Universidade Federal do Rio de Janeiro
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Controle e Estabilização para uma Classe de Sistemas de Boussinesq do tipo Benjamin-Bona-Mahony

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Tese
Orientador: Prof. Ademir Fernando Pazoto

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Tese de Doutorado apresentada ao Programa de Pós-graduação do Instituto de Matemática, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários para a obtenção do título de Doutor em Ciências.

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Resumo

Nesse trabalho, provamos uma série de resultados sobre as propriedades de controle e estabilização para uma classe de sistema Boussinesq que acopla duas equações do tipo Benjamin-Bona-Mahony. Inicialmente, consideramos o sistema num intervalo limitado e mostramos que este não é espectralmente controlável se os controles atuarem num extremo do intervalo. No entanto, pode-se mostrar que é aproximadamente controlável. Em seguida, quando o modelo é considerado em um domínio periódico, propomos vários mecanismos dissipativos que nos levam a sistemas para os quais todas as trajetórias são atraídas pela origem desde que a continuação única de soluções fracas seja válida. Finalmente, os problemas de controle e estabilização são abordados para um sistema Boussinesq de ordem superior, em um domínio periódico. As propriedades de estabilidade são demonstradas quando operadores dissipativos generalizados são introduzidos em cada equação. Mais precisamente, as soluções do sistema linearizado decaem uniformemente ou não para zero, dependendo dos parâmetros desses operadores. No caso de decaimento uniforme, mostramos que a mesma propriedade é válida para o sistema não linear. No que diz respeito às propriedades de controle, se apenas um único controle interno for usado, a controlabilidade exata é estabelecida através do Hilbert Uniqueness Method. Se considerarmos dois controles, obtém-se um resultado de controlabilidade exato mais forte usando o método de momentos.

Palavras chave: Controlabilidade, sistema de Boussinesq , equação de Benjamin-Bona-Mahony, propriedade de continuação única.

Abstract

This work is devoted to prove a series of results concerning the control and stabilization properties for a class of Boussinesq system which couples two Benjamin-Bona-Mahony type equations. Initially, we consider the system posed on a bounded interval and show that it is not spectrally controllable if the controls act at one endpoint of the interval. However, it can be shown that it is approximately controllable. Next, when the model is posed on a periodic domain, we propose several dissipation mechanisms leading to systems for which all the trajectories are attracted by the origin provided that the unique continuation of weak solutions holds. Finally, the control and stabilization problems are addressed for a higher-order Boussinesq system, posed on a periodic domain. The stability properties are proved when generalized damping operators are introduced in each equation. More precisely, the solutions of the linearized system decay uniformly or not to zero, depending on the parameters of the damping operators. In the uniform decay case, we show that the same property holds for the nonlinear system. In what concerns the controllability properties, if only a single internal control is used, the exact controllability is established via the Hilbert Uniqueness Method. If we consider two controls, a stronger exact controllability result is obtained by using the moment method.

Key words: Controllability, Boussinesq system, Benjamin-Bona-Mahony equation, unique continuation property.

Contents

Chapter 1

General Introduction

It is common knowledge that nonlinear dispersive wave equations arise as models of various physical phenomena. Because of the range of their applications, and because their mathematical properties are interesting and subtle, since the latter half of the 1960s and in the 1970s the mathematical theory for such equations came to the fore as a major topic within nonlinear analysis. In what concerns the propagation of unidirectional, one-dimensional, small-amplitude long waves in nonlinear dispersive media, for example, the phenomenon is well approximated by the Benjamin-Bona-Mahony (BBM) equation

$$u_t + u_x - u_{xxt} + uu_x = 0.$$

The equation itself was initially put forward in [3] and [30] as an approximate description of long-crested, surface water waves and it is an alternative to the classical Korteweg-de Vries (KdV) equation,

$$u_t + u_x + u_{xxx} + uu_x = 0.$$

Both models are special cases of a broad class of evolution equations for which the theory associated to the pure initial-value problem is by now well developed, though there are still interesting open issues. By contrast, the theory for coupled systems of such equations is much less developed, though physicists and mathematicians were led to derive sets of equations to describe the dynamics of the water waves in some specific physical regimes. For instance, in [4, 5] the authors have derived and analyzed a four-parameter family of Boussinesq systems

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{xxt} = 0, \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = 0, \end{cases} \tag{1.1}$$

to approximate the motion of small amplitude long waves on the surface of an ideal fluid under the force of gravity in situations where the motion is sensibly two-dimensional. Here, the variable, x, is proportional to distance in the direction of propagation while t is proportional to elapsed time. The quantity $\eta(x,t) + h_0$ corresponds to the total depth of the liquid at the point x and at time t, where h_0 is the undisturbed water depth. The variable $w(x,t)$ represents the horizontal velocity at the point $(x,y) = (x, \theta h_0)$, at time t, where y is the vertical coordinate, with $y = 0$ corresponding to the channel bottom or sea bed. Thus, w is the horizontal velocity field at the height θh_0, where θ is a fixed constant in the interval $[0,1]$. The constants a, b, c, d satisfy the consistency conditions

$$a + b = \frac{1}{2}(\theta^2 - \frac{1}{3}), \quad c + d = \frac{1}{2}(1 - \theta^2) \geq 0. \tag{1.2}$$

Contrary to some classical wave models which assume that the waves travel only in one direction, system (1.1) is free of the presumption of unidirectionality and may have a wider range of applicability.

1.1 Problems and Main Results

1.1.1 Controllability of the Boussinesq system of BBM-BBM type on a bounded domain

A major concern on the mathematical side of the study of dispersive wave models has been to settle the issues of local and global well-posedness for the pure initial-value problem, and thus pertains to wave motion far from the ends of a channel or for very long-crested waves in field situations. In this context, a natural example arises when modeling the effect in a channel of a wave maker mounted at one end, or in modeling near-shore zone motions generated by waves propagating from deep water. The mathematical theory pertaining to the study of such boundary value problem is considerably less advanced, specially in what concerns the study of the controllability properties. Such properties can be useful, for example, to see whether the solutions can be driven to a given state at a given final time by means of a control acting on a endpoint of the channel.

In this chapter, we are mainly concerned with the study of the Boussinesq system from the control point of view. Consideration is given to an initial-boundary-value problem associated to linearized Boussinesq system (1.1) when the parameters given in (1.2) are such that $a = c = 0$. The resulting system couples two Benjamin-Bona-Mahony type equations and it is called purely BBM-type Boussinesq system. Our attention, in particular, is given to the following distributed control system:

$$\begin{cases} \eta_t + w_x - b\eta_{txx} = 0, & x \in (0, 2\pi), \quad t > 0 \\ w_t + \eta_x - dw_{txx} = 0, & x \in (0, 2\pi), \quad t > 0 \\ \eta(t, 0) = w(t, 0) = 0, & t > 0 \\ \eta(t, 2\pi) = f(t); w(t, 2\pi) = g(t), & t > 0 \\ \eta(0, x) = \eta^0(x), & x \in (0, 2\pi) \\ w(0, x) = w^0(x), & x \in (0, 2\pi). \end{cases} \tag{1.3}$$

In (1.3), the external forcing terms f and g are considered as control inputs. The purpose is to see whether one can force the solutions of the system to have certain desired properties by choosing appropriate control inputs acting at one end of the channel. More precisely, we are mainly concerned with the following problems which are fundamental in control theory:

Given $T > 0$, initial states (η^0, w^0) and terminal states (η^1, w^1) in a certain space, can one find appropriate control inputs f and g so that the system (1.3) *admits a solution (η, w) which satisfies $(\eta(0, \cdot), w(0, \cdot)) = (\eta^0, w^0)$ and $(\eta(T, \cdot), w(T, \cdot)) = (\eta^1, w^1)$?*

If one can always find a control input to guide the system described by (1.3) from any given initial state to any given terminal state, then the system is said to be **exactly controllable**. If any given initial state can be steered to $(0, 0)$, the system is said to be **null controllable**.

Given $T > 0$, $\varepsilon > 0$, initial states (η^0, w^0) and terminal states (η^1, w^1) in a certain space H, can one find appropriate control inputs f and g so that the system (1.3) *admits a solution (η, w) which satisfies $||(\eta(T,\cdot), w(T,\cdot)) - (\eta^0, w^0)||_H < \varepsilon$, for a certain space H?*

This means that the set of reachable states is dense in H and, in this case, the system is said to be **approximately controllable**.

Observe that exact controllability is essentially stronger notion than approximate controllability. In other words, exact controllability always implies approximate controllability. The converse statement is generally false.

Those questions were first investigated in [25] where periodic boundary conditions were considered. The space of the controllable data is determined for each value of the four parameters a, b, c and d. Then, some simple feedback controls are constructed for a particular choice of the parameters such that the resulting closed-loop systems are exponentially stable. Later on, in [9], it was discovered that whether the system of Boussinesq system of KdV-KdV type ($b = d = 0$), posed on a interval, is exactly controllable or not depends on the length of the spatial domain. When the system is controllable, the authors also proved that the solutions issuing from small data are globally defined and exponentially decreasing in the energy space. A similar result was obtained in [28].

Concerning the Boussinesq system of BBM-BBM type, the work [26] addresses the stabilization problem for the linearized system, posed on a bounded interval, when a localized damping term acts in one equation only. By considering Dirichlet boundary conditions it was proved that the energy associated to the model converges to zero as time goes to infinity. More recently in [27], on periodic domain, the stability properties was studied by introducing generalized damping operators in each equation. In this case, whether the solutions of the linearized system decay uniformly or not to zero depend on the parameters of the damping operators. In the uniform decay case, the same property holds for the nonlinear system. Let us also mention that a similar problem for the model posed on the whole real axis was studied in [13].

We begin our analysis by providing a negative result for the first problem introduced above: system (1.3) is not spectrally controllable if $(\eta^0, w^0) \in (H_0^1(0, 2\pi))^2$. This means that no finite linear nontrivial combination of eigenvectors of the operator associated with the state equations ($\mathcal{A} : (H_0^1(0, 2\pi))^2 \to (H_0^1(0, 2\pi))^2$) can be driven to zero in finite time by using controls $(f, g) \in (H^1(0, T))^2$. As it will become clear during our proofs, the bad control property comes from the existence of a limit point in the spectrum of the operator associated with the state equations, a phenomenon already noticed in [24] for the single BBM equation. To obtain the results we make use of the carefull spectral analysis developed in [26], which provides important developments to justify the use of eigenvector expansions for the solutions, as well as, the asymptotic behavior of the eigenvalues. Therefore, for the sake of completeness, we have included the analysis of [26] in an Appendix.

Nevertheless, we give a positive answer to the second problem mentioned above, i.e., it is possible to show that system (1.3) is approximately controllable for any $T > 0$. More precisely, there exist control inputs $(f, g) \in (H^1(0, T))^2$ such that the set of reachable states is dense in $(L^2(0, 2\pi))^2$, for any $(\eta^0, w^0) \in (H^{-1}(0, 2\pi))^2$ and $T > 0$. As in the first problem, to obtain the results we rely strongly on the carefully developed spectral analysis in [26] for the operator associated with the state equations. The main idea is to use the series expansion of the solution in terms of the eigenvectors of the operator in order to reduce the problem to a unique continuation problem (of the eigenvectors). In the present case, it can be solved by classical ODE methods.

We point out that a similar problem was studied in [24] for the scalar BBM equation from which we borrow some ideas. The proofs in [24] make use of the explicit Fourier series expansion of the solution in terms of the eigenvectors of the differential operator associated to the space variable. This approach does not apply directly in our case, since the eigenfunctions are not explicit and, therefore, our proofs require further developments. On the other hand, the program of the present work establish as a fact that model (1.3) inherits some interesting properties initially observed for the linear BBM equation.

In what concern the nonlinear model, the problem remains open, including for the BBM equation. At this respect, we note that the controllability properties of nonlinear systems are usually studied by linearizing the problem at an equilibrium state, by proving exact controllability results for this linear problem and by applying next the implicit function theorem. However, taking into account the negative results obtained in this paper for the linearized model (like nonspectral controllability) it is not possible to study the controllability properties of the full Boussinesq system of BBM type by using one of the classical techniques. To our knowledge, the only result on the subject was obtained in [32] for the BBM equation on the torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. The authors show that, when an internal control acting on a moving interval is applied in the BBM equation, it is locally exactly controllable in $H^s(\mathbb{T})$, for any $s > 0$, and globally exactly controllable in $H^s(\mathbb{T})$, for any $s > 1$, in a sufficiently large time depending on the H^s-norms of the initial and terminal states.

1.1.2 Large time behavior for the Boussinesq system of BBM-BBM type

In all the situations mentioned in the previous sections, it is often important to investigate the stability properties of the solutions when dissipative effects are generated by internal and boundary damping. The problem might be easy to solve when the underlying models have a strong enough intrinsic dissipative nature, but very often, as the cases we address here, the models are of conservative nature and the decay requires appropriate damping mechanisms. Obviously, for practical purposes, it is desirable to achieve this property with a minimal amount of damping both in what concerns its support and its intensity. Moreover, in the context of coupled systems, the damping mechanism has to be designed in an appropriate way in order to capture all the components of the system. For all these reasons the right choice of damping terms is far from being obvious and requires a careful analysis in each particular case.

In this chapter, we investigate such questions when the parameters given in (1.2) are such that $a = c = 0$. The resulting system couples two Benjamin-Bona-Mahony type equations and it is called purely BBM-type Boussinesq system. We consider either a distributed (localized) feedback law or a boundary feedback law.

We first consider the case in which a localized damping mechanism acts in one equation of the system:

$$\begin{cases} \eta_t + w_x - b\eta_{txx} = \mathcal{B}\eta & \text{for } x \in (0, 2\pi), \quad t > 0 \\ w_t + \eta_x - dw_{txx} = 0 & \text{for } x \in (0, 2\pi), \quad t > 0 \\ \eta(0, x) = \eta^0(x) & \text{for } x \in (0, 2\pi), \\ w(0, x) = w^0(x) & \text{for } x \in (0, 2\pi), \end{cases} \tag{1.4}$$

where $b, d > 0$, and $\mathcal{B}$ is a linear bounded and positive operator which will be effective only on an open subset ω of the interval $(0, 2\pi)$. A precise definition of the operator $\mathcal{B}$ will be given in the next sections, but in this case (1.4) is closed with periodic boundary conditions, i. e.,

$$\begin{cases} \eta(t, 0) = \eta(t, 2\pi); \; \eta_x(t, 0) = \eta_x(t, 2\pi) & \text{for } t > 0 \\ w(t, 0) = w(t, 2\pi); \; w_x(t, 0) = w_x(t, 2\pi) & \text{for } t > 0. \end{cases} \tag{1.5}$$

The natural energy associated to (1.4)-(1.5) is given by

$$E(t) = \frac{1}{2} \int_0^{2\pi} \left[b|\eta_x(t, x)|^2 + |\eta(t, x)|^2 + d|w_x(t, x)|^2 + |w(t, x)|^2 \right] dx$$

and if we multiply the first equation in (1.4) by η, the second one by w and integrate by part over $(0, 2\pi)$, we obtain (at least formally)

$$\frac{dE(t)}{dt} = - \int_0^{2\pi} \mathcal{B}\eta(t)\, \eta(t) dx. \tag{1.6}$$

So, the energy decreases along the trajectories of the system.

When $\mathcal{B} \equiv \mathbf{0}$, i.e., in the absence of an internal damping term, we study (1.4) with the following set of boundary conditions:

$$\begin{cases} \eta_{xt}(t, 0) = \dfrac{w(t, 0)}{2b} + \eta(t, 0) \, ; & \eta_{xt}(t, L) = \dfrac{w(t, L)}{2b} - \eta(t, L) \quad \text{for } t \geq 0 \\ w_{xt}(t, 0) = \dfrac{\eta(t, 0)}{2d} + w(t, 0) \, ; & w_{xt}(t, L) = \dfrac{\eta(t, L)}{2d} - w(t, L) \quad \text{for } t \geq 0. \end{cases} \tag{1.7}$$

In this case, the energy associated to (1.4)-(1.7) satisfies the following energy dissipation law

$$\frac{dE(t)}{dt} = -b \left(|\eta(t, L)|^2 + |\eta(t, 0)|^2 \right) - d \left(|w(t, L)|^2 + |w(t, 0)|^2 \right). \tag{1.8}$$

Hence, $E(t)$ is decreasing and the boundary conditions play the role of a feedback damping mechanism.

In each case, we can ask weather $E(t)$ is asymptotically stable, as $t \to \infty$. The problem was first addressed in [31] for the scalar BBM equation and the conclusion is that all trajectories are indeed attracted by the origin provided that the unique continuation property holds for the conservative equation. We remark that the unique continuation property for the BBM equation is still an open problem. Moreover, since the underlying Cauchy problem is a characteristic one, we can not expect to apply Carleman-type estimates or the classical Holmgren uniqueness theorem. In order to overcome this difficulty, in [32] the authors introduced a moving control and derived with such a control both the exact controllability and the exponential stability of the full BBM equation. They also proved a conditional unique continuation result by assuming that the initial data is small in the L^∞-norm and it has a nonnegative mean value.

The program of the present work is carried out for the particular choice of damping effect entering in (1.6) and (1.8) and aims to establish as a fact that the corresponding models inherit the interesting qualitative properties initially observed by Rosier for the BBM equation. Following the approach developed in [31], we first prove the global wellposedness of the systems (1.4)-(1.5) and the convergence towards a solution which is null on a band. Then, from the unique continuation property obtained for finite energy solutions of the conservative system, it follows that the origin is asymptotically stable for the damped BBM-BBM model. Similar conclusions remains valid for system (1.4)-(1.7) and, as it will become clear during our proofs, the boundary conditions play an important role and allow us to apply the same unique continuation argument. Here, the proof of the unique continuation property makes use of the explicit Fourier series expansion of the solution in terms of the eigenvectors of the differential operator associated to the space variable. Concerning the existence of a solution, it is established by converting (1.4)-(1.5) and (1.4)-(1.7) into integral equations and applying the contraction-mapping principle. The regularity then follows from the fact that solutions of the integral equations are exactly as smooth as the data affords. At this respect, it is important to note that identities (1.6) and (1.8) do not provide any global (in time) *a priori* bounds for the solutions of the nonlinear system. Consequently, it does not lead to the existence of a global (in time) solution in the energy space. The same lack of *a priori* bounds occurs when higher order Sobolev norms are considered (e. g. H^s-norm). Since the main focus of this paper is on the asymptotic behavior of the solutions when the time goes to infinity, a global (in time) existence result is necessary.

The stabilization problem for the linearized Boussinesq system of BBM-BBM was also studied in [26], when the model is posed on a bounded interval. By considering Dirichlet boundary conditions and introducing a localized damping term in one equation it was proved that the energy associated to the model converges to zero as time goes to infinity. In periodic case, the stability properties was studied in [27] by introducing generalized damping operators in each equation. In this case, weather the solutions of the linearized system decay uniformly or not to zero depend on the parameters of the damping operators. In the uniform decay case, the same property holds for the nonlinear system. We also refer to [25] for a rather complete picture of the control properties of (1.1) on a periodic domain with a locally supported forcing term. As an application of the established exact controllability results, some feedback controls are constructed for some particular choice of the parameters such that the resulting closed-loop systems are exponentially stable. Later on, the boundary stabilization problem for the Boussinesq system of KdV-KdV type ($b = d = 0$) was studied in [28] and [10]. The authors proved that the system is locally exponentially stable in the energy space for solutions issuing from small data.

1.1.3 Stabilization for higher-order Boussinesq system with generalized damping on a periodic domain

Higher-order systems in the form

$$\begin{cases} \eta_t - b\eta_{txx} + b_2\eta_{txxxx} & = -w_x - (\eta w)_x - aw_{xxx} - (a+b-\frac{1}{3})(\eta w_x)_x - a_2 w_{xxxxx}, \\ w_t - dw_{txx} + d_2 w_{txxxx} & = -\eta_x - c\eta_{xxx} - ww_x - c(ww_x)_{xx} - (\eta\eta_{xx})_x \\ & \quad +(c+d-1)w_x w_{xx} + (c+d)\eta_x\eta_{xxx} - c_2\eta_{xxxxx}, \end{cases} \tag{1.9}$$

were also derived in [4, 5]. These systems are formally second-order approximations of the full, two-dimensional Euler equations. The constants a, b, c, d, a_2, b_2, c_2, d_2 satisfy (1.2) and

$$a_2 - b_2 = -\frac{1}{2}(\theta^2 - \frac{1}{3})b + \frac{5}{24}(\theta^2 - \frac{1}{5})^2, \quad c_2 - d_2 = \frac{1}{2}(1-\theta^2)c + \frac{5}{24}(1-\theta^2)(\theta^2 - \frac{1}{5}), \tag{1.10}$$

where, as before, $\theta \in [0,1]$.

In this chapter, attention is a given to a particular subclass of linear higher-order regularized long-wave systems that have

$$b,\ d,\ b_2,\ d_2 > 0;\ a,\ c < 0 \quad \text{or} \quad b,\ d,\ b_2,\ d_2 > 0;\ a = c \geq 0. \tag{1.11}$$

Adding damping mechanisms is often important in obtaining good agreement between experimental observations and the prediction of theoretical models describing the propagation of waves in nonlinear dispersive media (see, for instance, [7]). To address the issue, we will consider a general class of damping operator, with nonnegative symbol. Our purpose is to investigate the dissipative effects generated by these operators in model (1.9), posed on a periodic domain. More precisely, we consider the following system

$$\begin{cases} \eta_t + w_x - b\eta_{txx} + b_2\eta_{txxxx} + aw_{xxx} + \beta_1 M_{\alpha_1}\eta = -(\eta w)_x & \\ -(a+b-\frac{1}{3})(\eta w_x)_x, & \text{for } x \in (0,2\pi),\ t>0 \\ w_t + \eta_x - dw_{txx} + d_2 w_{txxxx} + c\eta_{xxx} + \beta_2 M_{\alpha_2} w = -ww_x - c(ww_x)_{xx} & \\ -(\eta\eta_{xx})_x + (c+d-1)w_x w_{xx} + (c+d)\eta_x\eta_{xxx} & \text{for } x \in (0,2\pi),\ t>0 \\ \eta(0,x) = \eta^0(x) & \text{for } x \in (0,2\pi) \\ w(0,x) = w^0(x) & \text{for } x \in (0,2\pi), \end{cases} \tag{1.12}$$

with the periodic boundary conditions

$$\begin{cases} \dfrac{\partial^r \eta}{\partial x^r}(t,0) = \dfrac{\partial^r \eta}{\partial x^r}(t,2\pi) \quad \text{for } t>0, \quad 0 \leq r \leq 3, \\[2ex] \dfrac{\partial^q w}{\partial x^q}(t,0) = \dfrac{\partial^q w}{\partial x^q}(t,2\pi) \quad \text{for } t>0, \quad 0 \leq q \leq 3, \end{cases} \tag{1.13}$$

where β_1, $\beta_2 \geq 0$, α_1, $\alpha_2 \in [0,4]$, and the operators M_{α_j} are Fourier multiplier operators defined in terms of their Fourier coefficients as follows:

$$M_{\alpha_j} : H_p^{s+4}(0,2\pi) \to H_p^s(0,2\pi),$$
$$M_{\alpha_j}\left(\sum_{k\in\mathbb{Z}} \widehat{a}_k e^{ikx}\right) = \sum_{k\in\mathbb{Z}} (1+k^2)^{\frac{\alpha_j}{2}} \widehat{a}_k e^{ikx} \qquad (j=1,2). \tag{1.14}$$

This type of damping was used to describe several dissipative dispersive phenomena (see, for instance, [8, 12, 19]). The operators M_{α_j} are, in some sense, similar to fractional derivative operators. Indeed, for a periodic function $h(x) = \sum_{k\in\mathbb{Z}^*} a_k e^{ikx}$, the Weyl fractional derivative operator of order $\alpha \geq 0$ applied to h is defined by (see [34])

$$W_x^\alpha h(x) = \sum_{k\in\mathbb{Z}^*} (ik)^\alpha a_k e^{ikx}.$$

Consequently, the Fourier coefficients of $M_\alpha h$ and $W_t^\alpha h$ behave in the same manner for large k.

With the notation introduced above, we consider the operator $\mathcal{H}$ as follows

$$\mathcal{H}\left(\sum_{k\in\mathbb{Z}} \widehat{a}_k e^{ikx}\right) = \sum_{k\in\mathbb{Z}} \sqrt{\frac{w_1}{w_2}} \widehat{a}_k e^{ikx},$$

where $w_1 = \frac{1-ak^2}{1+bk^2+b_2k^4}$ and $w_2 = \frac{1-ck^2}{1+dk^2+d_2k^4}$. Then, the energy associated to (1.12)-(1.13) is given by

$$E[\eta,w](t) = \int_0^{2\pi} \left(\left|(I - b\partial_x^2 + b_4\partial_x^2)^{1/2}\eta(t,x)\right|^2 + \left|(I - b\partial_x^2 + b_4\partial_x^2)^{1/2}\mathcal{H}w(t,x)\right|^2 \right) dx, \tag{1.15}$$

and, we obtain (see (4.63))

$$\begin{aligned} \frac{d}{dt}E[\eta,w](t) &\leq -\beta_1\|\eta\|^2_{H_p^{\frac{\alpha_1}{2}}(0,2\pi)} - C\beta_2\|w\|^2_{H_p^{\frac{\alpha_2}{2}}(0,2\pi)} - \int_0^{2\pi} (\eta w)_x(t)\eta(t)dx \\ &- (a+b-\frac{1}{3})\int_0^{2\pi} (\eta w_x)_x(t)\eta(t)dx - c\int_0^{2\pi} (ww_x)_{xx}(t)w(t)dx - \int_0^{2\pi} (\eta\eta_{xx})_x(t)w(t)dx \\ &+ \int_0^{2\pi} (c+d-1)w_x w_{xx}(t)w(t)dx + (c+d)\int_0^{2\pi} \eta_x\eta_{xxx}(t)w(t)dx. \end{aligned} \tag{1.16}$$

Inequality (1.16) shows that, if $\beta_1, \beta_2 \geq 0$, the terms $M_{\alpha_1}\eta$ and $M_{\alpha_2}w$ play the role of feedback damping mechanisms. For the linearized system we obtain that the energy (1.6) is non increasing. However, for the full system (1.12) the right hand side of (1.16) does not have a definite sign. Therefore, the study of the asymptotic behavior of solutions becomes a more difficult task. The following questions arise: Does $E(t) \to 0$, as $t \to \infty$? If it is the case, can we give its decay rate? The same questions can be addressed concerning the behavior of the H^s-norm (the Sobolev norm of order $s \in \mathbb{R}$) of η and w.

Firstly, we analyze the linearized system. Through a detailed spectral analysis, we obtain the following results:

- If $\alpha_1 = \alpha_2 = 4$ and $\beta_1, \beta_2 > 0$, we prove the exponential decay of solutions in the H^s-setting, for any $s \in \mathbb{R}$, and $(\eta^0, w^0) \in H^s \times H^s$ (see Theorem 4.1.5).
- If $\max\{\alpha_1,\ \alpha_2\} \in [0, 4)$, $\beta_1, \beta_2 \geq 0$ and $\beta_1^2+\beta_2^2 > 0$, we obtain a polynomial decay rate of solutions in the H^s-setting, by considering smoother initial data, more precisely, $(\eta^0, w^0) \in H^{s+q} \times H^{s+q}$, for $q > 0$ (see Theorem 4.1.6).

Secondly, the exponential decay estimate obtained in the first case is combined with the contraction mapping theorem in a convenient weighted space to prove the global well-posedness together with the exponential stability property of the nonlinear system (1.12) by considering small data (see Theorem 4.2.1).

As pointed out in section 1.1.2, the problem was only addressed in [26] when $b_2 = d_2 = a = c = 0$.

1.1.4 Controllability for higher-order linear Boussinesq system on a periodic domain

The main task of this chapter is to study the controllability for system (1.9) by means of some localized control actions. More precisely, we will consider the following nonhomogeneous systems

$$\begin{cases} \eta_t + w_x - b\eta_{txx} + b_2\eta_{txxxx} + aw_{xxx} + a_1 w_{xxxxx} = f(t,x) & \text{for } x \in (0, 2\pi),\ \ t > 0, \\ w_t + \eta_x - dw_{txx} + d_2 w_{txxxx} + c\eta_{xxx} + c_1\eta_{xxxxx} = g(t,x) & \text{for } x \in (0, 2\pi),\ \ t > 0, \end{cases} \tag{1.17}$$

with the periodic boundary conditions

$$\begin{cases} \dfrac{\partial^r \eta}{\partial x^r}(t, 0) = \dfrac{\partial^r \eta}{\partial x^r}(t, 2\pi) & \text{for } t > 0, \quad 0 \leq r \leq r_0, \\[2ex] \dfrac{\partial^q w}{\partial x^q}(t, 0) = \dfrac{\partial^q w}{\partial x^q}(t, 2\pi) & \text{for } t > 0, \quad 0 \leq q \leq q_0 \end{cases} \tag{1.18}$$

and the initial conditions

$$\eta(0, x) = \eta^0(x), \qquad w(0, x) = w^0(x) \quad \text{for } x \in (0, 2\pi). \tag{1.19}$$

The number of a boundary conditions depends on the values of the parameters. The forcing functions $f(t, x)$ and $g(t, x)$, which will be considered as control inputs, are assumed to be supported in ω, a nonempty open subinterval of a $(0, 2\pi)$. We will be mainly interested in the following problem for system (1.17)-(1.19).

Problem *(Exact controllability): Given $T > 0$, the initial state (η^0, w^0) and the terminal state (η^1, w^1) in a appropriate space, can one find controls f and g in a suitable space such that* (1.17) *admits a unique solution $(\eta(t, x), w(t, x))$ satisfying the boundary conditions* (1.18) *and*

$$(\eta(0, x), w(0, x)) = (\eta^0, w^0), \quad (\eta(T, x), w(T, x)) = (\eta^1, w^1)?$$

Depending on the values of its parameters, system (1.17) couples two equations that may be of KdV or BBM types of orders 5. It is therefore interesting to see to which extent the controllability properties of each equation are maintained. It is not unusual that in the first case (KdV case) some good controllability properties are proved whereas in the second case (BBM case) there are no such controllability properties. What happens in the last case is *a priori* less clear. We will however prove that the system is controllable in that case. Different approaches will be used to establish the exact controllability depending on whether we employ a single control input or two control inputs. If only a single control action is used, the exact controllability will be established via the Hilbert Uniqueness Method (HUM) (cf. [20]). If both control actions are used, a stronger exact controllability result will be obtained by using the classical moment method (cf. [33]). Our analysis is inspired by the results obtained in [25], where the same problem was studied by considering $b_2 = d_1 = a_1 = c_1 = 0$ in (1.17)-(1.19). Summarizing, our main results read as follows:

- *Assume that the parameter* $a_1 \neq 0$ *and* $T > \frac{2\pi}{\gamma}$, *where* γ *will be defined later. Let* $s \in \mathbb{R}$ *and define* n_1 *by*

$$n_1 = \begin{cases} 2, & \textit{if} \quad b_2 = 0, \quad b \neq 0, \\ 0, & \textit{if} \quad b_2 = b = 0, \\ 4, & \textit{if} \quad b_2 \neq 0. \end{cases}$$

  Then, for any given initial state (η^0, w^0) *and the terminal state* (η^1, w^1) *in* $[H_p^s(0, 2\pi)]^2$, *there exist* $(f, g) \in [L^2(0, T; H_p^{s-n_1}(0, 2\pi))]^2$, *such that the system* (1.17)-(1.19) *admits a unique solution* $(\eta, w) \in [C([0, T]; H_p^s(0, 2\pi))]^2$ *satisfying*

$$\eta(T, \cdot) = \eta^1(\cdot) \quad \textit{and} \quad w(T, \cdot) = w^1(\cdot) \quad \textit{in} \quad H_p^s(0, 2\pi).$$

  (see Theorem 5.2.1).

- *Assume that* $b_2 = d_1 = b = d = 0$. *Then, there exist a time* $T > 0$ *and a subspace* $\mathcal{V} \subset L_p^2(0, 2\pi) \times H_p^s(0, 2\pi)$, *defined in Theorem 5.3.3, such that, for given*

$$(\eta^0, w^0) \in \mathcal{V}, \quad (\eta^T, w^T) \in \mathcal{V},$$

  one can find a control input $f \in L^2((0, T) \times (0, 2\pi))$, *such that* (1.17)-(1.19) *with* $g = 0$ *admits a unique solution*

$$(\eta, w) \in C([0, T]; \mathcal{V})$$

  satisfying

$$(\eta(T, \cdot), w(T, \cdot)) = (\eta^T, w^T) \quad \textit{in} \quad \mathcal{V}.$$

  (see Theorem 5.3.3).

Chapter 2

Controllability of the Boussinesq system of BBM-BBM type on a bounded domain

In this chapter we are concerned with a Boussinesq system of Benjamin-Bona-Mahony type equation, posed on a bounded interval, modelling the the two-way propagation of surface waves in a uniform horizontal channel filled with an irrotational, incompressible and inviscid liquid under the influence of gravitation. The main focus is on the boundary controllability property, which corresponds to the question of whether the solutions can be driven to a given state at a given final time by means of controls acting at one endpoint of the interval. We first show that the equation is not spectrally controllable. This means that, no finite linear combination of eigenfunctions associated to the state equations, other than zero, can be steered to zero. Although the system is not spectrally controllable it can be shown that it is approximately controllable, i.e., any state can be steered arbitrarily close to another state. It gives the possibility of steering the system to the states which form the dense subspace in the state space.

2.1 Global well-posedness

In this section we present the well-posedness results needed to study the control system (1.3). We state results for both homogeneous and nonhomogeneous system.

2.1.1 The homogeneous system

Let us first consider the homogeneous system

$$\begin{cases} \eta_t + w_x - b\eta_{txx} = 0, & x \in (0, 2\pi), \ \ t > 0 \\ w_t + \eta_x - dw_{txx} = 0, & x \in (0, 2\pi), \ \ t > 0 \\ \eta(t, 0) = \eta(t, 2\pi) = 0, & t > 0 \\ w(t, 0) = w(t, 2\pi) = 0, & t > 0 \\ \eta(0, x) = \eta^0(x), & x \in (0, 2\pi) \\ w(0, x) = w^0(x), & x \in (0, 2\pi). \end{cases} \tag{2.1}$$

System (2.1) can be written as an abstract evolution equation in $(H_0^1(0,2\pi))^2$ as follows

$$\begin{cases} U_t + \mathcal{A}U = 0 \\ U(0) = U^0, \end{cases} \tag{2.2}$$

where $U = \begin{pmatrix} \eta \\ w \end{pmatrix}$, $U_0 = \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} \in (H_0^1(0,2\pi))^2$ and $\mathcal{A}$ is the operator associated to the state equations and belongs to $\mathcal{L}((H_0^1(0,2\pi))^2)$:

$$\mathcal{A} = \begin{pmatrix} 0 & (I - b\partial_x^2)^{-1}\,\partial_x \\ (I - d\partial_x^2)^{-1}\,\partial_x & 0 \end{pmatrix}. \tag{2.3}$$

Recall that, for $\alpha > 0$, the operator $(I - \alpha\partial_x^2)^{-1}$ is defined in the following way:

$$(I - \alpha\partial_x^2)^{-1}\varphi = v \Leftrightarrow \begin{cases} v - \alpha v_{xx} = \varphi \\ v(0) = v(2\pi) = 0. \end{cases}$$

Then, if $\varphi \in L^2(0,2\pi)$, we have that there exists a unique $v \in H^2(0,2\pi) \cap H_0^1(0,2\pi)$ verifying the above equation and $(I - \alpha\partial_x^2)^{-1} : L^2(0,2\pi) \to L^2(0,2\pi)$ is a well-defined, compact operator.

From the classical semigroup theory we have the following well-posedness result:

THEOREM 2.1.1. *Let $b, d > 0$. For any $U^0 \in (H_0^1(0,2\pi))^2$, system* (2.1) *has a unique classical solution $U \in \mathcal{C}(\mathbb{R}; (H_0^1(0,2\pi))^2)$. Moreover, $U \in \mathcal{C}^\omega(\mathbb{R}, (H_0^1(0,2\pi))^2)$, the class of analytic functions in $t \in \mathbb{R}$ with values in $H_0^1(0,2\pi)$.*

Proof. According to Theorem 6.2.2 of the Appendix 6.2, the operator $\mathcal{A}$ is skew-adjoint and, therefore, generates a group of isometries $\{S(t)\}_{t\in\mathbb{R}}$ in $(H_0^1(0,2\pi))^2$, which allows us to obtain the well-posedness result. The second part of the theorem follows from the fact that $\mathcal{A}$ is a compact operator in $(H_0^1(0,2\pi))^2$ (see, for instance, [Theorem 11.4.1, Chap. XI in [17]]). □

2.1.2 The nonhomogeneous system

In this subsection, attention will be given to the full system (1.3). We begin with the following result:

THEOREM 2.1.2. *Let $b, d > 0$. For any $\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} \in \left(H_0^1(0,2\pi)\right)^2$ and $\begin{pmatrix} f \\ g \end{pmatrix} \in \left(\mathcal{C}_0^2(0,\infty)\right)^2$ system* (1.3) *has a unique solution $U \in \mathcal{C}([0,\infty; (H_0^1(0,2\pi))^2)$.*

Proof. Let $\phi_1, \phi_2 \in \mathcal{C}^\infty([0,2\pi])$ be functions, such that $\phi_1(0) = \phi_2(0) = 0$ and $\phi_1(2\pi) = \phi_2(2\pi) = -1$. If we consider the change of functions

$$\begin{pmatrix} z \\ \varphi \end{pmatrix} = \begin{pmatrix} \eta \\ w \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(t)\phi_1(x) \\ g(t)\phi_2(x) \end{pmatrix}, \tag{2.4}$$

where $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{C}([0,\infty);(H_0^1(0,2\pi))^2)$ is the solution of the system

$$\begin{cases} u_t + v_x - bu_{txx} = 0, & x \in (0,2\pi), \ \ t>0 \\ v_t + u_x - dv_{txx} = 0, & x \in (0,2\pi), \ \ t>0 \\ u(t,0) = v(t,0) = 0, & t>0 \\ u(t,2\pi) = v(t,2\pi) = 0, & t>0 \\ u(0,x) = \eta^0(x), v(0,x) = w^0(x), & x \in (0,2\pi), \end{cases} \tag{2.5}$$

given by Theorem 2.1.1, the pair $\begin{pmatrix} z \\ \varphi \end{pmatrix}$ solves the problem

$$\begin{cases} z_t + \varphi_x - bz_{txx} = F, & x \in (0,2\pi), \ \ t>0 \\ \varphi_t + z_x - d\varphi_{txx} = G, & x \in (0,2\pi), \ \ t>0 \\ z(t,0) = \varphi(t,0) = 0, & t>0 \\ z(t,2\pi) = \varphi(t,2\pi) = 0, & t>0 \\ z(0,x) = \varphi(0,x) = 0, & x \in (0,2\pi), \end{cases} \tag{2.6}$$

where

$$\begin{pmatrix} F(t,x) \\ G(t,x) \end{pmatrix} = \begin{pmatrix} f'(t)\left(\phi_1(x) - b\phi_1''(x)\right) + g(t)\phi_2'(x) \\ g'(t)\left(\phi_2(x) - d\phi_2''(x)\right) + f(t)\phi_1'(x) \end{pmatrix} \in \left[C([0,\infty] \times [0,2\pi])\right]^2.$$

With the notation introduced in the previous section, system (2.6) can be written as an abstract evolution equation as follows

$$\begin{cases} W_t + \mathcal{A}W = \mathcal{H} \\ W(0) = 0, \end{cases}$$

where $W = \begin{pmatrix} z \\ \varphi \end{pmatrix}$ and $\mathcal{H} = \mathcal{A}_0 \begin{pmatrix} F \\ G \end{pmatrix} \in L^1\left(0,\infty;(H_0^1(0,2\pi))^2\right)$, being $\mathcal{A}_0 : (H_0^1(0,2\pi))^2 \longrightarrow (H_0^1(0,2\pi))^2$ defined by

$$\mathcal{A}_0 = \begin{pmatrix} (I - b\partial_x^2)^{-1} & 0 \\ 0 & (I - d\partial_x^2)^{-1} \end{pmatrix}. \tag{2.7}$$

Since $\mathcal{A}$ generates a group of isometries in $(H_0^1(0,2\pi))^2$ we have that system (2.6) has a unique solution $W = \begin{pmatrix} z \\ \varphi \end{pmatrix} \in \mathcal{C}\left([0,\infty);H_0^1(0,2\pi)\right)$. Then, returning to (2.4) we conclude the proof. □

Using the previous well-posedness results we will study solutions of the system (1.3) in the sense of transposition:

Definition 2.1.1. *Let* $\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} \in \left(H^{-1}(0,2\pi)\right)^2$ *and* $\begin{pmatrix} f \\ g \end{pmatrix} \in \left(H^1(0,T)\right)^2$. *A solution (by transposition) of the system* (1.3) *is a pair* $\begin{pmatrix} \eta \\ w \end{pmatrix} \in L^2(0,T;(L^2(0,2\pi))^2)$, *such that,* *any* $\begin{pmatrix} h \\ k \end{pmatrix} \in L^1(0,T;(L^2(0,2\pi))^2)$, *satisfies*

$$\int_0^T \int_0^{2\pi} (\eta h + wk)\, dxdt + \left\langle \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix}, \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \right\rangle_{(H^{-1}(0,2\pi))^2, \left(H_0^1(0,2\pi)\right)^2} = -b\int_0^T f(t)u_{xt}(t,2\pi)dt - \\ - d\int_0^T g(t)v_{xt}(t,2\pi)dt, \tag{2.8}$$

where $\begin{pmatrix} u \\ v \end{pmatrix}$ *is solution of the adjoint system*

$$\begin{cases} u_t + v_x - bu_{txx} = h, & x \in (0,2\pi), \ \ t > 0 \\ v_t + u_x - dv_{txx} = k, & x \in (0,2\pi), \ \ t > 0 \\ u(t,0) = v(t,0) = 0, & t > 0 \\ u(t,2\pi) = v(t,2\pi) = 0, & t > 0 \\ u(T,x) = v(T,x) = 0, & x \in (0,2\pi). \end{cases} \tag{2.9}$$

The existence of solutions for system (2.9) can be proved following the arguments used in the proof of Theorem 2.1.2. Moreover, due to the regularizing effect of the operator $(I - \alpha\partial_x^2)^{-1}$, $\alpha > 0$, we obtain the following result:

THEOREM 2.1.3. *If* $\begin{pmatrix} h \\ k \end{pmatrix} \in L^1(0,T;(L^2(0,2\pi))^2)$, *system* (2.9) *has a unique solution* $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{C}([0,T];(H_0^1(0,2\pi))^2)$. *Moreover,*

$$\|(u_t,v_t)\|_{L^1\left(0,T;(H_0^1\cap H^2(0,2\pi))^2\right)} \le C\|(h,k)\|_{L^1(0,T;(L^2(0,2\pi))^2)}, \tag{2.10}$$

for some constant $C > 0$.

Proof. System (2.9) can be written as an abstract evolution equation as follows

$$\begin{cases} W_t + \mathcal{A}W = \mathcal{F} \\ W(T) = 0, \end{cases}$$

where $W = \begin{pmatrix} u \\ v \end{pmatrix}$, $\mathcal{A} : (H_0^1(0,2\pi))^2 \longrightarrow (H_0^1 \cap H^2(0,2\pi))^2$ is given by (2.3) and $\mathcal{F} = \mathcal{A}_0 \begin{pmatrix} h \\ k \end{pmatrix} \in L^1\left(0,\infty;(H_0^1 \cap H^2(0,2\pi))^2\right)$, with $\mathcal{A}_0 : (L^2(0,2\pi))^2 \longrightarrow (H_0^1 \cap H^2(0,2\pi))^2$ defined by (2.7).

Since $\mathcal{A}$ generates a group of isometries in $(H_0^1(0,2\pi))^2$ we have that the system (2.9) has a unique solution $W = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{C}([0,\infty);(H_0^1(0,2\pi))^2)$. Moreover, by using the equations in (2.9), we deduce that $\begin{pmatrix} u_t \\ v_t \end{pmatrix} \in L^1(0,\infty;(H_0^1 \cap H^2(0,2\pi))^2)$. Indeed, in order to obtain estimate (2.10), we multiply the first equation in (2.9) by u, the second one by v and integrate by parts on $(0,2\pi)$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|(u(t,\cdot),v(t,\cdot))\|^2_{(H^1_0(0,2\pi))^2}=\int_0^{2\pi}(hu+kv)\;dx. \tag{2.11}$$

Integrating the above identity from t up to T, and using Young's inequality, it follows that

$$\begin{aligned}\frac{1}{2}\|(u(t,\cdot),v(t,\cdot))\|^2_{\left(H^1_0(0,2\pi)\right)^2}&\leq\|h\|_{L^1(0,T;L^2(0,2\pi))}\|u\|_{\mathcal{C}([0,T];L^2(0,2\pi))}+\|k\|_{L^1(0,T;L^2(0,2\pi))}\|v\|_{\mathcal{C}([0,T];L^2(0,2\pi))}\\ &\leq\frac{1}{4}\|(u,v)\|^2_{\mathcal{C}([0,T];(L^2(0,2\pi))^2)}+C\|(h,k)\|^2_{L^1(0,T;(L^2(0,2\pi))^2)}.\end{aligned} \tag{2.12}$$

Then, from (2.11) and (2.12) we get

$$\|(u,v)\|_{\mathcal{C}([0,T];(H^1_0(0,2\pi))^2)}\leq C\|(h,k)\|_{L^1(0,T;(L^2(0,2\pi))^2)}, \tag{2.13}$$

for some $C>0$. On the other hand, due to the regularizing effect of the operator $(I-\partial_x^2)^{-1}$, it follows that $(I-b\partial_x^2)^{-1}h(t,\cdot)$, $(I-d\partial_x^2)^{-1}k(t,\cdot)\in H^1_0\cap H^2(0,2\pi)$ and the operator $\mathcal{A}$ takes values in $(H^1_0\cap H^2(0,2\pi))^2$, which is compactly embedded in $(H^1_0(0,2\pi))^2$. Thus, combining (2.13) and the equations in (2.9), it follows that

$$\|(u_t(t,\cdot),v_t(t,\cdot))\|_{(H^1_0\cap H^2(0,2\pi))^2}\leq C(\|(u,v)\|_{\mathcal{C}\left([0,T];(H^1_0(0,2\pi))^2\right)}+\|(h(t,\cdot),k(t,\cdot))\|_{(L^2(0,2\pi))^2}),$$

where C is a positive constant. Then, integrating the inequality above we obtain (2.10). □

The next theorem establishes the existence and uniqueness of solutions for system (1.3) in the sense of transposition.

THEOREM 2.1.4. *Let* $\begin{pmatrix}\eta^0\\ w^0\end{pmatrix}\in\left(H^{-1}(0,2\pi)\right)^2$ *and* $\begin{pmatrix}f\\ g\end{pmatrix}\in\left(H^1(0,T)\right)^2$. *Then, there exists a unique solution* $\begin{pmatrix}\eta\\ w\end{pmatrix}\in\mathcal{C}\left([0,T];(L^2(0,2\pi))^2\right)$ *of system* (1.3) *which verifies* (2.8).

Proof. The result is proved in two steps. We first use the Riesz representation theorem to prove the existence of a solution in $L^1(0,T;(L^2(0,2\pi))^2)$. Then, the continuity in the time variable is proved by using density arguments.

We start by introducing the linear operator $T:L^1(0,T;(L^2(0,2\pi))^2)\longrightarrow\mathbb{R}$ as follows

$$\begin{aligned}T\left((h,k)\right)=&-\left\langle\begin{pmatrix}\eta^0\\ w^0\end{pmatrix},\begin{pmatrix}u(0)\\ v(0)\end{pmatrix}\right\rangle_{(H^{-1}(0,2\pi))^2,\left(H^1_0(0,2\pi)\right)^2}-b\int_0^T f(t)u_{xt}(t,2\pi)dt-\\ &-d\int_0^T g(t)v_{xt}(t,2\pi)dt,\end{aligned}$$

where $\begin{pmatrix} u \\ v \end{pmatrix}$ is a solution of (2.9).

Note that T is well defined and is continuous. Indeed, proceeding as in the proof of Theorem 2.1.3 we obtain identity (2.11). Then, integrating over $(0,T)$, from (2.13) it follows that

$$\|(u(0),v(0))\|_{\left(H_0^1(0,2\pi)\right)^2} \leq C\|(h,k)\|_{L^1(0,T;(L^2(0,2\pi))^2)}, \tag{2.14}$$

for some constant $C>0$. On the other hand, from the Sobolev embedding and estimate (2.10) the following estimate holds

$$\left| b\int_0^T f(t)u_{xt}(t,2\pi)dt + d\int_0^T g(t)v_{xt}(t,2\pi)dt \right| \leq C\|(f,g)\|_{(H^1(0,T))^2}\|(h,k)\|_{L^1(0,T;(L^2(0,2\pi))^2)}, \tag{2.15}$$

where $C>0$. Finally, (2.14) and (2.15) allow to conclude that

$$T \in \mathcal{L}(L^1(0,T;(L^2(0,2\pi))^2);\mathbb{R}).$$

Then, by Riesz representation theorem, we obtain the existence and uniqueness of $(\eta,w)\in L^\infty(0,T;(L^2(0,2\pi))^2)$ satisfying (2.8). Moreover,

$$\begin{aligned} \|(\eta,w\|_{[L^\infty(0,T;L^2(0,2\pi)]^2} =& \|T\|_{\mathcal{L}(L^1(0,T;(L^2(0,2\pi))^2);\mathbb{R})} \\ &\leq C\left(\|(\eta^0,w^0)\|_{(H^{-1}(0,2\pi))^2} + \|(f,g)\|_{(H^1(0,T))^2}\right). \end{aligned} \tag{2.16}$$

By using density arguments, starting with more regular data, we can also get the regularity in the time variable. Indeed, since $(f,g)\in\left(H^1(0,T)\right)^2$ and $(\eta^0,w^0)\in\left(H^{-1}(0,2\pi)\right)^2$, there exist sequences $(f_n,g_n)\in\left(\mathcal{D}(0,T)\right)^2$ and $(\eta_n^0,w_n^0)\in\left(\mathcal{D}(0,2\pi)\right)^2$, such that, as $n\to\infty$,

$$\begin{aligned} (f_n,g_n) &\longrightarrow (f,g) \quad \text{in } \left(H^1(0,T)\right)^2 \\ \left(\eta_n^0,w_n^0\right) &\longrightarrow \left(\eta^0,w^0\right) \quad \text{in } \left(H^{-1}(0,2\pi)\right)^2. \end{aligned}$$

Let us denote by (η^n,w^n) the solution of the system (1.3), corresponding to the data (f_n,g_n) and (η_n^0,w_n^0), given by Theorem 2.1.2. Then, $(\eta^n,w^n)\in\mathcal{C}([0,T];(L^2(0,2\pi)))$ and, for each $n\in\mathbb{N}$, the solution (η^n,w^n) satisfies (2.8). Thus, if (η,w) is a solution by transposition of (1.3), it follows that $(\eta^n,w^n)-(\eta,w)$ is a solution by transposition with data $(f_n,g_n)-(f,g)$ and $(\eta_n^0,w_n^0)-(\eta^0,w^0)$. From estimate (2.16), we obtain

$$\|(\eta^n-\eta,w^n-w)\|_{L^\infty(0,T;(L^2(0,2\pi))^2)} \leq C(\|(\eta_n^0-\eta^0,w_n^0-w^0)\|_{(H^{-1}(0,2\pi))^2}+\|(f_n-f,g_n-g)\|_{(H^1(0,T))^2}).$$

When $n\to\infty$, from the above inequality we deduce that $(\eta^n,w^n)\to(\eta,w)$ in $L^\infty(0,T;(L^2(0,2\pi))^2)$ and, since $(\eta^n,w^n)\in\mathcal{C}([0,T];(L^2(0,2\pi))^2)$, it follows that $(\eta,w)\in\mathcal{C}([0,T];(L^2(0,2\pi))^2)$.

□

2.2 Controllability

In this section we study some boundary controllability properties of the system (1.3). We start with the following characterization of a control driving system (1.3) to the rest. This kind of result is already classic for dispersive systems (see, for instance, [24]).

LEMMA 2.2.1. *The initial data* $\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} \in (H^{-1}(0,2\pi))^2$ *is controllable to zero in time* $T>0$ *with controls* $\begin{pmatrix} f \\ g \end{pmatrix} \in (H^1(0,T))^2$ *if and only if*

$$\left\langle \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix}, \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \right\rangle_{(H^{-1}(0,2\pi))^2, \left(H_0^1(0,2\pi)\right)^2} = -b\int_0^T f(t)u_{xt}(t,2\pi)dt - \int_0^T g(t)v_{xt}(t,2\pi)dt, \tag{2.17}$$

for any solution of the adjoint system

$$\begin{cases} u_t + v_x - bu_{txx} = 0, & x \in (0,2\pi),\ 0<t<T \\ v_t + u_x - dv_{txx} = 0, & x \in (0,2\pi),\ 0<t<T \\ u(t,0) = v(t,0) = 0, & 0<t<T \\ u(t,2\pi) = v(t,2\pi) = 0, & 0<t<T \\ u(T,x) = u^T(x), & x \in (0,2\pi) \\ v(T,x) = v^T(x), & x \in (0,2\pi), \end{cases} \tag{2.18}$$

with $\begin{pmatrix} u^T \\ v^T \end{pmatrix} \in (H_0^1(0,2\pi))^2$.

Proof. Remark that the change of variables $t \to T-t$ and $x \to L-x$ reduce system (2.18) to (1.3) with $f \equiv g \equiv 0$. Then, we can apply to $\begin{pmatrix} u \\ v \end{pmatrix}$ all the well-posedness results obtained in Section 2.

We first prove the result for regular solutions. The less regular framework can be proved using density arguments as in the proof of Theorem 2.1.4. Let (η, w) be a solution of (1.3) and (u,v) solution of (2.18). Integration by parts leads to

$$\begin{aligned} 0 = \int_0^T\int_0^{2\pi} (\eta_t + w_x - b\eta_{xxt})udxdt + \int_0^T\int_0^{2\pi} (w_t + \eta_x - dw_{xxt})vdxdt = \\ -\int_0^{2\pi} (\eta^0 u(0) + \eta_x^0 u_x(0))dx - \int_0^{2\pi} (w^0 v(0) + w_x^0 v_x(0))dx \\ + \int_0^{2\pi} (\eta(T)u(T) + \eta_x(T)u_x(T))dx + \int_0^{2\pi} (w(T)v(T) + w_x(T)v_x(T))dx \\ +b\int_0^T f(t)u_{xt}(t,2\pi)dt + d\int_0^T g(t)v_{xt}(t,2\pi)dt. \end{aligned}$$

Consequently, by the density of $H_0^1(0,2\pi)$ in $H^{-1}(0,2\pi)$, we can pass the limit in the identity above to obtain

$$\begin{aligned} \left\langle \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix}, \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \right\rangle_{(H^{-1}(0,2\pi))^2, (H_0^1(0,2\pi))^2} + b\int_0^T f(t)u_{xt}(t,2\pi)dt + d\int_0^T g(t)v_{xt}(t,2\pi)dt \\ = \left\langle \begin{pmatrix} \eta(T) \\ w(T) \end{pmatrix}, \begin{pmatrix} u(T) \\ v(T) \end{pmatrix} \right\rangle_{(H^{-1}(0,2\pi))^2, (H_0^1(0,2\pi))^2}. \end{aligned}$$

Hence, $\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix}$ is controllable to zero in time $T>0$ if and only if (2.17) holds. □

The next result is devoted to show that system (1.3) is not spectrally controllable. This means that no nontrivial finite linear combination of eigenvectors of the operator $\mathcal{A}$ defined in (2.3) can be driven to zero in finite time by using controls $f, g \in H^1(0,T)$.

THEOREM 2.2.1. *No eigenfunction of the operator $\mathcal{A}$ can be driven to zero in finite time.*

Proof. We first note that, according to Theorem 6.2.2 of the Appendix, the operator $\mathcal{A}$ has a sequence of purely imaginary eigenvalues $(\mu_n^j)_{n\in\mathbb{Z}^*, j\in\{1,2\}}$. Moreover, the corresponding eigenfunctions $(\Phi_n^j)_{n\in\mathbb{Z}^*, j\in\{1,2\}}$ form an orthogonal basis of $(H_0^1(0,2\pi))^2$.

For each $k \neq 0$, let us consider $\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} = \Phi_k^j = \begin{pmatrix} \varphi_k^j \\ \\ v_k^j \end{pmatrix}$, $j = 1,2$, eigenfunctions of the operator $\mathcal{A}$. Then, from the proof of Theorem 6.2.2 we have that, for each eigenvalue $\mu = \mu_k^j$, the functions $\begin{pmatrix} \varphi_k^1 \\ \\ v_k^1 \end{pmatrix}$ and $\begin{pmatrix} \varphi_k^2 \\ \\ v_k^2 \end{pmatrix}$ solve the problems

$$\begin{cases} \varphi^1 - b\varphi_{xx}^1 + \mu v_x^1 = 0, & x \in (0,2\pi) \\ v^1 - dv_{xx}^1 + \mu\varphi_x^1 = 0, & x \in (0,2\pi) \\ \varphi^1(0) = 0,\ \varphi_x^1(0) = 1 \\ v^1(0) = 0,\ v_x^1(0) = \gamma \end{cases} \tag{2.19}$$

and

$$\begin{cases} \varphi^2 - b\varphi_{xx}^2 + \mu v_x^2 = 0, & x \in (0,2\pi) \\ v^2 - dv_{xx}^2 + \mu\varphi_x^2 = 0, & x \in (0,2\pi) \\ \varphi^2(0) = 0,\ \varphi_x^2(0) = \gamma \\ v^2(0) = 0,\ v_x^2(0) = 1, \end{cases} \tag{2.20}$$

with $\gamma = \gamma_n^1$ and $\gamma = \gamma_n^2$, respectively. We also note that, according to Theorem 6.2.2, $|\gamma_n^1| \leq \dfrac{\delta}{|n|}$ and $|\gamma_n^2| \leq \dfrac{\delta}{|n|}$, for a given positive δ.

In a similar way, if we also consider

$$\begin{pmatrix} u^T \\ v^T \end{pmatrix} = \begin{cases} \Phi_n^j & n \neq k \\ 0 & n = k, \end{cases}$$

the corresponding solution of (2.18) can be written as

$$\begin{pmatrix} u \\ v \end{pmatrix} = e^{-\lambda_n^j(T-t)}\Phi_n^j, \qquad \text{where} \qquad \lambda_n^j = \frac{1}{\mu_n^j},$$

and μ_n^j are the eigenvalues of the operator $\mathcal{A}$ given by Theorem 6.2.2. Moreover,

$$\lim_{n\to\infty} \lambda_n^j = 0.$$

On the other hand, since the sequence $(\Phi_n^j)_{n\in\mathbb{Z}^*, j\in\{1,2\}}$ forms an orthonormal basis of $(H_0^1(0,2\pi))^2$, we get

$$\left\langle \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix}, \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \right\rangle_{\left(H_0^1(0,2\pi)\right)^2} = \delta_{n,k}^j e^{\lambda_n^j T},\ j = 1,2.$$

Thus, if $\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix}$ is controllable to zero in time $T > 0$, from (2.17) it follows that

$$\int_0^T \lambda_n^j e^{-\lambda_n^j (T-t)} \left(bf(t) \varphi_{n,x}^j(2\pi) + dg(t) v_{n,x}^j(2\pi) \right) dt = \delta_{n,k}^j e^{\lambda_n^j T}, \ j = 1, 2. \tag{2.21}$$

For $j = 1$, the identity above can be written as follows

$$\int_{\frac{-T}{2}}^{\frac{T}{2}} h(t) e^{-\lambda_n^1 \left(\frac{T}{2} - t \right)} dt = \delta_{n,k}^1 e^{\lambda_n^1 T}, \tag{2.22}$$

where

$$h(t) = bf\left(t + \tfrac{T}{2}\right) \varphi_{n,x}^1(2\pi) + dg\left(t + \tfrac{T}{2}\right) v_{n,x}^1(2\pi).$$

Since $h \in L^2\left(\frac{-T}{2}, \frac{T}{2}\right)$, if we define $F : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$F(z) = \int_{\frac{-T}{2}}^{\frac{T}{2}} h(t) e^{izt} dt.$$

From Paley-Wiener theorem, we have that F is an entire function. Moreover, since $\lim_{n\to\infty} \lambda_n^j = 0$, it follows that F is zero on a set with a finite accumulation points. Then, $F \equiv 0$ and, consequently,

$$bf\,(t)\, \varphi_{n,x}^1(2\pi) + dg\,(t)\, v_{n,x}^1(2\pi) = 0, \quad \forall t \in [0, T]. \tag{2.23}$$

For $j = 2$, we can use (2.21) and proceed in a similar way to obtain

$$bf\,(t)\, \varphi_{n,x}^2(2\pi) + dg\,(t)\, v_{n,x}^2(2\pi) = 0, \quad \forall t \in [0, T]. \tag{2.24}$$

Thus, from (2.23) and (2.24) we deduce that f and g should satisfy the system

$$\begin{cases} bf\,(t)\, \varphi_{n,x}^1(2\pi) + dg\,(t)\, v_{n,x}^1(2\pi) = 0 \\ \\ bf\,(t)\, \varphi_{n,x}^2(2\pi) + dg\,(t)\, v_{n,x}^2(2\pi) = 0. \end{cases} \tag{2.25}$$

The next steps are devoted to analyze carefully the coefficients of the system (2.25). In order to do that, we first consider the solution of the following problems

$$\begin{cases} -b\widetilde{\varphi}_{xx}^1 + \widetilde{\mu}\widetilde{v}_x^1 = 0, & x \in (0, 2\pi) \\ -d\widetilde{v}_{xx}^1 + \widetilde{\mu}\widetilde{\varphi}_x^1 = 0, & x \in (0, 2\pi) \\ \widetilde{\varphi}^1(0) = 0, \ \widetilde{\varphi}_x^1(0) = 1 \\ \widetilde{v}^1(0) = 0, \ \widetilde{v}_x^1(0) = \gamma, \end{cases} \tag{2.26}$$

and

$$\begin{cases} -b\widetilde{\varphi}_{xx}^2 + \widetilde{\mu}\widetilde{v}_x^2 = 0, & x \in (0, 2\pi) \\ -d\widetilde{v}_{xx}^2 + \widetilde{\mu}\widetilde{\varphi}_x^2 = 0, & x \in (0, 2\pi) \\ \widetilde{\varphi}^2(0) = 0, \ \widetilde{\varphi}_x^2(0) = \gamma \\ \widetilde{v}^2(0) = 0, \ \widetilde{v}_x^2(0) = 1. \end{cases} \tag{2.27}$$

For each $\mu = \widetilde{\mu_n}$, where $\widetilde{\mu}_n = \sqrt{bd}ni$ ($n \in \mathbb{Z}^*$), the solutions of (2.26) and (2.27) are given by formula (6.11) of the Appendix and will be denoted by

$$\widetilde{\Phi}_n^1 = \begin{pmatrix} \widetilde{\varphi}_n^1 \\ \\ \widetilde{v}_n^1 \end{pmatrix} \quad \text{and} \quad \widetilde{\Phi}_n^2 = \begin{pmatrix} \widetilde{\varphi}_n^2 \\ \\ \widetilde{v}_n^2 \end{pmatrix},$$

respectively. Then, we have that

$$\begin{pmatrix} \widetilde{\varphi}_{n,x}^1(2\pi) \\ \\ \widetilde{v}_{n,x}^1(2\pi) \end{pmatrix} = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \widetilde{\varphi}_{n,x}^2(2\pi) \\ \\ \widetilde{v}_{n,x}^2(2\pi) \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix}. \tag{2.28}$$

From (2.28), Proposition 6.1.2 and Theorem 6.2.2, the coefficients of the system (2.25) can be estimated as follows

$$|\varphi_{n,x}^1(2\pi) - 1| = |\varphi_{n,x}^1(2\pi) - \widetilde{\varphi}_{n,x}^1(2\pi)| \leq \frac{C}{|\mu_n|}(1 + |\gamma|) + C\left(|\mu_n^1 - \widetilde{\mu_n}| + |\gamma|\right) \leq$$
$$\leq \frac{C}{|\widetilde{\mu_n}|}\left(1 + \frac{\delta}{|n|}\right) + \frac{C}{|n|} + \frac{C\delta}{|n|},$$

for some constant $C > 0$. From the estimate above, we conclude that $\varphi_{n,x}^1(2\pi) \sim 1$. Performing similar computations, we get

$$v_{n,x}^1(2\pi) \sim 0, \quad \varphi_{n,x}^2(2\pi) \sim 0, \quad v_{n,x}^2(2\pi) \sim 1.$$

Finally, we deduce that the determinant of the coefficients of the system (2.25) satisfies

$$\begin{vmatrix} \varphi_{n,x}^1(2\pi) & v_{n,x}^1(2\pi) \\ \varphi_{n,x}^2(2\pi) & v_{n,x}^2(2\pi) \end{vmatrix} \sim 1.$$

Hence $f \equiv g \equiv 0$ is the unique solution of the system (2.25), which contradicts (2.21) and the proof ends. □

Remark 2.2.1. *Taking into account the properties of the operator $\mathcal{A}$ mentioned in the proof of Theorem 2.2.1, the following holds:*

- *Each eigenvalue of the operator $\mathcal{A}$ has geometric multiplicity at most two, i. e., there is no eigenvalue that corresponds to three linear independent eigenfunction. Indeed, suppose that there exist* $\begin{pmatrix} \varphi_n^1 \\ v_n^1 \end{pmatrix}, \begin{pmatrix} \varphi_n^2 \\ v_n^2 \end{pmatrix}, \begin{pmatrix} \varphi_n^3 \\ v_n^3 \end{pmatrix}$ *linear independent eigenfunctions that correspond to the same eigenvalue μ. Let*

$$\begin{pmatrix} \psi_n \\ z_n \end{pmatrix} = \alpha \begin{pmatrix} \varphi_n^1 \\ v_n^1 \end{pmatrix} + \beta \begin{pmatrix} \varphi_n^2 \\ v_n^2 \end{pmatrix} + \gamma \begin{pmatrix} \varphi_n^3 \\ v_n^3 \end{pmatrix},$$

and $\alpha, \beta, \gamma \in \mathbb{C}$, not simultaneously zero, such that

$$\begin{cases} \psi_n(0) = \alpha\varphi_{n,x}^1(0) + \beta\varphi_{n,x}^2(0) + \gamma\varphi_{n,x}^3(0) = 0 \\ z_n(0) = \alpha v_{n,x}^1(0) + \beta v_{n,x}^2(0) + \gamma v_{n,x}^3(0) = 0. \end{cases}$$

Under these conditions, it follows that $\begin{pmatrix} \psi_n \\ z_n \end{pmatrix}$ *solves the initial value problem*

$$\begin{cases} \psi - b\psi_{xx} + \mu z_x = 0, & x \in (0, 2\pi) \\ z - dz_{xx} + \mu\psi_x = 0, & x \in (0, 2\pi) \\ \psi(0) = \psi_x(0) = 0 \\ z(0) = z_x(0) = 0. \end{cases}$$

Then, $\begin{pmatrix} \psi \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$*, which allows us to conclude the claim.*

- $\lambda = 0$ *is not an eigenvalue of the operator* $\mathcal{A}$*. In fact, if this is the case we obtain* $\begin{pmatrix} \eta \\ w \end{pmatrix}$ *satisfying*

$$\begin{cases} (I - b\partial_x^2)^{-1} w_x = 0, & x \in (0, 2\pi), \\ (I - d\partial_x^2)^{-1} \eta_x = 0, & x \in (0, 2\pi), \\ \eta(0) = w(0) = 0, \\ \eta(2\pi) = w(2\pi) = 0. \end{cases} \tag{2.29}$$

Due to the properties of the operator $(I - \alpha\partial_x^2)^{-1}$*, for* $\alpha > 0$*, we deduce that* $\eta_x \equiv w_x \equiv 0$*, i.e.,* η *and* w *are constant functions. Then, from the boundary conditions, it follows that* $\eta \equiv w \equiv 0$*.*

We shall pass now to study the approximate controllability of systems (1.3). In order to make that precise, we introduce the following definition.

DEFINITION 2.2.1. *System* (1.3) *is approximately controllable in time* T *if, for every initial data* $\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} \in (H^{-1}(0, 2\pi))^2$*, the set of reachable states*

$$R\left(\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix}, T\right) = \left\{\begin{pmatrix} \eta(T, x) \\ w(T, x) \end{pmatrix} : \begin{pmatrix} f \\ g \end{pmatrix} \in (H^1(0, T))^2\right\}$$

is dense in $(L^2(0, 2\pi))^2$*.*

We have the following result:

THEOREM 2.2.2. *System* (1.3) *is approximately controllable in time* $T > 0$ *with controls in* $H^1(0, T)$*.*

Proof. Due to the linearity of the system under consideration, it is sufficient to prove the result for any $T > 0$ and $\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus, we are going to prove that the set $R\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, T\right)$ is dense in $(L^2(0, 2\pi))^2$.

Let $\begin{pmatrix} \eta \\ w \end{pmatrix} \in C([0, T], (L^2(0, 2\pi))^2)$ the corresponding solution of the system (1.3) given by Theorem 2.1.4 and $\begin{pmatrix} u \\ v \end{pmatrix}$ solution of the adjoint system (2.18). Then, it follows that

$$\left\langle \begin{pmatrix} \eta(T,x) \\ w(T,x) \end{pmatrix}, \begin{pmatrix} u^T \\ v^T \end{pmatrix} \right\rangle_{\left(H_0^1(0,2\pi)\right)^2} = b\int_0^T f(t)u_{xt}(t,2\pi)dt + d\int_0^T g(t)v_{xt}(t,2\pi)dt. \tag{2.30}$$

Assume that $R\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, T\right)$ is not dense in $(H_0^1(0,2\pi))^2$. In this case, there exists $\begin{pmatrix} u^T \\ v^T \end{pmatrix} \in (H_0^1(0,2\pi))^2$, $\begin{pmatrix} u^T \\ v^T \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, satisfying

$$\left\langle \begin{pmatrix} \eta(T,x) \\ w(T,x) \end{pmatrix}, \begin{pmatrix} u^T \\ v^T \end{pmatrix} \right\rangle_{\left(H_0^1(0,2\pi)\right)^2} = 0, \quad \forall \begin{pmatrix} f \\ g \end{pmatrix} \in (H^1(0,T))^2.$$

Consequently, from (2.30) we obtain

$$\left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} u_{xt}(\cdot,2\pi) \\ v_{xt}(\cdot,2\pi) \end{pmatrix} \right\rangle_{(L^2(0,T))^2} = 0, \quad \forall \begin{pmatrix} f \\ g \end{pmatrix} \in (H^1(0,T))^2.$$

Thus,

$$\begin{pmatrix} u_{xt}(t,2\pi) \\ v_{xt}(t,2\pi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \forall\, t \in (0,T). \tag{2.31}$$

On the other hand, since $\mathcal{A}$ is a skew adjoint operator in $(H_0^1(0,2\pi))^2$, it has a sequence of eigenvalues $(\lambda_n)_{n\geq 1} \subset i\mathbb{R}^*$ with geometric multiplicity at most two (see Remark 2.2.1). The corresponding eigenfunctions form an orthonormal basis for $(H_0^1(0,2\pi))^2$, which we denote by

$$(\Phi_n)_{n\geq 1} \cup \{\Phi_n^1, \Phi_n^2\}_{n\geq 1},$$

where $\Phi_n = \begin{pmatrix} \varphi_n \\ v_n \end{pmatrix}$ and $\Phi_n^j = \begin{pmatrix} \varphi_n^j \\ v_n^j \end{pmatrix}$, $j = 1, 2$, correspond to a simple and double eigenvalue, respectively.

Then, if $\begin{pmatrix} u^T \\ v^T \end{pmatrix} \in (H_0^1(0,2\pi))^2$, we have

$$\begin{pmatrix} u^T \\ v^T \end{pmatrix} = \sum_{\substack{n\geq 1 \\ \lambda_n simple}} a_n \Phi_n + \sum_{\substack{n\geq 1 \\ \lambda_n double}} a_n^1 \Phi_n^1 + a_n^2 \Phi_n^2$$

and the corresponding solution $\begin{pmatrix} u \\ v \end{pmatrix}$ can be written as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{\substack{n\geq 1 \\ \lambda_n\, simple}} a_n e^{-\lambda_n(T-t)} \Phi_n + \sum_{\substack{n\geq 1 \\ \lambda_n\, double}} (a_n^1 \Phi_n^1 + a_n^2 \Phi_n^2) e^{-\lambda_n(T-t)}. \tag{2.32}$$

Thus, from (2.31)-(2.32) it follows that

$$0 = u_{xt}(t, 2\pi) = \sum_{\substack{n\geq 1 \\ \lambda_n \, double}} \left(a_n^1 \varphi_{n,x}^1(2\pi) + a_n^2 \varphi_{n,x}^2(2\pi)\right) \lambda_n e^{-\lambda_n (T-t)} + \\ + \sum_{\substack{n\geq 1 \\ \lambda_n simple}} a_n \varphi_{n,x}(2\pi) \lambda_n e^{-\lambda_n (T-t)}.$$

Since u is an analytic function (see Theorem 2.1.1), we can integrate the identity above over $(-S, S)$, for any $S > 0$. Then, for each $m \in \mathbb{Z}^*$ we deduce that

$$0 = \lim_{s\to+\infty} \frac{1}{2S} \int_{-S}^{S} u_{xt}(s, 2\pi) e^{-\lambda_m s} ds =$$

$$= \begin{cases} \left(a_n^1 \varphi_{n,x}^1(2\pi) + a_n^2 \varphi_{n,x}^2(2\pi)\right) \lambda_n e^{-\lambda_n T}, & \text{if } \lambda_n \text{ is double} \\ \\ a_n \varphi_{n,x}(2\pi) \lambda_n e^{-\lambda_n T}, & \text{if } \lambda_n \text{ is simple .} \end{cases} \tag{2.33}$$

From (2.31) we have that $v_{xt}(t, 2\pi) = 0$. Therefore, we can use (2.32) and proceed in a similar way to obtain

$$0 = \begin{cases} \left(a_n^1 v_{n,x}^1(2\pi) + a_n^2 v_{n,x}^2(2\pi)\right) \lambda_n e^{-\lambda_n T}, & \text{if } \lambda_n \text{ is double} \\ \\ a_n v_{n,x}(2\pi) \lambda_n e^{-\lambda_n T}, & \text{if } \lambda_n \text{ is simple .} \end{cases} \tag{2.34}$$

Assume that λ_n is double. In this case, if we consider

$$\begin{pmatrix} \psi_n \\ z_n \end{pmatrix} = \begin{pmatrix} a_n^1 \varphi_n^1 + a_n^2 \varphi_n^2 \\ a_n^1 v_n^1 + a_n^2 v_n^2 \end{pmatrix} = a_n^1 \begin{pmatrix} \varphi_n^1 \\ v_n^1 \end{pmatrix} + a_n^2 \begin{pmatrix} \varphi_n^2 \\ v_n^2 \end{pmatrix},$$

from (2.33) and (2.34) we have that $\begin{pmatrix} \psi_n \\ z_n \end{pmatrix}$ and $\begin{pmatrix} \psi_{n,x}(2\pi) \\ z_{n,x}(2\pi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ solve the initial-value problem

$$\begin{cases} \psi - b\psi_{xx} + \mu z_x = 0, & x \in (0, 2\pi) \\ z - dz_{xx} + \mu \psi_x = 0, & x \in (0, 2\pi) \\ \psi(2\pi) = \psi_x(2\pi) = 0 \\ z(2\pi) = z_x(2\pi) = 0. \end{cases} \tag{2.35}$$

Then, by uniqueness, $\begin{pmatrix} \psi_n \\ z_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. On the other hand, since $\begin{pmatrix} \varphi_n^1 \\ v_n^1 \end{pmatrix}$ and $\begin{pmatrix} \varphi_n^2 \\ v_n^2 \end{pmatrix}$ are linearly independent, it follows that $a_n^1 = a_n^2 = 0$.

If λ_n is simple, from (2.33) and (2.34) we obtain

$$a_n \Phi_{n,x}(2\pi) = a_n \begin{pmatrix} \varphi_{n,x}(2\pi) \\ v_{n,x}(2\pi) \end{pmatrix} = 0.$$

If $\Phi_{n,x}(2\pi) = 0$, we can proceed as in the previous case to conclude that $\Phi_n = 0$. Since no eigenfunction can be identically zero, we have obtained a contradiction. Therefore, $a_n = 0$.

Thus, from (2.32) it follows that

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{\substack{n\geq 1 \\ \lambda_n \, simple}} a_n e^{-\lambda_n(T-t)}\Phi_n + \sum_{\substack{n\geq 1 \\ \lambda_n \, double}} (a_n^1\Phi_n^1 + a_n^2\Phi_n^2)e^{-\lambda_n(T-t)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and, in particular, $\begin{pmatrix} u^T \\ v^T \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This is also a contradiction and the proof ends. □

Chapter 3

Large time behavior for the Boussinesq system of BBM-BBM type

In this chapter we are concerned with a Boussinesq system of Benjamin-Bona-Mahony type modelling the two-way propagation of surface waves in a uniform horizontal channel filled with an irrotational, incompressible and inviscid liquid under the influence of gravitation. We propose several dissipation mechanisms leading to systems for which one has both the global existence of solutions and a nonincreasing energy. Following the analysis developed in [31] we prove that all the trajectories are attracted by the origin provided that the unique continuation of weak solutions holds.

3.1 Unique Continuation Property

In this section we study some unique continuation properties for the following system

$$\begin{cases} \eta_t + w_x - b\eta_{txx} = 0 & \text{for } x \in (0, 2\pi), \ \ t > 0 \\ w_t + \eta_x - dw_{txx} = 0 & \text{for } x \in (0, 2\pi), \ \ t > 0 \\ \eta(t, 0) = \eta(t, 2\pi); \ \ \eta_x(t, 0) = \eta_x(t, 2\pi) & \text{for } t > 0 \\ w(t, 0) = w(t, 2\pi); \ \ w_x(t, 0) = w_x(t, 2\pi) & \text{for } t > 0 \\ \eta(0, x) = \eta^0(x) & \text{for } x \in (0, 2\pi) \\ w(0, x) = w^0(x) & \text{for } x \in (0, 2\pi), \end{cases} \tag{3.1}$$

where $b, d > 0$, that will be used to obtain our main results. The proofs depend on some global well-posedness obtained in [26] by using the Fourier approach. Therefore, for the sake of completeness, we include such results in this section.

We first introduce a few notations. Given any $v \in L^2(0, 2\pi)$ and $k \in \mathbb{Z}$, we denote by $\widehat{v}_k$ the $k^{th}-$Fourier coefficient of v,

$$\widehat{v}_k = \frac{1}{2\pi} \int_0^{2\pi} v(x) e^{-ikx} \, \mathrm{d}x,$$

and, for any $m \in \mathbb{N}$, we define the space

$$H_p^m(0, 2\pi) = \left\{ v \in L^2(0, 2\pi) \;\middle|\; v = \sum_{k \in \mathbb{Z}} \widehat{v}_k e^{ikx}, \ \sum_{k \in \mathbb{Z}} |\widehat{v}_k|^2 (1 + k^2)^m < \infty \right\},$$

which is a Hilbert space with respect to the inner product

$$(v, w)_m = \sum_{k\in\mathbb{Z}} \widehat{v}_k \overline{\widehat{w}_k}(1+k^2)^m. \tag{3.2}$$

The norm corresponding to (3.2) is denoted by $\|\ \|_m$. It can be seen that

$$H_p^m(0,2\pi) = \left\{ v \in H^m(0,2\pi) \;\middle|\; \frac{\partial^r v}{\partial x^r}(0) = \frac{\partial^r v}{\partial x^r}(2\pi),\ 0 \le r \le m-1 \right\},$$

where $H^m(0,2\pi)$ stands for the classical Sobolev space of exponent m in $(0,2\pi)$. We can extend the definition of $H_p^m(0,2\pi)$ to the case $m = s \ge 0$, a nonnegative real number, by setting

$$H_p^s(0,2\pi) = \left\{ v = \sum_{k\in\mathbb{Z}} \widehat{v}_k e^{ikx} \in H^s(0,2\pi) \;\middle|\; \sum_{k\in\mathbb{Z}} |\widehat{v}_k|^2 (1+k^2)^s < \infty \right\}. \tag{3.3}$$

For any nonnegative real number s, $H_p^s(0,2\pi)$ can also be seen as a Hilbert space with respect to the inner product defined by (3.2) with m replaced by s. In particular, for any $v \in H_p^s(0,2\pi)$,

$$\|v\|_s = \left(\sum_{k\in\mathbb{Z}} |\widehat{v}_k|^2 (1+k^2)^s \right)^{\frac{1}{2}}.$$

As pointed out in [26], for $s < 0$ we define the space $H_p^s(0,2\pi)$ as the topological dual of $H_p^{-s}(0,2\pi)$:

$$H_p^s(0,2\pi) = \left(H_p^{-s}(0,2\pi) \right)'.$$

Riesz representation theorem ensures that any $v \in H_p^0(0,2\pi) = L^2(0,2\pi)$ can be identified with an element $w_v \in \left(H_p^0(0,2\pi) \right)'$ such that

$$w_v(z) = \int_0^{2\pi} z(x) v(x)\, \mathrm{d}x \qquad \left(z \in H_p^0(0,2\pi) \right).$$

Traditionally, the same notation is used for v and w_v (the spaces $\left(H_p^0(0,2\pi) \right)'$ and $H_p^0(0,2\pi)$ are identified). Given $s < 0$, any element $w \in H_p^s(0,2\pi)$ can be uniquely expanded as follows

$$w = \sum_{k\in\mathbb{Z}} \widehat{w}_k e^{ikx}, \tag{3.4}$$

where $\widehat{w}_k = \frac{1}{2\pi} w\left(e^{-ikx} \right)$ for each $k \in \mathbb{Z}$. The slight abuse of notation in (3.4) (the element w on the left hand side is not a function of x and the exponential function e^{ikx} on the right hand side is actually the representant of this L^2-function in the dual space) is compensated by the fact that expansion (3.4) looks exactly like one corresponding to an element in a space H^s with positive exponent s. On the other hand, the following map is a duality product between $H_p^s(0,2\pi)$ and $H_p^{-s}(0,2\pi)$, for any $s \ge 0$,

$$\langle v, w \rangle_s = \sum_{k\in\mathbb{Z}} \widehat{v}_k \widehat{w}_{-k} \qquad \left(v \in H_p^s(0,2\pi),\ w \in H_p^{-s}(0,2\pi) \right). \tag{3.5}$$

Consequently, if $s < 0$, the space $H_p^s(0, 2\pi)$ can also be defined by (3.3) and can be viewed as a Hilbert space with respect to the inner product (3.2) with m replaced by s.

Under the considerations above, for $\alpha > 0$ we can define the operator $(I - \alpha\partial_x^2)_p^{-1}$ in the following way:

$$(I - \alpha\partial_x^2)_p^{-1}\varphi = v \Leftrightarrow \begin{cases} v - \alpha v_{xx} = \varphi & \text{in } (0, 2\pi), \\ v(0) = v(2\pi), & v_x(0) = v_x(2\pi). \end{cases} \tag{3.6}$$

Since for any $\varphi \in L^2(0, 2\pi)$, the elliptic equation from above has a unique solution $v \in H_p^2(0, 2\pi)$, the operator $(I - \alpha\partial_x^2)_p^{-1}$ is a well-defined, compact operator in $L^2(0, 2\pi)$.

Given $s \in \mathbb{R}$, let us introduce the Hilbert space

$$V^s = H_p^s(0, 2\pi) \times H_p^s(0, 2\pi), \tag{3.7}$$

endowed with the inner product defined by

$$\langle (f_1, f_2), (g_1, g_2) \rangle = b(f_1, g_1)_s + d(f_2, g_2)_s. \tag{3.8}$$

Let us remark that system (3.1) can be written in the following vectorial form

$$\begin{pmatrix} \eta \\ w \end{pmatrix}_t (t) + A \begin{pmatrix} \eta \\ w \end{pmatrix} (t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \eta \\ w \end{pmatrix} (0) = \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix}, \tag{3.9}$$

where A is the linear compact operator in V^s defined by

$$A = \begin{pmatrix} 0 & (I - b\partial_x^2)_p^{-1}\partial_x \\ (I - d\partial_x^2)_p^{-1}\partial_x & 0 \end{pmatrix}. \tag{3.10}$$

Thus, if we assume that the initial data in (3.1) are given by

$$(\eta^0, w^0) = \sum_{k\in\mathbb{Z}} \left(\widehat{\eta}_k^0, \widehat{w}_k^0\right) e^{ikx}, \tag{3.11}$$

then, at least formally, the solution of (3.1) can be written as

$$(\eta, w)(t, x) = \sum_{k\in\mathbb{Z}} (\widehat{\eta}_k(t), \widehat{w}_k(t)) e^{ikx}, \tag{3.12}$$

where $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ fulfill

$$\begin{cases} (1 + bk^2)(\widehat{\eta}_k)_t + ik\widehat{w}_k = 0, & t \in (0, T), \\ (1 + dk^2)(\widehat{w}_k)_t + ik\widehat{\eta}_k = 0, & t \in (0, T), \\ \widehat{\eta}_k(0) = \widehat{\eta}_k^0, & \widehat{w}_k(0) = \widehat{w}_k^0. \end{cases} \tag{3.13}$$

We have the following result:

Lemma 3.1.1. *(see [26]) Let*

$$\lambda_k^{\pm} = \frac{\pm|k|i}{\sqrt{(1+bk^2)(1+dk^2)}} \qquad (k \in \mathbb{Z}^*). \tag{3.14}$$

The solution $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ *of* (3.13) *is given by*

$$\begin{cases} \widehat{\eta}_k(t) &= \dfrac{1}{2}\left[\left(\widehat{\eta}_k^0 + \sqrt{\dfrac{1+dk^2}{1+bk^2}}\widehat{w}_k^0\right)e^{-\lambda_k^+ t} + \left(\widehat{\eta}_k^0 - \sqrt{\dfrac{1+dk^2}{1+bk^2}}\widehat{w}_k^0\right)e^{-\lambda_k^- t}\right], \\ \widehat{w}_k(t) &= \dfrac{1}{2}\left[\left(\sqrt{\dfrac{1+bk^2}{1+dk^2}}\widehat{\eta}_k^0 + \widehat{w}_k^0\right)e^{-\lambda_k^+ t} - \left(\sqrt{\dfrac{1+bk^2}{1+dk^2}}\widehat{\eta}_k^0 - \widehat{w}_k^0\right)e^{-\lambda_k^- t}\right], \end{cases} \tag{3.15}$$

if $k \neq 0$ *and*

$$\begin{cases} \widehat{\eta}_0(t) &= \widehat{\eta}_0^0, \\ \widehat{w}_0(t) &= \widehat{w}_0^0. \end{cases} \tag{3.16}$$

Using Lemma 3.1.1 it was proved that the operator A generates an analytic semigroup in V^s.

Theorem 3.1.1. *(see [26]) The family of linear operators* $(S(t))_{t\geq 0}$ *defined by*

$$S(t)(\eta^0, w^0) = \sum_{k\in\mathbb{Z}}(\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx} \qquad \left((\eta^0, w^0) \in V^s\right), \tag{3.17}$$

where the coefficients $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ *are given by* (3.15)-(3.16)*, is an analytic semigroup in* V^s *and verifies the following estimate, for each* $s \in \mathbb{R}$,

$$\|S(t)(\eta^0, w^0)\|_{V^s} \leq M\|(\eta^0, w^0)\|_{V^s} \qquad \left((\eta^0, w^0) \in V^s\right), \tag{3.18}$$

where M *is a positive constant. Moreover, its infinitesimal generator is the operator* $(D(A), A)$*, where* $D(A) = V^s$ *and* A *is given by* (3.10).

From Theorem 3.1.1 and the semigroup theory, we obtain the following global well-posedness result:

Theorem 3.1.2. *(see [26]) Let* $T > 0$ *and* $s \in \mathbb{R}$. *For each* $(\eta^0, w^0) \in V^s$ *and* $(f, g) \in L^1(0, T; V^s)$*, there exists a unique solution* $(\eta, w) \in W^{1,1}([0,T]; V^s)$ *of the system*

$$\begin{pmatrix} \eta \\ w \end{pmatrix}_t (t) + A\begin{pmatrix} \eta \\ w \end{pmatrix}(t) = \begin{pmatrix} f \\ g \end{pmatrix}, \quad \begin{pmatrix} \eta \\ w \end{pmatrix}(0) = \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix}, \tag{3.19}$$

which verifies the variation of constants formula

$$\begin{pmatrix} \eta \\ w \end{pmatrix}(t) = S(t)\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} + \int_0^t S(t-s)\begin{pmatrix} f \\ g \end{pmatrix}(s)\, ds. \tag{3.20}$$

Moreover, if $(f, g) \equiv (0, 0)$ *it follows that* $(\eta, w) \in \mathcal{C}^\omega(\mathbb{R}, V^s)$*, the class of analytic functions in* $t \in \mathbb{R}$ *with values in* V^s.

The main results of this section reads as follows:

THEOREM 3.1.3. *Let (η, w) solution of system (3.1) given by Theorem 3.1.2. Suppose that there exist an open set $\Omega \subset [0, 2\pi]$ and $T > 0$ such that*

$$\eta(t,x) = 0, \qquad \forall\,(t,x) \in (0,T) \times \Omega. \tag{3.21}$$

Then,

$$(\eta, w) = (0,0) \qquad in \quad \mathbb{R} \times (0, 2\pi). \tag{3.22}$$

Proof. From Lemma 3.1.1, it follows that the solution (η, w) can be written as

$$\begin{cases} \eta(t,x) = \displaystyle\sum_{k\in\mathbb{Z}} \widehat{\eta}_k(t) e^{ikx} = \sum_{k\in\mathbb{Z}} \left(a_k^+ e^{-\lambda_k^+ t} + a_k^- e^{-\lambda_k^- t}\right) e^{ikx} \\ w(t,x) = \displaystyle\sum_{k\in\mathbb{Z}} \widehat{w}_k(t) e^{ikx} = \sum_{k\in\mathbb{Z}} \left(b_k^+ e^{-\lambda_k^+ t} + b_k^- e^{-\lambda_k^- t}\right) e^{ikx}, \end{cases} \tag{3.23}$$

where $a_k^+ = \frac{1}{2}\left(\widehat{\eta}_k^0 + \sqrt{\frac{1+dk^2}{1+bk^2}}\,\widehat{w}_k^0\right)$ and $a_k^- = \frac{1}{2}\left(\widehat{\eta}_k^0 - \sqrt{\frac{1+dk^2}{1+bk^2}}\,\widehat{w}_k^0\right)$. Since the solution (η, w) is an analytic function of t, from (3.21) we deduce that

$$\eta(t,x) = 0, \quad \forall\ (t,x) \in\ \mathbb{R} \times \Omega.$$

Consequently, for any $S > 0$ and $x \in \Omega$, if we multiply $\eta(t,x)$ by $e^{\lambda_k^+ t}$ and integrate between $-S$ and S, from (3.23) we obtain

$$\begin{aligned} 0 &= \lim_{S\to\infty} \frac{1}{2S} \int_{-S}^{S} \left(\sum_{k\in\mathbb{Z}} \left(a_k^+ e^{-\lambda_k^+ t} + a_k^- e^{-\lambda_k^- t}\right) e^{ikx} \right) e^{\lambda_j^+ t} dt \\ &= a_k^+ e^{ikx} + a_{-k}^+ e^{-ikx} \text{ in } \Omega. \end{aligned} \tag{3.24}$$

On the other hand, if we multiply $\eta(t,x)$ by $e^{\lambda_k^- t}$, similar computations yield

$$\begin{aligned} 0 &= \lim_{S\to\infty} \frac{1}{2S} \int_{-S}^{S} \left(\sum_{k\in\mathbb{Z}} \left(a_k^+ e^{-\lambda_k^+ t} + a_k^- e^{-\lambda_k^- t}\right) e^{ikx} \right) e^{\lambda_j^- t} dt \\ &= a_k^- e^{ikx} + a_{-k}^- e^{-ikx} \text{ in } \Omega. \end{aligned} \tag{3.25}$$

Since both functions on the left hand side are analytic in x, it follows that

$$a_k^{\pm} e^{ikx} + a_{-k}^{\pm} e^{-ikx} = 0 \text{ in } [0, 2\pi].$$

By using the orthogonality of $\{e^{ikx}\}_{k\in\mathbb{Z}}$ and $\{e^{-ikx}\}_{k\in\mathbb{Z}}$ in $[0, 2\pi]$, we deduce that $a_k^{\pm} = a_{-k}^{\pm} = 0$. This implies directly that $\widehat{\eta}_k^0 = \widehat{w}_k^0 = 0$ for any $k \in \mathbb{Z}$. Hence, $(\eta, w) = (0,0)$ in $\mathbb{R} \times (0, 2\pi)$. □

As consequence of Theorem 3.1.3, we have the following result:

Theorem 3.1.4. *Let* (η, w) *be solution of system* (3.1) *given by Theorem 3.1.2. Suppose that there exist an open set* $\Omega \subset [0, 2\pi]$ *and* $T > 0$ *such that*

$$\eta_x(t,x) = 0, \qquad \forall\, (t,x) \in (0,T) \times \Omega. \tag{3.26}$$

Then,

$$(\eta, w) = (c_1, c_2) \qquad in \quad \mathbb{R} \times (0, 2\pi), \tag{3.27}$$

for some constants c_1 *and* c_2.

Proof. From Lemma 3.1.1, we have that

$$\begin{cases} \eta(t,x) = \sum\limits_{k\in\mathbb{Z}} \widehat{\eta}_k(t) e^{ikx} = \sum\limits_{k\in\mathbb{Z}} \left(a_k^+ e^{-\lambda_k^+ t} + a_k^- e^{-\lambda_k^- t} \right) e^{ikx} \\ w(t,x) = \sum\limits_{k\in\mathbb{Z}} \widehat{w}_k(t) e^{ikx} = \sum\limits_{k\in\mathbb{Z}} \left(b_k^+ e^{-\lambda_k^+ t} + b_k^- e^{-\lambda_k^- t} \right) e^{ikx}, \end{cases} \tag{3.28}$$

where $a_k^+ = \frac{1}{2}\left(\widehat{\eta}_k^0 + \sqrt{\frac{1+dk^2}{1+bk^2}}\widehat{w}_k^0\right)$ and $a_k^- = \frac{1}{2}\left(\widehat{\eta}_k^0 - \sqrt{\frac{1+dk^2}{1+bk^2}}\widehat{w}_k^0\right)$. Then, proceeding as in the proof of Theorem 3.1.3, from (3.26) and (3.28) the following identities holds

$$\begin{aligned} k a_k^+ e^{ikx} + k a_{-k}^+ e^{-ikx} &= 0 \text{ in } [0, 2\pi], \\ k a_k^- e^{ikx} + k a_{-k}^- e^{-ikx} &= 0 \text{ in } [0, 2\pi], \end{aligned}$$

for any $k \in \mathbb{Z}^*$. From the orthogonality of $\{e^{ikx}\}_{k\in\mathbb{Z}}$ and $\{e^{-ikx}\}_{k\in\mathbb{Z}}$ in $[0, 2\pi]$, it follows that $a_k^\pm = a_{-k}^\pm = 0$, $\forall\, k \in \mathbb{Z}^*$. This implies directly $\widehat{\eta}_k^0 = \widehat{w}_k^0 = 0$, for any $k \in \mathbb{Z}^*$. Hence, $(\eta, w) = (c_1, c_2)$ in $\mathbb{R} \times (0, 2\pi)$, for some $c_1, c_2 \in \mathbb{R}$. □

3.2 Internal Stabilization

This section is devoted to prove the asymptotic behavior of the solutions of following system

$$\begin{cases} \eta_t + w_x - b\eta_{txx} + \mathcal{B}\eta = 0 & \text{for } x \in (0, 2\pi),\ \ t > 0 \\ w_t + \eta_x - dw_{txx} = 0 & \text{for } x \in (0, 2\pi),\ \ t > 0 \\ \eta(t,0) = \eta(t,2\pi);\ \eta_x(t,0) = \eta_x(t,2\pi) & \text{for } t > 0 \\ w(t,0) = w(t,2\pi);\ w_x(t,0) = w_x(t,2\pi) & \text{for } t > 0 \\ \eta(0,x) = \eta^0(x) & \text{for } x \in (0, 2\pi) \\ w(0,x) = w^0(x) & \text{for } x \in (0, 2\pi), \end{cases} \tag{3.29}$$

where b, $d > 0$ and $\mathcal{B} : H_p^s(0, 2\pi) \longrightarrow H_p^s(0, 2\pi)$ is a bounded operator. More precisely, let

$$\begin{cases} a \in C_p^\infty(0, 2\pi) \text{ a nonnegative function on } (0, 2\pi) \\ \text{with } a(x) > 0 \text{ on a given open set } \Omega_1 \subset (0, 2\pi). \end{cases} \tag{3.30}$$

We analyze the following cases for the operator $\mathcal{B}$:

$$\mathcal{B}\varphi = a(x)\varphi \quad \text{and} \quad \mathcal{B}\varphi = (a(x)\varphi_x)_x.$$

3.2.1 Internal stabilization with the feedback $\mathcal{B}\varphi = a(x)\varphi$

We first prove that the system is well-posed. This is done by using a fixed point argument, therefore we first write the solution of (3.29) in its integral form

$$\begin{cases} \eta(t) = \eta^0 - \displaystyle\int_0^t \left(1 - b\partial_x^2\right)^{-1} (\partial_x w + a\eta)(\tau) d\tau, \\ w(t) = w^0 - \displaystyle\int_0^t \left(1 - d\partial_x^2\right)^{-1} \partial_x \eta(\tau) d\tau, \end{cases} \tag{3.31}$$

where $(1 - \alpha\partial_x^2)^{-1} f$ denotes, for $f \in L^2(0, 2\pi)$ and $\alpha > 0$, the unique solution $v \in H_p^2(0, 2\pi)$ of the elliptic equation $(1 - \alpha\partial_x^2)\, v = f$. Moreover, for any $s \geq 0$,

$$\| \left(1 - \alpha\partial_x^2\right)_p^{-1} f\|_{H_p^s(0,2\pi)} \leq C\|f\|_{H_p^s(0,2\pi)} \quad \text{and} \quad \| \left(1 - \alpha\partial_x^2\right)_p^{-1} \partial_x f\|_{H_p^s(0,2\pi)} \leq C\|f\|_{H_p^s(0,2\pi)}, \tag{3.32}$$

for all $\alpha > 0$, where C is a positive constant.

We have the following result:

THEOREM 3.2.1. *Let $s \geq 0$. For any $(\eta^0, w^0) \in \left[H_p^s(0, 2\pi)\right]^2$, there exists $T > 0$ and a unique solution (η, w) of (3.29) with $\mathcal{B}\varphi = a(x)\varphi$ in the class $\left[\mathcal{C}([0, T]; H_p^s(0, 2\pi))\right]^2$. If $s = 1$, the solution exists for every $T > 0$. Moreover, the map $\mathcal{F}$ defined as follows*

$$\begin{aligned} \mathcal{F} : \left[H_p^s(0, 2\pi)\right]^2 &\longrightarrow \left[\mathcal{C}([0, T]; H_p^s(0, 2\pi))\right]^2 \\ (\eta^0, w^0) &\mapsto (\eta, w) \end{aligned}$$

is Lipschitz continuous.

Proof. In order to apply a fixed point argument, for any $(\eta^0, w^0) \in \left[H_p^s(0, 2\pi)\right]^2$, we introduce the operator

$$\Gamma(\eta, w)(t) := (\eta^0, w^0) - \left(\int_0^t \left(1 - b\partial_x^2\right)^{-1} (\partial_x w + a\eta)(\tau) d\tau, \int_0^t \left(1 - d\partial_x^2\right)^{-1} \partial_x \eta(\tau) d\tau\right).$$

Let $0 < \beta \leq T$, to be chosen later. Then, for each $(\eta^1, w^1), (\eta^2, w^2) \in \left[\mathcal{C}([0, \beta]; H_p^s(0, 2\pi)\right]^2$, from (3.32) it follows that

$$\begin{aligned}
&\|\Gamma(\eta^1, w^1) - \Gamma(\eta^2, w^2)\|_{\left[\mathcal{C}([0,\beta];H_p^s(0,2\pi)\right]^2} = \sup_{0 \leq t \leq \beta} \| \left(\Gamma(\eta^1, w^1) - \Gamma(\eta^2, w^2)\right)(t)\|_{\left[H_p^s(0,2\pi)\right]^2} \\
&\leq \int_0^\beta \left(\|(\left(1 - b\partial_x^2\right)^{-1} \partial_x(w^1 - w^2))(\tau)\|_{H_p^s(0,2\pi)} + \|(\left(1 - b\partial_x^2\right)^{-1} (a(\eta^1 - \eta^2)))(\tau)\|_{H_p^s(0,2\pi)}\right) d\tau \\
&+ \int_0^\beta \|(\left(1 - d\partial_x^2\right)^{-1} \partial_x(\eta^1 - \eta^2))(\tau)\|_{H_p^s(0,2\pi)} d\tau \\
&\leq C\beta \left(\|(\eta^1 - \eta^2, w^1 - w^2)\|_{\left[\mathcal{C}([0,\beta];H_p^s(0,2\pi)\right]^2} + \|(a(\eta^1 - \eta^2), w^1 - w^2)\|_{\left[\mathcal{C}([0,\beta];H_p^s(0,2\pi)\right]^2}\right) \\
&\leq C\beta\|(\eta^1, w^1) - (\eta^2, w^2)\|_{\left[\mathcal{C}([0,\beta];H_p^s(0,2\pi)\right]^2},
\end{aligned}$$

where C is a positive constant. Choosing $\beta > 0$ satisfying $C\beta \leq \frac{1}{2}$, from the estimate above we obtain

$$\|\Gamma(\eta^1, w^1) - \Gamma(\eta^2, w^2)\|_{[\mathcal{C}([0,\beta];H_p^s(0,2\pi)]^2} \leq \frac{1}{2}\|(\eta^1, w^1) - (\eta^2, w^2)\|_{[\mathcal{C}([0,\beta];H_p^s(0,2\pi)]^2}. \quad (3.33)$$

Let

$$(\eta, w) \in B_R(0) = \{(\eta, w) \in \left[\mathcal{C}([0,\beta]; H_p^s(0, 2\pi)\right]^2 : \|(\eta, w)\|_{[\mathcal{C}([0,\beta];H_p^s(0,2\pi)]^2} \leq R\},$$

where $R = 2\|(\eta^0, w^0)\|_{[H_p^s(0,2\pi)]^2}$. From (3.33), we obtain the following estimate

$$\begin{aligned}\|\Gamma(\eta, w)\|_{[C([0,\beta];H_p^s(0,2\pi)]^2} &\leq \|(\eta^0, w^0)\|_{[H_p^s(0,2\pi)]^2} + \|\Gamma(\eta, w) - \Gamma(0,0)\|_{[C([0,\beta];H^s(0,2\pi)]^2} \\ &\leq \|(\eta^0, w^0)\|_{[H_p^s(0,2\pi)]^2} + \frac{1}{2}\|(\eta, w)\|_{[\mathcal{C}([0,\beta];H_p^s(0,2\pi)]^2} \leq R, \quad (3.34)\end{aligned}$$

which allows us to conclude that

$$\Gamma : B_R(0) \subseteq \left[\mathcal{C}([0,\beta]; H_p^s(0, 2\pi)\right]^2 \longrightarrow B_R(0).$$

Hence, $\Gamma : B_R(0) \to B_R(0)$ is a contraction and, by Banach fixed-point theorem, we obtain a unique $(\eta, w) \in B_R(0)$ which solves the integral equation (3.31) for all $t \in (0, \beta)$. Since the choice of β is independent of (η^0, w^0), the standard continuation extension argument yields that the solution (η, w) belongs to $\left[\mathcal{C}([0, T]; H_p^1(0, 2\pi)\right]^2$ (see (1.6)).

Finally, in order to prove that the map $\mathcal{F}$ is Lipschitz continuous, we can proceed as in the proof of (3.33). Indeed, for any $(\eta^{0,1}, w^{0,1}), (\eta^{0,2}, w^{0,2}) \in \left[H_p^s(0, 2\pi)\right]^2$ if we consider the corresponding solutions (η^1, w^1) and (η^2, w^2), respectively, it follows that

$$\begin{aligned}&\|\mathcal{F}(\eta^{0,1}, w^{0,1}) - \mathcal{F}(\eta^{0,2}, w^{0,2})\|_{[\mathcal{C}([0,T];H_p^s(0,2\pi)]^2} = \|(\eta^1, w^1) - (\eta^2, w^2)\|_{[\mathcal{C}([0,T];H_p^s(0,2\pi)]^2} \\ &\leq \|(\eta^{0,1}, w^{0,1}) - (\eta^{0,2}, w^{0,2})\|_{[H_p^s(0,2\pi)]^2} + \frac{1}{2}\|(\eta^1, w^1) - (\eta^2, w^2)\|_{[\mathcal{C}([0,T];H_p^s(0,2\pi)]^2}.\end{aligned}$$

Since $(\eta^1 - \eta^2, w^1 - w^2)$ also solves the problem with initial data $(\eta^{0,1} - \eta^{0,2}, w^{0,1} - w^{0,2})$, we deduce that

$$\|\mathcal{F}(\eta^{0,1}, w^{0,1}) - \mathcal{F}(\eta^{0,2}, w^{0,2})\|_{[\mathcal{C}([0,T];H_p^s(0,2\pi)]^2} \leq 2\|(\eta^{0,1}, w^{0,1}) - (\eta^{0,2}, w^{0,2})\|_{[H_p^s(0,2\pi)]^2}. \quad (3.35)$$

The proof is complete. □

In what concerns the stabilization result, the following holds:

THEOREM 3.2.2. *For any* $(\eta^0, w^0) \in \left[H_p^1(0, L)\right]^2$, *the solution* (η, w) *of* (3.29) *given by Theorem 3.2.1 satisfies*

$$\begin{aligned}&(\eta(t), w(t)) \to (0, 0) \quad \text{weakly in } \left[H_p^1(0, L)\right]^2, \\ &(\eta(t), w(t)) \to (0, 0) \quad \text{strongly in } \left[H_p^s(0, L)\right]^2, \quad \text{for all } s < 1,\end{aligned}$$

as $t \to \infty$.

Proof. When $s = 1$, we can use Theorem 3.2.1 and the equations of the system (3.29) to deduce that $\eta_t = -(1 - b\partial_x^2)^{-1}(\partial_x w + a(x)\eta)$ and $w_t = -(1 - d\partial_x^2)^{-1}\partial_x \eta$ belong to $\mathcal{C}([0,T]; H_p^2(0,2\pi))$. Consequently, each term of both equations belongs to $\mathcal{C}([0,T]; L^2(0,2\pi))$. Thus, we can multiply the first equation in (3.29) by η, the second one by w and integrate by parts to obtain

$$\frac{d}{dt}\|(\eta(t), w(t))\|^2_{[H_p^1(0,2\pi)]^2} + \int_0^{2\pi} a(x)|\eta(s,x)|^2 dx = 0. \quad (3.36)$$

Integrating (3.36), we get

$$\frac{1}{2}\|(\eta(t), w(t))\|^2_{[H_p^1(0,2\pi)]^2} - \frac{1}{2}\|(\eta^0, w^0)\|^2_{[H_p^1(0,2\pi)]^2} + \int_0^t \int_0^{2\pi} a(x)|\eta(s,x)|^2 dxds = 0. \quad (3.37)$$

Identity (3.36) shows that the map $t \longmapsto \|(\eta(t), w(t))\|_{[H_p^1(0,2\pi)]^2}$ is nonincreasing and

$$\|(\eta(t), w(t))\|_{[H_p^1(0,2\pi)]^2} \leq \|(\eta^0, w^0)\|_{[H_p^1(0,2\pi)]^2}, \text{ for all } \quad t \geq 0. \quad (3.38)$$

Hence, there exists $l \in \mathbb{R}^+$, such that

$$\lim_{t\to+\infty} \|(\eta(t), w(t))\|_{[H_p^1(0,2\pi)]^2} = l.$$

Moreover, from (3.38) we infer the existence of a sequence $t_n \to +\infty$, such that

$$(\eta(t_n), w(t_n)) \rightharpoonup (\widetilde{\eta_0}, \widetilde{w_0}) \quad \text{weakly in} \quad \left[H_p^1(0,2\pi)\right]^2, \quad (3.39)$$

for some $(\widetilde{\eta_0}, \widetilde{w_0}) \in \left[H_p^1(0,2\pi)\right]^2$, and proceeding as in the proof of (3.37) we obtain

$$\|(\eta(t_{n+1}), w(t_{n+1}))\|^2_{[H_p^1(0,2\pi)]^2} - \|(\eta(t_n), w(t_n))\|^2_{[H_p^1(0,2\pi)]^2} + 2\int_{t_n}^{t_{n+1}} \int_0^{2\pi} a(x)|\eta(t,x)|^2 dxdt = 0.$$

Consequently,

$$\lim_{n\to+\infty} \int_{t_n}^{t_{n+1}} \int_0^{2\pi} a(x)|\eta(t,x)|^2 dxdt = 0. \quad (3.40)$$

On the other hand, from (3.39) and the Sobolev embedding, for any $s \in [0,1)$ we obtain the following convergence

$$(\eta(t_n), w(t_n)) \to (\widetilde{\eta_0}, \widetilde{w_0}) \quad \text{strongly in} \quad \left[H_p^s(0,2\pi)\right]^2. \quad (3.41)$$

Since the couple $(\eta(t_n + t, x), w(t_n + t, x))$ is solution of the system (3.29) with initial data $(\eta(t_n), w(t_n))$, from (3.35) and (3.41) we get

$$(\eta(t_n + \cdot), w(t_n + \cdot)) \to (\widetilde{\eta}, \widetilde{w}) \quad \text{in} \quad \left[C\left([0,T]; H_p^s(0,2\pi)\right)\right]^2, \quad \text{as} \quad n \to +\infty, \quad (3.42)$$

where $(\widetilde{\eta}, \widetilde{w}) \in \left[C\left([0,T]; H_p^1(0,2\pi)\right)\right]^2$ denotes the solution with initial data $(\widetilde{\eta^0}, \widetilde{w^0})$. The convergence above combined to (3.40) yields

$$\int_0^T \int_0^{2\pi} a(x)|\widetilde{\eta}(t,x)|^2 dxdt = 0. \quad (3.43)$$

Thus, $(\widetilde{\eta},\widetilde{w}) \in \left[C\left([0,T];H^1_p(0,2\pi)\right)\right]^2$ solves

$$\begin{cases} \widetilde{\eta}_t + \widetilde{w}_x - b\widetilde{\eta}_{txx} = 0 & \text{for } x \in (0,2\pi),\ \ t \in (0,T) \\ \widetilde{w}_t + \widetilde{\eta}_x - d\widetilde{w}_{txx} = 0 & \text{for } x \in (0,2\pi),\ \ t \in (0,T) \\ \widetilde{\eta}(t,0) = \widetilde{\eta}(t,2\pi);\ \widetilde{\eta}_x(t,0) = \widetilde{\eta}_x(t,2\pi) & \text{for } t \in (0,T) \\ \widetilde{w}(t,0) = \widetilde{w}(t,2\pi); \widetilde{w}_x(t,0) = \widetilde{w}_x(t,2\pi) & \text{for } t \in (0,T) \\ \widetilde{\eta}(0,x) = \widetilde{\eta}^0(x) & \text{for } x \in (0,2\pi) \\ \widetilde{w}(0,x) = \widetilde{w}^0(x) & \text{for } x \in (0,2\pi), \end{cases} \tag{3.44}$$

and (3.43) allows us to conclude that

$$\widetilde{\eta}(t,x) = 0, \quad \text{in} \quad (t,x) \in (0,T) \times \Omega_1,$$

for Ω_1 defined in (3.30). Finally, from Theorem 3.1.3 we have $(\widetilde{\eta_0},\widetilde{w_0}) = (0,0)$ and, as $t \to \infty$, the following holds

$$\begin{aligned} (\eta(t),w(t)) &\to (0,0) \quad \text{weakly in } \left[H^1_p(0,2\pi)\right]^2, \\ (\eta(t),w(t)) &\to (0,0) \quad \text{strongly in } \left[H^s_p(0,2\pi)\right]^2, \text{ for all } \quad s \in [0,1), \end{aligned}$$

which completes the proof. □

3.2.2 Internal stabilization with the feedback $\mathcal{B}\varphi = (a(x)\varphi_x)_x$

We first prove that the system (3.29) is well-posed. In order to do that, we argue as in the proof of Theorem 3.2.1 to obtain the following result:

THEOREM 3.2.3. *Let $s \geq 0$. For any $(\eta^0,w^0) \in \left[H^s_p(0,2\pi)\right]^2$, there exists $T > 0$ and a unique solution (η,w) of system (3.29) with $\mathcal{B}\varphi = (a(x)\varphi_x)_x$ in the class $\left[\mathcal{C}([0,T];H^s_p(0,2\pi))\right]^2$. If $s = 1$, the solution exists for every $T > 0$. Moreover, the map*

$$(\eta^0,w^0) \in \left[H^s_p(0,2\pi)\right]^2 \longrightarrow (\eta,w) \in \left[\mathcal{C}([0,T];H^s_p(0,2\pi))\right]^2$$

is Lipschitz continuous.

Proof. We proceed as in the proof of Theorem 3.2.1 applying a fixed point argument. Therefore, for any $(\eta^0,w^0) \in \left[H^s_p(0,2\pi)\right]^2$ we introduce the operator

$$\Gamma(\eta,w)(t) := \\ (\eta^0,w^0) - \left(\int_0^t \left(1-b\partial_x^2\right)^{-1}(\partial_x w + (a(x)\varphi_x)_x)(\tau)d\tau, \int_0^t \left(1-d\partial_x^2\right)^{-1}\partial_x\eta(\tau)d\tau\right).$$

In order to prove that Γ contracts in a ball of the space $\left[\mathcal{C}([0,T];H^s_p(0,2\pi))\right]^2$, instead of (3.32), we use the following estimate

$$\|\left(1-\alpha\partial_x^2\right)^{-1}\partial_x(au_x)\|_{H^s_p(0,2\pi)} \leq C\|u\|_{H^s_p(0,2\pi)}, \tag{3.45}$$

valid for $s \geq 0$ and for any $\alpha > 0$, where C is a positive constant. Taking (3.45) into account, the proof can be done arguing as in the proof of Theorem 3.2.1. Therefore, we omit the details. □

Remark 3.2.1. *From Theorem 3.2.3 we have the following conservation laws*

$$\frac{d}{dt}\int_0^{2\pi} \eta(t,x)dx = 0 \quad \text{and} \quad \frac{d}{dt}\int_0^{2\pi} w(t,x)dx = 0,$$

which are obtained by integrating the equations of the system with respect to x. Consequently,

$$\int_0^{2\pi} \eta(t,x)dx = \int_0^{2\pi} \eta^0(x)dx \quad \text{and} \quad \int_0^{2\pi} w(t,x)dx = \int_0^{2\pi} w^0(x)dx.$$

With the global wellposedness in hands, we prove the stabilization result.

Theorem 3.2.4. *For any $(\eta^0, w^0) \in \left[H_p^1(0,L)\right]^2$, the solution (η, w) of (3.29) given by Theorem 3.2.3 satisfies*

$$\begin{aligned} &(\eta(t), w(t)) \to ([\eta^0],[w^0]) \quad \text{weakly in } \left[H_p^1(0,L)\right]^2, \\ &(\eta(t), w(t)) \to ([\eta^0],[w^0]) \quad \text{strongly in } \left[H_p^s(0,L)\right]^2, \quad \text{for all } s<1, \end{aligned}$$

as $t \to \infty$, where $[f] := \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(x)dx$.

Proof. We first remark that, if $\varphi \in H_p^1(0,2\pi)$, from (3.30)

$$\langle -(a\varphi_x)_x, \varphi_x\rangle_{H_p^{-1}\times H_p^1} = \langle a\varphi_x, \varphi_x\rangle_{L^2\times L^2}.$$

Thus, we can proceed as in the proof of (3.36), to obtain

$$\frac{d}{dt}\|(\eta(t), w(t))\|^2_{\left[H_p^1(0,2\pi)\right]^2} + \int_0^{2\pi} a(x)|\eta_x(t,x)|^2 dx = 0. \tag{3.46}$$

Moreover, arguing as in the proof of Theorem 3.2.2, we obtain $(\widetilde{\eta_0}, \widetilde{w_0}) \in \left[H_p^1(0,2\pi)\right]^2$ and a sequence $t_n \to +\infty$, such that

$$(\eta(t_n), w(t_n)) \rightharpoonup (\widetilde{\eta_0}, \widetilde{w_0}) \quad \text{in } \left[H_p^1(0,2\pi)\right]^2, \tag{3.47}$$

$$(\eta(t_n), w(t_n)) \to (\widetilde{\eta_0}, \widetilde{w_0}) \quad \text{strongly in} \quad \left[H_p^s(0,2\pi)\right]^2, \tag{3.48}$$

and

$$(\eta(t_n+\cdot), w(t_n+\cdot)) \to (\widetilde{\eta}, \widetilde{w}) \quad \text{in} \quad \left[C\left([0,T]; H_p^s(0,2\pi)\right)\right]^2, \tag{3.49}$$

for any $s<1$, where $(\widetilde{\eta}, \widetilde{w}) \in \left[\mathcal{C}\left([0,T]; H_p^1(0,2\pi)\right)\right]^2$ denotes the solution of (3.29) with initial data $(\widetilde{\eta_0}, \widetilde{w_0})$.

From (3.49) it follows that

$$(\eta(t_n+\cdot), w(t_n+\cdot)) \text{ is bounded in } \left[L^2\left(0,T; H_p^s(0,2\pi)\right)\right]^2.$$

Then, we can extract a subsequence (if necessary), satisfying

$$(\eta(t_n+\cdot), w(t_n+\cdot)) \rightharpoonup (\widetilde{\eta}, \widetilde{w}) \quad \text{in} \quad \left[L^2\left(0,T; H_p^1(0,2\pi)\right)\right]^2. \tag{3.50}$$

On the other hand, from (3.46) we get

$$\|(\eta(t_{n+1}),w(t_{n+1}))\|^2_{[H^1_p(0,2\pi)]} - \|(\eta(t_n),w(t_n))\|^2_{[H^1_p(0,2\pi)]^2} + 2\int_{t_n}^{t_{n+1}}\int_0^{2\pi} a(x)|\eta_x(t,x)|^2 dxdt = 0,$$

which leads to

$$\lim_{n\to+\infty}\int_{t_n}^{t_{n+1}}\int_0^{2\pi} a(x)|\eta_x(t,x)|^2 dxdt = 0, \tag{3.51}$$

since $\|\cdot\|_{[H^1_p(0,2\pi)]^2}$ is nonincreasing, and therefore has a limit, as $t\to\infty$ (see (3.46)). By combining (3.50) and (3.51), we deduce that

$$\int_0^T\int_0^{2\pi} a(x)|\widetilde{\eta}_x(t,x)|^2 dxdt \le \liminf_{n\to\infty}\int_{t_n}^{t_{n+1}}\int_0^{2\pi} a(x)|\eta_x(t,x)|^2 dxdt = 0. \tag{3.52}$$

Therefore $(\widetilde{\eta},\widetilde{w})$ solves

$$\begin{cases} \widetilde{\eta}_t + \widetilde{w}_x - b\widetilde{\eta}_{txx} = 0 & \text{for } x\in(0,2\pi),\ \ t\in(0,T) \\ \widetilde{w}_t + \widetilde{\eta}_x - d\widetilde{w}_{txx} = 0 & \text{for } x\in(0,2\pi),\ \ t\in(0,T) \\ \widetilde{\eta}(t,0) = \widetilde{\eta}(t,2\pi);\ \widetilde{\eta}_x(t,0) = \widetilde{\eta}_x(t,2\pi) & \text{for} t\in(0,T) \\ \widetilde{w}(t,0) = \widetilde{w}(t,2\pi); \widetilde{w}_x(t,0) = \widetilde{w}_x(t,2\pi) & \text{for } t\in(0,T) \\ \widetilde{\eta}(0,x) = \widetilde{\eta}^0(x) & \text{for } x\in(0,2\pi) \\ \widetilde{w}(0,x) = \widetilde{w}^0(x) & \text{for } x\in(0,2\pi), \end{cases} \tag{3.53}$$

and (3.52) allows us to conclude that

$$\widetilde{\eta}_x(t,x) = 0,\ \forall\ (t,x)\in(0,T)\times\Omega_1,$$

for Ω_1 defined in (3.30). Thus, from Theorem 3.1.4 we have that $(\widetilde{\eta},\widetilde{w}) = (c_1,c_2)$ on $(0,T)\times(0,2\pi)$ for some $c_1,c_2\in\mathbb{R}$. From the Remark 3.2.1 and (3.47)-(3.48) it follows that

$$(c_1,c_2) = ([\eta_0],[w_0]),$$

and

$$(\eta(t),w(t)) \to ([\eta_0],[w_0]) \quad \text{weakly in } \left[H^1_p(0,2\pi)\right]^2,$$
$$(\eta(t),w(t)) \to ([\eta_0],[w_0]) \quad \text{strongly in } \left[H^s_p(0,2\pi)\right]^2, \quad \text{for all} \quad s\in[0,1).$$

□

3.3 Boundary stabilization

This section is devoted to study the boundary stabilization of the Boussinesq system posed on a bounded domain. More precisely, we consider the following initial boundary value problem

$$\begin{cases} \eta_t + w_x - b\eta_{txx} = 0 & \text{for } x \in (0,L), \ \ t \geq 0 \\ w_t + \eta_x - dw_{txx} = 0 & \text{for } x \in (0,L), \ \ t \geq 0 \\ \eta_{xt}(t,0) = \dfrac{w(t,0)}{2b} + \eta(t,0) & \text{for } t \geq 0 \\ \eta_{xt}(t,L) = \dfrac{w(t,L)}{2b} - \eta(t,L) & \text{for } t \geq 0 \\ w_{xt}(t,0) = \dfrac{\eta(t,0)}{2d} + w(t,0) & \text{for } t \geq 0 \\ w_{xt}(t,L) = \dfrac{\eta(t,L)}{2d} - w(t,L) & \text{for } t \geq 0 \\ \eta(0,x) = \eta^0(x) & \text{for } x \in (0,L) \\ w(0,x) = w^0(x) & \text{for } x \in (0,L). \end{cases} \tag{3.54}$$

If we multiply the first equation in (3.54) by η, the second one by w and integrate by parts over $(0,L)$, we obtain (at least formally)

$$\frac{d}{dt}\frac{1}{2}\|(\eta(t),w(t))\|^2_{[H^1(0,L)]^2} = -b\left(|\eta(t,L)|^2 + |\eta(t,0)|^2\right) - d\left(|w(t,L)|^2 + |w(t,0)|^2\right). \tag{3.55}$$

Hence, $\|(\eta(t),w(t))\|_{[H^1(0,L)]^2}$ is nonincreasing and the boundary conditions play the role of a feedback damping mechanism.

Before going into the stabilization problem, we first establish the following well-posedness result for (3.54):

THEOREM 3.3.1. *Let $s \in (1/2, 5/2)$ and $(\eta^0, w^0) \in [H^s(0,L)]^2$. For any $(\eta^0, w^0) \in [H^s(0,2\pi)]^2$, there exists a unique solution (η, w) of system (3.54) in $[\mathcal{C}([0,T]; H^s(0,2\pi))]^2$. Moreover, the map*

$$(\eta^0, w^0) \in [H^s(0,2\pi)]^2 \longrightarrow (\eta, w) \in [\mathcal{C}([0,T]; H^s(0,2\pi))]^2$$

is Lipschitz continuous.

Proof. The proof will be done by using a fixed point argument. Therefore, in order to write the problem as an integral equation, we set $(\widehat{\eta}, \widehat{w}) = (\eta_t, w_t)$ and remark that $(\widehat{\eta}, \widehat{w})$ solves the elliptic problem

$$\left(\left(1 - b\partial_x^2\right)\widehat{\eta}, \left(1 - d\partial_x^2\right)\widehat{w}\right) = (-w_x, -\eta_x), \quad x \in (0,L), \tag{3.56}$$

$$(\widehat{\eta}_x(0), \widehat{w}_x(0)) = (a_1, a_2), \tag{3.57}$$

$$(\widehat{\eta}_x(L), \widehat{w}_x(L)) = (a_3, a_4), \tag{3.58}$$

with

$$(a_1, a_2) = \left(\frac{w(t,0)}{2b} + \eta(t,0), \frac{\eta(t,0)}{2d} + w(t,0)\right),$$

$$(a_3, a_4) = \left(\frac{w(t,L)}{2b} - \eta(t,L), \frac{\eta(t,L)}{2d} - w(t,L)\right).$$

Note that the solution $(\widehat{\eta}, \widehat{w})$ of (3.56)-(3.58) may be written as

$$(\widehat{\eta}, \widehat{w}) = (h_1 + g_1, h_2 + g_2),$$

where

$$(g_1(x), g_2(x)) = \left(a_1 x + \frac{a_3 - a_1}{2L} x^2, a_2 x + \frac{a_4 - a_2}{2L} x^2\right),$$

and

$$(h_1, h_2) = \left(\left(1 - b\partial_x^2\right)_N^{-1} \left(-w_x - \left(1 - b\partial_x^2\right) g_1\right), \left(1 - d\partial_x^2\right)_N^{-1} \left(-\eta_x - \left(1 - d\partial_x^2\right) g_2\right)\right)$$

is a solution of

$$\begin{gathered}\left(\left(1 - b\partial_x^2\right) h_1, \left(1 - d\partial_x^2\right) h_2\right) = \left(-w_x - \left(1 - b\partial_x^2\right) g_1, -\eta_x - \left(1 - d\partial_x^2\right) g_2\right) \\ (h_{1,x}(0), h_{2,x}(0)) = (h_{1,x}(L), h_{2,x}(L)) = (0, 0),\end{gathered}$$

where, for any $\alpha > 0$, $(1 - \alpha\partial_x^2)_N$ denotes the elliptic operator with Neumann boundary conditions. Thus,

$$\eta_t = \widehat{\eta} = -\left(1 - b\partial_x^2\right)_N^{-1} (w_x) + \left(1 - \left(1 - b\partial_x^2\right)_N^{-1} \left(1 - b\partial_x^2\right)\right) g_1, \tag{3.59}$$

and

$$w_t = \widehat{w} = -\left(1 - d\partial_x^2\right)_N^{-1} (\eta_x) + \left(1 - \left(1 - d\partial_x^2\right)_N^{-1} \left(1 - d\partial_x^2\right)\right) g_2. \tag{3.60}$$

We remark that $\left((1 - \alpha\partial_x^2)_N^{-1} \circ \partial_x\right) (H^s(0, L)) \subset H^s(0, L)$ for $1/2 < s < 5/2$ and

$$\| \left(1 - b\alpha\partial_x^2\right)_N^{-1} f_x\|_{H^s(0,L)} \leq C\|f\|_{H^s(0,L)}, \tag{3.61}$$

for any $\alpha > 0$, where C is a positive constant.

Taking the above considerations into account, for any $(\eta^0, w^0) \in [H^s(0, L)]^2$ we introduce the operator

$$\Gamma(\eta, w)(t) := (\Gamma_1\eta(t), \Gamma_2 w(t)),$$

where

$$\begin{aligned}\Gamma_1\eta(t) :=& \eta^0 + \int_0^t \left(-\left(1 - b\partial_x^2\right)_N^{-1} (w_x)(\tau)\right) d\tau \\ &+ \int_0^t \left(1 - \left(1 - b\partial_x^2\right)_N^{-1} \left(1 - b\partial_x^2\right)\right) \left[\frac{w(\tau, 0)}{2b} + \eta(\tau, 0)\right] x d\tau \\ &+ \int_0^t \left(1 - \left(1 - b\partial_x^2\right)_N^{-1} \left(1 - b\partial_x^2\right)\right) \left[\frac{w(\tau, L)}{2b} - \eta(\tau, L) - \frac{w(\tau, 0)}{2b} - \eta(\tau, 0)\right] x^2 d\tau\end{aligned}$$

and

$$\begin{aligned}\Gamma_2 w(t) :=& w^0 + \int_0^t \left(-\left(1-d\partial_x^2\right)_N^{-1}(\eta_x)(\tau)\right) d\tau \\ &+ \int_0^t \left(1-\left(1-d\partial_x^2\right)_N^{-1}\left(1-d\partial_x^2\right)\right)\left[\frac{\eta(\tau,0)}{2d}+w(\tau,0)\right] x d\tau \\ &+ \int_0^t \left(1-\left(1-b\partial_x^2\right)_N^{-1}\left(1-b\partial_x^2\right)\right)\left[\frac{\eta(\tau,L)}{2d}-w(\tau,L)-\frac{\eta(\tau,0)}{2d}-w(\tau,0)\right] x^2 d\tau.\end{aligned}$$

Then, we seek (η, w) as a fixed point of the integral equation

$$(\eta, w)(t) = \Gamma(\eta, w)(t). \tag{3.62}$$

Using (3.61), the Sobolev embedding $H^s(0,L) \hookrightarrow \mathcal{C}([0,L])$ and proceeding as in the proof of Theorem 3.2.1, we have that

$$\|\left(\Gamma(\eta^1, w^1) - \Gamma(\eta^2, w^2)\right)\|_{[\mathcal{C}([0,T];H^s(0,L)]^2} \le \frac{1}{2}\|(\eta^1, w^1) - (\eta^2, w^2)\|_{[\mathcal{C}([0,T];H^s(0,L)]^2}, \tag{3.63}$$

and

$$\|\Gamma(\eta, w)\|_{[\mathcal{C}([0,T];H^s(0,L)]^2} \le \|(\eta^0, w^0)\|_{[H^s(0,L)]^2} + \frac{1}{2}\|(\eta, w)\|_{[\mathcal{C}([0,T];H^s(0,L)]^2}. \tag{3.64}$$

Then, for the choice $R = 2\|(\eta^0, w^0)\|_{[H^s(0,L)]^2}$, estimates (3.63) and (3.64) allow us to conclude that $\Gamma : B_R(0) \subseteq [\mathcal{C}([0,T]; H^s(0,2\pi)]^2 \longrightarrow B_R(0)$ is a contraction, hence it admits a unique fixed point $(\eta, w) \in B_R(0)$ which solves (3.62). If $s = 1$, from (3.55) we deduces that the solution exists for every $T > 0$. The continuity of the flow map follows from (3.64). This completes the proof. □

The stabilization result reads as follows:

THEOREM 3.3.2. *For any* $(\eta^0, w^0) \in \left[H^1(0,L)\right]^2$, *the solution* (η, w) *of* (3.54) *given by Theorem 3.3.1 satisfies*

$$\begin{aligned} &(\eta(t), w(t)) \to (0,0) \quad \text{weakly in } \left[H^1(0,L)\right]^2, \\ &(\eta(t), w(t)) \to (0,0) \quad \text{strongly in } \left[H^s(0,L)\right]^2 \quad \text{for all } s<1, \end{aligned} \tag{3.65}$$

as $t \to \infty$.

Proof. From (3.59) and (3.60), we deduce that $\eta_t, w_t \in \mathcal{C}(\mathbb{R}; H^2(0,L))$, so that (3.55) is valid. Thus, the map $t \mapsto \|(\eta(t), w(t))\|_{[H^1(0,L)]^2}$ is nonincreasing and admits a nonnegative limit, as $t \to \infty$. Proceeding as in the proof of Theorem 3.2.2, we obtain the existence of $(\widetilde{\eta_0}, \widetilde{w_0}) \in \left[H^1(0,L)\right]^2$ and a sequence $t_n \to +\infty$, such that

$$(\eta(t_n), w(t_n)) \rightharpoonup (\widetilde{\eta_0}, \widetilde{w_0}) \quad \text{weakly in } \left[H^1(0,L)\right]^2, \tag{3.66}$$

$$(\eta(t_n), w(t_n)) \to (\widetilde{\eta_0}, \widetilde{w_0}) \quad \text{strongly in } \left[H^s(0,L)\right]^2, \tag{3.67}$$

and

$$(\eta(t_n+\cdot), w(t_n+\cdot)) \to (\widetilde{\eta}, \widetilde{w}) \quad \text{in} \quad [\mathcal{C}([0,T]; H^s(0,L))]^2, \tag{3.68}$$

for $s<1$, where $(\widetilde{\eta}, \widetilde{w}) \in [C([0,T]; H^1(0,L))]^2$ denotes the solution of (3.54) with initial data $(\widetilde{\eta_0}, \widetilde{w_0})$. From (3.55) we obtain

$$\begin{aligned}
&\|(\eta(t_{n+1}), w(t_{n+1}))\|^2_{[H^1(0,L)]^2} - \|(\eta(t_n), w(t_n))\|^2_{[H^1(0,L)]^2} = \\
&-2b\int_{t_n}^{t_{n+1}} (|\eta(t,L)|^2 + |\eta(t,0)|^2)dt - 2d\int_{t_n}^{t_{n+1}} (|w(t,L)|^2 + |w(t,0)|^2)dt,
\end{aligned}$$

which allows us to conclude that

$$\lim_{n\to+\infty}\left(b\int_{t_n}^{t_{n+1}} (|\eta(t,L)|^2 + |\eta(t,0)|^2)dt + d\int_{t_n}^{t_{n+1}} (|w(t,L)|^2 + |w(t,0)|^2)dt\right) = 0. \tag{3.69}$$

Thus,

$$b\int_0^T (|\widetilde{\eta}(t,L)|^2 + |\widetilde{\eta}(t,0)|^2)dt + d\int_0^T (|\widetilde{w}(t,L)|^2 + |\widetilde{w}(t,0)|^2)dt = 0 \tag{3.70}$$

and therefore

$$\widetilde{\eta}(t,L) = \widetilde{\eta}(t,0) = \widetilde{w}(t,L) = \widetilde{w}(t,0) = 0, \quad t \in (0,T).$$

Let $(\widehat{\eta}, \widehat{w})$ be the extension by zero of $(\widetilde{\eta}, \widetilde{w})$ for $x \in (-a,a)\setminus(0,L)$, where $(-a,a) \supset (0,L)$ is a interval. Then, $(\widehat{\eta}, \widehat{w})$ solves

$$\begin{cases}
\widehat{\eta}_t + \widehat{w}_x - b\widehat{\eta}_{txx} = 0 & \text{for } x \in (-a,a), \ t \in (0,T) \\
\widehat{w}_t + \widehat{\eta}_x - d\widehat{w}_{txx} = 0 & \text{for } x \in (-a,a), \ t \in (0,T) \\
\widehat{\eta}(t,-a) = \widehat{\eta}(t,a) = 0; \ \widehat{\eta}_x(t,-a) = \widehat{\eta}_x(t,a) = 0 & \text{for } t \in (0,T) \\
\widehat{w}(t,-a) = \widehat{w}(t,a) = 0; \ \widehat{w}_x(t,-a) = \widehat{w}_x(t,a) = 0 & \text{for } t \in (0,T) \\
\widehat{\eta}(0,x) = \widehat{\eta}^0(x) & \text{for } x \in (-a,a) \\
\widehat{w}(0,x) = \widehat{w}^0(x) & \text{for } x \in (-a,a),
\end{cases} \tag{3.71}$$

and satisfies

$$(\widehat{\eta}(t,x), \widehat{w}(t,x)) = (0,0) \text{ for } (t,x) \in (0,T) \times ((-a,a) \setminus (0,L)),$$

where

$$\widehat{\eta}^0(x) = \begin{cases} \widetilde{\eta_0}(x) & x \in (0,L), \\ 0 & x \in (-a,a) \setminus (0,L) \end{cases}$$

and

$$\widehat{w}^0(x) = \begin{cases} \widetilde{w_0}(x) & x \in (0,L), \\ 0 & x \in (-a,a) \setminus (0,L). \end{cases}$$

We remark that Theorems 3.1.3 and 3.1.2 can be proved for a domain of the form $(-a,a)$. Therefore, since $(\widehat{\eta}^0, \widehat{w}^0) \in [H_0^1(-a,a)]^2$, from Theorem 3.1.2 it follows that $(\widehat{\eta}, \widehat{w}) \in [C^\omega([0,T]; H_0^1(-a,a))]^2$, and from Theorem 3.1.3 we deduce that $(\widehat{\eta}^0, \widehat{w}^0) = (0,0)$. Hence, $(\widetilde{\eta_0}, \widetilde{w_0}) = (0,0)$.

Finally, from (3.66) and (3.67) we have that, as $t \to \infty$,

$$\begin{aligned} (\eta(t), w(t)) &\to (0,0) \quad \text{weakly in } \left[H^1(0,L)\right]^2, \\ (\eta(t), w(t)) &\to (0,0) \quad \text{strongly in } \left[H^s(0,L)\right]^2, \quad \text{for all} \quad s < 1. \end{aligned}$$

□

Chapter 4

Stabilization for higher-order Boussinesq system with generalized damping on a periodic domain

This chapter is devoted to analyze the following system

$$\begin{cases} \eta_t + w_x - b\eta_{txx} + b_2\eta_{txxxx} + aw_{xxx} + \beta_1 M_{\alpha_1}\eta = -(\eta w)_x \\ -(a+b-\frac{1}{3})(\eta w_x)_x, & \text{for } x \in (0,2\pi),\ \ t>0 \\ w_t + \eta_x - dw_{txx} + d_2 w_{txxxx} + c\eta_{xxx} + \beta_2 M_{\alpha_2} w = -ww_x - c(ww_x)_{xx} \\ -(\eta\eta_{xx})_x + (c+d-1)w_x w_{xx} + (c+d)\eta_x\eta_{xxx} & \text{for } x \in (0,2\pi),\ \ t>0 \\ \dfrac{\partial^r \eta}{\partial x^r}(t,0) = \dfrac{\partial^r \eta}{\partial x^r}(t,2\pi) & \text{for } t>0, \quad 0 \le r \le 3, \\ \dfrac{\partial^q w}{\partial x^q}(t,0) = \dfrac{\partial^q w}{\partial x^q}(t,2\pi) & \text{for } t>0, \quad 0 \le q \le 3 \\ \eta(0,x) = \eta^0(x), \quad w(0,x) = w^0(x) & \text{for } x \in (0,2\pi), \end{cases} \tag{4.1}$$

where $b,\ d, b_2, d_2 > 0$, $\beta_1,\ \beta_2 \ge 0$, $\alpha_1,\ \alpha_2 \in [0,4]$, $a, c < 0$ or $a = c > 0$ and M_{α_j} are Fourier multiplier operators defined as follows

$$\begin{gathered} M_{\alpha_j} : H_p^{s+4}(0,2\pi) \to H_p^s(0,2\pi), \\ M_{\alpha_j}\left(\sum_{k\in\mathbb{Z}} \widehat{a}_k e^{ikx}\right) = \sum_{k\in\mathbb{Z}} (1+k^2)^{\frac{\alpha_j}{2}} \widehat{a}_k e^{ikx} \qquad (j=1,2). \end{gathered} \tag{4.2}$$

By means of spectral analysis and Fourier expansion, we prove that the solutions of the linearized system decay uniformly or not to zero, depending on the parameters of the damping operators. In the uniform decay case, we show that the same property holds for the nonlinear system.

We first study the linearized system.

4.1 The linearized system

The aim of this section is to study the main properties of the linearized model corresponding to (4.1). More precisely, we consider the following system

$$\begin{cases} \eta_t + w_x - b\eta_{txx} + b_2\eta_{txxxx} + aw_{xxx} + \beta_1 M_{\alpha_1}\eta = 0 & \text{for } x \in (0,2\pi),\ \ t>0 \\ w_t + \eta_x - dw_{txx} + d_2 w_{txxxx} + c\eta_{xxx} + \beta_2 M_{\alpha_2} w = 0 & \text{for } x \in (0,2\pi),\ \ t>0 \\ \dfrac{\partial^r \eta}{\partial x^r}(t,0) = \dfrac{\partial^r \eta}{\partial x^r}(t,2\pi) & \text{for } t>0, \quad 0 \le r \le 3, \\ \dfrac{\partial^q w}{\partial x^q}(t,0) = \dfrac{\partial^q w}{\partial x^q}(t,2\pi) & \text{for } t>0, \quad 0 \le q \le 3 \\ \eta(0,x) = \eta^0(x), \quad w(0,x) = w^0(x) & \text{for } x \in (0,2\pi). \end{cases} \tag{4.3}$$

We prove the well-posedness and stabilization results.

4.1.1 Well-posedness

Given $s \in \mathbb{R}$, let us introduce the Hilbert space

$$V^s = H^s_p(0,2\pi) \times H^s_p(0,2\pi), \tag{4.4}$$

endowed with the inner product defined by

$$\langle (f_1,f_2),(g_1,g_2)\rangle = (f_1,g_1)_s + (\mathcal{H}f_2,\mathcal{H}g_2)_s, \tag{4.5}$$

and the operator $\mathcal{H}$ defined in the following way

$$\mathcal{H}\left(\sum_{k\in\mathbb{Z}} \widehat{a}_k e^{ikx}\right) = \sum_{k\in\mathbb{Z}} \sqrt{\frac{w_1}{w_2}}\,\widehat{a}_k e^{ikx},$$

where $w_1 = \frac{1-ak^2}{1+bk^2+b_2k^4}$ and $w_2 = \frac{1-ck^2}{1+dk^2+d_2k^4}$. Let us remark that system (4.3) can be written in the following vectorial form

$$\begin{pmatrix} \eta \\ w \end{pmatrix}_t (t) + A \begin{pmatrix} \eta \\ w \end{pmatrix}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \eta \\ w \end{pmatrix}(0) = \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix}, \tag{4.6}$$

where A is the linear compact operator in V^s defined by

$$A = \begin{pmatrix} \beta_1 \left(I - b\partial_x^2 + b_2\partial_x^4\right)_p^{-1} M_{\alpha_1} & \left(I - b\partial_x^2 + b_2\partial_x^4\right)_p^{-1}\left(\partial_x + a\partial_x^3\right) \\ \left(I - d\partial_x^2 + d_2\partial_x^4\right)_p^{-1}\left(\partial_x + c\partial_x^3\right) & \beta_2\left(I - b\partial_x^2 + d_2\partial_x^4\right)_p^{-1} M_{\alpha_2} \end{pmatrix}. \tag{4.7}$$

We pass now to study the existence of solutions to (4.3). If we assume that the initial data in (4.3) are given by

$$(\eta^0, w^0) = \sum_{k\in\mathbb{Z}} \left(\widehat{\eta}^0_k, \widehat{w}^0_k\right) e^{ikx}, \tag{4.8}$$

then, at least formally, the solution of (4.3) can be written as

$$(\eta, w)(t,x) = \sum_{k\in\mathbb{Z}} (\widehat{\eta}_k(t), \widehat{w}_k(t)) e^{ikx}, \tag{4.9}$$

where $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ fulfills

$$\begin{cases} (1+bk^2+b_2k^4)(\widehat{\eta}_k)_t + ik(1-ak^2)\widehat{w}_k + \beta_1(1+k^2)^{\frac{\alpha_1}{2}}\widehat{\eta}_k = 0, \quad t\in(0,T), \\ (1+dk^2+d_2k^4)(\widehat{w}_k)_t + ik(1-ck^2)\widehat{\eta}_k + \beta_2(1+k^2)^{\frac{\alpha_2}{2}}\widehat{w}_k = 0, \quad t\in(0,T), \\ \widehat{\eta}_k(0) = \widehat{\eta}_k^0, \qquad \widehat{w}_k(0) = \widehat{w}_k^0. \end{cases} \tag{4.10}$$

The following results, whose proof can be found in [1], are needed for this study.

Proposition 4.1.1. *Let A a 2×2 matrix with eigenvalues $\lambda_1 \neq \lambda_2$. If*

$$Q_1 = \frac{A-\lambda_2 I}{\lambda_1-\lambda_2}; \qquad Q_2 = \frac{A-\lambda_1 I}{\lambda_2-\lambda_1},$$

then,

(i) $A = \lambda_1 Q_1 + \lambda_2 Q_2$;

(ii) $Q_1^2 = Q_1$; $Q_2^2 = Q_2$; $Q_2Q_1 = Q_1Q_2 = 0$;

(iii) $A^k = \lambda_1^k Q_1 + \lambda_2^k Q_2, \forall k \in \mathbb{N}$;

(iv) $e^{At} = e^{\lambda_1 t}Q_1 + e^{\lambda_2 t}Q_2$.

Proposition 4.1.2. *Let A a 2×2 matrix with eigenvalues $\lambda_0 = \lambda_1 = \lambda_2$, and*

$$Q = A - \lambda_0 I.$$

Then, $e^{At} = (I + tQ)\, e^{\lambda_0 t}$.

We have the following result.

Lemma 4.1.1. *The eigenvalues of the operator A defined by (4.7) are given by*

$$\lambda_k^{\pm} = \frac{1}{2}\left(\frac{\beta_1(1+k^2)^{\frac{\alpha_1}{2}}}{1+bk^2+b_2k^4} + \frac{\beta_2(1+k^2)^{\frac{\alpha_2}{2}}}{1+dk^2+d_2k^4} \pm 2|k|\sqrt{w_1w_2}\sqrt{e_k^2-1}\right) \quad (k\in\mathbb{Z}^*), \tag{4.11}$$

where

$$e_k = \frac{1}{2k\sqrt{(1-ak^2)(1-ck^2)}}\times \left(\beta_1(1+k^2)^{\frac{\alpha_1}{2}}\sqrt{\frac{1+dk^2+d_2k^4}{1+bk^2+b_2k^4}} - \beta_2(1+k^2)^{\frac{\alpha_2}{2}}\sqrt{\frac{1+bk^2+b_2k^4}{1+dk^2+d_2k^4}}\right) \tag{4.12}$$

and $\zeta_k = e_k - \sqrt{e_k^2 - 1}$ $(k \in \mathbb{Z}^*)$. *The solution* $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ *of* (4.10) *is given by*

$$\begin{cases} \widehat{\eta}_k(t) &= \frac{1}{1-\zeta_k^2}\left[\left(e^{-\lambda_k^+ t} - \zeta_k^2 e^{-\lambda_k^- t}\right)\widehat{\eta}_k^0 + i\sqrt{\frac{w_1}{w_2}}\zeta_k\left(e^{-\lambda_k^+ t} - e^{-\lambda_k^- t}\right)\widehat{w}_k^0\right], \\ \\ \widehat{w}_k(t) &= \frac{1}{1-\zeta_k^2}\left[\left(e^{-\lambda_k^- t} - \zeta_k^2 e^{-\lambda_k^+ t}\right)\widehat{w}_k^0 + i\sqrt{\frac{w_2}{w_1}}\zeta_k\left(e^{-\lambda_k^+ t} - e^{-\lambda_k^- t}\right)\widehat{\eta}_k^0\right], \end{cases} \tag{4.13}$$

if $|e_k| \neq 1$ *and* $k > 0$,

$$\begin{cases} \widehat{\eta}_k(t) &= \frac{1}{1-\zeta_k^2}\left[\left(e^{-\lambda_k^- t} - \zeta_k^2 e^{-\lambda_k^+ t}\right)\widehat{\eta}_k^0 - i\sqrt{\frac{w_1}{w_2}}\zeta_k\left(e^{-\lambda_k^+ t} - e^{-\lambda_k^- t}\right)\widehat{w}_k^0\right], \\ \\ \widehat{w}_k(t) &= \frac{1}{1-\zeta_k^2}\left[\left(e^{-\lambda_k^+ t} - \zeta_k^2 e^{-\lambda_k^- t}\right)\widehat{w}_k^0 - i\sqrt{\frac{w_2}{w_1}}\zeta_k\left(e^{-\lambda_k^+ t} - e^{-\lambda_k^- t}\right)\widehat{\eta}_k^0\right], \end{cases} \tag{4.14}$$

if $|e_k| \neq 1$ *and* $k < 0$,

$$\begin{cases} \widehat{\eta}_k(t) &= \left[\left(1 - |k|\sqrt{w_1 w_2}t\right)\widehat{\eta}_k^0 - ikw_1 t\widehat{w}_k^0\right]e^{-\lambda_k^+ t}, \\ \\ \widehat{w}_k(t) &= \left[-ikw_2 t\widehat{\eta}_k^0 + \left(1 + |k|\sqrt{w_1 w_2}t\right)\widehat{w}_k^0\right]e^{-\lambda_k^+ t}, \end{cases} \tag{4.15}$$

if $|e_k| = 1$ *and* $k \neq 0$, *and finally,*

$$\begin{cases} \widehat{\eta}_0(t) &= \widehat{\eta}_0^0 e^{-\beta_1 t}, \\ \\ \widehat{w}_0(t) &= \widehat{w}_0^0 e^{-\beta_2 t}. \end{cases} \tag{4.16}$$

Proof. It is easy to see that (4.10) is equivalent to

$$\begin{pmatrix} \widehat{\eta}_k \\ \\ \widehat{w}_k \end{pmatrix}_t (t) + A(k)\begin{pmatrix} \widehat{\eta}_k \\ \\ \widehat{w}_k \end{pmatrix}(t) = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \widehat{\eta}_k \\ \\ \widehat{w}_k \end{pmatrix}(0) = \begin{pmatrix} \widehat{\eta}_k^0 \\ \\ \widehat{w}_k^0 \end{pmatrix},$$

where

$$A(k) = \begin{pmatrix} \frac{\beta_1(1+k^2)^{\frac{\alpha_1}{2}}}{1+bk^2+b_2k^4} & \frac{ik(1-ak^2)}{1+bk^2+b_2k^4} \\ \\ \frac{ik(1-ck^2)}{1+dk^2+d_2k^4} & \frac{\beta_2(1+k^2)^{\frac{\alpha_2}{2}}}{1+dk^2+d_2k^4} \end{pmatrix}.$$

Hence, the solution of (4.10) is given by

$$\begin{pmatrix} \widehat{\eta}_k \\ \\ \widehat{w}_k \end{pmatrix}(t) = e^{-A(k)\,t}\begin{pmatrix} \widehat{\eta}_k^0 \\ \\ \widehat{w}_k^0 \end{pmatrix}. \tag{4.17}$$

The eigenvalues $\lambda_k^{\pm}$ of the matrix $A(k)$ are

$$\lambda_k^{\pm} = \frac{1}{2}\left(\frac{\beta_1(1+k^2)^{\frac{\alpha_1}{2}}}{1+bk^2+b_2k^4} + \frac{\beta_2(1+k^2)^{\frac{\alpha_2}{2}}}{1+dk^2+d_2k^4}\right) \pm \tag{4.18}$$

$$\pm\frac{1}{2}\sqrt{\left(\frac{\beta_1(1+k^2)^{\frac{\alpha_1}{2}}}{1+bk^2+b_2k^4} - \frac{\beta_2(1+k^2)^{\frac{\alpha_2}{2}}}{1+dk^2+d_2k^4}\right)^2 - \frac{4k^2(1-ak^2)(1-ck^2)}{(1+bk^2+b_2k^4)(1+dk^2+d_2k^4)}},$$

$k \in \mathbb{Z}^*$, that can be rewriten as (4.11).

Let us analyze the following cases:

(i) **Case** $|e_k| \neq 1$ **and** $k \neq 0$**.**

We have that, $\lambda_k^+ \neq \lambda_k^-$. Let

$$Q_1 = \frac{A(k) - \lambda_k^- I}{\lambda_k^+ - \lambda_k^-}. \tag{4.19}$$

Since

$$\frac{\frac{\beta_1(1+k^2)^{\frac{\alpha_1}{2}}}{1+bk^2+b_2k^4} - \lambda_k^-}{\lambda_k^+ - \lambda_k^-} = \frac{1}{2}\left(1 + \frac{sgn(k)e_k}{\sqrt{e_k^2-1}}\right), \tag{4.20}$$

$$\frac{\frac{\beta_2(1+k^2)^{\frac{\alpha_2}{2}}}{1+dk^2+d_2k^4} - \lambda_k^-}{\lambda_k^+ - \lambda_k^-} = \frac{1}{2}\left(1 - \frac{sgn(k)e_k}{\sqrt{e_k^2-1}}\right), \tag{4.21}$$

$$\frac{\frac{ik(1-ak^2)}{1+bk^2+b_2k^4}}{\lambda_k^+ - \lambda_k^-} = i\frac{sgn(k)\sqrt{\frac{w_1}{w_2}}}{2\sqrt{e_k^2-1}}, \tag{4.22}$$

and

$$\frac{\frac{ik(1-ck^2)}{1+dk^2+d_2k^4}}{\lambda_k^+ - \lambda_k^-} = i\frac{sgn(k)\sqrt{\frac{w_2}{w_1}}}{2\sqrt{e_k^2-1}}, \tag{4.23}$$

from (4.20)-(4.23) and (4.19) we obtain that

$$Q_1 = \begin{pmatrix} \frac{1}{2}\left(1 + \frac{sgn(k)e_k}{\sqrt{e_k^2-1}}\right) & i\frac{sgn(k)\sqrt{\frac{w_1}{w_2}}}{2\sqrt{e_k^2-1}} \\ i\frac{sgn(k)\sqrt{\frac{w_2}{w_1}}}{2\sqrt{e_k^2-1}} & \frac{1}{2}\left(1 - \frac{sgn(k)e_k}{\sqrt{e_k^2-1}}\right) \end{pmatrix} \tag{4.24}$$

$$= \begin{cases} \dfrac{1}{1-\zeta_k^2}\begin{pmatrix} 1 & i\sqrt{\frac{w_1}{w_2}}\zeta_k \\ i\sqrt{\frac{w_2}{w_1}}\zeta_k & -\zeta_k^2 \end{pmatrix} & \text{if} \quad k > 0, \\ \dfrac{1}{1-\zeta_k^2}\begin{pmatrix} -\zeta_k^2 & -i\sqrt{\frac{w_1}{w_2}}\zeta_k \\ -i\sqrt{\frac{w_2}{w_1}}\zeta_k & 1 \end{pmatrix} & \text{if} \quad k < 0. \end{cases}$$

Similarly,

$$Q_2 = \begin{pmatrix} \frac{1}{2}\left(1 - \frac{sgn(k)e_k}{\sqrt{e_k^2-1}}\right) & -i\frac{sgn(k)\sqrt{\frac{w_1}{w_2}}}{2\sqrt{e_k^2-1}} \\ -i\frac{sgn(k)\sqrt{\frac{w_2}{w_1}}}{2\sqrt{e_k^2-1}} & \frac{1}{2}\left(1 + \frac{sgn(k)e_k}{\sqrt{e_k^2-1}}\right) \end{pmatrix} \tag{4.25}$$

$$= \begin{cases} \dfrac{1}{1-\zeta_k^2}\begin{pmatrix} -\zeta_k^2 & -i\sqrt{\frac{w_1}{w_2}}\zeta_k \\ -i\sqrt{\frac{w_2}{w_1}}\zeta_k & 1 \end{pmatrix} & \text{if} \quad k > 0, \\ \dfrac{1}{1-\zeta_k^2}\begin{pmatrix} 1 & i\sqrt{\frac{w_1}{w_2}}\zeta_k \\ i\sqrt{\frac{w_2}{w_1}}\zeta_k & -\zeta_k^2 \end{pmatrix} & \text{if} \quad k < 0, \end{cases}$$

where $\zeta_k = e_k - \sqrt{e_k^2 - 1}$. On the other hand, from Proposition 4.1.1 we have that

$$e^{-A(k)t} = e^{-\lambda_k^+ t}Q_1 + e^{-\lambda_k^- t}Q_2. \tag{4.26}$$

Thus, from (4.26) and (4.17) the solution of (4.10) is given by (4.13) and (4.14) in the respective cases.

(ii) **Case $|e_k| = 1$ and $k \neq 0$.**

In this case,

$$\lambda_k^+ = \lambda_k^- = \frac{1}{2}\left(\frac{\beta_1(1+k^2)^{\frac{\alpha_1}{2}}}{1+bk^2+b_2k^4} + \frac{\beta_2(1+k^2)^{\frac{\alpha_2}{2}}}{1+dk^2+d_2k^4}\right).$$

Observe that, from (4.18), we have

$$\frac{1}{2}\left(\frac{\beta_1(1+k^2)^{\frac{\alpha_1}{2}}}{1+bk^2+b_2k^4} - \frac{\beta_2(1+k^2)^{\frac{\alpha_2}{2}}}{1+dk^2+d_2k^4}\right) = |k|\sqrt{w_1w_2}.$$

Let

$$Q = \lambda_k^+ - A(k) = \begin{pmatrix} \frac{1}{2}\left(\frac{\beta_2(1+k^2)^{\frac{\alpha_2}{2}}}{1+dk^2+d_2k^4} - \frac{\beta_1(1+k^2)^{\frac{\alpha_1}{2}}}{1+bk^2+b_2k^4}\right) & \frac{-ik(1-ak^2)}{1+bk^2+b_2k^4} \\ \frac{-ik(1-ck^2)}{1+dk^2+d_2k^4} & \frac{1}{2}\left(\frac{\beta_1(1+k^2)^{\frac{\alpha_1}{2}}}{1+bk^2+b_2k^4} - \frac{\beta_2(1+k^2)^{\frac{\alpha_2}{2}}}{1+dk^2+d_2k^4}\right) \end{pmatrix} \tag{4.27}$$

$$= \begin{pmatrix} -|k|\sqrt{w_1w_2} & -ikw_1 \\ -ikw_2 & |k|\sqrt{w_1w_2} \end{pmatrix}.$$

Hence, we infer from (4.27) and Proposition 4.1.2 that

$$e^{-A(k)t} = e^{-\lambda_k^+ t}(I+tQ) = e^{-\lambda_k^+ t}\begin{pmatrix} 1-t|k|\sqrt{w_1w_2} & -ikw_1t \\ -ikw_2t & 1+|k|\sqrt{w_1w_2}t \end{pmatrix}. \tag{4.28}$$

Furthermore, from (4.28) and (4.17) we have that the solution of (4.10) is given by (4.15).

(iii) **Case $k = 0$.**
It is a direct consequence of (4.10).

□

REMARK 4.1.1. *Firstly, we note that $\lambda_k^\pm = \lambda_{-k}^\pm$ and the following holds:*

- *If $e_k < 1$, then the eigenvalues $\lambda_k^\pm$ are complex numbers.*
- *If $e_k \geq 1$, then the eigenvalues $\lambda_k^\pm$ are real numbers and $\lambda_k^+ \geq \lambda_k^-$.*

Let us analyze more closely the eigenvalues $\lambda_k^\pm$ given by (4.11). In the sequel l, M and C denote generic positive constant which may change from one row to another.

We have the following result.

PROPOSITION 4.1.3. *Let $\alpha_1 < 4$ or $\alpha_2 < 4$ and $|e_k| \geq 1$. We suppose that, if $\alpha_j = \max\{\alpha_1, \alpha_2\}$, then $\beta_j > 0$. There exists a constant $l_1 > 0$, such that*

$$\lambda_k^- \geq \begin{cases} \frac{l_1}{|k|^{\max\{\alpha_1,\,\alpha_2\}}} & \text{if } \alpha_1+\alpha_2 \leq 6, \quad \max\{\alpha_1,\,\alpha_2\} > 3, \\ \frac{l_1}{|k|^{4-\min\{\alpha_1,\,\alpha_2\}}} & \text{if } \alpha_1+\alpha_2 > 6, \quad \max\{\alpha_1,\,\alpha_2\} > 3, \\ \frac{l_1}{|k|^{4-\max\{\alpha_1,\,\alpha_2\}}} & \text{if } \max\{\alpha_1,\,\alpha_2\} \leq 3. \end{cases} \tag{4.29}$$

Proof. From (4.18), λ_k^- can be written as

$$\lambda_k^- = \frac{1}{2}\sqrt{w_1 w_2}\left(r+s-\sqrt{(r-s)^2-4k^2}\right) = 2\sqrt{w_1 w_2}\left(\frac{rs+k^2}{r+s+\sqrt{(r-s)^2-4k^2}}\right) \tag{4.30}$$

$$\sim \frac{2}{k^2}\sqrt{\frac{ac}{b_2 d_2}}\left(\frac{rs+k^2}{r+s+\sqrt{(r-s)^2-4k^2}}\right),$$

where

$$r = \frac{1}{\sqrt{(1-ak^2)(1-ck^2)}}\beta_1(1+k^2)^{\frac{\alpha_1}{2}}\sqrt{\frac{1+dk^2+d_2k^4}{1+bk^2+b_2k^4}},$$

and

$$s = \frac{1}{\sqrt{(1-ak^2)(1-ck^2)}}\beta_2(1+k^2)^{\frac{\alpha_2}{2}}\sqrt{\frac{1+bk^2+b_2k^4}{1+dk^2+d_2k^4}}.$$

From the relations above we obtain that

$$rs \sim \frac{\beta_1\beta_2}{ac}|k|^{\alpha_1+\alpha_2-4}. \tag{4.31}$$

Note that $(r+s)$ has order $|k|^{\max\{\alpha_1,\,\alpha_2\}-2}$ and $(r-s)^2$ has order $|k|^{2(\max\{\alpha_1,\,\alpha_2\}-2)}$. Let us analyze the order of $(r-s)^2-4k^2$:

(i) If $\max\{\alpha_1,\,\alpha_2\} > 3$, then $2(\max\{\alpha_1,\,\alpha_2\}-2) > 2$. Hence, $((r-s)^2-4k^2)$ has order $|k|^{2(\max\{\alpha_1,\,\alpha_2\}-2)}$. Furthermore, $\left(r+s+\sqrt{(r-s)^2-4k^2}\right)$ has order $|k|^{\max\{\alpha_1,\,\alpha_2\}-2}$.

(ii) If $\max\{\alpha_1,\,\alpha_2\} \leq 3$, then $2(\max\{\alpha_1,\,\alpha_2\}-2) \leq 2$. Thus, $((r-s)^2-4k^2)$ has order $|k|^2$ and $\left(r+s+\sqrt{(r-s)^2-4k^2}\right)$ has order $|k|$.

Moreover, from (4.31), if $\alpha_1+\alpha_2 \leq 6$, we deduce that $(rs+k^2)$ has order $|k|^2$ and, if $\alpha_1+\alpha_2 > 6$, $(rs+k^2)$ has order $|k|^{\alpha_1+\alpha_2-4}$. Therefore, from (4.30), we have the following cases:

- If $\max\{\alpha_1, \alpha_2\} > 3$ and $\alpha_1 + \alpha_2 \leq 6$, then

$$\lambda_k^- \sim \frac{2}{k^2}\sqrt{\frac{ac}{b_2 d_2}}\left(\frac{rs + k^2}{r + s + \sqrt{(r-s)^2 - 4k^2}}\right) \sim \frac{l_1}{k^2}\left(\frac{k^2}{|k|^{\max\{\alpha_1, \alpha_2\}-2}}\right) \tag{4.32}$$
$$= l_1 \frac{k^2}{|k|^{\max\{\alpha_1, \alpha_2\}}} \geq \frac{l_1}{|k|^{\max\{\alpha_1, \alpha_2\}}}.$$

- If $\max\{\alpha_1, \alpha_2\} > 3$ and $\alpha_1 + \alpha_2 > 6$, then

$$\lambda_k^- \sim \frac{2}{k^2}\sqrt{\frac{ac}{b_2 d_2}}\left(\frac{rs + k^2}{r + s + \sqrt{(r-s)^2 - 4k^2}}\right) \sim \frac{l_1}{k^2}\left(\frac{|k|^{\alpha_1+\alpha_2-4}}{|k|^{\max\{\alpha_1, \alpha_2\}-2}}\right) \tag{4.33}$$
$$= \frac{l_1}{|k|^{4+\max\{\alpha_1, \alpha_2\}-(\alpha_1+\alpha_2)}} = \frac{l_1}{|k|^{4-\min\{\alpha_1, \alpha_2\}}}.$$

- If $\max\{\alpha_1, \alpha_2\} \leq 3$, we obtain that $\alpha_1 + \alpha_2 \leq 6$ and

$$\lambda_k^- \sim \frac{2}{k^2}\sqrt{\frac{ac}{b_2 d_2}}\left(\frac{rs + k^2}{r + s + \sqrt{(r-s)^2 - 4k^2}}\right) \sim \frac{l_1}{k^2}\left(\frac{k^2}{|k|}\right) \tag{4.34}$$
$$= l_1 \frac{|k|^{3-\max\{\alpha_1, \alpha_2\}}}{|k|^{4-\max\{\alpha_1, \alpha_2\}}} \geq \frac{l_1}{|k|^{4-\max\{\alpha_1, \alpha_2\}}}.$$

□

REMARK 4.1.2. *If* $\alpha_1 = \alpha_2 = 4$, *then* $\lim_{k\to\infty} \lambda_k^- = \min\{\frac{\beta_1}{b_2}, \frac{\beta_2}{d_2}\}$ *and* $\lim_{k\to\infty} \Re(\lambda_k^\pm) = \frac{1}{2}\left(\frac{\beta_1}{b_2} + \frac{\beta_2}{d_2}\right)$. *In this case,*

$$\lambda_k^- \sim \frac{C}{k^4}\left(\frac{\beta_1\beta_2}{ac}k^4 + k^2\right) \geq \begin{cases} \frac{l_2}{k^2} & \text{if } \beta_1\beta_2 = 0, \\ l_2 & \text{if } \beta_1\beta_2 > 0, \end{cases}$$

for some positive constant l_2.

REMARK 4.1.3. *If* $|e_k| < 1$, *we have that*

$$\Re(\lambda_k^\pm) \sim \frac{1}{2}\left(\frac{\beta_1}{b_2}|k|^{\alpha_1-4} + \frac{\beta_2}{d_2}|k|^{\alpha_2-4}\right).$$

Hence, we obtain the following cases:

- *If $\alpha_1 < 4$ and $\alpha_2 < 4$ (we suppose that, if $\alpha_j = \max\{\alpha_1, \alpha_2\}$, then $\beta_j > 0$), there exists a constant $l_2 > 0$, such that*

$$\begin{aligned}\Re(\lambda_k^{\pm}) &\geq \frac{l_2}{|k|^{4-\max\{\alpha_1,\alpha_2\}}}\left(\frac{\beta_1}{b_2}|k|^{\alpha_1-\max\{\alpha_1,\alpha_2\}} + \frac{\beta_2}{d_2}|k|^{\alpha_2-\max\{\alpha_1,\alpha_2\}}\right)\\ &\geq \frac{l_2\beta_j}{|k|^{4-\max\{\alpha_1,\alpha_2\}}}.\end{aligned}$$

- *If $\alpha_1 = \alpha_2 = 4$, then there exists a constant $l_2 > 0$, such that*

$$\Re(\lambda_k^{\pm}) \geq l_2.$$

Let us analyze more closely the case of double eigenvalue.

LEMMA 4.1.2. *With the notation from Lemma 4.1.1, we have that:*

(i) There exists only a finite number of values $k \in \mathbb{Z}$ with the property that $|e_k| = 1$.

(ii) There exists a subsequence $(e_{k_m})_{m\geq 1}$ of $(e_k)_{k\geq 1}$, such that $\lim_{k_m\to\infty} |e_{k_m}| = 1$ if and only if one of the following cases holds

(C_1) *$\alpha_1 = \alpha_2 = 3$ and* $\frac{1}{\sqrt{ac}}\left(\beta_1\sqrt{\frac{d}{b}} - \beta_2\sqrt{\frac{b}{d}}\right) = 2,$

(C_2) *$3 = \alpha_1 > \alpha_2$ and* $\beta_1 = 2\sqrt{\frac{acb_2}{d_2}},$

(C_3) *$3 = \alpha_2 > \alpha_1$ and* $\beta_2 = 2\sqrt{\frac{acd_2}{b_2}}.$

(iii) If $\lim_{k\to\infty} |e_k| = 1$, there exists a positive constant M, such that $\frac{1}{|k||\lambda_k^-|} \leq M.$

Proof. For the first part of the Lemma, let us suppose that we have an infinite number of different values $(k_m)_{m\geq 1} \subset \mathbb{N}$, such that $e_{k_m} = 1$. Without loss of generality, we may assume that $\lim_{m\to\infty} k_m = \infty$. We have the following cases:

- If $\alpha_1 > \alpha_2$, then

$$1 = \lim_{m\to\infty} e_{k_m} = \frac{\beta_1}{2}\sqrt{\frac{d_2}{b_2}} \lim_{m\to\infty} \frac{(1+k_m^2)^{\frac{\alpha_1}{2}}}{\sqrt{(1-ak_m^2)(1-ck_m^2)}k_m},$$

which implies that $\alpha_1 = 3$ and $\beta_1 = 2\sqrt{\frac{acb_2}{d_2}}$. Then,

$$-\beta_2\frac{(1+k_m^2)^{\frac{\alpha_2}{2}}}{2k_m\sqrt{(1-ak_m^2)(1-ck_m^2)}}\sqrt{\frac{1+bk_m^2+b_2k_m^4}{1+dk_m^2+d_2k_m^4}} =$$

$$1 - \frac{(1+k_m^2)^{\frac{3}{2}}}{k_m}\sqrt{\frac{acb_2(1+dk_m^2+d_2k_m^4)}{d_2(1+bk_m^2+b_2k_m^4)}} = \frac{\frac{l_{k_m}-p_{k_m}}{l_{k_m}}}{1+\sqrt{\frac{p_{k_m}}{l_{k_m}}}}, \qquad (4.35)$$

where,

$$l_{k_m} = d_2 k_m^2 (1 - a k_m^2)(1 - c k_m^2)(1 + b k_m^2 + b_2 k_m^4),$$
$$p_{k_m} = a c b_2 \left(1 + k_m^2\right)^3 (1 + d k_m^2 + d_2 k_m^4).$$

We have that $\dfrac{l_{k_m} - p_{k_m}}{l_{k_m}} = \dfrac{1}{k_m^2} \dfrac{\sum_{j=0}^{3} \frac{\theta_j^1}{k_m^{2j}}}{\sum_{j=0}^{5} \frac{\theta_j^2}{k_m^{2j}}}$, where θ_j^1, θ_j^2 are constants that depend only on the parameters a, b, d, b_2, d_2 and c.

Moreover, $\theta_0^2 = a c b_2 d_2 \neq 0$, $\lim_{m\to\infty} \dfrac{\sum_{j=0}^{3} \frac{\theta_j^1}{k_m^{2j}}}{\sum_{j=0}^{5} \frac{\theta_j^2}{k_m^{2j}}} = \dfrac{\theta_0^1}{\theta_0^2}$ and $\lim_{m\to\infty} \sqrt{\frac{p_{k_m}}{l_{k_m}}} = 1$.

Hence, from (4.35) we obtain

$$-\beta_2 \frac{k_m \left(1 + k_m^2\right)^{\frac{\alpha_2}{2}}}{2\sqrt{(1 - a k_m^2)(1 - c k_m^2)}} \sqrt{\frac{1 + b k_m^2 + b_2 k_m^4}{1 + d k_m^2 + d_2 k_m^4}} = \frac{\dfrac{\sum_{j=0}^{3} \frac{\theta_j^1}{k_m^{j}}}{\sum_{j=0}^{5} \frac{\theta_j^2}{k_m^{j}}}}{1 + \sqrt{\frac{p_{k_m}}{l_{k_m}}}}. \tag{4.36}$$

Furthermore,

$$\lim_{m\to\infty} \left(-\beta_2 \frac{k_m \left(1 + k_m^2\right)^{\frac{\alpha_2}{2}}}{2\sqrt{(1 - a k_m^2)(1 - c k_m^2)}} \sqrt{\frac{1 + b k_m^2 + b_2 k_m^4}{1 + d k_m^2 + d_2 k_m^4}} \right) = \frac{\theta_0^1}{2\theta_0^2}. \tag{4.37}$$

Thus , if $\alpha_2 \geq 1$, (4.37) implies that $\beta_2 = \theta_0^1 = 0$. If $\alpha_2 < 1$, from (4.37) we obtain $\theta_0^1 = 0$. Then, from (4.36) we deduce that

$$\lim_{m\to\infty} \left(-\beta_2 \frac{k_m^3 \left(1 + k_m^2\right)^{\frac{\alpha_2}{2}}}{2\sqrt{(1 - a k_m^2)(1 - c k_m^2)}} \sqrt{\frac{1 + b k_m^2 + b_2 k_m^4}{1 + d k_m^2 + d_2 k_m^4}} \right) = \frac{\theta_1^1}{2\theta_0^2}, \tag{4.38}$$

which implies that $\beta_2 = \theta_1^1 = 0$. However, e_{k_m} can be written as

$$e_{k_m} = \frac{(1 + k_m^2)^{\frac{3}{2}}}{k_m \sqrt{(1 - a k_m^2)(1 - c k_m^2)}} \sqrt{\frac{a c b_2 (1 + d k_m^2 + d_2 k_m^4)}{d_2 (1 + b k_m^2 + b_2 k_m^4)}}. \tag{4.39}$$

Therefore, $e_{k_m} = 1$ is equivalent to a eighth order equation in k_m which has at most eight solutions. We have obtain a contradiction and, thus, this case is not possible.

- The case $\alpha_1 < \alpha_2$ may be treated as before, and we obtain the same conclusion.
- If $\alpha_1 = \alpha_2$ we obtain that $\lim_{m\to\infty} e_{k_m} = 1$ if and only if $\alpha_1 = \alpha_2 = 3$ and $\dfrac{1}{\sqrt{ac}} \left(\beta_1 \sqrt{\dfrac{d}{b}} - \beta_2 \sqrt{\dfrac{b}{d}} \right) = 2$. However, in this case e_{k_m} is given by

$$e_{k_m} = \frac{(1+k_m^2)^{\frac{3}{2}}}{2k_m\sqrt{(1-ak_m^2)(1-ck_m^2)}}\left(\beta_1\sqrt{\frac{1+dk_m^2+d_2k_m^4}{1+bk_m^2+b_2k_m^4}} - \beta_2\sqrt{\frac{1+bk_m^2+b_2k_m^4}{1+dk_m^2+d_2k_m^4}}\right).$$

Therefore, $e_{k_m} = 1$ is equivalent to a fourteenth order equation in k_m which has at most fourteen solutions. We have again obtained a contradiction. Hence, there exists only a finite number of values $k \in \mathbb{Z}$ with the property that $|e_k| = 1$.

The second part of the Lemma follows as before, by analyzing the similar three cases.

For the third part of Lemma, we consider the following cases:

- If (C_1) holds, $\alpha_1 = \alpha_2 = 3$ and $\frac{1}{\sqrt{ac}}\left(\beta_1\sqrt{\frac{d}{b}} - \beta_2\sqrt{\frac{b}{d}}\right) = 2$. Then, from Proposition 4.1.3, we obtain a constant $l_1 > 0$, such that $|\lambda_k^-| \geq \frac{l_1}{|k|}$. Thus, there exists $M = \frac{1}{l_1}$ satisfying $\frac{1}{|k||\lambda_k^-|} \leq M$.
- For the cases (C_2) and (C_3) we proceed as in the case above.

□

Remark 4.1.4. *When we have complex eigenvalues, if* $\lim_{k\to\infty}|e_k| = 1$, *from Remark 4.1.3, there exist a positive constant* M, *such that* $\frac{1}{|k||\Re(\lambda_k^\pm)|} \leq M$.

Since $\Re(\lambda_k^\pm) \geq \lambda_k^- > 0$, for $|e_k| \geq 1$, in the sequel we consider

$$|\Re(\lambda_k^-)| := |\lambda_k^-|.$$

We have the following result.

Theorem 4.1.1. *There exists a constant* $M > 0$, *such that the solution* $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ *of* (4.10) *verifies the following estimate,*

$$|\widehat{\eta}_k(t)|^2 + \frac{w_1}{w_2}|\widehat{w}_k(t)|^2 \leq M\left(|\widehat{\eta}_k^0|^2 + \frac{w_1}{w_2}|\widehat{w}_k^0|^2\right)e^{-2t|\Re(\lambda_k^-)|} \quad (t \geq 0, \quad k \in \mathbb{Z}). \tag{4.40}$$

Proof. We have to analyze two different cases.

- If there exists no subsequence $(e_{k_m})_{m\geq 1}$ of $(e_k)_{k\geq 1}$, such that $\lim_{k_m\to\infty}|e_{k_m}| = 1$, then

$$\begin{aligned}\frac{1+|\zeta_k|+|\zeta_k|^2}{|1-\zeta_k^2|} &= \frac{|1-|\zeta_k||^2}{|1-\zeta_k^2|} + \frac{3|\zeta_k|}{|1-\zeta_k^2|} = \frac{|1-|\zeta_k||}{|1+|\zeta_k||} + \frac{3|\zeta_k|}{|1-\zeta_k^2|} \\ &\leq 1 + \frac{3}{2\sqrt{|e_k|^2-1}} \leq M,\end{aligned}$$

for some constant $M > 0$. Thus,

$$\limsup_{|k|\to\infty} \frac{1+|\zeta_k|+|\zeta_k|^2}{|1-\zeta_k^2|} \leq M. \tag{4.41}$$

Hence, from (4.13)-(4.14) and (4.41) we have that

$$\begin{aligned} |\widehat{\eta}_k(t)|^2 &\leq e^{-2t|\Re(\lambda_k^-)|}\left(\left|\frac{1+|\zeta_k|^2}{|1-\zeta_k^2|}\right|^2 |\widehat{\eta}_k^0)|^2 + 2\left|\frac{|\zeta_k|}{|1-\zeta_k^2|}\right|^2 \frac{w_1}{w_2}|\widehat{w}_k^0|^2\right) \\ &\leq M^2 e^{-2t|\Re(\lambda_k^-)|}\left(|\widehat{\eta}_k^0|^2 + \frac{w_1}{w_2}|\widehat{w}_k^0|^2\right), \end{aligned} \tag{4.42}$$

and

$$\begin{aligned} |\widehat{w}_k(t)|^2 &\leq e^{-2t|\Re(\lambda_k^-)|}\left(\left|\frac{1+|\zeta_k|^2}{|1-\zeta_k^2|}\right|^2 |\widehat{w}_k^0|^2 + 2\left|\frac{|\zeta_k|}{|1-\zeta_k^2|}\right|^2 \frac{w_2}{w_1}|\widehat{\eta}_k^0|^2\right) \\ &\leq M^2 e^{-2t|\Re(\lambda_k^-)|}\left(|\widehat{w}_k^0|^2 + \frac{w_2}{w_1}|\widehat{\eta}_k^0|^2\right). \end{aligned} \tag{4.43}$$

We multiply (4.43) by $\dfrac{w_1}{w_2}$ and add the resulting estimate, hand to hand, to (4.42) and obtain (4.40).

- Suppose that exists a subsequence $(e_{k_m})_{m\geq 1}$ of $(e_k)_{k\geq 1}$, such that $\lim_{k_m\to\infty}|e_{k_m}| = 1$. We claim that there exists a constant $M > 0$, such that

$$\left|\frac{e^{-\lambda_{k_m}^+ t} - e^{-\lambda_{k_m}^- t}}{|1-\zeta_{k_m}^2|}\right| \leq M e^{-t|\Re(\lambda_{k_m}^-)|}. \tag{4.44}$$

Suppose that it was proved. Then, from (4.13)-(4.14) we obtain

$$\begin{aligned} &|\widehat{\eta}_{k_m}(t)| \\ &\leq \left|\frac{1}{1-\zeta_{k_m}^2}\left(\left(e^{-\lambda_{k_m}^+ t} - e^{-\lambda_{k_m}^- t}\right) + \left(1-\zeta_{k_m}^2\right) e^{-\lambda_{k_m}^- t}\right)\widehat{\eta}_{k_m}^0\right| \\ &+ \left|\frac{i}{1-\zeta_{k_m}^2}\sqrt{\frac{w_1}{w_2}}\zeta_{k_m}\left(e^{-\lambda_{k_m}^+ t} - e^{-\lambda_{k_m}^- t}\right)\widehat{w}_{k_m}^0\right| \\ &\leq \left(M e^{-t|\Re(\lambda_{k_m}^-)|} + e^{-t|\Re(\lambda_{k_m}^-)|}\right)|\widehat{\eta}_{k_m}^0| + \sqrt{\frac{w_1}{w_2}}|\zeta_{k_m}| M e^{-t|\Re(\lambda_{k_m}^-)|}|\widehat{w}_{k_m}^0| \\ &\leq M e^{-t|\Re(\lambda_{k_m}^-)|}\left(|\widehat{\eta}_{k_m}^0| + \sqrt{\frac{w_1}{w_2}}|\widehat{w}_{k_m}^0|\right). \end{aligned}$$

Hence, from (4.44) it follows that

$$|\widehat{\eta}_{k_m}(t)|^2 \leq M^2 e^{-2t|\Re(\lambda_{k_m}^-)|}\left(|\widehat{\eta}_{k_m}^0|^2 + \frac{w_1}{w_2}|\widehat{w}_{k_m}^0|^2\right). \tag{4.45}$$

Similarly, from (4.13)-(4.14) we get

$$|\widehat{w}_{k_m}(t)|^2 \leq M^2 e^{-2t|\Re(\lambda^-_{k_m})|}\left(|\widehat{w}^0_{k_m}|^2 + \frac{w_2}{w_1}|\widehat{\eta}^0_{k_m}|^2\right). \tag{4.46}$$

Combining (4.46) and (4.45) we obtain (4.40).
Now, we prove the claim (4.44). Since $\lim_{k\to\infty}\left(\lambda^+_k - \lambda^-_k\right) = 0$, there exists a positive constant M, such that

$$|e^{-\left(\lambda^+_k-\lambda^-_k\right)t} - 1| \leq M|\lambda^+_{k_m} - \lambda^-_{k_m}|t. \tag{4.47}$$

Thus, from (4.47) we have that

$$\begin{aligned}\left|\frac{e^{-\lambda^+_{k_m}t} - e^{-\lambda^-_{k_m}t}}{|1-\zeta^2_{k_m}|}\right| &\leq \frac{Me^{-t|\Re(\lambda^-_{k_m})|}|k_m|\sqrt{w_1w_2}\left|\sqrt{e^2_{k_m}-1}\right|t}{\left|\sqrt{e^2_{k_m}-1}\left(e_{k_m}-\sqrt{e^2_{k_m}-1}\right)\right|} \\ &\sim \frac{Mte^{-t|\Re(\lambda^-_{k_m})|}}{|k_m|\left|\left(e_{k_m}-\sqrt{e^2_{k_m}-1}\right)\right|} \leq \frac{Mte^{-t|\Re(\lambda^-_{k_m})|}}{|k_m|},\end{aligned} \tag{4.48}$$

where M is a positive constant. From the L'Hôpital rule we deduce that

$$te^{-t|\Re(\lambda^-_{k_m})|} \sim \frac{e^{-t|\Re(\lambda^-_{k_m})|}}{|\Re(\lambda^-_{k_m})|}.$$

Hence, from (4.48) we obtain

$$\left|\frac{e^{-\lambda^+_{k_m}t} - e^{-\lambda^-_{k_m}t}}{|1-\zeta^2_{k_m}|}\right| \leq \frac{Me^{-t|\Re(\lambda^-_{k_m})|}}{|k_m||\Re(\lambda^-_{k_m})|} \tag{4.49}$$

As $\lim_{k_m\to\infty}|e_{k_m}| = 1$, from the Lemma 4.1.2, there exists a positive constant M, such that $\dfrac{1}{|k_m||\Re(\lambda^-_{k_m})|} \leq M$. Thus, from (4.49) we have

$$\left|\frac{e^{-\lambda^+_{k_m}t} - e^{-\lambda^-_{k_m}t}}{|1-\zeta^2_{k_m}|}\right| \leq Me^{-t|\Re(\lambda^-_{k_m})|},$$

for some constant $M > 0$.

□

The following result gives the semigroup associated to our linear problem.

THEOREM 4.1.2. *The family of linear operators $(S(t))_{t\geq 0}$ defined by*

$$S(t)(\eta^0, w^0) = \sum_{k\in\mathbb{Z}}(\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx} \qquad \left((\eta^0, w^0) \in V^s\right), \tag{4.50}$$

where the coefficients $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ *are given by* (4.13)-(4.16), *is a semigroup in* V^s *and verifies the following estimate, for each* $s \in \mathbb{R}$,

$$\|S(t)(\eta^0, w^0)\|_{V^s} \leq M\|(\eta^0, w^0)\|_{V^s} \qquad \left((\eta^0, w^0) \in V^s\right), \tag{4.51}$$

where M *is a positive constant.*

Proof. From Theorem 4.1.1, there exists a constant $M > 0$, such that

$$\left\|\sum_{k\in\mathbb{Z}}(\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx}\right\|^2_{V^s} = \sum_{k\in\mathbb{Z}}\left(|\widehat{\eta}_k(t)|^2 + \frac{w_1}{w_2}|\widehat{w}_k(t)|^2\right)(1+k^2)^s$$

$$\leq M^2\sum_{k\in\mathbb{Z}}\left(|\widehat{\eta}^0_k|^2 + \frac{w_1}{w_2}|\widehat{w}^0_k|^2\right)(1+k^2)^s = \; M^2\left\|\left(\eta^0, w^0\right)\right\|^2_{V^s}.$$

Then, $(S(t))_{t\geq 0}$ is a well-defined linear and continuous operator and satisfies (4.51). It is easy to see that $S(0) = I$, $S(t_1)\circ S(t_2) = S(t_1+t_2)$ for any $t_1, t_2 \in \mathbb{R}^+$ and, in addition, from (4.13)-(4.16) and the analysis developed in Theorem 4.1.1, we obtain that

$$\|S(t)(\eta^0, w^0) - (\eta^0, w^0)\|^2_{V^s} \leq C\sum_{k\in\mathbb{Z}}\Psi_k^2(t)\left(|\widehat{\eta}^0_k|^2 + \frac{w_1}{w_2}|\widehat{w}^0_k|^2\right)(1+k^2)^s,$$

where

$$\Psi_k(t) = \max\left\{\left|\frac{e^{-\lambda_k^+ t} - \zeta_k^2 e^{-\lambda_k^- t}}{1-\zeta_k^2} - 1\right|, \left|\frac{e^{-\lambda_k^- t} - \zeta_k^2 e^{-\lambda_k^+ t}}{1-\zeta_k^2} - 1\right|, \left|\frac{e^{-\lambda_k^+ t} - e^{-\lambda_k^- t}}{1-\zeta_k^2}\right||\zeta_k|\right\}.$$

Consequently $\lim_{t\to 0} S(t)(\eta^0, w^0) = (\eta^0, w^0)$ in V^s and the proof is complete.

□

THEOREM 4.1.3. *The infinitesimal generator of the semigroup* $(S(t))_{t\geq 0}$ *is a bounded operator* $(D(-A), -A)$, *where* $D(-A) = V^s$ *and* A *is given by* (4.7).

Proof. We show that

$$\lim_{t\to 0}\frac{S(t)(\eta^0, w^0) - (\eta^0, w^0)}{t} = -A(\eta^0, w^0), \tag{4.52}$$

if and only if $(\eta^0, w^0) \in V^s$.

This is equivalent to show that the derivative in zero of the series $\sum_{k\in\mathbb{Z}}(\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx}$, where $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ is given by (4.13)-(4.16), is convergent to $-A(\eta^0, w^0)$ in V^s if and only if $(\eta^0, w^0) \in V^s$.

If we denote by

$$\mathcal{S}_N(t) = \sum_{|k|\leq N}(\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx},$$

a partial sum of the series, a straightforward computation which takes into account (4.10) shows that

$$\left[\mathcal{S}_N\right]_t(0) = -A\left(\mathcal{S}_N\right)(0). \tag{4.53}$$

Let $(D(B), B)$ the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$. If $(\eta^0, w^0) \in D(B)$, from (4.53) we obtain that

$$B(\eta^0, w^0) = \lim_{t\to 0} \frac{S(t)(\eta^0, w^0) - (\eta^0, w^0)}{t} = \left[\sum_{k\in\mathbb{Z}} (\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx}\right]_t (0) \tag{4.54}$$
$$= \lim_{N\to\infty} \left[\mathcal{S}_N\right]_t (0) = \lim_{N\to\infty} -A\left(\mathcal{S}_N\right)(0) = -A(\eta^0, w^0).$$

Hence, $(\eta^0, w^0) \in D(-A) = V^s$ and $B(\eta^0, w^0) = -A(\eta^0, w^0)$, for any $(\eta^0, w^0) \in D(B)$. On the other hand, let $(\eta^0, w^0) \in D(-A) = V^s$. We we have to show that the series $\left[\sum_{k\in\mathbb{Z}} (\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx}\right]_t (0)$ is convergent. This is equivalent to show that

$$\left[\mathcal{S}_N\right]_t (0) = \left[\sum_{|k|\leq N} (\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx}\right]_t (0)$$

is a Cauchy sequence. Indeed,

$$\| \left[\mathcal{S}_{N+p}\right]_t (0) - \left[\mathcal{S}_N\right]_t (0)\|^2_{V^s} = \sum_{N\leq |k|\leq N+p} \left(|\widehat{\eta}_{k,t}(0)|^2 + \frac{w_1}{w_2}|\widehat{w}_{k,t}(0)|^2\right)(1+k^2)^s. \tag{4.55}$$

From (4.10) we deduce that

$$|\widehat{\eta}_{k,t}(0)|^2 = \left|\frac{\beta_1(1+k^2)^{\frac{\alpha_1}{2}}}{1+bk^2+b_2k^4}\right|^2 |\widehat{\eta}_k(0)|^2 + k^2 w_1^2 |\widehat{w}_k(0)|^2 \tag{4.56}$$
$$\leq M|\widehat{\eta}_k(0)|^2 + k^2 w_1^2 |\widehat{w}_k(0)|^2$$

and

$$|\widehat{w}_{k,t}(0)|^2 = k^2 w_2^2 |\widehat{\eta}_k(0)|^2 + \left|\frac{\beta_2(1+k^2)^{\frac{\alpha_2}{2}}}{1+dk^2+d_2k^4}\right|^2 |\widehat{w}_k(0)|^2 \tag{4.57}$$
$$\leq k^2 w_2^2 |\widehat{\eta}_k(0)|^2 + M|\widehat{w}_k(0)|^2,$$

where M is a positive constant depending only on $\alpha_1, \alpha_2, \beta_1, \beta_2, b, b_2, d$ and d_2. Then, from (4.56) and (4.57) we have that

$$|\widehat{\eta}_{k,t}(0)|^2 + \frac{w_1}{w_2}|\widehat{w}_{k,t}(0)|^2 \leq k^2 w_1^2 w_2^2 \left(|\widehat{\eta}_k(0)|^2 + \frac{w_1}{w_2}|\widehat{w}_k(0)|^2\right) + M\left(|\widehat{\eta}_k(0)|^2 + \frac{w_1}{w_2}|\widehat{w}_k(0)|^2\right) \tag{4.58}$$
$$\leq M\left(|\widehat{\eta}_k(0)|^2 + \frac{w_1}{w_2}|\widehat{w}_k(0)|^2\right).$$

Therefore, from (4.55) and (4.58) we obtain the following estimate,

$$\| \left[\mathcal{S}_{N+p}\right]_t (0) - \left[\mathcal{S}_N\right]_t (0)\|^2_{V^s} \leq M \sum_{N\leq |k|\leq N+p} \left(|\widehat{\eta}_k(0)|^2 + \frac{w_1}{w_2}|\widehat{w}_k(0)|^2\right)(1+k^2)^s \tag{4.59}$$
$$= M \sum_{N\leq |k|\leq N+p} \left(|\widehat{\eta}_k^0|^2 + \frac{w_1}{w_2}|\widehat{w}_k^0|^2\right)(1+k^2)^s,$$

and as $(\eta^0, w^0) \in D(-A) = V^s$,

$$[\mathcal{S}_N]_t\,(0) = \left[\sum_{|k|\le N} (\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx}\right]_t (0)$$

is a Cauchy sequence. Thus,

$$-A(\eta^0, w^0) = \lim_{N\to\infty} -A\,(\mathcal{S}_N)\,(0) = \lim_{N\to\infty} [\mathcal{S}_N]_t\,(0) = \left[\sum_{k\in\mathbb{Z}}(\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx}\right]_t (0)$$
$$= \lim_{t\to 0} \frac{S(t)(\eta^0, w^0) - (\eta^0, w^0)}{t} = B(\eta^0, w^0).$$

Hence, $(\eta^0, w^0) \in D(B)$ and $-A(\eta^0, w^0) = B(\eta^0, w^0)$, for any $(\eta^0, w^0) \in D(-A) = V^s$.

□

REMARK 4.1.5. *In fact much more can be said about the regularity of solutions of* (4.3). *Since* (4.3) *is linear and* $-A$ *is a bounded operator, we can easily deduce that* $(\eta, w) \in \mathcal{C}^w\left([0,\infty); V^s\right)$, *where* $\mathcal{C}^w\left([0,\infty); V^s\right)$ *represents the class of the analytic functions defined in* $[0,\infty)$ *with values in* V^s. *Indeed, for* $t_0 \in [0,\infty)$

$$\left\| \sum_{n=0}^{\infty} \frac{d^n}{dt^n}(\eta, w)(t_0)\frac{(t-t_0)^n}{n!} \right\|_{V^s} \le \sum_{n=0}^{\infty} \frac{|t-t_0|^n}{n!} \left\| \frac{d^n}{dt^n}(\eta, w)(t_0) \right\|_{V^s}$$
$$\le \|(\eta, w)(t_0)\|_{V^s} \sum_{n=0}^{\infty} \frac{|t-t_0|^n}{n!} \|A\|^n_{\mathcal{L}(V^s)} < \infty.$$

Hence, the series $\sum_{n=0}^{\infty} \frac{d^n}{dt^n}(\eta, w)(t_0)\frac{(t-t_0)^n}{n!}$ *is (absolutely) convergent and*

$$(\eta, w)(t) = exp\left(-A(t-t_0)\right)(\eta, w)(t_0) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!}(-A)^n(\eta, w)(t_0)$$
$$= \sum_{n=0}^{\infty} \frac{d^n}{dt^n}(\eta, w)(t_0)\frac{(t-t_0)^n}{n!}.$$

As a direct consequence of the Theorems 4.1.2 and 4.1.3 and the general theory of the evolution equations (see, for instance, [11]), we have the following existence and uniqueness result:

THEOREM 4.1.4. *Let* $T > 0$ *and* $s \in \mathbb{R}$. *For any* $(\eta^0, w^0) \in V^s$ *and* $(f, g) \in L^1\left(0, T; V^s\right)$, *there exists a unique solution* $(\eta, w) \in W^{1,1}\left([0,T]; V^s\right)$ *of the system*

$$\begin{pmatrix} \eta \\ w \end{pmatrix}_t (t) + A \begin{pmatrix} \eta \\ w \end{pmatrix} (t) = \begin{pmatrix} f \\ g \end{pmatrix}, \quad \begin{pmatrix} \eta \\ w \end{pmatrix} (0) = \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix}, \tag{4.60}$$

which verifies the constant variation formula

$$\begin{pmatrix} \eta \\ w \end{pmatrix} (t) = S(t) \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} + \int_0^t S(t-s) \begin{pmatrix} f \\ g \end{pmatrix} (s)\,\mathrm{d}s. \tag{4.61}$$

4.1.2 Asymptotic behavior

In this section we study the behavior of the solutions of system (4.3), as the time goes to infinity. In order to have a dissipative system, we assume that

$$\beta_1 \geq 0, \quad \beta_2 \geq 0, \quad \beta_1^2 + \beta_2^2 > 0. \tag{4.62}$$

Multiplying both sides of the first equation in (4.10) by $\overline{\widehat{\eta}_k}$ and the second equation by $\overline{\widehat{w}_k}$ if $a = c > 0$ or by $\left(\frac{1-ak^2}{1-k^2}\right)\overline{\widehat{w}_k}$ if $a < 0$, $c < 0$, and then adding the resulting first equation to the conjugate of the resulting second equation, we obtain

$$\begin{aligned}&\frac{d}{dt}(1 + bk^2 + b_2k^4)\left(|\widehat{\eta}_k|^2 + \frac{w_1(k)}{w_2(k)}|\widehat{w}_k|^2\right)\\ &= -2\beta_1(1+k^2)^{\frac{\alpha_1}{2}}|\widehat{\eta}_k|^2 - 2\beta_2(1+k^2)^{\frac{\alpha_2}{2}}\left(\frac{1-ak^2}{1-k^2}\right)|\widehat{w}_k|^2,\end{aligned} \tag{4.63}$$

for $k \in \mathbb{Z}$. Thus, if we define

$$E[\eta, w](t) = \int_0^{2\pi}\left(\left|(I - b\partial_x^2 + b_4\partial_x^2)^{1/2}\eta(t,x)\right|^2 + \left|(I - b\partial_x^2 + b_4\partial_x^2)^{1/2}\mathcal{H}w(t,x)\right|^2\right)dx, \tag{4.64}$$

then, from (4.63), we get

$$\frac{d}{dt}E[\eta, w](t) \leq -C\left(\|\eta\|^2_{H_p^{\frac{\alpha_1}{2}}(0,2\pi)} + \|w\|^2_{H_p^{\frac{\alpha_2}{2}}(0,2\pi)}\right), \tag{4.65}$$

for any $t \geq 0$ and some positive constant $C > 0$, depending only on β_1, β_2, a and c.

Firstly, we analyze the cases in which the solutions of (4.3) decay exponentially to zero. We recall that the solutions to (4.3) *decay exponentially in* V^s if there exist two positive constants M and μ, such that

$$\|S(t)(\eta^0, w^0)\|_{V^s} \leq Me^{-\mu t}\|(\eta^0, w^0)\|_{V^s} \qquad \left(t \geq 0, \quad (\eta^0, w^0) \in V^s\right). \tag{4.66}$$

We have the following result.

THEOREM 4.1.5. *The solutions of* (4.3) *decay exponentially in* V^s *if and only if* $\alpha_1 = \alpha_2 = 4$ *and* β_1, $\beta_2 > 0$. *Moreover,* μ *from* (4.66) *is given by*

$$\mu = \inf_{k\in\mathbb{Z}}\left\{\left|\Re(\lambda_k^-)\right|\right\}, \tag{4.67}$$

where the eigenvalues λ_k^- *are given by* (4.11).

Proof. Firstly, let $\alpha_1 = \alpha_2 = 4$ and β_1, $\beta_2 > 0$. In this case, Remarks 4.1.2 and 4.1.3 ensure that the eigenvalues λ_k^- are uniformly bounded away from the real axis:

$$|\Re(\lambda_k^-)| \geq D > 0 \qquad (k \in \mathbb{Z}),$$

where D, is a positive number, depending on the parameters β_1, β_2, α_1, α_2, b, d, b_2 and d_2. Thus, there exists $\mu = \inf_{k\in\mathbb{Z}} \left\{ \left|\Re(\lambda_k^-)\right| \right\}$, and from the Theorem 4.1.1, we obtain that

$$\begin{aligned} |\widehat{\eta}_k(t)|^2 + \frac{w_1}{w_2}|\widehat{w}_k(t)|^2 &\leq M\left(|\widehat{\eta}_k^0|^2 + \frac{w_1}{w_2}|\widehat{w}_k^0|^2\right) e^{-2t|\Re(\lambda_k^-)|} \\ &\leq Me^{-2t\mu}\left(|\widehat{\eta}_k^0|^2 + \frac{w_1}{w_2}|\widehat{w}_k^0|^2\right), \end{aligned}$$

for some constant $M > 0$, which implies (4.66).

On the other hand, we suppose that $\alpha_2 < \alpha_1 < 4$. Then, from the Proposition 4.1.3, there exist $l > 0$ and $\delta > 0$, such that $|\Re(\lambda_k^-)| \geq \dfrac{l}{|k|^\delta}$. Thus, from Theorem 4.1.1 we have that

$$\begin{aligned} |\widehat{\eta}_k(t)|^2 + \frac{w_1}{w_2}|\widehat{w}_k(t)|^2 &\leq Me^{-2t|\Re(\lambda_k^-)|}\left(|\widehat{\eta}_k^0|^2 + \frac{w_1}{w_2}|\widehat{w}_k^0|^2\right) \\ &\leq Me^{\frac{-2tl}{|k|^\delta}}\left(|\widehat{\eta}_k^0|^2 + \frac{w_1}{w_2}|\widehat{w}_k^0|^2\right). \end{aligned}$$

Hence, the decay rate cannot be exponential. Similarly, if $\alpha_1 < \alpha_2 < 4$. Therefore, $\alpha_1 = \alpha_2 = 4$. Now, we suppose that $\beta_1\beta_2 = 0$. From Remark 4.1.2, there exists a constant $l > 0$, such that $|\Re(\lambda_k^-)| \geq \dfrac{l}{|k|^2}$. We infer that the same conclusion holds. Therefore, $\beta_1,\ \beta_2 > 0$.

□

Now, we analyze the decay rate of solutions in the remaining cases. Since we know from Theorem 4.1.5 that we do not have an exponential decay, we can only expect a polynomial decay if the initial data have additional smoothness properties. We have the following result:

THEOREM 4.1.6. *Suppose that* (4.62) *holds and* $\alpha_1, \alpha_2 \in [0, 4)$. *Let* $\delta > 0$ *be defined by*

$$\delta = \begin{cases} 4 - \max\{\alpha_1,\ \alpha_2\} & \text{if} \quad \max\{\alpha_1,\ \alpha_2\} \leq 3, \\ \max\{\alpha_1,\ \alpha_2\} & \text{if} \quad \max\{\alpha_1,\ \alpha_2\} > 3, \quad \alpha_1 + \alpha_2 \leq 6 \\ 4 - \min\{\alpha_1,\ \alpha_2\} & \text{if} \quad \max\{\alpha_1,\ \alpha_2\} > 3, \quad \alpha_1 + \alpha_2 > 6. \end{cases} \tag{4.68}$$

Then, there exists $M > 0$, *such that the solutions of* (4.3) *satisfy*

$$\|S(t)(\eta^0, w^0)\|_{V^s} \leq \frac{M}{(1+t)^{\frac{q}{\delta}}}\|(\eta^0, w^0)\|_{V^{s+q}} \qquad \left(t \geq 0, \quad (\eta^0, w^0) \in V^{s+q}\right), \tag{4.69}$$

where $s \in \mathbb{R}$ *and* $q > 0$.

Proof. We use an argument developed in [21]. Firstly, we remark that it is sufficient to prove the result for t sufficiently large. From Proposition 4.1.3, there exists a constant $l > 0$, such that

$$|\Re(\lambda_k^{\pm})| \geq \frac{l}{|k|^{\delta}} \qquad (k \in \mathbb{Z}^*). \tag{4.70}$$

From Theorem 4.1.1 and (4.70) we deduce that

$$\begin{aligned} ||S(t)(\eta^0, w^0)||_{V^s}^2 &\leq M^2 \sum_{k\in\mathbb{Z}} (1+k^2)^s e^{-2t\min\{|\Re(\lambda_k^+)|, |\Re(\lambda_k^-)|\}} \left(|\hat{\eta}_k^0|^2 + \frac{w_1}{w_2}|\hat{w}_k^0|^2 \right) \\ &= M^2 \left(e^{-\min\{\beta_1,\beta_2\}t} \left(|\hat{\eta}_0^0|^2 + \frac{w_1}{w_2}|\hat{w}_0^0|^2 \right) \right. \\ &\left. + \sum_{k\in\mathbb{Z}^*} \frac{1}{(1+k^2)^q} e^{\frac{-2lt}{|k|^{\delta}}} (1+k^2)^{s+q} \left(|\hat{\eta}_k^0|^2 + \frac{w_1}{w_2}|\hat{w}_k^0|^2 \right) \right). \end{aligned} \tag{4.71}$$

Let us analyze $e^{\frac{-2l}{|k|^{\delta}}}$. As $\lim_{k\to\infty} e^{\frac{-2l}{|k|^{\delta}}} = 1$, there exists a constant $M > 0$, such that $e^{\frac{-2l}{|k|^{\delta}}} \geq M$, for all $|k| \geq k_0$ and some $k_0 \in \mathbb{N}$. Moreover, if $1 \leq |k| \leq k_0$, then $e^{-2l} \leq e^{\frac{-2l}{|k|^{\delta}}}$.

Hence, we obtain that

$$\begin{aligned} &\sum_{k\in\mathbb{Z}^*} \frac{1}{(1+k^2)^q} e^{\frac{-2lt}{|k|^{\delta}}} (1+k^2)^{s+q} \left(|\hat{\eta}_k^0|^2 + \frac{w_1}{w_2}|\hat{w}_k^0|^2 \right) \\ &\leq e^{2l} \sum_{1\leq|k|\leq k_0} \frac{1}{(1+k^2)^q} e^{\frac{-2l(t+1)}{|k|^{\delta}}} (1+k^2)^{s+q} \left(|\hat{\eta}_k^0|^2 + \frac{w_1}{w_2}|\hat{w}_k^0|^2 \right) \\ &+ \frac{1}{M} \sum_{k_0\leq|k|} \frac{1}{(1+k^2)^q} e^{\frac{-2l(t+1)}{|k|^{\delta}}} (1+k^2)^{s+q} \left(|\hat{\eta}_k^0|^2 + \frac{w_1}{w_2}|\hat{w}_k^0|^2 \right) \\ &\leq M \sum_{k\in\mathbb{Z}^*} \frac{1}{(1+k^2)^q} e^{\frac{-2l(t+1)}{|k|^{\delta}}} (1+k^2)^{s+q} \left(|\hat{\eta}_k^0|^2 + \frac{w_1}{w_2}|\hat{w}_k^0|^2 \right). \end{aligned} \tag{4.72}$$

Let us study the term $E_k(t) = \frac{1}{(1+k^2)^q} e^{\frac{-2l(t+1)}{|k|^{\delta}}}$ for $k \in \mathbb{Z}^*$. Firstly, we remark that $x \leq e^{x-1}$ for all $x \geq 0$. Then, given $\varsigma > 0$ the following inequality holds true

$$x^{\varsigma} e^{-x} \leq c(\varsigma) := \varsigma^{\varsigma} e^{-\varsigma} \qquad (x \geq 0). \tag{4.73}$$

By using (4.73) with $x = \frac{2l(t+1)}{|k|^{\delta}}$ and $\varsigma = \frac{2q}{\delta}$ we deduce that, there exists a constant $C(q, \delta, l) > 0$, such that

$$e^{-\frac{2l(t+1)}{|k|^{\delta}}} \leq \frac{\left(\frac{2q}{\delta}\right)^{\frac{2q}{\delta}} e^{\frac{-2q}{\delta}} |k|^{2q}}{(2l(t+1))^{\frac{2q}{\delta}}} \leq C(q, \delta, l) \frac{(1+k^2)^q}{(t+1)^{\frac{2q}{\delta}}}.$$

From the last estimate, we obtain for each $k \in \mathbb{Z}^*$

$$|E_k(t)| \leq \frac{C(q,\delta,l)}{(t+1)^{\frac{2q}{\delta}}} \qquad (t \geq 0). \tag{4.74}$$

Therefore, from (4.71), (4.72) and (4.74) we have that

$$||S(t)(\eta^0, w^0)||_{V^s}^2 \leq M^2 \left(e^{-\min\{\beta_1,\beta_2\}t} \left(|\hat{\eta}_0^0|^2 + \frac{w_1}{w_2}|\hat{w}_0^0|^2 \right) + \frac{1}{(t+1)^{\frac{2q}{\delta}}} ||(\eta^0, w^0)||_{V^{s+q}}^2 \right).$$

□

4.2 The nonlinear system

We are now in a position to prove the well-posedness and the stabilization for the solutions of the nonlinear system (4.1) issued from small initial data, when the linearized system is exponentially stable, i.e., under the hypothesis of Theorem 4.1.5. The proof will be done by using a fixed point argument. Therefore, the applications of the following lemma, proved in [5], will be needed:

LEMMA 4.2.1. *Let $s \geq -1$. There exists a constant $C > 0$, depending only on s, such that*

$$||fg||_{H_p^s(0,2\pi)} \leq C||f||_{H_p^{s+1}(0,2\pi)}||g||_{H_p^{s+1}(0,2\pi)},$$

for any $f, g \in H_p^{s+1}(0, 2\pi)$.

REMARK 4.2.1. *We write (4.1) in its integral form*

$$\begin{pmatrix} \eta \\ w \end{pmatrix}_t (t) + A \begin{pmatrix} \eta \\ w \end{pmatrix}(t) + N \begin{pmatrix} \eta \\ w \end{pmatrix}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \eta \\ w \end{pmatrix}(0) = \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix}, \tag{4.75}$$

where N is defined by

$$N(\eta, w)$$

$$= \begin{pmatrix} (I - b\partial_x^2 + b_2\partial_x^4)_p^{-1} \left[(\eta w)_x - b(\eta w)_{xxx} + (a+b-1)(\eta w_{xx})_x \right] \\ (I - b\partial_x^2 + d_2\partial_x^4)_p^{-1} \left[ww_x + c(ww_x)_{xx} + (\eta\eta_{xx})_x - (c+d-1)w_x w_{xx} - (c+d)ww_{xxx} \right] \end{pmatrix}, \tag{4.76}$$

and A is the compact operator defined by (4.7). Thus, we obtain that the solution of (4.75) is given by

$$(\eta, w)(t) = S(t)(\eta^0, w^0) - \int_0^t S(t-\tau)N(\eta, w)(\tau)\,\mathrm{d}\tau, \tag{4.77}$$

where $\{S(t)\}_{t\geq 0}$ is the semigroup defined in Theorem 4.1.2.

The main result of this section reads as follows:

THEOREM 4.2.1. *Let $s \geq 0$ and suppose that $\beta_1, \beta_2 > 0$ and $\alpha_1 = \alpha_2 = 4$. There exists $r > 0$, $C > 0$ and $\mu > 0$, such that, for any $(\eta^0, w^0) \in V^s$, satisfying*

$$||(\eta^0, w^0)||_{V^s} \leq r,$$

the system (4.1) *admits a unique solution $(\eta, w) \in \mathcal{C}([0,\infty); V^s)$ which verifies*

$$\|(\eta(t), w(t))\|_{V^s} \leq Ce^{-\mu t}\|(\eta^0, w^0)\|_{V^s} \quad (t \geq 0)\,. \tag{4.78}$$

Moreover, μ may be taken as in (4.67).

Proof. We remark that the hypothesis of Theorem 4.1.5 are verified and there exist $M, \mu > 0$, such that (4.66) holds true. In order to use a fixed point argument, we define the space

$$Y_{s,\mu} = \{(\eta, w) \in \mathcal{C}([0,\infty); V^s) : e^{\mu t}(\eta, w) \in \mathcal{C}([0,\infty); V^s)\},$$

with the norm

$$||(\eta, w)||_{Y_{s,\mu}} := \sup_{0 \leq t < \infty} ||e^{\mu t}(\eta, w)(t)||_{V^s},$$

and the function $\Gamma : Y_{s,\mu} \to Y_{s,\mu}$ by

$$\Gamma(\eta, w)(t) = S(t)(\eta^0, w^0) - \int_0^t S(t-\tau)N(\eta, w)(\tau)\,\mathrm{d}\tau.$$

From Lemma 4.2.1, we deduce that

$$||N(\eta_1, w_1)||_{V^s} \leq C||(\eta_1, w_1)||^2_{V^s}, \tag{4.79}$$

and

$$||N(\eta_1, w_1) - N(\eta_2, w_2)||_{V^s} \leq C(||(\eta_1, w_1)||_{V^s} + ||(\eta_2, w_2)||_{V^s})||(\eta_1, w_1) - (\eta_2, w_2)||_{V^s}, \tag{4.80}$$

for any $(\eta_1, w_1), (\eta_2, w_2) \in V^s$ and for some $C > 0$. Then, combining the estimates above and Theorem 4.1.5, we obtain

$$\begin{aligned} ||\Gamma(\eta, w)(t)||_{V^s} &\leq Me^{-\mu t}||(\eta^0, w^0)||_{V^s} + M\int_0^t e^{-\mu(t-\tau)}||N(\eta, w)(\tau)||_{V^s}\,\mathrm{d}\tau \\ &\leq Me^{-\mu t}||(\eta^0, w^0)||_{V^s} + MCe^{-\mu t}\sup_{0 \leq \tau \leq t}||e^{\mu\tau}(\eta, w)(\tau)||^2_{V^s}, \end{aligned} \tag{4.81}$$

for any $t \geq 0$ and some positive constants M and C. Thus, if we take $(\eta, w) \in B_R(0)$ where

$$B_R(0) = \left\{(\eta, w) \in Y_{s,\mu}; ||(\eta, w)||_{Y_{s,\mu}} \leq R\right\},$$

from (4.81) we conclude that

$$||\Gamma(\eta, w)||_{Y_{s,\mu}} \leq M||(\eta^0, w^0)||_{V^s} + MC||(\eta, w)||^2_{Y_{s,\mu}} \leq Mr + MCR^2. \tag{4.82}$$

A similar calculations shows that, for any $(\eta_1, w_1), (\eta_2, w_2) \in B_R(0)$, we have that

$$
\begin{aligned}
&|| \left(\Gamma(\eta_1, w_1) - \Gamma(\eta_2, w_2)\right)(t)||_{V^s} \leq Me^{-\mu t} \sup_{0\leq\tau\leq t} ||e^{\mu\tau} \left(N(\eta_1, w_1) - N(\eta_2, w_2)\right)(\tau)||_{V^s} \\
&\leq MCe^{-\mu t} \sup_{0\leq\tau\leq t} ((||(\eta_1, w_1)(\tau)||_{V^s} + ||(\eta_2, w_2)(\tau)||_{V^s})||e^{\mu\tau} \left((\eta_1, w_1) - (\eta_2, w_2)\right)(\tau)||_{V^s}) \\
&\leq 2RMC \sup_{0\leq\tau\leq t} ||e^{\mu\tau} \left((\eta_1, w_1) - (\eta_2, w_2)\right)(\tau)||_{V^s}.
\end{aligned}
$$

Therefore,

$$
||\Gamma(\eta_1, w_1) - \Gamma(\eta_2, w_2)||_{Y_{s,\mu}} \leq 2RMC||(\eta_1, w_1) - (\eta_2, w_2)||_{Y_{s,\mu}}. \tag{4.83}
$$

By choosing $R = 2Mr$ and $r \leq \dfrac{1}{8CM^2}$, from (4.82) and (4.83) we deduce that the map

$$
\Gamma : B_R(0) \subseteq Y_{s,\mu} \longrightarrow B_R(0)
$$

is a contraction, hence it admits a unique fixed point $(\eta, w) \in B_R(0)$ which solves the integral equation (4.75). Moreover,

$$
||e^{\mu t}(\eta, w)(t)||_{V^s} \leq R = 2Mr \qquad (t \geq 0).
$$

The proof of the Theorem is complete.

□

Chapter 5

Controllability for higher-order linear Boussinesq system on a periodic domain

Considered in this chapter is a Boussinesq systems of the form

$$\begin{cases} \eta_t + w_x - b\eta_{txx} + b_2\eta_{txxxx} + aw_{xxx} + a_1 w_{xxxxx} = f(t,x) & \text{for } x \in (0,2\pi),\ \ t>0, \\ w_t + \eta_x - dw_{txx} + d_2 w_{txxxx} + c\eta_{xxx} + c_1\eta_{xxxxx} = g(t,x) & \text{for } x \in (0,2\pi),\ \ t>0 \end{cases} \tag{5.1}$$

with periodic boundary conditions

$$\begin{cases} \dfrac{\partial^r \eta}{\partial x^r}(t,0) = \dfrac{\partial^r \eta}{\partial x^r}(t,2\pi) & \text{for } t>0, \quad 0 \le r \le r_0, \\[2ex] \dfrac{\partial^q w}{\partial x^q}(t,0) = \dfrac{\partial^q w}{\partial x^q}(t,2\pi) & \text{for } t>0, \quad 0 \le q \le q_0 \end{cases} \tag{5.2}$$

and initial condition

$$\eta(0,x) = \eta^0(x), \qquad w(0,x) = w^0(x) \quad \text{for } x \in (0,2\pi). \tag{5.3}$$

The number of boundary conditions depends on the values of the parameters.

Its well-posedness in a suitable classical Banach space will be investigated in the next section. Then, in sections 5.2 and 5.3, considering f and g as control inputs, we will study its control problems. In particular, exact controllability will be established in section 5.2 with two control inputs while section 5.3 will be devoted to study the system with a single control acting only on a subdomain $\omega \subset (0,2\pi)$.

5.1 Well-posedness

Assume that the initial data in (5.3) and the forcing terms in (5.1) are given by

$$(\eta^0, w^0) = \sum_{k\in\mathbb{Z}} \left(\widehat{\eta}^0_k, \widehat{w}^0_k\right) e^{ikx}, \qquad (f,g)(t) = \sum_{k\in\mathbb{Z}} \left(\widehat{f}_k(t), \widehat{g}_k(t)\right) e^{ikx}.$$

At least formally, the solution of (5.1)-(5.3) may be written as

$$(\eta, w)(t,x) = \sum_{k\in\mathbb{Z}} (\widehat{\eta}_k(t), \widehat{w}_k(t)) e^{ikx},$$

where $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ fulfill

$$\begin{cases} (1+bk^2+b_2k^4)(\widehat{\eta}_k)_t + ik(1-ak^2+a_1k^4)\widehat{w}_k = \widehat{f}_k, & t\in(0,T), \\ (1+dk^2+d_2k^4)(\widehat{w}_k)_t + ik(1-ck^2+c_1k^4)\widehat{\eta}_k = \widehat{g}_k, & t\in(0,T), \\ \widehat{\eta}_k(0) = \widehat{\eta}_k^0, \qquad \widehat{w}_k(0) = \widehat{w}_k^0. \end{cases} \tag{5.4}$$

We have the following result.

LEMMA 5.1.1. *The solution $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ of (5.4) is given by*

$$\begin{cases} \widehat{\eta}_k(t) &= cos[k\sigma(k)t]\widehat{\eta}_k^0 - i\sqrt{\frac{w_1}{w_2}} sin[k\sigma(k)t]\widehat{w}_k^0 + \int_0^t \frac{cos[k\sigma(k)(t-s)]}{1+bk^2+b_2k^4}\widehat{f}_k(s)ds \\ & -i\sqrt{\frac{w_1}{w_2}}\int_0^t \frac{sin[k\sigma(k)(t-s)]}{1+dk^2+d_2k^4}\widehat{g}_k(s)ds, \\ \widehat{w}_k(t) &= -i\sqrt{\frac{w_2}{w_1}} sin[k\sigma(k)t]\widehat{\eta}_k^0 + cos[k\sigma(k)t]\widehat{w}_k^0 - i\sqrt{\frac{w_1}{w_2}}\int_0^t \frac{sin[k\sigma(k)(t-s)]}{1+bk^2+b_2k^4}\widehat{f}_k(s)ds \\ & +\int_0^t \frac{cos[k\sigma(k)(t-s)]}{1+dk^2+d_2k^4}\widehat{g}_k(s)ds, \end{cases} \tag{5.5}$$

where, $\quad w_1 = \dfrac{1-ak^2+a_1k^4}{1+bk^2+b_2k^4}, \quad w_2 = \dfrac{1-ck^2+c_1k^4}{1+dk^2+d_2k^4}$ *and* $\quad \sigma(k) = \sqrt{w_1w_2}.$

Proof. The system (5.4) is equivalent to

$$\begin{pmatrix} \widehat{\eta}_k \\ \widehat{w}_k \end{pmatrix}_t (t) + ikA(k)\begin{pmatrix} \widehat{\eta}_k \\ \widehat{w}_k \end{pmatrix}(t) = \begin{pmatrix} \frac{\widehat{f}_k}{1+bk^2+b_2k^4} \\ \frac{\widehat{g}_k}{1+dk^2+d_2k^4} \end{pmatrix}, \quad \begin{pmatrix} \widehat{\eta}_k \\ \widehat{w}_k \end{pmatrix}(0) = \begin{pmatrix} \widehat{\eta}_k^0 \\ \widehat{w}_k^0 \end{pmatrix},$$

where

$$A(k) = \begin{pmatrix} 0 & w_1 \\ w_2 & 0 \end{pmatrix}.$$

Hence, the solution of (5.4) is given by

$$\begin{pmatrix} \widehat{\eta}_k \\ \widehat{w}_k \end{pmatrix}(t) = e^{-iktA(k)}\begin{pmatrix} \widehat{\eta}_k^0 \\ \widehat{w}_k^0 \end{pmatrix} + \int_0^t e^{-ik(t-s)A(k)}\begin{pmatrix} \frac{\widehat{f}_k(s)}{1+bk^2+b_2k^4} \\ \frac{\widehat{g}_k(s)}{1+dk^2+d_2k^4} \end{pmatrix} ds. \tag{5.6}$$

The eigenvalues of the matrix $A(k)$ are $\pm\sigma(k)$.

Under the above considerations, let

$$Q_1 = \frac{-ikA(k) + ik\sigma(k)}{2ik\sigma(k)} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{\frac{w_1}{w_2}} \\ -\sqrt{\frac{w_2}{w_1}} & 1 \end{pmatrix} \tag{5.7}$$

and

$$Q_2 = \frac{-ikA(k) - ik\sigma(k)}{-2ik\sigma(k)} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{\frac{w_1}{w_2}} \\ \sqrt{\frac{w_2}{w_1}} & 1 \end{pmatrix}. \tag{5.8}$$

Then, according to Proposition 4.1.1 given in the Chapter 3, we have that

$$e^{-ikA(k)t} = e^{ik\sigma(k)t}Q_1 + e^{-ik\sigma(k)t}Q_2 = \begin{pmatrix} cos[k\sigma(k)t] & -i\sqrt{\frac{w_1}{w_2}}sin[k\sigma(k)t] \\ -i\sqrt{\frac{w_2}{w_1}}sin[k\sigma(k)t] & cos[k\sigma(k)t] \end{pmatrix}. \tag{5.9}$$

Consequently, from (5.6) and (5.9), we deduce that the solution of (5.4) is given by (5.5). □

Let us introduce the number $l \in \mathbb{Z}$ with the property that

$$\sqrt{\frac{w_1}{w_2}} \sim C|k|^l, \quad \text{when} \quad |k| \to \infty, \tag{5.10}$$

where C is a positive constant not depending on k. For each $s \in \mathbb{R}$, we define the space

$$V^s = H_p^s(0, 2\pi) \times H_p^{s+l}(0, 2\pi),$$

endowed with the inner product defined by

$$\langle (f_1, f_2), (g_1, g_2) \rangle = (f_1, g_1)_s + (\mathcal{H}f_2, \mathcal{H}g_2)_s,$$

and the operator $\mathcal{H}$ is defined in the following way

$$\mathcal{H}\left(\sum_{k\in\mathbb{Z}} \widehat{a}_k e^{ikx}\right) = \sum_{k\in\mathbb{Z}} \sqrt{\frac{w_1}{w_2}}\widehat{a}_k e^{ikx}.$$

The following result gives the C_0 group associated to our problem.

THEOREM 5.1.1. *The family of linear operators $(S(t))_{t\in\mathbb{R}}$ defined by*

$$S(t)(\eta^0, w^0) = \sum_{k\in\mathbb{Z}} (\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx} \qquad \left((\eta^0, w^0) \in V^s\right), \tag{5.11}$$

where the Fourier coefficients $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ *are given by*

$$\begin{cases} \widehat{\eta}_k(t) & = cos[k\sigma(k)t]\widehat{\eta}_k^0 - i\sqrt{\frac{w_1}{w_2}} sin[k\sigma(k)t]\widehat{w}_k^0, \\ \\ \widehat{w}_k(t) & = -i\sqrt{\frac{w_2}{w_1}} sin[k\sigma(k)t]\widehat{\eta}_k^0 + cos[k\sigma(k)t]\widehat{w}_k^0, \end{cases} \tag{5.12}$$

is a group of isometries in V^s*, for any* $s \in \mathbb{R}$*.*

Proof. First, let us prove that $S(t)$ is a well-defined linear and continuous operator for any $t \in \mathbb{R}$. If $(\eta^0, w^0) = \sum_{k\in\mathbb{Z}} (\widehat{\eta}_k^0, \widehat{w}_k^0)\, e^{ikx} \in V^s$, then we claim that the series $\sum_{k\in\mathbb{Z}} (\widehat{\eta}_k(t), \widehat{w}_k(t))\, e^{ikx}$ converges in $C([0,\infty), V^s)$. This is equivalent to say that the sequence

$$\mathcal{P} = \left(\sum_{|k|\le N} (\widehat{\eta}_k(t), \widehat{w}_k(t))\, e^{ikx} \right)_{N\ge 1}$$

is a Cauchy sequence in $C([0,\infty), V^s)$. From (5.12), we obtain

$$\begin{aligned} \sup_{t\in[0,\infty)} \left\| \sum_{N\le|k|\le N+p} (\widehat{\eta}_k(t), \widehat{w}_k(t)) e^{ikx} \right\|_{V^s}^2 &= \sup_{t\in[0,\infty)} \sum_{N\le|k|\le N+p} \left(|\widehat{\eta}_k(t)|^2 + \frac{w_1}{w_2} |\widehat{w}_k(t)|^2 \right) (1+k^2)^s \\ &= \sum_{N\le|k|\le N+p} \left(|\widehat{\eta}_k^0|^2 + \frac{w_1}{w_2} |\widehat{w}_k^0|^2 \right) (1+k^2)^s. \end{aligned}$$

Thus, $\mathcal{P}$ is a Cauchy sequence in $C([0,\infty), V^s)$. Hence, the operator $S(t)$ is well-defined in V^s and $S(\cdot)(\eta^0, w^0) \in C([0,\infty), V^s)$. Moreover, since

$$\left\| \sum_{|k|\le N} (\widehat{\eta}_k(t), \widehat{w}_k(t)) e^{ikx} \right\|_{V^s}^2 = \sum_{|k|\le N} \left(|\widehat{\eta}_k^0|^2 + \frac{w_1}{w_2} |\widehat{w}_k^0|^2 \right) (1+k^2)^s,$$

we have that $(S(t))_{t\in\mathbb{R}}$ is a family of linear and continuous operators which are also isometries. It easy to see that $S(0) = I$, $S(t) \circ S(s) = S(t+s)$ for any $t, s \in \mathbb{R}$ and $\lim_{t\to 0} S(t)(\eta^0, w^0) = (\eta^0, w^0)$ in V^s. Therefore, $(S(t))_{t\in\mathbb{R}}$ is a group.

□

Let the number $\widetilde{e} \in \mathbb{Z}$ such that

$$\sqrt{w_1 w_2} \sim C|k|^{\widetilde{e}}, \quad \text{when} \quad |k| \to \infty, \tag{5.13}$$

where C is a positive constant not depending on k. Then, we have the following result:

THEOREM 5.1.2. *The infinitesimal generator of the group* $(S(t))_{t\in\mathbb{R}}$ *is the bounded operator* $(D(-\mathcal{A}), -\mathcal{A})$ *in* V^s*, where* $D(-\mathcal{A}) = V^{s+(1+\max\{-1,\widetilde{e}\})}$ *and* $\mathcal{A}$ *is given by*

$$\mathcal{A} = \begin{pmatrix} 0 & (I - b\partial_x^2 + b_2\partial_x^4)_p^{-1} (\partial_x + a\partial_x^3 + a_1\partial_x^5) \\ \\ (I - d\partial_x^2 + d_2\partial_x^4)_p^{-1} (\partial_x + c\partial_x^3 + c_1\partial_x^5) & 0 \end{pmatrix}. \tag{5.14}$$

Proof. We show that

$$\lim_{t\to 0}\frac{S(t)(\eta^0,w^0)-(\eta^0,w^0)}{t}=-\mathcal{A}(\eta^0,w^0), \tag{5.15}$$

if and only if $(\eta^0,w^0)\in V^{s+(1+\max\{-1,\widetilde{e}\})}$. This is equivalent to show that the derivative in zero of the series $\sum_{k\in\mathbb{Z}}(\widehat{\eta}_k(t),\widehat{w}_k(t))e^{ikx}$, where $(\widehat{\eta}_k(t),\widehat{w}_k(t))$ is given by (5.12), is convergent to $-\mathcal{A}(\eta^0,w^0)$ in V^s if and only if $(\eta^0,w^0)\in V^{s+(1+\max\{-1,\widetilde{e}\})}$.

If we denote by

$$\mathcal{S}_N(t)=\sum_{|k|\le N}(\widehat{\eta}_k(t),\widehat{w}_k(t))e^{ikx},$$

a partial sum of the series, a straightforward computation which takes into account (5.12), shows that

$$\left[\mathcal{S}_N\right]_t(0)=-\mathcal{A}\left(\mathcal{S}_N\right)(0). \tag{5.16}$$

Let $(D(\mathcal{B}),\mathcal{B})$ the infinitesimal generator of the group $(S(t))_{t\in\mathbb{R}}$. If $(\eta^0,w^0)\in D(\mathcal{B})$, we have that

$$\begin{aligned}\mathcal{B}(\eta^0,w^0)&=\lim_{t\to 0}\frac{S(t)(\eta^0,w^0)-(\eta^0,w^0)}{t}=\left[\sum_{k\in\mathbb{Z}}(\widehat{\eta}_k(t),\widehat{w}_k(t))e^{ikx}\right]_t(0) \\ &=\lim_{N\to\infty}\left[\mathcal{S}_N\right]_t(0)=\lim_{N\to\infty}-\mathcal{A}\left(\mathcal{S}_N\right)(0)=-\mathcal{A}(\eta^0,w^0).\end{aligned} \tag{5.17}$$

Hence, $(\eta^0,w^0)\in D(-\mathcal{A})=V^{s+(1+\max\{-1,\widetilde{e}\})}$ and $\mathcal{B}(\eta^0,w^0)=-\mathcal{A}(\eta^0,w^0)$ for any $(\eta^0,w^0)\in D(\mathcal{B})$.

On the other hand, let $(\eta^0,w^0)\in D(-\mathcal{A})=V^{s+(1+\max\{-1,\widetilde{e}\})}$. We show that the series

$$\left[\sum_{k\in\mathbb{Z}}(\widehat{\eta}_k(t),\widehat{w}_k(t))e^{ikx}\right]_t(0)$$

is convergent. This is equivalent to show that

$$\left[\mathcal{S}_N\right]_t(0)=\left[\sum_{|k|\le N}(\widehat{\eta}_k(t),\widehat{w}_k(t))e^{ikx}\right]_t(0)$$

is a Cauchy sequence. Indeed, from (5.12) we obtain that

$$\begin{aligned}\|\left[\mathcal{S}_{N+p}\right]_t(0)-\left[\mathcal{S}_N\right]_t(0)\|_{V^s}^2&=\sum_{N\le|k|\le N+p}\left(|\widehat{\eta}_{k,t}(0)|^2+\frac{w_1}{w_2}|\widehat{w}_{k,t}(0)|^2\right)(1+k^2)^s \\ &=\sum_{N\le|k|\le N+p}k^2\sigma(k)^2\left(|\widehat{\eta}_k(0)|^2+\frac{w_1}{w_2}|\widehat{w}_k(0)|^2\right)(1+k^2)^s \\ &\le C\sum_{N\le|k|\le N+p}k^{2(1+e)}\left(|\widehat{\eta}_k(0)|^2+\frac{w_1}{w_2}|\widehat{w}_k(0)|^2\right)(1+k^2)^s,\end{aligned} \tag{5.18}$$

where C is a positive constant. As $(\eta^0, w^0) \in D(-\mathcal{A}) = V^{s+(1+\max\{-1,\widetilde{e}\})}$, then

$$[\mathcal{S}_N]_t(0) = \left[\sum_{|k|\le N} (\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx}\right]_t (0)$$

is a Cauchy sequence. Thus,

$$-\mathcal{A}(\eta^0, w^0) = \lim_{N\to\infty} -\mathcal{A}(\mathcal{S}_N)(0) = \lim_{N\to\infty} [\mathcal{S}_N]_t(0) = \left[\sum_{k\in\mathbb{Z}} (\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx}\right]_t (0)$$
$$= \lim_{t\to 0} \frac{S(t)(\eta^0, w^0) - (\eta^0, w^0)}{t} = \mathcal{B}(\eta^0, w^0).$$

Hence, $(\eta^0, w^0) \in D(\mathcal{B})$ and $-\mathcal{A}(\eta^0, w^0) = \mathcal{B}(\eta^0, w^0)$ for any $(\eta^0, w^0) \in D(-\mathcal{A}) = V^{s+(1+\max\{-1,\widetilde{e}\})}$.

□

System (5.1)-(5.3) may be written in the following form

$$\begin{pmatrix} \eta \\ w \end{pmatrix}_t (t) + \mathcal{A}\begin{pmatrix} \eta \\ w \end{pmatrix}(t) = \begin{pmatrix} f^* \\ g^* \end{pmatrix}, \quad \begin{pmatrix} \eta \\ w \end{pmatrix}(0) = \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix}, \tag{5.19}$$

where $(f^*, g^*)(t) = \sum_{k\in\mathbb{Z}} \left(\frac{\widehat{f}_k(t)}{1+bk^2+b_2k^4}, \frac{\widehat{g}_k(t)}{1+dk^2+d_2k^4}\right) e^{ikx}$.

As a direct consequence of the Theorems 5.1.1 and 5.1.2 and the general theory of the evolution equations (see, for instance, [11]), we have the following existence and uniqueness result:

Theorem 5.1.3. *Let $T > 0$ and $s \in \mathbb{R}$. If $(\eta^0, w^0) \in V^s$ and $(f^*, g^*) \in L^1(0, T; V^s)$, then* (5.19) *admits a unique solution*

$$(\eta, w) \in C^1([0,T]; V^{s+(1+\max\{-1,\widetilde{e}\})}) \cap C([0,T]; V^s).$$

Moreover, there exists a positive constant $C > 0$, depending only on s, such that

$$\|(\eta, w)\|_{C([0,T];V^s)} \le C\left(\|(f^*, g^*)\|_{L^1(0,T;V^s)} + \|(\eta^0, w^0)\|_{V^s}\right). \tag{5.20}$$

5.2 Linear systems with two control inputs

In this section we study the controllability properties of the following linear system with two control inputs:

$$\begin{cases} \eta_t + w_x - b\eta_{txx} + b_2\eta_{txxxx} + aw_{xxx} + a_1 w_{xxxxx} = f(t,x) & \text{for } x \in (0,2\pi), \;\; t \in (0,T), \\ w_t + \eta_x - dw_{txx} + d_2 w_{txxxx} + c\eta_{xxx} + c_1\eta_{xxxxx} = g(t,x) & \text{for } x \in (0,2\pi), \;\; t \in (0,T), \\ \dfrac{\partial^r \eta}{\partial x^r}(t,0) = \dfrac{\partial^r \eta}{\partial x^r}(t,2\pi) & \text{for } t \in (0,T), \quad 0 \le r \le r_0, \\ \dfrac{\partial^q w}{\partial x^q}(t,0) = \dfrac{\partial^q w}{\partial x^q}(t,2\pi) & \text{for } t \in (0,T), \quad 0 \le q \le q_0, \\ \eta(0,x) = \eta^0(x), \qquad w(0,x) = w^0(x) & \text{for } x \in (0,2\pi). \end{cases} \tag{5.21}$$

We assume throughout this section that

$$b = d, \quad b_2 = d_2, \quad a = c, \quad a_1 = c_1. \tag{5.22}$$

Consider the change of variables

$$\eta = v + u \quad \text{and} \quad w = v - u.$$

In terms of these new variables, the equations in (5.21) become

$$\begin{cases} v_t + v_x - bv_{txx} + b_2 v_{txxxx} + av_{xxx} + a_1 v_{xxxxx} = f^*(t,x) & \text{for } x \in (0,2\pi), \;\; t \in (0,T), \\ u_t + u_x - bu_{txx} + b_2 u_{txxxx} + au_{xxx} + a_1 u_{xxxxx} = g^*(t,x) & \text{for } x \in (0,2\pi), \;\; t \in (0,T), \\ \dfrac{\partial^r v}{\partial x^r}(t,0) = \dfrac{\partial^r v}{\partial x^r}(t,2\pi) & \text{for } t \in (0,T), \quad 0 \le r \le r_0, \\ \dfrac{\partial^q u}{\partial x^q}(t,0) = \dfrac{\partial^q u}{\partial x^q}(t,2\pi) & \text{for } t \in (0,T), \quad 0 \le q \le q_0, \\ v(0,x) = v^0(x), \qquad u(0,x) = u^0(x) & \text{for } x \in (0,2\pi), \end{cases} \tag{5.23}$$

with

$$f^* = \frac{f+g}{2}, \quad g^* = \frac{f-g}{2}.$$

Let $a \in C_p^\infty(0,2\pi)$ with $a \neq 0$. We take $f^*(t,x)$ in (5.23) to have the following form

$$f^*(t,x) = a(x)h(t,x)$$

where

$$h(t,x) = a(x) \sum_{j=-\infty}^{\infty} f_j q_j(t) e^{ijx}, \tag{5.24}$$

with f_j and $q_j(t)$ to be determined later. Then, we have the following result that will be needed in our proofs:

Lemma 5.2.1. *Let* $m_{j,k} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} a^2(x)e^{i(j-k)x}dx, \quad j,k = \pm1,\pm2,\cdots$. *For any given finite sequence of nonzero integers* k_j, $j = 1,2,3,\ldots n$, *let*

$$A_n = \begin{pmatrix} m_{k_1,k_1} & \cdots & m_{k_1,k_n} \\ m_{k_2,k_1} & \cdots & m_{k_2,k_n} \\ \vdots & \vdots & \vdots \\ m_{k_n,k_1} & \cdots & m_{k_n,k_n} \end{pmatrix}. \tag{5.25}$$

Then, A_n *is an invertible* $n \times n$ *hermitian matrix.*

Proof. (See also [25].) Let $\alpha_j \in \mathbb{C}$, $j = 1,2,3,\ldots n$, such that

$$\sum_{j=1}^{n} \alpha_j m_{k_j,k_l} = 0, \quad \text{for} \quad l = 1,2,3,\ldots n. \tag{5.26}$$

Since

$$m_{k_j,k_l} = \frac{1}{2\pi}\left\langle a(\cdot)e^{ik_j(\cdot)}, a(\cdot)e^{ik_l(\cdot)}\right\rangle,$$

where $\langle,\rangle$ denote the inner product in the space $L^2(0,2\pi)$, from (5.26) we obtain that

$$\left\langle \sum_{j=1}^{n} \alpha_j a(\cdot)e^{ik_j(\cdot)}, a(\cdot)e^{ik_l(\cdot)} \right\rangle = 0, \quad \text{for} \quad l = 1,2,3,\ldots n.$$

Thus,

$$\left\langle \sum_{j=1}^{n} \alpha_j a(\cdot)e^{ik_j(\cdot)}, \sum_{l=1}^{n} \alpha_l a(\cdot)e^{ik_l(\cdot)} \right\rangle = 0,$$

which implies that

$$\sum_{j=1}^{n} \alpha_j a(x)e^{ik_jx} = 0.$$

Since $\left\{e^{ik_jx}\right\}_{j=1}^{n}$ form an orthonormal basis of *Span* $\left\{e^{ik_jx}\right\}$ in $L^2(0,2\pi)$ and $a \neq 0$, we get $\alpha_j = 0$, for $j = 1,2,3,\ldots n$. □

Now we return to the study of the controllability of system (5.23). Consider the following equation:

$$\begin{cases} v_t + v_x - bv_{txx} + b_2v_{txxxx} + av_{xxx} + a_1v_{xxxxx} = a(x)h & \text{for } x \in (0,2\pi), \ t \in (0,T), \\ v(0,x) = v^0(x), & \text{for } x \in (0,2\pi). \end{cases} \tag{5.27}$$

Then, the solution v of the equation in (5.27) can be written as

$$v(t,x) = \sum_{k=-\infty}^{\infty} \widehat{v}_k(t)e^{ikx},$$

where $\widehat{v}_k(t)$ solves

$$\frac{d}{dt}\widehat{v}_k(t) + ik\sigma(k)\widehat{v}_k(t) = \frac{1}{1+bk^2+b_2k^4}\sum_{j=-\infty}^{\infty} f_j q_j(t) m_{j,k}, \tag{5.28}$$

with $\sigma(k) = \dfrac{1-ak^2+a_1k^4}{1+bk^2+b_2k^4}$ and $m_{j,k}$ given by Lemma 5.2.1.

Let $\lambda_k = ik\sigma(k)$ the eigenvalues of the operator $\left((I - b\partial_x^2 + b_2\partial_x^4)_p^{-1}(\partial_x + a\partial_x^3 + a_1\partial_x^5)\right)$ and $\gamma > 0$ satisfying

$$\liminf_{k\to\infty} |\lambda_{k+1} - \lambda_k| \geq \gamma. \tag{5.29}$$

We have the following controllability result for (5.27).

Proposition 5.2.1. *Assume that the parameter $a_1 \neq 0$ and $T > \frac{2\pi}{\gamma}$, for γ given in (5.29). Let $s \in \mathbb{R}$ and n_1 given by*

$$n_1 = \begin{cases} 2, & \text{if} \quad b_2 = 0, \quad b \neq 0, \\ 0, & \text{if} \quad b_2 = b = 0, \\ 4, & \text{if} \quad b_2 \neq 0. \end{cases} \tag{5.30}$$

Then, for any given initial state $v^0 \in H_p^s(0,2\pi)$ and the terminal state $v^T \in H_p^s(0,2\pi)$, there exists a control $h \in L^2(0,T;H_p^{s-n_1}(0,2\pi))$, such that (5.27) admits a unique solution $v \in C([0,T];H_p^s(0,2\pi))$ satisfying

$$v(T,x) = v^T(x).$$

Moreover, there exists a constant $C > 0$, depending only on T and s, such that

$$\|h\|_{L^2(0,T;H_p^{s-n_1}(0,2\pi))} \leq C\left(\|v^0\|_{H_p^s(0,2\pi)} + \|v^T\|_{H_p^s(0,2\pi)}\right). \tag{5.31}$$

Proof. From (5.28) we have that

$$\widehat{v}_k(T)e^{ik\sigma(k)T} - \widehat{v}_k(0) = \frac{1}{1+bk^2+b_2k^4}\sum_{j=-\infty}^{\infty} f_j m_{j,k}\int_0^T e^{ik\sigma(k)\tau} q_j(\tau)d\tau. \tag{5.32}$$

It may occur that the eigenvalues

$$\lambda_k = ik\sigma(k), \quad k \in \mathbb{Z},$$

are not all different. If we count only the distinct values, we obtain the sequence $(\lambda_k)_{k\in\mathbb{I}}$, where $\mathbb{I} \subset \mathbb{Z}$ has the property that $\lambda_{k_1} \neq \lambda_{k_2}$ for any $k_1, k_2 \in \mathbb{I}$ with $k_1 \neq k_2$. For each $k_1 \in \mathbb{Z}$ set

$$\mathcal{I}(k_1) = \{k \in \mathbb{Z}; k\sigma(k) = k_1\sigma(k_1)\}$$

and $m(k_1) = |\mathcal{I}(k_1)|$ (the number of elements in $\mathcal{I}(k_1)$). We have the following properties of $m(k_1)$:

- $m(k_1) \le 5$. This is a consequence of the fact that $m(k_1)$ is less that the number of entire roots of the equation $x\sigma(x) = \alpha$, where α is an arbitrary real number.
- If $a_1 \neq 0$, then $k\sigma(k) \to \pm\infty$ as $k \to \pm\infty$. Hence, there exists $k^* \in \mathbb{N}$, such that $k \in \mathbb{I}$ for $|k| > k^*$. This is a consequence of the fact that the function $x\sigma(x)$ is strictly increasing for $|x|$ large enough.

Thus, there are only finite many integers in $\mathbb{I}$, saying k_j, $j = 1, \dots n$, such that one can find another integer $k \neq k_j$ with $\lambda_k = \lambda_{k_j}$. Let

$$\mathbb{I}_j = \left\{ k \in \mathbb{Z}; k \neq k_j, \lambda_k = \lambda_{k_j} \right\}, \quad j = 1, \dots n.$$

Then,

$$\mathbb{Z} = \mathbb{I} \cup \mathbb{I}_1 \cup \dots \cup \mathbb{I}_n.$$

Note that $\mathbb{I}_j$ contains at most four integers, for $m(k_j) \le 5$. We write

$$\mathbb{I}_j = \left\{ k_{j,1}, k_{j,2}, k_{j,3}, k_{j,m(k_j)-1} \right\}, \quad j = 1, \dots n$$

and rewrite k_j as $k_{j,0}$. Let

$$p_k(t) := e^{-ik\sigma(k)t}, \quad k = 0, \pm 1, \pm 2, \dots$$

Then, the set

$$\mathcal{P} := \{ p_k(t); k \in \mathbb{I} \}$$

forms a Riesz basis for $\mathcal{P}_T = \overline{\text{Span}\mathcal{P}}$, in $L^2(0, T)$ if

$$T > \frac{2\pi}{\gamma}.$$

Let $\mathcal{L} := \{ q_j(t); j \in \mathbb{I} \}$ be the unique dual Riesz basis for $\mathcal{P}$ in $\mathcal{P}_T$; that is, the functions in $\mathcal{L}$ are the unique elements of $\mathcal{P}_T$ such that

$$\langle q_j, p_k \rangle = \int_0^T q_j(t) \overline{p_k(t)} dt = \delta_{kj}, \quad k, j \in \mathbb{I}.$$

In adition, we choose

$$q_k = q_{k_{j,0}} \quad \text{if} \quad k \in \mathbb{I}_j.$$

For any $k \in \mathbb{Z}$, we obtain that:

- If $k \in \mathbb{I} \setminus \{k_1, ..., k_n\}$, then

$$\sum_{r \in \mathbb{Z}} f_r m_{r,k} \int_0^T e^{ik\sigma(k)\tau} q_r(\tau) d\tau \tag{5.33}$$

$$= \sum_{r \in \mathbb{I} \setminus \{k_1, \dots, k_n\}} f_r m_{r,k} \langle q_r, p_k \rangle + \sum_{l=1}^{n} f_{k_{l,0}} m_{k_{l,0},k} \left\langle q_{k_{l,0}}, p_k \right\rangle$$

$$+ \sum_{j=1}^{n} \sum_{l=1}^{m(k_j)-1} f_{k_{j,l}} m_{k_{j,l},k} \left\langle q_{k_{j,l}}, p_k \right\rangle$$

$$= f_k m_{k,k} + 0 + 0 = f_k m_{k,k}.$$

Thus, from (5.32) and (5.33) it follows that

$$\widehat{v}_k(T)e^{ik\sigma(k)T} - \widehat{v}_k(0) = \frac{1}{1+bk^2+b_2k^4} f_k m_{k,k} \quad \text{for} \quad k \in \mathbb{I} \setminus \{k_1, ..., k_n\}. \tag{5.34}$$

- If $k = k_{j,0}$, $j = 1, ..., n$, then

$$\begin{aligned}
&\sum_{r\in\mathbb{Z}} f_r m_{r,k_{j,0}} \int_0^T e^{ik_{j,0}\sigma(k_{j,0})\tau} q_r(\tau)d\tau \\
&= \sum_{r\in\mathbb{I}\setminus\{k_1,...,k_n\}} f_r m_{r,k_{j,0}} \left\langle q_r, p_{k_{j,0}} \right\rangle + \sum_{l=1}^{n} f_{k_{l,0}} m_{k_{l,0},k_{j,0}} \left\langle q_{k_{l,0}}, p_{k_{j,0}} \right\rangle \\
&+ \sum_{s=1}^{n} \sum_{l=1}^{m(k_s)-1} f_{k_{s,l}} m_{k_{s,l},k_{j,0}} \left\langle q_{k_{s,l}}, p_{k_{j,0}} \right\rangle \\
&= 0 + f_{k_{j,0}} m_{k_{j,0},k_{j,0}} + \sum_{l=1}^{m(k_j)-1} f_{k_{j,l}} m_{k_{j,l},k_{j,0}} = \sum_{l=0}^{m(k_j)-1} f_{k_{j,l}} m_{k_{j,l},k_{j,0}}.
\end{aligned} \tag{5.35}$$

Hence, from (5.32)and (5.35) we have that

$$\widehat{v}_{k_{j,0}}(T)e^{ik_{j,0}\sigma(k_{j,0})T} - \widehat{v}_{k_{j,0}}(0) = \frac{1}{1+bk_{j,0}^2+b_2k_{j,0}^4} \sum_{l=0}^{m(k_j)-1} f_{k_{j,l}} m_{k_{j,l},k_{j,0}}, \tag{5.36}$$

since $k = k_{j,0}$, $j = 1, ..., n$.

- Now, suppose that $k = k_{j,q}$, for $j = 1, ..., n$, and $q = 1, .., m(k_j) - 1$.
Since $k_{j,q} \in \mathbb{I}_j$, then $\lambda_{k_{j,q}} = \lambda_{k_{j,0}}$. Thus,

$$e^{i\lambda_{k_{j,q}}\sigma(\lambda_{k_{j,q}})} = e^{i\lambda_{k_{j,0}}\sigma(\lambda_{k_{j,0}})}, \quad j = 1, ..., n, \quad q = 1, .., m(k_j) - 1.$$

Therefore,

$$\begin{aligned}
&\sum_{r\in\mathbb{Z}} f_r m_{r,k_{j,q}} \int_0^T e^{ik_{j,q}\sigma(k_{j,q})\tau} q_r(\tau)d\tau = \\
&= \sum_{r\in\mathbb{I}\setminus\{k_1,...,k_n\}} f_r m_{r,k_{j,q}} \left\langle q_r, p_{k_{j,0}} \right\rangle + \sum_{l=1}^{n} f_{k_{l,0}} m_{k_{l,0},k_{j,q}} \left\langle q_{k_{l,0}}, p_{k_{j,0}} \right\rangle \\
&+ \sum_{s=1}^{n} \sum_{l=1}^{m(k_s)-1} f_{k_{s,l}} m_{k_{s,l},k_{j,q}} \left\langle q_{k_{s,l}}, p_{k_{j,q}} \right\rangle \\
&= 0 + f_{k_{j,0}} m_{k_{j,0},k_{j,q}} + \sum_{l=1}^{m(k_j)-1} f_{k_{j,l}} m_{k_{j,l},k_{j,q}} = \sum_{l=0}^{m(k_j)-1} f_{k_{j,l}} m_{k_{j,l},k_{j,q}}.
\end{aligned} \tag{5.37}$$

From (5.32) and (5.37), we deduce that

$$\widehat{v}_{k_{j,q}}(T)e^{ik_{j,0}\sigma(k_{j,0})T} - \widehat{v}_{k_{j,q}}(0) = \frac{1}{1+bk_{j,q}^2+b_2k_{j,q}^4}\sum_{l=0}^{m(k_j)-1} f_{k_{j,l}}m_{k_{j,l},k_{j,q}}, \tag{5.38}$$

since $k = k_{j,q}$, $j = 1, ..., n$, and $q = 1, .., m(k_j) - 1$.

Hence, from (5.34), (5.36) and (5.38) we have the following system

$$\begin{cases} \widehat{v}_k(T)e^{ik\sigma(k)T} - \widehat{v}_k(0) = \dfrac{1}{1+bk^2+b_2k^4} f_k m_{k,k} \quad \text{if} \quad k \in \mathbb{I}\setminus\{k_1, ..., k_n\}, \\ \\ \widehat{v}_{k_{j,q}}(T)e^{ik_{j,0}\sigma(k_{j,0})T} - \widehat{v}_{k_{j,q}}(0) = \dfrac{1}{1+bk_{j,q}^2+b_2k_{j,q}^4}\displaystyle\sum_{l=0}^{m(k_j)-1} f_{k_{j,l}}m_{k_{j,l},k_{j,q}}, \\ \\ \text{if} \quad k = k_{j,q}, \quad j = 1, ..., n, \quad \text{and} \quad q = 0, .., m(k_j) - 1. \end{cases} \tag{5.39}$$

Finally, for given initial state $v^0 = \sum_{k\in\mathbb{Z}} \widehat{v}_k^0 e^{ikx}$ and terminal state $v^T = \sum_{k\in\mathbb{Z}} \widehat{v}_k^T e^{ikx}$, with $\widehat{v}_k^0$ and $\widehat{v}_k^T$ replaced by $\widehat{v}_k(0)$ and $\widehat{v}_k(T)$, respectively, from Lemma 5.2.1, system (5.39) admits a unique solution $\overrightarrow{f}(\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots)$. Since

$$m_{k,k} = \frac{1}{2\pi}\int_0^{2\pi} a^2(x)dx =: \mu \neq 0,$$

from the first identity of (5.39), we get

$$f_k = \frac{1+bk^2+b_2k^4}{\mu}\left(\widehat{v}_k(T)e^{ik\sigma(k)T} - \widehat{v}_k(0)\right) \quad \text{if} \quad k \in \mathbb{I}\setminus\{k_1, ..., k_n\}.$$

Moreover, from the second identity of (5.39), for any $j = 1, ..., n$, it follows that

$$\mathcal{F}^j = \left[\mathcal{B}^j\right]^{-1}\mathcal{V}^j,$$

where

$$\mathcal{F}^j = \begin{pmatrix} f_{k_{j,0}} \\ f_{k_{j,1}} \\ \vdots \\ f_{k_{j,m(k_j)-1}} \end{pmatrix},$$

$$\mathcal{V}^j = \begin{pmatrix} \left(1+bk_{j,0}^2+b_2k_{j,0}^4\right)\left(\widehat{v}_{k_{j,0}}(T)e^{ik_{j,0}\sigma(k_{j,0})T}-\widehat{v}_{k_{j,0}}(0)\right) \\ \left(1+bk_{j,1}^2+b_2k_{j,1}^4\right)\left(\widehat{v}_{k_{j,1}}(T)e^{ik_{j,0}\sigma(k_{j,0})T}-\widehat{v}_{k_{j,1}}(0)\right) \\ \vdots \\ \left(1+bk_{j,m(k_j)-1}^2+b_2k_{j,m(k_j)-1}^4\right)\left(\widehat{v}_{k_{j,m(k_j)-1}}(T)e^{ik_{j,0}\sigma(k_{j,0})T}-\widehat{v}_{k_{j,m(k_j)-1}}(0)\right) \end{pmatrix},$$

$$\mathcal{B}^j = \begin{pmatrix} m_{k_{j,0},k_{j,0}} & m_{k_{j,1},k_{j,0}} \cdots & m_{k_{j,m(k_j)-1},k_{j,0}} \\ m_{k_{j,0},k_{j,1}} & m_{k_{j,1},k_{j,1}} \cdots & m_{k_{j,m(k_j)-1},k_{j,1}} \\ \vdots & \vdots & \vdots \\ m_{k_{j,0},k_{j,m(k_j)-1}} & m_{k_{j,1},k_{j,m(k_j)-1}} \cdots & m_{k_{j,m(k_j)-1},k_{j,m(k_j)-1}} \end{pmatrix}.$$

Observe that the existence of $[\mathcal{B}^j]^{-1}$ is guaranteed by Lemma 5.2.1.

Therefore, from (5.24) and (5.30), the following estimate holds

$$\|h\|^2_{L^2(0,T;H_p^{s-n_1}(0,2\pi))} = \int_0^T \|a(x)\sum_{k\in\mathbb{Z}} f_k q_k(t)e^{ikx}\|^2_{H_p^{s-n_1}(0,2\pi)}dt \tag{5.40}$$

$$\leq C\int_0^T \sum_{k\in\mathbb{Z}} |f_k q_k(t)|^2(1+k^2)^{s-n_1}dt = C\sum_{k\in\mathbb{Z}}|f_k|^2\|q_k\|^2_{L^2(0,T)}(1+k^2)^{s-n_1}$$

$$\leq C\left(\sum_{|k|\leq k^*}|f_k|^2(1+k^2)^{s-n_1}+\sum_{|k|>k^*}|f_k|^2(1+k^2)^{s-n_1}\right)$$

$$\leq C\sum_{|k|\leq k^*}\|\left[\mathcal{B}^j\right]^{-1}\|^2\left|(1+bk^2+b_2k^4)\left(\widehat{v}_k(T)e^{ik_j\sigma(k_j)T}-\widehat{v}_k(0)\right)\right|^2(1+k^2)^{s-n_1}$$

$$+C\sum_{|k|>k^*}\left|\frac{1+bk^2+b_2k^4}{\mu}\left(\widehat{v}_k(T)e^{ik\sigma(k)T}-\widehat{v}_k(0)\right)\right|^2(1+k^2)^{s-n_1}$$

$$\leq C\sum_{k\in\mathbb{Z}}|1+bk^2+b_2k^4|^2\left(|\widehat{v}_k(T)|^2+|\widehat{v}_k(0)|^2\right)(1+k^2)^{s-n_1}$$

$$\leq C\left(\|v^0\|^2_{H_p^s(0,2\pi)}+\|v^T\|^2_{H_p^s(0,2\pi)}\right).$$

□

REMARK 5.2.1. *Similarly, for given initial state u^0 and terminal state u^T in $H_p^s(0,2\pi)$, we obtain the same result for the second equation of (5.23). Indeed, choose*

$$g^*(t,x) = a^2(x)\sum_{j=-\infty}^{\infty} g_j q_{-j}(t)e^{ijx}.$$

Hence, system (5.23) *admits a unique solution* $(v(t,x),u(t,x))$ *satisfying*

$$(v(T,x),u(T,x))=(v^T(x),u^T(x)),$$

for $x\in(0,2\pi)$.

Proposition 5.2.1 and Remark 5.2.1 lead to the following controllability result for the system (5.21).

THEOREM 5.2.1. *Assume that the parameter* $a_1\neq 0$ *and* $T>\frac{2\pi}{\gamma}$, *for* γ *given in* (5.29). *Let* $s\in\mathbb{R}$ *and we define* n_1 *by*

$$n_1=\begin{cases}2, & \text{if}\quad b_2=0,\quad b\neq 0,\\ 0, & \text{if}\quad b_2=b=0,\\ 4, & \text{if}\quad b_2\neq 0.\end{cases}$$

Then, for any given initial state (η^0,w^0) *and the terminal state* (η^1,w^1) *in* $[H_p^s(0,2\pi)]^2$, *there exist* $(f,g)\in[L^2(0,T;H_p^{s-n_1}(0,2\pi))]^2$, *such that the system* (5.21) *admits a unique solution* $(\eta,w)\in[C([0,T];H_p^s(0,2\pi))]^2$ *satisfying*

$$\eta(T,\cdot)=\eta^1(\cdot)\quad\text{and}\quad w(T,\cdot)=w^1(\cdot)\quad\text{in}\quad H_p^s(0,2\pi).$$

Proof. It is immediate, since $(\eta,w)=(v+u,v-u)$ and $(f,g)=\left(\dfrac{f^*+g^*}{2},\dfrac{f^*-g^*}{2}\right)$. □

5.3 Linear systems with a single control input

In this section we study the control of the following system with a single control input:

$$\begin{cases}\eta_t+w_x+aw_{xxx}+a_1w_{xxxxx}=Qh & \text{for } x\in(0,2\pi),\ \ t\in(0,T),\\ w_t+\eta_x+c\eta_{xxx}+c_1\eta_{xxxxx}=0 & \text{for } x\in(0,2\pi),\ \ t\in(0,T),\\ \dfrac{\partial^r\eta}{\partial x^r}(t,0)=\dfrac{\partial^r\eta}{\partial x^r}(t,2\pi) & \text{for } t\in(0,T),\quad 0\le r\le r_0,\\ \dfrac{\partial^q w}{\partial x^q}(t,0)=\dfrac{\partial^q w}{\partial x^q}(t,2\pi) & \text{for } t\in(0,T),\quad 0\le q\le q_0,\\ \eta(0,x)=\eta^0(x),\qquad w(0,x)=w^0(x) & \text{for } x\in(0,2\pi),\end{cases}\tag{5.41}$$

where the operator Q is defined by

$$[Qh]\,(t,x)=q(x)h(t,x),$$

and $q\in L^2(0,2\pi)$ is a given non-negative function supported in ω, and such that $q(x)>C$ on a (nonempty) open set $\omega'\subset\omega$, $C>0$ being some constant.

The control problem will be solved by using the Hilbert Uniqueness Method (HUM) introduced by J.-L. Lions [20]. Therefore, we consider the following backward initial boundary value problem of the homogeneous adjoint system of (5.41):

$$\begin{cases} \xi_t + u_x + cu_{xxx} + c_1 u_{xxxxx} = 0 & \text{for } x \in (0, 2\pi), \;\; t \in (0, T), \\ u_t + \xi_x + a\xi_{xxx} + a_1 \xi_{xxxxx} = 0 & \text{for } x \in (0, 2\pi), \;\; t \in (0, T), \\ \dfrac{\partial^r \xi}{\partial x^r}(t, 0) = \dfrac{\partial^r \xi}{\partial x^r}(t, 2\pi) & \text{for } t \in (0, T), \quad 0 \le r \le r_0, \\ \dfrac{\partial^q u}{\partial x^q}(t, 0) = \dfrac{\partial^q u}{\partial x^q}(t, 2\pi) & \text{for } t \in (0, T), \quad 0 \le q \le q_0, \\ \xi(T, x) = \xi^T(x), \qquad u(T, x) = u^T(x) & \text{for } x \in (0, 2\pi). \end{cases} \tag{5.42}$$

Let

$$\widetilde{w}_1 = \widetilde{w}_1(k) = 1 - ck^2 + c_1 k^4, \quad \widetilde{w}_2 = \widetilde{w}_2(k) = 1 - ak^2 + a_1 k^4.$$

We note that the eigenvalues of the state operator are given by $\lambda_k = ik\sigma(k)$, where $\sigma(k) = \sqrt{w_1 w_2}$. Then, if $c_1 \neq 0$ and $a_1 \neq 0$, it follows that there exists $\gamma > 0$, such that

$$\liminf_{k \to \infty} |\lambda_{k+1} - \lambda_k| \ge \gamma. \tag{5.43}$$

Introduce the number $\widetilde{l} \in \mathbb{Z}$ with the property that

$$\sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_2}} \sim C|k|^{\widetilde{l}}, \quad \text{as} \quad |k| \to \infty,$$

where C is a positive constant not depending on k. For each $s \in \mathbb{R}$, we define the space

$$\widetilde{V}^s = H_p^s(0, 2\pi) \times H_p^{s+\widetilde{l}}(0, 2\pi),$$

endowed with the inner product defined by

$$\langle (f_1, f_2), (g_1, g_2) \rangle = (f_1, g_1)_s + (\widetilde{\mathcal{H}} f_2, \widetilde{\mathcal{H}} g_2)_s,$$

and the operator $\mathcal{H}$ is defined in the following way

$$\widetilde{\mathcal{H}} \left(\sum_{k \in \mathbb{Z}} \widehat{a}_k e^{ikx} \right) = \sum_{k \in \mathbb{Z}} \sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_2}} \widehat{a}_k e^{ikx}.$$

Letting $t' = T - t$, $x' = 2\pi - x$, one can easily see that (5.42) is (5.1)-(5.3) with f and g being zero and a and a_1 changed by c and c_1, respectively. Consequently, we have the following result.

THEOREM 5.3.1. *Let $T>0$ and $s\in\mathbb{R}$. If $(\xi^T,u^T)\in\widetilde{V}^s$, then* (5.42) *admits a unique solution (ξ,u) in $C^1([0,T];\widetilde{V}^{s+(1+\max\{-1,\widetilde{e}\})})\cap C\left([0,T];\widetilde{V}^s\right)$. Moreover, there exists a positive constant $C>0$, depending only on s, such that*

$$\|((\xi,u)\|_{C\left([0,T];\widetilde{V}^s\right)}\leq C\|(\xi^T,u^T)\|_{\widetilde{V}^s}. \tag{5.44}$$

If

$$(\xi^T,u^T)=\sum_{k\in\mathbb{Z}}(\widehat{\xi}^T_k,\widehat{u}^T_k)e^{ikx},$$

then the solution of (5.42) *may be written as*

$$(\xi,u)(t,x)=\sum_{k\in\mathbb{Z}}(\widehat{\xi}_k(t),\widehat{u}_k(t))e^{ikx},$$

where

$$\begin{cases} \widehat{\xi}_k(t) &= \frac{1}{2}\left(\widehat{\xi}^T_{-k}-\sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_2}}\widehat{u}^T_{-k}\right)e^{ik\sigma(k)(T-t)}+\frac{1}{2}\left(\widehat{\xi}^T_{-k}+\sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_2}}\widehat{u}^T_{-k}\right)e^{-ik\sigma(k)(T-t)}, \\ \widehat{u}_k(t) &= \frac{1}{2}\left(\widehat{u}^T_{-k}-\sqrt{\frac{\widetilde{w}_2}{\widetilde{w}_1}}\widehat{\xi}^T_{-k}\right)e^{ik\sigma(k)(T-t)}+\frac{1}{2}\left(\widehat{u}^T_{-k}+\sqrt{\frac{\widetilde{w}_2}{\widetilde{w}_1}}\widehat{\xi}^T_{-k}\right)e^{-ik\sigma(k)(T-t)}. \end{cases} \tag{5.45}$$

If we define

$$\begin{cases} \mathcal{V}=\left\{(\eta,w)\in L^2(0,2\pi)\times H^l_p(0,2\pi):\widehat{w}_0=0\right\}, \\ \widetilde{V}^0_{*,0}=\left\{(\xi,u)\in L^2(0,2\pi)\times H^{\widetilde{l}}_p(0,2\pi):\widehat{u}_0=0\right\}, \end{cases} \tag{5.46}$$

we obtain that $\mathcal{V}$ and $\widetilde{V}^0_{*,0}$ are closed subspaces of V^0 and $\widetilde{V}^0$, respectively. Then, from Theorem 5.1.1, the group $(S(t))_{t\in\mathbb{R}}$ is well-defined in those spaces.

We also define the duality product

$$\langle(\eta,w),(\xi,u)\rangle_{\mathcal{V},\widetilde{V}^0_{*,0}}=\sum_{k\in\mathbb{Z}}\left(\widehat{\eta}_k\overline{\widehat{\xi}_k}+\widehat{w}_k\overline{\widehat{u}_k}\right)=\int_0^{2\pi}\eta(x)\overline{\xi(x)}dx+\langle w,u\rangle_l, \tag{5.47}$$

that will play on important role in our approach. Taking (5.47) into account, the following proposition presents an equivalent condition for the controllability of (5.41).

PROPOSITION 5.3.1. *The initial data $(\eta^0,w^0)\in\mathcal{V}$ is controllable to zero in time $T>0$ with control $h\in L^2((0,T)\times(0,2\pi))$ if and only if*

$$\left\langle(\eta^0,w^0),(\xi(0),u(0))\right\rangle_{\mathcal{V},\widetilde{V}^0_{*,0}}+\int_0^T\int_0^{2\pi}[Qh](t,x)\overline{\xi(t,x)}dxdt=0, \tag{5.48}$$

for any $(\xi^T,u^T)\in\widetilde{V}^0_{*,0}$, where (ξ,u) is the solution of (5.42).

Proof. We first prove (5.48) for regular data. Multiplying the first and the second equation in (5.41) by $\overline{\xi}$ and $\overline{u}$ respectively, integrating by parts over the domain $(0,T)\times(0,2\pi)$ and adding the resulting relations we have that

$$\int_0^T\int_0^{2\pi}[Qh](t,x)\overline{\xi(t,x)}dxdt=\int_0^T\int_0^{2\pi}[\eta_t+w_x+aw_{xxx}+a_1w_{xxxxx}]\,\overline{\xi}dxdt$$
$$+\int_0^T\int_0^{2\pi}[w_t+\eta_x+c\eta_{xxx}+c_1\eta_{xxxxx}]\,\overline{u}dxdt$$
$$=\int_0^{2\pi}\left[\eta\overline{\xi}+w\overline{u}\right]_0^T dx=\left\langle(\eta(T),w(T)),(\xi^T,u^T)\right\rangle_{\mathcal{V},\widetilde{V}^0_{*,0}}-\left\langle(\eta^0,w^0),(\xi(0),u(0))\right\rangle_{\mathcal{V},\widetilde{V}^0_{*,0}}.$$

Hence, by a density argument we conclude that $(\eta^0,w^0)\in\mathcal{V}$ is controllable to zero in time $T>0$ if and only if (5.48) holds.

□

The variational equality (5.48) has a solution if and only if there exists a constant $C>0$, such that the following observation inequality holds for any $(\xi^T,u^T)\in\widetilde{V}^0_{*,0}$

$$\|(\xi(0),u(0))\|^2_{\widetilde{V}^0_{*,0}}\le C\int_0^T\int_0^{2\pi}[Q\xi](t,x)\overline{\xi(t,x)}dxdt. \tag{5.49}$$

Then, we have the following result:

THEOREM 5.3.2. *There exist a time $T>0$ and a constant $C>0$, such that, for any $(\xi^T,u^T)\in\widetilde{V}^0_{*,0}$, the corresponding solution (ξ,u) of (5.42) satisfies the inequality*

$$\|(\xi^T,u^T)\|^2_{\widetilde{V}^0_{*,0}}\le C\int_0^T\int_{\omega'}|\xi(t,x)|^2dxdt. \tag{5.50}$$

Proof. Let $(\xi^T,u^T)=\sum_{k\in\mathbb{Z}}(\widehat{\xi}^T_k,\widehat{u}^T_k)e^{ikx}$. The corresponding solution of (5.42) is given by

$$(\xi,u)(t,x)=\sum_{k\in\mathbb{Z}}(\widehat{\xi}_k(t),\widehat{u}_k(t))e^{ikx},$$

with $(\widehat{\xi}_k(t),\widehat{u}_k(t))$ satisfying (5.45). On the other hand,

$$\xi(t,x)=\frac{1}{2}\left[\sum_{k\in\mathbb{Z}}\left(\widehat{\xi}^T_{-k}-\sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_2}}\widehat{u}^T_{-k}\right)e^{ik\sigma(k)(T-t)}e^{ikx}+\sum_{k\in\mathbb{Z}}\left(\widehat{\xi}^T_{-k}+\sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_2}}\widehat{u}^T_{-k}\right)e^{-ik\sigma(k)(T-t)}e^{ikx}\right]$$
$$=\frac{1}{2}\sum_{k\in\mathbb{Z}}\left[\left(\widehat{\xi}^T_{-k}-\sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_2}}\widehat{u}^T_{-k}\right)e^{ikx}+\left(\widehat{\xi}^T_{k}+\sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_2}}\widehat{u}^T_{k}\right)e^{-ikx}\right]e^{ik\sigma(k)(T-t)}$$
$$=\frac{1}{2}\sum_{k_1\in\mathbb{I}}\sum_{k\in\mathcal{I}(k_1)}\left[\left(\widehat{\xi}^T_{-k}-\sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_2}}\widehat{u}^T_{-k}\right)e^{ikx}+\left(\widehat{\xi}^T_{k}+\sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_2}}\widehat{u}^T_{k}\right)e^{-ikx}\right]e^{ik_1\sigma(k_1)(T-t)},$$

where $\mathbb{I} \subset \mathbb{Z}$ has the property that $\lambda_{k_1} \neq \lambda_{k_2}$ for any $k_1, k_2 \in \mathbb{I}$ with $k_1 \neq k_2$, and for each $k_1 \in \mathbb{Z}$

$$\mathcal{I}(k_1) = \{k \in \mathbb{Z}; k\sigma(k) = k_1\sigma(k_1)\}$$

and $m(k_1) = |\mathcal{I}(k_1)|$. By using a generalization of Inghams's inequality (see [2] and [18]), from (5.43), we deduce that, for any $T > \frac{2\pi}{\gamma}$, there exists a constant $C > 0$, such that

$$\int_{\Omega'} \int_0^T |\xi(t,x)|^2 dt dx \geq C \int_{\Omega'} \sum_{k_1 \in \mathbb{I}} \left| \sum_{k \in \mathcal{I}(k_1)} \left(a_k e^{-ikx} + b_k e^{ikx} \right) \right|^2 dx, \tag{5.51}$$

where $a_k = \widehat{\xi}_k^T + \sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_2}} \widehat{u}_k^T, \quad b_k = \widehat{\xi}_{-k}^T - \sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_2}} \widehat{u}_{-k}^T$. As, $\liminf_{|k|\to\infty} |\lambda_k| = \infty$, there exists $k^* \in \mathbb{N}$, large enough, such that $m(k_1) = 1$ for all $|k_1| > k^*$. Hence, we obtain that

$$\sum_{k \in \mathcal{I}(k_1)} \left(a_k e^{-ikx} + b_k e^{ikx} \right) = a_{k_1} e^{-ik_1 x} + b_{k_1} e^{ik_1 x}, \quad \text{for all} \quad |k_1| > k^*.$$

Thus, if we set $\Omega' = (\alpha, \beta)$, the following holds:

$$\begin{aligned}
&\int_{\Omega'} \sum_{\substack{k_1 \in \mathbb{I} \\ |k_1| > k^*}} \left| \sum_{k \in \mathcal{I}(k_1)} \left(a_k e^{-ikx} + b_k e^{ikx} \right) \right|^2 dx = \int_{\Omega'} \sum_{\substack{k_1 \in \mathbb{I} \\ |k_1| > k^*}} \left| a_{k_1} e^{-ik_1 x} + b_{k_1} e^{ik_1 x} \right|^2 dx \qquad (5.52) \\
&= \sum_{\substack{k_1 \in \mathbb{I} \\ |k_1| > k^*}} \left[\int_{\Omega'} \left(|a_{k_1} e^{-ik_1 x}|^2 + |b_{k_1} e^{ik_1 x}|^2 \right) dx + 2 \int_\alpha^\beta \Re \left(\overline{a_{k_1}} b_{k_1} e^{2ik_1 x} \right) dx \right] \\
&= \sum_{\substack{k_1 \in \mathbb{I} \\ |k_1| > k^*}} \left\{ |\Omega'| \left(|a_{k_1}|^2 + |b_{k_1}|^2 \right) + 2\Re \left[\overline{a_{k_1}} b_{k_1} \left(\frac{e^{2ik_1\beta} - e^{2ik_1\alpha}}{2ik_1} \right) \right] \right\} \\
&\geq \sum_{\substack{k_1 \in \mathbb{I} \\ |k_1| > k^*}} \left[|\Omega'| \left(|a_{k_1}|^2 + |b_{k_1}|^2 \right) - \frac{1}{k^*} \left(|a_{k_1}|^2 + |b_{k_1}|^2 \right) \right] \geq C \sum_{\substack{k_1 \in \mathbb{I} \\ |k_1| > k^*}} \left(|a_{k_1}|^2 + |b_{k_1}|^2 \right) \\
&= C \sum_{\substack{k_1 \in \mathbb{I} \\ |k_1| > k^*}} \sum_{k \in \mathcal{I}(k_1)} \left(|a_k|^2 + |b_k|^2 \right),
\end{aligned}$$

for some positive constant $C > 0$. For $|k_1| \leq k^*$, let us consider the seminorm on the sequences of numbers $(a_k, b_k)_{k \in \mathcal{I}(k_1)}$ given by

$$\left| (a_k, b_k)_{k \in \mathcal{I}(k_1)} \right|_* = \left[\int_{\Omega'} \left| \sum_{k \in \mathcal{I}(k_1)} \left(a_k e^{-ikx} + b_k e^{ikx} \right) \right|^2 dx \right]^{1/2}. \tag{5.53}$$

Observe that (5.53) is a norm. Indeed, suppose that $\left|(a_k, b_k)_{k\in\mathcal{I}(k_1)}\right|_* = 0$. Then,

$$\sum_{k\in\mathcal{I}(k_1)} \left(a_k e^{-ikx} + b_k e^{ikx}\right) = 0 \quad \text{in} \quad \Omega',$$

and by analytic continuation, we have that

$$\sum_{k\in\mathcal{I}(k_1)} \left(a_k e^{-ikx} + b_k e^{ikx}\right) = 0 \quad \text{in} \quad [0, 2\pi].$$

By using the orthogonality of $\left\{e^{ikx}\right\}$ and $\left\{e^{-ikx}\right\}$ in $[0, 2\pi]$, we deduce that $a_k = b_k = 0$, for any $k \in \mathcal{I}(k_1)$. Therefore, we obtain that $\left|(,)\right|_*$ is a norm on the sequences of numbers $(a_k, b_k)_{k\in\mathcal{I}(k_1)}$. Moreover, as $m(k_1) = |\mathcal{I}(k_1)| \leq 10$ and $|k_1| \leq k^*$, we obtain that there exists a constant $C > 0$, such that

$$C \sum_{\substack{k_1\in\mathbb{I}\\|k_1|\leq k^*}} \sum_{k\in\mathcal{I}(k_1)} \left(|a_k|^2 + |b_k|^2\right) = C \sum_{\substack{k_1\in\mathbb{I}\\|k_1|\leq k^*}} \left\|(a_k, b_k)_{k\in\mathcal{I}(k_1)}\right\|^2 \tag{5.54}$$

$$\leq \sum_{\substack{k_1\in\mathbb{I}\\|k_1|\leq k^*}} \left|(a_k, b_k)_{k\in\mathcal{I}(k_1)}\right|_*^2 = \sum_{\substack{k_1\in\mathbb{I}\\|k_1|\leq k^*}} \int_{\Omega'} \left|\sum_{k\in\mathcal{I}(k_1)} \left(a_k e^{-ikx} + b_k e^{ikx}\right)\right|^2 dx.$$

From (5.51)-(5.54), we have that there exists a constant $C > 0$, such that

$$\int_{\Omega'}\int_0^T |\xi(t,x)|^2 dt dx \geq C \sum_{k_1\in\mathbb{I}} \sum_{k\in\mathcal{I}(k_1)} \left(|a_k|^2 + |b_k|^2\right) \tag{5.55}$$

$$= C \sum_{k\in\mathbb{Z}} \left(|\widehat{\xi}_k^T + \sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_2}} \widehat{u}_k^T|^2 + |\widehat{\xi}_{-k}^T - \sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_2}} \widehat{u}_{-k}^T|^2 \right)$$

$$\geq C \sum_{k\in\mathbb{Z}} \left(|\widehat{\xi}_k^T + \sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_2}} \widehat{u}_k^T|^2 + |\widehat{\xi}_k^T - \sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_2}} \widehat{u}_k^T|^2 \right)$$

$$\geq C \sum_{k\in\mathbb{Z}} \left(|\widehat{\xi}_k^T|^2 + \frac{\widetilde{w}_1}{\widetilde{w}_2} |\widehat{u}_k^T|^2 \right) = C\|(\xi^T, u^T\|_{\widetilde{V}_{*,0}^0}^2$$

□

Since system (5.41) is conservative and time reversible, the null controllability is equivalent to the exact controllability. Moreover, since $(S(t))_{t\in\mathbb{R}}$ is a group of isometries in $\widetilde{V}_{*,0}^0$, from Theorem 5.3.2 and the definition of the operator Q we obtain (5.49). Hence, the following result holds:

THEOREM 5.3.3. *There exists a time $T > 0$, such that, for given*

$$(\eta^0, w^0) \in \mathcal{V}, \quad (\eta^T, w^T) \in \mathcal{V},$$

one can find a control input $h \in L^2((0,T) \times (0,2\pi))$*, such that* (5.41) *admits a unique solution*

$$(\eta, w) \in C([0,T]; \mathcal{V})$$

satisfying

$$(\eta(0,\cdot), w(0,\cdot)) = (\eta^0, w^0) \quad (\eta(T,\cdot), w(T,\cdot)) = (\eta^T, w^T) \quad \textit{in} \quad \mathcal{V}.$$

Moreover, there exists a constant $C > 0$*, such that*

$$\|h\|_{L^2((0,T)\times(0,2\pi))} \leq C \left(\|(\eta^0, w^0)\|_{\mathcal{V}} + \|(\eta^T, w^T)\|_{\mathcal{V}} \right).$$

Chapter 6

Appendix

The results presented in this section were obtained in [26]. For the sake of completeness, they are included in this work.

6.1 Study of some initial value problems

This section is devoted to present some explicit formulae and properties of a family of initial value problems depending on several parameters. These results allow us to obtain the asymptotic behavior of the eigenvalues and eigenfunctions of the differential operator associated to (1.3) in Chapter 1. Firstly, we study the properties of the following simple initial value problem, where $\sigma \in \mathbb{C}^*$ is a complex nonzero parameter:

$$\begin{cases} -b\varphi_{xx} + \sigma_1 v_x = f, \quad x \in (0, 2\pi) \\ -dv_{xx} + \sigma_1 \varphi_x = g, \quad x \in (0, 2\pi) \\ \varphi(0) = \varphi^0, \ \varphi_x(0) = \varphi^1 \\ v(0) = v^0, \ v_x(0) = v^1. \end{cases} \tag{6.1}$$

In (6.1) and in the remaining part of the thesis b and d denote two positive real numbers. We have the following result.

LEMMA 6.1.1. *Given* $\begin{pmatrix} \varphi^0 \\ \varphi^1 \\ v^0 \\ v^1 \end{pmatrix} \in \mathbb{C}^4$ *and* $\begin{pmatrix} f \\ g \end{pmatrix} \in (L^2(0, 2\pi))^2$ *there exists a unique* *solution* $\begin{pmatrix} \varphi \\ v \end{pmatrix}$ *of problem* (6.1) *given by the following formula*

$$\begin{pmatrix} \varphi(x) \\ \\ v(x) \end{pmatrix} = \begin{pmatrix} \varphi^0 + \dfrac{\sqrt{bd}}{\sigma}\sinh\left(\dfrac{\sigma x}{\sqrt{bd}}\right)\varphi^1 + \dfrac{d}{\sigma}\left(\cosh\left(\dfrac{\sigma x}{\sqrt{bd}}\right) - 1\right)v^1 \\ \\ v^0 + \dfrac{b}{\sigma}\left(\cosh\left(\dfrac{\sigma x}{\sqrt{bd}}\right) - 1\right)\varphi^1 + \dfrac{\sqrt{bd}}{\sigma}\sinh\left(\dfrac{\sigma x}{\sqrt{bd}}\right)v^1 \end{pmatrix} -$$

$$- \begin{pmatrix} \dfrac{1}{\sigma}\displaystyle\int_0^x \left[\sqrt{\frac{d}{b}}\sinh\left(\frac{\sigma(x-s)}{\sqrt{bd}}\right)f(s) + \left(\cosh\left(\frac{\sigma(x-s)}{\sqrt{bd}}\right) - 1\right)g(s)\right]\mathrm{d}s \\ \\ \dfrac{1}{\sigma}\displaystyle\int_0^x \left[\left(\cosh\left(\frac{\sigma(x-s)}{\sqrt{bd}}\right) - 1\right)f(s) + \sqrt{\frac{b}{d}}\sinh\left(\frac{\sigma(x-s)}{\sqrt{bd}}\right)g(s)\right]\mathrm{d}s \end{pmatrix}. \tag{6.2}$$

In the remaining part of the thesis C denotes a positive constant that may change from one line to another, but it is independent of the parameter σ and the initial data. We define the set

$$Z = \left\{ z \in \mathbb{C} \,:\, |z| \geq \frac{1}{2},\ |\Re(z)| \leq 1 \right\}, \tag{6.3}$$

and we show that the following estimates for the solution $\begin{pmatrix} \varphi \\ v \end{pmatrix}$ of (6.1) hold if $\sigma \in Z$.

Lemma 6.1.2. *Let* $\begin{pmatrix} \varphi \\ v \end{pmatrix}$ *be the solution of* (6.1)*. There exists a positive constant* $C > 0$ *such that the following estimates hold for any* $x \in (0, 2\pi)$ *and* $\sigma \in Z$*:*

$$\begin{aligned} |\varphi(x)| &\leq |\varphi^0| + \frac{C}{|\sigma|}\left[|\varphi^1| + |v^1| + \int_0^x |f(s)|\,\mathrm{d}s + \int_0^x |g(s)|\,\mathrm{d}s\right], \\ |v(x)| &\leq |v^0| + \frac{C}{|\sigma|}\left[|\varphi^1| + |v^1| + \int_0^x |f(s)|\,\mathrm{d}s + \int_0^x |g(s)|\,\mathrm{d}s\right], \\ \max\{|\varphi_x(x)|, |v_x(x)|\} &\leq C\left[|\varphi^1| + |v^1| + \int_0^x |f(s)|\,\mathrm{d}s + \int_0^x |g(s)|\,\mathrm{d}s\right]. \end{aligned} \tag{6.4}$$

Now, we consider system (6.1) with $f \equiv g \equiv 0$

$$\begin{cases} -b\varphi_{xx} + \sigma v_x = 0, & x \in (0, 2\pi) \\ -dv_{xx} + \sigma\varphi_x = 0, & x \in (0, 2\pi) \\ \varphi(0) = \varphi^0,\ \ v(0) = v^0 \\ \varphi_x(0) = \varphi^1,\ \ v_x(0) = v^1, \end{cases} \tag{6.5}$$

and the following system

$$\begin{cases} \xi - b\xi_{xx} + \sigma\zeta_x = 0, & x \in (0, 2\pi) \\ \zeta - d\zeta_{xx} + \sigma\xi_x = 0, & x \in (0, 2\pi) \\ \xi(0) = \xi^0,\ \xi_x(0) = \xi^1, \\ \zeta(0) = \zeta^0,\ \zeta_x(0) = \zeta^1, \end{cases} \tag{6.6}$$

for which we have he following result:

PROPOSITION 6.1.1. *There exists a positive constant $C > 0$ such that, for any $\sigma \in Z$ and any $\begin{pmatrix} \xi \\ \zeta \end{pmatrix}$ and $\begin{pmatrix} \varphi \\ v \end{pmatrix}$ solutions of* (6.6) *and* (6.5), *respectively, with the same initial condition $\begin{pmatrix} \xi_0 \\ \xi_1 \\ \zeta_0 \\ \zeta_1 \end{pmatrix}$, the following estimate holds*

$$\|\xi - \varphi\|_{L^\infty} + \|\zeta - v\|_{L^\infty} \leq \frac{C}{|\sigma|}\left[|\xi_0| + |\zeta_0| + \frac{1}{|\sigma|}(|\xi_1| + |\zeta_1|)\right]. \tag{6.7}$$

Finally, the difference between the solutions of (6.6) and (6.1) are given by the following result.

PROPOSITION 6.1.2. *Let $\begin{pmatrix} \varphi \\ v \end{pmatrix}$ and $\begin{pmatrix} \xi \\ \zeta \end{pmatrix}$ solutions of* (6.1) *and* (6.6), *respectively, with $f \equiv g \equiv 0$. Then, there exists a positive constant $C > 0$, such that*

$$\begin{aligned} |\xi_x(x) - \varphi_x(x)| + |\zeta_x(x) - v_x(x)| \leq \frac{C}{|\sigma_1|}(|\xi^1| + |\zeta^1|) + C|\sigma_1 - \sigma|\left(|\varphi^1| + |v^1|\right) + \\ + C\left[|\xi^1 - \varphi^1| + |\zeta^1 - v^1| + |\xi^0| + |\zeta^0|\right]. \end{aligned} \tag{6.8}$$

6.2 Spectral analysis of the operator $\mathcal{A}$ introduced in the Chapter 2

Given $b, d > 0$, we define the operators $\mathcal{A},\ \mathcal{B} : (H_0^1(0, 2\pi))^2 \to (H_0^1(0, 2\pi))^2$ given by

$$\mathcal{A} = -\begin{pmatrix} 0 & \left(I - b\partial_x^2\right)^{-1}\partial_x \\ \left(I - d\partial_x^2\right)^{-1}\partial_x & 0 \end{pmatrix},\ \mathcal{B} = -\begin{pmatrix} 0 & \left(-b\partial_x^2\right)^{-1}\partial_x \\ \left(-d\partial_x^2\right)^{-1}\partial_x & 0 \end{pmatrix}. \tag{6.9}$$

Recall that, for $\alpha > 0$, the operator $(I - \alpha\partial_x^2)^{-1}$ is defined in the following way:

$$(I - \alpha\partial_x^2)^{-1}\varphi = v \Leftrightarrow \begin{cases} v - \alpha v_{xx} = \varphi \\ v(0) = v(2\pi) = 0. \end{cases}$$

Then, if $\varphi \in L^2(0,2\pi)$, we have that there exists a unique $v \in H^2(0,2\pi) \cap H_0^1(0,2\pi)$ verifying the above equation and $(I-\alpha\partial_x^2)^{-1} : L^2(0,2\pi) \to L^2(0,2\pi)$ is a well-defined, compact operator. Similarly, for $\alpha > 0$, the operator $(-\alpha\partial_x^2)^{-1} : L^2(0,2\pi) \to L^2(0,2\pi)$ defined by

$$(-\alpha\partial_x^2)^{-1}\varphi = v \Leftrightarrow \begin{cases} -\alpha v_{xx} = \varphi \\ v(0) = v(2\pi) = 0, \end{cases}$$

is a well-defined, compact operator in $L^2(0,2\pi)$.

In this section and the rest of the paper, $\mu \in \mathbb{C}$ is called *eigenvalue of the operator* $\mathcal{A}$ *(*$\mathcal{B}$*)* if there exists a nontrivial vector $\Phi = \begin{pmatrix} \varphi \\ v \end{pmatrix} \in (H_0^1(0,2\pi))^2$, called *eigenfunction corresponding to* μ, such that $\mu\mathcal{A}\Phi = \Phi$ ($\mu\mathcal{B}\Phi = \Phi$, respectively). The following two theorems are devoted to the spectral analysis of theses operators.

Theorem 6.2.1. *The eigenvalues of the operator* $\mathcal{B}$ *are* $\widetilde{\mu}_n = \sqrt{bd}n\,i$ *with* $n \in \mathbb{Z}^*$. *Each eigenvalue* $\widetilde{\mu}_n$ *is double and has two independent eigenfunctions given by*

$$\widetilde{\Phi}_n^1 = \frac{b}{\widetilde{\mu}_n}\begin{pmatrix} \sqrt{\frac{d}{b}}\sinh\left(\frac{\widetilde{\mu}_n x}{\sqrt{bd}}\right) \\ \cosh\left(\frac{\widetilde{\mu}_n x}{\sqrt{bd}}\right) - 1 \end{pmatrix}, \; \widetilde{\Phi}_n^2 = \frac{d}{\widetilde{\mu}_n}\begin{pmatrix} \cosh\left(\frac{\widetilde{\mu}_n x}{\sqrt{bd}}\right) - 1 \\ \sqrt{\frac{b}{d}}\sinh\left(\frac{\widetilde{\mu}_n x}{\sqrt{bd}}\right) \end{pmatrix} \quad (n \in \mathbb{Z}^*). \tag{6.10}$$

Moreover, the sequence $(\widetilde{\Phi}_n^j)_{n\in\mathbb{Z}^*, j\in\{1,2\}}$ *forms an orthonormal basis of* $(H_0^1(0,2\pi))^2$.

Proof. By using Lemma 6.1.1, with $\varphi^0 = v^0 = 0$ and $f = g = 0$, we deduce that $\begin{pmatrix} \varphi \\ v \end{pmatrix}$ is an eigenfunction of $\mathcal{B}$ corresponding to the eigenvalue μ if and only if

$$\begin{pmatrix} \varphi(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{bd}}{\mu}\sinh\left(\frac{\mu x}{\sqrt{bd}}\right)\varphi^1 + \frac{d}{\mu}\left(\cosh\left(\frac{\mu x}{\sqrt{bd}}\right) - 1\right)v^1 \\ \frac{b}{\mu}\left(\cosh\left(\frac{\mu x}{\sqrt{bd}}\right) - 1\right)\varphi^1 + \frac{\sqrt{bd}}{\mu}\sinh\left(\frac{\mu x}{\sqrt{bd}}\right)v^1 \end{pmatrix}, \tag{6.11}$$

and

$$\begin{vmatrix} \frac{\sqrt{bd}}{\mu}\sinh\left(\frac{2\pi\mu}{\sqrt{bd}}\right) & \frac{d}{\mu}\left(\cosh\left(\frac{2\pi\mu}{\sqrt{bd}}\right) - 1\right) \\ \frac{b}{\mu}\left(\cosh\left(\frac{2\pi\mu}{\sqrt{bd}}\right) - 1\right) & \frac{\sqrt{bd}}{\mu}\sinh\left(\frac{2\pi\mu}{\sqrt{bd}}\right) \end{vmatrix} = 0. \tag{6.12}$$

From (6.12) it follows that the eigenvalues $(\widetilde{\mu}_n)_{n\in\mathbb{Z}^*}$ are the roots of the equation

$$\exp\left(\frac{2\pi\mu}{\sqrt{bd}}\right) = 1$$

and $\widetilde{\Phi}_n^1$ and $\widetilde{\Phi}_n^2$ given by (6.10) are two independent eigenfunctions corresponding to $\widetilde{\mu}_n$. $\square$

We pass to analyze the spectral properties of the operator $\mathcal{A}$. The main difference with respect to $\mathcal{B}$ is that we do not have an explicit representation formula as (6.10) for the eigenfunctions of $\mathcal{A}$. In order to complete the task, we use a strategy which combines two dimensional versions of the shooting method and Rouché's Theorem.

THEOREM 6.2.2. *The eigenvalues of the operator $\mathcal{A}$ are purely imaginary numbers $(\mu_n^j)_{n\in\mathbb{Z}^*,j\in\{1,2\}}$ with the property that*

$$\mu_n^j = \widetilde{\mu}_n + \mathcal{O}\left(\frac{1}{|n|}\right) \qquad (n\in\mathbb{Z}^*,\ j\in\{1,2\}). \tag{6.13}$$

Moreover, to each eigenvalue μ_n^j corresponds an eigenfunction Φ_n^j given by

$$\Phi_n^j = \widetilde{\Phi}_n^j + \mathcal{O}\left(\frac{1}{|n|^2}\right) \qquad (n\in\mathbb{Z}^*,\ j\in\{1,2\}), \tag{6.14}$$

with the property that the sequence $(\Phi_n^j)_{n\in\mathbb{Z}^*,j\in\{1,2\}}$ forms an orthogonal basis of $(H_0^1(0,2\pi))^2$.

Proof. Let us first remark that $\mathcal{A}$ is a compact skew-adjoint operator in $(H_0^1(0,2\pi))^2$. Indeed, this follow immediately by taking into account the definition of $\mathcal{A}$ in (6.9) and that the following relations hold for any $\varphi_j, v_j\in\mathcal{D}(0,2\pi)$ and $j=1,2$,

$$\left\langle \mathcal{A}\begin{pmatrix}\varphi_1\\ v_1\end{pmatrix}, \begin{pmatrix}\varphi_2\\ v_2\end{pmatrix}\right\rangle_{H_0^1} = -\int_0^{2\pi}\partial_x(I-b\partial_x^2)^{-1}v_{1,x}\varphi_{2,x}\,\mathrm{d}x - \int_0^{2\pi}\partial_x(I-d\partial_x^2)^{-1}\varphi_{1,x}v_{2,x}\,\mathrm{d}x$$

$$= \int_0^{2\pi}(I-b\partial_x^2)^{-1}v_{1,x}\varphi_{2,xx}\,\mathrm{d}x + \int_0^{2\pi}(I-d\partial_x^2)^{-1}\varphi_{1,x}v_{2,xx}\,\mathrm{d}x$$

$$= \int_0^{2\pi}v_{1,x}\partial_x(I-b\partial_x^2)^{-1}\varphi_{2,x}\,\mathrm{d}x + \int_0^{2\pi}\varphi_{1,x}\partial_x(I-d\partial_x^2)^{-1}v_{2,x}\,\mathrm{d}x = -\left\langle \begin{pmatrix}\varphi_1\\ v_1\end{pmatrix}, \mathcal{A}\begin{pmatrix}\varphi_2\\ v_2\end{pmatrix}\right\rangle_{H_0^1}.$$

It follows that $\mathcal{A}$ has a sequence of purely imaginary eigenvalues tending to infinity. In order to localize these eigenvalues, let us define, for given $\delta>0$ and $N\in\mathbb{N}$, the sets

$$D_n(\delta) = \left\{(\mu,\gamma)\in\mathbb{C}^2 \;\middle|\; |\mu-\widetilde{\mu}_n|^2+|\gamma|^2\le\frac{\delta^2}{n^2}\right\},\quad \Gamma_n(\delta)=\partial D_n(\delta) \qquad (|n|>N),$$

$$D_N = \left\{(\mu,\gamma)\in\mathbb{C}^2 \;\middle|\; |\Re\mu|\le 1,\ |\Im\mu|\le\sqrt{bd}\left(N+\frac{1}{2}\right), |\gamma|\le 1\right\},\quad \Gamma_N=\partial D_N.$$

Also, let us define the maps $F^j, G^j:\mathbb{C}^2\to\mathbb{C}^2$, $j\in\{1,2\}$, given by

$$F^j(\mu,\gamma) = \begin{pmatrix}\varphi^j(\mu,\gamma,2\pi)\\ v^j(\mu,\gamma,2\pi)\end{pmatrix}, \qquad G^j(\mu,\gamma) = \begin{pmatrix}\widetilde{\varphi}^j(\mu,\gamma,2\pi)\\ \widetilde{v}^j(\mu,\gamma,2\pi)\end{pmatrix}, \tag{6.15}$$

where $\begin{pmatrix}\varphi^1(\mu,\gamma,\cdot)\\ v^1(\mu,\gamma,\cdot)\end{pmatrix}$, $\begin{pmatrix}\varphi^2(\mu,\gamma,\cdot)\\ v^2(\mu,\gamma,\cdot)\end{pmatrix}$, $\begin{pmatrix}\widetilde{\varphi}^1(\mu,\gamma,\cdot)\\ \widetilde{v}^1(\mu,\gamma,\cdot)\end{pmatrix}$ and $\begin{pmatrix}\widetilde{\varphi}^2(\mu,\gamma,\cdot)\\ \widetilde{v}^2(\mu,\gamma,\cdot)\end{pmatrix}$ are solutions of the following initial values problems

$$\begin{cases}\varphi^1 - b\varphi^1_{xx} + \mu v^1_x = 0, & x\in(0,2\pi)\\ v^1 - dv^1_{xx} + \mu\varphi^1_x = 0, & x\in(0,2\pi)\\ \varphi^1(0)=0,\ \varphi^1_x(0)=1\\ v^1(0)=0,\ v^1_x(0)=\gamma,\end{cases} \tag{6.16}$$

$$\begin{cases} \varphi^2 - b\varphi^2_{xx} + \mu v^2_x = 0, & x \in (0, 2\pi) \\ v^2 - dv^2_{xx} + \mu\varphi^2_x = 0, & x \in (0, 2\pi) \\ \varphi^2(0) = 0, \ \varphi^2_x(0) = \gamma \\ v^2(0) = 0, \ v^2_x(0) = 1, \end{cases} \tag{6.17}$$

$$\begin{cases} -b\widetilde{\varphi}^1_{xx} + \mu\widetilde{v}^1_x = 0, & x \in (0, 2\pi) \\ -d\widetilde{v}^1_{xx} + \mu\widetilde{\varphi}^1_x = 0, & x \in (0, 2\pi) \\ \widetilde{\varphi}^1(0) = 0, \ \widetilde{\varphi}^1_x(0) = 1 \\ \widetilde{v}^1(0) = 0, \ \widetilde{v}^1_x(0) = \gamma, \end{cases} \tag{6.18}$$

$$\begin{cases} -b\widetilde{\varphi}^2_{xx} + \mu\widetilde{v}^2_x = 0, & x \in (0, 2\pi) \\ -d\widetilde{v}^2_{xx} + \mu\widetilde{\varphi}^2_x = 0, & x \in (0, 2\pi) \\ \widetilde{\varphi}^2(0) = 0, \ \widetilde{\varphi}^2_x(0) = \gamma \\ \widetilde{v}^2(0) = 0, \ \widetilde{v}^2_x(0) = 1, \end{cases} \tag{6.19}$$

respectively.

According to Theorem 6.2.1, $\widetilde{\mu}$ is an eigenvalue of $\mathcal{B}$ if and only if $G^1_0(\widetilde{\mu}, 0) = 0$ or $G^2_0(\widetilde{\mu}, 0) = 0$. Moreover, from definition (6.15) and (6.16)-(6.17), we deduce that μ is an eigenvalue of $\mathcal{A}$ if and only if there exists $\gamma \in \mathbb{C}$ such that $F^1_0(\mu, \gamma) = 0$ or $F^2_0(\mu, \gamma) = 0$. Hence, we have reduced the problem of finding the eigenvalues of $\mathcal{A}$ to the problem of determining the zeros of the maps $(F^j)_{j=1,2}$. We'll analyze only the zeros of the map F^1, for those of F^2 the study being similar. Firstly note that the maps F^1 and G^1 are analytic and the following estimates hold

$$\left|F^1(\mu, \gamma) - G^1(\mu, \gamma)\right| \le \frac{C_1}{|\mu|^2} \qquad \left(|\Re\mu| \le 1, \ |\gamma| \le 1, \ |\mu| \ge \frac{1}{2}\right), \tag{6.20}$$

$$\left|G^1(\mu, \gamma)\right| \ge \frac{C_2\delta}{|\mu|^2} \qquad \left((\mu, \gamma) \in \Gamma_n(\delta)\right), \tag{6.21}$$

for some positive constants C_1, C_2 and $|n| > N$. Indeed, since $\mu \in Z$ and $|\gamma| \le 1$, (6.20) is a consequence of Proposition 6.1.1. On the other hand, (6.21) follows from the fact that there exists $C > 0$ such that the following estimate holds

$$\min\left\{\left|\sinh\left(\frac{2\pi\mu}{\sqrt{bd}}\right)\right|, \left|\cosh\left(\frac{2\pi\mu}{\sqrt{bd}}\right) - 1\right|\right\} \ge \frac{C\delta}{|\mu|} \qquad \left(\mu \in \mathbb{C}, \ |\mu - \sqrt{bd}n\,i| = \frac{\delta}{|n|}\right), \tag{6.22}$$

which is a direct consequence of Lemma 6.1.1. It follows from the multidimensional version of Rouché's Theorem [22, Theorem 1] (see, also, [23, Theorem 3]) that there exists $\delta > 0$ and $N \in \mathbb{N}$ such that the maps F^1 and G^1 have the same number of zeros in $D_n(\delta)$ for each $|n| > N$. Since G^1 vanishes once in $(\widetilde{\mu}_n, 0)$ in $D_n(\delta)$, it follows that F^1 has a unique zero (μ^1_n, γ^1_n) in $D_n(\delta)$. Thus, we have proved the existence of the eigenvalues $(\mu^1_n)_{|n|>N}$ of $\mathcal{A}$ and the corresponding asymptotic estimates (6.13). From the analysis of the map F^2, we get the existence of a family of zeros $(\mu^2_n, \gamma^2_n)_{|n|>N}$ from which we obtain the other sequence of eigenvalues $(\mu^2_n)_{|n|>N}$ of $\mathcal{A}$ and the corresponding asymptotic estimate. The existence of the eigenvalues $(\mu^1_n)_{|n|\le N}$ and $(\mu^2_n)_{|n|\le N}$ is obtained in a similar way, therefore we omit the details.

Let us pass to the analysis of the eigenfunctions. To each eigenvalues μ^j_n corresponds a unique normalized eigenfunction Φ^j_n which verifies (2.29) with $\gamma = \gamma^1_n$ or (6.17) with

$\gamma = \gamma_n^2$, respectively. Since $|\gamma_n^1| \leq \frac{\delta}{n}$, $|\gamma_n^2| \leq \frac{\delta}{n}$ and $|\mu_n^j - \widetilde{\mu}_n| \leq \frac{\delta}{n}$, for any $|n| > N$, from Proposition 6.1.1 and Lemma 6.1.2, we deduce that (6.14) is verified. Finally, since $\mathcal{A}$ is a skew-adjoint operator, these eigenfunctions are orthogonal in $(H_0^1(0, 2\pi))^2$. The proof of the theorem is complete. $\square$

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