liedtke allows us to prove that supersingular elliptic K3 surfaces also have infinitely many fibers for which the inequality in (1) is strict over its field of unirationality. The reader should notice that this is special to the positive characteristic setting since there are no unirational non-geometrically rational surface in characteristic zero.

Chapter 4 is dedicated to presenting some examples where the rank jump can be found and sometimes described. The most natural example are rational surfaces. Indeed, in characteristic 0 these are the only unirational surfaces (over an algebraically closed field). In positive characteristic, the K3s mentioned above provide non-rational unirational surfaces. This allows us to find chains of elliptic surfaces $\mathscr{E} \leftarrow \mathscr{E}' \leftarrow \mathscr{E}''$ such that there exists a rank jump between them, implying that, in such cases, there are infinitely many fibers of $\mathscr{E}$ such that $\operatorname{rank} \mathscr{E}_t(K) > \operatorname{rank} \mathscr{E}(C) + 3$. We also considered the case of a Frobenius base change from a rational elliptic surface, since it always yields a unirational surface. When the resulting surface is K3, we know it is supersingular and can compute its rank. It turns out that no surface obtained in this way has rank strictly bigger than the rank of the rational elliptic surface that originated it. These computations were made using the Shioda-Tate formula and the possible configurations of singular fibers on a rational elliptic surface, in the Appendix we listed all of supersingular K3 surfaces that arises by this process.

Contents

Chapter 1

Preliminaries

In this chapter we introduce elliptic surfaces and discuss their basic properties. For a complete treatment of the subject we refer to [38] and [48].

1.1 Elliptic Surfaces

Let k be a field. An abelian variety over k is an algebraic variety that is also a complete algebraic group over k. An abelian variety of dimension one is called an elliptic curve. Equivalently, an elliptic curve is a smooth, projective, geometrically connected curve of genus 1 with a k-rational point. This thesis' focus lies on elliptic surfaces, which can be seen as 1-dimensional algebraic families of elliptic curves.

Definition 1.1. Let C be a smooth, projective, geometrically connected curve over an algebraically closed field k. An elliptic surface over C is a smooth projective surface S together with a surjective morphism $\pi : S \to C$ satisfying:

1. All but finitely many fibers are smooth curves of genus 1,
2. $S \to C$ is minimal in the sense that no fiber of S contains a (-1)-curve, and
3. $S \to C$ possess a section $o : C \to S$.

Remark. In this text we save the terminology *genus 1 fibration* for surfaces as above not necessarily having a section as in item 3.

These surfaces do not need to be minimal as algebraic surfaces, so exceptional curves can appear, but not as components of the fibers. A section for the elliptic surface S is a morphism $\sigma : C \to S$, defined over k, satisfying $\pi \circ \sigma = \mathrm{Id}_C$. The property $3)$ allows us to associate to S an elliptic curve E over the function field $k(C)$ of C: its generic fiber $E := S_\eta$ is a genus 1 curve over $k(C)$ and the restriction of the section o to the generic point of C gives a k-rational point in E, hence we get a Weierstrass equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \qquad a_i \in k(C).$$

In fact, there exists a correspondence between the sections of $S \to C$ and the points on the elliptic curve E: given a section $\sigma : C \to S$, the restriction $\sigma(C) \cap S_\eta$ is a point in E and, given a point $P \in E$, the restriction of π to the closure of P in S is a birational morphism onto a non-singular curve. By Zariski Main Theorem, it is an isomorphism, and thus has an inverse σ_P, which is a section of $S \to C$.

Conversely, given an elliptic curve E over $k(C)$ we can ask whether there exists an elliptic surface $S \to C$ with E as its generic fiber and satisfying the property that its rational points extend to sections of S. The answer is positive and we will discuss it in the next chapter.

Isotrivial and constant surfaces Since we are mostly interested in the variation of the fibers, we rule out two special kinds of elliptic surfaces that do not show much geometric variation. Let $S \to C$ be an elliptic surface and let E be its generic fiber.

- S is said to be *constant* if $E \simeq_k E' \times k(C)$, where E' is an elliptic curve defined over k, equivalently E can be defined by a Weierstrass equation with coefficients in k. This implies that S is isomorphic, as a surface, to the product $E' \times_k C$.

- S is said to be *isotrivial* if there exists a finite extension K' of $k(C)$ such that E becomes constant over K'. This is equivalent to the j-invariant of E being an element of k.

It is a consequence of the definition that in isotrivial elliptic surfaces the smooth fibers are all isomorphic over k.

1.2 Singular Fibers and the Shioda-Tate Formula

The singular fibers of an elliptic surface over perfect fields have been classified by Kodaira and Néron. Tate developed an algorithm to describe the singular fibers if the base field is perfect. The case of non-perfect fields was described by M. Szydlo in [43] and, for fields of characteristic 2 and

3, Tate's Algorithm can fail and new fiber types can appear. Since we are working in characteristic $\neq 2, 3$, we will only consider singular fibers of Kodaira's list, described in the following table. If the fiber is reducible, then every component is a smooth rational curve with self-intersection -2.

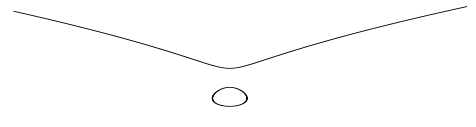
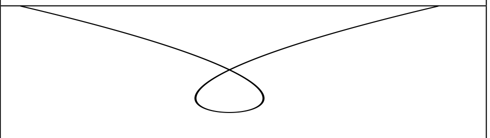
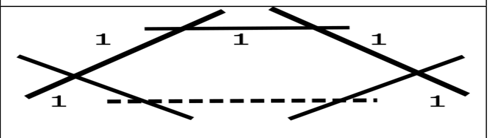
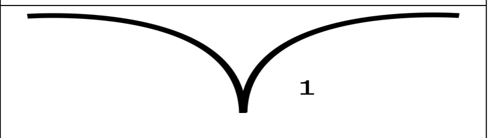
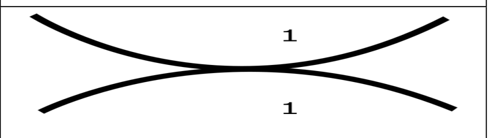
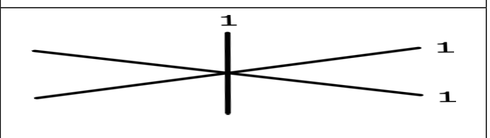
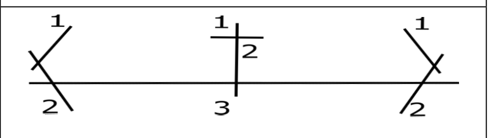
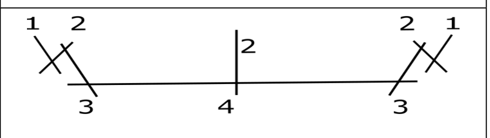
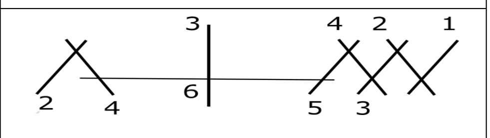
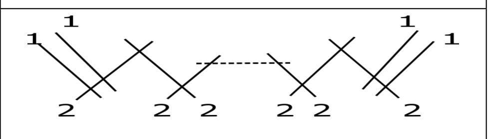
Type	m_v	Description
I_0 (non-singular)	1	
I_1	1	
I_n, $n \geq 2$	n	
II	1	
III	2	
IV	3	
IV^*	7	
III^*	8	
II^*	9	
I^*_n	n+5	

Table 1.1 - List of possible singular fibers.

If a fiber is reducible, each irreducible component is a smooth rational

(-2)-curve. The fibers of type I_n, $n \geq 1$, are called *multiplicative* (or semi-stable), whereas the others are called *additive* (or unstable). The names semi-stable and unstable correspond to the behavior of the fibers after pullback: semi-stable fibers remain semistable after pullback, while unstable fibers may become good (see Table 1.2).

Let $f : S \to C$ be an elliptic surface and $v \in C$. Denote by f_v the fiber over v. Consider the sets

$$\mathrm{Sing}(f) = \{v \in C \mid f_v \text{ is singular}\} \text{ and } \mathrm{Red}(f) = \{v \in C \mid f_v \text{ is reducible}\}.$$

For each $v \in C$, let $\Theta_{v,0}$ be the component of f_v intersecting the zero section. Thus we can write

$$f_v = \Theta_{v,0} + \sum_{i}^{m_v - 1} \mu_v \Theta_{v,i}.$$

where, m_v is the number of irreducible components of f_v, $\Theta_{v,i}$, $i \geq 1$, are the components of f_v not meeting the zero section and μ_v their multiplicity in f_v.

The group of divisors modulo algebraic equivalence of a smooth variety V is called the *Néron-Severi group*, denoted by $\mathrm{NS}(V)$. It follows from the Theorem of the Base that the $\mathrm{NS}(V)$ is finitely generated and torsion-free (see, for example, [9, V.1]). Its rank is called the *Picard number* of the variety V and denoted by $\rho = \rho(V)$. The group $\mathrm{NS}(V)$ becomes an indefinite integral lattice with respect to the intersection pairing of divisors.

Let us get back to the elliptic surface setting. In this situation, we have at least two curves that are independent in $\mathrm{NS}(S)$, namely the class of a fiber, which we will denote by F, and the class of the zero section $\langle O\rangle$. Let V be the direct sum $V = \bigoplus_{v\in\mathrm{Red}(f)}\langle\Theta_{v,i}\rangle_{i=1,\ldots,m_{v-1}}$. We define the *trivial sublattice* as $T := \langle F, (O)\rangle \oplus V$. Now, let $E(K)$ be the elliptic curve associated. Given a point in $P \in E(K)$, let $\bar{P}$ denote the associated section in $\mathrm{MW}(f)$.

Theorem 1.2 (Shioda-Tate Formula)**.** *The map* $P \to \bar{P} \pmod{T}$ *induces an isomorphism between* $E(K)$ *and* $\mathrm{NS}(S)/T$. *In particular, comparing ranks, we have the following formula, known as the Shioda-Tate formula*

$$\operatorname{rank} E(K) = \rho(S) - 2 - \sum_v (m_v - 1),$$

where m_v *denotes the number of irreducible components of the fiber* f_v *and the sum runs over all fibers of the fibration. (Note that it* f_v *is smooth then the summand is zero, thus the sum runs over the finitely many singular fibers.)*

Remark.: We should point out that the above formula, as well as the definitions used on it are considered over an algebraically closed field. Nevertheless, by considering the Galois action on the involved lattices, the same holds over the base field, so if $k = \bar{l}$, then

$$\operatorname{rank}\mathscr{E}(C/l) = \rho(\mathscr{E}/l) - \operatorname{rank} T^{\mathrm{Gal}(k/l)},$$

where $\mathscr{E}(C/l)$ is the groups of sections of $\mathscr{E} \to C$ defined over l and $\rho(S/l)$ is the rank of $\mathrm{NS}(\mathscr{E})^{\mathrm{Gal}(k/l)}$.

Base Change

A standard method for producing new elliptic surfaces from old ones is that of base change. More precisely, let $f : S \to B$ be an elliptic surface and let C be a projective curve mapping surjectively to C. Then the fiber product $S \times_B C$ provides a surface with a map to C whose fibers are elliptic curves.

$$\begin{tikzcd} & S \times_B C \arrow[dl] \arrow[dr] & \\ S \arrow[dr, "f"'] & & C \arrow[dl] \\ & B & \end{tikzcd}$$

Hence the surface $S \times_C B \to C$ is also an elliptic surface. Furthermore, if $\sigma : B \to C$ is a section of f, then $\sigma \times Id_C$ is a section of $S \times_B C \to C$, thus $S \times_B C$ inherits the sections of S (this will be developed with more details in Chapter 3). Geometrically, the surface can be obtained by the pull-back of the Weierstrass form of S using the map $C \to B$. For example, given an elliptic curve over a curve B over a finite field $\mathbb{F}_q$, the resulting surface induced by the q-Frobenius morphism $B \to B$ is given by raising the local parameter of B in the Weierstrass equation of S to the q-th power.

The effect of base curve change on the fibers depends on the ramification information of the base change. Let $C \to B$ be a base curve change, the fibers over which the map is not ramified are replaced by d copies of itself, where d is the degree of the map. Now let F be a fiber over which the map is ramified. Then its pull-back via the base change map is as follows: If a point in B of ramification index d has the fiber type

I_n, it is replaced by a fiber type I_{dn}. The behavior of the unstable fibers depends on the congruency class of the ramification index modulo some small integers, as described in the table bellow, from [28].

$$I_n^* \begin{cases} I_{dn} & d \equiv 0 \pmod 2 \\ I_{dn}^* & d \equiv 1 \pmod 2 \end{cases}$$

$$II \begin{cases} I_0 & d \equiv 0 \pmod 6 \\ II & d \equiv 1 \pmod 6 \\ IV & d \equiv 2 \pmod 6 \\ I_0^* & d \equiv 3 \pmod 6 \\ IV^* & d \equiv 4 \pmod 6 \\ II^* & d \equiv 5 \pmod 6 \end{cases} \qquad II^* \begin{cases} I_0 & d \equiv 0 \pmod 6 \\ II^* & d \equiv 1 \pmod 6 \\ IV^* & d \equiv 2 \pmod 6 \\ I_0^* & d \equiv 3 \pmod 6 \\ IV & d \equiv 4 \pmod 6 \\ II & d \equiv 5 \pmod 6 \end{cases}$$

$$III \begin{cases} I_0 & d \equiv 0 \pmod 4 \\ III & d \equiv 1 \pmod 4 \\ I_0^* & d \equiv 2 \pmod 4 \\ III^* & d \equiv 3 \pmod 4 \end{cases} \qquad III^* \begin{cases} I_0 & d \equiv 0 \pmod 4 \\ III^* & d \equiv 1 \pmod 4 \\ I_0^* & d \equiv 2 \pmod 4 \\ III & d \equiv 3 \pmod 4 \end{cases}$$

$$IV \begin{cases} I_0 & d \equiv 0 \pmod 3 \\ IV & d \equiv 1 \pmod 3 \\ IV^* & d \equiv 2 \pmod 3 \end{cases} \qquad IV^* \begin{cases} I_0 & d \equiv 0 \pmod 3 \\ IV^* & d \equiv 1 \pmod 3 \\ IV & d \equiv 2 \pmod 3 \end{cases}$$

Table 1.2 - Effect of base curve change on singular fibers

Alterations As mentioned before, a step of Silverman's proof does not really need the birationality hypothesis, so we can use a weaker form of resolution of singularities and work with a smooth variety. Given a morphism $f : W \to V$ of algebraic varieties we say f is a *modification* if it is birational and proper. A morphism $f : V \to W$ is called a *resolution of singularities* of W if it is a modification and V is non-singular. A theorem of Hironaka says that if V is an algebraic variety defined over an algebraically closed field of characteristic 0 has a resolution of singularities. Unfortunately, this theorem is not available as of today in positive characteristic, nevertheless a weaker version proved by de Jong [15] is enough for our needs: We say a morphism $f : W \to V$ is an *alteration* if it is surjective, generically finite and proper. Generically finite means that there is some non-empty set $U \subset V$ such that $f^{-1}(U) \to U$ is finite.

Theorem 1.3. *[15] Let K be a field and V an algebraic variety over K. There exists an alteration $f : W \to V$, where W is a nonsingular variety.*

1.3 Rational Elliptic Surfaces

Let $f : S \to C$ be an elliptic surface such that S is rational, i.e., S is a smooth projective rational surface over k and $f : S \to C$ is a relatively minimal elliptic fibration with a section $o : C \to S$. Since the elliptic fibration induces an inclusion of function fields $k(C) \to k(S)$ and $k(S)$ is a rational function field, we conclude that is $k(C)$ purely transcendental function field of degree 1 over k and $C \simeq \mathbb{P}^1_k$.

Example 1.3.1. *Let f, g denote cubic polynomials. Assume that f and g do not have a common factor. Consider the cubic pencil*

$$\mathcal{V} = \{\alpha f + \beta g = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^1$$

If $char \neq 2, 3$, a general member of the pencil os a curve of genus 1. Fixing a rational point on the pencil yields a section, thus the pencil is itself an elliptic surface. Another way this can be seen is to onsider the rational map $\pi : \mathbb{P}^2 \to \mathbb{P}^1$ given by $(f : g)$. Note that π is not defined in the base points of the pencil $\mathcal{V}$. By blowing-up the base points of the pencil on $\mathbb{P}^2$ we get a (rational) surface S, isomorphic to $\mathcal{V}$, equipped with a genus 1 fibration to $\mathbb{P}^1$

$$\begin{tikzcd} S \arrow[d] \arrow[dr] & \\ \mathbb{P}^2 \arrow[r, dashed] & \mathbb{P}^1 \end{tikzcd}$$

Example 1.3.2. *Consider the elliptic modular surface of level 5, denoted by $R_{5,5}$. It is the surface obtained by blowing $\mathbb{P}^2$ at the base points of the pencil given by the cubics $y(x-z)(y-z)$ and $zy(x-y)$. Over $\mathbb{Q}$, $R_{5,5}$ has a Weierstrass equation*

$$y^2 = x^3 + A(t)x + B(t),$$

where

$$A(t) = -\frac{1}{3}t^4 + 4t^3 - \frac{14}{3}t^2 - 4t - \frac{1}{3} \text{ and } B(t) = \frac{2}{27}t^6 - \frac{4}{3}t^5 + \frac{50}{9}t^4 + \frac{50}{9}t^2 + \frac{4}{3}t + \frac{2}{27}.$$

The elliptic surface $R_{5,5}$ has discriminant

$$\Delta = 2^{12}t^5(t^2 - 11t - 1)$$

Thus, $R_{5,5}$ has two fibers of type I_5, over 0 and ∞, and two fibers of type I_1 in the splitting field of $t^2 - 11t - 1$. The reduction modulo 5 yields the surface with Weierstrass equation

$$y^2 = x^3 + (3t^4 + 4t^3 + 2t^2 + t + 3)x + (t^6 + 2t^5 + 3t + 1)$$

Which has two singular fibers of type I_5 but the two fibers of type I_1 become a fiber of type II, since $t^2 - 11t - 1 = (t+2)^2$ become inseparable in characteristic $p = 5$ (and only in characteristic $p = 5$).

The method used to obtain the surface above, by considering a pencil of cubics, is actually enough to describe all Rational elliptic surfaces (over an algebraically closed field). The following theorem can be found on [7] for arbitrary characteristic.

Theorem 1.4. *[7, Theorem 5.6.1] Let $f : X \to \mathbb{P}^1$ be a rational elliptic surface over an algebraically closed field. Then there exists a birational morphism $\pi : X \to \mathbb{P}^2$ such that the composition of rational maps $f \circ \pi^{-1} : \mathbb{P}^2 \to \mathbb{P}^1$ is given by a pencil of cubics. Conversely, if $\mathbb{P}^2 \to \mathbb{P}^1$ is a rational map given by such pencil, then its minimal resolution of indeterminacy points is a rational elliptic surface.*

1.4 K3 surfaces

We focus here on K3 surfaces, the primary type of surface we apply our main theorem. We refer to [13] for a comprehensive treatment of the subject. A K3 surface is a minimal surface that is regular (i.e. its irregularity $h^{1,0} = H^1(\mathcal{O}_X)$ is zero) and has trivial canonical bundle. The main invariants of a K3 surface S are: $\omega_S \simeq \mathcal{O}_S, p_g = 1, h^{0,1} = h^{1,0} = b_1 = 0, \chi(\mathcal{O}_S) = 2, c_2 = 24$ and $b_2 = 22$. The Hodge diamond of a general K3 surface is given by

$$
\begin{array}{ccccc}
 & & h^{0,0} & & \\
 & h^{1,0} & & h^{0,1} & \\
h^{2,0} & & h^{1,1} & & h^{0,2} \\
 & h^{2,1} & & h^{1,2} & \\
 & & h^{2,2} & &
\end{array}
\qquad
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

In characteristic zero one knows that the Picard number ρ of S satisfies $\rho \leq 20$ and complex surfaces with Picard number attaining 20 are called *singular* [1]. On the other hand, Tate and Shioda found examples of K3 surfaces in positive characteristic satisfying $\rho = 22$ (Artin proved that there exists no K3 surface with $\rho = 21$, see [1]), such surfaces are called supersingular (or Shioda-Supersingular).

Remark. A general surface is said to be *supersingular* if the étale cohomology group $H^1(S, \mathbb{Q}_l)$, for l not equal to the characteristic, is generated by divisors on S, so the Picard number equals the second Betti

[1]This concept os singularity has no relation with the singularities of the surface.

number. This is the concept known as *Shioda-Supersingularity*. Artin [1] proved that the formal Brauer group of Shioda-Supersingular K3 surfaces are of infinite height, these surfaces are called *Artin-supersingular*. From the progress of Tate conjectures for K3 surfaces we know (Charles [Ch13], Madapusi Pera [MP13], and Maulik [Mau12]) that a K3 surface in odd characteristic is Artin-supersingular if and only if it is Shioda-supersingular.

K3 Elliptic Surfaces Every K3 elliptic surface is fibered over $\mathbb{P}^1$. Indeed, given an elliptic fibration with section $S \to C$ with C smooth, then Leray Spectral sequence gives an injection $H^1(C, \mathcal{O}_C) \hookrightarrow H^1(S, \mathcal{O}_S) = 0$, so $C \simeq \mathbb{P}^1$. Not every K3 admits an elliptic fibration and a given surface can have non-isomorphic fibrations, as shown in the following proposition.

Proposition 1.5. *[13, Proposition 11.1.3] Over an algebraically closed field of characteristic $\neq 2, 3$, a K3 surface Xadmits an elliptic fibration if and only if there exists a non-trivial line bundle L such that $L^2 > 0$. In particular, if $\rho(X) \geq 5$ then X admits an elliptic fibration. Furthermore, each surface admits at most finitely many non-isomorphic elliptic fibrations.s*

Example 1.4.1. *([13, Example 1.1.3]) Consider the surface $X_4 \subset \mathbb{P}^3$ given by the equation*

$$X: \; x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$$

In characteristic $\neq 2$, X is smooth. In fact, any smooth quartic in $\mathbb{P}^3$ is a K3 *surface: The long exact sequence induced from*

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

on $\mathbb{P}^3$ *implies that* X *has regularity* 0*, since* $H^1(\mathbb{P}^3, \mathcal{O}) = H^2(\mathbb{P}^3, \mathcal{O}(-4)) = 0$ *and the adjunction formula implies that the canonical bundle is trivial. (see the reference above for more details). A line* l *on the Fermat surface induces can be used to define a line bundle* $L := \mathcal{O}(1) \otimes \mathcal{O}(-l)$ *satisfying* $L^2 = 0$*, thus, by the proposition above,* X_4 *admits an elliptic fibration.*

Chapter 2

Silverman's Theorem in positive characteristic

As discussed in Section 1.1, the set of sections of $\mathscr{E} \to C$ is a finitely generated abelian group, isomorphic to the group of points of the elliptic curve given by the generic fiber of $\mathscr{E} \to C$ over the function field $k(C)$. The goal of this chapter is to compare the ranks of the fibers of the fibration with the rank of $\mathscr{E}(C)$. Silverman [41] dealt with the more general case of families of abelian varieties. His celebrated Specialization Theorem shows that given a smooth variety that is a family of abelian varieties fibered over a curve C, then the rank of all but finitely many fibers is greater or equal to the rank of the generic fiber. Clearly, Silverman's theorem applies in the context of elliptic surfaces. On the other hand, if instead of starting with a smooth family $\mathcal{A} \to C$ we start with an abelian variety $A/k(C)$, the specialization can be defined and we wonder if Silverman's relation still holds. The natural answer is yes since we expect every variety over a function field of a curve C to have a model as a fibered variety over C. For example, in Chapter 1 we briefly mentioned that given an elliptic curve E over the function field K of some curve C, one can obtain an elliptic surface $\mathscr{E} \to C$ such that the generic fiber $\mathscr{E}_\eta$ is isomorphic to E, so Silverman's theorem can be applied as stated. However, Silverman's proof does not apply to higher dimensional abelian varieties in positive characteristic. In this chapter we prove that Silveman's theorem still holds even if we do not assume the existence of a smooth model for variety whose generic fiber is an abelian variety in positive characteristic.

2.1 Models

As already mentioned, given an elliptic surface $\mathscr{E} \to C$, its generic fiber is an elliptic curve over $k(C)$. In this section we deal with the converse problem of finding a suitable fibered surface $\mathscr{E} \to C$ for a given elliptic curve over the function field of C. The idea is to start with the Weierstrass equation of $E/k(C)$ and consider the surface associated to it, the works of Lipman [25] and Shafarevich [36] thus yield an elliptic surface as defined in the last section. We refer to [26, Chapters 9 and 10] for a detailed account of the subject.

The more general problem is to consider a curve S, normal, connected and projective over K, the function field of a 1-dimensional Dedekind scheme C [1]. Then a model for S over C gives a way to see S 'as a surface' over C.

[1]A Dedekind scheme S is a 0 or 1-dimensional Noetherian scheme, i.e., a scheme X such that is a union of finite affine open X_i such that $\mathcal{O}(X_i)$ is a noetherian ring.

Definition 2.1. A normal fibered surface $\mathcal{S} \to C$ together with an isomorphism $f : \mathcal{S}_\eta \simeq S$ is called a model of S over C. If $\mathcal{S}$ is regular we say that the model $\mathcal{S} \to C$ is regular.

One can think of a model as a way to extend S surjectively over C. Actually, given a variety V/K, we can work with models of V/K: it is a scheme $\mathcal{V}$, surjective and flat over C, together with an isomorphism $\mathcal{V}_\eta \simeq V$. One may ask what properties $\mathcal{V}$ can have or, more specifically, to what properties of V translate in $\mathcal{V}$.

There are two classical models: the *minimal regular model for curves* of genus ≥ 1 and *Néron Models* for abelian varieties. Elliptic curves lie in the intersection between these two classes. We briefly mention the first theory. Given a curve C over K with $p_a(C) \geq 1$, there exists a regular model which is minimal with respect to birational maps. Unlike smooth curves, two generic fibered surfaces can be birational without being isomorphic, and we say $\mathcal{S} \to C$ is *minimal* if any birational morphism $\mathcal{C}' \to \mathcal{C}$ of fibered surfaces over S is in fact an isomorphism. This is equivalent to say that $\mathcal{C}$ has no exceptional divisors on the fibers. The idea to obtain the minimal regular model is to eliminate the denominators and normalize, obtaining a normal model of S over C, after desingularization a regular model is obtained and contracting the exceptional divisors yields the minimal regular model. A full proof of the existence of the minimal regular model and its uniqueness for curves of positive genus can be found on [26, Section 10.1].

Let us go back to the elliptic curve context. Given an elliptic curve over the function field $K(C)$. E admits a Weierstrass equation, which defines an equation of a surface over $K(C)$. Working with such an equation yields a model $\mathcal{W} \to C$, known as *Weierstrass Model* of E. The following results can be found on [48].

Proposition 2.2. *Given an elliptic curve $E/K(C)$, there exists a normal surface $\mathcal{W}$ over K and a morphism $\pi_0 : \mathcal{W} \to C$ with the following properties: $\mathcal{W}$ is normal, absolutely irreducible, and projective over K, π_0 is surjective, each of its fibers is isomorphic to an irreducible plane cubic, and its generic fiber is isomorphic to E.*

A process of desingularization of the surface $\mathcal{W}$ above, given by Tate's algorithm, yieds the desired elliptic surface associated to E.

Theorem 2.3. *There exists a sequence of blow-ups yielding a birational morphism $\mathscr{E} \to \mathcal{W}$ where the surface $\mathscr{E}$ and the induced map $\pi : \mathscr{E} \to C$ satisfy the properties of elliptic surface on Definition 1.1. i.e., $\mathscr{E}$ is a smooth, absolutely irreducible, relatively minimal and projective surface over K and the generic fiber of π is isomoprhic to E.*

We should point out that the surface obtained above does not need to be the minimal regular model of $E/K(C)$, see **[?]** for a comprehensive exposition of the subject.

Now, for a higher dimensional abelian variety A, there are no Weierstrass equations that one can work with to obtain such models. In any case, there exists a projective and flat C-scheme $\mathcal{A} \to C$ satisfying

$$\mathrm{Sec}_k(\mathcal{A} \to C) \simeq A(K),$$

but $\mathcal{A}$ may not be regular. Over fields of characteristic 0, resolution of singularities yields a regular projective scheme $\mathcal{A}'$ and a birational morphism $\mathcal{A}' \to \mathcal{A}$. So, if we start with an abelian variety A/K and K is a function field of characteristic 0, we can find a smooth, irreducible, projective variety $\mathcal{A}$ equipped with a flat morphism $\mathcal{A} \to C$. Silverman's Specialization Theorem is proved in this setting. If we move to the positive characteristic case we can start with a variety $\mathcal{A}$ as above and consider the generic fiber. On the other hand, if we start with an abelian variety A/K, resolution of singularities is not known, so we cannot guarantee the existence of such $\mathcal{A}$, unless we are in dimension one (in this case the minimal regular model is enough).

Nevertheless, with minor modifications, the proof of Silverman's theorem works if we work with $\mathcal{A}$ projective and normal. This is the subject of the next section.

2.2 Specialization in positive characteristic

Let K be a global field and C be a curve defined over K. Consider an abelian variety A defined over the function field $K(C)$ of C and let $\mathcal{A} \to C$ be a model for $A/K(C)$, by model we mean a proper and flat scheme $\mathcal{A}$ over C given by a morphism $\pi : \mathcal{A} \to C$ whose generic fiber $\mathcal{A}_\eta = \mathcal{A} \times_C \mathrm{Spec}(K(C))$ is isomorphic to $A/K(C)$ and the natural map $\mathcal{A}(C) \to A(K(C))$ is an isomorphism, in other words, every section $\mathrm{Spec}(K(C)) \to A$ extends to a section $C \to \mathcal{A}$. Except for a finite number of points in $C(K)$, the fiber $\mathcal{A}_t := \pi^{-1}(t)$ is an abelian variety defined over K and we have a natural group homomorphism

$$\sigma_t : A(K(C)) \to \mathcal{A}_t(K),$$

given by intersecting the section $\sigma \in A(K(C))$ with the fiber $\mathcal{A}_t$. The map σ_t is called specialization at t.

Definition 2.4. Let $\mathcal{K}/K$ be an primitive extension[2]. Let A be an abelian variety over $\mathcal{K}$. The $\mathcal{K}/K-$trace of A, denoted by $(\mathrm{Tr}_{\mathcal{K}/K}(A)), \tau_{A,\mathcal{K}/K})$, is the final object of the category of pairs (B, f), where B is abelian variety over K and $f : B_{\mathcal{K}} \to A$ is a K-morphism.

The $\mathcal{K}/K$-trace satisfies the following universal property: every K-morphism $\lambda : B_{\mathcal{K}} \to A$, where B is an abelian variety over K, factors through $\tau_{A,\mathcal{K}/K}$. In our setting $\mathcal{K} = K(C)$. The theorem of Lang-Néron asserts that the quotient group $A(K(C))/\operatorname{Tr}_{K(C)/K}(A)(K)$ is finitely generated (for more details on this, check [5]). In our case, K is a global field, so, by the Mordell-Weil theorem, $\operatorname{Tr}_{K(C)/K}(A)(K)$ is itself finitely generated. Thus we conclude that also $A(K(C))$ is finitely generated, say, with rank r_η. On the other hand, again by the Mordell-Weil theorem, for those $t \in C(K)$ such that $\mathcal{A}_t$ is an abelian variety, $\mathcal{A}_t(K)$ is again a finitely generated group, let r_t denote its rank.

A theorem of Silverman (see [41]) states that, if we start with a flat family of abelian varieties $\mathcal{A} \to C$ such that $\mathcal{A}/K$ is smooth and $\operatorname{Tr}_{K(C)/K}(A) = 0$ then the specialization map is injective except for finitely many rational

[2] I.e., the algebraic closure of K in $\mathcal{K}$ is purely inseparable

points of C. In particular, except for finitely many $t \in C(K)$, we have

$$r_\eta \leq r_t. \tag{2.1}$$

As discussed in the previous section, to adapt this result to our setting it would be enough to take a model $\mathcal{A} \to C$ for $A/K(C)$ such that $\mathcal{A}$ is smooth over K. Since we are interested in the behavior of almost all fibers (meaning we can ignore finitely many fibers) we can make birational transformations on the model $\mathcal{A}$ without interfering in the resulting nature of the fibers. In the characteristic 0 case, that is, if the global field we started with is a number field, or a function field over a number field, then Hironaka's resolution of singularities yields a smooth model $\mathcal{A} \to C$ for $A/K(C)$ and Silverman's theorem can be applied.

However, if K is itself a function field of a curve defined over a finite field, resolution of singularities is not available and we can only assume the model $\mathcal{A} \to C$ to be at most projective and normal. We include here the minor changes that must be made in order for Silverman's proof to work in our setting. The main difference is that we will need to work with Cartier divisors instead of the usual Weil divisors. In another step we can drop down the birationality hypothesis between models for A and consider alterations, a weaker form of resolution of singularities.

The first detail to take into account is that of extending divisors from the generic fiber to the whole model. In the smooth case we can work with Weil divisors on A and they can be extended to $\mathcal{A}$ by taking the Zariski closure. Since $\mathcal{A}$ may not be smooth, we will have to deal with Cartier divisors, or line bundles. C. Pépin [32] proved that, in the local case, we can modify birationally a variety so that line bundles extend from the generic fiber to the whole variety. Given a finite number of fibers we can glue the modified varieties in order to obtain a global extension.

Lemma 2.5. *Let $A/K(C)$ be an abelian variety, $\mathcal{A} \to C$ a model for A and $L_i \in \mathrm{Pic}(A), i \in I$, a finite number of line bundles on A. Then, there exists a proper model $\mathcal{A}' \to C$ for A and $\mathcal{L}_i \in \mathrm{Pic}(\mathcal{A}')$ such that $\mathcal{L}_i|_{\mathcal{A}'_\eta} = L_i$.*

Proof. We can assume that C is affine, say $C = \mathrm{Spec}\, R$, so $\mathcal{A} = \mathrm{Proj}\, B$ for some finitely generated R-algebra B, in this situation $\mathcal{A}_\eta = \mathrm{Proj}(B \otimes_R K)$, where K is the field of fractions of R. Since A is projective, Cartier divisors can be given on a finite number of open sets, which are the complements of zeroes of polynomials in $B \otimes_R K$ and after clearing denominators can be seen as polynomials in $B \otimes_R R_f$ for some localization R_f of R, thus defining open sets in $A \times_R \mathrm{Spec}\, R_f$, the same idea holds for rational functions on the generic fiber and we can extend them to open sets $A \times_R \mathrm{Spec}\, R_g$. Intersecting these sets we conclude that Cartier divisors can be extended over an open set of the base curve.

Now, if we take the intersection U of the open sets of the base curve over which each L_i can be extended, we are left with finitely many points in C, so it is enough to find a model of $\mathcal{A}$ over these points such that every L_i is extended to these points and glue L_i on them so that it will be defined over all $\mathcal{A}_U$. Since the resulting local model does not depend on the line bundle we started with, we can work with only one line bundle L, extending to a line bundle $\mathcal{L}_U$ over $\mathcal{A}_U$. So we must prove that L can be extended to the remaining finitely many points of $C \setminus U$. Let $t \in C \setminus U$ and consider $\mathcal{A}_{(t)} = \mathcal{A} \times_C \operatorname{Spec}(\mathcal{O}_{C,t})$, which has generic fiber defined over the field of fractions of $\mathcal{O}_{C,t}$. By the main result of [32], there exists a proper $\mathcal{O}_{C,t}$-scheme $\mathcal{A}''_{(t)}$ which is semi-factorial, *i.e.*, the restriction map $\operatorname{Pic}(\mathcal{A}''_{(t)}) \to \operatorname{Pic}((\mathcal{A}''_{(t)})_\eta) = \operatorname{Pic}((\mathcal{A}_{(t)})_\eta)$ is surjective, so L extends to a line bundle $\mathcal{L}''_{(t)}$ on $\mathcal{A}'_{(t)}$. Now, using [3, Sec. 1.2, Lemma 5], there exists a non-empty open subset C' of C, which we can assume to intersect $C \setminus U$ only at t. There also exists a C'-scheme $\mathcal{A}'_{(t)}$ such that $\mathcal{A}''_{(t)} = \mathcal{A}_{(t)} \times_C \operatorname{Spec}(\mathcal{O}_{C,t})$. Again, after reducing C' if necessary, we can extend $\mathcal{L}''_{(t)}$ to a line bundle $\mathcal{L}'_{(t)}$ on $\mathcal{A}'_{(t)}$, and each of these models over $\mathcal{O}_{C,t}$ have the same generic fiber, so we can glue them and obtain a C-scheme $\mathcal{A}'$, which is a model for $A/K(C)$. This model comes equipped with a line bundle $\mathcal{L}'$, obtained by gluing the line bundles $\mathcal{L}'_{(t)}$ with $\mathcal{L}_U$, restricting to L in the generic fiber of $\mathcal{A}'$. Since properness is a local property we conclude that $\mathcal{A}' \to C$ is proper, as desired. $\square$

Remark: Chow's Lemma (see [9, Ex. II.4.10]) and normalisation imply that we can assume $\mathcal{A}' \to C$ normal and projective.

In order to obtain the injectivity of the specialization map we present here the results of Silverman, the first two theorems deal with the variation of the canonical height between the fibers of the elliptic surface and the third is the injectivity result (see [41, §3, 4 and 5]).

Let $\mathcal{L}$ be a line bundle on $\mathcal{A}$ and L its restriction to $\mathcal{A}_\eta = A$. Consider $[-1]^*L$. By Lemma 2.5 it can be extended to a line bundle $\mathcal{L}'$ on $\mathcal{A}$. Denote $\mathcal{L}^{(\mathrm{s})} = \mathcal{L} \otimes \mathcal{L}'$ and $\mathcal{L}^{(\mathrm{as})} = \mathcal{L} \otimes \mathcal{L}'^{-1}$. In analogy with abelian varieties, we say that $\mathcal{L}$ is *symmetric* if $\mathcal{L}^{(\mathrm{as})} = 0$ and *anti-symmetric* if $\mathcal{L}^{(\mathrm{s})} = 0$. One should be attentive to the fact that $[-1]$ is not defined on A. Nevertheless, when restricted to the generic fiber, the equality $\mathcal{L}^{\otimes 2} = \mathcal{L}^{(\mathrm{s})} \otimes \mathcal{L}^{(\mathrm{as})}$ gives the usual decomposition of L as a sum of a symmetric and an anti-symmetric line bundle on A.

From the Weil height machine (see [20, Chapter 3]) we have that

$$h_{\mathcal{A},\mathcal{L}} = \frac{1}{2} h_{\mathcal{A},\mathcal{L}^{(\mathrm{s})}} + \frac{1}{2} h_{\mathcal{A},\mathcal{L}^{(\mathrm{as})}} + O(1).$$

On the other hand, we can glue the canonical heights defined on the smooth fibers $\mathcal{A}_t$, $t \in C^0$, defining a canonical height on $\pi^{-1}(C^0) =: U$, $\hat{h}_{\mathcal{A},\mathcal{L}} : U(\bar{K}) \to \mathbb{R}$ and similarly $\hat{h}_{\mathcal{A},\mathcal{L}^{(\mathrm{s})}}, \hat{h}_{\mathcal{A},\mathcal{L}^{(\mathrm{as})}} : U(\bar{K}) \to \mathbb{R}$. Again by the Weil height machine we have the equality

$$\hat{h}_{\mathcal{A},\mathcal{L}} = \hat{h}_{\mathcal{A},\mathcal{L}^{(\mathrm{s})}} + \hat{h}_{\mathcal{A},\mathcal{L}^{(\mathrm{as})}}.$$

Thus, for $P \in U(\bar{K})$, we have

$$|\hat{h}_{\mathcal{A},\mathcal{L}}(P) - h_{\mathcal{A},\mathcal{L}}(P)| \leq \frac{1}{2} |\hat{h}_{\mathcal{A},\mathcal{L}^{(\mathrm{s})}}(P) + \hat{h}_{\mathcal{A},\mathcal{L}^{(\mathrm{as})}}(P) - h_{\mathcal{A},\mathcal{L}^{(\mathrm{s})}}(P) - h_{\mathcal{A},\mathcal{L}^{(\mathrm{as})}}(P) + O(1)|.$$

Now, if we have bounds like

$$|\hat{h}_{\mathcal{A},\mathcal{S}}(P) - h_{\mathcal{A},\mathcal{S}}(P)| \leq c h_{C,\xi}(\pi(P)) + O(1),$$

for $\mathcal{S} = \mathcal{L}^{(\mathrm{s})}, \mathcal{L}^{(\mathrm{as})}$, where c is a constant not depending on P, the same holds for $\mathcal{L}$. This is the content of the following theorem.

Theorem 2.6. *Fix an ample divisor $\xi \in \mathrm{Div}_{\bar{K}}(C)$. There exist a non-empty open subset $U' \subset U$ and a constant $c = c(\mathcal{L}, \pi)$ such that for all $P \in U'(\bar{K})$,*

$$|\hat{h}_{\mathcal{A},\mathcal{L}}(P) - h_{\mathcal{A},\mathcal{L}}(P)| < ch_{C,\xi}(\pi(P)) + O(1).$$

Proof. By the above remark, it is enough to prove the theorem when $\mathcal{L}$ is either symmetric or anti-symmetric. Let's assume that $\mathcal{L}$ is symmetric and make the necessary remarks in the anti-symmetric case. Denote by D the Weil divisor on $\mathcal{A}_\eta$ corresponding to the restriction of $\mathcal{L}$ to the generic fiber.

The doubling morphism [2] on the generic fiber extends to a rational map that coincides with the doubling morphism $[2]_t$ when restricted to smooth fibers $\mathcal{A}_t$. Take $\phi : \mathcal{A}' \to \mathcal{A}$ to be an alteration of $\mathcal{A}$, as defined in 1.2, such that $[2] \circ \phi$ extends to a morphism $\psi : \mathcal{A}' \to \mathcal{A}$. Denote by π' the composition of $\pi \circ \phi$.

$$\begin{tikzcd}
\mathcal{A}' \arrow[d, "\phi"'] \arrow[drr, "\psi"] \arrow[ddd, bend right=60, "\pi'"'] & & & \\
\mathcal{A} \arrow[rr, dashed, "{[2]}"] \arrow[dd, "\pi"] & & \mathcal{A} & \\
& U \arrow[ul, hook'] \arrow[rr, "{[2]}"'] & & U \arrow[ul, hook'] \\
C & & &
\end{tikzcd}$$

Consider the line bundle $\mathcal{L}' := \psi^*\mathcal{L} - 4\phi^*\mathcal{L} \in \mathrm{Pic}(\mathcal{A}')$. Since $\mathcal{A}'$ is regular, $\mathcal{L}'$ corresponds to a Weil divisor D'. Let $\equiv$ denote the linear equivalence of divisors. On the generic fiber $\mathcal{A}'_\eta$,

$$D' = \psi^* D - 4\phi^* D = \phi^*([2]^* D - 4D) \equiv 0.$$

The equivalence $\equiv$ follows from the fact that we have assumed $\mathcal{L}$ to be symmetric - in the anti-symmetric case we should have defined $\mathcal{L}' := \psi^*\mathcal{L} - 2\phi^*\mathcal{L}$ - so D' is principal on $\mathcal{A}'_\eta$ and we can choose another divisor D'' equivalent to D' such that $\pi'(\mathrm{Supp}\ D'')$ is a closed proper subset of C, so it is a finite subset, say $\{P_1, \dots, P_r\}$. This implies that we can pick a divisor $\theta = \sum n_i P_i$ such that $\pi'^*\theta \pm D''$ are both positive.

Given $P \in U(\bar{K})$ such that $P \notin \phi(\mathrm{Supp}\ D'')$, any $P' \in \phi^{-1}(P)$ satisfies $P' \notin \mathrm{Supp}\ D'' \supset \mathrm{Supp}\ (\pi^*\theta \pm D'')$, so, by the Weil height machine, we have

$$h_{\mathcal{A}', \pi'^*\theta \pm D''}(P') > O(1), \tag{2.2}$$

since $\pi'^*\theta \pm D''$ are positive. Moreover

$$h_{\mathcal{A}', \pi'^*\theta \pm D''}(P') = h_{\mathcal{A}', \pi'^*\theta}(P') \pm h_{\mathcal{A}', D''}(P') + O(1).$$

Hence, by (2.2)

$$\begin{aligned} |h_{\mathcal{A}', D''}(P')| &< h_{\mathcal{A}', \pi'^*\theta}(P') + O(1) \\ &< h_{C,\theta}(\pi'(P')) + O(1) \\ &= h_{C,\theta}(\pi(P)) + O(1). \end{aligned}$$

On the other hand,

$$\begin{aligned} |h_{\mathcal{A},\mathcal{L}}([2]P) - 4h_{\mathcal{A},\mathcal{L}}(P)| &= |h_{\mathcal{A},\mathcal{L}}(\psi(P')) - 4h_{\mathcal{A},\mathcal{L}}(\phi(P'))| \\ &= |h_{\mathcal{A}',\psi*\mathcal{L}-4\phi*}(P')| + O(1) \\ &= |h_{\mathcal{A}',\mathcal{L}''}(P')| + O(1). \end{aligned}$$

Thus, we have the inequality

$$|h_{\mathcal{A},\mathcal{L}}([2]P) - 4h_{\mathcal{A},\mathcal{L}}(P)| < h_{C,\theta}(\pi(P)) + O(1).$$

This holds for every P such that for any $P' \in \phi^{-1}(P)$, we have $P' \notin \mathrm{Supp}\ D''$. But $\pi'(\mathrm{Supp}\ D'')$ is a proper closed subset of S, thus $\mathrm{Supp}\ D''$ has only finitely many fibers as components, defining U' as the complement in U of these fibers, the inequality holds for any $P \in U'$. To conclude, given any ample divisor $\xi \in C$, there exists c such that

$$h_{C,\theta} < ch_{C,\xi} + O(1),$$

thus, for $P \in U'(\bar{K})$

$$|h_{\mathcal{A},\mathcal{L}}([2]P) - 4h_{\mathcal{A},\mathcal{L}}(P)| < ch_{C,\xi}(\pi(P)) + O(1).$$

□

Now we state the other results

Theorem 2.7. *Let K be a global field and C/K a nonsingular projective curve with function field $K(C)$. Let $A/K(C)$ be an abelian variety with K(C)/K-trace* *zero and $\mathcal{A} \to C$ a normal and projective model for $A/K(C)$. Fix a section $P \in A(C)$, a line bundle $L \in \mathrm{Pic}(A)$ and an extension $\mathcal{L}$ of L to $\mathcal{A}$, given by Lemma 2.5. Then*

$$\lim_{t \in C^0(\bar{K}), h_C(t) \to \infty} \frac{\hat{h}_{\mathcal{A}_t, \mathcal{L}_t}(P_t)}{h_C(t)} = \hat{h}_{\mathcal{A}_\eta, \mathcal{L}_\eta}(P_\eta)$$

Proof. See [41, Theorem B]. The only observation we make is that, in the last paragraph, we should take n prime to p in order to have $\hat{h}([n]P) = n^2\hat{h}(P)$. □

We are now able to state Silverman's Specialization theorem.

Theorem 2.8. *[Silverman Specialization Theorem] Let $\sigma_t : \mathcal{A}(C) \to \mathcal{A}_t(\bar{K})$ be the specialization map. If the $K(C)/K$-trace of A is zero, then*

$$\{t \in C^0(\bar{K}) \mid \sigma_t \textit{ is not injective}\}$$

is a set of bounded height in $C(\bar{K})$. In particular, for finite extensions L of K, σ_t is injective for all but finitely many $t \in C^0(L)$.

Proof. See [41, Theorem B]. □

Note that the theorem assumes that A has $K(C)/K$-trace zero, just as Theorem 2.7, but for our purposes we can drop off this hypothesis. As we are dealing mostly with elliptic curves, the first option is to consider only non-constant ones, because they have trace zero (See [5, Exemple 2.2]). On the other hand, since we are only interested in the behavior of the rank, a weak version of the theorem suffices. Indeed, if we let $A_0 =$

$\mathrm{Tr}_{K(C)/K}(A)$, then $(A_0)_{K(C)}$ is isogenous to an abelian subvariety A' of A and the rank inequality (2.1) holds for A', since $(A_0)_{K(C)}(K(C)) = A_0(K)$ and the rank is preserved by isogenies. Now, there exists an abelian subvariety B, whose trace is zero, such that $A' \times B$ is isogenous to A. By having no constant part, Silverman's Specialization Theorem 2.8 can be applied directly to B. Therefore, inequality (2.1) for both A' and B imply the same inequality for A.

Chapter 3

Rank jump in characteristic p

In this section,

k	= $\mathbb{F}_q$ is a finite field of characteristic p
k_n	= $\mathbb{F}_{q^n}$ is the degree n extension of k
$K = k(C)$	is a global field of characteristic p
X/K	is a curve over K
$\psi : \mathcal{X} \to C$	is a smooth projective model for X/K over C
$\mathcal{K}$	= function field of X/K
$E/\mathcal{K}$	is a non-constant elliptic curve
$\mathscr{E} \to X$	elliptic surface over K with generic fiber E

Let $\pi : \mathscr{E} \to B$ be an elliptic surface. By Theorem 2.8, except for finitely many $t \in B(K)$, we have $\operatorname{rank} \mathscr{E}_t(K) \geq \operatorname{rank} \mathscr{E}_\eta(\mathcal{K})$. In [35], Salgado proves that if K is a number field and $\mathscr{E}$ is a unirational elliptic surface, then there exists infinitely many points on $B(K)$ such that $\operatorname{rank} \mathscr{E}_t(K) \geq \operatorname{rank} \mathscr{E}_\eta(\mathcal{K}) + 1$. In this chapter we prove the analogous result in the positive characteristic setting.

Theorem 3.1. *Let K be a global field of positive characteristic $p > 0$ and $\mathscr{E} \to C$ be an elliptic fibration over K such that $\mathscr{E}$ is unirational, then there are infinitely many points $t \in C(K)$ such that* $\operatorname{rank} \mathscr{E}_t(K) \geq \operatorname{rank} \mathrm{MW}(\mathscr{E} \to C/K) + 1.$

In positive characteristic there exists non-rational unirational surfaces, so we get new classes of (elliptic) surfaces satisfying the hypothesis of the theorem. For instance,

Corollary 3.2. *If S is a supersingular $K3$ surface with a non-trivial elliptic fibration $S \to \mathbb{P}^1$, then there exists a base change $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that*

$$\operatorname{rank} \mathrm{MW}(S \times_f \mathbb{P}^1 \to \mathbb{P}^1) \geq \operatorname{rank} \mathrm{MW}(S \to \mathbb{P}^1) + 1.$$

Corollary 3.2 is a consequence of Theorem 3.1 and a recent result from Liedtke (see [22]) proving that every supersingular $K3$ surface is a non-rational unirational surface and if the Artin invariant is ≤ 9, it admits a jacobian elliptic fibration. A key step in the proof of theorem 3.1 is to find bounds for the extensions generated by division points of rational points in the elliptic curve associated to $\mathscr{E}$. This is also the point which differs most from Salgado's proof, since division by multiples of the characteristic behaves in a different manner. We isolate these results in the next section and summarize them in Proposition 3.6.

3.1 Bounding the degree of $\frac{1}{n}P_0$ & Kummer Theory

Let $E/\mathcal{K}$ be a non-isotrivial elliptic curve, $P_0 \in E(\mathcal{K})$ and $P \in E(\overline{\mathcal{K}})$. Assume that $[n]P = P_0$ for some integer $n \geq 1$. The goal of this section is to give an upper bound for n depending on the degree of the field of definition of P.

Torsion We first deal with the case where P is a torsion point. It is known that, given a non-isotrivial elliptic curve over a function field K of one variable, its L-rational torsion, where L is an extension of the base field, is bounded by a constant depending on the degree h_L, say

$$\#E(L)_{\mathrm{tors}} \leq f(h_L),$$

for some monotone increasing function f (see [33, Theorem 1.5]). Thus, if we let m denote the order of P and L the field of definition of P, we have an inequality

$$m \leq \#E_{\mathrm{tors}}(L) \leq f(h_L) \tag{3.1}$$

bounding m by a function on the degree of the field L.

However, since we are dealing with an elliptic curve E over $\mathcal{K}$, a function field on two variables of a finite field, we will have to go through a process of specialization in order to apply the argument above.

Specialization argument Let v be a place of good reduction of E. Denote by $k(v)$ the residue field of v and E_v the reduction modulo v of E. Then we have the reduction modulo v map $E(\mathcal{K}) \to E_v(k(v))$ and a short exact sequence

$$0 \longrightarrow E_1(\mathcal{K}) \longrightarrow E(\mathcal{K}) \longrightarrow E_v(k(v)) \to 0$$

Theorem 3.3 (see [11, Theorems C.2.5, C.2.6]). *The kernel of the reduction map, $E_1(\mathcal{K})$, is isomorphic to the formal group associated to the maximal ideal $\mathfrak{m}_v$ and thus has no prime-to-p torsion, so the reduction map is injective in the prime-to-p torsion.*

Now, if $\mathcal{L}$ is a finite extension of $\mathcal{K}$, and $l(v)$ the residue field associated to an extension of v to $\mathcal{L}$, then, by (3.1) we can bound the prime-to-p torsion of $E_v(l(v))$ by the degree of $l(v)/k(v)$, which can be bounded by the degree of $L/\mathcal{K}$ and, again, by the injectivity of the reduction map away from p, we bound the prime-to-p torsion of $E(\mathcal{L})$ by the degree of $\mathcal{L}$. And, as desired, the order of a torsion point can thus be bounded by the degree of the field generated by its coordinates.

In order to extend these bounds to all torsion points we will have to exclude some places of good reduction (this corresponds to ignore some points in the base curve). Since E is non-isotrivial, thus ordinary, and the specialization map is injective except for a finite number of fibers, then these fibers are also ordinary, so the number of places of supersingular reduction is finite and their p^n-torsion is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. Since the reduction map is already surjective we obtain an isomorphism $E[p^n](\overline{\mathcal{K}}) \simeq$

$E_v[p^n](\overline{k(v)})$, thus bounds for p^n-torsion points depending on the degree of extensions $l(v)/k(v)$ induce bounds for p^n-torsion points depending on the degree of $\mathcal{L}/\mathcal{K}$.

Now suppose P has infinite order. Write $n = p^\alpha n'$, with $(p, n') = 1$. Then we have the field inclusions $\mathcal{K} = \mathcal{K}(P_0) \subset \mathcal{K}([n']P) \subset \mathcal{K}(P)$, so

$$[\mathcal{K}(P) : \mathcal{K}] = [\mathcal{K}(P) : \mathcal{K}([n']P)][\mathcal{K}([n']P) : \mathcal{K}],$$

thus we can deal with the prime-to-p and p-power cases separatedly.

Inseparable Case Let $P_0 \in E(\mathcal{K}) \setminus [p]E(\mathcal{K})$ and $P \in E(\bar{\mathcal{K}})$ satisfy $[p^n]P = P_0$, $n \in \mathbb{N}$. The condition $P_0 \notin [p]E(\mathcal{K})$ is automatic if we assume that n is minimal with the property that $[p^n]P \in E(\mathcal{K})$, this will be enough for our application. We prove the more generic fact that, except for a finite number of points of points in $E(\mathcal{K})$ (those with coordinates in the base finite field), given $P_i \in E(\overline{\mathcal{K}})$ satisfying $[p]P_i = P_{i-1}$ for $i \geq 1$, there exists a constant c, independent of P_i, such that $[\mathcal{K}(P_i) : \mathcal{K}]_{\text{insep}} \geq cp^i$.

We deal first with the case $P_0 \notin E(\mathcal{K}^p)$. In this situation the equality holds. Write $\mathcal{K}_i := \mathcal{K}(P_i)$. Since we can extend the base field, without loss of generality, we assume that $G := E[p](\overline{\mathcal{K}}) \subset E(\mathcal{K})$, then the usual projection $f : E \to E/G =: E_1$ is separable, defined over $\mathcal{K}$ and has kernel G, let $\hat{f} : E_1 \to E$ be its dual, so $\hat{f} \circ f = [p]$. Recall that $[p]$ can also be factored as $V \circ \mathrm{Frob}$, where $\mathrm{Frob} : E \to E^{(p)}$ denotes the usual Frobenius and $V = \widehat{\mathrm{Frob}}$ its dual, the Verschiebung map.

$$\begin{tikzcd}
 & & E_1^{(p)} \arrow[dd, "\phi"] \\
 & E_1 \arrow[ur, "\mathrm{Frob}_1"] \arrow[dr, "\hat{f}"] & \\
E \arrow[ur, bend left, "f"] \arrow[rr, "{[p]}"] \arrow[dr, bend right, "\mathrm{Frob}"'] & & E \\
 & E^{(p)} \arrow[ur, bend right, "V"'] &
\end{tikzcd}$$

Since we assumed that E is non-isotrivial, we have that $G \simeq \mathbb{Z}/p\mathbb{Z}$, thus $\ker f = G = \ker[p]$, so $\hat{f}$ is a purely inseparable map and we can factor it as $\hat{f} = \phi \circ \mathrm{Frob}_1$, where $\mathrm{Frob}_1 : E_1 \to E_1^{(p)}$ is the Frobenius of E_1 and ϕ is the separable part, which is an isomorphism since $\hat{f}$ is purely inseparable.

Since $\hat{f} \circ f(P_1) = P_0$ and $P_0 \notin E(\mathcal{K}^p)$, in particular, $P_0 \notin \hat{f}(E_1(\mathcal{K}))$, we conclude that the coordinates of $f(P_1)$ are p^{th}-roots of elements in $\mathcal{K}$ that are not p^{th} powers, so $\mathcal{K}(f(P_1))$ is inseparable of degree 1 over $\mathcal{K}$. Moreover, being f separable, $\mathcal{K}_1/\mathcal{K}(f(P_1))$ is separable, thus $[\mathcal{K}_1 : \mathcal{K}]_{\text{insep}} = p$. Now we proceed by induction. We will need the following result

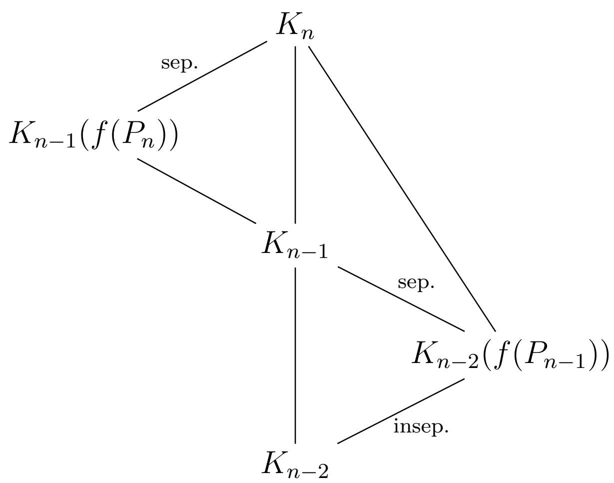
Lemma 3.4. *Let K be a function field of positive characteristic p and F/K a purely inseparable extension of degree p. If L is a separable extension of F, then L^p is contained in a separable extension of K.*

Proof. Since F/K is purely inseparable of degree p we can assume that $F = K(\alpha)$ and the minimal polynomial of α is given by $f(x) = x^p - a$ for some $a \in K$. Now, by the primitive element theorem, since L/K is separable, $L = F(\beta)$ for some β separable over F, thus $L = K(\alpha, \beta)$. Then

$$L^p = K^p(\alpha^p, \beta^p) = K^p(a, \beta^p) = K^p(\beta^p) \subset K(\beta^p).$$

Write the minimal polynomial of β over F as $p_{\beta,F}(x) = x^n + \sum_{i=0}^{n-1} a_i x^i$, with $a_i \in F = K(\alpha)$, then $g(x) = x^n + a_{n-1}^p x^{n-1} + \cdots + a_1^p x + a_0^p$ has β^p as a root. Since $a_i \in K(\alpha)$, $a_i^p \in K$, so β^p is separable over K. Thus L^p is contained in a separable extension of K. $\square$

Assume, by induction, that $[\mathcal{K}(P_i) : \mathcal{K}]_{\text{insep}} = p^i$ for $i \in \{1, \ldots, n-1\}$.



If $P_{n-1} \notin E(\mathcal{K}_{n-1}^p)$ then the same argument from the case $i = 1$ works and we conclude that $[\mathcal{K}_n : \mathcal{K}]_{\text{insep}} = p^n$. We claim that the $P_{n-1} \in E(\mathcal{K}_{n-1}^p)$ cannot occur. Indeed, if this is the case then $f(P_n) \in E_1(\mathcal{K}_{n-1})$, so

$$\mathcal{K}_{n-1} = \mathcal{K}_{n-1}(f(P_n)).$$

In particular, $\mathcal{K}_n/\mathcal{K}_{n-1}(f(P_{n-1}))$ would be separable. By the Lemma 3.4

and the induction hypothesis, this would imply that $\mathrm{Frob}(P_n)$ has coordinates in a separable extension of $\mathcal{K}_{n-2}$.

On the other hand, $P_{n-1} = V(\mathrm{Frob}(P_n))$, and since V is a separable map, we conclude that the coordinates of P_{n-1} are separable over $\mathcal{K}_{n-2}$, which is a contradiction with the induction hypothesis that $[\mathcal{K}_{n-1} : \mathcal{K}_{n-2}]_{\text{insep}} = p$. We thus obtain the bound $[\mathcal{K}_n : \mathcal{K}] \geq [\mathcal{K}_n : \mathcal{K}]_{\text{insep}} = p^n$

Now, assume that $P_0 \in E(\mathcal{K}^p)$, so $\hat{f}(P_1) \in E(\mathcal{K})$, thus $\mathcal{K}_1/\mathcal{K}$ is separable of degree $\leq p$. We claim that $P_1 \notin E(\mathcal{K}_1^p)$. Indeed, in this case the p^{th}-roots of the coordinates of P_1 would be contained in $\mathcal{K}_1$. Since we assumed the coodinates of P_0 were not in the finite base field, the same holds for P_1, so the p^{th}-roots of its coordinates belonging to $\mathcal{K}_1$ contradicts the fact that $\mathcal{K}_1 = \mathcal{K}(P_1)$. Therefore we have $P_1 \notin E(\mathcal{K}_1^p)$ and, repeating the argument from (3.1), we again obtain $[\mathcal{K}_n : \mathcal{K}] \geq [\mathcal{K}_n : \mathcal{K}]_{\text{insep}} \geq (1/p)p^n$.

Separable Case Let m be the smallest positive integer such that there exists $P_1 \in E(\mathcal{K})$ and $T \in E_{tors}$ satisfying

$$[m]P = P_1 + T.$$

Note that, since $[n]P = P_0$, such m exists and $m \leq n$. In fact, given an integer $d \geq 0$

$$[n - dm]P = [n]P - [dm]P = (P_0 - [d]P_1) - [d]T \in E(\mathcal{K}) + E_{\text{tors}},$$

and by the minimality of m we conclude that $m|n$, thus m is also prime with p. The same argument from [35] tells us that P_1 is indivisible by m.

Let $m' = mm_1$, where m_1 is the order of T, let T' be such that $[m]T' = T$ and write $P' = P - T'$, so $\mathcal{K}(P', E_{m'}) = \mathcal{K}(P, E_{m'})$. We have the following inequality

$$[\mathcal{K}(P) : \mathcal{K}] \geq [\mathcal{K}(P, E_{m'}) : \mathcal{K}(E_{m'})] = [\mathcal{K}(P', E_{m'}) : \mathcal{K}(E_{m'})].$$

We are now in the following situation: $[m]P' = P_1 \in E(\mathcal{K})$ indivisible and $T' \in E_{m'}$ and we want to estimate $[\mathcal{K}(P, E_{m'}) : \mathcal{K}(E_{m'})]$. This was done for number fields in [10] using ideas of [34]. Since we are in the case where p does not divide m, the same argument using Galois Theory works in our situation.

Lemma 3.5. *Let A be an abelian variety over $\mathcal{K}$. Then there exists a constant $c_0 = c_0(E, \mathcal{K})$ such that, if P is an indivisible point of infinite order in $A(\mathcal{K})$ and if $m|n$ are positive integers with $(p, m) = 1$, then*

$$\left[\mathcal{K}\left(A_n, \frac{1}{m}P\right) : \mathcal{K}(A_n)\right] \geq c_0 m$$

Proof. See [10, Appendix 2]. The only remark we make is that Falting's results regarding the endomorphism ring and the semisimplicity of the torsion subgroups must be replaced by analogues in positive characteristic, these were proved by Zahrin and Moret-Bailly and can be found, for example, in [49]. □

We conclude this section by grouping together the results above in the following proposition.

Proposition 3.6. *Let E be a non-isotrivial elliptic curve over a function field $\mathcal{K}$ of positive characteristic p. Let $P_0 \in E(\mathcal{K})$ such that the coordinates of P_0 are not contained in the base field, then there exists a constant $c = c(E)$ such that for every $P \in E(\overline{\mathcal{K}})$ satisfying $[n]P = P_0$ we have $[\mathcal{K}(P) : \mathcal{K}] \geq cn$, assuming n is minimal with the property that $[n]P \in E(\mathcal{K})$. In other words, if h denotes the degree of $\mathcal{K}(P)$ over $\mathcal{K}$, there exists f_h such that $n \leq f_h$.*

3.2 Rank jump in characteristic p

The goal in this section is to extend Salgado's results (see [35]) for elliptic surfaces over global fields of positive characteristic.

Let X be a curve over $K = \mathbb{F}_q(C)$ and E be an elliptic curve over the function field $\mathcal{K}$ of X. Let $\pi : \mathscr{E} \to X$ be an elliptic surface associated to $E/\mathcal{K}$. By Silverman's specialization (Theorem 2.8) we have

$$\operatorname{rank}(\mathscr{E}(\mathcal{K})) \leq \operatorname{rank}(\mathscr{E}_t(K)),$$

for all but finitely many $t \in X(K)$. We prove that, if $\mathscr{E}/K$ is a unirational surface, then for infinitely many $t \in X(K)$,

$$\operatorname{rank}(\mathscr{E}_t(K)) \geq \operatorname{rank}(\mathscr{E}(\mathcal{K})) + 1. \tag{3.2}$$

The idea is to pick a finite cover $\phi : Y \to X$ and consider the base change

$$\begin{tikzcd} \mathscr{E} \times_X Y = \mathscr{E}_Y \arrow[r, "p_1"] \arrow[d, "\pi_Y"'] & \mathscr{E} \arrow[d, "\pi"] \\ Y \arrow[r, "\phi"'] & X \end{tikzcd}$$

If we can find such Y satisfying

$$\operatorname{rank}(\mathscr{E}_Y(K(Y))) \geq \operatorname{rank}(\mathscr{E}(K(X))) + 1, \tag{3.3}$$

again, by Silverman's specialization, for all but finitely many $t \in Y(K)$

$$\operatorname{rank}(\mathscr{E}_Y(K(Y))) \leq \operatorname{rank}((\mathscr{E}_Y)_t(K)).$$

Since $(\mathscr{E}_Y)_t(K) \simeq \mathscr{E}_{\phi(t)}(K)$, for these t, we have

$$\operatorname{rank} \mathscr{E}_{\phi(t)}(K) = \operatorname{rank}((\mathscr{E}_Y)_t(K)) \geq \operatorname{rank}(\mathscr{E}_Y(K(Y))) \geq \operatorname{rank}(\mathscr{E}(\mathcal{K})) + 1$$

thus, for the points $\phi(t) \in X(K)$ we have the desired inequality. Since we are interested in the case where (3.2) is satisfied for infinitely many points, we consider only curves Y such that $Y(K)$ is infinite. In the conclusion we will only deal with rational curves but the main results are valid for any curve Y with infinitely many rational points, thus Y is either a rational curve of an elliptic curve with positive rank.

Base Change Consider a curve $i_Y : Y' \subset \mathscr{E}$ that is not contained in a fiber, so $\pi(Y')$ is not a single point, thus $\pi|_{Y'} : Y' \to X$ is a surjective morphism of algebraic curves. Consider $\nu : Y \to Y'$ the normalization of

Y, so we have a finite cover $\phi = \pi \circ i_Y \circ \nu : Y \to X$.

Each section $\sigma \in \mathrm{Sec}(\mathscr{E})$ induces a section in $\mathscr{E}_Y$ given by

$$\sigma \times \mathrm{Id}_Y : X \times_X Y = Y \to \mathscr{E}_Y = \mathscr{E} \times_X Y,$$

on the other hand, we have a new section

$$\tau_Y = (\nu \circ i_Y, \mathrm{Id}_Y) : Y \to \mathscr{E}_Y = \mathscr{E} \times_X Y.$$

It can happen, though, that the new section τ_Y is linearly dependent with the old ones, in this case, there are sections σ_i and $n_i, n \in \mathbb{Z}$ such that

$$\sum [n_i](\sigma_i)_Y + [n]\tau = 0.$$

This can be translated in the following criteria.

Lemma 3.7. *An irreducible curve $Y \subset \mathscr{E}$ that is not a component of a fibre induces a new section on $\mathscr{E}_Y$ independent of the old ones if and only if for every section $Y_0 \in \mathrm{Sec}(\mathscr{E})$ and every $n \in \mathbb{N}^*$, Y is not a component of $[n]^{-1}(Y_0)$.*

Instead of searching for a curve that verifies the above condition, we prove that for curves in a numerical family, its enough to consider only a finite number of sections Y_0 and finitely many positive integers n. This is the content of the following Proposition, proved in [35]. Since we dealt with the problem of inseparability in the preceding section, the proof goes verbatim to the cited reference. We include it in here for the sake of completeness.

Proposition 3.8 (cf. [35, Proposition 4.2]). *Let $\mathscr{E} \to X$ be an elliptic surface defined over K, let $\mathscr{Y}$ be a numerical family of curves inside of $\mathscr{E}$. Then there exists $n_0 = n_0(\mathscr{Y})$ and a finite subset $\Sigma_0(\mathscr{Y}) \subset \mathrm{Sec}(\mathscr{E})$ such that for Y in the family $\mathscr{Y}$, the new section induced by Y in $\mathscr{E}_Y$ is linearly dependent of the old ones if and only if $[n]Y = Y_0$, for some $Y_0 \in \Sigma_0(\mathscr{Y})$ and $n \leq n_0(\mathscr{Y})$.*

Proof. Take Y in $\mathscr{Y}$, if it induces a section that is linearly dependent with the old ones, then $[n]\tau = \sigma_Y$ for some $\sigma \in \mathrm{Sec}\,\mathscr{E}$, so $[n]Y = Y_0$ for $Y_0 = \sigma(X)$, we can assume n minimal with this property. Kummer theory gives an upper bound for n depending on the degree of curves in $\mathscr{Y}$ and Néron-Tate height theory shows that there are only finitely many possibilities for such Y_0. Therefore linear dependence can only happen in this setting.

Let P_0 be the point in $E(\mathcal{K})$ corresponding to a section Y_0 of $\mathscr{E}$, if $P \in E(\bar{\mathcal{K}})$ satisfy $[n]P = P_0$ and h denotes the degree of the field generated by P, i.e. $h = [\mathcal{K}(P) : \mathcal{K}]$, then, by Proposition 3.6 we get $n \leq f(h) =: n_0$.

Now, fix a set of generators $\{Y_1, \ldots, Y_r\}$ for the Mordell-Weil group of $\mathscr{E}$. For a fixed n write $n_i = (([n]Y).Y_i)$, these numbers depend only on the numerical class of Y. Write $m_0 = ([n]Y.O)$, where O denotes the zero-section and $l_{v,j_v} = ([n]Y.\Theta_{v,j_v})$. Consider the set

$$\Sigma_0 = \{Y_0 \in \mathrm{Sec}(\mathscr{E}) \mid (Y_0.Y_i) = n_i, (Y_0.\Theta_{v,j_v}) = l_{v,j_v} \text{ and} (Y_0.O) = m_0\}$$

The heights of the sections of $\mathscr{E}$ depend only on the intersection numbers with the zero section, the generators of the Mordell-Weil group and the components of the fibers, thus the constraints on the intersection numbers of the sections in Σ_0 imply that this is a section of bounded height, so

Σ_0 is finite, thus there are finitely many possible sections Y_0 such that $Y \subset [n]^{-1}Y_0$. $\square$

The problem now is reduced to find an infinite number of rational curves in the same numerical class and use the theorem to conclude that only a finite number of them will yield a section that is linearly dependent with the old ones. For this we assume that $\mathscr{E}$ is a unirational surface, so there exists a dominant rational map $\psi : \mathbb{P}^2 \to \mathscr{E}$. Let L be the family of lines passing through the origin of $\mathbb{P}^2$ and L^ψ its image on $\mathscr{E}$ by ψ. Then L^ψ is a family of rational lines on $\mathscr{E}$, in particular they are all numerically equivalent. Thus the theorem implies that there exists a positive integer $n_0(L^\psi)$ and a finite subset of sections $\sigma_0(L^\psi)$ such that if $C \in L^\psi \setminus [1, n_0(L^\psi)] \cdot \Sigma_0(L^\psi)$, where

$$[1, n_0(L^\psi)] \cdot \Sigma_0(L^\psi) = \left\{ n \cdot \sigma(B) \mid n \in \{1, \ldots, n_0(L^\psi)\} \text{ and } \sigma \in \Sigma_0(L^\psi) \right\},$$

then the change of basis $\mathscr{E} \to \mathscr{E}_C$ yields an elliptic surface satisfying (3.3), so, by specialization, we have infinitely many $t \in B(K)$ such that $\operatorname{rank} \mathscr{E}_t(B(K)) \geq \operatorname{rank} \mathscr{E}(k(B)) + 1$ and this proves Theorem 3.1.

Chapter 4

Applications

A direct application of Theorem 3.1 consists of considering unirational elliptic surfaces. Recall that a surface S is called *unirational* if there exists a dominant rational map $f : \mathbb{P}^2 \dashrightarrow S$. In algebraically closed fields of characteristic zero, Castelnuovo's theorem implies that unirational surfaces are rational, see, for example, [2]. On the other hand, in positive characteristic, a non-zero pluricanonical form may vanish under pull-back, so being dominated by a rational surface does not imply that S is rational. [1] Indeed, the result still holds if the function field extension induced by f is separable, in this case we say S is separably rational. However, if S is inseparably rational, there exists unirational surfaces that are not rational.

Many non-rational unirational surfaces have been found since then. Zariski [50] gave examples of such surfaces in every positive characteristic. In this chapter we study surfaces with an elliptic fibration for which

[1] For higher dimensional varieties, even in characteristic 0 there are examples of unirational varieties that are not rational. Some examples are given in [30], for every dimension $n \geq 3$.

Theorem 3.1 can be applied. In particular, we find all possible fiber configurations for supersingular K3 arising as the pull-back of rational elliptic surfaces by Frobenius morphism.

K3 surfaces as base changes of rational elliptic surfaces A nice way to obtain K3 surfaces is to consider base curve changes from rational elliptic surfaces: Let $R \to \mathbb{P}^1$ be a rational elliptic surface and let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a finite degree morphism. The singular fibers of the resulting surface $S = R \times_{\mathbb{P}^1} \mathbb{P}^1$ can be described from the ones of R, through the process described in Section 1.2 so, computing the Euler numbers we can restrict ourselves to base changes yielding K3 surfaces

Example 4.0.1. *Let S be the rational elliptic surface with fiber type $I_9 + 3I_1$. It can be described as the pencil generated by the cubics Q and Q', where Q' has a node at q and Q intersects Q at q with multiplicity 9, for example*

$$\begin{aligned} Q' : \quad x^3 + y^3 - xyz &= 0 \\ Q : \quad xz^2 + x^2y + y^2z &= 0. \end{aligned}$$

Thus, the pencil of cubics has equation

$$\mu(x^3 + y^3 - xyz) + \lambda(xz^2 + x^2y + y^2z) = 0, \quad (\mu : \lambda) \in \mathbb{P}^1.$$

Which has the Weierstrass equation

$$E : y^2 = x^3 + \left(\frac{1}{2}t^3 - \frac{1}{48}\right)x + \left(\frac{1}{4}t^6 - \frac{1}{24}t^3 + \frac{1}{864}\right).$$

The discriminant of E is given by $-27t^{12} + t^9$, and the singular fibers are over 0 (of type I_9) and over $\zeta_3^n \frac{1}{3}$, with $n = 0, 1, 2$ and ζ_3 a primitive cubic root of unity (all of type I_1).

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a degree-two morphism branched over 0 and $\frac{1}{3}$ and let X be the surface induced by the pullback of S by f, denote its generic fiber by E'. Then X is a $K3$ surface with an elliptic fibration to $\mathbb{P}^1$ and has the following singular fibers: an I_{18} over 0, an I_2 fiber over $\frac{1}{3}$ and four I_1 fibers, over the pre-images of $\zeta_3 \frac{1}{3}$ and $\zeta_3^2 \frac{1}{3}$. Let T be the sublattice of $\mathrm{NS}(X)$ generated by the zero section, a fiber F and the irreducible components of fibers,

$$T = \langle (O), F \rangle \oplus \bigoplus_{v \in R} T_v = \langle (O), F \rangle \oplus T_0 \oplus T_{\frac{1}{3}}$$

thus $\operatorname{rank} T = 2 + m_0 + m_{\frac{1}{3}} = 2 + 17 + 1 = 20$. The Shioda-Tate Formula gives a bound on the rank of E':

$$\operatorname{rank} E' = \rho(X) - \operatorname{rank} T = \rho(X) - 20$$

Since the Picard number of a K3 surface cannot be 21 we have two options: either $\rho(X) = 20$ or $\rho(X) = 22$. Notice that if $\rho(X) = 20$, then E has rank zero and the Mordell-Weil group of $X \to \mathbb{P}^1$ is finite, otherwise $\rho(X) = 22$, i.e. X is supersingular, implying that $MW(X \to \mathbb{P}^1)$ has either rank 0 to 2.

Base change from a rational elliptic surface is not the only way to produce supersingular K3 surfaces. Let n be coprime with $p = char(K)$.

The Fermat surface of degree n is given by

$$X_n : x_1^n + x_2^n + x_3^n + x_4^n = 0.$$

The surface X_n is non-singular and irreducible if $n > 2$. It is rational if $n = 3$, K3 if $n = 4$ and of general type if $n \geq 5$. The Fermat surface X_4 is unirational if $p \not\equiv 3 \pmod 4$ and Tate (see [45]) showed that $\rho(X_4) = 22$. In [39] Shioda gave a condition for X_n to be unirational and proved that any unirational surface is supersingular. The Jacobian fibration of the Fermat Surface X_4 has six singular fibers of type I_4 and thus cannot be considered as the a base change from some rational surface through Frobenius pullback, since $p \neq 2$. (see [24] for more details). The Fermat quartic is actually a particular example of a Kummer surface:

Example 4.0.2. *Let A be an abelian surface over K a let $\iota : A \to A$ be the automorphism $x \mapsto -x$, thus the group $G = \{\mathrm{Id}_A, \iota\}$ acts on A and, if $char(K) \neq 2$, has 16 fixed points, which gives 16 rational double point singularities on A/G. The blow-up of A/G of these points yields a K3 surface (see [2, Theorem 10.6 and Remark 10.7(b)]). The surfaces given by this process are called Kummer surfaces and often denoted by $\mathrm{Km}(A)$. The Picard number of $\mathrm{Km}(A)$ is given by $16 + \rho(A)$. If $A = E \times E'$ is a product of elliptic curves we have*

$$\rho(\mathrm{Km}(A)) = 16 + 2 + \operatorname{rank} \operatorname{Hom}(E, E').$$

In particular, if $E = E'$ is a supersingular elliptic curve, then $\operatorname{Hom}(E, E') = \operatorname{End}(E)$ is an order in a quaterion algebra [42, V.3], thus $\operatorname{rank}(\mathrm{Km}(A)) = 22$.

In [40], Shioda proved that a Kummer surface in characteristic $p > 2$ is unirational if and only if it is supersingular. In this same paper Shioda (and several others) conjectured that such result holds for any K3 surface. Some special cases where proved by Rudakov and Shafarevich, Pho, Shimada and others. The full conjecture was proved by Liedtke for characteristic $p \geq 5$ in [22]. Which we briefly describe here.

Fields of definition Let us make a remark about the fields of definition. Since supersingular elliptic curves are defined over finite fields, Kummer surfaces associated to product of supersingular elliptic curves are indeed supersingular K3 surfaces, but they can be defined over $\mathbb{F}_{p^2}$. A supersingular abelian surface is isogenous to a product of supersingular elliptic curves but such isogeny is defined over the algebraic closure of the field of definition, thus there exist supersingular abelian surfaces that cannot be defined over finite fields. The Kummer surfaces associated to these surfaces yield supersingular K3 surfaces that are not defined over finite fields, so the elliptic fibrations are 'honest' fibrations over $\mathbb{P}^1$ over function fields.

Liedtke's work Given a supersingular K3 surface X, the discriminant of its Néron-Severi lattice is equal to $-p^{2\sigma_0}$ for some $\sigma_0 \in \{1, \ldots, 10\}$. The integer σ_0 is called the *Artin invariant* of X. Given two varieties X and Y of the same dimension, an isogeny of degree n is a dominant, rational and generically finite map $X \dashrightarrow Y$ of degree n. A purely inseparable isogeny of height h is an isogeny that is purely inseparable of degree p^h.

Liedtke's main result is the theorem below

Theorem 4.1. *[See [22], Main theorem] Given any two supersingular K3 surfaces X and X' with Artin invariant σ_0 and σ_0', then there exist dominant and rational maps*

$$X \dashrightarrow X' \dashrightarrow X$$

both of which are purely inseparable and generically finite of degree $p^{2\sigma_0+2\sigma_0'-4}$.

Since it is already known that supersingular Kummer surfaces are unirational, given a K3 surface X, the existence of such a dominant rational map from a Kummer surface to X implies that X is unirational. In the course of the proof, Liedtke determines which supersingular $K3$ have a jacobian elliptic fibration:

Theorem 4.2. *Let X be a supersingular K3 surface with Artin invariant σ_0 in characteristic $p \geq 5$.*

1. *If $\sigma_0 \leq 9$ then X admits a jacobian elliptic fibration.*

2. *If $\sigma_0 = 10$, then X does not admit a jacobian elliptic fibration.*

The following elliptic surface is another example supersingular K3 surface with an elliptic fibration which cannot be given as a pull back of a rational elliptic fibration:

Example 4.0.3. *In [37], the authors proved that the there exists a unique elliptic surface with an I_{19} fiber. This is a K3 surface whose equation can be given as*

$$X : y^2 = x^3 + (t^4 + t^3 + 3t^2 + 1)x^2 + 2(t^3 + t^2 + 2t)x + t^2 + t + 1.$$

If the characteristic $p \neq 2$ is not a quadratic residue $\pmod{19}$ or $p = 19$, then X defines a supersingular surface. If $p \neq 19$, then X cannot arise as a Frobenius pull-back of a rational surface, since the fiber I_{19} would then arise from a fiber of type I_n with $19 = pn$. If $p = 19$ then X is indeed the 19-Frobenius pull back of a surface with singular fiber configuration $[I_1, II, III^*]$.

Example 4.0.4. *Recall that a surface X is an Enriques surface if $q(X) = 0$, $p_g(X) = 0$ and $2K_X = 0$. Enriques surfaces can be obtained by taking the quotient of a $K3$ surface by a fixed-point-free involution, this involution is usually called an Enriques involution. On the other hand, every Enriques surface has a double cover which is '$K3$-like'. In the complex scenario, the study of the effect of the double cover (and the fixed-point-free involution) in the associated Néron-Severi lattices gives a criterion for the existence of an Enriques quotient for a generic $K3$. The proof uses the Torelli theorem for K3 surfaces, proved by A. Ogus, a survey on this subject can be found on [23]. Since the theorem holds for supersingular $K3$ surfaces, the result can be extended for such $K3$ surfaces in positive characteristic. Using this idea, Jang verified (see [17]) that a supersingular $K3$ surface in characteristic $p = 19$ or $p > 23$ has an Enriques involution if and only if its Artin invariant is less than 6.*

Now, given an Enriques surface E, recall that E has a genus 1 fibration that is not elliptic, since it has non-reduced fibers. Nevertheless, the jacobian of such Enriques surface is a rational elliptic surface.

Proposition 4.3. *Let $\mathscr{E}^{(p)} \to \mathbb{P}^1$ be a K3 elliptic surface in characteristic p arising as a Frobenius pull-back from a rational elliptic surface $\mathscr{E}$. Then $\mathscr{E}^{(p)} \to \mathbb{P}^1$ is supersingular and there is no rank growth from $\mathscr{E}$ to $\mathscr{E}^{(p)}$.*

Proof. Since $\mathscr{E}$ is a rational elliptic surface, it's generic fiber has a Weierstrass equation of the form

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t),$$

with $\deg a_i(t) \leq i$.

The Frobenius base change amounts to raising the parameter t of the base curve to the p-power. Thus, $\mathscr{E}^{(p)}$ is given by the equation

$$y^2 + a_1(t^p)xy + a_3(t^p)y = x^3 + a_2(t^p)x^2 + a_4(t^p)x + a_6(t^p).$$

Since $t \to t^{1/p}$ is purely inseparable, we have an embedding

$$k(x, y, t^{1/p}) \hookrightarrow k(x, y, t) \hookrightarrow k(\mathbb{P}^2),$$

so $\mathscr{E}^{(p)}$ is unirational, therefore supersingular by Liedtke's theorem 4.1. Since $\mathscr{E}^{(p)}$ is supersingular, its Néron-Severi group has rank 22. Together with the contribution of the fibers we can compute the respective ranks and check that there is no rank growth. Tables **A.1-6** in Appendix **A** describe the change in the configuration of bad fibers in the base change for elliptic surfaces over fields of characteristic $p \leq 19$. For $p > 19$, the possible singular fibers configurations are given in Table **A.7**, according to the congruence class of p: If $p \equiv 3 \pmod 4$ then C_3 is a possible configuration, if $p \equiv 2 \pmod 3$ then C_4 is also a possible configuration, if, furthermore, $p \equiv 5 \pmod 6$ then C_2 and C_3 also appear as possible fiber configurations. $\square$

Example 4.0.5. *Consider the elliptic surface $\mathscr{E} \to \mathbb{P}^1$ over the algebraic closure of $\mathbb{F}_p(a)$ given by*

$$y^2 + xy = x^3 - at,$$

$\mathscr{E}$ is a rational surface with two fibers of type I_1 and one fiber of type II^*. By Shioda-Tate formula $\mathscr{E}$ has rank 0. From Table **A.3** *we see that, for $p = 11$, the elliptic surface $\mathscr{E}^{(p)} \to \mathbb{P}^1$ given as the pull-back by the Frobenius map $t \mapsto t^{11}$ yields a unirational K3 surface with singular fibers I_{11}, I_{11} and II. A Weierstrass equation for $\mathscr{E}^{(p)}$ is given by:*

$$y^2 + xy = x^3 - at^p$$

The K3 surface $\mathscr{E}^{(p)}$ has singular fibers I_{11}, I_{11} and II and rank 0. The discriminant and j-invariant of $\mathscr{E}^{(p)}$ are given as follows:

$$\Delta = 8a^2t^{22} + at^{11} \text{ and } j = \frac{1}{8a^2t^{22} + at^{11}}.$$

Now, consider the specializations $t \mapsto a^{p^{n-1}}$, for $n \in \mathbb{N}$. The fibers are given by the elliptic curves

$$\mathscr{E}^{(p)}_{a^n}/\mathbb{F}_p(a) \quad : \quad y^2 + xy = x^3 - a^{p^n+1}.$$

D. Ulmer proved in [47] that the elliptic curves $\mathscr{E}^{(p)}_{a^n}$ have rank at least $\frac{(p^n-1)}{2n}$ for $n \in \mathbb{N}$, so there exist infinitely many fibers such that

$$0 = \mathrm{MW}(\mathscr{E}^{(p)} \to \mathbb{P}^1) < \frac{p^n - 1}{2n} = \operatorname{rank} \mathscr{E}^{(p)}_{a^n},$$

as expected by Theorem 3.1.

A jump in the rank from the rational to a supersingular K3 Let us consider an example of a rational elliptic surface $\mathscr{E} \to \mathbb{P}^1$ such that the rank jump is 3. The idea is to consider a base change from $\mathscr{E}$ to a K3 surface $\mathscr{E}'$ such that the rank jump between $\mathscr{E}$ and $\mathscr{E}'$ is 2, i.e.

$$\operatorname{rank} \mathscr{E}'(\mathbb{P}^1) = \operatorname{rank} \mathscr{E}(\mathbb{P}^1) + 2.$$

Now, if we could find an example where the resulting $K3$ is supersingular and unirational over k, then Theorem 3.1 can be applied yielding another elliptic surface $\mathscr{E}'' \to C$ such that

$$\operatorname{rank} \mathscr{E}''(C) \geq \operatorname{rank} \mathscr{E}'(\mathbb{P}^1) + 1 = \operatorname{rank} \mathscr{E}(\mathbb{P}^1) + 3.$$

So, we have infinitely many points $t \in \mathbb{P}^1$ such that

$$\operatorname{rank} \mathscr{E}_t(k) \geq \operatorname{rank} \mathscr{E}(\mathbb{P}^1) + 3.$$

If we take $\mathscr{E}' \to \mathbb{P}^1$ to be the Frobenius pullback of $\mathscr{E}$, Proposition 4.3 assures that the resulting K3 surface will be supersingular, on the other hand it shows that these base changes do not yield any rank jump. We must then restrict ourselves to non-Frobenius base change. Let us consider the base changes yielding a configuration of singular fibers whose trivial lattice big rank, we consider the following cases:

$\boxed{\operatorname{rank} T(\mathscr{E}') = 22}$ In this case Shioda-Tate formula implies that $\operatorname{rank} \mathscr{E}'(\mathbb{P}^1) = 0$, since $\rho(\mathscr{E}') \leq 22$. So $\mathscr{E}'$ is supersingular but there is no rank jump.

$\boxed{\operatorname{rank} T(\mathscr{E}') = 21}$ Since $\rho(\mathscr{E}') \neq 21$, we should have $\rho(\mathscr{E}') = 22$, so $\operatorname{rank}(\mathscr{E}') = 1$. Using the possible singlar fiber configurations for the rational elliptic surfaces $\mathscr{E}$ and the effect of base change, we found that the ones yielding a K3 $\mathscr{E}'$ with $\operatorname{rank} T(\mathscr{E}') = 21$ already satisfy $\operatorname{rank} \mathscr{E}(\mathbb{P}^1) = 1$, so the section of $\mathscr{E}' \to \mathbb{P}^1$ is inherited from $\mathscr{E}$, thus there is no rank jump.

$\boxed{\operatorname{rank} T(\mathscr{E}') = 20}$ From $\rho(\mathscr{E}') \neq 21$ we conclude that $\mathscr{E}'$ has rank either 0 or 2. If the rank of the rational elliptic surface $\mathscr{E}$ under $\mathscr{E}'$ is already 2, then $\mathscr{E}'$ is supersingular but its rank is also 2. For example, the p-Frobenius pullback of a rational elliptic surface with three singular fibers of type IV yield a K3 elliptic surface with three IV^* fibers whenever $p \equiv 2 \pmod 3$. In this case $\operatorname{rank} T(\mathscr{E}') = 20$, but the initial rational elliptic surface have rank 2, so $\mathscr{E}'$ is a supersingular K3 with rank 2.

On the other hand, if $\operatorname{rank} \mathscr{E}(C) = 0$, then there is room for a jump in the rank. We find such a surface. We present it in the following example.

Example 4.0.6. *Take $\mathscr{E} \to \mathbb{P}^1$ to be the rational elliptic surface over $K = \mathbb{F}_p(a)$ with fiber configuration $[I_6, I_3, I_2, I_1]$. One such surface is given by the Weierstrass equation* [2]

$$\mathscr{E}: \ y^2 + txy + ty = x^3 + (t+1)x^2 + tx$$

From Shioda-Tate formula $\mathscr{E}(\mathbb{P}^1)$ has rank 0. Now take the base field with characteristic 5 and consider the base change $t \mapsto (t-3)^2$ yields a K3 surface $\mathscr{E}'$, a

[2]We remark that, even though $\mathscr{E}$ can be defined as a surface over $\mathbb{F}_p$, it is considered here as surface over the function field $\mathbb{F}_p(a)$, since the rank of specialization only make sense in this context.

Weierstrass equation for $\mathscr{E}'$ is given by

$$y^2 + (t^2 + 4t + 4)xy + (t^2 + 4t + 4)y = x^3 + (t^2 + 4t)x^2 + (t^2 + 4t + 4)x$$

MAGMA *[51] confirms that the rank of $\mathscr{E}'$ is 2 (over the algebraically closed field) and, indeed, we could find sections like $P = (t, t+1)$ and $Q = (4t + 1, 4t + 2)$. $2P$ is given by*

$$2P = \left(\frac{4t^4 + 2t^3 + t^2 + 3t}{t^4 + 3t^3 + 3t^2 + 3t + 1}, \frac{3t^6 + 4t^5 + t^4 + 2t^3 + 3t^2 + 3t}{t^6 + 2t^5 + t^4 + 3t^3 + t^2 + 2t + 1} \right)$$

So, by the Nagell-Lutz Theorem (this is classical in characteristic 0, see, for example, [42], a generalization for positive characteristic was proved in [6]), we conclude that P is non-torsion [3]. Therefore, $\mathscr{E}'$ is a supersingular K3 surface and there is a rank jump from $\mathscr{E}$ to $\mathscr{E}'$. So, by Corollary 3.2, there exists a base change $C \to \mathbb{P}^1$ such that the associated pullback yields a surface $\mathscr{E}''$ satisfying $\operatorname{rank} \mathscr{E}''(C) \geq \operatorname{rank} \mathscr{E}(\mathbb{P}^1) + 3 \geq 3$, *so there are infinitely many fibers $\mathscr{E}_t$ of the initial rational elliptic surface such that* $\operatorname{rank} \mathscr{E}_t(K) \geq 3$.

Remark. The base change above in other characteristics may have different behavior. For example, in characteristic $p = 19$. The base change $t \mapsto (t-8)^2$ on the rational elliptic surface with no free part

$$\mathscr{E} :\ y^2 + txy + t = x^3 + (t+1)x^2 + tx$$

[3] The section Q is also non-torsion but it is not independent with P since $3P = -3Q + T$, where $T = (4, 0)$ is a 2-torsion point.

yields the K3 surface given by

$$\mathscr{E}' : \; y^2 + (t^2+3t+7)xy + (t^2+3t+7)y = x^3 + (t^2+3t+8)x^2 + (t^2+3t+7)x,$$

which MAGMA predicts to also have rank 0.

Appendix A

Supersingular K3 arising from rational

In this Appendix we describe the effect on the configuration of singular fibers of the pullback p-Frobenius morphism on the base curve of the rational elliptic surfaces $\mathscr{E} \to \mathbb{P}^1$ yielding K3 surfaces $\mathscr{E}^{(p)} \to \mathbb{P}^1$. The algorithm bellow describes how the list was obtained. We list the ingredients:

- **L** is the list of possible singular fiber configurations for rational elliptic surfaces. Such list can be obtained, for example, from [29] or [31]. For $\operatorname{char} k \neq 2, 3$, the list of singular fibers coincide with the ones in characteristic zero, Jarvis *et. al* [18] has a description for these cases.

- Each **FiberConfig** is the list of singular fibers in a particular rational elliptic surface (RES) in **L**, described in Table 1.1.

- The function **Euler** computes the Euler number of a RES with a given singular fiber configuration, defined as the alternating sum of the Betti numbers. From the formula $e(\mathscr{E}) = \sum_{v \in C} e(F_v)$, where F_v

is the fiber on $\mathscr{E}$ over v, and from the Euler numbers of the fibers,

$$e(F_v) = \begin{cases} 0 & \text{if } F_v \text{ is smooth,} \\ m_v & \text{if } F_v \text{ is multiplicative,} \\ m_v + 1 & \text{if } F_v \text{ is additive,} \end{cases}$$

where m_v isthe number of irreducible fibers of F_v, we can compute $e(S)$. (See [7] or [38] for more details).

- The **BaseChange** function returns singular fibers on $\mathscr{E}^{(p)}$ originating from the ones on $\mathscr{E}$, this is described in Table 1.2.

- The function **Trivial** gives the rank of the trivial lattice, given by the formula $2 - \sum_{v \in C}(m_v - 1)$, as seen on Section 1.2.

The algorithm is given by:

Input: List L of config. for RES, prime p
Output: List of config. of K3 arising as Frobenius pull back from RES and their rank

```
Lo ← [ ]
for FiberConf in L do
    if Euler(FiberConf) = 24 then
        NewFiberConf ← [BaseChange(Fiber, p) for Fiber in
          FiberConf ]
        rk ←22 - Trivial(NewFiberConf)
        Lo.append([NewFiberConf, rk])
    end
end
return Lo
```

$p = 5$

Config. of singular fibers of $\mathscr{E}$	Config. of singular fibers of $\mathscr{E}^{(p)}$	Rank
$[III^*, I_2, I_1]$	$[III^*, I_{10}, I_5]$	0
$[III^*, II, I_1]$	$[III^*, II^*, I_5]$	1
$[III^*, 3I_1]$	$[III^*, 3I_5]$	1
$[IV^*, I_3, I_1]$	$[IV, I_{15}, I_5]$	0
$[IV^*, I_2, II]$	$[IV, I_{10}, II^*]$	1
$[IV^*, I_2, 2I_1]$	$[IV, I_{10}, 2I_5]$	1
$[IV^*, 2II]$	$[IV, 2II^*]$	2
$[IV^*, II, 2I_1]$	$[IV, II^*, 2I_5]$	2
$[IV^*, 4I_1]$	$[IV, 4I_5]$	2
$[I_2^*, III, I_1]$	$[I_{10}^*, III, I_5]$	1
$[I_1^*, IV, I_1]$	$[I_5^*, IV^*, I_5]$	1
$[I_1^*, III, I_2]$	$[I_5^*, III, I_{10}]$	1
$[I_1^*, III, II]$	$[I_5^*, III, II^*]$	2
$[I_1^*, III, 2I_1]$	$[I_5^*, III, 2I_5]$	2
$[I_0^*, IV, II]$	$[I_0^*, IV^*, II^*]$	2
$[I_0^*, IV, 2I_1]$	$[I_0^*, IV^*, 2I_5]$	2
$[I_0^*, III, I_2, I_1]$	$[I_0^*, III, I_{10}, I_5]$	2
$[I_0^*, III, II, I_1]$	$[I_0^*, III, II^*, I_5]$	3
$[I_0^*, III, 3I_1]$	$[I_0^*, III, 3I_5]$	3
$[3IV]$	$[3IV^*]$	2
$[IV, 2III, I_2]$	$[IV^*, 2III, I_{10}]$	3
$[IV, 2III, 2I_1]$	$[IV^*, 2III, 2I_5]$	4
$[I_3, 3III]$	$[I_{15}, 3III]$	3
$[3III, I_2, I_1]$	$[3III, I_{10}, I_5]$	4
$[3III, 3I_1]$	$[3III, 3I_5]$	5

Table A.1: Configurations for 5-Frobenius base change

$p = 7$

Config. of singular fibers of $\mathscr{E}$	Config. of singular fibers of $\mathscr{E}^{(p)}$	Rank
$[II^*, 2I_1]$	$[II^*, 2I_7]$	0
$[III^*, I_2, I_1]$	$[III, I_{14}, I_7]$	0
$[III^*, 3I_1]$	$[III, 3I_7]$	1
$[IV^*, III, I_1]$	$[IV^*, III^*, I_7]$	1
$[IV^*, I_2, II]$	$[IV^*, I_{14}, II]$	1
$[IV^*, II, 2I_1]$	$[IV^*, II, 2I_7]$	2
$[I_2^*, 2II]$	$[I_{14}^*, 2II]$	2
$[I_1^*, IV, I_1]$	$[I_7^*, IV, I_7]$	1
$[I_1^*, III, II]$	$[I_7^*, III^*, II]$	2
$[I_1^*, 2II, I_1]$	$[I_7^*, 2II, I_7]$	3
$[I_0^*, IV, 2I_1]$	$[I_0^*, IV, 2I_7]$	2
$[I_0^*, 2III]$	$[I_0^*, 2III^*]$	2
$[I_0^*, III, II, I_1]$	$[I_0^*, III^*, II, I_7]$	3
$[I_0^*, I_2, 2II]$	$[I_0^*, I_{14}, 2II]$	3
$[I_0^*, 2II, 2I_1]$	$[I_0^*, 2II, 2I_7]$	4

Table A.2: Configurations for 7-Frobenius base change

$p = 11$

Config. of singular fibers of $\mathscr{E}$	Config. of singular fibers of $\mathscr{E}^{(p)}$	Rank
$[II^*, 2I_1]$	$[II, 2I_{11}]$	0
$[III^*, II, I_1]$	$[III, II^*, I_{11}]$	1
$[IV^*, III, I_1]$	$[IV, III^*, I_{11}]$	1
$[IV^*, 2II]$	$[IV, 2II^*]$	2
$[I_0^*, IV, II]$	$[I_0^*, IV^*, II^*]$	2
$[I_0^*, 2III]$	$[I_0^*, 2III^*]$	2
$[3IV]$	$[3IV^*]$	2

Table A.3: Configurations for 11-Frobenius base change

$p = 13$

Config. of singular fibers of $\mathscr{E}$	Config. of singular fibers of $\mathscr{E}^{(p)}$	Rank
$[III^*, II, I_1]$	$[III^*, II, I_{13}]$	1
$[IV^*, III, I_1]$	$[IV^*, III, I_{13}]$	1
$[I_1^*, III, II]$	$[I_{13}^*, III, II]$	2
$[I_0^*, III, II, I_1]$	$[I_0^*, III, II, I_{13}]$	3

Table A.4: Configurations for 13-Frobenius base change

$p = 17$

Config. of singular fibers of $\mathscr{E}$	Config. of singular fibers of $\mathscr{E}^{(p)}$	Rank
$[IV^*, III, I_1]$	$[IV, III, I_{17}]$	1
$[IV^*, 2II]$	$[IV, 2II^*]$	2
$[I_0^*, IV, II]$	$[I_0^*, IV^*, II^*]$	2
$[3IV]$	$[3IV^*]$	2

Table A.5: Configurations for 17-Frobenius base change

$p = 19$

Config. of singular fibers of $\mathscr{E}$	Config. of singular fibers of $\mathscr{E}^{(p)}$	Rank
$[III^*, II, I_1]$	$[III, II, I_{19}]$	1
$[I_0^*, 2III]$	$[I_0^*, 2III^*]$	2

Table A.6: Configurations for 19-Frobenius base change

For $p > 19$, the possible fiber configurations are given below

	Config. of singular fibers of $\mathscr{E}$	Config. of singular fibers of $\mathscr{E}^{(p)}$	Rank
C_1	$[IV^*, 2II]$	$[IV, 2II^*]$	2
C_2	$[I_0^*, IV, II]$	$[I_0^*, IV^*, II^*]$	2
C_3	$[I_0^*, 2III]$	$[I_0^*, 2III^*]$	2
C_4	$[3IV]$	$[3IV^*]$	2

Table A.7: Configurations for p-Frobenius base change, with $p > 19$.

Bibliography

[1] Artin, M. *Supersingular K3 surfaces*, Ann. Sci. École Norm. Sup. (4) 7, 543-567 (1974).

[2] Badescu, L. *Algebraic surfaces*. Universitext, Springer-Verlag, New York, 2001.

[3] Bosch, S.; Lütkebohmert, W.; Raynaud, M. *Néron models*. Ergebnisseer Mathematik und ihrer Grenzgebiete (3), 21. Springer-Verlag, Berlin, 1990. x+325 pp.

[4] Conrad, B. *Chow's K/k-image and K/k-trace, and the Lang-Néron theorem*. Enseign. Math. (2) 52 (2006), no. 1-2, 37-108

[5] Conrad, B. *Minimal models for elliptic curves*. November, 2015. Preprint available at `http://math.stanford.edu/ conrad/papers/minimalmodel.pdf`

[6] Conrad, B.; Conrad, K.; Helfgott, H. *Root numbers and ranks in positive characteristic*, Advances in Mathematics 198 (2005), pp. 684-731

[7] Cossec, F. R., Dolgachev, I. V. *Enriques surfaces. I.* Progress in Math., vol. 76. Birkhäuser, Basel (1989)

[8] Dokchitser, T. *Notes on the Parity Conjecture.* Elliptic curves, Hilbert modular forms and Galois deformations, 135–198, Adv. Courses Math. CRM Barcelona, Birkhaüser/Springer, Basel, 2013.

[9] Hartshorne, R. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[10] Hindry, M. *Autour d'une conjecture de Serge Lang*. Invent. Math. 94 (1988), no. 3, 575-603

[11] Hindry, M.; Silverman, J. H. *Diophantine geometry. An introduction*. Graduate Texts in Mathematics, 201. Springer-Verlag, New York, 2000. xiv+558 pp.

[12] Hindry, M.; Pacheco, A.; Wazir, R. *Fibrations et conjecture de Tate.* J. Number Theory 112 (2005), no. 2, 345-368.

[13] Huybrechts, D. *Lectures on K3 Surfaces.* Cambridge Studies in Advanced Mathematics (2016)

[14] Ito, H. *On automorphisms of supersingular K3 surfaces.* Osaka J. Math. 34 (1997), 717-724

[15] de Jong, A. J. *Smoothness, semi-stability and alterations*. Inst. Hautes Études Sci. Publ. Math. No. 83 (1996), 51-93.

[16] de Jong, A. J. *Stacks Project*. (http://stacks.math.columbia.edu/)

[17] Jang, J. *An Enriques involution of a supersingular K3 surface over odd characteristic*. Int Math Res Notices, 11 (2014), 3158-3175

[18] Jarvis, T. *et al. Classification of Singular fibers on rational elliptic surfaces in charactersitic three.* Comm. in Algebra, 33 (2005) 4533–4566,

[19] Katsura, T. *Unirational elliptic surfaces in characteristic* p. Tôhoku Math. Journal 33 (1981), 521-553.

[20] Lang, S. *Fundamentals of Diophantine Geometry*. Springer Science & Business Media, (1983)

[21] Liedtke, C. *Algebraic surfaces in positive characteristic.* Birational geometry, rational curves, and arithmetic, 229-292, Springer, New York, 2013

[22] Liedtke, C. *Supersingular K3 surfaces are unirational.* Inventiones Mathematicae, 200 (2013), 979-1014

[23] Liedtke, C. *Lectures on Supersingular K3 Surfaces and the Crystalline Torelli Theorem.* In: Faber C., Farkas G., van der Geer G. (eds) K3 Surfaces and Their Moduli. Progress in Mathematics, vol 315. Birkhäuser, Cham (2016)

[24] Liedtke, C.; Ito, H. *Elliptic K3 surfaces with p^n-torsion sections.* J. Algebraic Geom. 22 (2013), no. 1, 105–139

[25] Lipman, J. *Desingularization of two-dimensional schemes* Ann. Math. 107 (1978), 151-207.

[26] Liu, Q. *Algebraic geometry and arithmetic curves.* Oxford Graduate Texts in Mathematics, 6. Oxford Science Publications. Oxford University Press, Oxford, 2002. xvi+576 pp.

[27] Milne J. S. *On a conjecture of Artin and Tate*. Ann. of Math. (2) 102 (1975), 517-533.

[28] Miranda, R. *The basic theory of elliptic surfaces*. Dottorato di ricerca in matematica, ETS Editrice, Pisa (1989).

[29] Miranda, R. *Persson's list of singular fibers for a rational elliptic surface* Math. Z. 205, (1990), 191-211.

[30] Michael, A.; Mumford, D. *Some elementary examples of unirational varieties which are not rational.* Proceedings of the London Mathematical Society 3.1 (1972), 75-95.

[31] Persson, U. *Configurations of Kodaira fibers on rational elliptic surfaces.* Mathematische Zeitschrift 205.1 (1990) 1-47.

[32] Pépin, C. *Modèles semi-factoriels et modèles de Néron*. Math. Ann. 355 (2013), no. 1, 147-185.

[33] Poonen, B. *Gonality of modular curves in characteristic p*. Math. Res. Letters 14, 2007, p. 691-701.

[34] Ribet, K. *Kummer theory on extensions of abelian varieties by tori.* Duke Math. J. 46 (1979), no. 4, 745-761.

[35] Salgado, C. *On the rank of the fibers of rational elliptic surfaces*. Algebra Number Theory 6 (2012), no. 7, 1289-1314.

[36] Shafarevich, I. *Lectures on minimal models and birational transformations of two dimensional schemes* Tata Inst. Fundam. Res., 37, Bombay, 1966.

[37] Schütt, M. *The maximal singular fibres of elliptic K3 surfaces.* Arch. Math. (Basel) 87, Nr. 4 (2006), pp. 309–319.

[38] Schütt, M.; Shioda, T. *Elliptic surfaces.* Algebraic geometry in East Asia—Seoul 2008, 51-160, Adv. Stud. Pure Math., 60, Math. Soc. Japan, Tokyo, 2010.

[39] Shioda, T. *An example of unirational surfaces in characteristic p.* Math. Ann., 211 (1974), 233-236.

[40] Shioda, T. *Some Results on Unirationality of Algebraic Surfaces.* Math. Ann., 230 (1977), 153-168

[41] Silverman, J. H. *Heights and the specialization map for families of abelian varieties*. J. Reine Angew. Math. 342 (1983), 197-211.

[42] Silverman, J. H. *The Arithmetic of Elliptic Curve*. GTM 106, Springer (1986)

[43] Szydlo, M. *Elliptic fibers over non-perfect residue fields.* J. Number Theory 104 (2004), 75-99

[44] Tate, J. T. *On the conjectures of Birch and Swinnerton-Dyer and a geometric analog*. Séminaire bourbaki, vol. 9, 1966, pp. Exp. No. 306, 415-440

[45] Tate, J. *Algebraic cycles and poles of zeta functions.* In Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), pages 93-110. Harper & Row, New York, 1965

[46] Trihan, F.; Wuthrich, C. *Parity conjectures for elliptic curves over global fields of positive characteristic.* Compositio Math. 147 (2011) 1105-1128

[47] Ulmer, D. *Elliptic curves with large rank over function fields.* Annals of Mathematics 155 (2002), 295–315

[48] Ulmer, D. *Elliptic curves over function fields.* Arithmetic of L-functions, 211-280, IAS/Park City Math. Ser., 18, Amer. Math. Soc., Providence, RI, 2011.

[49] Zarhin, Yu. *Abelian varieties over fields of finite characteristic.* Cent. Eur. J. Math. 12 (2014), no. 5, 659-674.

[50] Zariski, O. *On Castelnuovo's criterion of rationality* $p_a = P_2 = 0$ *of an algebraic surface*. Illinois Journal of Mathematics, 2 (1958) 303–315

[51] Bosma, W.; Cannon, J.; Playoust, C. *The Magma algebra system. I. The user language.* J. Symbolic Comput., 24 (1997), 235–265.