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Sharp isoperimetric inequalities for small volumes in complete noncompact Riemannian manifolds of bounded geometry involving the scalar curvature

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Sob a orientação do

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Tese apresentada ao Programa de Pós-Graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio de Janeiro como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo: Damos um teorema de comparação isopemétrico para pequenos volumes em uma variedade Riemanniana n-dimensional (M^n, g) com C^3-geometria limitada em certo sentido envolvendo a curvatura escalar. Com C^3-geometria limitada, se o supremo da curvatura escalar satisfaz $S_g < n(n-1)k_0$ para certo $k_0 \in \mathbb{R}$, então para pequenos volumes o perfil isoperimétrico de (M^n, g) é menor ou igual que o perfil isoperimétrico do espaço forma completo simplesmente conexo de curvatura seccional constante k_0. Neste trabalho generalizamos o Teorema 2 de [Dru02c] no qual o mesmo resultado é provado no caso quando (M^n, g) é uma variedade compacta. Como consequência de nosso trabalho, damos uma expansão assintótica em series de Puiseux até o segundo termo não trivial do perfil isoperimétrico para pequenos volumes, generalizando a expansão de [Nar14b]. Finalmente, como corolário do nosso resultado isoperimétrico de comparação, provamos para pequenos volumes que a Conjetura de Aubin-Cartan-Hadamard é valida para qualquer dimensão n no caso especial de variedades com C^3-geometria limitada e $S_g < n(n-1)k_0$. Damos também duas provas do fato que uma região isoperimétrica de pequeno volume é de pequeno diametro. A primeira é feita com a hipótese de geometria limitada suave, isto é, raio de injetividade positivo e curvatura de Ricci limitada por baixo. A segunda é feita assumindo a existência de um limite por cima da curvatura seccional, raio de injetividade positivo e limite por baixo da curvatura de Ricci.

Palavras chaves: Comparação isoperimétrica, desigualdades isoperimétricas, pequenos volumes, geometria limitada, Conjetura de Aubin-Cartan-Hadamard, conjuntos de perímetro finito, geometria métrica, cálculo das variações, teoria geométrica da medida, desigualdades de Sobolev em variedades, curvatura escalar, equações diferenciais parciais em variedades, fórmula de monotonocidade, varifolds.

Abstract: We provide an isoperimetric comparison theorem for small volumes in a n-dimensional Riemannian manifold (M^n, g) with C^3 bounded geometry in a suitable sense involving the scalar curvature function. Under C^3 bounds of the geometry, if the supremum of scalar curvature function $S_g < n(n-1)k_0$ for some $k_0 \in \mathbb{R}$, then for small volumes the isoperimetric profile of (M^n, g) is less then or equal to the isoperimetric profile of the complete simply connected space form of constant sectional curvature k_0. This work generalizes Theorem 2 of [Dru02c] in which the same result was proved in the case where (M^n, g) is assumed to be compact. As a consequence of our result we give an asymptotic expansion in Puiseux series up to the second nontrivial term of the isoperimetric profile function for small volumes, generalizing our earlier asymptotic expansion [Nar14b]. Finally, as a corollary of our isoperimetric comparison result, it is shown that for small volumes the Aubin-Cartan-Hadamard's Conjecture is true in any dimension n in the special case of manifolds with C^3 bounded geometry, and $S_g < n(n-1)k_0$. Two different intrinsic proofs of the fact that an isoperimetric region of small volume is of small diameter. The first under the assumption of mild bounded geometry, i.e., positive injectivity radius and Ricci curvature bounded below. The second assuming the existence of an upper bound of the sectional curvature, positive injectivity radius, and a lower bound of the Ricci curvature.

Key Words: Isoperimetric comparison, isoperimetric inequalities, small volumes, bounded geometry, Aubin-Cartan-Hadamard's conjecture, finite perimeter sets, metric geometry, calculus of variations, geometric measure theory, Sobolev's inequalities on manifolds, scalar curvature, partial differential equations on manifolds, monotonicity formula, varifolds.

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Introduction

A Cartan-Hadamard manifold is a complete, simply connected Riemannian manifold of nonpositive sectional curvature, the name of this type of manifolds comes from the Cartan-Hadamard Theorem (see A.1.2) which asserts that all Cartan-Hadamard manifolds are diffeomorphic to a Euclidean space via the exponential map at any point. In comparison geometry arises the natural question.

Question 1. *Any Cartan-Hadamard manifold satisfies an Euclidean isoperimetric inequality?*

Conjecture 0.0.1 (Aubin-Cartan-Hadamard conjecture)**.** *Let (M^n, g) be a complete, simply connected, n-dimensional manifold, whose sectional curvatures satisfy inequality $Sec_M \leq k_0 \leq 0$, for some constant $k_0 \leq 0$. Then the sharp isoperimetric inequality holds true*

$$Area_g(\partial\Omega) \geq Area_{g_{k_0}}(\partial B),$$

Where B is a geodesic ball on the complete and simply connected space $\mathbb{M}_{k_0}$ whose sectional curvatures are equal to k_0, and $Vol_g(\Omega) = Vol_{g_{k_0}}(B)$. If equality holds in (1), then Ω is isometric to the geodesic ball of volume $Vol_g(\Omega)$ in $\mathbb{M}_{k_0}$.

Observe that in the case $k_0 = 0$ this is equivalent to say that the isoperimetric profile I_M of M is bounded from below by the isoperimetric profile $I_{\mathbb{M}_0}$ of the complete and simply connected space form $\mathbb{M}_0 = \mathbb{R}^n$ whose sectional curvatures are equal to 0, then for all $\Omega \subset M$ with smooth boundary with volume $Vol_g(\Omega)$ satisfies

$$Area_g(\partial\Omega) \geq I_{\mathbb{M}_0}(Vol_g(\Omega)) = K(n,1)^{-1} Vol_g(\Omega)^{\frac{n-1}{n}}, \tag{1}$$

Where $K(n,1) := \frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{1/n} = c_n^{-1}$.

Furthermore by the work of Federer and Fleming [HF60], the inequality (1) is equivalent to the sharp inequality

$$\left(\int_M |u|^{\frac{n}{n-1}} dv_g\right)^{\frac{n-1}{n}} \leq K(n,1) \int_M |\nabla u| dv_g, \tag{2}$$

for every $u \in W^{1,1}(M)$. (For this assertion see Appendix G.1).

The history of the conjecture starts in the case $n = 2$ and $k_0 = 0$, with the work made in 1926 by Weil [Wei26], this proof uses conformal representation and the theory of harmonic functions, thus answering a question of Paul Lévy addressed during a Hadamard seminar at the Collège de France. Independently this result was obtained by Beckenbach and Radó [BR33], using a relation between subharmonic functions and surfaces of negative curvature. Both articles capitalizing a result of Carleman [Car21] of 1921, who proved the inequality $L^2 \geq 4\pi A$ for every simply-connected rectifiable piece of a minimal surface. Later Bol [Bol41] established the case when $n = 2$ and $k_0 \neq 0$, improving a technic of interior parallels, for his proof in english can be consulted Theorem 1 and 2 of [Ban83]. The Conjecture 0.0.1 has been proved by C. Croke [Cro84] in 1984 for the case $n = 4$ with $k_0 = 0$, using the Santalo's formula, in his proof he found the inequalities $Area_g(\partial\Omega) \geq \tilde{C}(n)^{-\frac{1}{n}} Vol_g(\Omega)^{\frac{n-1}{n}}$, the value of the constant for $n \geq 3$ is

$$\tilde{C}(n) = \frac{\omega_{n-2}^{n-2}}{\omega_{n-1}^{n-1}} \left(\int_0^{\frac{\pi}{2}} \cos^{\frac{n}{n-2}}(t) \sin^{n-2}(t) dt\right)^{n-2}. \tag{3}$$

$\tilde{C}(n)$ is optimal being the sharp constant only when $n = 4$, that is $\tilde{C}(4)^{\frac{1}{4}} = K(4,1)$. Recently Kloeckner and Kuperberg in [BK13] extend to the case $n = 4$ with $k_0 > 0$, in addition they make a very interesting and natural question.

Question 2 (Question 4.1 of [BK13])**.** *If M is a Cartan-Hadamard manifold and Ω minimizes $Area_g(\partial\Omega)$ for some fixed value of $Vol_g(\Omega)$, then is it convex? Is it a topological ball?.*

The answer to his question is given by J. Hass [Has16], where he assert that the answer to both parts of the question is no in dimensions two and three. Furthermore, he gives examples of Cartan-Hadamard manifolds in which the isoperimetric region need not even be connected.

Later in 1992, Kleiner [Kle92] proved Conjecture 0.0.1 in dimension $n = 3$ with $k_0 \leq 0$, where he uses only the fact that the dimension is three in an application of Gauss-Bonnet formula over the two-dimensional boundary of an isoperimetric domain to prove that $\max_{\partial\Omega} H_{\partial\Omega} \geq H_{k_0}(Area(\partial\Omega))$, where Ω is a compact set, $\partial\Omega$ is $C^{1,1}$, and H_{k_0} is the mean curvature in the model space $\mathbb{M}^3_{k_0}$ of a geodesic ball B with $Area(\partial\Omega) = Area(\partial B)$. In 2005 Ritoré [Rit05] gives a different proof of the result of Kleiner, and another proof is made by Schulze [Sch08] using the mean curvature flow.

A different approach begins with Morgan and Johnson [MJ00] in 2000, they prove a compact version of the conjecture with an additional assumption on the Gauss-Bonnet-Chern integrand in even dimensions (of course when M is compact we do not have necessarily that the manifold is simply connected and with nonpositive curvature), and they restrict to the case of small volume. The Bol-Fiala inequality says that for a smooth Riemannian surface of Gauss curvature $Sec_g \leq k_0$ the perimeter P and area A of a disc satisfy

$$P^2 \geq 4\pi A - k_0 A^2.$$

In the Proposition 5.2 and Theorem 5.3 of [MJ00] they prove a generalization to arbitrary regions of sufficiently small volume that coincides with the Bol-Fiala inequality when considering disks. For example if we consider Figure 1 the surface given by two units spheres connected by a thin cylinder, we can easily see that small discs can have perimeter P satisfying $P^2 = 4\pi A - k_0 A^2$, while sections of the cylinder can have perimeter $2L_0$, where L_0 is the lengths of a simple suitable closed geodesic in the cylinder.

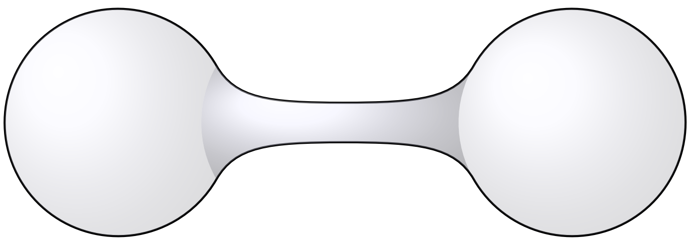


Figure 1: Bol-Fiala inequality from discs to arbitrary regions, $P^2 \geq \min\{(2L_0)^2, 4\pi A - k_0 A^2\}$.

Another reason for considering to take small volumes, is because in this Theorem 2.2 of

[MJ00], they prove that in a smooth, compact Riemannian manifold, the least perimeter enclosure of small volume is a (nearly round) sphere.

In dimension $n \geq 5$ the Aubin-Cartan-Hadamard conjecture is still open. At our knowledge the only previous partial results in any dimension n with the sharp constant, but restricted to the small volume regime, are Theorem 4.4 of [MJ00] which require additional assumptions on the Gauss-Bonnet-Chern integrand in even dimension, Theorem 2 of [Dru02c] in case of compact manifolds and Corollary 2 of [MFN15] in case of noncompact manifolds with C^2-locally asymptotically bounded geometry at infinity (compare Definition 1.0.8) that is the noncompact version of Theorem 4.4 of [MJ00], but this requires a sectional curvature comparison rather than a scalar curvature one. Our Corollary 1 extends these partial results in any dimension to domains Ω of small volume inside a Cartan-Hadamard manifold (M, g) having C^3-locally asymptotically strong bounded geometry smooth at infinity. The difference between our result and Theorem 2 of [Dru02c] is that we relax the assumption on the manifold M of being compact and replace it by just requiring that M have C^3-locally asymptotically strong bounded geometry smooth at infinity. For a more exhaustive treatment about the state of the art of the Aubin-Cartan-Hadamard's conjecture we suggest the reading of the very good surveys of Olivier Druet [Dru10] available online, Section 3.2 of Manuel Ritoré in [RS10], and the very recent and interesting paper [BK13]. Let us state here Theorem 1 of [Dru02c].

Theorem 0.0.2 (Theorem 1 of [Dru02c])**.** *Let (M^n, g) be a complete Riemannian manifold, with $n \geq 2$, $x \in M$ such that there exists $k_0 \in \mathbb{R}$ satisfying $Sc_g(x) < n(n-1)k_0$. Then there exists $r_x > 0$ such that for every finite perimeter set Ω contained in the geodesic ball of center x and radius r_x,*

$$\mathcal{P}_g(\Omega) > \mathcal{P}_{g_{k_0}}(B), \tag{4}$$

where B is a ball enclosing a volume $v = V_g(\Omega)$ in the model simply connected space form $(\mathbb{M}^n_{k_0}, g_{k_0})$ of constant sectional curvature k_0.

It should be seen from the proof of the preceding result and Theorems 1-3 of this paper that a lower bound of the optimal r_x is continuous with respect to $Sc_g(x)$ and C^3 convergence of metrics, so if M is compact there exists $r := \inf\{r_x : x \in M\} > 0$ such that the conclusion of the Theorem 0.0.2 holds for any Ω contained in a ball of radius r. Unfortunately the radius r_x could go to zero when x tends to infinity in an arbitrary noncompact complete Riemannan manifold. Hence some extra assumptions on the geometry at infinity of M are needed to allow us to find such a positive uniform lower bound r. Actually using the last equation at page 2353 of [Dru02c] and reasoning by contradiction it appears evident from the proof that to have C^3-locally asymptotically strong bounded geometry smooth at infinity, gives such a lower bound. A necessary condition to have $r > 0$ is that the volume of balls of a fixed radius for example $r/2$ does not vanish when the centers go to infinity. This is a non-collapsing condition that for example follows assuming $Ric_g \geq (n-1)kg$ for some $k \in \mathbb{R}$ and positivity of the injectivity radius. Thus it seems natural to make these assumptions in our Theorem 1. Actually, in other parts of the proof we will need to strengthen a little more our assumptions on the geometry of M and we are lead to assume that M have strong bounded geometry in the sense of our Definition 1.0.3. To obtain our main result about small volumes, namely Theorem 3, in first we prove a global isoperimetric comparison for small diameters in Theorem 1 when M has C^3-locally asymptotically bounded geometry at infinity (see Definition 1.0.3). Given granted the proof of Theorem 1 we then generalize it proving Theorem 3 by geometric measure theory and Gromov-Hausdorff convergence of manifolds. The proof of Theorem 1 goes along the same lines of Theorem 1 of [Dru02c]. So the main ingredients used in its proof are results about local optimal Sobolev inequalities in $W^{1,p}$ via PDE techniques when $p > 1$ which are easier to obtain than when $p = 1$. After the limit problem when $p \to 1^+$ is studied. These local optimal Sobolev inequalities in $W^{1,p}$ are combined with an asymptotic analysis of solutions of quasi-elliptic equations involving the p-Laplacian when the parameter $p \to 1^+$. The importance of the scalar curvature when studying sharp Sobolev inequalities on Riemannian manifolds was first observed by Olivier Druet in [Dru98], later by Hebey in [Heb02] and appears evident when deducing Theorem 1 [Dru02c] from Proposition 1 of [Dru02c]. The modifications required to pass from the proof of the results contained in [Dru02c] and ours are highly nontrivials, so to make the paper self-contained we wrote the entire proof of our Theorem 1 in Section 2.1. Theorem 3 is a consequence of Theorem 1 using techniques of geometric measure theory, say the theory of sets of finite perimeter, comparison geometry, metric geometry and Gromov-

Hausdorff convergence of manifolds. The proof follows the scheme traced by the proof of Theorem 2 of [Dru02c], however the required changes in the proof are highly nontrivial and original. The two main difficulties that are encountered when one tries to apply the proof of Theorem 2 of [Dru02c] (working only for compact manifolds) to our more general context consist in the fact that existence of isoperimetric regions for every volume in a noncompact Riemannian manifold is no longer guaranteed and that one needs to prove that isoperimetric regions of small volumes are also of small diameter. For an account of results on the by now classical problem of existence of isoperimetric regions (see Definition 1.0.9) in complete Riemannian manifolds the reader is referred to [Nar14a], [MN16] and the references therein. Our approach to solve this difficulty is to use the theory of generalized existence and generalized compactness developed by the first author in [Nar14a], [MN15], and replace genuine isoperimetric regions in M by generalized isoperimetric regions lying in some pointed limit manifold. This is possible because the hypotheses of Theorem 1 of [MN15] are automatically fulfilled in the context of C^3-locally asymptotic strong bounded geometry smooth at infinity that we consider here. To finish the proof of Theorem 3 we need to prove that in a limit manifold an isoperimetric region having small volumes have also a small diameter. With this aim in mind we replace the proof of [MJ00] based on Nash's isometric embeddings by another intrinsic one. We carry out this task proving a little more general result in Lemma 3.1.9, which asserts that if just the *Ricci* curvature is bounded below and the injectivity radius is positive, isoperimetric regions of small volumes are of small diameter. In this proof we don't need to use any monotonicity formula; this fact constitutes a novelty with respect to the existing literature and in particular to the classical extrinsic proof of [MJ00]. Our proof is completely intrinsic and uses a cut and paste argument inspired by Proposition 2.5 of [Nar14b] (which works only for manifolds of strong bounded geometry) adapted to the case of weak bounded geometry (see Definition 1.0.1) joint with others non trivial intrinsic arguments aimed to encompass some technical difficulties of geometric measure theory, which arise when passing from the Euclidean space $\mathbb{R}^n$ to an arbitrary Riemannian manifold (M^n, g) without using Nash's isometric embedding theorem. The arguments of the proof permit also to give an effective estimate of the constants of Lemma 3.1.9 as functions of the bounds of the geometry of

(M^n, g), which is new in the literature. The main result of this paper is Theorem 3. As corollaries of our main Theorem 3 we get immediately Corollary 1 that is a special case of the Aubin-Cartan-Hadamard's Conjecture and the expansion of the isoperimetric profile in Puiseux series given by Corollary 2. We include also in Section 5.1.1 another purely intrinsic proof that for small volumes isoperimetric regions are of small diameter based on a monotonicity formula for varifolds of bounded generalized mean curvature which allows us to use an argument inspired from the correspondent extrinsic proof of [MJ00] and combining it with our cut and paste argument to give finally Lemma 5.1.2. The monotonicity formula that we use here is an adaptation of Theorem 2.1 and Proposition 2.2 of [Lel12] to our intrinsic Riemannian context via Hessian comparison theorems for the distance function. At our knowledge this is the first time that such an intrinsic approach appears in the literature, although being a very natural one. The applications of this methods are wide and opens the doors for extending in a rigorous way to a Riemannian ambient manifold the geometric measure theory known in $\mathbb{R}^n$, without using the Nash's isometric embedding theorem. As a final remark we have that all the constants involved in our statements of Section 1 but the Druet radius are effectively computed in terms of the minimal bounds on the geometry that we are assuming.

Chapter 1

Definitions and Results

In this chapter we fix the notations used all along the subsequent treatment, in addition we give the preliminary results that we use. Furthermore we give in the appendix a survey about comparison geometry Appendix A and convergence of manifolds Appendix B for completeness's sake.

In the sequel we always assume that all the Riemannian manifolds M^n considered are smooths with smooth Riemannian metric g. We denote by V_g the canonical Riemannian measure induced on M by g, and by A_g the $(n-1)$-Hausdorff measure associated to the canonical Riemannian length space metric d of M, that we also denote by $\mathcal{H}^{n-1}_g$. When it is already clear from the context, explicit mention of the metric g will be suppressed. We will denote by Ric_g the Ricci tensor of (M, g), by Sec_g the sectional curvature of (M, g), Sc_g the scalar curvature function, $S_g := \sup_{x\in M}\{Sc_g(x)\}$ and by $\mathbb{M}^n_k$ the simply connected space form endowed with the standard metric of constant sectional curvature $k \in \mathbb{R}$ that we denote by g_k, by $inj_{(M,g)}$ the injectivity radius of M, for any $D \subseteq M$, $diam_g(D)$ the diameter of D in the metric space (M, g), dv_g the Riemannian measure with respect to the metric g, $B_{(M,g)}(p, r)$ is the open geodesic ball of M centered at p and of radius $r > 0$. In what follows we will consider as a key object the set of all finite perimeter sets (see Definition 2.1.2) of M that we will denote by $\tilde{\tau}$. So a little technical discussion is in order here. By classical results of geometric measure theory (see Proposition 12.19 and Formula (15.3) of [Mag12]) we know that if E is a set of locally finite perimeter in M, then $spt(\nabla\chi_E) = \{x \in M^n : 0 < V_g(E \cap B(x, r)) < \omega_n r^n, \forall r > 0\} \subseteq \partial E$, furthermore there exists an equivalent Borel set F (i.e., $V_g(E\Delta F) = 0$) such that $spt(\nabla\chi_F) = \partial F = \overline{\partial^* F}$, where $\partial^* F$ is the reduced boundary of F. It is not too hard to show that if E has C^1 boundary, then $\partial^* E = \partial E$, where ∂E is the topological boundary of E. De Giorgi's structure theorem (compare Theorem 15.9 of [Mag12]) guarantees that for every set E of locally finite perimeter, $A_g(\partial^* E) = \mathcal{H}_g^{n-1}(\partial^* E) = \mathcal{P}_g(E)$. Hence without loss of generality we will adopt the assumption that all the locally finite perimeter sets considered in this text satisfy $\overline{\partial^* E} = \partial E$. It is worth to mention that the results in the book [Mag12] are stated and proved in $\mathbb{R}^n$ but they are valid mutatis mutandis also in an arbitrary complete Riemannian manifold, the required details could be easily provided using the work about BV-functions on a Riemannian manifold accomplished in [MPPP07].

Definition 1.0.1. *A complete Riemannian manifold (M, g), is said to have* ***weak bounded geometry****, if there exists a constant $k \in \mathbb{R}$, such that $Ric_M \geq k(n-1)$ (i.e., $Ric_M \geq k(n-1)g$ in the sense of quadratic forms) and $V_g(B_{(M,g)}(p, 1)) \geq v_0 > 0$, for some positive constant v_0, where $B_{(M,g)}(p, r)$ is the open geodesic ball of M centered at p and of radius $r > 0$.*

Remark 1.0.1. *In this paper we differ from the nomenclature used by the first author in his preceding works. What we call here weak bounded geometry is what is called, in all previous articles of the first author, just bounded geometry.*

Definition 1.0.2. *A complete Riemannian manifold (M, g), is said to have* ***mild bounded geometry****, if there exists a constant $k \in \mathbb{R}$, such that $Ric_M \geq k(n-1)$ (i.e., $Ric_M \geq k(n-1)g$ in the sense of quadratic forms) and $inj_M > 0$, where inj_M is the injectivity radius of M.*

Remark 1.0.2. *It is known that mild bounded geometry implies weak bounded geometry, but the converse is not true. For more details about this point the reader is referred to Remark 2.5 of [MN16] and to the references therein.*

Definition 1.0.3. *A complete Riemannian manifold (M, g), is said to have* ***strong bounded geometry****, if there exists a positive constant $K > 0$, such that $|Sec_M| \leq K$ and $inj_M \geq i_0 > 0$ for some positive constant i_0. Sometimes we will use the condition*

$\Lambda_1 \leq Sec_M \leq \Lambda_2$, *for some given constants* $\Lambda_1, \Lambda_2 \in \mathbb{R}$ *instead of* $|Sec_M| \leq K$ *to express that* M *have a two sided bound on the sectional curvature.*

Remark 1.0.3. *It turns out that it is easy to check that strong bounded geometry implies mild bounded geometry, with the converse being not true in general.*

Definition 1.0.4. *For any* $m \in \mathbb{N}$, $\alpha \in [0,1]$, *a sequence of pointed smooth complete Riemannian manifolds is said to* ***converge in the pointed*** $C^{m,\alpha}$***, respectively*** C^m ***topology to a smooth manifold*** M *(denoted by* $(M_i, g_i, p_i) \to (M, g, p)$*), if for every* $R > 0$ *we can find a domain* Ω_R *with* $B_g(p, R) \subseteq \Omega_R \subseteq M$, *a natural number* $\nu_R \in \mathbb{N}$, *and* C^{m+1} *embeddings* $F_{i,R} : \Omega_R \to M_i$, *for large* $i \geq \nu_R$ *such that* $B_{g_i}(p_i, R) \subseteq F_{i,R}(\Omega_R)$ *and* $F^*_{i,R}(g_i) \to g$ *on* Ω_R *in the* $C^{m,\alpha}$, *respectively* C^m *topology.*

Definition 1.0.5 (Page 308 of [Pet06])**.** *A subset* A *of a Riemannian n-manifold* M *has* ***bounded*** $C^{m,\alpha}$ ***norm on the scale of*** r, $||A||_{C^{m,\alpha},r} \leq Q$, *if every point* p *of* M *lies in an open set* U *with a chart* ψ *from the Euclidean* r*-ball into* U *such that*

(i): *For all* $p \in A$ *there exists* U *such that* $B(p, \frac{1}{10}e^{-Q}r) \subseteq U$.

(ii): $|D\psi| \leq e^Q$ *on* $B(0, r)$ *and* $|D\psi^{-1}| \leq e^Q$ *on* U.

(iii): $r^{|j|+\alpha}||D^j g||_\alpha \leq Q$ *for all multi indices* j *with* $0 \leq |j| \leq m$, *where* g *is the matrix of functions of metric coefficients in the* ψ *coordinates regarded as a matrix on* $B(0, r)$.

We write that $(M, g, p) \in \mathcal{M}^{m,\alpha}(n, Q, r)$, *if* $||M||_{C^{m,\alpha},r} \leq Q$.

Definition 1.0.6. *For any* $m \in \mathbb{N}$ *and* $\alpha \in [0,1]$, *we say that a smooth Riemannian manifold* (M^n, g) *has* $C^{m,\alpha}$***-locally asymptotically weak bounded geometry****, if has weak bounded geometry and if for every diverging sequence of points* (p_j), *there exists a subsequence* (p_{j_l}) *and a pointed smooth manifold* $(M_\infty, g_\infty, p_\infty)$ *with* g_∞ *a smooth Riemannian metric such that the* $C^{m,\alpha}$ *norm is finite and the sequence of pointed manifolds* $(M, g, p_{j_l}) \to (M_\infty, g_\infty, p_\infty)$, *in* $C^{m,\alpha}$*-topology. When* $\alpha = 0$ *we write* C^m *instead of* $C^{m,0}$.

Remark 1.0.4. *The condition of being smooth at infinity is used just in the last equation* (4.1) *of the proof of Theorem 3 when we apply Theorem 1 to a possibly limit manifold* (M_∞, g_∞) *that even in strong bounded geometry is a* $C^{3,\beta}$ *differentiable manifold but with* *a metric that is just* $C^{1,\beta}$ *and no more regular. There are examples of this phenomenon as explained in Example* 1.8 *of [Pet87a]. Actually the limit metric is* $W^{2,p}$ *for any* $p > 1$*, as showed in [Nik91]. This last regularity result is not enough strong to allow the use of the arguments of the proof of Theorem 1 in* (M_∞, g_∞)*.*

This last remark justifies the following definitions.

Definition 1.0.7. *We say that a smooth Riemannian manifold* (M^n, g) *is* ***smooth at infinity****, if for every diverging sequence of points* (p_j)*, there exists a subsequence* (p_{j_l}) *and a pointed smooth manifold* $(M_\infty, g_\infty, p_\infty)$ *with* g_∞ *of class* C^∞*, such that* $(M^n, g, p_{j_l}) \to (M_\infty, g_\infty, p_\infty)$ *in the pointed Gromov-Hausdorff topology. We say that a smooth Riemannian manifold* (M^n, g) *has* ***strong bounded geometry smooth at infinity****, if it is of strong bounded geometry and is smooth at infinity. We say that* (M^n, g) *has* $C^{m,\alpha}$*-locally asymptotically strong bounded geometry smooth at infinity, if it is of strong bounded geometry, smooth at infinity, and has* $C^{m,\alpha}$***-locally asymptotic bounded geometry****.*

Definition 1.0.8. *We say that a smooth Riemannian manifold* (M^n, g) *has* $C^{m,\alpha}$***-locally asymptotically strong bounded geometry smooth at infinity****, if it has strong bounded geometry, smooth at infinity, and has* $C^{m,\alpha}$*-locally asymptotically weak bounded geometry.*

Remark 1.0.5. *Observe that by Theorems* 76 *and* 72 *of [Pet06] or Theorem* 4.4 *of [Pet87a] it is easily seen that to have strong bounded geometry smooth at infinity implies to have* $C^{1,\beta}$*-locally asymptotic weak bounded geometry, for any* β*.*

We have now all the definitions needed to state our results.

Theorem 1 (Small diameters in C^3-locally asymptotically strong bounded geometry smooth at infinity)**.** *Let* (M^n, g) *be a complete Riemannian manifold with* $n \geq 2$ *and with* C^3*-locally asymptotically strong bounded geometry smooth at infinity. Let us assume that there exists a real constant* $k_0 \in \mathbb{R}$ *such that* $S_g < n(n-1)k_0$*. Then there exists* $d = d(n, k, k_0, inj_M, S_g) > 0$*, which depends only on* n, k, k_0, inj_M, S_g *such that for every* $\Omega \subseteq M^n$ *finite perimeter set with diameter* $\mathrm{diam}_g(\Omega) \leq d$ *holds*

$$\mathcal{P}_g(\Omega) > \mathcal{P}_{g_{k_0}}(B), \tag{1.1}$$

where $B \subseteq \mathbb{M}^n_{k_0}$ *is a geodesic ball having* $V_{g_{k_0}}(B) = V_g(\Omega)$. *Moreover we have the following lower bound on the greatest* d *for which* (1.1) *holds, namely* $d = d(n, k, k_0, inj_M, S_g, r_\varepsilon(M, g))$ *could be chosen to be equal to*

$$\min\left\{C(n,k)^{-\frac{1}{n}}\left\{\frac{nK(n,1)^2}{2(n+2)C_0(n,k_0)}\left[n(n-1)k_0 - S_g\right]\right\}^{\frac{1}{4}}, r_\varepsilon(M,g), 1\right\}, \tag{1.2}$$

with $\varepsilon = n(n-1)k_0 - S_g > 0$ *see equation* (2.96) *and Definition 2.1.4 for the exact meaning of the constants involved here.*

Remark 1.0.6. *Strict inequality is necessary in the assumptions of the preceding theorem because as pointed out in* [Dru02c] *Theorem 1 is false if we have just* $Ric_g \leq (n-1)k_0$ *and not* $S_g < n(n-1)k_0$. *The comparison result is false also on* $S^2 \times S^2$, *as noticed in* [MJ00], *compare again* [Dru02c] *page* 2352.

A first consequence of Theorem 1 is the following result whose proof is much more simpler than that of our main Theorem 3.

Theorem 2 (Small volumes à la Bérard-Meyer)**.** *Let* (M^n, g) *be a complete Riemannian manifold,* $n \geq 2$, *and with* C^3*-locally asymptotically strong bounded geometry smooth at infinity. Let us assume that there exists a real constant* $k_0 \in \mathbb{R}$ *such that* $S_g < n(n-1)k_0$. *Then for every* $\varepsilon > 0$ *there exists a positive constant* $\tilde{v}_0 = \tilde{v}_0(M, \varepsilon) > 0$ *such that for every* $\Omega \subseteq M$ *finite perimeter set with* $V_g(\Omega) \leq \tilde{v}_0$ *holds*

$$\mathcal{P}_g(\Omega) > (1-\varepsilon)\mathcal{P}_{g_{k_0}}(B), \tag{1.3}$$

where $B \subseteq \mathbb{M}^n_{k_0}$ *is a geodesic ball having* $V_{g_{k_0}}(B) = V_g(\Omega)$.

Remark 1.0.7. *This gives a refinement of the classical result of Bérard-Meyer in [BM82], provided that one assumes stronger assumptions on the bounds of the geometry of* (M, g), *i.e.,* C^3*-locally asymptotically strong bounded geometry smooth at infinity. Of course Theorem 2 follows immediately from the stronger Theorem 3.*

In the next theorem we refine the results contained in Theorem 1, replacing the assumption of small diameter with that of small volume. The price to pay to have this stronger result is that the proof of Theorem 3 is much more involved.

Theorem 3 (Sharp small volumes in C^3-locally asymptotically strong bounded geometry smooth at infinity)**.** *Let* (M^n, g) *be a complete Riemannian manifold,* $n \geq 2$*, with* C^3*-locally asymptotically strong bounded geometry smooth at infinity. Let us assume that there exists a real constant* $k_0 \in \mathbb{R}$ *such that* $S_g < n(n-1)k_0$*. Then there exists a positive constant* $\tilde{v}_0 = \tilde{v}_0(n, k, k_0, inj_{M,g}, S_g, d) > 0$ *such that for every* $\Omega \subseteq M$ *finite perimeter set with* $V_g(\Omega) \leq \tilde{v}_0$ *it holds*

$$\mathcal{P}_g(\Omega) > \mathcal{P}_{g_{k_0}}(B), \tag{1.4}$$

where $B \subseteq \mathbb{M}^n_{k_0}$ *is a geodesic ball having* $V_{g_{k_0}}(B) = V_g(\Omega)$*. Moreover* $\tilde{v}_0$ *can be chosen as an arbitrary number*

$$0 < \tilde{v}_0 \leq \min\left\{v^*, \left(\frac{d}{\mu^*}\right)^n\right\}, \tag{1.5}$$

where d *is as in* (1.2) *and* v^*, μ^* *are given by Lemma 3.1.9.*

A particular case of the more general situation considered in Theorem 3 gives a positive answer to a special case of the Aubin-Cartan-Hadamard's Conjecture for small volumes as stated in the following corollary.

Corollary 1 (Aubin's Conjecture in C^3-locally asymptotically strong bounded geometry smooth at infinity for small volumes)**.** *Let* (M^n, g) *be a Cartan-Hadamard manifold,* $n \geq 2$ *with* C^3*-locally asymptotically strong bounded geometry smooth at infinity, and* $S_g < 0$*. Then there exists a positive constant* $\tilde{v}_0 = \tilde{v}_0(n, k, k_0, inj_{M,g}, S_g, d) > 0$ *such that for every* $\Omega \subseteq M$ *finite perimeter set with* $V_g(\Omega) \leq \tilde{v}_0$ *it holds*

$$\mathcal{P}_g(\Omega) > \mathcal{P}_{g_{k_0}}(B), \tag{1.6}$$

where $B \subseteq \mathbb{M}^n_{k_0}$ *is a geodesic ball having* $V_{g_{k_0}}(B) = V_g(\Omega)$*. Moreover* $\tilde{v}_0$ *can be chosen as an arbitrary number*

$$0 < \tilde{v}_0 \leq \min\left\{v^*, \left(\frac{d}{\mu^*}\right)^n\right\}, \tag{1.7}$$

where d *is as in* (1.2) *and* v^*, μ^* *are given by Lemma 3.1.9.*

As a last consequence of Theorem 3 we get Corollary 2 which gives an asymptotic expansion of the isoperimetric profile in Puiseux series up to the second non trivial order generalizing previous results of [Nar14b]. Before to state the corollary we recall here the definition of the isoperimetric profile.

Definition 1.0.9. *Let (M^n, g) be an arbitrary Riemannian manifold. For every $v \in]0, V_g(M)[$ we define $I_{M,g}(v) := \inf\{\mathcal{P}_g(\Omega)\}$, where the infimum is taken over the family of finite perimeter subsets $\Omega \subseteq M$ having fixed volume $V_g(\Omega) = v$ that will be denoted in the sequel $\tilde{\tau}_v$. If there exists a finite perimeter set Ω satisfying $V_g(\Omega) = v$, $I_{M,g}(V_g(\Omega)) = A_g(\partial\Omega) = \mathcal{P}_g(\Omega)$ such an Ω will be called an **isoperimetric region**, and we say that $I_{M,g}(v)$ is **achieved**.*

Corollary 2 (Asymptotic expansion of the isoperimetric profile)**.** *If (M, g) have C^3-locally asymptotically strong bounded geometry smooth at infinity, then*

$$I_{M,g}(v) = c_n v^{\frac{(n-1)}{n}} \left(1 - \gamma_n S_g v^{\frac{2}{n}}\right) + O\left(v^{\frac{4}{n}}\right),$$

when $v \to 0^+$, where $S_g := \sup_{x \in M}\{Sc_g(x)\}$ and $\gamma_n = \frac{1}{2n(n+2)\omega_n^{\frac{2}{n}}}$ is a positive dimensional constant. Here ω_n is the volume of a geodesic ball of radius 1 in $\mathbb{R}^n$.

Remark 1.0.8. *The preceding corollary roughly speaking means that up to the second nontrivial term the asymptotic expansion of $I_{M,g}$ coincides with $I_{\mathbb{M}^n_k, g_k}$, where $n(n-1)k = S_g$.*

Remark 1.0.9. *If for some $\alpha > 0$, (M, g) have C^3-locally asymptotically strong bounded geometry smooth at infinity and moreover have $C^{3,\alpha}$ bounded geometry at some scale r as in Definition 1.0.5, then (M, g) satisfies the assumptions of Theorems 1, 3 and Corollaries 1, 2. Furthermore in this special case the proof of Theorem 3 does not need the use of Lemma 2.1.4 and Corollary 2.1.1 but only the use of the statement of Theorem 1.*

Chapter 2

Sobolev inequalities and the proof of Theorem 1

Many tools are necessary to the understanding of this chapter. So we collected all the needed background material in the various appendix sections. In appendix C one can find a very helpful little introduction to the Sobolev Spaces in Riemannian manifolds, including the correspondents theorems about the Sobolev embeddings, without proof. In addition in Appendix D is given the proof of the existence of solutions by the Generalized Scalar Curvature equation, which is now a classical but not at all an easy result. Also we write the Concentration-Compactness Lemma of Lions in Appendix E. Several relations between the function u_p and v_p given all along this chapter can be found in Appendix F. Finally in Appendix G.2 we wrote de computations leading to expansion in Puiseux series of the area of geodesic balls in function of the enclosed volumes near zero, up to the second nontrivial term.

2.1 Sharp Local Isoperimetric Inequalities using Sobolev Inequalities

In this section we closely follow the proof of Theorem 1 of [Dru02c]. We just make the needed changes to get the proof of our Theorem 1. First we set some notations and make the definitions that will be required in the sequel. By ξ we denote the standard Euclidean metric of $\mathbb{R}^n$. For every $1 \leq p < n$, $K(n,p) > 0$ is the best constant in the Sobolev inequalities on $(\mathbb{R}^n, \xi)$ defined as

$$K(n,p)^{-p} := \inf_{u \not\equiv 0, u \in C_c(\mathbb{R}^n)} \left\{ \frac{\int_{\mathbb{R}^n} |\nabla u|_\xi^p dv_\xi}{\left(\int_{\mathbb{R}^n} |u|^{p^*} dv_\xi\right)^{\frac{p}{p^*}}} \right\}, \tag{2.1}$$

where $p^* := \frac{np}{n-p}$ is the critical Sobolev's exponent. The explicit value of $K(n,p)$ is computed in [Aub76], [Tal76] namely

$$\begin{aligned} K(n,1) &:= \frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{1/n} = c_n^{-1}, \\ K(n,p) &:= \frac{1}{n}\left(\frac{n(p-1)}{n-p}\right)^{1-1/p}\left(\frac{\Gamma(n+1)}{\Gamma(n/p)\Gamma(n+1-n/p)\omega_{n-1}}\right)^{1/n}. \end{aligned}$$

However the only property of $K(n,p)$ that we will use is that

$$\lim_{p \to 1^+} K(n,p) = K(n,1) = \frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{1/n}.$$

We will use frequently the L^p and the $W^{1,p}$ norm on M defined by

$$||u||_{p,g} := \left(\int_M |u|^p dv_g\right)^{\frac{1}{p}},$$

$$||u||_{1,p,g} := ||u||_{p,g} + ||\nabla_g u||_{p,g},$$

for any function u belonging respectively to $L^p(M,g)$ and $W^{1,p}(M,g)$. When $1 \leq p < n$ we will need to work inside $W_\xi^{1,p}(\mathbb{R}^n)$ that will denote the standard Sobolev space defined as the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm

$$||u||_{1,p,\xi} := \left(\int_{\mathbb{R}^n} |\nabla_\xi u|_\xi^p dv_\xi\right)^{\frac{1}{p}}. \tag{2.2}$$

Definition 2.1.1. *Let (M,g) be a Riemannian manifold of dimension n, $U \subseteq M$ an open subset, $\mathfrak{X}_c(U)$ the set of smooth vector fields with compact support on U. Given a function $u \in L^1(M,g)$, define the variation of u by*

$$|Du|(M) := \sup\left\{\int_M u \, div_g(X) dv_g : X \in \mathfrak{X}_c(M), ||X||_\infty \leq 1\right\}, \tag{2.3}$$

where $||X||_\infty := \sup\{|X_p|_g : p \in M\}$ and $|X_p|_g$ is the norm of the vector X_p in the metric g on T_pM. We say that a function $u \in L^1(M,g)$, has ***bounded variation****, if $|Du|(M) < \infty$ and we define the set of all functions of bounded variations on M by $BV(M,g) := \{u \in$*

$L^1(M,g)$: $|Du|(M) < +\infty\}$. A function $u \in L^1_{loc}(M)$ has ***locally bounded variation in*** *M, if for each open set $U \subset\subset M$,*

$$|Du|(U) := \sup\left\{\int_U u div_g(X)dv_g : X \in \mathfrak{X}_c(U), \|X\|_{\infty,g} \leq 1\right\} < \infty,$$

and we define the set of all functions of locally bounded variations on M by $BV_{loc}(M) := \{u \in L^1_{loc}(M) : |Du|(U) < +\infty, U \subset\subset M\}$. So for any $u \in BV(M,g)$, we can associate a vector Radon measure on M $\nabla^g u$ with total variation $|\nabla^g u|$.

Definition 2.1.2. *Let (M,g) be a Riemannian manifold of dimension n, $U \subseteq M$ be an open subset, $\mathfrak{X}_c(U)$ the set of smooth vector fields with compact support in U. Given $E \subset M$ measurable with respect to the Riemannian measure, the* ***perimeter of*** *E* ***in*** *U, $\mathcal{P}_g(E,U) \in [0,+\infty]$, is*

$$\mathcal{P}_g(E,U) := \sup\left\{\int_U \chi_E div_g(X)dv_g : X \in \mathfrak{X}_c(U), ||X||_\infty \leq 1\right\}, \tag{2.4}$$

where $||X||_\infty := \sup\{|X_p|_g : p \in M\}$ and $|X_p|_g$ is the norm of the vector X_p in the metric g on T_pM. If $\mathcal{P}_g(E,U) < +\infty$ for every open set $U \subset\subset M$, we call E a ***locally finite perimeter set****. Let us set $\mathcal{P}_g(E) := \mathcal{P}_g(E,M)$. Finally, if $\mathcal{P}_g(E) < +\infty$ we say that E* ***is a set of finite perimeter****. We will use also the following notation $\mathcal{P}_g(E,F) := |\nabla\chi_E|_g(F)$ for every Borel set $F \subseteq M$.*

Before to prove Theorem 1 we prove Proposition 1 which is sufficient to prove Theorem 1. We postpone the proof of this last fact to the end of this section. In the proof of Proposition 1 we make frequent use of de Moser's iterative scheme, so we give an ad-hoc version of it in the following lemma which is suitable for our applications. In fact we borrowed the arguments of the proof from Proposition 8.15 of [GT01] and Théorème 2.3 of [Dru02a] in which only the case $p = 2$ is treated.

Definition 2.1.3. *Let us define the* ***p-Laplacian*** *of a C^2 function u defined on a Riemannian manifold (M^n,g) as the partial differential operator $\Delta_{p,g}u := -div_g(|\nabla_g u|_g^{p-2}\nabla_g u)$.*

Lemma 2.1.1 (Ad-hoc De Giorgi-Nash-Moser)**.** *Let (M^n,g) be a complete Riemannian manifold, $n \geq 2$, $1 < p < n$, and $v \in W^{1,p}_g(M) \cap L^\infty(M,g)$ with $0 \leq v \leq 1$, satisfying*

$$\Delta_{p,g}v \leq \Lambda v^{p^*-1}, \tag{2.5}$$

in the sense of distributions, where $\Lambda > 0$ *is independent of* p*. Then for any* x *in* M*, and for any* $\delta > 0$ *it holds*

$$||v||_{L^\infty(B_g(x,\delta/2))} \leq C \left(\int_{B_g(x,\delta)} v^{p^*} dv_g \right)^{\frac{1}{p^*}}, \tag{2.6}$$

where $C = C(x, M, g, n, \delta) > 0$ *does not depend on* p*.*

Remark 2.1.1. *Substituting the condition* $0 \leq v \leq 1$*, by* $\|v\|_{L^q(B(x,2\delta))} < K$ *for a suitable value of* K*, and* $q > p^*$*, get the same result, the proof is based on the Moser iterative scheme applied to* (2.5)*. See for example Lemmas* 3.1 *and* 3.2 *of* [*AL99*]*.*

Remark 2.1.2. *The constant* C *in strong bounded geometry, the preceding lemma could be chosen in such a way that* $C = C(n, \Lambda_1, \Lambda_2, inj_M)$*.*

Proof. Consider the inequality $\Delta_{p,g} v \leq \Lambda v^{p^*-1}$ in M, and $v \leq 1$, for some positive constant Λ independent of p. Consider a non-negative $\eta \in C_c^\infty(B_g(x,\delta))$ such that for $0 < r < s \leq \delta$ satisfies

i. $0 \leq \eta \leq 1$,

ii. $\eta \equiv 1$, in $B_g(x,r)$,

iii. $\eta \equiv 0$, in $B_g(x,\delta) \setminus B_g(x,s)$,

iv. $|\nabla_g \eta|_g \leq \frac{C_0}{s-r}$, where C_0 depends only on the geometry of (M,g) or on the bounds of the geometry in case M satisfy some condition of bounded geometry, for example in strong bounded geometry C_0 depends on $n, \Lambda_1, \Lambda_2, inj_M$.

It is straightforward to check that (2.5) is true in a weak form when we multiply it by any test function belonging to the functional space $W_0^{1,p}(B_g(x,\delta))$. Now multiply Equation (2.5) by the valid test function $\eta^p v^{k+1}$, for $0 < k \leq p^* - p$, and integrate it by parts over $B_g(x,s)$, this leads to

$$\int_{B_g(x,s)} |\nabla_g v|_g^{p-2} \langle \nabla_g v, \nabla_g(\eta^p v^{k+1}) \rangle_g dv_g \leq \int_{B_g(x,s)} \Lambda v^{p^*+k} \eta^p dv_g. \tag{2.7}$$

Let $w = v^{\frac{k+p}{p}}$, then $|\nabla_g w|_g^p = \left(\frac{k+p}{p}\right)^p v^k |\nabla_g v|_g^p$.

We observe that

$$
\begin{aligned}
&\int_{B_g(x,s)} |\nabla_g v|_g^{p-2} \langle \nabla_g v, \nabla_g(\eta^p v^{k+1})\rangle_g dv_g \\
&\quad =(k+1)\int_{B_g(x,s)} |\nabla_g v|_g^p v^k \eta^p dv_g + \int_{B_g(x,s)} |\nabla_g v|_g^{p-2} v^{k+1} \langle \nabla_g v, \nabla_g(\eta^p)\rangle_g dv_g \\
&\quad =(k+1)\left(\frac{k+p}{p}\right)^{-p}\int_{B_g(x,s)} |\eta\nabla_g w|_g^p dv_g + \int_{B_g(x,s)} |\nabla_g v|_g^{p-2} v^{k+1} \langle \nabla_g v, \nabla_g(\eta^p)\rangle_g dv_g \\
&\quad \geq(k+1)\left(\frac{k+p}{p}\right)^{-p}\int_{B_g(x,s)} |\eta\nabla_g w|_g^p dv_g - \int_{B_g(x,s)} |\nabla_g v|_g^{p-1} v^{k+1} |\nabla_g \eta^p|_g dv_g \\
&\quad =(k+1)\left(\frac{k+p}{p}\right)^{-p}\int_{B_g(x,s)} |\eta\nabla_g w|_g^p dv_g - p\int_{B_g(x,s)} |\nabla_g v|_g^{p-1} v^{k+1} \eta^{p-1} |\nabla_g \eta|_g dv_g \\
&\quad =(k+1)\left(\frac{k+p}{p}\right)^{-p}\int_{B_g(x,s)} |\eta\nabla_g w|_g^p dv_g - p\int_{B_g(x,s)} \left(|\nabla_g \eta|_g v^{\frac{k+p}{p}}\right)\left(|\nabla_g v|_g^{p-1}\eta^{p-1} v^{\frac{k(p-1)}{p}}\right) dv_g \\
&\quad =(k+1)\left(\frac{k+p}{p}\right)^{-p}\int_{B_g(x,s)} |\eta\nabla_g w|_g^p dv_g - p\left(\frac{k+p}{p}\right)^{1-p}\int_{B_g(x,s)} (w|\nabla_g \eta|_g)(\eta|\nabla_g w|_g)^{p-1} dv_g,
\end{aligned}
$$

where we applied the Cauchy-Schwarz's inequality. Later we have to use in the second integral on the right the Young's inequality in the following form

$$ab \leq \frac{(\theta^{-1}a)^p}{p} + \frac{(p-1)(\theta b)^{\frac{p}{p-1}}}{p},$$

with $\theta \in]0,+\infty[$. Set $a = w|\nabla_g \eta|_g$, $b = (\eta|\nabla_g w|_g)^{p-1}$, and choose $\theta > 0$ such that

$$\left(\frac{k+p}{p}\right)^{1-p}(p-1)\theta^{\frac{p}{p-1}} = \frac{1}{2}(k+1)\left(\frac{k+p}{p}\right)^{-p}, \tag{2.8}$$

we get

$$
\begin{aligned}
\int_{B_g(x,s)} |\nabla_g v|_g^{p-2}\langle \nabla_g v, \nabla_g(\eta^p v^{k+1})\rangle_g dv_g \quad &\geq \quad \frac{(k+1)}{2}\left(\frac{k+p}{p}\right)^{-p}\int_{B_g(x,s)} |\eta\nabla_g w|_g^p dv_g \\
&- \quad 2^{p-1}\left(\frac{p-1}{k+1}\right)^{p-1}\int_{B_g(x,s)} |w\nabla_g \eta|_g^p dv_g.
\end{aligned} \tag{2.9}
$$

Combining (2.7) and (2.9) leads to

$$
\begin{aligned}
\int_{B_g(x,s)} |\eta\nabla_g w|_g^p dv_g \leq & C_1\int_{B_g(x,s)} \eta^p v^{k+p^*} dv_g + C_2\int_{B_g(x,s)} |w\nabla_g \eta|_g^p dv_g \\
=& C_1\int_{B_g(x,s)} (\eta w)^p v^{p^*-p} dv_g + C_2\int_{B_g(x,s)} |w\nabla_g \eta|_g^p dv_g,
\end{aligned} \tag{2.10}
$$

where $C_1(p) = \frac{2}{k+1}\left(\frac{k+p}{p}\right)^p \Lambda$, and $C_2(p) = 2^p(p-1)^{(p-1)}$.

Independently we have the following computations

$$\int_{B_g(x,s)} |\nabla_g(\eta w)|_g^p dv_g \leq 2^{p-1}\left(\int_{B_g(x,s)} |\eta\nabla_g w|_g^p dv_g + \int_{B_g(x,s)} |w\nabla_g \eta|_g^p dv_g\right), \tag{2.11}$$

and by the Sobolev embedding we get

$$\begin{aligned}\left(\int_{B_g(x,s)} |\,\eta w|^{p^*} dv_g\right)^{\frac{p}{p^*}} &\leq C(n,p)\int_{B_g(x,s)} |\nabla_g(\eta w)|_g^p dv_g,\\ &\overset{(2.11)}{\leq} C_3\int_{B_g(x,s)} |\eta\nabla_g w|_g^p dv_g + C_3\int_{B_g(x,s)} |w\nabla_g\eta|_g^p dv_g\\ &\overset{(2.10)}{\leq} C_4\int_{B_g(x,s)} (\eta w)^p v^{p^*-p} dv_g + C_5\int_{B_g(x,s)} |w\nabla_g\eta|_g^p dv_g,\end{aligned}$$

where $C_3 = 2^{p-1}C(n,p)$, $C_4 = C_1C_3$, and $C_5 = C_2C_3 + C_3$.

But since $v \leq 1$, $0 \leq \eta \leq 1$ over $B_g(x,s)$, and $|\nabla_g\eta|_g \leq \frac{C_0}{s-r}$ we have that

$$\left(\int_{B_g(x,s)} |\eta w|^{p^*} dv_g\right)^{\frac{p}{p^*}} \leq \left[C_4 + C_5\left(\frac{C_0}{s-r}\right)^p\right]\int_{B_g(x,s)} |w|^p dv_g.$$

On the other hand we have $\left(\frac{k+p}{p}\right)^p \leq (k+1)^p$ for $k > 0$ and $1 < p < n$, then $C_1 \leq 2(k+1)^{n-1}\Lambda$, $C_2 \leq 2^n(n-1)^{n-1}$, $C_3 \leq 2^{n-1}C(n)$, $C_4 \leq 2^n(k+1)^{n-1}\Lambda C(n)$, $C_5 \leq 2^{n-1}(2^n(n-1)^{n-1}(k+1)^{n-1}+1)C(n)$. Then

$$C_4 + C_5\left(\frac{C_0}{s-r}\right)^p \leq 2^{n-1}C(n)\left(2(k+1)^{n-1}\Lambda + (2^n(n-1)^{n-1}+1)\left(\frac{C_0}{s-r}\right)^p\right).$$

Thus setting

$$B_0 = 2^{n-1}C(n)\left(2(k+1)^{n-1}\Lambda + (2^n(n-1)^{n-1}+1)\left(\frac{C_0}{s-r}\right)^p\right),$$

we get

$$\left(\int_{B_g(x,r)} |v|^{\frac{p^*(k+p)}{p}} dv_g\right)^{\frac{p}{p^*(k+p)}} \leq B_0^{\frac{1}{k+p}}\left(\int_{B_g(x,s)} |v|^{k+p} dv_g\right)^{\frac{1}{k+p}}. \tag{2.12}$$

Now we want to use the Moser's iterative scheme. Let us call

$$F(t,\rho) = \left(\int_{B_g(x,\rho)} v^t dv_g\right)^{1/t},$$

by the inequality (2.12) we get that

$$F\left((k+p)\frac{p^*}{p}, r\right) \leq B_0^{\frac{1}{k+p}} F((k+p), s). \tag{2.13}$$

Choose k_0 such that $(k_0+p) = p^*$, $s_0 = \delta$ and define for every $i \geq 1$

$$(k_i+p) = \frac{p^*}{p}(k_{i-1}+p) = \left(\frac{p^*}{p}\right)^i (k_0+p),\, s_i = \frac{\delta}{2} + \frac{\delta}{2^{i+1}}.$$

Make $k = k_i$, $s = s_i$ and $r = s_{i+1}$. Note that $s_i - s_{i+1} = \frac{\delta}{2^{i+2}}$, furthermore we get $k_i \to +\infty$ when $i \to +\infty$, because

$$k_{i+1} - k_i = (k_0 + p)\left(\frac{p^*}{p}\right)^i \left[\frac{p^*}{p} - 1\right] > 0.$$

Now we apply this to (2.13), and we obtain

$$F\left(\frac{k_i + p}{p}p^*, s_{i+1}\right) = F((k_{i+1} + p), s_{i+1}) \leq B_i^{\frac{1}{k_i+p}} F((k_i + p), s_i).$$

Then making the iteration yields

$$F((k_{i+1} + p), s_{i+1}) \leq \prod_{j=0}^{i} B_j^{\frac{1}{k_j+p}} F((k_0 + p), s_0) = \prod_{j=0}^{i} B_j^{\frac{1}{k_j+p}} F(p^*, \delta).$$

Taking $i \to \infty$ we have $s_i \to \frac{\delta}{2}$, $k_i \to +\infty$, and $||v||_{L^t} \to ||v||_\infty$ thus the expression above becomes

$$\|v\|_{L_g^\infty(B_g(x,\delta/2))} \leq \prod_{i=0}^{+\infty} B_i^{\frac{1}{k_i+p}} \left(\int_{B_g(x,\delta)} v^{p^*} dv_g\right)^{\frac{1}{p*}}. \tag{2.14}$$

It remains to prove the convergence of $\prod_{i=0}^{+\infty} B_i^{\frac{1}{k_i+p}}$ to a constant independent of p.

Since for p sufficiently close to 1 we can get that $\frac{p^*}{p} = \frac{n}{n-p} \leq 2$, we have

$$(k_i + 1) < (k_i + p) = \left(\frac{p^*}{p}\right)^i (k_0 + p) \leq 2^i(k_0 + n),$$

and making the choice $\tilde{\delta} = \min\{1, \delta\}$, we get

$$\begin{aligned}
B_i &= 2^{n-1}C(n)\left(2(k_i+1)^{n-1}\Lambda + (2^n(n-1)^{n-1}+1)\left(\frac{C_0}{s_{i+1}-s_i}\right)^p\right) \\
&\leq 2^{n-1}C(n)\left(2(k_0+n)^n 2^{in}\Lambda + (2^n(n-1)^{n-1}+1)\left(\frac{2^{(i+2)^p}C_0^p}{\delta^p}\right)\right) \\
&\leq 2^{in}2^{n-1}C(n)\left(2(k_0+n)^n\Lambda + (2^n(n-1)^{n-1}+1)2^{2n}C_0^n\tilde{\delta}^n\right) \\
&= 2^{in}\tilde{C}(n).
\end{aligned}$$

As it is easy to see from the definition of k_i we have

$$\frac{1}{2^i(k_0+n)} \leq \frac{1}{k_i+p} \leq \frac{1}{k_i+1}.$$

Let us define $\alpha_i := \frac{1}{2^i(k_0+n)}$, if $B_i < 1$ and $\alpha_i := \frac{1}{k_i+1}$, if $B_i \geq 1$, in any case we have

$$B_i^{\frac{1}{k_i+p}} \leq B_i^{\alpha_i}.$$

Then passing to the infinite products

$$\begin{aligned}\prod_{i=0}^{+\infty} B_i^{\frac{1}{k_i+p}} &\leq \prod_{i=0}^{+\infty} B_i^{\alpha_i} \leq \left(\prod_{i=0}^{+\infty} \tilde{C}^{\alpha_i}\right)\left(\prod_{i=0}^{+\infty} (2^{in})^{\alpha_i}\right) \\ &= \left(\tilde{C}^{\sum_{i=0}^{+\infty} \alpha_i}\right)\left(2^{n\sum_{i=0}^{+\infty} i\alpha_i}\right).\end{aligned}$$

Notice that

$$\sum_{i=0}^{+\infty} \alpha_i \quad \text{and} \quad \sum_{i=0}^{+\infty} i\alpha_i,$$

are convergent series. Then for values of p close to 1 we have

$$\|v\|_{L_g^{\infty}(B_g(x,\delta/2))} \leq C\left(\int_{B_g(x,\delta)} v^{p^*} dv_g\right)^{\frac{1}{p^*}}, \tag{2.15}$$

where C does not depend on p. □

The following proposition is proved in the original paper [Dru02c].

Proposition 2.1.1 (Proposition [Dru02c] page 2353)**.** *Let (M^n, g) be a complete Riemannian manifold of dimension $n \geq 2$. Let $x_0 \in M$, and $\varepsilon > 0$, let us define $\alpha_\varepsilon := \dfrac{n}{n+2} Sc_g(x_0) + \varepsilon$. Then for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that for any u in $C_c^\infty(B_g(x_0, r_\varepsilon))$ we have that*

$$\|u\|^2_{\frac{n}{n-1},g} \leq K(n,1)^2 \left(\|\nabla u\|^2_{1,g} + \alpha_\varepsilon \|u\|^2_{1,g}\right). \tag{2.16}$$

The preceding proposition justifies the following definition.

Definition 2.1.4. *Let (M, g) be a Riemannian manifold. Let us define $r^*_\varepsilon(M, g, x) \in [0, +\infty]$ as the supremum of all $r > 0$ such that (2.16) is satisfied. Of course we put by definition $r^*_\varepsilon(M, g, x) = 0$, if there is no such positive r_ε. We call $r^*_\varepsilon(M, g, x)$ the **Druet radius of** (M, g) **at** x. Let us define $r^*_\varepsilon(M, g) \in [0, +\infty]$ as the infimum of $r^*_\varepsilon(M, g, x)$ taken over all $x \in M$. We call $r^*_\varepsilon(M, g)$ the **Druet radius of** (M, g).*

Proposition 1. *Let (M, g) be a complete Riemannian manifold of dimension $n \geq 2$ with C^3-locally asymptotically strong bounded geometry smooth at infinity. For any $\varepsilon > 0$ there exists $r_\varepsilon = r_\varepsilon(M, g) > 0$ such that for any point $x_0 \in M$, any function $u \in C_c^\infty(B_g(x_0, r_\varepsilon))$, we have*

$$||u||^2_{\frac{n}{n-1},g} \leq K(n,1)^2 \left(||\nabla_g u||^2_{1,g} + \alpha_{\varepsilon,g}||u||^2_{1,g}\right), \tag{2.17}$$

where $\alpha_{\varepsilon,g} = \frac{n}{n+2} Sc_g(x_0) + \varepsilon$.

Remark 2.1.3. *We notice that the constant $r_\varepsilon = r_\varepsilon(M,g) > 0$ is obtained by contradiction and that the proof does not give an explicit effective lower bound on it.*

By Proposition 2.1.1, if (M^n, g) is complete then for any $x \in M$ we have $r^*_\varepsilon(M,g,x) > 0$. By Proposition 1 if (M^n, g) has C^3-locally asymptotic strong bounded geometry smooth at infinity, then we have $r^*_\varepsilon(M,g) > 0$. We want to study now a little bit of stability properties of Druet's radius with respect to the convergence of manifolds.

Lemma 2.1.2. *Suppose to have a sequence of pointed complete smooth Riemannian manifolds $(M_i, g_i, p_i) \to (M_\infty, g_\infty, p_\infty)$ in C^0 topology with $(M_\infty, g_\infty, p_\infty)$ smooth and $Sc_{g_i}(p_i) \to Sc_{g_\infty}(p_\infty)$. Then*

$$\liminf_{i\to+\infty} r^*_\varepsilon(M_i, g_i, p_i) \leq r^*_\varepsilon(M_\infty, g_\infty, p_\infty). \tag{2.18}$$

Remark 2.1.4. *The assumptions made in the preceding lemma are automatically fulfilled if (M,g) has C^2-asymptotically bounded geometry smooth at infinity and a fortiori also under the assumptions of Theorems 1, 3.*

Proof. In first, if $\liminf_{i\to+\infty} r^*_\varepsilon(M_i, g_i, p_i) = 0$, then there is nothing to prove. Now, suppose that $\liminf_{i\to+\infty} r^*_\varepsilon(M_i, g_i, p_i) = l > 0$, fix $0 < r < l$, then there exists $i_r \in \mathbb{N}$ such that for all $i \geq i_r$ it holds $r^*_\varepsilon(M_i, g_i, p_i) \geq r$. Consider $p_\infty \in M_\infty$, $u \in C^\infty_c(B_{g_\infty}(p_\infty, r))$, $B := B_{g_\infty}(p_\infty, r+1)$. Notice that $B_{g_\infty}(p_\infty, r) \subseteq B$. Take the sequence of diffeomorphisms $\phi_i : B \to M_i$ of Definition 1.0.4. Then by C^0 convergence we have that for every $u \in C^\infty_c(B_{g_\infty}(p_\infty, r))$ it holds

$$||\nabla_{g_\infty} u||^2_{1,g_\infty} = \lim_{i\to+\infty} ||\nabla_{g_i}(u \circ (\phi_i)^{-1})||^2_{1,g_i}, \tag{2.19}$$

$$||u||^2_{\frac{n}{n-1},g_\infty} = \lim_{i\to+\infty} ||u \circ (\phi_i)^{-1}||^2_{\frac{n}{n-1},g_i}, \tag{2.20}$$

$$||u||^2_{1,g_\infty} = \lim_{i\to+\infty} ||u \circ (\phi_i)^{-1}||^2_{1,g_i}, \tag{2.21}$$

and by $Sc_{g_i}(p_i) \to Sc_{g_\infty}(p_\infty)$ we get

$$\alpha_{\varepsilon,g_\infty} = \frac{n}{n+2} Sc_{g_\infty}(p_\infty) + \varepsilon = \lim_{i\to+\infty} \alpha_{\varepsilon,g_i} = \lim_{i\to+\infty} \frac{n}{n+2} Sc_{g_i}(\phi_i(p_\infty)) + \varepsilon. \tag{2.22}$$

Fix a r' satisfying $r < r' < l$, our choice of r and the fact that ϕ_i is an almost isometry imply that there exists an $\tilde{i} := \tilde{i}_{r',r} \in \mathbb{N}$ such that for every $i \geq \tilde{i}$, $\phi_i(B_{g_\infty}(p_\infty, r)) \subseteq$

$B_{g_i}(p_i, r')$, so $u \circ (\phi_i)^{-1} \in C_c^\infty(B_{g_i}(p_i, r'))$, and it holds

$$||u \circ (\phi_i)^{-1}||^2_{\frac{n}{n-1}, g_i} \le K(n,1)^2 \left(||\nabla_{g_i}(u \circ (\phi_i)^{-1})||^2_{1,g_i} + \alpha_{\varepsilon, g_i} ||u \circ (\phi_i)^{-1}||^2_{1,g_i} \right), \tag{2.23}$$

whenever $u \in C_c^\infty(B_{g_\infty}(p_\infty, r))$. Taking the limit in the preceding equation and using (2.19), (2.20), (2.21), (2.22) we obtain for every $u \in C_c^\infty(B_{g_\infty}(p_\infty, r))$ it is true that

$$\|u\|^2_{\frac{n}{n-1}, g_\infty} \le K(n,1)^2 \left(\|\nabla u\|^2_{1,g_\infty} + \alpha_\varepsilon \|u\|^2_{1,g_\infty} \right). \tag{2.24}$$

From (2.31) readily follows that $r^*_\varepsilon(M_\infty, g_\infty, p_\infty) \ge r$, for every $r < l$ which implies that $r^*_\varepsilon(M_\infty, g_\infty, p_\infty) \ge l$ and this finish the proof. □

The following lemma is a straightforward consequence of the definitions.

Lemma 2.1.3. *Suppose to have a sequence of pointed complete smooth Riemannian manifolds $(M_i, g_i, p_i) \to (M_\infty, g_\infty, p_\infty)$ in C^0 topology with $(M_\infty, g_\infty, p_\infty)$, then for every point $x_0 \in M_\infty$ there exists a sequence of points $\tilde{p}_i$ that depends on x_0, such that $d_{g_i}(\tilde{p}_i, p_i)$ is a uniformly bounded sequence and $(M_i, g_i, \tilde{p}_i) \to (M_\infty, g_\infty, x_0)$ in C^0 topology.*

Combining these last two lemmas one easily prove the following.

Lemma 2.1.4. *Suppose to have a sequence of pointed complete smooth Riemannian manifolds $(M_i, g_i, p_i) \to (M_\infty, g_\infty, p_\infty)$ in C^0 topology with $(M_\infty, g_\infty, p_\infty)$ smooth and $Sc_{g_i} \circ (\phi_i)^{-1} \to Sc_{g_\infty}$ where $(\phi_i)_i$ is one of the diffeomorphisms sequence of Definition 1.0.4 . Then*

$$\liminf_{i \to +\infty} r^*_\varepsilon(M_i, g_i) \le r^*_\varepsilon(M_\infty, g_\infty). \tag{2.25}$$

However we give here a proof of Lemma 2.1.4.

Proof. In first, if $\liminf_{i\to+\infty} r^*_\varepsilon(M_i, g_i) = 0$, then there is nothing to prove. Now, suppose that $\liminf_{i\to+\infty} r^*_\varepsilon(M_i, g_i) = l > 0$, fix $0 < r < l$, then there exists $i_r \in \mathbb{N}$ such that for all $i \ge i_r$ it holds $r^*_\varepsilon(M_i, g_i) \ge r$. Consider $x_0 \in M_\infty$, $u \in C_c^\infty(B_{g_\infty}(x_0, r))$, $B := B_{g_\infty}(p_\infty, d_{g_\infty}(p_\infty, x_0) + r + 1)$. Notice that $B_{g_\infty}(x_0, r) \subseteq B$. Take the sequence of diffeomorphisms $\phi_i : B \to M_i$ of Definition 1.0.4. Then by C^0 convergence we have that for every $u \in C_c^\infty(B_{g_\infty}(x_0, r))$ it holds

$$||\nabla_{g_\infty} u||^2_{1,g_\infty} = \lim_{i\to+\infty} ||\nabla_{g_i}(u \circ (\phi_i)^{-1})||^2_{1,g_i}, \tag{2.26}$$

$$||u||^2_{\frac{n}{n-1},g_\infty} = \lim_{i\to+\infty} ||u\circ(\phi_i)^{-1}||^2_{\frac{n}{n-1},g_i}, \tag{2.27}$$

$$||u||^2_{1,g_\infty} = \lim_{i\to+\infty} ||u\circ(\phi_i)^{-1}||^2_{1,g_i}, \tag{2.28}$$

and by $Sc_{g_i} \to Sc_{g_\infty}$ we get

$$\alpha_{\varepsilon,g_\infty} = \frac{n}{n+2}Sc_{g_\infty}(x_0)+\varepsilon = \lim_{i\to+\infty}\alpha_{\varepsilon,g_i} = \lim_{i\to+\infty}\frac{n}{n+2}Sc_{g_i}(\phi_i(x_0))+\varepsilon. \tag{2.29}$$

Fix an r' satisfying $r<r'<l$, our choice of r and the fact that ϕ_i is an almost isometry imply that there exists an $\tilde{i} := \tilde{i}_{r',r} \in \mathbb{N}$ such that for every $i \geq \tilde{i}$, $\phi_i(B_{g_\infty}(x_0,r)) \subseteq B_{g_i}(\phi_i(x_0),r')$, so $u\circ(\phi_i)^{-1} \in C^\infty(B_{g_i}(\phi_i(x_0),r'))$, and it holds

$$||u\circ(\phi_i)^{-1}||^2_{\frac{n}{n-1},g_i} \leq K(n,1)^2\left(||\nabla_{g_i}(u\circ(\phi_i)^{-1})||^2_{1,g_i}+\alpha_{\varepsilon,g_i}||u\circ(\phi_i)^{-1}||^2_{1,g_i}\right), \tag{2.30}$$

whenever $u \in C^\infty_c(B_{g_\infty}(x_0,r))$. Taking the limit in the preceding equation and using (2.28), (2.27), (2.26), (2.29) we obtain for every $u\in C^\infty_c(B_{g_\infty}(x_0,r))$ it is true that

$$\|u\|^2_{\frac{n}{n-1},g_\infty} \leq K(n,1)^2\left(\|\nabla u\|^2_{1,g_\infty}+\alpha_{\varepsilon,g_\infty}\|u\|^2_{1,g_\infty}\right). \tag{2.31}$$

From (2.31) readily follows that $r^*_\varepsilon(M_\infty,g_\infty,x_0)\geq r$, for every $r<l$ which implies that $r^*_\varepsilon(M_\infty,g_\infty,x_0)\geq l$, for every $x_0\in M_\infty$ which in turn proves (2.25). □

We state here a Corollary of Lemma 2.1.4 that will be used into the proof of Theorem 1. The proof is immediate and is left to the reader.

Corollary 2.1.1. *Let (M,g) be a complete Riemannian manifold of dimension $n\geq 2$ with C^3-locally asymptotically strong bounded geometry smooth at infinity, $p_i\to\infty$ and $(M,g,p_i)\to(M_\infty,g_\infty,p_\infty)$. Then for every $\varepsilon>0$ we have $r^*_\varepsilon(M_\infty,g_\infty)\geq r^*_\varepsilon(M,g)>0$.*

Proof of Proposition 1. For any $x_0\in M$, for any $r>0$, any $p>1$ and any $\varepsilon>0$, set

$$\lambda_{p,r,g}(x_0) := \inf_{\substack{u\in C^\infty_c(B_g(x_0,r))\\ u\not\equiv 0}} \frac{\left(\int_{B_g(x_0,r)}|\nabla_g u|^p dv_g\right)^{2/p}+\alpha_\varepsilon\left(\int_{B_g(x_0,r)}|u|^p dv_g\right)^{2/p}}{\left(\int_{B_g(x_0,r)}|u|^{p^*}dv_g\right)^{2/p^*}},$$

where $B_g(x_0,r)\subseteq M$ is the geodesic ball (M,g) centered at $x_0\in M$ and of radius $r>0$. We will argue the theorem by contradiction. With this aim in mind suppose that there exists $\varepsilon_0>0$ such that for every $r>0$ there exists a point $x_{0,r}$ depending on r such that it holds

$$\lambda_{1,r,g}(x_{0,r}) < K(n,1)^{-2}.$$

As it is easy to check from the very definition of $\lambda_{p,r,g}(x_{0,r})$, we have that $\limsup_{p\to 1^+}\lambda_{p,r,g}(x_{0,r}) \le \lambda_{1,r,g}(x_{0,r})$, which implies that for any $r > 0$, there exists $p_r(x_{0,r}) > 1$ such that

$$\lambda_{p_r,r,g}(x_{0,r}) < K(n,1)^{-2}\left(\frac{n-p_r(x_{0,r})}{p_r(x_{0,r})(n-1)}\right),\ \lambda_{p_r,r,g}(x_{0,r}) < K(n,p_r(x_{0,r}))^{-2}. \tag{2.32}$$

We may assume that $r \searrow 0$ and choose $p_r(x_{0,r})$ decreasing when r is decreasing. Then inverting this sequence we get a sequence $p > 1$ going to 1^+ a sequence $r_p > 0$ going to 0^+ as p goes to 1^+, and a sequence of points $x_{0,p} := x_{0,r_p} \in M$ which verify (2.32). Notice here that in general the sequence of points $x_{0,p}$ could go to infinity when $p \to 1^+$. This is the main difficulty we encounter in adapting the original proof of Theorem 1 of [Dru02c] in case of noncompact ambient manifolds. Set $\alpha_p := \dfrac{n}{n+2}Sc_g(x_{0,p}) + \varepsilon_0$. Now up to a subsequence we can assume that

$$\lim_{p\to 1^+}\alpha_p = \frac{n}{n+2}l_1 + \varepsilon_0, \tag{2.33}$$

for some $l_1 \in [S_{inf,g}, S_g]$, where $S_{inf,g} := \inf\{Sc_g(x) : x \in M\}$ and $S_g := \sup\{Sc_g(x) : x \in M\}$. It is worth to note that $S_{inf,g}$ and S_g are finite real numbers, because Sc_g is bounded from below by $n(n-1)k$ and from above by $n(n-1)k_0$. The second equation in (2.32) can be written as $\lambda_{p,r_p}(x_{0,p}) < K(n,p)^{-2}$, and by Theorems 1.1, 1.3, 1.5 of [Dru00], we have the existence of a minimizer u_p which satisfies

$$\begin{aligned} C_p\Delta_{p,g}u_p + \alpha_p\|u_p\|_{p,g}^{2-p}u_p^{p-1} &= \lambda_p u_p^{p^*-1}, \quad \text{in } B_g(x_{0,p},r_p),\\ u_p &\in C^{1,\eta}(B_g(x_{0,p},r_p)), \quad \text{for some } \eta > 0,\\ u_p > 0, \quad \text{in } B_g(x_{0,p},r_p), \quad u_p &= 0, \quad \text{in } \partial B_g(x_{0,p},r_p), \end{aligned} \tag{2.34}$$

$$\int_{B_g(x_{0,p},r_p)} u_p^{p^*}dv_g = 1, \tag{2.35}$$

$$\lambda_p < K(n,p)^{-2}, \quad \lambda_p < K(n,1)^{-2}\left(\frac{n-p}{p(n-1)}\right)^2, \tag{2.36}$$

$$C_p := \left(\int_{B_g(x_{0,p},r_p)} |\nabla_g u_p|_g^p dv_g\right)^{\frac{2-p}{p}}, \tag{2.37}$$

where $\lambda_p := \lambda_{p,r_p}(x_{0,p})$, $\Delta_{p,g}$ is the p-Laplacian with respect to g, defined by $\Delta_{p,g}u := -div_g(|\nabla_g u|^{p-2}\nabla_g u)$, with $\nabla_g u$ being the gradient of u with respect to the metric g. As a consequence of (2.34) we have

$$\|\nabla_g u_p\|_{p,g}^2 + \alpha_p\|u_p\|_{p,g}^2 = \lambda_p\|u_p\|_{p^*,g}^{p^*}. \tag{2.38}$$

The strategy that we will adopt to go head in this proof is concerned with the study of the sequence (u_p) as $p \to 1^+$. With this aim in mind, let x_p be a point in $B_g(x_{0,p}, r_p)$ where u_p achieves its maximum (x_p tends to infinity, iff $x_{0,p}$ tends to infinity) and we define

$$u_p(x_p) = \mu_p^{1-\frac{n}{p}}.$$

Observing that $u_p(x_p)^{p^*} = \mu_p^{-n}$ we get

$$1 = \int_{B_g(x_{0,p},r_p)} u_p^{p^*} dv_g \leq V_g(B_g(x_{0,p}, r_p))\mu_p^{-n} \leq C_0(n,k) r_p^n \mu_p^{-n}, \tag{2.39}$$

where the last inequality is due to Bishop-Gromov. Since r_p goes to 0, from (2.39) we conclude that μ_p goes to 0 as p goes to 1^+, moreover $\mu_p = O(r_p)$ and the constant $C_0 = C_0(n,k)$ is uniform with respect to p, i.e., is uniform with respect to the location of $x_{0,p}$ inside M. Analogously, applying Hölder's inequalities, with $q = \frac{n}{n-p} > 1$ and Bishop-Gromov yields

$$\lim_{p\to 1^+} \int_{B_g(x_{0,p},r_p)} u_p^p dv_g \leq \lim_{p\to 1^+} \left\{ \int_{B_g(x_{0,p},r_p)} u_p^{p^*} dv_g \right\}^{\frac{1}{q}} V_g(B_g(x_{0,p}, r_p))^{\frac{1}{q'}} = 0, \tag{2.40}$$

here q' denotes the conjugate exponent of q, i.e., $\frac{1}{q} + \frac{1}{q'} = 1$.

Step 1 In this first step we want to show the validity of the two following equations

$$\lim_{p\to 1^+} \lambda_p = K(n,1)^{-2}, \tag{2.41}$$

and

$$\lim_{p\to 1^+} \int_{B_g(x_{0,p},r_p)} |\nabla_g u_p|_g^p dv_g = K(n,1)^{-1}. \tag{2.42}$$

By Theorem 7.1 of [Heb99], it follows that for all $\varepsilon > 0$ there exists $B_\varepsilon = B_\varepsilon(n,k,p,inj_{M,g}) > 0$ such that for any $p > 1$,

$$\begin{aligned}\left(\int_{B_g(x_{0,p},r_p)} u_p^{p^*} dv_g\right)^{2\frac{n-1}{n}} &\leq (K(n,1)+\varepsilon)^2 \left(\int_{B_g(x_{0,p},r_p)} \left|\nabla_g \left(u_p^{\frac{p(n-1)}{n-p}}\right)\right|_g dv_g\right)^2 \\ &+ B_\varepsilon \left(\int_{B_g(x_{0,p},r_p)} u_p^{\frac{p(n-1)}{n-p}} dv_g\right)^2,\end{aligned}$$

which gives with (2.38), (2.35) and Hölder's inequalities

$$\begin{aligned}1 &\leq (K(n,1)+\varepsilon)^2 \left(\frac{p(n-1)}{n-p}\right)^2 (\lambda_p - \alpha_p \|u_p\|_p^2) + B_\varepsilon \|u_p\|_p^2 \\ &\leq (K(n,1)+\varepsilon)^2 \left(\frac{p(n-1)}{n-p}\right)^2 (\lambda_p - \alpha_0 \|u_p\|_p^2) + B_\varepsilon \|u_p\|_p^2,\end{aligned}$$

where $\alpha_0 := \frac{n}{n+2}n(n-1)k + \varepsilon_0 \leq \alpha_p$. This combined with (2.40) give

$$1 \leq \left(1 + \varepsilon K(n,1)^{-1}\right)^2 \liminf_{p\to 1^+} \left(\lambda_p K(n,1)^2\right).$$

Since this inequality is true for every $\varepsilon > 0$, letting $\varepsilon \to 0$ we obtain $\liminf_{p\to 1^+} \lambda_p \geq K(n,1)^{-2}$. Using the fact that $\lambda_p < K(n,p)^{-2}$, and $\lambda_p < K(n,1)^{-2}\left(\dfrac{n-p}{p(n-1)}\right)^2$, we conclude that

$$K(n,1)^{-2} \leq \liminf_{p\to 1^+} \lambda_p \leq \liminf_{p\to 1^+} K(n,p)^{-2},$$

and thus (2.41) is proved.

Remark 2.1.5. *Until this point we have just used the assumption of mild bounded geometry, i.e., Ricci bounded below and positive injectivity radius.*

Now, it easily seen that (2.42) is an obvious consequence of (2.34), (2.35), (2.40), and (2.41).

Remark 2.1.6. *To prove rigorously* (2.42) *we will use that the scalar curvature is bounded from both sides and Ricci curvature is bounded below and the injectivity radius is positive.*

To prove the equality (2.42), we use (2.38) and the fact that under the assumptions of Proposition 1, we have $n(n-1)k \leq Sc_g < n(n-1)k_0$. It follows immediately that (α_p) is a bounded sequence, i.e., $\alpha_p = O(1)$. Taking $p \to 1^+$, in (2.38) and (2.35) we obtain that

$$\lim_{p\to 1^+} \|\nabla_g u_p\|_{p,g}^2 + \alpha_p \|u_p\|_{p,g}^2 = \lim_{p\to 1^+} \lambda_p \|u_p\|_{p^*,g}^{p^*} = \lim_{p\to 1^+} \lambda_p,$$

then using the above result, we can conclude that

$$\lim_{p\to 1^+} \|\nabla_g u_p\|_{p,g} = K(n,1)^{-1}.$$

Step 2 Let $\Omega_p := \mu_p^{-1} \exp_{x_p}^{-1}(B_g(x_{0,p}, r_p)) \subset T_{x_p}M \cong \mathbb{R}^n$, the metric $g_p(x) := \exp_{x_p}^* g(\mu_p x)$ for $x \in \Omega_p$, and the function given by $v_p(x) = \mu_p^{\frac{n}{p}-1} u_p(\exp_{x_p}(\mu_p x))$ for $x \in \Omega_p$, $v_p(x) = 0$ in $x \in \mathbb{R}^n \setminus \Omega_p$. It is worth mentioning that the definition of v_p for p close to 1 is well posed and does not give any problem, since we suppose that the injectivity radius of M is strictly positive and $r_p \to 0$ as $p \to 1^+$. Then substituting in (2.34) we obtain v_p satisfies

$$C_p \Delta_{p,g_p} v_p + \alpha_p \mu_p^2 \|v_p\|_{p,g_p}^{2-p} v_p^{p-1} = \lambda_p v_p^{p^*-1}, \text{in } \Omega_p, \tag{2.43}$$

with $v_p = 0$ in $\partial\Omega_p$, and

$$\int_{\Omega_p} v_p^{p^*} dv_g = 1. \tag{2.44}$$

Thus v_p satisfies also

$$\|\nabla_{g_p} v_p\|_{p,g_p}^2 + \alpha_p \mu_p^2 \|v_p\|_{p,g_p}^2 = \lambda_p \|v_p\|_{p^*,g_p}^{p^*} = \lambda_p. \tag{2.45}$$

Unfortunately the sequence (v_p) is not bounded in $W_\xi^{1,1}(\mathbb{R}^n)$ so we need to take another auxiliary sequence $(\tilde{v}_p)$ related in some way to the previous one and is bounded in $W_\xi^{1,1}(\mathbb{R}^n)$. We do this because we are interested in a limit function v_0 that realizes the minimum of the problem at infinity and so it is expected to be the characteristic function of a ball. To realize this strategy we look for powers of the function v_p. As we will see later a suitable choice is the following

$$\tilde{v}_p(x) = v_p(x)^{\frac{p(n-1)}{n-p}}. \tag{2.46}$$

It is useful to recall here that for every $x \in M$ the exponential map exp_x is a bi-Lipschitz map of an open geodesic ball centered at x having radius inj_x over a ball of $\mathbb{R}^n$ having the same radius, with Lipschitz constant L_x that in general depend on x, however by the Rauch's comparison Theorem we know that L_x can be bounded by a constant that depends just on $n, \Lambda_1, \Lambda_2, inj_M$, which in turn permit to conclude that under our assumption of strong bounded geometry the constants L_x are uniformly bounded with respect to x by a positive constant that depends only on the bounds on the geometry, namely $n, \Lambda_1, \Lambda_2, inj_M$. Hence using the Cartan's expansion of the metric g_p close to x_p we can show the existence of a positive constant $C = C(n, \Lambda_1, \Lambda_2, inj_M) > 0$, such that for any $x \in \Omega_p$,

$$(1 - C\mu_p^2|x|^2)dv_{g_p} \leq dv_\xi \leq (1 + C\mu_p^2|x|^2)dv_{g_p}. \tag{2.47}$$

From this we conclude that there exists another constant again denoted by $C = C(n, \Lambda_1, \Lambda_2, inj_M) > 1$ such that

$$dv_{g_p} \geq \left(1 - \frac{1}{C}\mu_p^2\right) dv_\xi, \tag{2.48}$$

$$|\nabla_\xi v_p|_\xi^p dv_\xi \leq (1 + C\mu_p^2)|\nabla_{g_p} v_p|_{g_p}^p dv_{g_p}, \tag{2.49}$$

where ξ is the Euclidean metric. Equations (2.47), (2.48), (2.49) with (2.42), (2.44) and Hölder's inequalities leads to

$$\lim_{p\to 1^+} \frac{\int_{\mathbb{R}^n} |\nabla_\xi \tilde{v}_p|_\xi dv_\xi}{\left(\int_{\mathbb{R}^n} \tilde{v}_p^{\frac{n}{n-1}} dv_\xi\right)^{\frac{n-1}{n}}} = K(n,1)^{-1}. \tag{2.50}$$

To show this observe that by (2.47), (2.48), (2.49) $\left(\int_{\mathbb{R}^n} \tilde{v}_p^{\frac{n}{n-1}} dv_\xi\right)^{\frac{n-1}{n}} \sim \int_{\Omega_p} v_p^{p^*} dv_g = 1$, when $p \to 1^+$. To see what happens to the numerator of (2.50) just look at (2.51) below

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla_\xi \tilde{v}_p|_\xi dv_\xi &= \int_{\mathbb{R}^n} \frac{p(n-1)}{n-p} v_p^{\frac{n(p-1)}{n-p}} |\nabla_\xi v_p|_\xi dv_\xi \\
&\le \frac{p(n-1)}{n-p} \left\{\int_{\mathbb{R}^n} v_p^{p^*} dv_\xi\right\}^{\frac{p}{p-1}} \left\{\int_{\mathbb{R}^n} |\nabla_\xi v_p|_\xi^p dv_\xi\right\}^{\frac{1}{p}} \\
&\le \frac{p(n-1)}{n-p}(1 + C\mu_p^2)||\nabla_{g_p} v_p||_{p,g_p} \\
&= \frac{p(n-1)}{n-p}(1 + C\mu_p^2)||\nabla_g u_p||_{p,g} \to K(n,1)^{-1}.
\end{aligned} \tag{2.51}$$

The last equality is a consequence of (2.42) and the following calculation

$$\begin{aligned}
\int_{\Omega_p} |\nabla_{g_p} v_p(x)|_{g_p}^r dv_{g_p} &= \int_{\Omega_p} |\nabla_{g_p}(\mu_p^{\frac{n-p}{p}} u_p(\exp_{x_p}(\mu_p x)))|_{g_p}^r dv_{g_p}(x) \\
&= \mu_p^{(\frac{n-p}{p})r} \int_{\mu_p^{-1} \exp_{x_p}^{-1}(B_g(x_{0,p}, r_p))} |\nabla_{g_p} u_p(\exp_{x_p}(\mu_p x))|_{g_p}^r dv_{g_p}(x) \\
&= \mu_p^{\frac{n-p}{p}r} \mu_p^{-n} \mu_p^r \int_{\exp_{x_p}^{-1}(B_g(x_{0,p}, r_p))} |\nabla_{g_p} u_p(\exp_{x_p}(x))|_{g_p}^r dv_{g_p}(\mu_p^{-1} x) \\
&= \mu_p^{\frac{n(r-p)}{p}} \int_{\exp_{x_p}^{-1}(B_g(x_{0,p}, r_p))} |\nabla_{g_p} u_p(\exp_{x_p}(x))|_{g_p}^r dv_{g_p}(\mu_p^{-1} x) \\
&= \mu_p^{\frac{n(r-p)}{p}} \int_{B_g(x_{0,p}, r_p)} |\nabla_g u_p(x)|_g^r dv_g,
\end{aligned}$$

from which follows

$$||\nabla_{g_p} v_p||_{r,g_p}^r = \mu_p^{\frac{n(r-p)}{p}} ||\nabla_g u_p||_{r,g}^r.$$

Remember here that $r_p \to 0$ as $p \to 1^+$. Notice that by (2.2), $(\tilde{v}_p)$ is bounded in $W_\xi^{1,1}(\mathbb{R}^n)$. Thus there exists $v_0 \in BV_{loc}(\mathbb{R}^n)$ such that

$$\lim_{p\to 1^+} \tilde{v}_p = v_0, \quad \text{strictly in } BV_{loc}(\mathbb{R}^n),$$

this means that $\tilde{v}_p \to v_0$ in $L^1_{loc}(\mathbb{R}^n)$ and $||\nabla \tilde{v}_p||_{1,\xi}(K) \to |Dv_0|(K)$, $\forall K \subset\subset \mathbb{R}^n$. For a proof of this fact see Thm. 3.23 of [AFP00]. If we apply the concentration-compactness principle of P.L. Lions ([Lio84], [Lio85], see also [Str08] for an exposition in book form) to $|v_p|^{p^*}dv_\xi$, four situations may occur: compactness, concentration, dichotomy or vanishing. Dichotomy is classically forbidden by (2.50). To be convinced of this fact the reader could mimic the proof of Theorem 4.9 of [Str08]. Concentration without compactness cannot happen since $\sup_{\Omega_p} v_p = v_p(0) = 1$. As for vanishing, since v_p is bounded in L^∞, by applying Moser's iterative scheme (see for instance Theorem 1 [Ser64]) to (2.43), one gets the existence of some $C = C(n, \alpha_p C_p^{-1}\mu_p^2||v_p||_p^{2-p}, C_p^{-1}\lambda_p, ||g||_{0,r}) > 0$ such that for any $p > 1$,

$$1 = \sup_{\Omega_p\cap B_{g_p}(0,1/2)} v_p \leq C\left(\int_{\Omega_p\cap B_{g_p}(0,1)} v_p^{p^*}dv_{g_p}\right)^{\frac{1}{p^*}}, \tag{2.52}$$

where $||g||_{0,r}$ is the norm defined at page 308 of [Pet06] (see Definition 1.0.5). Since a careful analysis of the proof of Theorem 1 of [Ser64] combined with (2.37), (2.40), (2.41), (2.42), (which imply, by a change of variables in the integrals, that $\alpha_p\mu_p^2||v_p||_p^{2-p} = \alpha_p\mu_p^p||u_p||^{2-p} \to 0$, thanks to the fact that $\alpha_p \to \frac{n}{n+2}l_1 + \varepsilon_0 \in \mathbb{R}$, hence α_p is uniformly bounded), and the C^0 convergence of the metric tensor due to Theorems 72 and 76 of [Pet06], when $p \to 1^+$, shows that C is uniformly bounded with respect to p. Thus vanishing cannot happen. Another way to see that our problem have no vanishing is to apply directly Lemma 2.1.1 with $g = g_p$ and $v = v_p$, this is justified because in equation (2.43) we know that $\alpha_p\mu_p^2\|v_p\|_{p,g_p}^{2-p} \to 0$, and $C_p^{-1}\lambda_p \to K(n,1)^{-1}$. Then for p close to 1, we can consider that v_p satisfies the following inequality in the sense of distributions

$$\Delta_{p,g}v_p \leq \Lambda v_p^{p^*-1},$$

where Λ depends only on n. Compactness implies that $|v_p|^{p^*}dv_\xi \to |v_0|^{\frac{n}{n-1}}dv_\xi$ that is $\|\tilde{v}_p\|_{\frac{n}{n-1}} \to \|v_0\|_{\frac{n}{n-1}}$. To see this we observe that by the compactness case of the concentration-compactness principle we have that for all $\varepsilon > 0$ there exist $R_\varepsilon > 0$ and $p_\varepsilon > 0$ such that

$$1-\varepsilon \leq \int_{B_\xi(0,R_\varepsilon)} v_p^{p^*}dv_\xi \leq 1+\varepsilon, \quad p \leq p_\varepsilon,$$

passing to the limit when $p \to 1^+$ yields $\int_{\mathbb{R}^n} v_0^{\frac{n}{n-1}} = 1$, since

$$1-\varepsilon \leq \int_{B_\xi(0,R_\varepsilon)} v_0^{\frac{n}{n-1}}dv_\xi = \lim_{p\to 1^+}\int_{B_\xi(0,R_\varepsilon)} v_p^{p^*}dv_\xi \leq 1+\varepsilon,$$

and

$$\int_{\mathbb{R}^n} v_0^{\frac{n}{n-1}} dv_\xi = \lim_{\varepsilon\to 0^+} \int_{B_\xi(0,R_\varepsilon)} v_0^{\frac{n}{n-1}} dv_\xi. \tag{2.53}$$

It is clear that $\|\tilde{v}_p\|_{\frac{n}{n-1}}$ is bounded by all $p > 1$, on the other hand, as is well known $L^{\frac{n}{n-1}}(\mathbb{R}^n)$ is a reflexive Banach space thus $\tilde{v}_p \rightharpoonup v_0$ weakly in $L^{\frac{n}{n-1}}(\mathbb{R}^n)$. A classical result ensures that weak convergence and convergence of norms as in (2.53) gives $\tilde{v}_p \to v_0$ strongly in $L^{\frac{n}{n-1}}(\mathbb{R}^n)$.

Since we have that $\int_{\mathbb{R}^n} |\nabla \tilde{v}_p| dv_\xi \to \int_{\mathbb{R}^n} |\nabla v_0| dv_\xi = K(n,1)^{-1}$. Then v_0 is a minimizer for the $W^{1,1}$ Euclidean Sobolev inequality which verifies

$$\int_{\mathbb{R}^n} v_0^{\frac{n}{n-1}} dv = 1.$$

Thus there exists $y_0 \in \mathbb{R}^n$, $\lambda_0 > 0$ and $R_0 > 0$ such that

$$v_0 = \lambda_0 \mathbf{1}_{B_\xi(y_0,R_0)}, \tag{2.54}$$

where $\mathbf{1}_{B_\xi(y_0,R_0)}$ denotes the characteristic function of the Euclidean ball $B_\xi(y_0,R_0)$, and moreover, since $v_p \leq 1$ in Ω_p we obtain by pointwise convergence a.e. dv_ξ that $0 \leq \lambda_0 \leq 1$. On the other hand $v_p \leq 1$ and the strong convergence in $L^{\frac{n}{n-1}}(\mathbb{R}^n)$ give that for all $q \geq \frac{n}{n-1}$, $\tilde{v}_p \to v_0$ strongly in $L^q(\mathbb{R}^n)$. Therefore

$$\lambda_0^q V_\xi(B_\xi(y_0,R_0)) = \lim_{p\to 1^+} \int_{\mathbb{R}^n} \tilde{v}_p^{\frac{n}{n-1}} dv_\xi = 1, \ \forall q \geq \frac{n}{n-1}.$$

Taking the limit when $q \to +\infty$ we deduce that λ_0 cannot be strictly less than 1, thus we get $\lambda_0 = 1$. So we have

$$V_\xi(B_\xi(y_0,R_0)) = \frac{\omega_{n-1}}{n} R_0^n = 1. \tag{2.55}$$

Up to changing x_p into $\exp_{x_p}(\mu_p y_0)$ in the definition of v_p, Ω_p and g_p, we may assume that $y_0 = 0$. In particular we have

$$\lim_{p\to 1} \tilde{v}_p = \mathbf{1}_{B_\xi(0,R_0)}, \quad \text{strongly in } L^{\frac{n}{n-1}}(\mathbb{R}^n), \tag{2.56}$$

and that for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\lim_{p\to 1} \int_{\mathbb{R}^n} |\nabla \tilde{v}_p|_\xi^p \varphi dv_\xi = \int_{\partial B_\xi(0,R_0)} \varphi d\sigma_\xi, \tag{2.57}$$

where $d\sigma_\xi$ is the $(n-1)$-dimensional Riemannian measure of $\partial B_\xi(0,R_0)$ induced by the metric ξ of $\mathbb{R}^n$. Consider the extremal functions $V_p \in W^{1,p}(\mathbb{R}^n)$ for $K(n,p)^{-p}$ defined below

$$V_p(x) = \left(1 + \left(\frac{|x|}{R_0}\right)^{\frac{p}{p-1}}\right)^{1-\frac{n}{p}}, \quad x \in \mathbb{R}^n, \tag{2.58}$$

a simple application of the concentration-compactness principle, using (2.57), gives

$$\lim_{p\to 1^+} \int_{\mathbb{R}^n} |\nabla_\xi(\tilde{v}_p - V_p)|_\xi dv_\xi = 0. \tag{2.59}$$

Applying again the Moser's iterative scheme Lemma 2.1.1 to (2.43) with the help of (2.56), we also get that for any $R > R_0$,

$$\lim_{p\to 1^+} \sup_{\Omega_p \setminus B_{g_p}(0,R)} v_p = 0. \tag{2.60}$$

The application of Moser's iterative scheme is possible in strong bounded geometry because of the same arguments leading to (2.52).

Step 3 In this step we want to obtain from the $L^{\frac{n}{n-1}}$-estimate (2.56) the pointwise estimates (2.72), (2.73) which gives estimates on the decay rate to zero of $v_p(z)$ when $|z| \to +\infty$. For more details one can see for instance [Dru02b] and [Dru99]. With this aim in mind let us define

$$w_p(x) = |x|^{\frac{n}{p}-1} v_p(x), \tag{2.61}$$

and let $z_p \in \Omega_p$ be a point where w_p attains its maximum, i.e.,

$$w_p(z_p) = \|w_p\|_\infty. \tag{2.62}$$

Suppose by contradiction that

$$\lim_{p\to 1} \|w_p\|_\infty = \lim_{p\to 1} w_p(z_p) = +\infty. \tag{2.63}$$

Now we set

$$\nu_p^{1-\frac{n}{p}} = v_p(z_p),$$

this implies by (2.63) that

$$\lim_{p\to 1} |z_p| v_p(z_p)^{\frac{p}{n-p}} = \lim_{p\to 1} \frac{|z_p|}{\nu_p} = +\infty. \tag{2.64}$$

Using the fact that $v_p \leq 1$ in Ω_p and (2.61), we conclude that

$$\lim_{p\to 1} |z_p| = +\infty. \tag{2.65}$$

Consider $\exp_{g_p,z_p}$ the exponential map associated to g_p at z_p, let $\tilde{\Omega}_p = \nu_p^{-1} \exp_{g_p,z_p}^{-1}(\Omega_p)$, the metric $\tilde{g}_p(x) = \exp_{g_p,z_p}^{*} g_p(\nu_p x)$ for $x \in \tilde{\Omega}_p$, and the function given by

$$\phi_p(x) = \nu_p^{\frac{n}{p}-1} v_p(\exp_{z_p}(\nu_p x)) \text{ for } x \in \tilde{\Omega}_p,\ \phi_p(x) = 0 \text{ in } x \in \tilde{\Omega}_p^c.$$

Then for $x \in B_\xi(0,1)$, by (2.64), and (2.65), we can prove that ϕ_p is uniformly bounded in $B_\xi(0,1)$, and verifies (2.68).

In fact, for $x \in B_\xi(0,1)$, and by the definition of the $\exp_{g_p,z_p}$ map we have

$$\nu_p \ \geq \ d_{g_p}(z_p, \exp_{z_p}(\nu_p x)),$$

using the triangular inequality get

$$\begin{aligned} |\exp_{z_p}(\nu_p x)| &\geq |z_p| - d_{g_p}(z_p, \exp_{z_p}(\nu_p x)), \\ &\geq |z_p| - \nu_p \\ &= |z_p| - \left(w_p(z_p)|z_p|^{1-\frac{n}{p}} \right)^{\frac{p}{p-n}} \\ &= \left(1 - w_p(z_p)^{\frac{p}{p-n}} \right) |z_p|. \end{aligned} \tag{2.66}$$

Since $w_p(z_p) \to +\infty$ when $p \to 1$ and $\frac{p}{p-n} < 0$ for values of p very close to 1, and since (2.62) and (2.63), are valid for $x \in B_\xi(0,1)$ we obtain

$$|\exp_{z_p}(\nu_p x)| \geq \frac{1}{2}|z_p|. \tag{2.67}$$

Rewriting in terms of w_p we get

$$\begin{aligned} \phi_p(x) &= \nu_p^{\frac{n}{p}-1} v_p(\exp_{z_p}(\nu_p x)) \\ &= \nu_p^{\frac{n}{p}-1} w_p(\exp_{z_p}(\nu_p x))|\exp_{z_p}(\nu_p x)|^{1-\frac{n}{p}}. \end{aligned}$$

Since $1 - \frac{n}{p} < 0$ for values of p close to 1, we obtain

$$\phi_p(x) \ \leq \ \nu_p^{\frac{n}{p}-1} w_p(\exp_{z_p}(\nu_p x)) \left(\frac{1}{2}|z_p| \right)^{1-\frac{n}{p}},$$

and since z_p is the maximum of w_p, we have $w_p(\exp_{z_p}(\nu_p x)) \leq w_p(z_p)$, thus

$$\phi_p(x) \ \leq \ 2^{\frac{n}{p}-1}|z_p|^{1-\frac{n}{p}} \nu_p^{\frac{n}{p}-1} w_p(z_p),$$

and we know by definition that $v_p(z_p)^{-1}w_p(z_p) = |z_p|^{\frac{n}{p}-1}$, we are lead to

$$\phi_p(x) \leq 2^{\frac{n}{p}-1},$$

that is, $\|\phi_p\|_{L^\infty(B_\xi(0,1))} \leq 2^{\frac{n}{p}-1}$. Substituting in (2.43) a straightforward computation gives that ϕ_p satisfies

$$C_p\Delta_{p,\tilde{g}_p}\phi_p + \alpha_p\mu_p^2\nu_p^2\|\phi_p\|_p^{2-p}\phi_p^{p-1} = \lambda_p\phi_p^{p^*-1}, \text{ in } \tilde{\Omega}_p, \tag{2.68}$$

and $\phi_p = 0$ in $\partial\tilde{\Omega}_p$. Since ϕ_p is uniformly bounded we can apply Moser's iterative scheme Lemma 2.1.1 to the equation (2.68) to get the existence of some $C > 0$ independent of p such that

$$1 = \phi_p(0) \leq \sup_{\tilde{\Omega}_p\cap B_\xi(0,1/2)} \phi_p \leq C\left(\int_{\tilde{\Omega}_p\cap B_\xi(0,1)} \phi_p^{p^*}\,dv_{\tilde{g}_p}\right)^{\frac{1}{p^*}}. \tag{2.69}$$

For a subsequent use remember that

$$\int_{\tilde{\Omega}_p\cap B_\xi(0,1)} \phi_p^{p^*}\,dv_{\tilde{g}_p} = \int_{\Omega_p\cap B_{g_p}(z_p,\nu_p)} v_p^{p^*}\,dv_{g_p}. \tag{2.70}$$

Again an application of the Moser's iterative scheme Lemma 2.1.1 is legitimate by the same arguments leading to (2.52). Therefore by (2.69) we get immediately that

$$\liminf_{p\to 1}\int_{B_{g_p}(z_p,\nu_p)\cap\Omega_p} v_p^{p^*}\,dv_{g_p} > 0. \tag{2.71}$$

By (2.69) and (2.70) given $R > 0$, we get $B_{g_p}(0,R)\cap B_{g_p}(z_p,\nu_p) = \emptyset$ because $|z_p| \to \infty$ when $p \to 1$. Furthermore

$$1 = \int_{\Omega_p} v_p^{p^*}\,dv_{g_p} = \int_{\Omega_p\cap B_{g_p(0,R)}} v_p^{p^*}\,dv_{g_p} + \int_{\Omega_p\setminus B_{g_p(0,R)}} v_p^{p^*}\,dv_{g_p},$$

on the other hand by (2.47), (2.48) we have

$$\begin{aligned}\left(1 - C\mu_p^2\right)\int_{B_\xi\left(0,\frac{R}{\mu_p}\right)} v_p^{p^*}\,dv_\xi &\leq \int_{\Omega_p\cap B_{g_p(0,R)}} v_p^{p^*}\,dv_{g_p}\\ &\leq \left(1 + C\mu_p^2\right)\int_{B_\xi\left(0,\frac{R}{\mu_p}\right)} v_p^{p^*}\,dv_\xi,\end{aligned}$$

taking the limit when $p \to 1$ in the last two equations, using (2.53), (2.56), (2.71) and remembering that $\mu_p \to 0$ when $p \to 1^+$ we get easily

$$0 < \liminf_{p\to 1^+}\int_{B_{g_p}(z_p,\nu_p)\cap\Omega_p} v_p^{p^*}\,dv_{g_p} \leq \liminf_{p\to 1^+}\int_{\Omega_p\setminus B_{g_p(0,R)}} v_p^{p^*}\,dv_{g_p} = 0,$$

which is the desired contradiction. Since this contradition comes from taking for granted (2.63), we are lead to negate (2.63) and to have the existence of some $C > 0$ such that for any $p > 1$, and for all $x \in \Omega_p$

$$w_p(x) = |x|^{\frac{n}{p}-1} v_p(x) \leq C. \tag{2.72}$$

In the same way, using (2.72), one proves thanks to (2.60) that for any $R > R_0$,

$$\lim_{p\to 1} \sup_{\Omega_p \setminus B_{g_p}(0,R)} |x|^{\frac{n}{p}-1} v_p(x) = 0. \tag{2.73}$$

To prove (2.73) we argue by contradiction so we suppose that there exist $y_p \in \Omega_p$ and $\delta > 0$ such that

$$\lim_{p\to 1} |y_p| = +\infty, \qquad \text{and} \;\; w_p(y_p) \geq \delta.$$

Define $v_p(y_p) = \nu_p^{1-\frac{n}{p}}$, and $\tilde{\Omega}_p = \nu_p^{-1} \exp_{y_p}^{-1}(\Omega_p)$. Observe that $w_p(y_p) = |y_p|^{\frac{n}{p}-1} \nu_p^{1-\frac{n}{p}} \geq \delta$. For $x \in \tilde{\Omega}_p$, let $\phi_p(x) = \nu_p^{\frac{n}{p}-1} v_p(\exp_{y_p}(\nu_p x))$ and $\phi_p(x) = 0$ in $x \in \tilde{\Omega}_p^c$, and $\tilde{g}_p(x) = \exp_{y_p}^* g_p(\nu_p x)$.

Now for any $x \in B_\xi\left(0, \frac{1}{2}\delta^{\frac{p}{n-p}}\right)$, by the same arguments that above, we get that $|\exp_{y_p}(\nu_p x)| \geq \frac{1}{2}|y_p|$. Then using (2.72), we get that

$$\begin{aligned} \phi_p(x) &= \nu_p^{\frac{n}{p}-1} v_p(\exp_{y_p}(\nu_p x)) = \nu_p^{\frac{n}{p}-1} w_p(\exp_{y_p}(\nu_p x)) |\exp_{y_p}(\nu_p x)|^{1-\frac{n}{p}} \\ &\leq C 2^{\frac{n}{p}-1} |y_p|^{1-\frac{n}{p}} \nu_p^{\frac{n}{p}-1} \leq C 2^{\frac{n}{p}-1} \delta^{-1}. \end{aligned}$$

That is $\|\phi_p\|_{L^\infty\left(B_\xi\left(0,\frac{1}{2}\delta^{\frac{p}{n-p}}\right)\right)} \leq C 2^{\frac{n}{p}-1}\delta^{-1}$, and by Moser's iterative scheme Lemma 2.1.1 we get that

$$1 = \phi_p(0) \leq \sup_{\tilde{\Omega}_p \cap B_\xi\left(0,\frac{1}{4}\delta^{\frac{p}{n-p}}\right)} \phi_p \leq C \left(\int_{\tilde{\Omega}_p \cap B_\xi\left(0,\frac{1}{2}\delta^{\frac{p}{n-p}}\right)} \phi_p^{p^*} dv_{\tilde{g}_p} \right)^{\frac{1}{p^*}}.$$

On the other hand, since

$$\int_{\tilde{\Omega}_p \cap B_\xi\left(0,\frac{1}{2}\delta^{\frac{p}{n-p}}\right)} \phi_p^{p^*} dv_{\tilde{g}_p} = \int_{\Omega_p \cap B_{g_p}\left(y_p,\frac{1}{2}\delta^{\frac{p}{n-p}}\nu_p\right)} v_p^{p^*} dv_{g_p},$$

using the same arguments as above we get that for $R > 0$ for p close to 1

$$B_{g_p}\left(y_p, \frac{1}{4}\delta^{\frac{p}{n-p}}\nu_p\right) \cap B_{g_p}(0,R) = \emptyset.$$

But for $R > R_0$, by (2.60) we have $\lim_{p\to 1}\sup_{\Omega_p\setminus B_{g_p}(0,R)} v_p = 0$, and

$$\Omega_p \cap B_{g_p}\left(y_p, \frac{1}{2}\delta^{\frac{p}{n-p}}\nu_p\right) \subset \Omega_p \setminus B_{g_p}(0,R),$$

thus

$$1 \leq \lim_{p\to 1} \sup_{\tilde{\Omega}_p\cap B_\xi\left(0,\frac{1}{2}\delta^{\frac{p}{n-p}}\right)} \phi_p = 0,$$

which is a contradiction.

Step 4 Unfortunately the pointwise estimates that we obtained in (2.72) is not enough to prove our crucial (2.91). For this reasons we need to improve it. This is the goal to achieve in this step 4, which culminate in the proof of (2.74) below. Consider the following operator

$$L_p u := C_p\Delta_{p,g_p}u + \alpha_p\mu_p^2\|v_p\|_{p,g_p}^{2-p}u^{p-1} - \lambda_p v_p^{p^*-p}u^{p-1}.$$

Choose $0 < \nu < n-1$ and put

$$G_p(x) = \theta_p|x|^{-\frac{n-p-\nu}{p-1}},$$

where θ_p is some positive constant to be fixed later.

We will use the following relation for the p-Laplacian for radial functions that could be found in Lemma 1.2 of [Bie03] for an arbitrary Riemannian metric h

$$-\Delta_{p,h}u = -\Delta_{p,\xi}u + O(r)|\partial_r u|^{p-2}\partial_r u.$$

and we obtain

$$|x|^p\frac{L_pG_p(x)}{G_p(x)^{p-1}} \geq C_p\nu\left(\frac{n-p-\nu}{p-1}\right)^{p-1} - C\mu_p^2|x|^2 + \alpha_p\mu_p^2\|v_p\|_p^{2-p}|x|^p - \lambda_p|x|^p v_p^{p^*-p},$$

in $\Omega_p\setminus\{0\}$. Here C denotes some constant independent of p. Thanks to (2.40), (2.41), (2.42), (2.73) and the fact that $r_p \to 0$ as $p \to 1$, one gets that for any $R > R_0$, $L_pG_p(x) \geq 0$ in $\Omega_p \setminus B_{g_p}(0,R)$ for p close enough to 1. On the other hand,

$$L_pv_p = 0 \quad \text{on } \Omega_p,$$

in the sense of distributions. At last, it is not too hard to check with (2.60) that

$$v_p \leq G_p \quad \text{on } \partial B_{g_p}(0,R),$$

if we take $\theta_p = R^{\frac{n-p-\nu}{p-1}}$. Now we may apply the maximum principle as stated for instance in Lemma 3.4 of [AL99] to get,

$$v_p(y) \leq \left(\frac{R}{|y|}\right)^{\frac{n-p-\nu}{p-1}} \quad \text{in } \Omega_p \setminus B_{g_p}(0,R),$$

for p close enough to 1. This inequality obviously holds on $B_{g_p}(0,R)$ and so we have finally obtained for any $n-1 > \nu > 0$ and any $R > R_0$, a constant $C(R,\nu) > 0$ such that for any $p > 1$ and any $y \in \Omega_p$,

$$\left(\frac{|y|}{R}\right)^{\frac{n-p-\nu}{p-1}} v_p(y) \leq C(R,\nu). \tag{2.74}$$

Step 5 We give in this Step the final arguments to conclude the proof of our Proposition 1. We apply the $W_\xi^{1,1}(\mathbb{R}^n)$ Euclidean Sobolev inequality to $\tilde{v}_p$:

$$\left(\int_{\Omega_p} \tilde{v}_p^{\frac{n}{n-1}} dv_\xi\right)^{\frac{n-1}{n}} \leq K(n,1) \int_{\Omega_p} |\nabla_\xi \tilde{v}_p|_\xi dv_\xi. \tag{2.75}$$

Recalling the Cartan expansion of g_p around 0, we have

$$dv_\xi = \left(1 + \frac{1}{6}\mu_p^2 Ric_g(x_p)_{ij} x^i x^j + o(\mu_p^2|x|^2)\right) dv_{g_p}, \tag{2.76}$$

where Ric_g denotes the Ricci curvature of g in the $\exp_{x_p}$-map.

Remark 2.1.7. *Notice that the reminder in the preceding equation a priori depends on the metric tensor up to the third derivatives at x_p for this reason we need in this proof to have C^3 bounds on the geometry of (M^n, g) if we want that in the sequel the third term and its integrals go to zero when $p \to 1^+$.*

Formula (2.76) is true because $Ric_{g_p}(0) = \mu_p^2 Ric_g(\exp_{x_p}(0)) = \mu_p^2 Ric_g(x_p)$. Thus, by (2.44) we obtain

$$\int_{\Omega_p} \tilde{v}_p^{\frac{n}{n-1}} dv_\xi = 1 + \frac{1}{6}\mu_p^2 Ric_g(x_p)_{ij} \int_{\Omega_p} x^i x^j v_p^{p^*} dv_{g_p} + o\left(\mu_p^2 \int_{\Omega_p} |x|^2 v_p^{p^*} dv_{g_p}\right).$$

To estimate the last term on the right hand side of the preceding equality we need to prove (2.77) and (2.78) below

$$\begin{aligned}
\int_{B_\xi(0,R_0)} x^i x^j dv_\xi &= \frac{\delta^{ij}}{n} \int_{B_\xi(0,R_0)} |x|^2 dv_\xi = \frac{\delta^{ij}}{n} \int_0^{R_0} \int_{\partial B_\xi(0,r)} r^2 d\sigma_\xi dr \\
&= \frac{\delta^{ij}}{n} \int_0^{R_0} r^{n+1} dr \int_{\partial B_\xi(0,1)} d\sigma_\xi \\
&= \frac{\delta^{ij}}{n(n+2)} \omega_{n-1} R_0^{n+2}.
\end{aligned} \tag{2.77}$$

Let $\beta_p = \frac{n-p-\nu}{p-1}$ and $R > \max\{1, R_0\}$, by (2.74) we obtain that

$$\begin{aligned}
\int_{\Omega_p \setminus B_\xi(0,R)} |x|^2 v_p^{p^*} dv_{g_p} &\leq CR^{p^*\beta_p} \int_{\Omega_p \setminus B_\xi(0,R)} |x|^{2-p^*\beta_p} dv_\xi \\
&\leq C(1 + C\mu_p^2) R^{p^*\beta_p} \int_{\mathbb{R}^n} |x|^{2-p^*\beta_p} dv_\xi \\
&\leq C(1 + C\mu_p^2)\omega_{n-1} R^{p^*\beta_p} \int_R^\infty \rho^{2-p^*\beta_p} \rho^{n-1} d\rho \\
&\leq C(1 + C\mu_p^2)\omega_{n-1} R^{p^*\beta_p} \left(\frac{\rho^{n+2-p^*\beta_p}}{n+2-p^*\beta_p} \right)\Big|_R^\infty \\
&= C(1 + C\mu_p^2)\omega_{n-1} R^{n+2} \tilde{\gamma}_{p,n} \to 0, \qquad (2.78)
\end{aligned}$$

where $\tilde{\gamma}_{p,n} := \dfrac{1}{p^*\beta_p - n - 2}$. Using (2.56), (2.77), and (2.78) we conclude that

$$\int_{\Omega_p} \tilde{v}_p^{\frac{n}{n-1}} dv_\xi = 1 + \frac{Sc_g(x_p)}{6n(n+2)} \omega_{n-1} R_0^{n+2} \mu_p^2 + o(\mu_p^2),$$

and the expression on the right hand side of (2.75) becomes

$$\left(\int_{\Omega_p} \tilde{v}_p^{\frac{n}{n-1}} dv_\xi \right)^{\frac{n-1}{n}} = 1 + \frac{(n-1)Sc_g(x_p)}{6n^2(n+2)} \omega_{n-1} R_0^{n+2} \mu_p^2 + o(\mu_p^2). \qquad (2.79)$$

Denote by l_2 the limit of the scalar curvature function at x_p, i.e.,

$$l_2 := \lim_{p \to 1^+} Sc_g(x_p) \in \mathbb{R}, \qquad (2.80)$$

which exists and is finite because in strong bounded geometry $|Sc_g(x)|$ is uniformly bounded with respect to $x \in M$. A fact that will be used often in the sequel is that thanks to the hypothesis of C^2 convergence of the metric g to the metric at infinity we have

$$l_2 = \lim_{p \to 1^+} Sc_g(x_p) = \lim_{p \to 1^+} Sc_g(x_{0,p}) = l_1, \qquad (2.81)$$

since $d_g(x_p, x_{0,p}) \leq r_p \to 0$, when $p \to 1^+$. By the Cartan expansion of g_p at 0, since $r_p \to 0$ as $p \to 1$, we also have

$$|\nabla_\xi \tilde{v}_p|_\xi^p = |\nabla_{g_p} \tilde{v}|_{g_p}^p \left(1 + \frac{\mu_p^2}{6} |\nabla_{g_p} \tilde{v}_p|_{g_p}^{-2} Rm_g(x_p)(\nabla_{g_p} \tilde{v}_p, x, x, \nabla_{g_p} \tilde{v}_p) + o(\mu_p^2 |x|^2) \right),$$

where Rm_g denotes the Riemann curvature of g in the $\exp_{x_p}$-map. Then, using (2.76), we get

$$\int_{\Omega_p} |\nabla_\xi \tilde{v}_p|_\xi dv_\xi = \int_{\Omega_p} |\nabla_{g_p}\tilde{v}_p|_{g_p} dv_{g_p} + \frac{\mu_p^2}{6} Ric_g(x_p)_{ij} \int_{\Omega_p} x^i x^j |\nabla_\xi \tilde{v}_p|_\xi dv_\xi$$
$$+ \frac{\mu_p^2}{6}\int_{\Omega_p} |\nabla_{g_p}\tilde{v}_p|_{g_p}^{-1} Rm_g(x_p)(\nabla_{g_p}\tilde{v}_p, x, x, \nabla_{g_p}\tilde{v}_p) dv_{g_p}$$
$$+ o\left(\mu_p^2 \int_{\Omega_p} |x|^2 |\nabla_{g_p}\tilde{v}_p|_{g_p} dv_{g_p}\right). \tag{2.82}$$

Let us now estimate the different terms of (2.82). First, by equation (2.43) and relation (2.36), we have

$$\begin{aligned}
\int_{\Omega_p} |\nabla_{g_p}\tilde{v}_p|_{g_p} dv_{g_p} &= \tilde{\gamma}_{p,n}^* \int_{\Omega_p} v_p^{\frac{n(p-1)}{n-p}} |\nabla_{g_p} v_p|_{g_p} dv_{g_p} \\
&\le \tilde{\gamma}_{p,n}^* \left(\int_{\Omega_p} v_p^{p^*} dv_{g_p}\right)^{\frac{p-1}{p}} \left(\int_{\Omega_p} |\nabla_{g_p} v_p|_{g_p}^p dv_{g_p}\right)^{\frac{1}{p}} \\
&\le \tilde{\gamma}_{p,n}^* \left(\lambda_p - \alpha_p \mu_p \|v_p\|_p^2\right)^{\frac{1}{2}} \\
&= \tilde{\gamma}_{p,n}^* \lambda_p^{\frac{1}{2}} \left(1 - \alpha_p \mu_p \lambda_p^{-1} \|v_p\|_p^2\right)^{\frac{1}{2}} \\
&\le K(n,1)^{-1} \left(1 - \alpha_p \mu_p^2 \lambda_p^{-1} \|v_p\|_p^2\right)^{\frac{1}{2}},
\end{aligned}$$

where $\tilde{\gamma}_{p,n}^* := \frac{p(n-1)}{n-p}$. Since, by (2.56) and (2.74), $\|v_p\|_{g_p,p} = 1 + o(1)$, we get

$$\int_{\Omega_p} |\nabla_{g_p}\tilde{v}_p|_{g_p} dv_{g_p} \le K(n,1)^{-1} - \frac{\alpha_p}{2} K(n,1)\mu_p^2 + o(\mu_p^2). \tag{2.83}$$

By Holder's inequalities, we have

$$\begin{aligned}
\int_{\Omega_p} |x|^2 |\nabla_{g_p}\tilde{v}_p|_{g_p} dv_{g_p} &= \tilde{\gamma}_{p,n}^* \int_{\Omega_p} v_p^{\frac{n(p-1)}{n-p}} |x|^2 |\nabla_{g_p} v_p|_{g_p} dv_{g_p} \\
&\le \tilde{\gamma}_{p,n}^* \left(\int_{\Omega_p} v_p^{p^*} dv_{g_p}\right)^{\frac{p-1}{p}} \left(\int_{\Omega_p} |x|^{2p} |\nabla_{g_p} v_p|_{g_p}^p dv_{g_p}\right)^{\frac{1}{p}} \\
&= \tilde{\gamma}_{p,n}^* \left(\int_{\Omega_p} |x|^{2p} |\nabla_{g_p} v_p|_{g_p}^p dv_{g_p}\right)^{\frac{1}{p}}.
\end{aligned}$$

Multiplying the equation (2.43) by $|x|^{2p} v_p$ and integrating by parts, one gets

$$\begin{aligned}
\int_{\Omega_p} |\nabla_{g_p} v_p|^{p-2} \langle \nabla_{g_p}(|x|^{2p} v_p), \nabla_{g_p} v_p\rangle dv_{g_p} &= C_p^{-1}\lambda_p \int_{\Omega_p} |x|^{2p} v_p^{p^*} dv_{g_p} \\
&- C_p^{-1}\alpha_p \mu_p^2 \|v_p\|_p^{2-p} \int_{\Omega_p} |x|^{2p} v_p^p dv_{g_p}.
\end{aligned}$$

By (2.74), every term on the right hand side of the preceding inequality is uniformly bounded with respect to p, then we conclude that

$$\int_{\Omega_p} |\nabla_{g_p} v_p|^{p-2} \langle \nabla_{g_p}(|x|^{2p} v_p), \nabla_{g_p} v_p \rangle dv_{g_p} \leq C, \tag{2.84}$$

for some $C > 0$ that does not depend on p. Furthermore by Cauchy-Schwarz's inequality we get that

$$\begin{aligned}
\int_{\Omega_p} |\nabla_{g_p} v_p|^{p-2} \langle \nabla_{g_p}(|x|^{2p} v_p), \nabla_{g_p} v_p \rangle dv_{g_p} &= \int_{\Omega_p} |x|^{2p} |\nabla_{g_p} v_p|^p \\
&+ \int_{\Omega_p} |\nabla_{g_p} v_p|^{p-2} v_p \langle \nabla_{g_p}(|x|^{2p}), \nabla_{g_p} v_p \rangle dv_{g_p} \\
&\geq \int_{\Omega_p} |x|^{2p} |\nabla_{g_p} v_p|^p dv_{g_p} \\
&- 2p \int_{\Omega_p} |x|^{2p-1} \nabla_{g_p}(|x|) |\nabla_{g_p} v_p|^{p-1} v_p dv_{g_p} \\
&= \int_{\Omega_p} |x|^{2p} |\nabla_{g_p} v_p|^p dv_{g_p} \\
&- 2p \int_{\Omega_p} |x|^{2p-1} |\nabla_{g_p} v_p|^{p-1} v_p dv_{g_p}.
\end{aligned}$$

Therefore we are lead to

$$\begin{aligned}
\int_{\Omega_p} |x|^{2p} |\nabla_{g_p} v_p|^p_{g_p} dv_{g_p} &\leq \int_{\Omega_p} |\nabla_{g_p} v_p|^{p-2}_{g_p} \langle \nabla_{g_p}(|x|^{2p} v_p), \nabla_{g_p} v_p \rangle_{g_p} dv_{g_p} \\
&+ C \int_{\Omega_p} |x|^{2p-1} |\nabla_{g_p} v_p|^{p-1}_{g_p} v_p dv_{g_p} \\
&\leq C \\
&+ C \left(\int_{\Omega_p} |x|^{2p} |\nabla_{g_p} v_p|^p_{g_p} dv_{g_p} \right)^{\frac{p-1}{p}} \left(\int_{\Omega_p} |x|^p v_p^p dv_{g_p} \right)^{\frac{1}{p}},
\end{aligned}$$

where C denotes some constants independent of p. By (2.74) we see easily that $\int_{\Omega_p} |x|^p v_p^p dv_{g_p}$, is uniformly bounded with respect to p. Then by Young's inequalities, one deduces that

$$\begin{aligned}
\int_{\Omega_p} |x|^{2p} |\nabla_{g_p} v_p|^p_{g_p} dv_{g_p} \leq & C + C \left(\int_{\Omega_p} |x|^{2p} |\nabla_{g_p} v_p|^p_{g_p} dv_{g_p} \right)^{\frac{p-1}{p}} \\
\leq & C + \frac{C^p}{p} + \frac{p-1}{p} \left(\int_{\Omega_p} |x|^{2p} |\nabla_{g_p} v_p|^p_{g_p} dv_{g_p} \right)^{\frac{p-1}{p} \frac{p}{p-1}},
\end{aligned}$$

and so

$$\int_{\Omega_p} |x|^{2p} |\nabla_{g_p} v_p|^p_{g_p} dv_{g_p} \leq \left(1 - \frac{p-1}{p} \right)^{-1} \left(C + \frac{C^p}{p} \right) \leq \tilde{C},$$

with $\tilde{C} > 0$ independent of p. That is

$$\int_{\Omega_p} |x|^{2p} |\nabla_{g_p} v_p|^p_{g_p} dv_{g_p} = O(1). \tag{2.85}$$

Now for some $R > R_0$, we get readily by (2.57) that

$$\begin{aligned}\int_{\Omega_p} |\nabla_\xi \tilde{v}_p|_\xi x^i x^j dv_\xi &= O\left(\int_{\Omega_p \setminus B_\xi(0,R)} |x|^2 |\nabla_\xi \tilde{v}_p|_\xi dv_\xi\right) \\ &+ \int_{\partial B_\xi(0,R_0)} x^i x^j d\sigma_\xi + o(1).\end{aligned}$$

By Hölder inequality we obtain

$$\int_{\Omega_p \setminus B_\xi(0,R)} |x|^2 |\nabla_{g_p} \tilde{v}_p|_{g_p} dv_{g_p} \leq \frac{p(n-1)}{n-p} \left(\int_{\Omega_p \setminus B_\xi(0,R)} |x|^{2p} |\nabla_{g_p} v_p|^p_g dv_{g_p}\right)^{\frac{1}{p}}.$$

Multiplying the equation (2.43) by $|x|^{2p} v_p$, integrating over $\Omega_p \setminus B_\xi(0,R) := \Omega_p^*$, and using Cauchy-Schwarz, Hölder inequality and later by Young inequality we obtain

$$\begin{aligned}\int_{\Omega_p^*} |x|^{2p} |\nabla_{g_p} v_p|^p_{g_p} dv_{g_p} &\leq \int_{\Omega_p^*} |\nabla_{g_p} v_p|^{p-2}_{g_p} \langle \nabla_{g_p}(|x|^{2p} v_p), \nabla_{g_p} v_p\rangle_{g_p} dv_{g_p} \\ &+ 2p \int_{\Omega_p^*} |x|^{2p-1} |\nabla_{g_p} v_p|^{p-1}_{g_p} v_p dv_{g_p} \\ &\leq \int_{\Omega_p^*} |\nabla_{g_p} v_p|^{p-2}_{g_p} \langle \nabla_{g_p}(|x|^{2p} v_p), \nabla_{g_p} v_p\rangle_{g_p} dv_{g_p} \\ &+ 2p \left(\int_{\Omega_p^*} |x|^{2p} |\nabla_{g_p} v_p|^p_{g_p} dv_{g_p}\right)^{\frac{p-1}{p}} \left(\int_{\Omega_p^*} |x|^p v_p^p dv_{g_p}\right)^{\frac{1}{p}} \\ &\leq \int_{\Omega_p^*} |\nabla_{g_p} v_p|^{p-2}_{g_p} \langle \nabla_{g_p}(|x|^{2p} v_p), \nabla_{g_p} v_p\rangle_{g_p} dv_{g_p} \\ &+ 2(p-1) \int_{\Omega_p^*} |x|^{2p} |\nabla_{g_p} v_p|^p_{g_p} dv_{g_p} + 2 \int_{\Omega_p^*} |x|^p v_p^p dv_{g_p}.\end{aligned}$$

At last we obtain that

$$\begin{aligned}0 \leq (3-2p) \int_{\Omega_p^*} |x|^{2p} |\nabla_{g_p} v_p|^p_{g_p} dv_{g_p} &\leq \int_{\Omega_p^*} |\nabla_{g_p} v_p|^{p-2}_{g_p} \langle \nabla_{g_p}(|x|^{2p} v_p), \nabla_{g_p} v_p\rangle_{g_p} dv_{g_p} \\ &+ 2 \int_{\Omega_p^*} |x|^p v_p^p dv_{g_p}.\end{aligned}$$

But when $p \to 1$, by (2.74), the terms on the right hand side go to 0, then we can conclude that

$$\int_{\Omega_p \setminus B_\xi(0,R)} |x|^2 |\nabla_\xi v_p|_\xi dv_\xi \to 0. \tag{2.86}$$

Thus for the second term on the right hand side of (2.82) we see that

$$\lim_{p\to 1} Ric_g(x_p)_{ij} \int_{\Omega_p} |\nabla_\xi \tilde{v}_p|_\xi x^i x^j dv_\xi = \frac{\omega_{n-1}}{n} R_0^{n+1} l_1. \tag{2.87}$$

Now, we look at the third term on the right hand side of (2.82). Since ∇V_p, V_p as in (2.58), and x are pointwise colinear vector fields, we have

$$Rm_g(x_p)(\nabla_{g_p}\tilde{v}_p, x, x, \nabla_{g_p}\tilde{v}_p) \leq C|x|^2 |\nabla_\xi \tilde{v}_p|_\xi |\nabla_\xi(\tilde{v}_p - V_p)|_\xi. \tag{2.88}$$

Now by (2.88), integrating over $\Omega_p \cap B_\xi(0,R) := \hat{\Omega}_p$ we have

$$\begin{aligned}
\int_{\hat{\Omega}_p} |\nabla_{g_p}\tilde{v}_p|_{g_p}^{-1} Rm_g(x_p)(\nabla_{g_p}\tilde{v}_p, x, x, \nabla_{g_p}\tilde{v}_p) dv_{g_p} &\leq \int_{\hat{\Omega}_p} C|\nabla_{g_p}\tilde{v}_p|_{g_p}^{-1} |x|^2 |\nabla_\xi \tilde{v}_p|_\xi |\nabla_{g_p}(\tilde{v}_p - V_p)|_\xi dv_{g_p} \\
&\leq C_R \left(1 + C\mu_p^2\right) \int_{\hat{\Omega}_p} |\nabla_\xi(\tilde{v}_p - V_p)|_\xi dv_\xi.
\end{aligned}$$

This last inequality combined with (2.59) yields to

$$\lim_{p\to 1} \int_{\Omega_p \cap B_\xi(0,R)} |\nabla_{g_p}\tilde{v}_p|_{g_p}^{-1} Rm_g(x_p)(\nabla_{g_p}\tilde{v}_p, x, x, \nabla_{g_p}\tilde{v}_p) dv_{g_p} = 0. \tag{2.89}$$

We want to estimate the integral of the same integrand function of (2.89) but outside $B_\xi(0,R)$, for this we have

$$\begin{aligned}
\int_{\Omega_p \setminus B_\xi(0,R)} |\nabla_{g_p}\tilde{v}_p|_{g_p}^{-1} Rm_g(x_p)(\nabla_{g_p}\tilde{v}_p, x, x, \nabla_{g_p}\tilde{v}_p) dv_{g_p} &\leq \int_{\Omega_p \setminus B_\xi(0,R)} C_{\Lambda_2} |x|^2 |\nabla_{g_p}\tilde{v}_p|_{g_p}^{-1} |\nabla_{g_p}\tilde{v}_p|_{g_p}^2 dv_{g_p} \\
&\leq C_{\Lambda_2}(1 + C\mu_p^2) \int_{\Omega_p \setminus B_\xi(0,R)} |x|^2 |\nabla_{g_p}\tilde{v}_p|_\xi dv_\xi \\
&\overset{(2.86)}{\to} 0.
\end{aligned} \tag{2.90}$$

Combining (2.89) and (2.90) we conclude that

$$\lim_{p\to 1} \int_{\Omega_p} |\nabla_{g_p}\tilde{v}_p|_{g_p}^{-1} Rm_g(x_p)(\nabla_{g_p}\tilde{v}_p, x, x, \nabla_{g_p}\tilde{v}_p) dv_{g_p} = 0. \tag{2.91}$$

Finally, substituting in (2.75), using (2.81), and (2.79)-(2.85), we obtain,

$$\begin{aligned}
1 + \frac{(n-1)l_2}{6n^2(n+2)} \omega_{n-1} R_0^{n+2} \mu_p^2 &+ o(\mu_p^2) \\
&\leq K(n,1)\left[K(n,1)^{-1} - \frac{\alpha_p}{2} K(n,1)\mu_p^2\right] \\
&+ K(n,1)\left[\frac{\omega_{n-1}}{6n} R_0^{n+1} l_2 \mu_p^2\right] + o(\mu_p^2) \\
&= 1 - \frac{\alpha_p}{2} K(n,1)^2 \mu_p^2 \\
&+ \frac{K(n,1)\omega_{n-1}}{6n} R_0^{n+1} l_2 \mu_p^2 + o(\mu_p^2).
\end{aligned}$$

Since $\frac{\omega_{n-1}}{n} = \frac{1}{R_0^n}$, and $K(n,1) = \frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{\frac{1}{n}} = \frac{R_0}{n}$, a straightforward computation leads to

$$\frac{K(n,1)^2}{2}\left(\alpha_p - \frac{n}{n+2} l_2\right)\mu_p^2 + o(\mu_p^2) \leq 0.$$

This gives the desired contradiction by letting p go to 0, recalling here that $l_1 = l_2$ by (2.80) it holds

$$\frac{n}{n+2}l_1 - \varepsilon_0 + \frac{n}{n+2}l_2 = \lim_{p\to 1^+} \alpha_p - \frac{n}{n+2}l_2 = \varepsilon_0 > 0. \tag{2.92}$$

This ends the proof of Proposition 1. □

We are now ready to accomplish the proof of our global comparison theorem for small diameters in C^3-locally asymptotically strong bounded geometry. We use the same argument used in [Dru02c], for completeness's sake we write the details here as pointed out to us by Olivier Druet in a private communication.

Proof of Theorem 0.0.2. The Proposition at page 2353 of [Dru02c] rewritten in this text as Proposition 2.1.1 says that for any $\varepsilon > 0$, there exists $r_\varepsilon = r_\varepsilon(x_0, M, g) > 0$ such that if $\Omega \subset B_g(x_0, r_\varepsilon)$, then

$$V_g(\Omega)^{2\frac{n-1}{n}} \leq K(n,1)^2 A_g(\partial\Omega)^2 + K(n,1)^2 \left(\frac{n}{n+2}Sc_g(x_0) + \varepsilon\right) V_g(\Omega)^2.$$

By assumption we know that $Sc_g(x_0) < n(n-1)k_0$, so that applying the preceding inequality with

$$\varepsilon = \frac{n}{2(n+2)}\left[n(n-1)k_0 - Sc_g(x_0)\right] > 0,$$

we get that there exists $r > 0$, $r_\varepsilon(x_0, M, g) \geq r > 0$ such that if $\Omega \subset B_g(x_0, r)$, then

$$\begin{aligned} V_g(\Omega)^{2\frac{n-1}{n}} &\leq K(n,1)^2 A_g(\partial\Omega)^2 \\ &+ K(n,1)^2 \left(\frac{n}{n+2}n(n-1)k_0 - \frac{n}{2(n+2)}\varepsilon_0\right) V_g(\Omega)^2. \end{aligned} \tag{2.93}$$

where $\varepsilon_0 = n(n-1)k_0 - Sc_g(x_0) > 0$ fixed. Now let B_v be a small ball in the model space $(\mathbb{M}_{k_0}, g_{k_0})$ of constant sectional curvature k_0 and volume v, for any $V_0 > 0$ (small enough in the case of the sphere, i.e., $k_0 > 0$) there exists $C_0 = C_0(n, k_0, V_0) > 0$ such that for balls of volume $0 \leq v \leq V_0$ it holds

$$\begin{aligned} V_{g_{k_0}}(B_v)^{2\frac{n-1}{n}} &\geq K(n,1)^2 A_{g_{k_0}}(\partial B_v)^2 \\ &+ K(n,1)^2 \frac{n}{n+2} n(n-1)k_0 V_{g_{k_0}}(B_v)^2 \\ &- C_0 v^{2\frac{n+2}{n}}. \end{aligned} \tag{2.94}$$

If we assume that $V_g(\Omega) = V_{g_{k_0}}(B_v) = v \leq V_0$, we get that

$$\begin{aligned} K(n,1)^2 A_{g_{k_0}}(\partial B_v)^2 &+ K(n,1)^2 \frac{n}{n+2} n(n-1) k_0 v^2 - C_0 v^{2\frac{n+2}{n}} \\ &\leq v^{2\frac{n-2}{n}} \\ &\leq K(n,1)^2 A_g(\partial\Omega)^2 \\ &+ K(n,1)^2 \left(\frac{n}{n+2} n(n-1) k_0 - \frac{n}{2(n+2)} \varepsilon_0 \right) v^2 \end{aligned}$$

that is,

$$A_{g_{k_0}}(\partial B_v)^2 \leq A_g(\partial\Omega)^2 + C_0 K(n,1)^{-2} v^{2\frac{n+2}{n}} - \frac{n}{2(n+2)} \varepsilon_0 v^2.$$

If we choose $v < V_1 < V_0 < \min\{1, V_{g_{k_0}}\left(\mathbb{M}^n_{k_0}\right)\}$ [1], with the property that

$$C_0(n,k_0) K(n,1)^{-2} V_1^{2\frac{n+2}{n}} - \frac{n}{2(n+2)} \varepsilon_0 V_1^2 < 0,$$

which is always possible to find, then we get

$$A_{g_{k_0}}(\partial B_v) < A_g(\partial\Omega).$$

Thus there exists $V_1 = V_1(n, k_0, V_0, Sc_g(x_0)) > 0$ such that if $\Omega \subset B_g(x_0, r)$ with volume $V_{g_0}(B_v) = v < V_1$, then the comparison inequality (4) of the theorem holds. Now, up to lower a little bit r (depending on curvatures of M), we are sure that any domain $\Omega \subset B_g(x_0, r)$ has volume less than that of the ball $B_g(x_0, r)$ which can be chosen to be less than V_1 and the theorem is proved. □

The following proposition is an immediate consequence of Proposition 1 and the proof is left to the reader.

Proposition 2. *Let (M, g) be a complete Riemannian manifold of dimension $n \geq 2$ with C^3-locally asymptotically strong bounded geometry smooth at infinity. For any $\varepsilon > 0$ there exists $r_\varepsilon = r_\varepsilon(M, g) > 0$ such that for any point $x_0 \in M$, any function $u \in C_c^\infty(B_g(x_0, r_\varepsilon))$, we have*

$$||u||^2_{\frac{n}{n-1},g} \leq K(n,1)^2 \left(||\nabla_g u||^2_{1,g} + \alpha_\varepsilon ||u||^2_{1,g} \right), \tag{2.95}$$

where $\alpha_\varepsilon = \frac{n}{n+2} S_g + \varepsilon$ with $S_g := \sup_{x \in M}\{Sc_g(x)\} \in \mathbb{R}$.

[1] $\min\{1, +\infty\}$ is assumed to be equal to 1.

Remark 2.1.8. *We notice that the constant* $r_\varepsilon = r_\varepsilon(M,g) > 0$ *is obtained by contradiction and that the proof does not give an explicit effective lower bound on it.*

We are thus led to the following version of Theorem 1 of [Dru02c], namely our Theorem 1, in which an uniform estimate of a lower bound on r_x is obtained provided M is of C^3-locally asymptotically strong bounded geometry at infinity.

Proof of Theorem 1. We proceed as in the proof of Theorem 0.0.2 using our Proposition 2 instead of the Proposition at page 2353 of [Dru02c]. This gives the existence of an uniform $r_\varepsilon(M,g) > 0$ independent of x_0. With the help of Bishop-Gromov Theorem we have for every $0 \leq r \leq 1$.

$$V_g(B_g(x_0,r)) \leq V_{g_k}(B_{g_k}(x_0,r)) \leq C(n,k)r^n,$$

where k is a lower bound on the Ricci curvature of (M,g). So we can take a radius $r = r(n,k,k_0,V_0,S_g) > 0$ such that

$$r < \sqrt[n]{\frac{V_1(n,k_0,V_0,S_g)}{C(n,k)}}, \tag{2.96}$$

where V_1 is as in the proof of Theorem 0.0.2 with ε_0 replaced by $\tilde{\varepsilon}_0 = n(n-1)k_0 - S_g > 0$. Observing that we can take for example $V_0 < \min\{1, V_{g_{k_0}}\left(\mathbb{M}^n_{k_0}\right)\}$ fixed we obtain

$$r = r(n,k,k_0,S_g,r_{\tilde{\varepsilon}_0}(M,g)) := \min\left\{\sqrt[n]{\frac{V_1(n,k_0,V_0,S_g)}{C(n,k)}}, 1, r_{\tilde{\varepsilon}_0}(M,g)\right\} > 0.$$

□

Chapter 3

Mild bounded geometry and the proof of Theorem 2

3.1 In mild bounded geometry isoperimetric regions of small volume are of small diameter

In this section we work with just a fixed Riemannian metric g defined on M.

Lemma 3.1.1 (Lemma 3.2 of [Heb99]). *Let (M^n, g) be a smooth, complete Riemannian n-dimensional manifold with weak bounded geometry. There exist two positive constants $C_{Heb} = C_{Heb}(n, k, v_0) > 0$ and $\bar{v} := \bar{v}(n, k, v_0) > 0$, depending only on $n, k,$ and v_0, such that for any open subset Ω of M with smooth boundary and compact closure, if $V_g(\Omega) \leq \bar{v}$, then $C_{Heb} V_g(\Omega)^{\frac{n-1}{n}} < A_g(\partial\Omega)$.*

Remark 3.1.1. *After Theorem 1 of [MFN15] we know that we can extend the preceding lemma to an arbitrary finite perimeter set simply by approximating with open bounded with smooth boundary subsets having the same volume, or simply by the more classical approximation theorem of finite perimeter sets by open relatively compact with smooth boundary subsets.*

Let us introduce a crucial notion for the remaining part of this section.

Definition 3.1.1. *We say that a sequence (D_j) of finite perimeter sets, $D_j \subseteq M$, with* *finite volume* $V_g(D_j) \to 0$*, is called an* ***approximate isoperimetric sequence****, if*

$$\lim_{j\to\infty} \frac{\mathcal{P}_g(D_j)}{V_g(D_j)^{\frac{n-1}{n}}} = \lambda,$$

where $\lambda := \liminf_{v\to 0^+} \dfrac{I_{M,g}(v)}{v^{(n-1)/n}}$.

Remark 3.1.2. *Comparing with geodesic balls we have clearly that* $\lambda \leq c_n$*, where* c_n *is the Euclidean isoperimetric constant defined by* $I_{\mathbb{R}^n}(v) = c_n v^{\frac{n-1}{n}}$, $\forall v \in]0, V(M)[$.

Remark 3.1.3. *When* (M^n, g) *have weak bounded geometry then* $\lambda \geq C_{Heb}(n, k, v_0) > 0$*, because of Lemma* 3.2 *of [Heb99] reported here in Lemma 3.1.1 and the related Remark 3.1.1. Actually* λ *is the best constant appearing in Lemma 3.1.1.*

Remark 3.1.4. *When* (M^n, g) *have strong bounded geometry then* $\lambda = c_n$*, this is an easy consequence of the Théoréme of Appendice C at page 531 of [BM82]. We wrote an alternative proof of this last fact, based on Theorems 1 and 2, in our Theorem 3.1.1 below.*

We recall here three well known lemmas (see for instance Corollary 2.1 of [Nar14a]) that we will use often in the sequel.

Lemma 3.1.2. *Let* (M^n, g) *be a complete Riemannian manifold with weak bounded geometry. Then for each* $r > 0$ *there exists* $c_1 = c_1(n, k, r) > 0$ *such that* $V_g(B_M(p, r)) > c_1(n, k, r)v_0$*, where* $c_1(n, k, r) = \min\left\{\frac{r^n}{e^{\sqrt{(n-1)|k|}}}, 1\right\}$.

Lemma 3.1.3. *Let* (M^n, g) *with weak bounded geometry. Then there exist two positive constants* $C_1 = C_1(n, k) > 0$*,* $C_2 = C_2(n, k) > 0$ *such that for every* $0 < r < \bar{r} = \bar{r}(n, k) := \min\left\{1, e^{\frac{\sqrt{(n-1)|k|}}{n}}\right\}$ *we have*

$$v_0 C_1 r^n \overset{doubling+noncollapsing}{\leq} V_g(B_M(x, r)) \overset{Bishop-Gromov}{\leq} C_2 r^n, \tag{3.1}$$

where $C_1 = C_1(n, k) = \frac{1}{e^{\sqrt{(n-1)|k|}}}$.

Lemma 3.1.4. *Let* (M^n, g) *with weak bounded geometry. Then there exist two positive constants* $\bar{v}_1 = \bar{v}_1(n, k, v_0) > 0$ *and* $C_3 = C_3(n, k) > 0$*, such that for every* $0 < v < \bar{v}_1$ *we have*

$$\lambda \leq \frac{I_M(v)}{v^{\frac{n-1}{n}}} \leq C_3(n, k). \tag{3.2}$$

Here $\bar{v}_1 := \min\{1, \bar{v}\}$.

Lemma 3.1.5. *Let (M^n, g) be a complete Riemannian manifold with weak bounded geometry. There exists a positive constant $N = N(n, k, v_0) > 0$ such that, whenever D is a finite perimeter set with finite volume and $0 < R < \bar{R} = \bar{R}(n, k) := \min\left\{1, 2e^{\frac{\sqrt{(n-1)|k|}}{n}}, \frac{2}{7}\bar{r}\right\}$ there exists a partition $(D_l)_l$ of D, i.e, $D = \mathring{\cup}_l D_l$, where every D_l is a set of finite perimeter contained in a ball of radius R and such that*

$$\left(\sum_l \mathcal{P}_g(D_l)\right) - \mathcal{P}_g(D) \leq N(n, k, v_0)\frac{V_g(D)}{R}. \tag{3.3}$$

Proof. Let $(p_l)_{l\in\mathbb{N}}$ be a sequence of points of M such that $\left\{B_{(M,g)}(p_l, \frac{R}{4})\right\}$ is a maximal set of disjoint balls. It is straightforward to show that

$$M = \bigcup_l B_{(M,g)}\left(p_l, \frac{R}{2}\right).$$

Set $\mathcal{A} := \{p_l\}_{l\in\mathbb{N}}$. By coarea formula we can cut D with a ball of radius r_l centered at p_l, such that $\frac{R}{2} < r_l < R$ and

$$A_g(D \cap \partial B_{(M,g)}(p_l, r_l)) \leq \frac{2V_g(D)}{R}. \tag{3.4}$$

Consider $D \setminus (\bigcup_l \partial B_M(p_l, r_l)) = \mathring{\bigcup}_l D_l$. Then there exists a constant $\tilde{N} = \tilde{N}(n, k, v_0) > 0$ such that

$$\left(\sum_l \mathcal{P}_g(D_l)\right) - \mathcal{P}_g(D) \leq 4\tilde{N}\frac{V_g(D)}{R}.$$

Note that by a simple combinatorial argument, $\tilde{N}$ could be taken as an upper bound of the greatest number of disjoint balls of radius $\frac{R}{4}$ contained in a ball of radius $\frac{7}{4}R$. This upper bound depends only on n, k, v_0 since for every $x \in M$ by our assumption $R < \bar{R}$ and by Lemma 3.1.3 it holds

$$\begin{aligned}
\tilde{N}C_1(n,k)\left(\frac{R}{4}\right)^n v_0 \quad &\leq \quad \sum_{p_i\in B_M(x,\frac{7}{4}R)} V_g(B_M(p_i, \frac{R}{4}))\\
&\leq \quad V_g\left(B_M\left(x, \frac{7}{4}R\right)\right)\\
&\overset{Bishop-Gromov}{\leq} \quad V_{g_k}\left(B_{\mathbb{M}^n_k}\left(\frac{7}{4}R\right)\right) \leq C_2(n,k)\left(\frac{7}{4}R\right)^n,
\end{aligned}$$

where $C_1(n, k) = \frac{1}{e^{\sqrt{(n-1)|k|}}}$. Setting $N = 4\tilde{N}$ we finish the proof of the lemma. □

The following lemma, is the analog of Lemma 2.3 of [Nar14b], but with a different proof that makes it true under weaker assumptions on the bounded geometry of (M, g).

Lemma 3.1.6. *Let (M^n, g) be a complete Riemannian manifold with weak bounded geometry, and $D_j \subset M$ be a sequence of finite perimeter sets with finite volume. Then there exist a partition of D_j by finite perimeter sets of $D_j = \mathring{\bigcup}_l D_{j,l}$ and a sequence of radii R_j, such that* $\mathrm{diam}_g(D_{j,l}) \leq 2R_j$*, with* $\lim_{j\to\infty} \frac{V_g(D_j)^{1/n}}{R_j} = 0$*,* $R_j \to 0$*, and*

$$\limsup_{j\to\infty} \left[\left(\sum_l \mathcal{P}_g(D_{j,l})\right) - \mathcal{P}_g(D_j)\right] \frac{1}{V_g(D_j)^{\frac{n-1}{n}}} = 0. \tag{3.5}$$

Proof. It is enough to apply (3.3) with $D = D_j$ and $R = R_j := V_g(D_j)^\alpha$, with $0 < \alpha < \frac{1}{n}$. □

We are now ready to prove Theorem 2.

Proof of Theorem 2. Consider an arbitrary sequence of finite perimeter sets Ω_j such that $v_j := V_g(\Omega_j) \to 0$. By Lemma 3.1.6 we can find a partition of Ω_j satisfying (3.5). For sufficiently large j we have $R_j \leq r$ where $r := \frac{d}{2}$ and d is given by Theorem 1. Set $\eta_j := N(n,k,v_0)\frac{v_j}{R_j}$, with $R_j >> v_j^{\frac{1}{n}}$, from which it follows $\frac{v_j}{R_j} \to 0$, when $j \to +\infty$, we obtain

$$\frac{I_{\mathbb{M}^n_{k_0}}(v_j)}{v_j^{\frac{n-1}{n}}} - \frac{\eta_j}{v_j^{\frac{n-1}{n}}} \leq \frac{\sum_l I_{\mathbb{M}^n_{k_0}}(v_{j,l})}{v_j^{\frac{n-1}{n}}} - \frac{\eta_j}{v_j^{\frac{n-1}{n}}} \tag{3.6}$$

$$< \frac{\sum_l \mathcal{P}_g(\Omega_{j,l})}{v_j^{\frac{n-1}{n}}} - \frac{\eta_j}{v_j^{\frac{n-1}{n}}} \tag{3.7}$$

$$\leq \frac{\mathcal{P}_g(\Omega_j)}{v_j^{\frac{n-1}{n}}}, \tag{3.8}$$

where the first inequality is due to the strict subadditivity of $I_{\mathbb{M}^n_{k_0}}$, the second is due to Theorem 1 (because $\mathrm{diam}(\Omega_{j,l}) < d$ for j large enough), and the last inequality is due to Lemma 3.1.6. For all j large enough we have that

$$(1-\varepsilon)\frac{I_{\mathbb{M}^n_{k_0}}(v_j)}{v_j^{\frac{n-1}{n}}} \leq \frac{I_{\mathbb{M}^n_{k_0}}(v_j)}{v_j^{\frac{n-1}{n}}} - \frac{\eta_j}{v_j^{\frac{n-1}{n}}}, \tag{3.9}$$

thus

$$(1-\varepsilon)\frac{I_{\mathbb{M}^n_{k_0}}(v_j)}{v_j^{\frac{n-1}{n}}} < \frac{\mathcal{P}_g(\Omega_j)}{v_j^{\frac{n-1}{n}}},$$

The last inequality combined with (3.6)-(3.8) easily establish the validity of (1.3) and complete the proof of the theorem. □

Corollary 3.1.1. *Let (M^n, g) be a complete Riemannian manifold with C^3-locally asymptotically strong bounded geometry smooth at infinity. Then*

$$\liminf_{v\to 0^+} \frac{I_M(v)}{v^{\frac{(n-1)}{n}}} = c_n,$$

where c_n is the Euclidean isoperimetric constant defined by $I_{\mathbb{R}^n}(v) = c_n v^{\frac{n-1}{n}}$.

Proof. Take an arbitrary sequence $v_j \to 0$ and a sequence of positive real numbers $\varepsilon_j \to 0$, by the definition of $I_{M,g}$ we know that we can take a sequence of finite perimeter sets Ω_j, such that $V_g(\Omega_j) = v_j$ and

$$I_{M,g}(v_j) \leq \mathcal{P}_g(\Omega_j) \leq I_{M,g}(v_j) + \varepsilon_j. \tag{3.10}$$

Passing to the limit in (1.3) or using (3.9) combined with the asymptotic expansion of the perimeter of geodesic balls in the model simply connected space forms in function of the volume enclosed it follows that

$$c_n \leq \liminf_{j\to+\infty} \frac{\mathcal{P}_g(\Omega_j)}{v_j^{\frac{n-1}{n}}}. \tag{3.11}$$

With Inequality (3.10) and Inequality (3.11) in mind the corollary follows without any further difficulty. $\square$

Before to continue, let us state some results of independent interest that will be crucial in the proof of Lemma 3.1.9.

Theorem 3.1.1 (Selecting a large subdomain non effective). *Let (M^n, g) be a complete Riemannian manifold with weak bounded geometry, and $(D_j)_j$ is an approximate isoperimetric sequence. Then there exists another approximate isoperimetric sequence (D'_j) such that $\lim_{j\to\infty} \frac{V_g(D_j \triangle D'_j)}{V_g(D_j)} = 0$, $\lim_{j\to\infty} \frac{V_g(D'_j)}{V_g(D_j)} = 1$, $\lim_{j\to\infty} \dfrac{\mathcal{P}_g(D'_j)}{\mathcal{P}_g(D_j)} = 1$ and $diam_g(D'_j) \to 0$, when $j \to \infty$.*

Proof. We perform the same construction of a partition as in the proof of Lemma 3.1.5 applied to any D_j with a suitable radius R_j that we will choose later, and obtain a suitable partition $D_{j,l}$ of D_j a maximal family of points $\mathcal{A}_j$ such that

$$\left(\sum_l \mathcal{P}_g(D_{j,l})\right) - \mathcal{P}_g(D_j) \leq N(n, k, v_0)\frac{V_g(D_j)}{R_j}. \tag{3.12}$$

Set $v_j := V_g(D_j)$, by the definition of λ and of I_M, we get that for large j it holds

$$\mathcal{P}_g(D_{j,l}) \geq I_{M,g}(V_g(D_{j,l})) \geq \lambda v_{j,l}^{(n-1)/n}, \tag{3.13}$$

where $v_{j,l} := V_g(D_{j,l})$. Trivially for large j we have $V_g(D_{j,l}) \leq v_j \leq \bar{v}$ and the Euclidean type isoperimetric inequality for small volumes holds. This implies by Lemma 3.1.5 that

$$\frac{\sum_l \lambda V_g(D_{j,l})^{(n-1)/n}}{v_j^{(n-1)/n}} \leq \frac{\sum_l \mathcal{P}_g(D_{j,l})}{v_j^{(n-1)/n}} \leq \frac{\mathcal{P}_g(D_j)}{v_j^{\frac{n-1}{n}}} + N\frac{v_j^{\frac{1}{n}}}{R_j}. \tag{3.14}$$

Using the arguments of the combinatorial Lemma 2.3 of [Nar14b] applied to $f_{j,l} := \frac{V_g(D_{j,l})}{v_j}$, we get that $f_j^* := \max\{f_{j,1}, ..., f_{j,l_j}\}$ satisfies

$$\sum_l f_{j,l} f_j^{*-\frac{1}{n}} \leq \sum_l f_{j,l} f_{j,l}^{-\frac{1}{n}} = \sum_l f_{j,l}^{\frac{n-1}{n}} \leq \frac{1}{\lambda}\left[\frac{\mathcal{P}_g(D_j)}{v_j^{\frac{n-1}{n}}} + N\frac{v_j^{\frac{1}{n}}}{R_j}\right],$$

hence

$$f_j^* \geq \left[\frac{\lambda}{\frac{\mathcal{P}_g(D_j)}{v_j^{\frac{n-1}{n}}} + N\frac{v_j^{\frac{1}{n}}}{R_j}}\right]^n. \tag{3.15}$$

Without loss of generality we can assume that $f_j^* = f_{j,1}$ and so

$$V_g(D_{j,1}) \geq v_j \left[\frac{\lambda}{\frac{\mathcal{P}_g(D_j)}{v_j^{\frac{n-1}{n}}} + N\frac{v_j^{\frac{1}{n}}}{R_j}}\right]^n. \tag{3.16}$$

On the other hand we recall that by construction there exists a point $p_{D_j} \in \mathcal{A}_j$ depending on D_j such that $D_{j,1} \subseteq B_M(p_{D_j}, R_j)$. Fix an arbitrary sequence $\mu_j \to +\infty$ and set $R_j := \mu_j v_j^{\frac{1}{n}}$, $D'_j := B_M(p_{D_j}, R_j) \cap D_j$, $v'_j := V_g(D'_j)$, $l_{1,j} := \mathcal{P}_g(D_j, B_M(p_{D_j}, R_j))$, $l_{2,j} := \mathcal{P}_g(D_j) - l_{1,j}$, $A_j := \mathcal{P}_g(D'_j) = l_{1,j} + \frac{\Delta v_j}{R_j}$, and $\Delta v_j := v_j - v'_j$ we have $D_{j,1} \subseteq D'_j \subseteq D_j$ thus

$$\lim_{j\to+\infty} \frac{v'_j}{v_j} = 1, \tag{3.17}$$

$$\frac{\Delta v_j}{v_j} \leq 1 - \left[\frac{\lambda}{\frac{\mathcal{P}(D_j)}{v_j^{\frac{n-1}{n}}} + N\frac{v_j^{\frac{1}{n}}}{R_j}}\right]^n = 1 - \left[\frac{\lambda}{\frac{\mathcal{P}(D_j)}{v_j^{\frac{n-1}{n}}} + \frac{N}{\mu_j}}\right]^n, \tag{3.18}$$

$$\frac{\Delta v_j}{v_j} \to 0, \tag{3.19}$$

$$\frac{A_j}{\mathcal{P}_g(D_j)} \to 1, \tag{3.20}$$

$$\frac{l_{1,j}}{\mathcal{P}_g(D_j)} \to 1, \tag{3.21}$$

$$\frac{l_{2,j}}{v_j^{\frac{n-1}{n}}} \to 0. \tag{3.22}$$

□

Essentially Theorem 3.1.1 says that for small volumes approximate isoperimetric sequences have all the mass and perimeter that stay inside a ball of small radius. What will be proved further in Lemma 3.1.9 is that in fact the part outside this latter ball does not give any contribution and actually for small volumes is empty.

Definition 3.1.2. *Let (M^n, g) be a Riemannian manifold. We say that (M^n, g)* ***satisfy*** *(H), if there exists a positive constant $\lambda > 0$ such that*

$$\lim_{v\to 0^+} \frac{I_M(v)}{v^{\frac{n-1}{n}}} = \liminf_{v\to 0^+} \frac{I_M(v)}{v^{\frac{n-1}{n}}} = \limsup_{v\to 0^+} \frac{I_M(v)}{v^{\frac{n-1}{n}}} = \lambda. \tag{3.23}$$

In the next theorem we will give a little more refined proof of Theorem 3.1.1 having the advantage of being effective.

Theorem 3.1.2 (Selecting a large subdomain effective)**.** *Let (M^n, g) be a complete Riemannian manifold with weak bounded geometry and $\mu > 0$. Then there exists $\bar{v}_2 = \bar{v}_2(n, k, v_0, \mu) > 0$ such that for any finite perimeter set D with volume $v \leq \bar{v}_2$ there exists $p_D \in M$, and another finite perimeter set $D' := B_M(p_D, \mu v^{\frac{1}{n}}) \cap D \subseteq D$ such that*

$$\frac{V_g(D \triangle D')}{V_g(D)} \leq 1 - \left[\frac{\lambda}{\frac{\mathcal{P}_g(D)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}}\right]^n, \tag{3.24}$$

$$\frac{V_g(D')}{V_g(D)} \geq \left[\frac{\lambda}{\frac{\mathcal{P}_g(D)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}}\right]^n, \tag{3.25}$$

where λ is as in Definition 3.1.1. In particular $diam_g(D') \leq 2\mu v^{\frac{1}{n}}$.

Proof. First of all choose $\bar{v}_2 \leq \min\{\frac{\bar{R}}{\mu}, \bar{v}_1\}$, where $\bar{R}$ is as in Lemma 3.1.5 and $\bar{v}_1$ is as in Lemma 3.1.4. Then perform the same construction of a partition as in the proof of Lemma 3.1.5 applied to D with radius $R := \mu v^{\frac{1}{n}}$, and obtain a suitable partition $(D_l)_l$

containing a finite number l_D of components $D_j = \mathring{\cup} D_l$ joint with a maximal family of points $\mathcal{A}$ such that

$$\left(\sum_{l=1}^{l_D} \mathcal{P}_g(D_l)\right) - \mathcal{P}_g(D) \leq N(n, k, v_0)\frac{V_g(D)}{R}. \tag{3.26}$$

Set $v_l := V_g(D_l)$, by the definition of λ and of I_M, it holds

$$\mathcal{P}_g(D_l) \geq I_{M,g}(V_g(D_l)) \geq \lambda v_l^{\frac{n-1}{n}}. \tag{3.27}$$

Since $v \leq \bar{v}_2$ we have $V_g(D_l) \leq v \leq \bar{v}_2$ and the Euclidean type isoperimetric inequality for small volumes holds. This implies by Lemma 3.1.5 that

$$\frac{\sum_l \lambda V_g(D_l)^{(n-1)/n}}{v^{(n-1)/n}} \leq \frac{\sum_l \mathcal{P}_g(D_l)}{v^{(n-1)/n}} \leq \frac{\mathcal{P}_g(D)}{v^{\frac{n-1}{n}}} + N\frac{v^{\frac{1}{n}}}{R}. \tag{3.28}$$

Using the arguments of the combinatorial Lemma 2.3 of [Nar14b] applied to $\gamma_l := \dfrac{V_g(D_l)}{v}$, we get that $\gamma^* := \max\{\gamma_1, ..., \gamma_{l_D}\}$ satisfies

$$\sum_l \gamma_l {\gamma^*}^{-\frac{1}{n}} \leq \sum_l \gamma_l \gamma_l^{-\frac{1}{n}} = \sum_l \gamma_l^{\frac{n-1}{n}} \leq \frac{1}{\lambda}\left[\frac{\mathcal{P}_g(D)}{v^{\frac{n-1}{n}}} + N\frac{v^{\frac{1}{n}}}{R}\right],$$

hence

$$\gamma^* \geq \left[\frac{\lambda}{\frac{\mathcal{P}_g(D)}{v^{\frac{n-1}{n}}} + N\frac{v^{\frac{1}{n}}}{R}}\right]^n, \tag{3.29}$$

which implies

$$V_g(\tilde{D}_1') \geq v\left[\frac{\lambda}{\frac{\mathcal{P}_g(D)}{v^{\frac{n-1}{n}}} + N\frac{v^{\frac{1}{n}}}{R}}\right]^n, \tag{3.30}$$

where $\tilde{D}_1'$ is one of the connected components of the partition $(D_l)_l$ of D that satisfy $\frac{V_g(\tilde{D}_1')}{v} = \gamma^*$. On the other hand we recall that by construction there exists a point $p_D \in \mathcal{A}$ depending on D such that $\tilde{D}_1' \subseteq B_M(p_D, R)$. Set $D' := B_M(p_D, R) \cap D$, $v' := V_g(D')$, and $\Delta v := v - v'$ we have $\tilde{D}_1' \subseteq D' \subseteq D$ thus by (3.30)

$$\frac{v'}{v} \geq \left[\frac{\lambda}{\frac{\mathcal{P}_g(D)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}}\right]^n, \tag{3.31}$$

uniformly with respect to all finite perimeter sets D of volume v. Furthermore, it is also easily seen by (3.30) that

$$\frac{\Delta v}{v} \leq 1 - \left[\frac{\lambda}{\frac{\mathcal{P}_g(D)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}}\right]^n. \tag{3.32}$$

Remark 3.1.5. *At this stage we made the choice of not controlling the perimeter added cutting with a ball of radius R by a coarea formula argument. We recall that this is always possible (by coarea formula) up to take a slightly larger radius $R + \eta_D R$ for a suitable $0 < \eta_D < 1$.*

□

The following lemma have its own interest. Its proof is based on the adaptation of the arguments of the Deformation Lemma 4.5 of [GR13] and of formula (9) of [GMT83] also named Almgren's Lemma in some literature see for instance the book [Mag12].

Lemma 3.1.7 (Theorem 2.10 of [Giu84])**.** *Let Ω be a bounded open set in M^n with Lipschitz continuous boundary $\partial\Omega$ and let $f \in BV(\Omega)$. Then there exists a function $\phi \in L^1(\partial\Omega)$ such that for $\mathcal{H}^{n-1}$-almost all $x \in \partial\Omega$*

$$\lim_{\rho\to 0^+} \rho^{-n} \int_{B_M(x,\rho)\cap\Omega} |f(z) - \phi(x)| dv_g(z) = 0. \tag{3.33}$$

Moreover, for every $X \in \mathfrak{X}_0^1(M)$,

$$\int_\Omega f div_g X dv_g = -\int_\Omega \langle X, \nabla f\rangle_g dv_g + \int_{\partial\Omega} \phi\langle X, \nu\rangle_g d\mathcal{H}_g^{n-1}, \tag{3.34}$$

where ν is the unit outer normal to $\partial\Omega$ defined H_g^{n-1} a.e. on $\partial\Omega$.

The preceding lemma justifies rigourously the following definition.

Definition 3.1.3. *For every $f \in BV(M,g)$ every $D \subset M$ with Lipschitz continuous boundary ∂D we define the* ***trace of f on the boundary of D*** *as the function $\phi \in L^1(\partial D)$ of the preceding lemma and we will denote such a ϕ by $f_{|\partial D}$.*

We define a concept that will be useful in the sequel.

Definition 3.1.4 ([Mag12], Section 5.3, p. 62)**.** *For any given $t \in [0,1]$ and $E \subseteq M$ measurable with respect to the Riemannian measure V_g, we define the set $E^{(t)}$* ***of points of density t of E*** *as*

$$E^{(t)} := \{x \in M : \Theta^n(\chi_E V_g, x) = t\},$$

where $\Theta^n(\chi_E V_g, x)$ is the density of the measure $\chi_E V_g$ at the point x. For the formal rigorous definition of $\Theta^n(\chi_E V_g, x)$ the reader is referred to Definition 5.1.3.

Remark 3.1.6. *Notice that $E^{(t)}$ is a Borel measurable set for any $t \in [0,1]$.*

Lemma 3.1.8 (Deformation Lemma)**.** *Let (M^n, g) be a Riemannian manifold, $p \in M$, $B := B_{M,g}(p,r)$ a geodesic ball with $0 < r < inj_M$, $k \in \mathbb{R}$, $(n-1)k$ a lower bound on the Ricci curvature tensor inside B, $u : B \to [0,+\infty[$, $u : x \mapsto d_M(p,x)$, is the distance function to the point p, $E \subseteq M$ a set of locally finite perimeter in M. Then it holds*

$$\mathcal{P}_g(B, (M \setminus E)^{(1)}) \leq \mathcal{P}_g(E, B) + \frac{c}{r} V_g(B \setminus E), \tag{3.35}$$

where $c = c(n, k, inj_M) := 1 + (n-1)c_k(inj_M) > 0$ is a positive constant, $(M \setminus E)^{(1)}$ is the set of points of density 1 of $(M \setminus E)$, $t \mapsto c_k(t) := t \cot_k(t)$.

Proof. Applying Lemma 3.1.7 with $\Omega = B$, $f = \chi_{E^c}$, and $X := \varphi \frac{u}{r} \nabla u$, where $E^c := B \setminus E$ and $\varphi \in C_0^1(B_M(p, inj_M))$ with the property that $\{x \in M : d_M(x, B) \leq \varepsilon\} \subseteq \varphi^{-1}(1) \subseteq B_M(p, inj_M)$ for some small $\varepsilon > 0$ (observe that this choice of φ yields $X \in \mathfrak{X}_0^1(M)$) leads to

$$\begin{aligned}\int_B \chi_{E^c} div_g \left(\frac{u}{r}\nabla u\right) dv_g &= \int_{\partial B} \chi_{E^c}|_{\partial B} \frac{u}{r}\langle \nabla u, \nu_{ext}\rangle d\mathcal{H}_g^{n-1} \\ &- \int_B \left\langle \nabla \chi_{E^c}, \frac{u \nabla u}{r} \right\rangle dv_g.\end{aligned}$$

Now by standard comparison theorems on the Laplacian of the distance function it is not too hard to see that

$$\begin{aligned}\int_B \chi_{E^c} div_g \left(\frac{u}{r}\nabla u\right) dv_g &= \int_B \chi_{E^c} \left(\frac{\|\nabla u\|^2}{r} + \frac{u}{r} div_g(\nabla_g u)\right) dv_g \\ &\leq \int_B \chi_{E^c} \left(\frac{1}{r} + \frac{u(x)(n-1)}{r} \cot_k(u(x))\right) dv_g(x) \\ &\leq \int_B \chi_{E^c} \left(\frac{1}{r} + \frac{n-1}{r} c_k(r)\right) dv_g(x) \\ &\leq \frac{1}{r}\left(1 + (n-1)c_k(r)\right) v_g(E^c).\end{aligned}$$

On the other hand we know that

$$\int_B \left| \left\langle \nabla \chi_{E^c}, \frac{u}{r}\nabla u \right\rangle \right|_g dv_g \leq \mathcal{P}_g(E, B),$$

and

$$\mathcal{P}_g(B \setminus E, (M \setminus E)^{(1)}) + \mathcal{H}_g^{n-1}(\Gamma) = \int_{\partial B} \chi_{E^c}|_{\partial B} \frac{u}{r} \langle \nabla u, \nu_{ext}\rangle d\mathcal{H}_g^{n-1},$$

where $\Gamma := \{x \in \partial^* E^c \cap \partial^* B : \nu_{E^c}(x) = \nu_B(x)\}$ with ν_Ω being the measure theoretic outer normal to Ω, for any locally finite perimeter set Ω. Hence

$$\mathcal{P}_g(B, (M \setminus E)^{(1)}) - \mathcal{P}_g(E, B) \leq \frac{1}{r}\left(1 + (n-1)c_k(r)\right) V_g(E^c).$$

Notice that using the language of Theorem 16.3 [Mag12] we have of

$$\mathcal{P}_g(B, (M \setminus E)^{(1)}) = \mathcal{H}_g^{n-1}(\partial^* B \cap E^{(0)}).$$

From the last inequality it is easy to deduce (3.35), after the simple observation that $c_k(r) := r \cot_k(r)$ is a strictly increasing function in particular is bounded in $[0, inj_M]$. □

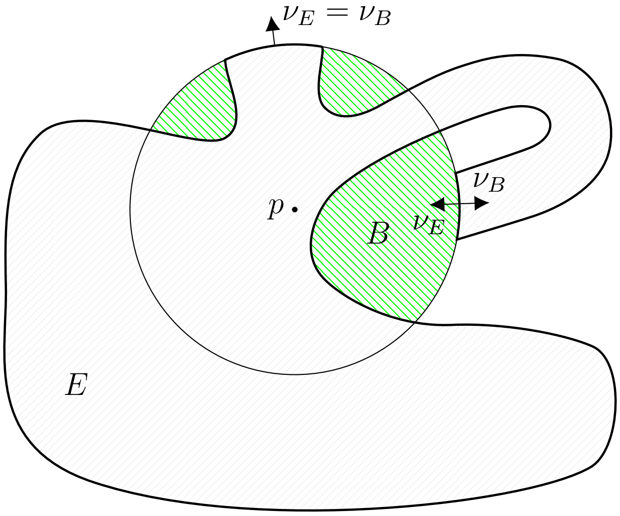


Figure 3.1: Illustration of the Deformation Lemma.

Remark 3.1.7. *It is worth to recall here that by Theorem* 1 *of [GMT83] (which immediately could be adapted to the Riemannian manifold because is a local theorem) an isoperimetric region have always nonempty interior as well as its complement but a lot of proofs of regularity do not give a satisfying and uniform estimates of the radius of the balls contained inside.*

The proof of the following lemma is based on the adaptation of the arguments of the Deformation Lemma 4.5 of [GR13] which in this context are given by our Lemma 3.1.8 combined with the arguments of Section 2 of [Nar14b] that are adapted here in Theorem 3.1.1 with the use of the Heintze-Karcher comparison Theorem of [HK78] combined with the proof of Lemma 3.8 of [Nar15] and Theorem 3 of [Nar14a].

Lemma 3.1.9. *Let (M^n, g) be a complete Riemannian manifold with mild bounded geometry satisfying the condition (H). Then there exist two positive constants $\mu^* = \mu^*(n, k, inj_M, \lambda) > 0$ and $v^* = v^*(n, k, inj_M, \lambda) > 0$ such that whenever $\Omega \subseteq M$ is an isoperimetric region of volume $0 \leq v \leq v^*$ it holds that*

$$\mathrm{diam}_g(\Omega) \leq \mu^* v^{\frac{1}{n}}.$$

Remark 3.1.8. *In mild bounded geometry v_0 depends on k and inj_M so in the preceding lemma we have that $\mu^* = \mu^*(n, k, inj_M, \lambda)$ and $v^* = v^*(n, k, inj_M, \lambda)$. In strong bounded geometry condition (H) is always fulfilled, moreover it is known that $\lambda = c_n$, hence in the preceding lemma when specialised to the case of strong bounded geometry we have actually $\mu^* = \mu^*(n, k, inj_M)$ and $v^* = v^*(n, k, inj_M)$. The construction made to prove the preceding lemma it is possible only because we assume positive injectivity radius. So the injectivity radius is hidden inside μ^* and v^* although it is tempting to prove Lemma 3.1.9 just assuming M with weak bounded geometry instead of mild bounded geometry.*

Remark 3.1.9. *As already observed, if M have strong bounded geometry, then always exists $\lim_{v\to 0^+} \frac{I_M(v)}{v^{\frac{n-1}{n}}} = \lambda$ and so in particular Lemma 3.1.9 applies to manifold with strong bounded geometry. Unfortunately, we still do not know wether the existence of $\lim_{v\to 0^+} \frac{I_M(v)}{v^{\frac{n-1}{n}}} = \lambda$ could be dropped or not in the statement of the preceding lemma. Obviously in weak bounded geometry or in mild bounded geometry, if one proves that $\lambda = c_n$, then automatically condition (H) is fulfilled.*

In fact the following questions are still open at the present stage of our knowledge.

Question 3. *If (M^n, g) is with weak bounded geometry, then M satisfy (H)?*

Question 4. *If (M^n, g) is with mild bounded geometry, then M satisfy (H)?*

Question 5. *If (M^n, g) is with weak bounded geometry or with mild bounded geometry, what is the sharp value of λ?*

Remark 3.1.10. *The main reason to assume positive injectivity radius in the preceding lemma is that we make a crucial use of Lemma 3.1.8 which in turn uses radial deformations which are well defined only locally at a point $x \in M$ inside a ball of radius less than*

inj_x. We will see later in the proof of Lemma 3.1.9 that we want to apply radial deformations with center at the point p^*_Ω defined further, but if $inj_M = 0$ we have no control about the size of $inj_{p^*_\Omega}$ (remember that p^*_Ω could go to infinity) and the volume that we can put inside $B_M(p^*_\Omega, inj_{p^*_\Omega})$. To avoid this problem of course it is enough to assume positive injectivity radius, but we still do not know whether this assumption could be dropped and replaced just by the noncollapsing of the volume of balls of radius 1.

Remark 3.1.11. *It is well known by that in mild bounded geometry $v_0 = v_0(n, k, inj_M) > 0$. Thus in the statement of Lemma 3.1.9 we can suppress the dependence of v^* on v_0.*

The geometric idea of the proof is not too complicated but unfortunately the writing turns out to be technical, because of the effective calculations of the constants involved. In first by an application of Theorem 3.1.1 we find a point $p_\Omega \in M$ and a controlled radius $\mu v^{\frac{1}{n}}$, such that almost all the volume of Ω is recovered inside the ball $B_{(M,g)}(p_\Omega, \mu v^{\frac{1}{n}})$. In second, we take a ball inside Ω of controlled volume and radius and show that these two balls cannot be disjointed when the volume tends to 0, so we take a bigger but still with controlled radius. Then we proceed by contradiction and suppose that there are points of Ω very far from p_Ω and adapt the arguments of the proof of Theorem 3 of [Nar14a], which were used to give a proof of the boundedness of the isoperimetric regions inside manifolds with weak bounded geometry.

Proof of Lemma 3.1.9. For simplicity of notations we consider just the case $k \leq 0$. When $k > 0$ the theorem of Bonnet-Myers ensures that M is compact and so the lemma is already proved in Theorem 2.2 of [MJ00]. Our proof works also without the restriction $k \leq 0$. For now on in this proof we assume that $k \leq 0$. The hypothesis (H) permits to us to define the quantities

$$\tilde{f}^*(n, k, v_0, \lambda, \mu) := \lim_{v \to 0^+} \tilde{f}(v, n, k, v_0, \lambda, \mu) = \left[\frac{\lambda}{\lambda + \frac{N}{\mu}}\right]^n, \tag{3.36}$$

$$f^*(n, k, v_0, \lambda, \mu) := \lim_{v \to 0^+} f(v, n, k, v_0, \lambda, \mu) = 1 - \left[\frac{\lambda}{\lambda + \frac{N}{\mu}}\right]^n, \tag{3.37}$$

where

$$\tilde{f}(v, n, k, v_0, \lambda, \mu, M) := \left[\frac{\lambda}{\frac{I_M(v)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}}\right]^n,$$

$$\hat{f}(v,n,k,v_0,\lambda,\mu,M) := 1 - \left[\frac{\lambda}{\frac{I_M(v)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}}\right]^n,$$

$$f(v,n,k,v_0,\lambda,\mu) := 1 - \left[\frac{\lambda}{\frac{I_{\mathbb{M}^n_k}(v)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}}\right]^n \geq \hat{f}(v,n,k,v_0,\lambda,\mu,M).$$

In the remaining part of this proof we will use frequently the two following crucial properties

$$\lim_{\mu\to+\infty} \tilde{f}^*(n,k,v_0,\lambda,\mu) = 1, \tag{3.38}$$

$$\lim_{\mu\to+\infty} f^*(n,k,v_0,\lambda,\mu) = 0. \tag{3.39}$$

Suppose until the end of the proof that Ω is an isoperimetric region of volume v. By Heintze-Karcher's theorem we have that in weak bounded geometry

$$c_7(n,k)v^{\frac{1}{n}} \geq inrad_g(\Omega) \geq \frac{v}{I_M(v)} \geq \frac{v}{I_{\mathbb{M}^n_k}(v)} \geq \frac{v}{c_2 v^{(n-1)/n}} = \tilde{c}_2 v^{\frac{1}{n}}, \tag{3.40}$$

for some positives constants $c_7 = c_7(n,k) > 0$ and $\tilde{c}_2 = \tilde{c}_2(n,k) > 0$, where $inrad_g(\Omega)$ is the radius of the largest ball contained in Ω. To see this in details the reader could consult Lemma 3.1 of [Nar15], furthermore we notice that for the needs of the proof we just need the existence of a ball contained in an isoperimetric region of a two sided controlled radius touching the boundary (in a smooth point) which is always possible when Ricci is bounded below by Heitze-Karcher. In first we observe that the hypothesis that M satisfy (H) permits to have (3.36)-(3.39) which in turn allows us to choose $\mu = \mu(n,k,v_0,\lambda) > 0$ large enough to satisfy simultaneously

$$\mu > \tilde{c}_2, \tag{3.41}$$

$$\mu > (C_1(n,k)v_0)^{-1/n}, \tag{3.42}$$

$$\mu > c_7(n,k), \tag{3.43}$$

$$(2f^*(n,k,v_0,\lambda,\mu))^{1/n} < \frac{\tilde{c}_2 C_{Heb}}{2(n+1)}, \tag{3.44}$$

where $C_{Heb} > 0$ is the constant appearing in Lemma 3.2 of [Heb99], rewrited in this text as Lemma 3.1.1. As it is easy to see, using (3.36) and (3.37) we can prove the existence of $v^* = v^*(n,k,v_0,inj_M,\lambda) > 0$ such that for every $v \leq v^*$ we have that the following conditions are satisfied

$$f(v,n,k,v_0,\lambda,\mu) < C_1(n,k)\tilde{c}_2 v_0, \tag{3.45}$$

$$f(v,n,k,v_0,\lambda,\mu)\leq 2f^*(n,k,v_0,\lambda,\mu), \tag{3.46}$$

$$r_v=4\mu v^{\frac{1}{n}}\leq\frac{1}{4}inj_M, \tag{3.47}$$

$$(1+(n-1)c_k(\mu v^{\frac{1}{n}}))\leq n+1, \tag{3.48}$$

$$c_1(n,k,\mu v^{\frac{1}{n}})<1, \tag{3.49}$$

$$v\leq\min\{1,\overline{v},\bar{v}_1,\bar{v}_2\}, \tag{3.50}$$

where $\overline{v}$ is obtained in Lemma 3.2 of [Heb99], or Lemma 3.1.1, i.e., such that for volumes smaller than $\overline{v}$ it holds $I_M(v)\geq\lambda v^{\frac{n-1}{n}}$. In the remaining part of this proof we always assume that $v\leq v^*$. Under this last assumption define $\tilde{C}_1:=\tilde{C}_1(n,k)$ such that $c_1(n,k,\mu v^{\frac{1}{n}})=\tilde{C}_1\mu^n v$. Consider an isoperimetric region Ω of $V_g(\Omega)=v$, the same construction of Theorem 3.1.1 applied to Ω gives the existence of $p_\Omega\in M$, (notice that the point p_Ω could be chosen satisfying the condition $p_\Omega\in\mathring{\Omega}$, but this is not relevant for the rest of our discussion) such that

$$\begin{aligned}\frac{V_g(B_M(p_\Omega,\mu v^{1/n})\cap\Omega)}{v} &= \frac{v_1(\Omega)}{v}\\ &\geq \left[\frac{\lambda}{\frac{\mathcal{P}_g(\Omega)}{v^{\frac{n-1}{n}}}+\frac{N}{\mu}}\right]^n\\ &\geq \tilde{f}(v,n,k,v_0,\lambda,\mu,M)=\left[\frac{\lambda}{\frac{I_M(v)}{v^{\frac{n-1}{n}}}+\frac{N}{\mu}}\right]^n.\end{aligned}$$

Consider $\Delta v=\Delta v(\Omega):=v-v_1(\Omega)$, observe that

$$\frac{\Delta v}{v}=\frac{v-v_1(\Omega)}{v}\leq 1-\left[\frac{\lambda}{\frac{\mathcal{P}_g(\Omega)}{v^{\frac{n-1}{n}}}+\frac{N}{\mu}}\right]^n\leq 1-\left[\frac{\lambda}{\frac{I_M(v)}{v^{\frac{n-1}{n}}}+\frac{N}{\mu}}\right]^n. \tag{3.51}$$

Observe that we can put inside Ω a geodesic ball

$$B_1(\Omega):=B_M(p^*_\Omega,inrad(\Omega))\subset\Omega.$$

We now show that $B_0(\Omega):=B_M(p_\Omega,\mu v^{1/n})$, cannot be disjoint from $B_1(\Omega)$. We prove this last assertion by contradiction. Indeed if it was the case we would have $B_1(\Omega)\subset\Omega\backslash B_0(\Omega)$, this would implies that $V_g(B_1(\Omega))\leq V_g(\Omega\backslash B_0(\Omega))=\Delta v$, and in turn by estimative (3.51)

$$C_1v_0\tilde{c}_2v\leq V_g(B_1(\Omega))\leq v\hat{f}(v,n,k,v_0,\lambda,\mu,M)\leq vf(v,n,k,v_0,\lambda,\mu),$$

which manifestly contradicts (3.45). Hence we necessarily have $B_1(\Omega)\cap B_0(\Omega)\neq\emptyset$. Thanks to our choice (3.43) the ball $B_1(\Omega)\subseteq B_{M,g}(p^*_\Omega,\mu v^{\frac{1}{n}})$, since $\mu v^{\frac{1}{n}}>c_7(n,k)v^{\frac{1}{n}}\geq inrad(\Omega)$. Moreover by our choice (3.42) the $V_g(B_M(p^*_\Omega,\mu v^{\frac{1}{n}}))>v$, hence there exists a radius $r^*_v<\mu v^{\frac{1}{n}}$ such that the ball $B:=B_{M,g}(p^*_\Omega,r^*_v)\subset B_2(\Omega):=B_{M,g}(p_\Omega,3\mu v^{\frac{1}{n}})$ have $V_g(B)=v$. Notice that B is just contained in $B_2(\Omega)$ and cannot be chosen as a proper subset of Ω. This guarantees that $V_g(B\setminus\Omega)>0$ and furthermore that

$$V_g(B\setminus\Omega)=V_g(B)-V_g(B\cap\Omega)\geq v-V_g(B_2\cap\Omega)\geq\Delta v, \tag{3.52}$$

because $V_g(B\setminus\Omega)=v-V_g(B_M(p^*_\Omega,r^*_v)\cap\Omega)$ but $V_g(B_M(p^*_\Omega,r^*)\cap\Omega)\leq v_1(\Omega)$ and (3.52) follows readily. Observe that by our choice (3.47) we have $B_M(p_\Omega,r_v)\subset B_M(p_\Omega,\frac{1}{4}inj_M)$. Assume the following notations

$$\tilde{d}_\Omega:=\sup_{x\in\Omega}\{d(x,p_\Omega)\},\quad d_\Omega=\tilde{d}_\Omega-r_v,\quad d_v:=\sup_{\Omega\in\tilde{\tau},V(\Omega)=v}\{d_\Omega\}.$$

For any $r>0$ let us define $V_\Omega(r):=V_g(\Omega\cap(M\setminus\bar{B}_r))=V_g(U_r)$ where $B_r:=\{y\in M: d_M(p_\Omega,y)<r\}$, p_Ω is given by Theorem 3.1.1, and $U_r=\Omega\cap(M\setminus\bar{B}_r)$. The function $V_\Omega(r)$ is monotone decreasing and $V_\Omega(r)\searrow 0$ as $r\to\infty$. Denote by $A_\Omega(r):=\mathcal{H}^{n-1}_g(\partial\Omega\cap(M\setminus\bar{B}_r))$. Coarea formula gives immediately

$$V_g(\Omega\cap(M\setminus\bar{B}_r))=\int_r^\infty\mathcal{H}^{n-1}_g(\Omega\cap\partial B_r)dr,$$

then

$$V'_\Omega(r)=-\mathcal{H}_g(\Omega\cap\partial B_r)=-\mathcal{H}^{n-1}_g(\Omega\cap\partial(M\setminus B_r)).$$

Consider any $r\geq 3\mu v^{\frac{1}{n}}$ and put all the volume $\Delta^*v:=V_\Omega(r)$ inside B, by choosing a concentric ball B_3 with $B_1\subset B_3\subset B\subset B_2$ of radius

$$\tilde{c}_2v^{\frac{1}{n}}\leq\rho_1=\rho_1(v,r)\leq\mu v^{\frac{1}{n}},$$

such that $V_g(B_3\setminus\Omega)=\Delta^*v$ (remember of (3.41)), then

$$F:=(B_3\cup\Omega)\setminus B_M(p_\Omega,r)=(B_M(p^*_\Omega,\rho_1(v,r))\cup\Omega)\setminus B_M(p_\Omega,r),$$

satisfies $V_g(F)=V_g(\Omega)$. The following picture illustrates well our construction

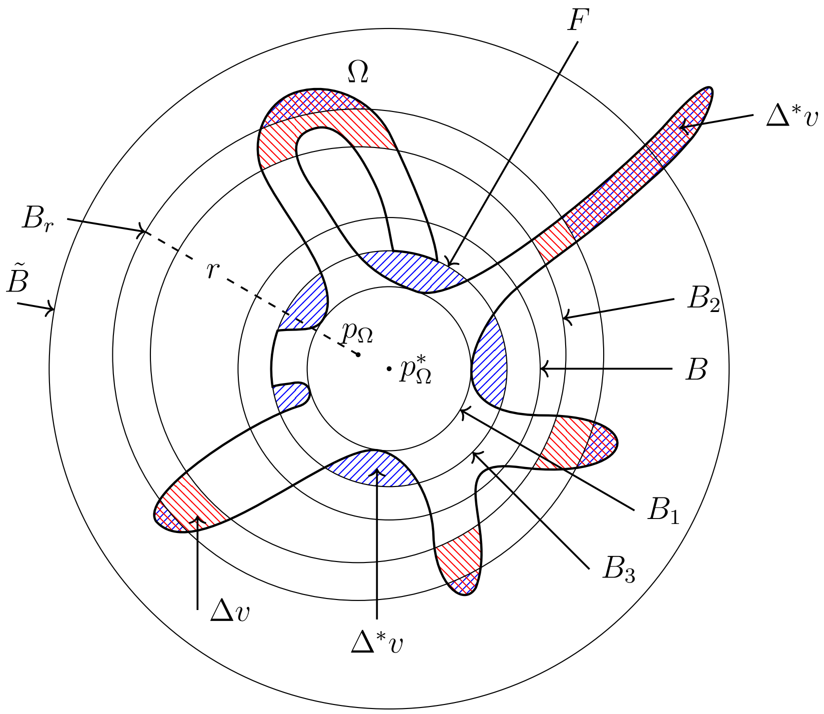


Figure 3.2: Construction of the competitor $F := (B_3 \cup \Omega) \setminus B_r$ used in the proof of Lemma 3.1.9. Here $\tilde{B} := B_M(p^*_\Omega, inj_M)$, $B_2 := B_M(p_\Omega, 3\mu v^{\frac{1}{n}})$, $B_r := B_M(p_\Omega, r)$.

From the fact that Ω is an isoperimetric region follows that $\mathcal{P}(\Omega) \leq \mathcal{P}(F)$, from an application of Lemma 3.1.8 with $E = \Omega \cap B_3$ inside the ball B_3, and from standard properties of finite perimeter set (compare Theorem 16.3 of [Mag12] and Remarks 2.13, 2.14 of [Giu84]) we have that for almost all $r \geq 3\mu v^{\frac{1}{n}}$ it holds

$$l_1(\Omega)(r) + A_\Omega(r) \leq l_1(\Omega)(r) - V'_\Omega(r) + (1 + (n-1)c_k(\rho_1))\frac{1}{\rho_1}V_\Omega(r), \tag{3.53}$$

where $l_1(\Omega)(r) := \mathcal{P}_g(\Omega, B_M(p_\Omega, r))$. This easily leads to

$$A_\Omega(r) \leq -V'_\Omega(r) + KV_\Omega(r), \tag{3.54}$$

where $K = K(n, k, v_0, v, \mu) = \tilde{C}_2(n, k)\frac{1}{v^{\frac{1}{n}}}$ for a suitable constant $\tilde{C}_2 = \tilde{C}_2(n, k) = \frac{n+1}{\tilde{c}_2(n,k)} > 0$. Notice that to obtain (3.53) we need to pay attention to the intersection of the reduced boundary of finite perimeter sets. For the details about this technical point the reader can consult Theorem 16.3 of [Mag12] or Remarks 2.13 and 2.14 of [Giu84]. Independently, by the Euclidean type isoperimetric inequality for small volumes of Lemma 3.2 of [Heb99] we have that for small volumes there exists a positive constant $\bar{v}_1 = \bar{v}_1(n, k, v_0) > 0$, such that if $v \leq \bar{v}_1$, then for every $r > 0$ it holds

$$C_{Heb} V_\Omega(r)^{(n-1)/n} \leq A_g(\partial U_r),$$

where $C_{Heb} = C_{Heb}(n, k, v_0) > 0$ is given by Lemma 3.2 of [Heb99] too. Thus for almost every $r > 0$ we have the following

$$\begin{aligned} -V_\Omega'(r) + A_\Omega(r) =& A_g(\Omega \cap \partial(M \setminus B_r)) + A_g(\partial\Omega \cap (M \setminus \bar{B}_r)) \\ \geq& A_g(\partial(\Omega \cap (M \setminus B_r))) \\ =& A_g(\partial U) \geq C_{Heb} V_\Omega(r)^{(n-1)/n}. \end{aligned} \tag{3.55}$$

Adding the two inequalities (3.54) and (3.55) we get that

$$2V_\Omega'(r) \leq K V_\Omega(r) - C_{Heb} V_\Omega(r)^{(n-1)/n}.$$

Using the fact that $n(V_\Omega^{1/n})' = V_\Omega^{\frac{1}{n}-1} V_\Omega'$ we can write the preceding inequality as

$$(V_\Omega^{1/n})'(r) \leq \frac{\tilde{C}_2}{2n} \left(\frac{V_\Omega(r)}{v} \right)^{1/n} - \frac{C_{Heb}}{2n},$$

for every $v \leq v^*$ and Ω such that $V_g(\Omega) = v$, where $C_{Heb} = C_{Heb}(n, k, v_0) > 0$ is the constant appearing in the isoperimetric inequality for small volumes of Lemma 3.2 of [Heb99] reported here as Lemma 3.1.1. Since $r \geq 3\mu v^{\frac{1}{n}}$, one have

$$(V_\Omega^{1/n})' \leq \frac{\tilde{C}_2}{2n} \left(2f^*(n, k, v_0, \lambda, \mu) \right)^{1/n} - \frac{C_{Heb}}{2n}.$$

By Theorem 3.1.1 and (3.44) we argue that

$$(V_\Omega^{1/n}(r))' \leq -C' = -\frac{C_{Heb}}{4n}. \tag{3.56}$$

It is worth to recall here that by Theorem 3 of [Nar14a], in weak bounded geometry $diam_g(\Omega) < +\infty$, because Ω is an isoperimetric region, and hence $d_\Omega := essup_{x \in \Omega} d_M(p_\Omega, x) = ||d_M(p_\Omega, \cdot)||_{L^\infty(\Omega)} < +\infty$. Furthermore we have the elementary relation $diam_g(\Omega) \leq 2d_\Omega$. Now, if we assume $r_v := 3\mu v^{\frac{1}{n}} < d_\Omega$, we can integrate (3.56) over the interval $[r_v, d_\Omega]$, and noting that $V_\Omega(r_v) \leq V_g(\Omega) = v$, $V_\Omega(d_\Omega) = 0$, we get

$$d_\Omega \leq \frac{1}{C'} V_\Omega(r_v)^{1/n} + r_v \leq \frac{v^{\frac{1}{n}}}{C'} + r_v = \left(\frac{1}{C'} + 3\mu \right) v^{\frac{1}{n}}.$$

From this last equation we easily a constant μ^* such that for every $v \leq v^*$ results

$$diam_g(\Omega) \leq \mu^*(n, k, inj_M, \lambda) v^{\frac{1}{n}},$$

which clearly proves the lemma. □

Chapter 4

Isoperimetric comparison and proof of Theorem 3

4.1 Proof of Theorem 3

Now we are in position to prove Theorem 3.

Proof of Theorem 3. If $|Sec_M| \leq K$ and $inj_M > 0$ then the assumptions of Theorem 76 of [Pet06] holds which implies that also the assumptions of Theorem 72 of [Pet06] are satisfied with $m = 1$, see also Theorem 4.4 of [Pet87a]. The problem here is that the limit metric space have an atlas of harmonic coordinates of class $C^{3,\alpha}$ with just a $C^{1,\alpha}$ limit metric. Unfortunately, to apply Theorem 1 to a limit manifold we need to have in the limit a smooth Riemannian manifold with a smooth Riemannian metric, for this reason we make a stronger assumption on (M^n, g) requiring that M have strong bounded geometry smooth at infinity. A fortiori M have also C^0-locally bounded geometry. This means by Theorem 1 of [Nar14a] or Theorem 1 of [MN15] that for every $v \in]0, V(M)[$ there exists a generalized isoperimetric region $\tilde{\Omega}_v$ contained in some smooth limit manifold (M_∞, g_∞) (that could even coincide with (M, g)). Now if we look at the limit manifold (M_∞, g_∞), by (iv) in Theorem 4.4 of [Pet87a] we learn that $inj_{(M_\infty, g_\infty)} \geq inj_{(M,g)} > 0$. Moreover, Theorem 10.7.1 of [BBI01] permits to conclude that (M_∞, g_∞) have sectional curvature bounded below by Λ_1. On the other hand, the property of being a *metric space of curvature* $\leq K$ (see Definition 1.2 at page 159 of [BH99], i.e., being locally a $Cat(K)$ space) pass to the limit in the pointed Gromov-Hausdorff convergence because distances pass to the Gromov-Hausdorff limit. This fact combined with Theorem $1A.6$ at page 173 of [BH99] implies that for a smooth Riemannian manifold to have sectional curvature bounded above by K is equivalent to satisfy the condition of having curvature bounded above in the sense of Alexandrov, that is, in the sense of Definition 1.2 of page 159 of [BH99]. From this easily follows that the sectional curvature of (M_∞, g_∞) (that exists because g_∞ is assumed at least C^2 or more regular by the assumption of strong bounded geometry smooth at infinity) is bounded from above by the same constant than the sectional curvature of M. Hence (M_∞, g_∞) have strong bounded geometry, in particular have also mild bounded geometry which gives the validity of Lemma 3.1.9 in (M_∞, g_∞), with a constant $v^*(M_\infty, g_\infty)$ that in principle depend on (M_∞, g_∞) but actually depends only on the bound of the geometry of (M_∞, g_∞) that are the same as (M, g) since they are transported to infinity. Furthermore Corollary 2.1.1 asserts that for a given $\varepsilon > 0$, $r_\varepsilon(M_\infty, g_\infty) \geq r_\varepsilon(M, g) > 0$, hence the same proof of Theorem 1 gives the existence of

$$d = \min\left\{C(n,k)^{-\frac{1}{n}}\left\{\frac{nK(n,1)^2}{2(n+2)C_0(n,k_0)}\left[n(n-1)k_0 - S_g\right]\right\}^{\frac{1}{4}}, r_\varepsilon(M,g), 1\right\},$$

with $\varepsilon = n(n-1)k_0 - S_g$ such that for every Ω with $diam_{g_\infty} \leq d$ and $V_g(\Omega) = v$ we have $\mathcal{P}_g(\Omega) > \mathcal{P}_{g_{k_0}}(B_v)$. On the other hand by (1.2) and the explicit estimates that Lemma 3.1.9 gives on $v^* = v^*(n, k, v_0)$ we argue that $d(M_\infty, g_\infty) \geq d > 0$ and $v^*(M_\infty, g_\infty) \geq v^* > 0$. At this stage an application of Lemma 3.1.9 gives that for $v \leq v^*$ any generalized isoperimetric region $\tilde{\Omega}_v \subseteq M_\infty$ with $V_{g_\infty}(\tilde{\Omega}_v) = v$ have $diam_{g_\infty}(\tilde{\Omega}_v) \leq \mu^* v^{\frac{1}{n}}$. Thus for small values of $v \leq \tilde{v}_0 := \min\{v^*, \left(\frac{d}{\mu^*}\right)^n\}$ we have $\mathrm{diam}_{g_\infty}(\tilde{\Omega}_v) \leq d$, where d is given by Theorem 1. Finally for every finite perimeter set $\Omega \subset M$ such that $V_g(\Omega) = v \leq \tilde{v}_0$, we conclude that

$$\mathcal{P}_g(\Omega) \geq I_{M,g}(v) = I_{M_\infty, g_\infty}(v) = \mathcal{P}_{g_\infty}(\tilde{\Omega}_v) > \mathcal{P}_{g_0}(B) = I_{\mathbb{M}^n_{k_0}}(v), \tag{4.1}$$

where the first inequality comes from the definition of $I_{M,g}$, the first equality comes from Theorem 1 of [Nar14a] where $\tilde{\Omega}_v$ is a generalized isoperimetric region of $V_{g_\infty}(\tilde{\Omega}_v) = v$, the second equality comes from the definition of $\tilde{\Omega}_v$ as a generalized isoperimetric region of volume v, the second (strict) inequality is due to an application of Theorem 1 to (M_∞, g_∞), and the last equality express simply the fact that the isoperimetric regions in space forms are the geodesic balls. This finish the proof of Theorem 3. □

4.2 Asymptotic expansion of the isoperimetric profile in C^3-bounded geometry

We prove in this last section the asymptotic expansion in Puiseux series up to the second nontrivial term stated in Corollary 2.

Proof of Corollary 2. We use Theorem 3 to prove the first of the following inequalities, then we compare with the area of a geodesic ball centered at x_0 and of enclosed volume v, proving the second inequality of

$$
\begin{aligned}
P_{g_{\frac{S_g}{n(n-1)}}}\left(B_{g_{\frac{S_g}{n(n-1)}},v}\right) &= c_n v^{\frac{(n-1)}{n}}\left(1-\gamma_n S_g v^{\frac{2}{n}}\right)+O_{\frac{S_g}{n(n-1)}}\left(v^{\frac{4}{n}}\right) \\
&< P_{g_{k_0}}\left(B_{g_{k_0},v}\right) \\
&= c_n v^{\frac{(n-1)}{n}}\left(1-\gamma_n n(n-1)k_0 v^{\frac{2}{n}}\right)+O_{k_0}\left(v^{\frac{4}{n}}\right) \\
&\leq I_M(v) \qquad (4.2)\\
&\leq I_{M_\infty}(v) \qquad (4.3)\\
&\leq c_n v^{\frac{(n-1)}{n}}\left(1-\gamma_n Sc_g(x_0) v^{\frac{2}{n}}\right)+O_{x_0}\left(v^{\frac{4}{n}}\right) \qquad (4.4)\\
&= c_n v^{\frac{(n-1)}{n}}\left(1-\gamma_n Sc_g v^{\frac{2}{n}}\right)+O_{x_0}\left(v^{\frac{4}{n}}\right) \\
&= P_g(B_g(x_0,v)),
\end{aligned}
$$

for every $x_0 \in M_\infty$, where x_0 is a maximum point of the scalar curvature function, $k_0 > S_g$, $v \leq \tilde{v}_0$, with $B_{g_{k_0},v}$ a ball in $\mathbb{M}^n_{k_0}$ such that $V_{g_{k_0}}(B_{k_0,v}) = v$ and $B_g(x_0,v)$ a ball of (M^n,g) having $V_g(B_g(x_0,v)) = v$. Observe that M_∞ could be coincident with M or is one of the limit pointed manifolds, in any case since the metrics at infinity are assumed to be smooth we always have an asymptotic expansion for small volumes of the perimeter of the geodesic balls up the third non trivial term by the Cartan expansion of the metric so $O_{x_0}\left(v^{\frac{4}{n}}\right) = f(x_0)v^{\frac{4}{n}}$ where $f(x_0)$ is an expression that depends on the metric up to the fourth order derivative. From this the corollary, indeed follows promptly. $\square$

Chapter 5

Intrinsic theory of Varifold in arbitrary Riemannian Manifolds

5.1 Small volumes implies small diameters, via an intrinsic monotonicity formula in Riemannian manifolds

5.1.1 An intrinsic monotonicity formula

Now we introduce the notations and concepts relative to varifolds that we need in the second proof of Lemma 3.1.9. In this respect we closely follow [All72] and [Lel12]. Just in this subsection, V will always denotes a varifold and μ_g the Riemannian measure of (M^n, g) instead of V_g used in the preceding sections.

Definition 5.1.1. *For any $0 \leq m \leq n$, we say that V is a m-**dimensional varifold in** M, if V is a nonnegative real extended valued (compare section 2.6 of [All72]) Radon measure on $G_m(M)$ the Grassmannian manifold whose underlying set is the union of the sets of m-dimensional subspaces of T_xM as x varies on M. For every $m \in \{0, ..., n\}$, we define $\mathbf{V}_m(M)$ to be the space of all m-dimensional varifolds on M endowed with the weak topology induced by $C^0_c(G_m(M))$ say the space of continuous compactly supported functions on $G_m(M)$ endowed with the compact open topology.*

Definition 5.1.2. *Let $V \in \mathbf{V}_m(M)$, g is a Riemannian metric on M, we say that the nonnegative Radon measure on M, $||V||$ is the* ***weight*** *of V, if $||V|| = \pi_{\#}(V)$, here π indicates the natural fiber bundle projection $\pi : G_m(M) \to M$, $\pi : (x, S) \mapsto x$, for every $(x, S) \in G_m(M)$, $x \in M$, $S \in G_m(T_x M)$,*

$$||V||(A) := V(\pi^{-1}(A)).$$

Notice that the notion of a varifold is independent of the choice of any Riemannian metric g on M. This reflects the phenomenon that on a differentiable manifold one can have a fixed submanifold but whose metric datas like volume, curvature, second fundamental form, etc. depends on the metric that we put on it. If we consider a varifold $V \in \mathbf{V}_m(M)$ we can construct without the help of a metric the support of $||V||$ that is a set contained in M, however starting from a set $E \subseteq M$ even a good one like a m-dimensional smooth submanifold of M, there is no canonical way to come back to a uniquely determined varifold $V \in \mathbf{V}_m(M)$, such that $Supp||V|| = E$. One way to proceed is to chose a metric g and to associate to a $\mathcal{H}^m_g$-countably m-rectifiable set E, the varifold $V_g(E) \in \mathbf{V}_m(M)$, where

$$V(E, g)(A) := \mathcal{H}^m_g(\{x \in E : (x, T_x E) \in A\}),\ \forall A \in G_m(M^n), \tag{5.1}$$

in this way the manifold associated is unique and canonical in the sense that depends only on the choice of the metric g. When (M^n, g) is $(\mathbb{R}^n, \xi)$ we find again the classical theory of varifolds as developed by Almgren, Allard et al. The way in which classically one proceed to study the theory of varifolds in Riemannian manifolds is well explained in [All72] and consists in embedding isometrically (M^n, g) in some higher dimensional Euclidean space via Nash's Theorem, and then using the existing theory on $\mathbb{R}^n$ of [All72]. The point of view that we will adopt here is an intrinsic one, without having to choose an isometric embedding. This is needed because in the Euclidean monotonicity formula will appear an upper bound of the second fundamental form of the particular isometric embedding chosen and it is not clear to us how to bound the second fundamental form of the isometric embedding just starting with intrinsic bounded geometry assumptions on the manifold (M, g). The intrinsic approach avoid this technical difficulty and permits to have a monotonicity formula which depends only on an upper bound of the sectional curvature.

This means that locally the geometric measure theory of $\mathbb{R}^n$ is mutatis mutandis the same as the corresponding theory developped on a Riemannian manifold, with just the constants involved depending on the bound of the sectional curvature. This is what one could expects since locally a Riemannian manifold is bi-Lipschitz equivalent to an Euclidean ball via the exponential map. The importance of making rigorous the details and the proofs is appears clear when we deal with problems in ambient manifolds with variable metric as in [Nar15].

Definition 5.1.3. *Let μ be a Borel regular measure on a locally compact Hausdorff topological space X. Define*

$$\Theta^m_*(\mu, a) := \liminf_{r\to 0^+} \frac{\mu(B(a,r))}{\omega_m r^m},$$

*the m-**lower density of** μ **at** $a \in M$,*

$$\Theta^{*m}(\mu, a) := \limsup_{r\to 0^+} \frac{\mu(B(a,r))}{\omega_m r^m},$$

*the m-**upper density of** μ **at** $a \in M$, and if*

$$\Theta^m_*(\mu, a) = \Theta^{*m}(\mu, a),$$

then we set

$$\Theta^m(\mu, a) := \Theta^{*m}(\mu, a) = \Theta^m_*(\mu, a) = \lim_{r\to 0^+} \frac{\mu(B(a,r))}{\omega_m r^m}.$$

*We call $\Theta^m(\mu, a)$ the m-**density of** μ **at** $a \in X$.*

According to [All72] we give the following definition for the first variation of a varifold.

Definition 5.1.4. *Let $V \in \mathbf{V}_m(M)$. Let $\mathfrak{X}_c(M)$ denotes the set of smooth vector fields on M with compact support, we denote by the linear function $\delta_g V(X) : \mathfrak{X}_c(M) \to \mathbb{R}$, the **first variation** of the varifold V in the direction of the vector field $X \in \mathfrak{X}_c(M)$, defined as follows*

$$\begin{aligned} \delta_g V(X) &:= \int_{\xi\in G_m(M)} \langle (\nabla^g X(\pi(\xi)) \circ \pi_S), \pi_S \rangle_g \, dV(\xi) \\ &:= \int_{\xi\in G_m(M)} \sum_{i=1}^n \left\langle \nabla^g_{\pi_S(e_i)} X, \pi_S(e_i) \right\rangle_g dV(\xi) \qquad (5.2) \\ &:= \int_{\xi\in G_m(M)} div_S X dV(\xi), \qquad (5.3) \end{aligned}$$

for every $X \in \mathfrak{X}_c(M)$*, where* $S \leq T_xM$ *is such that* $\xi = (x, S) \in G_m(M)$*, i.e., a* m*-dimensional subspace of* T_xM*,* π_S *is the orthogonal projection* $\pi_S : T_xM \to S$ *with respect to the metric* g*,* $(e_1, ..., e_n)$ *is an orthonormal basis of* $(T_{\pi(\xi)}M, g_{\pi(\xi)})$*, and* $div_S X = \sum_{i=1}^m \langle \nabla_{\tilde{e}_i} X, \tilde{e}_i \rangle_g$*, with* $\{\tilde{e}_1, \ldots, \tilde{e}_m\}$ *being an orthonormal basis over* S*.*

Remark 5.1.1. *The first variation is a metric concept and depends on* g*.*

Remark 5.1.2. *In the rest of this paper we adopt the convention to denote real variables with letters without subscripts and real constants by letters with subscripts.*

Following the treatment given in [Lel12], we give the next definition.

Definition 5.1.5 ([Lel12] Definition 1.1)**.** *Let* (M^n, g)*, be a Riemannian manifold,* $0 \leq m \leq n$*,* $m \in \mathbb{N}$*,* $\Gamma \subseteq M^n$ *a* m*-countably rectifiable set, and* $f : \Gamma \to \mathbb{N} \setminus \{0\}$ *a Borel map, we define a varifold* $V(\Gamma, f, g) \in \mathbf{V}_m(M)$ *as follows*

$$V(\Gamma, f, g)(A) := \int_{\{x \in \Gamma : (x, T_x\Gamma) \in A\}} f d\mathcal{H}^m_g, \ \forall A \in G_m(M^n). \tag{5.4}$$

We say that a varifold $V \in \mathbf{V}_m(M)$ *is a* m***-integral varifold****, if there exists a* m*-countably rectifiable set* $\Gamma \subseteq M$*, a Borel map* $f : \Gamma \to \mathbb{N} \setminus \{0\}$*, such that* $V = V(\Gamma, f, g)$*. The set of all* m*-integral manifolds of* (M^n, g) *will be denoted by* $\mathbf{IV}_m(M^n, g)$*.*

Definition 5.1.6 ([All72] Section 3.2)**.** *Let* (M^n, g_M) *and* (N^l, g_N) *be Riemannian manifolds and* $F : M^n \to N^l$ *be a smooth map. If* $V \in \mathbf{V}_m(M)$*, then* F *induce a natural Borel regular measure on* $G_m(N)$ *characterized by*

$$F_\#(V)(B) := \int_{\{(x,S):(F(x), dF_x(S)) \in B\}} |\Lambda_m dF_x \circ \pi_S|_{g_N} dV(x, S),$$

for any Borel subset B *of* $G_m(N)$*,* $|\Lambda_m dF_x \circ \pi_S|_{g_N} := tr_{g_N}(\Lambda_m dF_x \circ \pi_S) := \sum_{i=1}^m \langle dF_x \circ \pi_S(e_i), e_i \rangle_{g_N}$*, with* $(e_1, ..., e_n)$ *being an orthonormal basis of* S*. The measure* $F_\#(V)$ *is a varifold when* F *is a proper map, in this case* $F_\#(V)$ *is called the* **pushforward varifold of** V **by** F*.*

If $V(\Gamma, f, g)$ is an integral varifold in M, and $F : M \to N$ is a diffeomorphism then we have that $(F(\Gamma), f \circ F^{-1}, g_N)$ is an integral varifold in N that coincides with $F_\#(V)$.

Given a vector field $X \in \mathfrak{X}_c^1(M)$, the one-parameter family of diffeomorphisms generated by X is $\Phi_t(x) = \Phi(t,x)$ where $\Phi : \mathbb{R} \times M \to M$ is the unique solution of

$$\begin{cases} \frac{\partial \Phi}{\partial t} = X(\Phi), \\ \Phi(0,x) = x. \end{cases}$$

Proposition 5.1.1. *If $V \in \mathbf{IV}_m(M)$ and $X \in \mathfrak{X}_c^1(M)$, then the first variation of V along X is given by the following formula*

$$\delta V(X) = \frac{d}{dt}\Big|_{t=0} ||(\Phi_t)_\# V||(M) = \int_M div_{T_x\Gamma} X d||V||,$$

where Φ_t is the one-parameter family generated by X.

The proof of this fact is straightforward and goes along the same lines of Proposition 1.5 of [Lel12].

Definition 5.1.7. *We say that $V \in \mathbf{IV}_m(M)$ has **bounded generalized mean curvature**, if there exists a constant $C \geq 0$ such that*

$$|\delta V(X)| \leq C \int_M |X|_g d\|V\|, \quad \textit{for all } X \in \mathfrak{X}_c^1(M).$$

Since the map $X \mapsto \delta V(X)$ is linear the Riesz representation theorem and the Radon-Nikodym Theorem yield the following corollary.

Corollary 5.1.1. *If $V \in \mathbf{IV}_m(M)$ with bounded generalized mean curvature, then there is a bounded Borel map $H_g : M \to TM$ such that*

$$\delta V(X) = -\int_M \langle X, H_g \rangle_g d\|V\| \quad \textit{for all } X \in \mathfrak{X}_c^1(M).$$

*H_g will be called the **generalized mean curvature vector** of the varifold V and is defined up to sets of $\|V\|$-measure zero.*

To apply results from the general theory of varifolds to finite perimeter sets we need to observe that by De Giorgi's Structure Theorem (compare for instance Theorem 15.9 of [Mag12]) the reduced boundary of a locally finite set is a countably m-rectifiable set, so there is a natural integral varifold that could be associated to it. Inspired by Theorem 2.1 of [Lel12] and using the Hessian comparison theorem (compare [BM15], [HS74], [PRS06]) as in Lemma 3.6 of [HS74] we have the following monotonicity formula for the perimeter of a finite perimeter set with bounded generalized mean curvature.

Lemma 5.1.1 (Monotonicity Formula for reduced boundary)**.** *Let (M^n, g) be a Riemannian manifold $Sec_g \leq b$, for some constant $b \in \mathbb{R}$, $r_0 = r_0(b) > 0$ such that $r_0 \cot_b(r_0) > 0$, $r_0 < inj_x(M, g)$. Then for every $x \in \partial^*\Omega$ there exists a positive constant $0 < c = c(b, r_0, x) \leq 1$ such that $0 \leq r \leq r_0$ we have that $r \mapsto \mathcal{P}(\Omega, B_M(x, r))r^{-(n-1)}e^{\frac{\|H_g\|_\infty}{c}r}$ is monotone nondecreasing, $\Theta^{n-1}(\mathcal{H}_g^{n-1} \llcorner \partial^*\Omega, x)$ exists and holds the following inequality*

$$\mathcal{P}(\Omega, B(x, r)) \geq \omega_{n-1}\Theta^{n-1}(\mathcal{H}_g^{n-1} \llcorner \partial^*\Omega, x)r^{n-1}e^{-\frac{\|H_g\|_\infty}{c}r}, \tag{5.5}$$

where the Borel measure $\mathcal{H}_g^{n-1} \llcorner \partial^*\Omega$ is defined such that $\mathcal{H}_g^{n-1} \llcorner \partial^*\Omega(A) := \mathcal{H}_g^{n-1}(A \cap \partial^*\Omega)$ for every Borel set $A \subseteq M$. In particular when $inj_{(M,g)} > 0$, the constant $0 < c \leq 1$ could be chosen to be independent of x and depending just on b and $inj_{(M,g)}$.

Before to prove this lemma, we need the following results that are an adaptation to our context of Theorem 2.1 and Proposition 2.2 of [Lel12]. This generalization is made be possible by the comparison lemma for the partial divergence of the radial vector field given in Lemma 3.6 of [HS74].

Theorem 5.1.1 (Monotonicity Formula, Inequality)**.** *Let (M, g) be a complete Riemannian manifold with $Sec_g \leq b$, for some constant $b \in \mathbb{R}$, $V \in \mathbf{IV}_m(M)$ with bounded mean curvature vector H, fix $\xi \in M$, and $r_0 > 0$ such that $r_0 \cot_b(r_0) > 0$, $r_0 < inj_\xi(M, g)$. Then there exists a constant $c = c(b)$ satisfying $0 < c \leq 1$ such that for $0 < \sigma < \rho < r_0$, if we call $u(x) = r_\xi(x) = dist_{(M,g)}(x, \xi)$ we have that*

$$\frac{\|V\|(B_g(\xi, \rho))}{\rho^m} - \frac{\|V\|(B_g(\xi, \sigma))}{\sigma^m} \geq \frac{1}{c}\int_{B_g(\xi,\rho)} \frac{\langle H_g, u\nabla_g u\rangle_g}{m}\left(\frac{1}{m(r)^m} - \frac{1}{\rho^m}\right)d\|V\| + \\ + \frac{1}{c}\int_{B_g(\xi,\rho)\setminus B_g(\xi,\sigma)} \frac{|\nabla_g^\perp r|^2}{r^m}d\|V\|, \tag{5.6}$$

where $\nabla_g^\perp r = P_{T_x\Gamma^\perp}(\nabla_g r_\xi)$, and $m(r) := \max\{r_\xi(x), \sigma\}$.

Remark 5.1.3. *Notice that the optimal r_0 in the preceding theorem is given by the first positive zero of the function $t \mapsto t\cot_b(t)$, if $b > 0$, and $r_0 = +\infty$, if $b \leq 0$.*

Proof. Let $a = r_0$, $\gamma \in C_c^1(]-r_0, r_0[) = C_c^1(]-a, a[)$ such that $\gamma \equiv 1$ in a neighborhood of 0, and define the vector field $X_s(x) = \gamma\left(\frac{au(x)}{s}\right)u\nabla u$, where we suppressed the index ξ for simplicity of notation and we have defined $u(x) := r_\xi(x)$. By Corollary 5.1.1 it is easily seen that

$$-\int_M \langle X_s(x), H_g\rangle_g d\|V\|(x) = \int_M div_{T_x\Gamma}X_s(x)d\|V\|(x). \tag{5.7}$$

Notice that $T_x\Gamma$ exists a.e. $||V||$ since Γ is a countably m-rectifiable set. Fix $\{e_1,\dots,e_m\}$ an orthonormal basis of $T_x\Gamma$, and a completion $\{e_1,\dots,e_m,e_{m+1},\dots,e_n\}$ to an orthonormal basis of T_xM, write $\nabla_g^\perp u$ and $\nabla_g^\top u$ the orthogonal projections over $T_x\Gamma^\perp$ and $T_x\Gamma$ of the gradient vector field $\nabla_g u$ and remember that $\|\nabla_g u\|^2 = \|\nabla_g^\perp u\|^2 + \|\nabla_g^\top u\|^2 = 1$ this leads to

$$\begin{aligned} div_{T_x\Gamma} X_s(x) &= \sum_{i=1}^m \langle e_i, \nabla^g_{e_i} X_s(x)\rangle_g \\ &= \gamma\left(\frac{au}{s}\right) div^M_{T_x\Gamma}\left(u\nabla_g u\right) + \gamma'\left(\frac{au}{s}\right)\frac{au}{s}\sum_{j=1}^m \langle \nabla_g u, e_j\rangle^2 \\ &= \gamma\left(\frac{au}{s}\right) div^M_{T_x\Gamma}\left(u\nabla_g u\right) + \gamma'\left(\frac{au}{s}\right)\frac{au}{s}\|\nabla_g^\top u\|_g^2 \\ &= \gamma\left(\frac{au}{s}\right) div^M_{T_x\Gamma}\left(u\nabla_g u\right) + \frac{au}{s}\gamma'\left(\frac{au}{s}\right)\left(1-\|\nabla_g^\perp u\|_g^2\right). \end{aligned}$$

Now, inserting in (5.7) and dividing by s^{m+1} it follows

$$\begin{aligned} -\int_M \gamma\left(\frac{au}{s}\right)\frac{\langle H_g, u\nabla_g u\rangle_g}{s^{m+1}} d\|V\| = \int_M \frac{div^M_{T_x\Gamma}(u\nabla_g u)}{s^{m+1}}\gamma\left(\frac{au}{s}\right) d\|V\| \\ + \int_M \frac{au}{s^{m+2}}\gamma'\left(\frac{au}{s}\right)\left(1-\|\nabla_g^\perp u\|_g^2\right) d\|V\|. \end{aligned}$$

Integrating in s, between σ and ρ we obtain

$$\begin{aligned} -\int_\sigma^\rho\int_M \gamma\left(\frac{au}{s}\right)\frac{\langle H_g, u\nabla_g u\rangle_g}{s^{m+1}} d\|V\|ds &= \int_\sigma^\rho\int_M \frac{div^M_{T_x\Gamma}(u\nabla_g u)}{s^{m+1}}\gamma\left(\frac{au}{s}\right) d\|V\|ds \\ &+ \int_\sigma^\rho\int_M \frac{au}{s^{m+2}}\gamma'\left(\frac{au}{s}\right)\left(1-\|\nabla_g^\perp u\|_g^2\right) d\|V\|ds. \end{aligned}$$

Applying the Fubini's Thorem at all the terms of the preceding equality we get

$$\begin{aligned} -\int_M\left\{\int_\sigma^\rho \gamma\left(\frac{au}{s}\right)\frac{\langle H_g, u\nabla_g u\rangle_g}{s^{m+1}} ds\right\} d\|V\| = \int_M\left\{\int_\sigma^\rho \frac{div^M_{T_x\Gamma}(u\nabla_g u)}{s^{m+1}}\gamma\left(\frac{au}{s}\right) ds\right\} d\|V\| \\ + \int_M\left\{\int_\sigma^\rho \frac{au}{s^{m+2}}\gamma'\left(\frac{au}{s}\right)\left(1-\|\nabla_g^\perp u\|_g^2\right) ds\right\} d\|V\|. \end{aligned}$$

Notice that

$$\int_\sigma^\rho \frac{m}{s^{m+1}}\gamma\left(\frac{au}{s}\right) + \frac{au}{s^{m+2}}\gamma'\left(\frac{au}{s}\right) ds = -\int_\sigma^\rho \frac{d}{ds}\left(\frac{1}{s^m}\gamma\left(\frac{au}{s}\right)\right) ds,$$

and by the Rauch's comparison Theorem applied as in Lemma 3.6 of [HS74], (since $Sec_g \leq b$) and the very definition of the curvature tensor

$$div^M_{T_x\Gamma}(u\nabla u) \geq mu\cot_b(u) \geq mc,$$

where $0 < c = c(b) := r_0(b)\cot_b(r_0(b)) \leq 1$, if $b > 0$ and $c = 1$ otherwise. Remember the definition of $\cot_b(t) := \frac{s_b(t)}{c_b(t)}$, where $s_b(t) := \frac{\sin(\sqrt{b}t)}{\sqrt{b}}$, if $b > 0$, $s_b(t) := \frac{\sinh(\sqrt{b}t)}{\sqrt{b}}$, if $b < 0$, $s_b(t) := t$, if $b = 0$, and $c_b(t) := s_b'(t)$. Applying the Fundamental Theorem of Calculus we get

$$c\left(\frac{1}{\rho^m}\int_M \gamma\left(\frac{au}{\rho}\right)d\|V\| - \frac{1}{\sigma^m}\int_M \gamma\left(\frac{au}{\sigma}\right)d\|V\|\right)$$
$$\geq -\int_M \|\nabla^\perp u\|_g^2 \int_\sigma^\rho \frac{au}{s^{m+2}}\gamma'\left(\frac{au}{s}\right)dsd\|V\| + \int_M \langle H_g, u\nabla u\rangle_g \int_\sigma^\rho \frac{1}{s^{m+1}}\gamma\left(\frac{au}{s}\right)dsd\|V\|. \quad (5.8)$$

Integrating by parts yields

$$\begin{aligned}\int_\sigma^\rho \frac{au}{s^{m+2}}\gamma'\left(\frac{au}{s}\right)ds &= \int_\sigma^\rho -\frac{1}{s^m}\left[-\frac{au}{s^2}\gamma'\left(\frac{au}{s}\right)\right]ds = \int_\sigma^\rho -\frac{1}{s^m}\frac{d}{ds}\left(\gamma\left(\frac{au}{s}\right)\right)ds\\ &= -\frac{1}{s^m}\gamma\left(\frac{au}{s}\right)\Big|_\sigma^\rho - \int_\sigma^\rho \frac{m}{s^{m+1}}\gamma\left(\frac{au}{s}\right)ds\\ &= \frac{1}{\sigma^m}\gamma\left(\frac{au}{\sigma}\right) - \frac{1}{\rho^m}\gamma\left(\frac{au}{\rho}\right) - \int_\sigma^\rho \frac{m}{s^{m+1}}\gamma\left(\frac{au}{s}\right)ds.\end{aligned}$$

Replacing in (5.8) gives

$$\begin{aligned} c\left(\frac{1}{\rho^m}\int_M \gamma\left(\frac{au}{\rho}\right)d\|V\| - \frac{1}{\sigma^m}\int_M \gamma\left(\frac{au}{\sigma}\right)d\|V\|\right) \geq \\ \int_M \|\nabla_g^\perp u\|^2\left[\frac{1}{\rho^m}\gamma\left(\frac{au}{\rho}\right) - \frac{1}{\sigma^m}\gamma\left(\frac{au}{\sigma}\right) + \int_\sigma^\rho \frac{m}{s^{m+1}}\gamma\left(\frac{au}{s}\right)ds\right]d\|V\| + \\ \int_M \langle H_g, u\nabla_g u\rangle_g \int_\sigma^\rho \frac{1}{s^{m+1}}\gamma\left(\frac{au}{s}\right)dsd\|V\|. \end{aligned} \quad (5.9)$$

Now testing (5.9) with a sequence of cut-off functions $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $\gamma_n \to \chi_{]-a,a[}$ from below, by the dominated convergence theorem we conclude that we can insert $\gamma = \chi_{]0,a[}$ in (5.9), thus we get

$$\begin{aligned} &c\left(\frac{\|V\|(B_g(\xi,\rho))}{\rho^m} - \frac{\|V\|(B_g(\xi,\sigma))}{\sigma^m}\right) \geq \\ &\int_M \|\nabla_g^\perp u\|^2\left[\frac{1}{\rho^m}\chi_{]0,a[}\left(\frac{au}{\rho}\right) - \frac{1}{\sigma^m}\chi_{]0,a[}\left(\frac{au}{\sigma}\right) + \int_\sigma^\rho \frac{m}{s^{m-1}}\chi_{]0,a[}\left(\frac{au}{s}\right)ds\right]d\|V\| \\ &+ \int_M \langle H, u\nabla_g u\rangle_g \int_\sigma^\rho \frac{m}{s^{m+1}}\chi_{]0,a[}\left(\frac{au}{s}\right)dsd\|V\|. \end{aligned}$$

Finally observing that

$$\int_\sigma^\rho \frac{m}{s^{m+1}}\chi_{]0,a[}\left(\frac{au}{s}\right)ds = \int_\sigma^\rho \frac{m}{s^{m+1}}\chi_{]0,s[}(u)ds = \left[\frac{1}{m(r)^m} - \frac{1}{\rho^m}\right]\chi_{]0,\rho[}(u),$$

the monotonicity formula (5.5) follows easily. □

We list now some results that are important consequences of the monotonicity formula.

Corollary 5.1.2. *The function*

$$\rho \mapsto e^{\frac{\|H_g\|_\infty}{c}\rho}\rho^{-m}\|V\|(B_g(\xi,\rho)), \tag{5.10}$$

is monotone increasing.

Proof. Let $f(\rho) = \rho^{-m}\|V\|(B_g(\xi,\rho))$, then

$$\frac{f(\rho)-f(\sigma)}{\rho-\sigma} = \frac{\rho^{-m}\|V\|(B_g(\xi,\rho)) - \sigma^{-m}\|V\|(B_g(\xi,\sigma))}{\rho-\sigma}.$$

By the monotonicity formula (5.5) of Theorem 5.1.1 it is easily checked that

$$\frac{f(\rho)-f(\sigma)}{\rho-\sigma} \geq \frac{1}{c}\frac{1}{\rho-\sigma}\left[\int_{B_g(\xi,\rho)} \frac{\langle H_g, u\nabla_g u\rangle_g}{m}\left(\frac{1}{m(r)^m} - \frac{1}{\rho^m}\right) d\|V\| + \int_{B_g(\xi,\rho)\setminus B_g(\xi,\sigma)} \frac{|\nabla^\perp r|^2}{r^m} d\|V\|\right].$$

Hence

$$\begin{aligned}
\frac{f(\rho)-f(\sigma)}{\rho-\sigma} &\geq \frac{1}{c}\frac{1}{\rho-\sigma}\int_{B_g(\xi,\rho)} \frac{\langle H_g, u\nabla_g u\rangle_g}{m}\left(\frac{1}{m(r)^m} - \frac{1}{\rho^m}\right) d\|V\| \\
&\geq -\frac{\|H_g\|_\infty}{cm}\int_{B_g(\xi,\rho)} u(x)\frac{m(r)^{-m} - \rho^{-m}}{\rho-\sigma} d\|V\| \\
&\geq -\frac{\|H_g\|_\infty}{cm}\rho\|V\|(B_g(\xi,\rho))\frac{\sigma^{-m} - \rho^{-m}}{\rho-\sigma}.
\end{aligned}$$

Since $h : \rho \mapsto \rho^{-m}$ is convex, setting $\rho = \sigma + \varepsilon$ we get

$$\begin{aligned}
\frac{f(\rho)-f(\sigma)}{\rho-\sigma} &\geq -\frac{\|H_g\|_\infty}{cm}\rho\|V\|(B_g(\xi,\rho))\frac{\sigma^{-k} - (\sigma+\varepsilon)^{-k}}{\varepsilon} \\
&= \frac{\|H_g\|_\infty}{cm}\rho\|V\|(B_g(\xi,\rho))h'(\eta) \\
&\geq -\frac{\|H_g\|_\infty}{c}\rho\|V\|(B_g(\xi,\rho))\frac{1}{\sigma^{(m+1)}},
\end{aligned}$$

where $\eta \in]\sigma,\rho[$. Therefore

$$\frac{f(\sigma+\varepsilon)-f(\sigma)}{\varepsilon} \geq -\frac{\|H_g\|_\infty}{c} f(\sigma+\varepsilon)\frac{(\sigma+\varepsilon)^{m+1}}{\sigma^{m+1}}. \tag{5.11}$$

If ψ_δ is a standard non-negative mollifier, we can first take the convolution with ψ_δ integrating with respect to the variable σ, in both sides of (5.11) yields

$$\frac{f(\sigma+\varepsilon)-f(\sigma)}{\varepsilon} * \psi_\delta \geq -\frac{\|H_g\|_\infty}{c}\left(f(\sigma+\varepsilon)\frac{(\sigma+\varepsilon)^{k+1}}{\sigma^{k+1}} * \psi_\delta\right),$$

and only after letting $\varepsilon \downarrow 0$. We obtain in this way

$$(f * \psi_\delta)' \geq -\frac{\|H_g\|_\infty}{c}\,(f * \psi_\delta)\,.$$

Hence, multiplying by $e^{\frac{\|H_g\|_\infty}{c}\rho}$

$$e^{\frac{\|H_g\|_\infty}{c}\rho}\,(f * \psi_\delta)' + \frac{\|H_g\|_\infty}{c} e^{\frac{\|H_g\|_\infty}{c}\rho}\,(f * \psi_\delta) \geq 0,$$

or equivalently

$$\frac{d}{d\rho}\left(e^{\frac{\|H_g\|_\infty}{c}\rho}\,(f * \psi_\delta)\right) \geq 0.$$

Finally taking the limit when $\delta \to 0$ in the preceding inequality the result follows easily.

□

The following proposition is an interesting application of Theorem 5.1.1, whose proof goes along the same lines of the corresponding Euclidean one that the reader could find in Proposition 2.2 and Theorem 2.1 of [Lel12], after being established our intrinsic monotonicity formula.

Proposition 5.1.2. *Let (M^n, g) and $V \in \mathbf{IV}_m(M)$ be as in the preceding theorem. Then*

(i) the limit

$$\Theta^m(\|V\|, x) = \lim_{\rho \downarrow 0} \frac{\|V\|(B_g(x,\rho))}{\omega_m \rho^m}, \tag{5.12}$$

exists at every $x \in M$,

(ii) $x \mapsto \Theta^m(\|V\|, x)$ is upper semicontinuous in the variable x,

(iii) $\Theta^m(\|V\|, x) \geq 1$, for all $x \in Spt\|V\|$,

(iv)

$$\|V\|(B_g(x,\rho)) \geq \omega_m e^{-\frac{\|H_g\|_\infty}{c}\rho}\rho^m, \tag{5.13}$$

for all $x \in Spt(\|V\|)$ and for all $\rho < r_0$,

(v) $\mathcal{H}^m_g\,(Spt\|V\| \setminus \Gamma) = 0$.

Proof. (i) The existence of the limit is guaranteed by the monotonicity of

$$\rho \mapsto e^{\frac{\|H_g\|_\infty}{c}\rho}\rho^{-m}\|V\|(B_\rho(x)),$$

when $\rho \to 0$.

(ii) Fix $x \in M$ and $\varepsilon > 0$, Let $0 < 2\rho < r_0$ such that

$$e^{\frac{\|H_g\|_\infty}{c} r} \frac{\|V\|(B_g(x,r))}{r^{-m}\omega_m} \le \Theta^m(\|V\|, x) + \frac{\varepsilon}{2}, \quad \forall r < 2\rho. \tag{5.14}$$

If $\delta < \rho$ and $|x-y| < \delta$, then

$$\begin{aligned} \Theta^m(\|V\|, y) &:= \lim_{\rho\downarrow 0} \frac{\|V\|(B_g(y,\rho))}{\omega_m \rho^m} \le \lim_{\rho\downarrow 0} e^{\frac{\|H_g\|_\infty}{c}\rho} \frac{\|V\|(B_g(y,\rho))}{\omega_m \rho^m} \\ &\le \lim_{\rho\downarrow 0} e^{\frac{\|H_g\|_\infty}{c}(\rho+\delta)} \frac{\|V\|(B_g(x,\rho+\delta))}{\omega_m \rho^m} \\ &= \lim_{\rho\downarrow 0} e^{\frac{\|H_g\|_\infty}{c}(\rho+\delta)} \frac{\|V\|(B_g(x,\rho+\delta))}{\omega_m (\rho+\delta)^m} \left(\frac{\rho+\delta}{\rho}\right)^m \\ &\le \left(1 + \frac{\delta}{\rho}\right)^m \left[\Theta^m(\|V\|, x) + \frac{\varepsilon}{2}\right], \end{aligned}$$

where the last inequality is true because of (5.14). If $\delta > 0$ is small enough

$$\Theta^m(\|V\|, y) \le \Theta^m(\|V\|, x) + \varepsilon,$$

which proves the upper semicontinuity.

(iii) Since f is integer valued the set $\{\Theta^m(\|V\|, \cdot) \ge 1\}$ has full $||V||$-measure. Thus $\{\Theta^m(\|V\|, \cdot) \ge 1\}$ must be dense in $Spt(\|V\|)$ and so, for $\|V\|$-a.e. $x \in Spt(\|V\|)$ the inequality $\Theta^m(\|V\|, x) \ge 1$ follows from the upper semicontinuity.

(iv) Inequality (5.13) follows trivially from Theorem 5.1.1.

(v) Finally by the classical density theorems $\Theta^m(\|V\|, \cdot) = 0$ $\mathcal{H}^m$-a.e. on $M \setminus \Gamma$. Hence the result follows from (iii).

□

Proof of Lemma 5.1.1. As it is well known De Giorgi's Structure Theorem for finite perimeter sets have as consequence that the reduced boundary of every set of finite perimeter defines a rectifiable $(n-1)$-varifold of multiplicity $f \equiv 1$. Moreover we have that $Sec_g \le b$ for some $b \in \mathbb{R}$. Thus we can directly apply the results of the Proposition 5.1.2 namely Inequality (5.13) to get Inequality (5.5) which certainly proves the lemma. □

5.1.2 Small diameters implies small volumes. A simpler alternative proof via monotonicity formula.

Now we are ready to give a simpler alternative proof of the results contained in Lemma 3.1.9, proving Lemma 5.1.2 but making stronger assumptions on the way in which the geometry of (M^n, g) is bounded this proof makes a crucial use of the monotonicity formula obtained in Lemma 5.1.1. We start the proof as in the proof of 3.1.9. In first by an application of Theorem 3.1.1 we find a point $p_\Omega \in M$ and a controlled radius $\mu v^{\frac{1}{n}}$, such that almost all the volume of Ω is recovered inside the ball $B_{(M,g)}(p_\Omega, \mu v^{\frac{1}{n}})$. Then we proceed by contradiction and suppose that there are points p of $\partial^*\Omega$ very far from p_Ω. Then we cut by a ball centered at p and with radius $r \sim const.v^{\frac{1}{n}}$ and apply the monotonicity formula for the perimeter of an isoperimetric region inside this ball. This will lead to a contradiction.

Lemma 5.1.2. *Let (M^n, g) be a complete Riemannian manifold with positive injectivity radius $inj_{(M,g)} > 0$, $Ric_g \geq (n-1)k$, and $Sec_g \leq \Lambda_2$. Then there exist two positive constants $\mu^* = \mu^*(n, k, \Lambda_2, inj_M) > 0$ and $v^* = v^*(n, k, \Lambda_2, inj_M) > 0$ such that whenever $\Omega \subseteq M$ is an isoperimetric region of volume $0 \leq v \leq v^*$ it holds that*

$$\mathrm{diam}_g(\Omega) \leq \mu^* v^{\frac{1}{n}}.$$

Proof. First of all we want to mention that in this proof we will use the same notations used in the proof of Lemma 3.1.9. By Heintze-Karcher we know that the length of the mean curvature vector $H_{g,\Omega}$ (which is actually a non-negative constant) of $\partial^*\Omega$ is less than $\frac{C(n,k)}{\mu v^{\frac{1}{n}}}$ for sufficiently small $v \leq v^* = v^*(n, k, v_0)$ as in (3.40), where $C(n, k)$ is a positive constant depending only on n and on k. Fix $\mu > 0$ large enough such that

$$2^{\frac{n-1}{n}} C_3(n,k) \left\{ 1 - \left[\frac{c_n}{c_n + \frac{N}{\mu}} \right]^n \right\}^{\frac{n-1}{n}} < \omega_{n-1} \mu^{n-1} e^{-c'}, \tag{5.15}$$

where $c' = c'(n, k, \Lambda_2)$ is a positive constant satisfying $\frac{||H_{g,\Omega}||_\infty}{c} \mu v^{\frac{1}{n}} \leq c'$ and c is the constant appearing in (5.5). Then choose possibly a smaller $v^* > 0$ such that it holds also $v^* \leq \min\left\{ \bar{v}_2, \left(\frac{\Lambda_2 r_0}{\mu}\right)^n \right\}$, where $r_0 > 0$ is as in Theorem 5.1.1 and such that

$$1 - \left[\frac{\lambda}{\frac{\mathcal{P}_g(\Omega)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}} \right]^n = 1 - \left[\frac{\lambda}{\frac{I_M(v)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}} \right]^n \leq 2 \left\{ 1 - \left[\frac{c_n}{c_n + \frac{N}{\mu}} \right]^n \right\}, \tag{5.16}$$

for every $0 < v \leq v^*$. It follows from (3.24) of Theorem 3.1.2 that

$$\frac{\Delta v}{v} \leq 2\left\{1 - \left[\frac{c_n}{c_n + \frac{N}{\mu}}\right]^n\right\}. \tag{5.17}$$

Look at the following picture

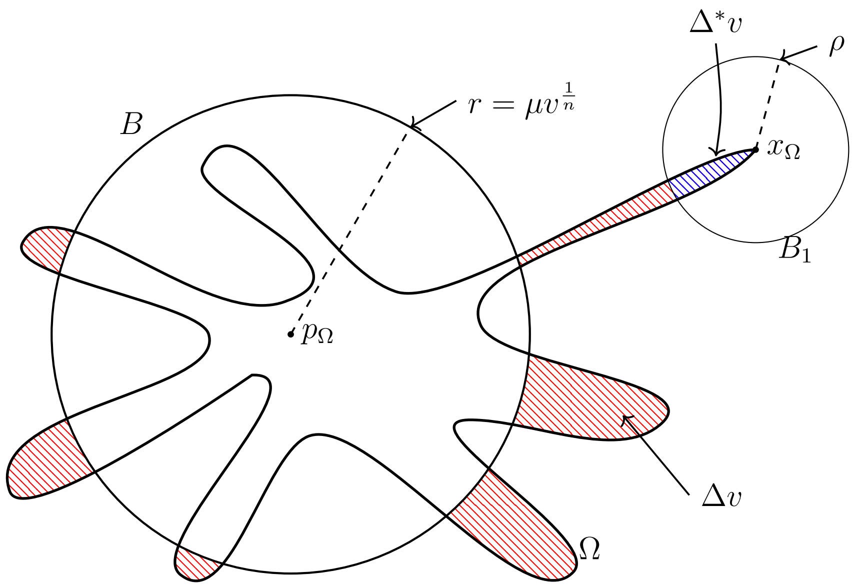


Figure 5.1: Construction used in the proof of Lemma 5.1.2. Here $B := B_g(p_\Omega, \mu v^{\frac{1}{n}})$, $B_1 := B_g(x_\Omega, \rho)$, where $\rho := \mu v^{\frac{1}{n}}$.

Assume the following notations

$$\tilde{d}_\Omega := \sup_{x\in\Omega}\{d(x, p_\Omega)\} = d(x_\Omega, p_\Omega), \quad d_\Omega = \tilde{d}_\Omega - r_v, \quad d_v := \sup_{\Omega\in\tilde{\tau}, V(\Omega)=v}\{d_\Omega\},$$

and assume by contradiction that $d := \inf_{0<v\leq v^*}\{d_v - \mu v^{\frac{1}{n}}\} > 0$. We can choose

$$0 < v < \min\left\{\left(\frac{d}{\mu}\right)^n, v^*\right\}. \tag{5.18}$$

Recalling that $||H_{g,\Omega}||_\infty \sim \frac{C}{\mu v^{\frac{1}{n}}}$, by the monotonicity formula of Lemma 5.1.1, i.e., Inequality (5.5) we obtain

$$\mathcal{P}_g(\Omega, B_g(x_\Omega, \mu v^{\frac{1}{n}}) \geq \omega_{n-1}(\mu v^{\frac{1}{n}})^{n-1} e^{-\frac{||H_{g,\Omega}||_\infty}{c}\mu v^{\frac{1}{n}}}, \tag{5.19}$$

on the other hand by Lemma 3.1.4 applied to $\Omega \setminus B$ we have that

$$C_3(n,k)(\Delta v)^{\frac{n-1}{n}} \geq \mathcal{P}_g(\Omega, B_g(x_\Omega, \mu v^{\frac{1}{n}})). \tag{5.20}$$

From the preceding two inequalities we obtain

$$C_3(n,k)\left(\frac{\Delta v}{v}\right)^{\frac{n-1}{n}} \geq \omega_{n-1}\mu^{n-1}e^{-c'}, \tag{5.21}$$

which by (5.17) implies

$$C_3\left(2\left\{1-\left[\frac{c_n}{c_n+\frac{N}{\mu}}\right]^n\right\}\right)^{\frac{n-1}{n}} \geq \omega_{n-1}\mu^{n-1}e^{-c'},$$

which in turn contradicts (5.15) and completes the proof. □

Appendix A

Comparison geometry

Since our study is made in Riemannian manifolds whose geometry is bounded in some sense, is essential to give the more relevant comparison theorems of Riemannian geometry. These theorems are among the basic ingredients of the analysis on manifolds with bounded geometry.

A.1 Comparison Theorems

Theorem A.1.1 (Bonnet-Myers Theorem). *Let (M^n, g) be a complete Riemannian manifold with $Ric_g \geq (n-1)k$, with $k > 0$. Then M is compact and* $\operatorname{diam}(M) \leq \dfrac{\pi}{\sqrt{k}}$. *In particular its fundamental group is finite.*

Theorem A.1.2 (Cartan-Hadamard theorem). *Let $(M{\cdot}g)$ be a complete Riemannian manifold with non-positive sectional curvature. Then for any p in M, the exponential map* $\exp_p$ *is a covering map. In particular, M is diffeomorphic to $\mathbb{R}^n$ provided it is simply connected.*

A simply proof of the Bonnet-Myers Theorem and Cartan-Hadamard Theorem can be found in [GHL12] page 162 and 163 respectively.

Theorem A.1.3 (Weak Rauch comparison theorem). *Let M^n a Riemannian manifold with $Sec_M \leq k$, where $k \in \mathbb{R}$ is constant, and $\gamma : [0, a] \to M$ is a geodesic parametrized by arc length. If J is a Jacobi field non trivial along of γ, with $J(0) = 0$ e $\langle J', \gamma' \rangle(0) = 0$,* *then*

$$|J(t)| \geq sn_k(t)|J'(0)|$$

for $0 \leq t \leq a$ *if* $k \leq 0$, *or* $0 \leq t \leq \min\left\{a, \frac{\pi}{\sqrt{k}}\right\}$ *se* $k > 0$.

The above Theorem is called by others authors like Jacobi Field Comparison Theorem.

Theorem A.1.4 (Rauch comparison theorem I)**.** *Let* M^n *and* $\tilde{M}^m$, $m \geq n$ *be Riemannian manifolds and normal geodesic* γ *(resp.,* $\tilde{\gamma}$*) of length* T *in* M *(resp.,* $\tilde{M}$*). Assume that* $t \in [0, T]$ *and* $X \in T_{\gamma(t)}M$ *(resp.,* $\tilde{X} \in T_{\tilde{\gamma}(t)}\tilde{M}$*) perpendicular to* $\gamma'(t)$ *(resp.,* $\tilde{\gamma}'(t)$*) are such that*

(i) $\tilde{\gamma}(t)$ *is not conjugate to* $\tilde{\gamma}(0)$ *along* $\tilde{\gamma}$ *for all* $0 < t \leq T$,

(ii) $Sec_M(\gamma'(t), X) \leq Sec_{\tilde{M}}(\tilde{\gamma}'(t), \tilde{X})$,

If J *(resp.,* $\tilde{J}$*) are Jacobi field along a normal geodesic* γ *(resp.,* $\tilde{\gamma}$*) not identically null, such that* $J(0) = \tilde{J}(0) = 0$ *and* $\langle J'(0), \gamma'(0)\rangle = \langle \tilde{J}'(0), \tilde{\gamma}'(0)\rangle = 0$, *then*

1. $F(t) := \dfrac{\|\tilde{J}(t)\|}{\|J(t)\|}$ *is monotone increasing for* $0 < t \leq T$, *and*
2. $\langle J'(t), J(t)\rangle \geq \dfrac{\|J(t)\|^2}{\|\tilde{J}(t)\|^2}\langle \tilde{J}'(t), \tilde{J}(t)\rangle$

Theorem A.1.5 (Rauch comparison theorem II)**.** *Let* M^n *and* $\tilde{M}^m$, $m \geq n$ *be complete Riemannian manifolds and normal geodesic* γ *(resp.,* $\tilde{\gamma}$*) of length* T *in* M *(resp.,* $\tilde{M}$*). Assume that* $t \in [0, T]$ *and* $X \in T_{\gamma(t)}M$ *(resp.,* $\tilde{X} \in T_{\tilde{\gamma}(t)}\tilde{M}$*) perpendicular to* $\gamma'(t)$ *(resp.,* $\tilde{\gamma}'(t)$*) are such that*

(i) $\tilde{\gamma}(t)$ *is not conjugate to* $\tilde{\gamma}(0)$ *along* $\tilde{\gamma}$ *for all* $0 < t \leq T$,

(ii) $Sec_M(\gamma'(t), X) \leq Sec_{\tilde{M}}(\tilde{\gamma}'(t), \tilde{X})$,

Let J *(resp.,* $\tilde{J}$*) be a Jacobi field along a normal geodesic* γ *(resp.,* $\tilde{\gamma}$*) such that* $\|J'(0)\| = \|\tilde{J}'(0)\|$, *then*

$$\|J(t)\| \geq \|\tilde{J}(t)\| \quad \forall t \in [0, T].$$

Furthermore, if $\|J(t_0)\| = \|\tilde{J}(t_0)\|$ *for some* $t_0 \in]0, T]$, *then for* $t \in]0, t_0]$ *we have*

$$\|J(t)\| = \|\tilde{J}(t)\|, \quad Sec_M(\gamma'(t), X) = Sec_{\tilde{M}}(\tilde{\gamma}'(t), \tilde{X}).$$

A direct application of A.1.4, gives the proof of the Cartan-Hadamard Theorem A.1.2 and of the followin theorem:

Theorem A.1.6 (Klingenberg). *Suppose that (M, g) is a Riemannian manifold with $0 < k \leq Sec_M \leq K$ and let L be half the length of the shortest closed geodesic in M. Then*

$$inj_M = \min\left\{\frac{\pi}{\sqrt{K}}, L\right\}.$$

Theorem A.1.7 (Bishop-Gromov). *Assume that $Ric \geq (n-1)k$ where k is any real number, and let $p \in M$. Then*

$$r \mapsto \frac{Vol(B^M(p,r))}{Vol(B^{\mathbb{M}_k}(r))}$$

is a non-increasing function which tends to 1 as $r \to 0$. In particular for any $r \geq 0$ we have $Vol(B(p,r) \leq Vol(B^{\mathbb{M}_k}(r))$, where $B^M(p,r)$ is a geodesic ball in M centered in p and radii r, and $B^{\mathbb{M}_k}(r)$ is a geodesic ball of radii r in the space form $\mathbb{M}_k$ of constant sectional curvature k.

Theorem A.1.8 (Bishop-Gunther). *Let M^n a complete Riemannian manifold oriented, $p \in M$ and $r > 0$ such that $B^M(p,r) \cap Cut(p) = \emptyset$. If $Sec_M \leq k$, then*

$$r \mapsto \frac{Vol(B^M(p,r))}{Vol(B^{\mathbb{M}_k}(r))}$$

is a non-decreasing function which tends to 1 as $r \to 0$. In particular for any $r \geq 0$ we have $Vol(B^M(p,R)) \geq Vol(B^{\mathbb{M}_k}(r))$.

If we define the function

$$sn_k(x) = \begin{cases} \frac{1}{\sqrt{k}}\sin(\sqrt{k}x) & \text{if } k > 0,\\ x & \text{if } k = 0,\\ \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}x) & \text{if } k < 0.\end{cases}$$

We call $cn_k(x) = sn_k'(x)$ and $ct_k(x) = \dfrac{sn_k'(x)}{sn_k(x)} = \dfrac{cn_k(x)}{sn_k(x)}$

Theorem A.1.9 (Laplacian comparison). *If (M^n, g) is a complete Riemannian manifold with $Ric \geq (n-1)k$, where $k \in \mathbb{R}$, and if $p \in M^n$, then for any $x \in M^n$ where $d_p(x)$ is smooth, we have*

$$\Delta d_p(x) \leq (n-1)\cot_k(d_p(x))$$

On the whole manifold, the Laplacian comparison theorem holds in the sense of distributions.

Theorem A.1.10 (Hessian comparison theorem)**.** *For $i = 1, 2$, let (M_i^n, g_i) be complete Riemannian manifolds, $\gamma_i : [0, L] \to M_i$ be geodesics parametrized by arc length such that γ_i does not intersect the $Cut(\gamma_i(0))$, and let $d_i := d(\cdot, \gamma_i(0))$. If for all $t \in [0, L]$ we have*

$$Sec_{M_1}(\gamma_1'(t), X_1) \leq Sec_{M_2}(\gamma_2'(t), X_2),$$

for all unit vectors $X_i \in T_{\gamma_i}(t)M_i$ perpendicular to $\gamma_i'(t)$, then

$$(\operatorname{Hess} d_1)_{\gamma_1(t)}(X_1, X_1) \geq (\operatorname{Hess} d_2)_{\gamma_2(t)}(X_2, X_2).$$

Corollary A.1.1. *With the same notations and hypotheses of the Hessian comparison theorem, we have that*

$$(\Delta d_1)(\gamma_1(t)) \geq (\Delta d_2)(\gamma_2(t)), \quad \forall 0 < t \leq L.$$

Definition A.1.1. *Let M be a Riemannian manifold, for $p \in M$, consider polar coordinates (ρ, v) on T_pM, where $\rho \in \mathbb{R}^+$ and $v \in T_pM$ is a unitary vector. For $q \in M$, the geodesic distance from p to q is given by $r(q) = \rho(\exp_p^{-1} q)$; $r(q)$ is well defined for $q \in \exp_p B(0, R)$. The vector field $X(q) = r(q)\nabla r(q)$ is called the radial vector field centered at p.*

Now we give a proof of the Comparison Lemma 3.5 of [HS74].

Lemma A.1.1. *Let $f : M^m \to N^n$ be an isometric immersion $p \in M$, and $r(\cdot) = d_N(\cdot, p)$, where d_N is the geodesic distance in N^n. Let $X = r\nabla^N r$ be the radial vector field centered at p. If $Sec_N \leq k$, then*

$$\operatorname{div}_M X(q) \geq mr(q)ct_k(r(q)).$$

Proof. Call $r = r_p = d_M(\cdot, p)$. Fix $0 < r_0 < \min\{\frac{\pi}{\sqrt{k}}, inj_M(p)\}$, and consider the geodesic ball $B = \{x \in M : d_M(x, p) < r_0\}$. It is clear that r is differentiable in $B \setminus \{p\}$. Now by the Hessian comparison Theorem we have that

$$\operatorname{Hess} r(v, v) \geq \frac{s_k'(r)}{s_k(r)} \left(1 - \langle \nabla r, v \rangle^2\right) = \cot_k(r) \left(1 - \langle \nabla r, v \rangle^2\right)$$

for all the points in $B \setminus \{p\}$, and vector fields $v : B \setminus \{p\} \to TN$ with $|v| = 1$.

For a vector field $Y : M \to TN$, the divergence of Y on M is given by

$$\operatorname{div}_M Y = \sum_{i=1}^{m} \langle D_{e_i} Y, e_i \rangle,$$

where $\{e_1, \ldots, e_m\}$ denotes a local orthonormal frame on M. Then for $X = r\nabla r$, we obtain that,

$$\begin{aligned}
\operatorname{div}_M X &= \sum_{i=1}^{m} \langle D_{e_i} X, e_i \rangle = \sum_{i=1}^{m} \left[\langle (D_{e_i} r) \nabla r, e_i \rangle + r \langle D_{e_i} \nabla r, e_i \rangle \right] \\
&= \sum_{i=1}^{m} \left[\langle \nabla r, e_i \rangle^2 + r \operatorname{Hess} r(e_i, e_i) \right] \\
&\geq \|\nabla^M r\| + r \cot_k(r) \sum_{i=1}^{m} \left(1 - \langle \nabla r, e_i \rangle^2 \right) \\
&= \|\nabla^M r\| + mr \cot_k(r) - \|\nabla^M r\| \\
&= mr \cot_k(r).
\end{aligned}$$

□

Appendix B

Convergence of Manifolds

B.1 Hausdorff distance

In this appendix we follow closely the presentation given in Chapter 7 of [BBI01].

Definition B.1.1 (Hausdorff distance)**.** *Let (X, d) be a metric space and A, B subsets of Z. One defines the Hausdorff distance between A and B to be*

$$\begin{aligned} d_H^X(A,B) &:= \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right\} \quad \text{(B.1)} \\ &= \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\} \\ &= \inf\left\{\varepsilon \geq 0 : A \subset U_\varepsilon(B), B \subset U_\varepsilon(A)\right\} \quad \text{(B.2)} \end{aligned}$$

where $U_\varepsilon(A) = \{z : d(z, A) \leq \varepsilon\}$.

The Hausdorff distance can present some pathologies like

1. The Hausdorff distance is not a metric on the subsets of X. In fact consider any dense proper subset B of X. Then $d_H^X(B, X) = 0$, this last because $U_\varepsilon(B) \supset X$ for all $\varepsilon > 0$.

2. d_H^Z is not always finite, for example $d_H^{\mathbb{R}}(\{0\}, \mathbb{R}) = \infty$. Or $X = \mathbb{R}^2$, $A = \{(x, y) : y = x\}$, $B = \{(x, y) : y = -x\}$.

Proposition B.1.1. *Let X be a metric space. Then*

1. d_H is a semi-metric on 2^X (the set of all subsets of X).

2. *$d_H(A, \overline{A}) = 0$ for any $A \subset X$ where $\overline{A}$ denotes the closure of A.*

3. *If A and B are closed subsets of X and $d_H(A, B) = 0$, then $A = B$.*

4. *If $A, B, C \subset X$ then $d_H(A, C) \leq d_H(A, B) + d_H(B, C)$.*

Then the set of closed subsets of X equipped with Hausdorff distance is a metric space.

B.2 Gromov-Hausdorff distance

Definition B.2.1. *Let X, Y be metric spaces. The Gromov-Hausdorff distance between them, denoted by $d_{GH}(X, Y)$, is defined by the following relation. For a $r > 0$, $d_{GH}(X, Y) < r$ if and only if there exist a metric space Z and subspaces X' and Y' of it which are isometric to X and Y respectively and such that $d_H^Z(X', Y') < r$. In other words, $d_{GH}(X, Y)$ is the infimum of positive r for which the above Z, X' and Y' exist.*

Another equivalent definition is the following

Definition B.2.2. *Let X, Y be metric spaces. Define*

$$d_{GH}(X, Y) = \inf_{Z,f,g} d_H^Z(f(X), g(Y))$$

where $f : X \to Z$ and $g : Y \to Z$ are isometric embeddings (distance preserving) into the metric spaces (Z, d).

Observation: $f(X), g(Y)$ (or similar X' and Y') in the above definition are regarded with the restriction of the metric of the ambient space Z, as opposed to the induced intrinsic metric.

For example, if X is a sphere with its standard Riemannian metric, one cannot take $Z = \mathbb{R}^3$ and $X' = S^2 \subset \mathbb{R}^3$ because X and X' would be only path-isometric but not isometric.

And it is obvious that if X and Y are isometric, then $d_{GH}(X, Y) = 0$.

Proposition B.2.1. *d_{GH} is a metric on the set of classes of isometric spaces.*

For example, if $X = S^1 \times [0, 1]$ and $Y = [0, 1]$ we can show that $d_{GH}(X, Y) = \dfrac{\pi}{2}$.

Remark B.2.1.

1. *If X, Y are compact, then $d_{GH}(X,Y) < \infty$. In fact let $Z = X\mathring{\cup}Y$ with the metric that satisfies $d(X,Y) = \sup\{\mathrm{diam}(X), \mathrm{diam}(Y)\}$*

2. *Two metric spaces with finite diameter can be separated by zero Gromov-Hausdorff distance without being isometric. For example $X = [0,1]$, $Y = [0,1] \cap \mathbb{Q}$. In fact, let $Z = X$, and as isometries the identities, then $d_H^Z(X,Y) = 0$.*

3. *Even for very simple domains in $\mathbb{R}^n$, the Gromov-Hausdorff distance is not realized by embeddings into Euclidean space.*

B.3 Gromov-Hausdorff Convergence

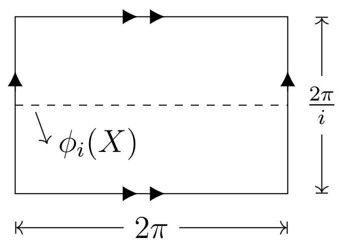
Definition B.3.1. *A sequence $\{X_n\}$ of compact metric spaces converge to a compact metric space X if $d_{GH}(X_n, X) \to 0$ as $n \to \infty$. We write $X_n \xrightarrow{GH} X$.*

Since d_{GH} is a metric, the limit is unique up to isometry.

Remark B.3.1. *For subspaces in the same metric space, Gromov-Hausdorff distance by definition is not greater than the Hausdorff distance. Thus Hausdorff convergence implies Gromov-Hausdorff convergence (but not reciprocally).*

For understand better how it works the Gromov-Hausdorff convergence, consider the following example taken from [Sor12].

Let $X_i = S^1 \times S^1_{1/i}$, and $X = S^1$, where $S^1_{1/i}$ is the sphere o radius $\frac{1}{i}$ we will prove that $d_{GH}(X_i, X) \to 0$. We take $Z_i = X_i$ and isometrically embeds X into Z_i by $\phi_i : X \to X_j$, $x \mapsto \left(x, \frac{\pi}{i}x\right)$.



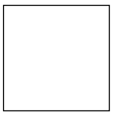
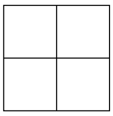
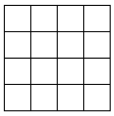
Then $d_{GH}(X_i, X) \le d_H^{Z_i}(X_i, \phi_i(X)) = \dfrac{\pi}{i} \to 0 \quad \text{as } i \to \infty.$

Others examples are:

1. Let (X, d) be a metric space, and for $\lambda > 0$, let λX denote $(X, \lambda d)$. If $\mathrm{diam}\, X < \infty$, then $\lambda X \xrightarrow{GH} \{\text{ a point }\}$ as $\lambda \to 0$.

2. $d_{GH}(\lambda X, \mu X) = \frac{1}{2}|\lambda - \mu|X.$

3. Consider the following sequence

$D_1 =$  , $D_2 =$  , $D_3 =$  $\xrightarrow{GH}$ D

where (D, d) is the unit square with the metric d given by $\|(x, y)\| = |x| + |y|$.

Gromov-Hausdorff limits of Riemannian manifolds are geodesic metric spaces. This means that the distance between any pair of points is equal to the length of the shortest curve between them. The shortest curve exist and is called a minimal geodesic.

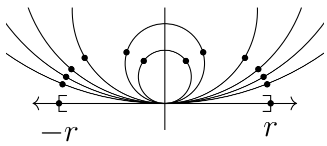
B.4 The noncompact case

When the spaces are non-compact, similarly to what happens for sequences of continuous functions it is very useful to introduce an analog notion to the uniform convergence on compact sets, namely Convergence of Pointed Spaces. Different equivalent definitions can be found in the literature we adopt here the definition of the book of M. Bridson and A. Haefliger [BH99].

Definition B.4.1. *Consider a sequence of metric spaces X_n with basepoints $x_n \in X_n$. The sequence of pointed spaces (X_n, x_n) is said to converge to (X, x) if for each $r > 0$ the sequence of closed balls $\overline{B}(x_n, r)$ (with induced metrics) converges to $\overline{B}(x_n, r) \subset X$ in the Gromov-Hausdorff metric.*

With the following example, we show the importance of choosing a sequence of pointed spaces. Consider the sequence of circles $C_i = \{x \in \mathbb{R}^2 : |x - ie_2| = i\} = \{x^2 + (y - i)^2 = i^2\} \subset \mathbb{R}^2$.

For fixed $r > 0$, the (closed) balls of radius r centered at the origin $0 \in C_i$, look similar to the corresponding ball in $\mathbb{R}$, i.e, the interval $[-r, r]$. We would like to be able to say that C_i converges to $\mathbb{R}$ in the limit as $i \to \infty$. However $d_{GH}(C_i, \mathbb{R}) = \infty$.



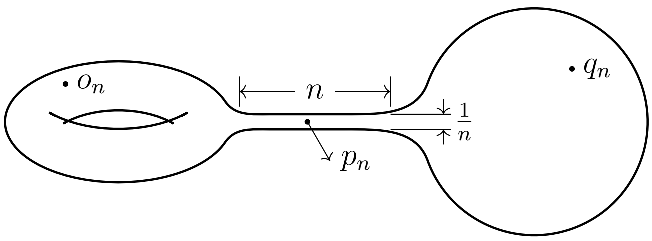
Then a $\overline{B}(0,r) \subset C_i \to [-r,r]$. We have that $\overline{B(0,r)}^{C_i} \xrightarrow{GH} \overline{B(0,r)}^{\mathbb{R}}$. There we expect that $d_{GH}(C_i,\mathbb{R}) \to 0$ but this don't is true, we have that $d_{GH}(C_i,\mathbb{R}) = \infty$ for all i. In fact, we know if $\operatorname{diam} X < \infty$ then

$$d_{GH}(X,Y) \geq \frac{1}{2}|\operatorname{diam} X - \operatorname{diam} Y|$$

make $X = C_i$ then $\operatorname{diam} C_i = \pi i$, $Y = \mathbb{R}$ then $\operatorname{diam} Y = \infty$, then for all i we have that

$$d_{GH}(C_i,\mathbb{R}) \geq \infty.$$

If we take now if the family pointed circles pointed $(C_i, 0)$, we have that $(C_i, 0) \xrightarrow{GH} (\mathbb{R}, 0)$. To see how the choose of the base point is important lets consider the sequence of intervals $\{[0, 2n]\}$. Depending on how one chooses the distinguished points we can have different limit spaces, that is, if we take the sequence of pointed spaces $\{[0, 2n], 0\}$ converge to the pointed space $(\mathbb{R}_{\geq 0}, 0)$. But the sequence $\{[0, 2n], n\}$ converge to the pointed space $(\mathbb{R}, 0)$. This example is due to K. Fukaya the reader can compare with Example 6.3 of [Fuk14]. Let X_n, o_n, p_n, and q_n as in the next figure



then we easily see that

1. $\lim_{n\to\infty}(X_n, o_n) = (\mathbb{T}^2 \setminus \{\text{point}\}, \text{point})$.

2. $\lim_{n\to\infty}(X_n, p_n) = (\mathbb{R}, 0)$.

3. $\lim_{n\to\infty}(X_n, q_n) = (\mathbb{S}^2 \setminus \{\text{point}\}, \text{point})$.

We remark that in this example the limit space does depend on the choice of base points. It's easy to see that if we choose any (M, g) compact n-dimensional Riemannian manifold and $p \in M$. Then

$$\lim_{R\to\infty}((M, Rg), p) = (\mathbb{R}^n, O)$$

for any $p \in M$.

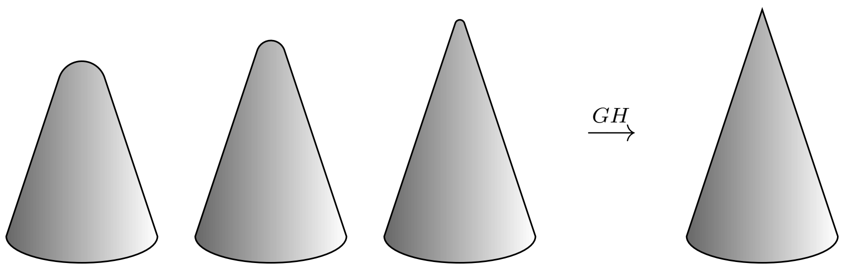
B.5 Convergence of Manifolds

We describe quickly the principal theorems that we can find to have control on the dimension and volume of the limit manifold. In 1967 J. Cheeger in his paper [Che70] proved the following finiteness theorem refined later by [Pet84].

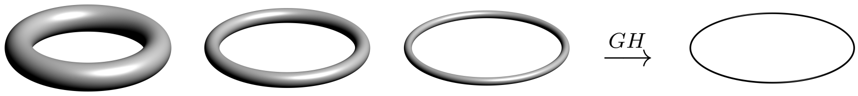
Theorem B.5.1 (Cheeger's Finiteness Theorem). *Let $\mathcal{M}(n, \Lambda, d, v)$ be the set of compact Riemannian manifolds of dimension n, sectional curvature $|Sec| < \Lambda$, diameter $\operatorname{diam}(M) < d$, and volume $Vol(M) > v$. Then there are only finitely many diffeomorphism types of manifolds in $\mathcal{M}(n, \Lambda, D, v)$.*

Theorem B.5.2 (Gromov's Precompactness Theorem). *Let $\mathcal{M}(n, k, v, D)$ be the set of compact manifolds of dimension n of volume greater than v, diameter less than D, and Ricci curvature greater than $(n-1)k$. Then $\mathcal{M}(n, k, v, D)$ is precompact in the pointed Gromov-Hausdorff topology.*

Consider a sequence of smooth truncated cones in $\mathbb{R}^3$ converging to a cone, equipped with the induced length space structure.



this example shows that only we can have pre-compactness, it can occurs that the limit metric space is not a Riemannian manifold. That is a singularity can arises in the limit, this happened on this example due to the fact that the curvature does not stay bounded. Then this make that we have to restrict our attention to sequence with bounded sectional curvature $|Sec| \leq k$. Now if we take a sequence of flat tori becoming thinner, reducing one of his radius, until eventually collapsing to S^1.



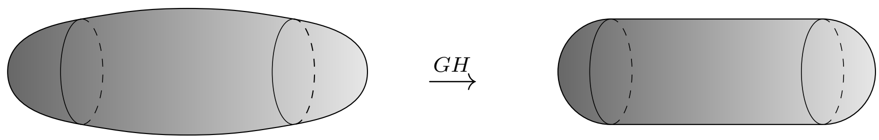
Here we satisfies the conditions of sectional curvature ($Sec = 0$) and uniform bounded diameter, but the limit space is of lower dimension, here the volume of the sequence goes to zero. M. Gromov in 1981 extended this result to prove precompactness in the Lipschitz topology.

Theorem B.5.3 (Gromov's $C^{1,1}$-precompactness Thm. 8.20 of [Gro07])**.** *With respect to the Lipchitz topology (and Gromov-Hausdorff topology) $\mathcal{M}(n, d, \Lambda, v)$ is relatively compact. Any sequence of manifolds $\{M_i\} \in \mathcal{M}(n, \Lambda, d, v)$ has a subsequence that converges to a differentiable manifold with a C^0 metric, and a $C^{1,1}$ distance function.*

In 1987 this theorem was improved independently by S. Peters [Pet87b] and Greene-Wu [GW88].

Theorem B.5.4 ($C^{1,\alpha}$-precompactness Greene-Wu-Peters)**.** *The space $\mathcal{M}(n, \Lambda, d, v)$ is precompact in the Lipschitz topology. Any sequence of manifolds $\{M_i\} \in \mathcal{M}(n, \Lambda, d, v)$ has a convergent subsequence to a differentiable manifold with a $C^{1,\alpha}$ metric.*

It is important to realize that, by nature, one cannot expect more than C^2 convergence or expect a C^2 limit for g. Consider the trivial example of a cylinder with two hemispherical caps.



That is, the cylinder with two hemispherical caps can be obtained as a limit of Gromov-Hausdorff of surfaces in $\mathcal{M}(2, \Lambda, d, v)$ but the limit is not C^2, otherwise the sectional curvature would have to be continuous, because it depends on the second derivative of the metric.

Remark B.5.1. *The hypotheses $Vol(M) \geq v$ can be replaced by $inj_M \geq i_0$ (See Theorem* 2.1 *and Corollary* 2.2 *of* [*Che70*]).

Definition B.5.1. *Let $\{(M_i, g_i)\}$ be a sequence of n-dimension Riemannian manifolds. The sequence converge in the $C^{1,\alpha}$-topology to a $C^{1,\alpha}$-manifold (M, g) if M is a C^∞ manifold such that for some fixed $C^{1,\alpha}$ atlas on M compatible with its C^∞ structure, g is $C^{1,\alpha}$, and there are diffeomorphism $\phi_i : M_i \to M$, for which $\phi_i^* g_i \to g$ with the $C^{1,\alpha}$ norm.*

The following theorem of Anderson can be viewed as a generalization of the above theorems.

Theorem B.5.5 (Theorem 1.1 of [And90])**.** *The class of compact, connected Riemannian n-manifolds M satisfying* $|Ric(M)| \leq k$*,* $inj(M) \geq i > 0$ *and* $\operatorname{diam}(M) \leq D$*, is precompact in the* $C^{1,\alpha}$*-topology.*

Appendix C

Sobolev Spaces and Sobolev Embeddings in Riemannian Manifolds

The following results and definitions can be found in [Heb99].

C.1 Sobolev Spaces in Riemannian Manifolds

Let (M, g) be a n-dimensional Riemannian manifold. Define the space $\mathcal{C}^{k,p}$ by

$$\mathcal{C}^{k,p}(M,g) := \left\{ u \in C^\infty(M) : \forall j = 0,\dots,k, \int_M |\nabla^j u|_g^p dv_g < +\infty \right\},$$

for a integer k, a real $p \geq 1$, $\nabla^k u$ denotes the k-th covariant derivative of u, $|\nabla^j u|_g$ the norm of $\nabla^k u$ is defined in a local chart by

$$|\nabla^j u|_g = g^{i_1 j_1} \cdots g^{i_k j_k} (\nabla^k u)_{i_1 \dots j_k} (\nabla^k u)_{j_1 \dots j_k},$$

here $(\nabla u)_i = \partial_i u$, $(\nabla^2 u)_{ij} = \partial_{ij} u - \Gamma_{ij}^k \partial_k u$. In local coordinates, $dv_g = \sqrt{\det(g_{ij})} dv_\xi$, and where dv_ξ stands for the Lebesgue's volume element of $\mathbb{R}^n$.

Definition C.1.1 (Sobolev space $W^{k,p}(M)$)**.** *Given (M^n, g) a smooth Riemannian manifold, denote by $W^{k,p}(M)$ the completion of $\mathcal{C}^{k,p}(M,g)$ with respect to the norm*

$$\|u\|_{W^{k,p}} = \sum_{j=0}^{k} \left(\int_M |\nabla_g^j u|_g^p dv_g \right)^{\frac{1}{p}}.$$

Observe that we can look at these spaces as subspaces of $L^p(M,g)$, where the L^p-norm is defined by

$$\|u\|_{p,g} := \left(\int_M |u|^p dv_g\right)^{\frac{1}{p}}.$$

Suppose that we have a compact Riemannian manifold M endowed with two different Riemannian metrics g_1 and g_2. It is easy to check that there exist a constant $C > 1$ such that

$$\frac{1}{C} g_1 \leq g_2 \leq C g_1,$$

on M, where the inequalities are understood in the sense of bilinear forms. With this in mind, it is straightforward to prove the following proposition.

Proposition C.1.1 (Prop 2.2 [Heb99] p. 22)**.** *If (M,g) is compact, $W^{k,p}(M,g)$ does not depend on the metric.*

Taking into account the preceding proposition we will not mention the dependence on the metric. More important, the following theorem allows us to work more easily with Sobolev spaces in Riemannian manifolds, as in the Euclidean context.

Theorem C.1.1 (Thm 2.4 [Heb99] p. 25)**.** *Let (M^n, g) be a smooth Riemannian manifold, and called $\mathcal{D}(M)$ the set of smooth functions with compact support in M. Then $\mathcal{D}(M)$ is dense in $W^{1,p}(M)$ for any $p \geq 1$.*

C.2 Sobolev Embeddings

As in the Euclidean case, we have interest in the Sobolev Embeddings in Riemannian manifolds. Initially we list the theorems for compact manifolds, and after for non-compact manifolds.

Lemma C.2.1 (Lemma 2.1 [Heb99] p. 26)**.** *Let (M^n, g) be a smooth Riemannian manifold. Suppose that $W^{1,1}(M,g) \subset L^{\frac{n}{n-1}}(M,g)$. Then for any real numbers $1 \leq p < q$, and any integers $0 \leq k < m$ such that $\frac{1}{q} = \frac{1}{p} - \frac{m-k}{n}$, $W^{m,p}(M,g) \subset W^{k,q}(M,g)$.*

Theorem C.2.1 (Thm. 2.6 p. 32, Thm. 2.7 p. 34, and Thm. 2.8 p. 35 of [Heb99])**.** *Let (M^n, g) be a smooth compact Riemannian manifold.*

1. *The Sobolev embeddings in their first part do hold on (M,g) in the sense that for any real numbers $1 \leq p < q$, and any integers $0 \leq k < m$, if $\frac{1}{q} = \frac{1}{p} - \frac{m-k}{n}$, $W^{m,p}(M) \subset W^{k,q}(M)$. In particular, for any $p \in [1,n[$ real, $W^{1,p}(M) \subset L^q(M)$, where $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$.*

2. *Let $p \geq 1$ real, and $k < m$ integers. If $\frac{1}{p} < \frac{m-k}{n}$, then $W^{m,p}(M) \subset C^m(M)$.*[1]

3. *If $p \geq 1$ real, and $0 < \lambda < 1$ real. If $\frac{1}{p} < \frac{1-\lambda}{n}$, then $W^{1,p}(M) \subset C^\lambda(M)$.* [2]

The following is a well known Sobolev compact embedding for compact manifolds.

Theorem C.2.2 (Rellich-Kondrakov Theorem)**.** *Let (M^n,g) be a smooth compact Riemannian manifold.*

1. *For any integers $j \geq 0$ and $m \geq 1$, any real number $p \geq 1$, and any real number q such that $1 \leq q < \frac{np}{n-mp}$, the embedding of $W^{j+m,p}(M)$ in $W^{j,q}(M)$ is compact. In particular, for any $p \in [1,n[$ real and any $q \geq 1$ such that $\frac{1}{q} > \frac{1}{p} - \frac{1}{n}$, the embedding of $H^{1,p}(M)$ in $L^q(M)$ is compact.*

2. *For $p > n$, the embedding of $W^{1,p}(M)$ in $C^\lambda(M)$ is compact for any $\lambda \in]0,1[$ such that $(1-\lambda)p > n$. In particular, the embedding of $H^{1,p}(M)$ in $C^0(M)$ is compact.*

For treat the non-compact case, we consider the following definition. Given (M,g) a smooth, complete Riemannian manifold, m an integer, and $p \geq 1$ real, we define $W_0^{m,p} =$ closure of $\mathcal{D}(M)$ in $W^{m,p}(M)$. In the Euclidean case, we have that

Proposition C.2.1. *For any m an integer and any $p > 1$ real, $W_0^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$.*

Fortunately for the case of a complete Riemannian manifold (M^n,g), this last result still holds for $m = 1$.

Theorem C.2.3 (Thm 3.1 p. 49 [Heb99])**.** *Given (M^n,g) a smooth complete Riemannian manifold we have that $W_0^{1,p}(M) = W^{1,p}(M)$, $p > 1$ real.*

The Sobolev embeddings in complete noncompact Riemannian manifolds are more restrictives. Additional assumptions on the geometry of the manifold are necessary to get the respective embeddings.

---

[1] Define the $\|\cdot\|_{C^m}$ on $C^m(M)$ by $\|u\|_{C^m} = \sum_{j=0}^m \max_{x\in M} |\nabla^j u(x)|$.

[2] $C^\lambda(M) = \{u \in C^0(M) : \|u\|_{C^\lambda} = \max_{x\in M}|u(x)| + \max_{x\neq y\in M} \frac{|u(x)-u(y)|}{d_g(x,y)^\lambda} < +\infty\}$.

Theorem C.2.4 (Prop. 3.6, Thm 3.2 p. 54, and Thm 3.4 p. 63 of [Heb99])**.** *Let* (M^n, g) *be a smooth complete Riemannian manifold, with Ricci bounded below, and positive injectivity radius, then*

1. *For any* $p \in [1, n[$ *real,* $W^{1,p}(M,g) \subset L^q(M,g)$ *where* $\frac{1}{q} = \frac{1}{q} - \frac{1}{n}$.
2. *Let* $p \geq 1$ *real,* $k < m$ *integers, if* $\frac{1}{q} < \frac{k-m}{n}$, *then* $W^{m,p}(M,g) \subset C^k_B(M,g)$. [3]
3. *If* $p \geq 1$ *real,* $0 < \lambda < 1$ *real. If* $\frac{1}{p} \leq \frac{1-\lambda}{n}$, *then* $W^{1,p}(M) \subset C^\lambda_B(M,g)$. [4]

Statements 1. *and* 3. *keep true if we replace positive injectivity radius by volume (respect to g) of unitary balls uniformly bounded below by a positive constant independent of their centers.*

---

[3] $C^m_B(M,g) = \{u \in C^m(M) : \|u\|_{C^m} = \sum_{j=0}^m \sup_{x\in M} |\nabla^j_g u(x)|_g < +\infty\}$.

[4] $C^\lambda_B(M,g) = \{u \in C^m(M,g) : \|u\|_{C^\lambda} = \sup_{x\in M} |u(x)| + \sup_{x\neq y\in M} \frac{|u(x)-u(y)|}{d_g(x,y)^\lambda} < +\infty\}$.

Appendix D

Existence of solutions for generalized scalar curvature equations

In this part we present some PDE's theorems needed to understand the proof of the proposition of the article [Dru02c] page 2353. The results can be found in [Dru00] or for instance [Heb99] page 85 for an exposition in book form of such ideas. In this appendix let (M^n, g) be a smooth compact Riemanian-manifold of dimension $n \geq 2$. Let $p \in]1, n[$, and a, f two smooth functions on M. We are concerned with the existence of positive solutions $u \in W^{1,p}(M)$ of the equation

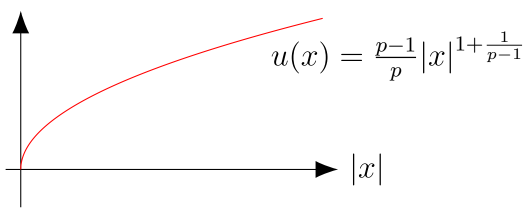
$$\Delta_{p,g} u + a(x)u^{p-1} = f(x)u^{p^*-1}, \tag{D.1}$$

called by Olivier Druet "generalized scalar curvature equation".

By regularity results any solution u of D.1 is $C^{1,\eta}$ for some $\eta \in]0, 1[$ (see Theorem 2.3 of [Dru00] for this assertion) this type of regularity is, in general, optimal for p arbitrary. For $(\mathbb{R}^n, \xi)$ the Euclidean space the function

$$u(x) = \frac{p-1}{p}|x|^{1+\frac{1}{p-1}}$$

is a solution of $\Delta_{p,\xi} u = -n$ in $\mathbb{R}^n$ and is $C^{1,\eta}$ with $\eta = \frac{1}{p-1}$.



Another issue here is that we want to work with $q = p^*$, where p^* is the critical exponent for the noncompact embedding of $W^{1,p}$ in L^{p^*}, unfortunately it is not possible to obtain solutions of (D.1) via simple variational arguments because the functional defined by

$$\frac{1}{p}\int_M |\nabla u|^p + \frac{1}{p}\int_M a|u|^p_{\frac{1}{p^*}} \int_M |u|^{p^*},$$

does not in general satisfy the Palais-Smale condition (see for example Chapter II, section 2 of [Str08] for a treatment of the Palais-Smale condition) and then it is not possible to obtain critical points of the functional. Consider the operator

$$L_{p,g} = \Delta_{p,g} u + a(x)|u|^{p-2}u,$$

and define the functional

$$I(u) = \int_M (L_{p,g}u)u dv_g = \int_M (|\nabla u|_g^p dv_g + a(x)|u|^p) dv_g,$$

and we say that $L_{g,p}$ is coercive if there exists some $\lambda > 0$ such that for any $u \in W^{1,p}(M)$ we have

$$\int_M |\nabla u|_g^p dv_g + \int_M a(x)|u|^p dv_g = I(u) \geq \lambda \int_M |u|^p dv_g. \tag{D.2}$$

When $a > 0$ then $L_{p,g}$ is immediately coercive. We also let

$$\Lambda_s = \left\{ u \in W^{1,p}(M) : \int_M f|u|^s dv_g = 1 \right\}, \qquad \mu_s = \inf_{u\in\Lambda_s} I(u),$$

$\Lambda := \Lambda_{p^*}$, and $\mu := \inf_{u\in\Lambda} I(u)$.

Observe that $L_{p,g}$ is coercive if there exists $\tilde{\lambda} > 0$ such that for all $u \in W^{1,p}(M)$ we have

$$\int_M |\nabla u|_g^p dv_g + \int_M a(x)|u|^p dv_g = I(u) \geq \tilde{\lambda}\|u\|_{W^{1,p}}. \tag{D.3}$$

It is easily checked that D.3 is equivalent to D.2. In fact, is straightforward to see that D.3 implies D.2. Reciprocally, assume D.2, and for $\beta > 0$ we write

$$\begin{aligned} I(u) =& \beta I(u) + (1-\beta)I(u) \geq \beta I(u) + (1-\beta)\lambda^{-1}\|u\|_p^p \\ &\geq \beta\|\nabla u\|_p^p - \beta\|a\|_{L^\infty}\|u\|_p^p + (1-\beta)\lambda^{-1}\|u\|_p^p \\ &\geq \beta\|\nabla u\|_p^p + \left((1-\beta)\lambda^{-1} - \beta\|a\|_{L^\infty}\right)\|u\|_p^p \\ &\geq \tilde{\beta}\left(\|\nabla u\|_p^p + \|u\|_p^p\right), \end{aligned}$$

where we choose $\tilde{\beta}$ such that $0 < \tilde{\beta} < \beta$, and $\tilde{\beta} < (1-\beta)\lambda^{-1} - \beta\|a\|_{L^\infty}$. This proves the equivalence. For the main result, we proceed in several steps.

Lemma D.0.1. *Let (M^n, g) be a smooth compact Riemannian manifold, $n \geq 2$, $p \in]1, n[$ some real number, and let a, f be smooth, real-valued functions on M. We assume that $L_{p,g}$ is coercive and that f is positive somewhere on M. For any $s \in]p, p^*[$ real, the equation*

$$\Delta_p u + a(x)u^{p-1} = \mu_s f(x) u^{p^*-1}$$

posseses a positive solution $u_s \in \Lambda_s \cap C^{1,\eta}$, $\eta \in]0, 1[$.

Proof. Let (u_i) a minimizing sequence in Λ_s for μ_s. Namely, $u_i \in \Lambda_s$, for any i, and

$$\lim_{i \to +\infty} I(u_i) = \mu_s.$$

Since $|\nabla |u|| = |\nabla u|$, without loss of generality, up to replacing u_i, by $|u_i|$, one can assume that the u_i's are non-negative. Since $L_{p,g}$ is coercive, (u_i) is a bounded sequence in $W^{1,p}(M)$. Up to the extraction of a subsequence, since $W^{1,p}(M)$ is reflexive, and by the Rellich- Kondrakov Theorem, we get the existence of some $u_s \in W^{1,p}(M)$ such that

$$u_q \rightharpoonup u \quad \text{in } W^{1,p}(M), \qquad u_q \to u \quad \text{in } L^s(M), \qquad u_q \to u \quad \text{a.e.}$$

One then gets that $u_s \geq 0$ a.e., and that $u_s \in \Lambda_s$. Moreover, the weak convergence in $W^{1,p}(M)$ implies that

$$I(u_s) \leq \liminf_{i \to +\infty} I(u_i).$$

Hence, $I(u_s) = \mu_s$. The fact that u_s is a minimizer for I on Λ_s, gives that u_s is a solution of the corresponding Euler's equation

$$\begin{cases} \Delta_{p,g} u_s + a(x) u_s^{p-1} = \mu_s f(x) u_s^{s-1}, & \text{on } M \\ \int_M f u_s^s dv_g = 1. \end{cases} \tag{D.4}$$

The result then easily follows from the maximum principles and regularity results. □

Lemma D.0.2 (Technical lemma)**.** *With the same notation as above, we have*

$$\limsup_{s \to p} \mu_s \leq \inf_{u \in \Lambda} I(u).$$

Proof. Let $\varepsilon > 0$ be given, and let $v \in \Lambda$, v non-negative and such that

$$I(v) \leq \inf_{u \in \Lambda} I(u) + \varepsilon.$$

For s close to p^* it holds

$$v_s = \left(\int_M f(x) v^s dv_g \right)^{-\frac{1}{s}} v,$$

makes sense and belongs to Λ_s. Hence $I(v_s) \geq \mu_s$. Noting that $I(v_s) \to I(v)$ as $s \to p^*$, one gets that

$$\limsup_{s \to p^*} \mu_s \leq I(v) \leq \inf_{u \in \Lambda} I(u) + \varepsilon.$$

The fact that such an inequality holds for any $\varepsilon > 0$ proves the above claim. □

In what follows, up to the extraction of a subsequence, we can assume that there exists $\lim_{s \to p^*} \mu_s$. We let

$$\mu = \lim_{s \to p^*} \mu_s.$$

Lemma D.0.3. *Let (M^n, g) be a smooth, compact Riemannian manifold, $n \geq 2$, $p \in]1, n[$ some real number, and let a, f be smooth, real-valued functions on M. We assume that $L_{p,g}$ is coercive and that f is positive somewhere on M. For any $s \in]p, p^*[$ real, let (u_s) be as in Lemma D.0.1 with the additional property that (μ_s) has a limit μ as $s \to p^*$. Suppose that a subsequence of (u_s) converges in some $L^k(M)$, $k > 1$, to a function $u \not\equiv 0$. Then $u \in C^{1,\eta}(M)$, $\eta \in]0, 1[$, u is positive, and*

$$\Delta_{p,g} u + a(x) u^{p-1} = \mu f(x) u^{p^*-1}.$$

In particular, $\mu > 0$ and, up to rescaling, u is a solution of (D.1).

Proof. It is easy to check that (u_q) is bounded in $W^{1,p}(M)$. Up to a subsequence we may then assume that for $s \to p^*$,

$$u_s \rightharpoonup u \quad \text{in } W^{1,p}(M), \qquad u_s \to u \quad \text{in } L^p(M), \qquad u_s \to u \quad \text{a.e.}$$

In particular, u is non-negative. Moreover, since $|\nabla u_s|_g$ is bounded in $L^p(M)$, we can assume that for $s \to p^*$,

$$|\nabla u_s|_g^{p-2} \nabla u_s \rightharpoonup w,$$

in $L^{\frac{p}{p-1}}(M)$. Similarly, we can assume that

$$u_s^{s-1} \rightharpoonup u^{p-1} \quad \text{in } L^{\frac{p}{p-1}}(M),$$

since (u_s^{s-1}) is bounded in $L^{\frac{p}{s-1}}(M) \subset L^{\frac{p}{p-1}}(M)$. Then by taking $s \to p^*$ in (D.4), we obtain

$$-\operatorname{div}_g(w) + a(x)u^{p-1} = \mu f(x)u^{p^*-1}.$$

It is obvious that $\mu f(x)u_s^{s-1} - a(x)u_s^{p-1}$ is bounded in $L^1(M)$. Thanks to [DH98, lemma 2], $w = |\nabla u|_g^{p-2}\nabla u$. Notice that the proof presented in [DH98] in the Euclidean context can easily be extended to the Riemannian context. Hence, u is solution of

$$\Delta_{p,g}u + a(x)u^{p-1} = \mu f(x)u^{p^*-1}.$$

By maximum principles and regularity results, one then gets that u is positive and that $u \in C^{1,\eta}(M)$ for some $\eta \in]0,1[$. Moreover, multiplying this equation by u and integrating over M we show that μ and $\int_M f(x)u^{p^*}dv_g$ are positive. This proves the lemma. □

As a general remark on this result, one can note that $\mu = \inf_{u\in\Lambda} I(u)$ and that u of Lemma D.0.3 belongs to Λ, so that u realizes the infimum of I on Λ. Indeed, multiplying the equation of Lemma D.0.3 by u and integrating the result over M, one gets that

$$\begin{aligned}
\mu \int_M f(x)u^{p^*}dv_g &= \int_M \left(|\nabla u|^p + a(x)u^p\right)dv_g \\
&\leq \liminf_{s\to p^*} \int_M \left(|\nabla u_s|^p + a(x)u_s^p\right)dv_g \\
&= \liminf_{s\to p^*} \mu_s.
\end{aligned}$$

Hence, $\int_M f(x)u^{p^*}dv_g \leq 1$. Let $v = u\left(\int_M f(x)u^{p^*}dv_g\right)^{-\frac{1}{p}}$. Then $v \in \Lambda$, according to what has been said above,

$$\mu \leq I(v) = \mu\left(\int_M f(x)u^{p^*}dv_g\right)^{1-\frac{p}{p^*}}.$$

As a consequence, $\int_M f(x)u^{p^*}dv_g \geq 1$, so that $\int_M f(x)u^{p^*}dv_g = 1$ and μ is the infimum of I on Λ. This proves the above claim.

From now on we assume that every subsequence of (u_s) which converges in $L^k(M)$, $k > 1$, converges to 0. Let $y \in M$, $\delta > 0$ and $\sigma \in C^\infty(M)$ such that $\sigma = 1$ in $B(y,\delta/2)$ and $\sigma = 0$ out of $B(y,\delta)$. Multiplying (D.4) by $\sigma^p u_s^k$, $1 < k < \frac{p^*}{p}$ and integrating over M, we get that

$$\frac{kp^p}{(k+p-1)^p}\int_M |\nabla\left(\sigma u_s^{\frac{k+p-1}{p}}\right)|^p dv_g \leq A + \mu_q \int_M \sigma^p f(x)u_s^{k+q-1}dv_g,$$

where A does not dependent on s. Analyzing the cases we get:

1. $f(y) < 0$. Taking δ small enough, we have

$$\int_M |\nabla \left(\sigma u_s^{\frac{k+p-1}{p}}\right)|^p dv_g \leq A\frac{(k+p-1)^p}{kp^p}.$$

2. $f(y) \geq 0$. Fairly standard computations, based on Aubin's inequalities [Aub76], lead to the following result. If

$$\mu f(y)^{\frac{p}{p^*}} K(n,p)^p \limsup_{s\to p^*} \left[\int_{B(y,\delta)} |f(x)| u_s^s dv_g\right]^{1-\frac{p}{p^*}} < 1$$

then we can find δ small enough and B independent of s such that

$$\int_M |\nabla \left(\sigma u_s^{\frac{k+p-1}{p}}\right)|^p dv_g \leq B.$$

Lemma D.0.4. *We still assume that every subsequence of (u_s) which converges in $L^k(M)$, $k > 1$, converges to 0. Moreover, we assume that there exists some $y \in M$, $\delta > 0$, σ as above and $C > 0$, $k > 1$ independent of s such that*

$$\int_M |\nabla \left(\sigma u_s^{\frac{k+p-1}{p}}\right)|^p dv_g \leq C.$$

Then

$$\limsup_{s\to p^*} \int_{B(y,\delta/2)} u_s^s dv_g = 0.$$

Proof. Assume by contradiction that

$$\limsup_{s\to p^*} \int_{B(y,\delta/2)} u_s^s dv_g > 0.$$

We have that

$$\int_{B(y,\delta/2)} u_s^s dv_g \leq C_1 \left(\int_{B(y,\delta/2)} u_s^{p^*} dv_g\right)^{\frac{s}{p^*}},$$

where C_1 is independent of s. Using Hölder's inequalities, we obtain

$$\left(\int_{B(y,\delta/2)} u_s^{p^*} dv_g\right)^{\frac{s}{p^*}} \leq C_2 \left(\int_{B(y,\delta/2)} u_s^{\frac{n(k+p-1)}{nk-p}} dv_g\right)^{\frac{(nk-p)s}{p^* n(k+p-1)}}$$

Hence

$$\limsup_{s\to p^*} \int_{B(y,\delta/2)} u_s^{\frac{n(k+p-1)}{nk-p}} dv_g > 0$$

and we get a contradiction with the fact that every subsequence of (u_s) which converges in $L^k(M)$, $k > 1$, converges to 0. This proves the lemma. □

Now we have the tools to prove the principal result.

Theorem D.0.1. *Let (M^n, g) be a compact Riemannian n-manifold, $n \geq 2$, $1 < p < n$, and let a, f be smooth real-valued functions on M. We assume that $L_{p,g}$ is coercive and that f is positive somewhere on M. If*

$$\left(\max_{x\in M} f(x)\right)^{\frac{p}{p^*}} \inf_{u\in\Lambda} I(u) < K(n,p)^{-p}$$

then equation (D.1), $\Delta_{p,g}u + a(x)u^{p-1} = f(x)u^{p^*-1}$, possesses a positive solution $u \in C^{1,\eta}(M)$, $\eta \in]0,1[$.

Proof. If for all $y \in M$ such that $f(y) > 0$, exist δ_y satisfying

$$\mu f(y)^{\frac{p}{p^*}} K(n,p)^p \limsup_{s\to p^*} \left[\int_{B(y,\delta)} |f(x)| u_s^s dv_g\right]^{1-\frac{p}{p^*}} < 1,$$

then, recovering M by a finite number of balls $B(y,\delta)$, lemma D.0.4 gives us a contradiction with the fact that

$$\int_M f(x) u_s^s dv_g = 1.$$

So there exists $y \in M$ such that $f(y) > 0$ and $\delta > 0$,

$$\mu f(y)^{\frac{p}{p^*}} K(n,p)^p \limsup_{s\to p^*} \left[\int_{B(y,\delta)} |f(x)| u_s^s dv_g\right]^{1-\frac{p}{p^*}} \geq 1.$$

Independently, since $\mu K(n,p)^p \left(\max_{x\in M} f(x)\right)^{\frac{p}{p^*}} < 1$, we obtain

$$\begin{aligned}\mu f(y)^{\frac{p}{p^*}} K(n,p)^p \limsup_{s\to p^*} &\left[\int_{B(y,\delta)} |f(x)| u_s^s dv_g\right]^{1-\frac{p}{p^*}} \\ \leq \limsup_{s\to p^*} &\left[\int_{B(y,\delta)} |f(x)| u_s^s | dv_g\right]^{1-\frac{p}{p^*}}\end{aligned}$$

But taking δ small enough such that $f(x) > 0$ in $B(y,\delta)$, we have

$$\limsup_{s\to p^*} \int_{B(y,\delta)} |f(x)| u_s^s dv_g \leq \limsup_{s\to p^*} \int_M |f(x)| u_s^s dv_g = 1$$

since, according to the previous discussion, (u_s^s) converges to 0 in $L^1(M)$ in a neighborhood of any x such that $f(x) \leq 0$. So we obtain a contradiction. The falsified hypothesis is: every subsequence of (u_s) which converges in $L^k(M)$, $k > 1$, converges to 0. By lemma D.0.3 this ends the proof of the theorem. □

Appendix E

The Concentration-Compactness Principle

In this Appendix we introduce the reader to The Concentration-Compactness Principle, developed by P.-L.Lions. It turns out to be a very powerful tool to help us to obtain the existence of a nontrivial extremal function that minimizes a functional inequality, under certain hypotheses. We follow closely the treatment presented in the Struwe's book [Str08].

Theorem E.0.1. [*Lemma-I p.* 39 *of* [Str08]] *Let* μ_m *be a sequence of probability measures on* $\mathbb{R}^n$ *such that* $\mu_m \geq 0$, $\int_{\mathbb{R}^n} d\mu_m = 1$. *Then there exists a subsequence* (μ_m) *such that one of the following three conditions holds:*

i. *(Compactness) There exists a sequence* $x_m \subset \mathbb{R}^n$ *such that for any* $\varepsilon > 0$ *there exists a* $R > 0$ *with the property that for all* m

$$\int_{B(x_m,R)} d\mu_m \geq 1 - \varepsilon.$$

ii. *(Vanishing) For all* $R > 0$ *we have*

$$\lim_{m\to\infty} \left(\sup_{x\in\mathbb{R}^n} \int_{B(x,R)} d\mu_m \right) = 0.$$

iii. *(Dichotomy) There exist* λ, $0 < \lambda < 1$, *such that for all* $\varepsilon > 0$, *there is a number* $R > 0$ *and a sequence* $(x_m) \subset \mathbb{R}^n$ *with the following property: Given* $R' > R$, *there* *are measures* μ_m^1, μ_m^2 *such that*

$$0 \le \mu_m^1 + \mu_m^2 \le \mu_m,$$

$$supp(\mu_m^1) \subset B(x_m, R), \;\; supp(\mu_m^2) \subset \mathbb{R}^n \setminus B(x_m, R'),$$

$$\limsup_{m\to\infty} \left(\left| \lambda - \int_{\mathbb{R}^n} d\mu_m^1 \right| + \left| (1-\lambda) - \int_{\mathbb{R}^n} d\mu_m^2 \right| \right) \le \varepsilon.$$

Remark E.0.1. *The above Theorem is true if we replace* $\mathbb{R}^n$ *by a complete Riemannian manifold* (M, g).

Theorem E.0.2. *[Lema-II Pag. 44 [Str08]] Let* $k \in \mathbb{N}$, $p \ge 1$, $kp < n$, $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$, *suppose that* $u_m \rightharpoonup u$ *weakly in* $W^{k,p}(\mathbb{R}^n)$ *and* $\mu_m = |\nabla^k u_m|^p dx \rightharpoonup \mu$, $\nu_m = |u_m|^q dx \rightharpoonup \nu$ *weakly in the measure sense where* μ *and* ν *are bounded non-negatives measures on* $\mathbb{R}^n$. *Then*

(i.) There exists some at most countable set J, *a family* $\{x^j; j \in J\}$ *of distinct points of* $\mathbb{R}^n$, *and a family* $\{\nu^{(j)}; j \in J\}$ *of positive numbers such that*

$$\nu = |u|^q dx + \sum_{j\in J} \nu^{(j)} \delta_{x^{(j)}},$$

where δ_x *is the Dirac-measure of mass 1 concentrated at* $x \in \mathbb{R}^n$.

(ii.) In addition we have

$$\mu \ge |\nabla^k u|^p dx + \sum_{j\in J} \mu^{(j)} \delta_{x^{(j)}}$$

for some family $\{\mu^{(j)}; j \in J\}$, $\mu^{(j)} > 0$ *satistying*

$$K(n,p)(\nu^{(j)})^{p/q} \le \mu^{(j)}, \forall j \in J.$$

In particular, $\sum_{j\in J} (\nu^{(j)})^{p/q} < \infty$.

Let us show in form of a lemma, that the measure $|v_p|^{p^*} dx$ does not have dichotomy.

Lemma E.0.1. *Consider the measure given by* $\nu_p = |v_p|^{p*} dx = |\tilde{v}_p|^{\frac{n}{n-1}} dx$. *Then we have that*

$$\lim_{p\to 1} \frac{\int_{\mathbb{R}^n} |\nabla \tilde{v}_p| dx}{\left(\int_{\mathbb{R}^n} \tilde{v}_p^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}}} = K(n,1)^{-1}.$$

Proof. After rescaling and normalizing the sequence v_p, we can suppose that $\int_{\mathbb{R}^n} v_p^{p^*} dx = 1$, and then we have only to prove that $\lim_{p\to 1} \int_{\mathbb{R}^n} |\nabla \tilde{v}_p| dx = K(n,1)^{-1}$. Consider a family of measures $\nu_p = |v_p|^{p*} dx$. If we have dichotomy, let $\lambda \in]0,1[$ be as in Theorem (E.0.1)-(iii.) and for $\varepsilon > 0$ determine $R > 0$, a sequence (x_p), and measures ν_p^1 , ν_p^2 as in that lemma such that

$$0 \leq \nu_p^1 + \nu_p^2 \leq \nu_p,$$

$$\operatorname{supp}(\nu_p^1) \subset B(x_p, R),\ \operatorname{supp}(\nu_p^2) \subset \mathbb{R}^n \setminus B(x_p, 2R),$$

$$\limsup_{p\to 1} \left(\left| \lambda - \int_{\mathbb{R}^n} d\nu_p^1 \right| + \left| (1-\lambda) - \int_{\mathbb{R}^n} d\nu_p^2 \right| \right) \leq \varepsilon.$$

Choosing a sequence $\varepsilon_p \to 0$ with corresponding $R_p > 0$ and x_p, upon passing to a subsequence $(\nu_p)_p$ if necessary, we can achieve that

$$\operatorname{supp}(\nu_p^1) \subset B(x_p, R_p),\ \operatorname{supp}(\nu_p^2) \subset \mathbb{R}^n \setminus B(x_p, 2R_p),$$

and

$$\limsup_{p\to 1} \left(\left| \lambda - \int_{\mathbb{R}^n} d\nu_p^1 \right| + \left| (1-\lambda) - \int_{\mathbb{R}^n} d\nu_p^2 \right| \right) = 0.$$

Moreover, in view of Theorem (E.0.1) we may suppose that $R_p \to \infty$ $(p \to 1)$. Choose $\phi \in C^\infty(B(2,0))$ with $0 \leq \phi \leq 1$ and such that $\phi \equiv 1$ in $B(1,0)$. For $p \to 1$, let $\phi_p(x) = \phi\left(\frac{x-x_p}{R_p}\right)$, then

$$|\nabla \tilde{v}_p| = |\nabla \tilde{v}_p| \phi_p + |\nabla \tilde{v}_p|(1-\phi_p).$$

By triangular inequality we have

$$\|(\nabla \tilde{v}_p)\phi_p\| \geq \|\nabla(\tilde{v}_p \phi_p)\| - \|\tilde{v}_p \nabla \phi_p\|.$$

Since $|\nabla \phi_p| \leq C R_p^{-1}$, we get

$$\|(\nabla \tilde{v}_p)\phi_p\| \geq \|\nabla(\tilde{v}_p \phi_p)\| - C R_p^{-1} \|\tilde{v}_p\|,$$

and similarly for $(1-\phi_p)$ instead of ϕ_p we get

$$\|(\nabla \tilde{v}_p)(1-\phi_p)\| \geq \|\nabla(\tilde{v}_p(1-\phi_p))\| - C R_p^{-1} \|\tilde{v}_p\|.$$

Let the annulus $A_p = B(2R_p, x_p) \setminus B(R_p, x_p)$. But by Hölder inequality

$$\begin{aligned} R_p^{-1}\|\tilde{v}_p\|_{L^1(A_p)} &\leq R_p^{-1}\left(V_\xi(A_p)\right)^{\frac{1}{n}}\|\tilde{v}_p\|_{L^{\frac{n}{n-1}}(A_p)} \leq C\|\tilde{v}_p\|_{L^{\frac{n}{n-1}}(A_p)} \\ &= C\left(\int_{A_p} v_p^{p^*}dx\right)^{\frac{(n-1)}{n}} \\ &\leq C\left[\int_{\mathbb{R}^n} d\nu_p - \left(\int_{\mathbb{R}^n} d\nu_p^1 + \int_{\mathbb{R}^n} d\nu_p^2\right)\right]^{\frac{(n-1)}{n}}. \end{aligned} \tag{E.1}$$

The constant C depend on n. Hence this last term tends to 0 as $p \to 1$. From (E.1) we thus obtain that $CR_p^{-1}\|\tilde{v}_p\| \leq o(1)$, where $o(1) \to 0$ $(p \to 1)$. By Sobolev's inequality we find

$$\begin{aligned} \|\nabla\tilde{v}_p\| &\geq \|\nabla(\tilde{v}_p\phi_p)\| + \|\nabla(\tilde{v}_p(1-\phi_p))\| + o(1) \\ &\geq K(n,1)^{-1}\left(\|\tilde{v}_p\phi_p\|_{\frac{n}{n-1}} + \|\tilde{v}_p(1-\phi_p)\|_{\frac{n}{n-1}}\right) + o(1) \\ &\geq K(n,1)^{-1}\left[\left(\int_{\mathbb{R}^n} v_p^{p^*}\phi_p^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} + \left(\int_{\mathbb{R}^n} v_p^{p^*}(1-\phi_p)^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\right] + o(1) \\ &\geq K(n,1)^{-1}\left[\left(\int_{B(R_p,x_p)} d\nu_p\right)^{\frac{n-1}{n}} + \left(\int_{\mathbb{R}^n\setminus B(2R_p,x_p)} d\nu_p\right)^{\frac{n-1}{n}}\right] + o(1) \\ &\geq K(n,1)^{-1}\left[\left(\int_{\mathbb{R}^n} d\nu_p^1\right)^{\frac{n-1}{n}} + \left(\int_{\mathbb{R}^n} d\nu_p^2\right)^{\frac{n-1}{n}}\right] + o(1) \\ &\geq K(n,1)^{-1}\left(\lambda^{\frac{n-1}{n}} + (1-\lambda)^{\frac{n-1}{n}}\right) - o(1), \end{aligned}$$

where $o(1) \to 0$ when $p \to 1$. But for $0 < \lambda < 1$ and $\frac{n-1}{n} < 1$ we have $\lambda^{\frac{n-1}{n}} + (1-\lambda)^{\frac{n-1}{n}} > 1$, contradicting the initial assumption that $\|\nabla\tilde{v}_p\| \to K(n,1)^{-1}$. □

Appendix F

F.1 Relations between u_p and v_p.

Since $v_p(x) = \mu_p^{\frac{n}{p}-1} u_p(\exp_{x_p}(\mu_p x))$, $\Omega_p := \mu_p^{-1}\exp_{x_p}^{-1}(B_g(x_{0,p}, r_p))$ and $g_p(x) := \exp_{x_p}^* g(\mu_p x)$ we have

$$\begin{aligned}
\int_{\Omega_p} v_p^r(x) dv_{g_p} &= \int_{\Omega_p} \mu_p^{\frac{n-p}{p}r} u_p^r(\exp_{x_p}(\mu_p x)) dv_{g_p}(x) \\
&= \mu_p^{\frac{n-p}{p}r} \int_{\mu_p^{-1}\exp_{x_p}(B_g(x_{0,p},r_p))} u_p^r(\exp_{x_p}(\mu_p x)) dv_{g_p}(x) \\
&= \mu_p^{\frac{n-p}{p}r} \mu_p^{-n} \int_{\exp_{x_p}(B_g(x_{0,p},r_p)} u_p^r(\exp_{x_p}(x)) dv_{g_p}(\mu_p^{-1}x) \\
&= \mu_p^{-(\frac{n}{r}-\frac{n-p}{p})r} \int_{B_g(x_{0,p},r_p)} (\exp_{x_p}^{-1})^*[u_p^r(\exp_{x_p}(x)) dv_{g_p}(\mu_p^{-1}x)] \\
&= \mu_p^{-\beta} \int_{B_g(x_{0,p},r_p)} u_p^r(x) dv_g(x),
\end{aligned}$$

where $\beta = \left(\frac{n}{r} - \frac{n-p}{p}\right) r$. Then

$$\|v_p\|_r^r = \mu_p^{-\beta} \|u_p\|_r^r. \tag{F.1}$$

Consider two particular cases. If $r = p^* = \frac{np}{n-p}$, then $\beta = 0$ and $\|v_p\|_{p^*} = \|u_p\|_{p^*}$. For $r = p$ we have that $\|v_p\|_p = \mu_p^{-1}\|u_p\|_p$, and since $\|u_p\|_p \to 0$ we obtain that

$$\mu_p^2 \|v_p\|_p^{2-p} = \mu_p^p \|u_p\|_p^{2-p} \to 0. \tag{F.2}$$

Now for the gradient we have that

$$\begin{aligned}\int_{\Omega_p} |\nabla v_p(x)|_{g_p}^r dv_{g_p} &= \int_{\Omega_p} |\nabla(\mu_p^{\frac{n-p}{p}} u_p(\exp_{x_p}(\mu_p x)))|^r dv_{g_p}(x)\\ &= \mu_p^{(\frac{n-p}{p})r} \int_{\mu_p^{-1}\exp_{x_p}^{-1}(B_g(0,r_p))} |\nabla u_p(\exp_{x_p}(\mu_p x))|^r dv_{g_p}(x)\\ &= \mu_p^{\frac{n-p}{p}r} \mu_p^{-n}\mu_p^r \int_{\exp_{x_p}^{-1}(B_g(0,r_p))} |\nabla u_p(\exp_{x_p}(x))|^r dv_{g_p}(\mu_p^{-1}x)\\ &= \mu_p^{\frac{n(r-p)}{p}} \int_{\exp_{x_p}^{-1}(B_g(0,r_p))} |\nabla u_p(\exp_{x_p}(x))|^r dv_{g_p}(\mu_p^{-1}x)\\ &= \mu_p^{\frac{n(r-p)}{p}} \int_{B_g(0,r_p)} |\nabla u_p(x)|^r dv_g,\end{aligned}$$

from which follows

$$\|\nabla_{g_p} v_p\|_r^r = \mu_p^{\frac{n(r-p)}{p}} \|\nabla_g u_p\|_r^r.$$

In particular for $r = p$ we have that

$$\|\nabla_{g_p} v_p\|_p = \|\nabla_g u_p\|_p. \tag{F.3}$$

To get the relation between their respective p-Laplacians, denote by $G_{p,ij} = [g_{p,ij}(x)]$ the matrix of $\exp_p^*(g)$ in the exponential chart $\exp_{x_p} : \mu_p\Omega_p \to M$, where $\mu_p\Omega_p = B_g(x_{0,p}, r_p)$, i.e.,

$$g_{p,ij}(x) = \left\langle d(exp)_{x_p}(x)(e_i), d(exp)_{x_p}(x)(e_j)\right\rangle_g (exp_p(x)),$$

since $g_p(x) = \exp_{x_p}^*(g)(\mu_p x)$, then the representative matrix of g_p that we denote by $\tilde{G}_p$ satisfies the relation $\tilde{G}_p(x) = G_p(\mu_p x)$. Now we calculate $\operatorname{div}_{g_p}(X)$ with $X = X^i\frac{\partial}{\partial x^i}$

$$\begin{aligned}\operatorname{div}_{g_p}(X)(x) &= \frac{1}{\sqrt{\det(\tilde{G}_p(x))}}\frac{\partial}{\partial x^i}\left(\sqrt{\det(\tilde{G}_p(x))}X^i(x)\right)\\ &= \frac{1}{\sqrt{\det(G_p(\mu_p x))}}\frac{\partial}{\partial x^i}\left(\sqrt{\det(G_p(\mu_p x))}X^i(x)\right)\\ &= \frac{\mu_p}{\sqrt{\det(G_p(y))}}\frac{\partial}{\partial y^i}\left(\sqrt{\det(G_p(y))}X^i(y)\right)\\ &= \mu_p \operatorname{div}_g(X)(y),\end{aligned}$$

where $y = \mu_p x$. Now since

$$\begin{aligned}\nabla_{g_p} u_p(\exp_{x_p}(\mu_p x)) &= \tilde{g}_p^{ij}(x)\frac{\partial u_p}{\partial x^i}(\mu_p x) = g_p^{ij}(\mu_p x)\frac{\partial u_p}{\partial x^i}(\mu_p x)\\ &= g_p^{ij}(y)\mu_p\frac{\partial u_p}{\partial y^i}(y) = \mu_p \nabla_g u_p(y),\end{aligned}$$

we obtain

$$\begin{aligned}\Delta_{p,g_p} v_p &= -\operatorname{div}_{g_p}(|\nabla_{g_p} v_p|^{p-2}\nabla_{g_p} v_p)\\ &= -\mu_p^{\left(\frac{n}{p}-1\right)(p-1)}\operatorname{div}_{g_p}(|\nabla_{g_p} u_p|^{p-2}\nabla_{g_p} u_p)(x)\\ &= \mu_p^{n-\frac{n}{p}+1}(-\operatorname{div}_g(|\nabla_g u_p|^{p-2}\nabla u_p)) = \mu_p^{n-\frac{n}{p}+1}\Delta_{p,g} u_p.\end{aligned}$$

F.2 Extremal functions for the Sobolev inequality

It is well known that the extremal functions for the Sobolev inequality in $\mathbb{R}^n$ are functions of the form

$$u(x) = \lambda(\mu + |x - x_0|^{\frac{p}{p-1}})^{1-\frac{n}{p}}, \quad p > 1.$$

The function given by $V_p(x) = \left(1 + \left(\frac{|x|}{R_0}\right)^{\frac{p}{p-1}}\right)^{1-n}$ play an important role when $p \to 1$.

Observe that

$$|\nabla V_p(x)| = \frac{n-p}{p-1}\left(\frac{1}{R_0}\right)^{\frac{p}{p-1}}\left(1 + \left(\frac{|x|}{R_0}\right)^{\frac{p}{p-1}}\right)^{-n}|x|^{\frac{1}{p-1}}.$$

Therefore we get that

$$\begin{aligned}\int_{\mathbb{R}^n}|\nabla V_p(x)|dv_\xi &= \omega_{n-1}\left(\frac{1}{R_0}\right)^{\frac{p}{p-1}}\frac{p(n-1)}{p-1}\int_0^\infty r^{\frac{1}{p-1}+n-1}\left(1+\left(\frac{r}{R_0}\right)^{\frac{p}{p-1}}\right)^{-\frac{n}{p}}dr\\ &= \omega_{n-1}\left(\frac{1}{R_0}\right)^{\frac{p}{p-1}}\frac{p(n-1)}{p-1}R_0^{\frac{1}{p-1}+n}\int_0^\infty \frac{r^{\frac{1}{p-1}+n-1}}{(1+r^{\frac{p}{p-1}})^{\frac{n}{p}}}dr\\ &= \omega_{n-1}R_0^{n-1}\frac{p(n-1)}{p-1}\frac{(p-1)}{p}\int_0^\infty \frac{t^{\frac{(p-1)(n-1)}{p}}}{(1+t)^{\frac{n}{p}}}dr\\ &= \omega_{n-1}R_0^{n-1}(n-1)\frac{\Gamma\left(n-\frac{n-1}{p}\right)\Gamma\left(\frac{2n-1}{p}-n\right)}{\Gamma\left(\frac{n}{p}\right)}\\ &= K(n,1)^{-1}(n-1)\frac{\Gamma\left(n-\frac{n-1}{p}\right)\Gamma\left(\frac{2n-1}{p}-n\right)}{\Gamma\left(\frac{n}{p}\right)} \to K(n,1)^{-1}.\end{aligned}$$

Here we used the **Beta Function** $B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, and the fact that $K(n,1)^{-1} = n\left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{n}} = \frac{n}{R_0}$.

F.3 The p-Laplacian in geodesic polar coordinates

Lemma F.3.1. *Let (r,θ) be geodesic polar coordinates and let $u = u(r)$ be a radial function. Then,*

$$\Delta_{p,g} u = \Delta_{p,\xi} u + O(r)|\partial_r u|^{p-2}\partial_r u.$$

Proof. Since $\partial_\theta u \equiv 0$ and in a geodesic coordinate system $g^{rr} = 1$ and $g^{r\theta} = 0$, we have

$$\begin{aligned}
-\Delta_{p,g} u =& \operatorname{div}_{p,g} |\nabla u|^{p-2}\nabla u \\
=& \partial_i \left(|\nabla u|^{p-2} g^{ij}\partial_j u\right) + \left(|\nabla u|^{p-2} g^{kj}\partial_j u\right)\Gamma^i_{ki} \\
=& \partial_i \left(|\nabla u|^{p-2} g^{ir}\partial_r u\right) + \left(|\nabla u|^{p-2} g^{kr}\partial_r u\right)\Gamma^i_{ki} \\
=& \partial_r \left(|\nabla u|^{p-2} g^{rr}\partial_r u\right) + \left(|\nabla u|^{p-2} g^{rr}\partial_r u\right)\Gamma^i_{ri} \\
=& \partial_r \left(|\partial_r u|^{p-2}\partial_r u\right) + |\partial_r u|^{p-2}\partial_r u \partial_r \ln(\sqrt{\det g}) \\
=& (p-1)|\partial_r u|^{p-2}\partial_{rr} u + |\partial_r u|^{p-2}\partial_r u \partial_r \ln(\sqrt{\det g}),
\end{aligned}$$

but we have that $\sqrt{\det g} = r^{n-1}J(r,\theta)$, where $J(r,\theta) = 1 + O(r^2)$, then

$$\begin{aligned}
\Delta_{p,g} u =& -(p-1)|\partial_r u|^{p-2}\partial_{rr} u - |\partial_r u|^{p-2}\partial_r u\left(\frac{n-1}{r} + \partial_r \ln J(r,\theta)\right) \\
=& -|\partial_r u|^{p-2}\left((p-1)\partial_{rr} u + \frac{n-1}{r}\partial_r u\right) - |\partial_r u|^{p-2}\partial_r u \frac{\partial_r J(r,\theta)}{J(r,\theta)} \\
=& \Delta_{p,\xi} u - |\partial_r u|^{p-2}\partial_r u \frac{\partial_r J(r,\theta)}{J(r,\theta)} \\
=& \Delta_{p,\xi} u + O(r)|\partial_r u|^{p-2}\partial_r u.
\end{aligned}$$

The last assertion is true because the Laplacian for radial functions is

$$-\Delta_{2,\xi} u = \partial_{rr} u + \frac{n-1}{r}\partial_r u,$$

and the p-Laplacian is

$$-\Delta_{p,\xi} u(r) = |\partial_r u|^{p-2}\left((p-1)\partial_{rr} u + \frac{n-1}{r}\partial_r u\right).$$

□

Now if $G_p(x) = \theta_p |x|^{-\frac{n-p-\nu}{p-1}} = \theta_p r^\alpha$, for $\alpha = -\dfrac{n-p-\nu}{p-1}$, then we have that

$$\begin{aligned}
\Delta_{p,\xi} G_p &= -|\partial_r G_p|^{p-2} \left((p-1)\partial_{rr} G_p + \frac{n-1}{r} \partial_r G_p \right) \\
&= -\theta_p^{p-1} |\alpha|^{p-2} r^{(\alpha-1)(p-2)} \left[(p-1)\alpha(\alpha-1) r^{\alpha-2} + (n-1)\alpha r^{\alpha-2} \right] \\
&= -\theta_p^{p-1} |\alpha|^{p-2} \alpha r^{\alpha(p-1)-p} \left[(p-1)(\alpha-1) + (n-1) \right] \\
&= -\theta_p^{p-1} |\alpha|^{p-2} \alpha r^{\alpha(p-1)-p} \left[\alpha(p-1) - p + n \right] \\
&= -\theta_p^{p-1} \nu |\alpha|^{p-2} \alpha r^{-n+\nu} \\
&= \theta_p^{p-1} \nu \left(\frac{n-p-\nu}{p-1} \right)^{p-1} r^{-n+\nu}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{|x|^p \Delta_{p,g} G_p(x)}{G_p^{p-1}} &= \nu \left(\frac{n-p-\nu}{p-1} \right)^{p-1} + O(|x|^2) \left(\frac{n-p-\nu}{p-1} \right)^{p-1} \theta_p^{p-1} \\
&\geq \nu \left(\frac{n-p-\nu}{p-1} \right)^{p-1} - C\mu_p^2 |x|^2. \qquad \text{(F.4)}
\end{aligned}$$

Appendix G

G.1 Sobolev embedding and the isoperimetric problem

If (M,g) is a complete Riemannian n-manifold of infinite volume, and $p \in [1,n[$ real. We say that the Euclidean-type Sobolev inequality of order p is valid if there exists a positive constant depending only on p and n, $C(n,p) > 0$ such that for any $u \in W^{1,p}(M)$, we have the following inequality

$$\left(\int_M |u|^q dv_g\right)^{\frac{p}{q}} \leq C(n,p) \int_M |\nabla u|^p dv_g,$$

where $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. This inequality holds true for the Euclidean Space. Thanks to the work of Croke [Cro84] this inequality with $p = 1$ holds true on any complete simply connected Riemannian manifold of nonpositive sectional curvature.

The Aubin-Cartan-Hadamard conjecture states that for Cartan-Hadamard n-dimensional manifolds, the sharp inequality holds, that is,

$$\left(\int_M |u|^{\frac{n}{n-1}} dv_g\right)^{\frac{n-1}{n}} \leq K(n,1) \int_M |\nabla u| dv_g,$$

for any $u \in \mathcal{D}(M)$. The explicit value of the constant $K(n,1)$ is $K(n,1) = \frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{\frac{1}{n}}$. But by the works of Federer and Fleming [HF60], this is equivalent to the following isoperimetric problem, for any smooth, bounded domain Ω on a Cartan-Hadamard n-dimensional manifold (M,g), holds the sharp isoperimetric inequality

$$A_g(\partial\Omega) \geq \frac{1}{K(n,1)} V_g(\Omega)^{\frac{n-1}{n}}.$$

We give now the equivalence of the assertions, following [Heb99].

Lemma G.1.1. *The sharp isoperimetric inequality* $A_g(\partial\Omega) \geq \frac{1}{K(n,1)} V_g(\Omega)^{\frac{n-1}{n}}$ *is valid if and only if the sharp functional inequality* $\left(\int_M |u|^{\frac{n}{n-1}} dv_g\right)^{\frac{n-1}{n}} \leq K(n,1) \int_M |\nabla u| dv_g$*, is valid.*

Proof. Let Ω a smooth bounded domain in (M, g), and for $\varepsilon > 0$, let u_ε consider the following function

$$u_\varepsilon(x) = \begin{cases} 1, & \text{if } x \in \Omega, \\ 1 - \frac{1}{\varepsilon} d_g(x, \partial\Omega), & \text{if } x \in M \setminus \Omega, \ d_g(x, \partial\Omega) < \varepsilon, \\ 0, & \text{if } x \in M \setminus \Omega, \ d_g(x, \partial\Omega) \geq \varepsilon. \end{cases}$$

By definition, u_ε, by all $\varepsilon > 0$ is a Lipschitz function, and satisfies,

$$\lim_{\varepsilon \to 0} \int_M u_\varepsilon^{\frac{n}{n-1}} dv_g = V_g(\Omega) \quad \text{and} \quad |\nabla u_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon} & \text{if } x \in M \setminus \overline{\Omega}, \ d_g(x, \partial\Omega) < \varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\lim_{\varepsilon \to 0} \int_M |\nabla u_\varepsilon| dv_g = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} V_g\left(\{x \notin \Omega : d_g(x, \partial\Omega) < \varepsilon\}\right) = A_g(\partial\Omega),$$

and then we have that

$$\inf_{u \in \mathcal{D}(M)} \frac{\int_M |\nabla u| dv_g}{\left(\int_M |u|^{\frac{n}{n-1}} dv_g\right)^{\frac{n-1}{n}}} \leq \inf_\Omega \frac{A_g(\partial\Omega)}{V_g(\Omega)^{\frac{n-1}{n}}}.$$

Now we prove the reverse inequality. Let $u \in \mathcal{D}(M)$, and for every $t \geq 0$ we defined $\Omega_t := \{x \in M : |u|(x) > t\}$, and $V(t) = V_g(\Omega_t)$.

Then we have the following

$$\begin{aligned} \|u\|_{\frac{n}{n-1}} &= \left(\int_M |u|^{\frac{n}{n-1}} dv_g\right)^{\frac{n-1}{n}} = \left(\int_0^\infty dv_g \int_0^{|u|} \frac{n}{n-1} t^{\frac{1}{n-1}} dt\right)^{\frac{n-1}{n}} \\ &= \left(\frac{n}{n-1} \int_0^\infty t^{\frac{1}{n-1}} dt \int_{\Omega_t} dv_g\right)^{\frac{n-1}{n}} = \left(\frac{n}{n-1} \int_0^\infty t^{\frac{1}{n-1}} V(t) dt\right)^{\frac{n-1}{n}}, \end{aligned}$$

and by the same way

$$\|u\|_1 = \int_0^\infty V(t) dt.$$

Now define the functions

$$F(s) = \int_0^s V(t)^{\frac{n-1}{n}} dt, \quad G(s) = \left(\frac{n}{n-1} \int_0^s t^{\frac{1}{n-1}} V(t) dt\right)^{\frac{n-1}{n}}.$$

By straightforward calculation we get that $F(0) = G(0)$, and since $V(s)$ is a decreasing function of s,

$$\begin{aligned} G'(s) &= \frac{n-1}{n}\left(\frac{n}{n-1}\right)^{\frac{n-1}{n}}\left(\int_0^s t^{\frac{1}{n-1}}V(t)dt\right)^{-\frac{1}{n}} s^{\frac{1}{n-1}}V(s) \\ &\leq \left(\frac{n}{n-1}\right)^{-\frac{1}{n}}\left(\int_0^s t^{\frac{1}{n-1}}dt\right)^{-\frac{1}{n}} s^{\frac{1}{n-1}}V(s)^{\frac{n-1}{n}} \\ &= V(s)^{\frac{n-1}{n}} = F'(s). \end{aligned}$$

This implies that

$$\int_0^\infty V(t)^{\frac{n-1}{n}}dt \geq \left(\frac{n}{n-1}\int_0^\infty t^{\frac{1}{n-1}}V(t)dt\right)^{\frac{n-1}{n}},$$

and by co-area formula for smooth functions we obtain that

$$\int_M |\nabla u| dv_g = \int_0^\infty \left(\int_{|u|^{-1}(t)} d\mathcal{H}^{n-1}\right) dt \geq \left(\inf_\Omega \frac{A_g(\partial\Omega)}{V_g(\Omega)^{\frac{n-1}{n}}}\right)\int_0^\infty V(t)^{\frac{n-1}{n}}dt.$$

Then combining the inequalities follows that

$$\begin{aligned} \int_M |\nabla u| dv_g &\geq \left(\inf_\Omega \frac{A_g(\partial\Omega)}{V_g(\Omega)^{\frac{n-1}{n}}}\right)\int_0^\infty V(t)^{\frac{n-1}{n}}dt \\ &\geq \left(\inf_\Omega \frac{A_g(\partial\Omega)}{V_g(\Omega)^{\frac{n-1}{n}}}\right)\left(\frac{n}{n-1}\int_0^\infty t^{\frac{1}{n-1}}V(t)dt\right)^{\frac{n-1}{n}} \\ &= \left(\inf_\Omega \frac{A_g(\partial\Omega)}{V_g(\Omega)^{\frac{n-1}{n}}}\right)\left(\int_M |u|^{\frac{n}{n-1}}dv_g\right)^{\frac{n-1}{n}}, \end{aligned}$$

taking the infimum over u, we get the result and the lemma is proved. □

G.2 Expansion of area in term of the enclosed volume of small geodesic balls

In this section we give the expansion of the area of a small ball in terms of the volume. In Gray [Gra12] we can found the following expansion of small balls in term of the radius,

$$A(r) = ar^{n-1} + br^{n+1} + O(r^{n+3}),$$

$$V(r) = Ar^n + Br^{n+2} + O(r^{n+4}),$$

where $a = \omega_{n-1}$, $b = -\omega_{n-1}\dfrac{Sc}{6n}$,

$A = \dfrac{\omega_{n-1}}{n}$, $B = -\dfrac{\omega_{n-1}}{n}\dfrac{Sc}{6(n+2)}$.

Inverting the series for $V(r)$ we get that

$$r(V) = A^{-\frac{1}{n}}V^{\frac{1}{n}} - \frac{1}{n}A^{-\frac{n+3}{n}}BV^{\frac{3}{n}} + O\left(V^{\frac{5}{n}}\right).$$

Then inserting this series in the expansion of $A(r)$ we have

$$A(V) = a\left(\frac{1}{A}\right)^{\frac{n-1}{n}} V^{\frac{n-1}{n}} + \frac{1}{n}(nAb - (n-1)aB)\left(\frac{1}{A}\right)^{\frac{2n+1}{n}} V^{\frac{n+1}{n}} + O(V^{\frac{n+5}{n}}).$$

The square of the above quantity is

$$A^2(V) = a^2\left(\frac{1}{A}\right)^{2\frac{n-1}{n}} V^{2\frac{n-1}{n}} + \frac{2a(nAb - (n-1)aB)}{n}\left(\frac{1}{A}\right)^{3} V^2 + O\left(x^{2\frac{n+1}{n}}\right).$$

We have that

$$a\left(\frac{1}{A}\right)^{\frac{n-1}{n}} = n\left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{n}}$$

$$\frac{1}{n}(nAb - (n-1)aB)\left(\frac{1}{A}\right)^{\frac{2n+1}{n}} = -\frac{Sc}{2(n+2)}\left(\frac{\omega_{n-1}}{n}\right)^{-\frac{1}{n}}$$

For the square of the perimeter:

$$a^2\left(\frac{1}{A}\right)^{2\frac{n-1}{n}} = n^2\left(\frac{\omega_{n-1}}{n}\right)^{\frac{2}{n}}$$

$$\frac{2a(nAb - (n-1)aB)}{n}\left(\frac{1}{A}\right)^{3} = -\frac{nSc}{n+2}$$

Finally we obtain that:

$$A(V) = n\left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{n}} V^{\frac{n-1}{n}} - \frac{Sc}{2(n+2)}\left(\frac{\omega_{n-1}}{n}\right)^{-\frac{1}{n}} V^{\frac{n+1}{n}} + O(V^{\frac{n+3}{n}}).$$

Since $K(n,1)^{-1} = n\left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{n}}$, replacing we get

$$A(V) = K(n,1)^{-1}V^{\frac{n-1}{n}}\left[1 - \frac{Sc}{2n(n+2)}\left(\frac{n}{\omega_{n-1}}\right)^{\frac{2}{n}} V^{\frac{2}{n}} + O(V^{\frac{4}{n}})\right].$$

And similarly for the square of the area

$$\begin{aligned} A^2(V) =& K(n,1)^{-2}V^{2\frac{n-1}{n}} - \frac{nSc}{(n+2)}V^2 + O(V^{2\frac{n+2}{n}}) \\ =& K(n,1)^{-2}V^{2\frac{n-1}{n}}\left[1 - \frac{nK(n,1)^2Sc}{(n+2)}V^{\frac{2}{n}} + O(V^{\frac{4}{n}})\right]. \end{aligned}$$

G.3 Counterexample, when $Ric \leq (n-1)k_0$ then isoperimetric comparison could fails

It is known that in normal polar coordinates the expression of the perimeter in function of the radius of a geodesic ball up to the third nontrivial term is given by the following equation

$$A(r) = \omega_{n-1} r^{n-1} \left(1 + A_1 r^2 + A_2 r^4 + O(r^6)\right),$$

where $A_1 = -\frac{Sc}{6n}$, $A_2 = \frac{5Sc^2 + 8\sum R_{ij}^2 - 3\sum R_{ijkl}^2 - 18\Delta Sc}{360n(n+2)}$.

Consider $\mathbb{S}^4(\sqrt{3})$, the 4-dimensional canonical round sphere of radius $\sqrt{3}$, with constant sectional curvature $k_0 = \frac{1}{3}$. One easy computation shows that in normal coordinates we have

$$R_{ijkl} = k_0(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),$$

then we have that

$$\begin{aligned}
\sum R_{ikjl}^2 &= k_0^2 \sum_{i,j,k,l} \delta_{ik}\delta_{jl} - 2\delta_{ik}\delta_{jl}\delta_{il}\delta_{jk} + \delta_{il}\delta_{jk} \\
&= 2k_0^2 \sum_{i,j,k,l} (\delta_{ik}\delta_{jl} - \delta_{ik}\delta_{jl}\delta_{il}\delta_{jk}) \\
&= 2k_0^2 \sum_{i,j,k} (\delta_{ik}\delta_{jj} - \delta_{ik}\delta_{ij}\delta_{jk}) \\
&= 2k_0^2 \sum_{i,j} (\delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ij}) \\
&= 2k_0^2 \sum_{i} (n\delta_{ii} - \delta_{ii}) \\
&= 2k_0^2 n(n-1) = 2\frac{1}{9}(4)(3) = \frac{8}{3}.
\end{aligned} \tag{G.1}$$

In a similar way we obtain the Ricci and the scalar curvature

$$R_{ij} = \sum R^k{}_{ikj} = k_0 \sum_k (\delta_{kk}\delta_{ij} - \delta_{kj}\delta_{ki}) = k_0(n\delta_{ij} - \delta_{ij}) = k_0(n-1)\delta_{ij},$$

$$Sc = \sum_{ij} k_0(n-1)\delta_{ij} = n(n-1)k_0 = (4)(3)\frac{1}{3} = 4, \tag{G.2}$$

thus

$$\sum_{ij} R_{ij}^2 = k_0^2(n-1)^2 \sum_{ij} \delta_{ij} = k_0^2 n(n-1)^2 = \frac{1}{9}(4)(3^2) = 4. \tag{G.3}$$

Now, we get the expansion of the Area of a small geodesic ball of $\mathbb{S}^4(\sqrt{3})$

$$A_1 = -\frac{Sc}{6n} = -\frac{1}{6},$$

$$A_2 = \frac{5Sc^2 + 8\sum R_{ij}^2 - 3\sum R_{ijkl}^2}{360n(n+2)} = \frac{5n^2(n-1)^2 + 8n(n-1)^2 - 6n(n-1)}{360n(n+2)}k_0^2$$
$$= \frac{n(n-1)(n+2)(5n-7)}{360n(n+2)}k_0^2 = \frac{(n-1)(5n-7)}{360}k_0^2 = \frac{13}{1080}.$$

Then for a small geodesic ball in $\mathbb{S}^4(\sqrt{3})$, this perimeter is given by

$$A_0(r) = 2\pi^2 r^3\left(1 - \frac{r^2}{6} + \frac{13r^4}{1080} + O(r^6)\right).$$

The same kind of computation leading to (G.1), (G.2) and (G.3) for $\mathbb{S}^2$, gives

$$\sum R_{ijkl}^2 = 2*2*1 = 4, \quad \sum R_{ij}^2 = 2*1 = 2, \quad Sc = 2.$$

Now, consider $\mathbb{S}^2 \times \mathbb{S}^2$. Then we have that

$$\sum R_{ijkl}^2 = 2*4 = 8, \quad \sum R_{ij}^2 = 2*2 = 4, \quad Sc = 4.$$

Therefore for a small geodesic ball in $\mathbb{S}^2 \times \mathbb{S}^2$, we have

$$A_1 = -\frac{Sc}{6n} = -\frac{1}{6},$$

$$A_2 = \frac{5Sc^2 + 8\sum R_{ij}^2 - 3\sum R_{ijkl}^2}{360n(n+2)} = \frac{5*16 + 8*4 - 3*8}{360*4*6} = \frac{11}{1080},$$
$$A(r) = 2\pi^2 r^3\left(1 - \frac{r^2}{6} + \frac{11r^4}{1080} + O(r^6)\right).$$

By direct comparison we have that $A(r) < A_0(r)$, but $Ric_{\mathbb{S}^2\times\mathbb{S}^2} = (n-1)k_0 g_{\mathbb{S}^2\times\mathbb{S}^2} = g_{\mathbb{S}^2\times\mathbb{S}^2}$.

In the expansion of the Area the expression $5Sc^2 + 8\sum R_{ij}^2 - 3\sum R_{ijkl}^2 - 18\Delta Sc$ shows that the isoperimetric comparison can fails if the manifold is Ricci flat but not flat. When $dim = 3$ by only purely algebraic considerations we get that Ricci flat implies that the manifold is flat. But more generally it is known by Fisher and Wolf [FW75] that if a compact manifold M admits a flat metric then any Ricci flat metric on M is flat. However Willmore in [Wil56] furnished the following example of a Riemannian metric Ricci flat but not flat with $dim = 4$

$$ds^2 = x^4(dx^2 + dy^2 + dz^2) + x^{-2}dt^2.$$

A direct computation shows that the non-null terms of Riemann tensor are

$$
\begin{aligned}
&R^1{}_{221} = R^1{}_{331} = -\tfrac{2}{x^2}, \qquad R^2{}_{121} = R^3{}_{131} = \tfrac{2}{x^2},\\
&R^2{}_{442} = R^3{}_{443} = -\tfrac{2}{x^8}, \qquad R^3{}_{232} = R^4{}_{141} = -\tfrac{4}{x^2},\\
&R^4{}_{242} = R^4{}_{343} = \tfrac{2}{x^2}, \qquad R^2{}_{332} = \tfrac{4}{x^2}, \quad R^1{}_{441} = \tfrac{4}{x^8}.
\end{aligned}
$$

Then we have that $Ric \equiv 0$, that is, a manifold which is Ricci flat but not flat.

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