Universidade Federal do Rio de Janeiro
Instituto de Matemática

Jorge Barbosa de Mello Junior

Canonical Heights, Arithmetical Degrees, and Dynamical Degrees on Varieties with Systems of Maps

Tese
Orientador: Amilcar Pacheco

Rio de Janeiro
Maio de 2017

Canonical Heights, Arithmetical Degrees, and Dynamical Degrees on Varieties with Systems of Maps

Jorge Barbosa de Mello Junior

Tese de Doutorado apresentada ao Programa de Pós-graduaçãoo do Instituto de Matemática, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Doutor em Matemática.

Orientador: Amilcar Pacheco

Rio de Janeiro
Maio de 2017

Canonical Heights, Arithmetical Degrees, and Dynamical Degrees on Varieties with Systems of Maps

Jorge Barbosa de Mello Junior

Orientador: Amilcar Pacheco

Tese submetida ao Corpo Docente do Instituto de Matemática Rio de Janeiro - UFRJ, como parte dos requisitos necessários para obtenção do grau de Doutor em Matemática.

Aprovada por:

Presidente, Prof. Amilcar Pacheco - IM/UFRJ

Prof. Ilir Snopche - IM/UFRJ

Prof. Cecilia Salgado - IM/UFRJ

Prof. Hemar Teixeira Godinho - MAT/UnB

Prof. Herivelto Borges Filho - ICMC/USP

Suplente, Prof. Oliver Lorscheid - IMPA

Rio de Janeiro
Dezembro de 2016

"Toda a lei se resume num só mandamento: Ame o seu próximo como a si mesmo"

Agradecimentos

Agradeço primeiro a Deus, por esta oportunidade de trabalho e pesquisa, e seus milagres de bondade que me trouxeram e me fortaleceram até aqui, ao autor da vida eu devoto este trabalho, bem como agradeço a todos meus irmãos em Cristo espalhados pelo mundo, que tanto me ensinaram sobre o sentido da vida e do amor e cooperaram para este trabalho, em especial agradeço a Marcelo e Jéssica Rodrigues, e a Bartow e Marianne Wyllie. Agradeço à minha esposa Beatriz por me amar e se deixar ser amada por mim, o meu obrigado por ser razão e inspiração em minha vida, sem minha amada eu não saberia como viver, minha fortaleza de amor que me preencheu e trouxe vida, eu gostaria de lhe dedicar esta tese. Agradeço a meus pais pelo sacrifício e doação, pelo cuidado, por me ensinarem com atitudes sobre o amor, amizade, e por serem exemplo e força pra mim, eu gostaria de dedicar-lhes esta tese, embora isso não seja capaz de retribuir à altura ou chegar perto de tudo o que fizeram e continuam fazendo por mim e meus irmãos. De fato todo este trabalho é um dentre outros frutos diretos do heroísmo e compaixão protetora das vidas compartilhadas de Ana Maria e Jorge Barbosa, eles me inspiram. Agradeço a meus irmãos Luciana e Felipe, pela amizade, pelos momentos juntos, conversas, divertimentos, pelo amor que inspira, pelas lutas e alegrias juntos, que me fortaleceram para fazer este trabalho, pelo companheirismo e abraços, e beijos, vocês são partes de amor dentro de mim que me impulsionam para o melhor e sempre serão, dedico-lhes este trabalho. Também agradeço aos demais familiares e amigos pelo carinho e amor.

Meu muito obrigado para Capes pelo suporte e apoio, e para a UFRJ pela oportunidade de trabalho e por tanto contribuir em minha formação. Agradeço ao meu orientador Amilcar Pacheco, por sua paciência, ensinos e ajuda, sem as quais este trabalho não seria possível, bem como ao meu coorientador Ilir, por toda a ajuda. Agradeço a todos os professores de matemática que me influenciaram na vida. Agradeço aos membros da banca, que aceitaram participar e contribuir com este trabalho.

Abstract

We define arithmetical and dynamical degrees for dynamical systems with several rational maps on projective varieties, study their properties and relations, and prove the existence of a canonical height function associated with divisorial relations in the Néron-Severi Group over Global fields of characteristic zero, when the rational maps are morphisms. We define canonical heights on projective varieties over Number fields for systems of several rational maps, show cases where points of height zero might lie in a non-dense set, and exhibit effective lower bounds for heights of points with dense orbit in the torus, with the bounds depending on the arithmetic of the points and on the maps. We study variation of Kawaguchi's canonical height on families of varieties and how to see its local components as intersection numbers.

Keywords: Canonical Heights, Rational Points, Preperiodic Points, Weil Local Heights, Algebraic Varieties, Multiplicative groups, Rational Maps, Lower Bounds.

Contents

Introduction

Weil heights play one of the key roles in Diophantine geometry, and particular Weil heights that enjoy nice properties, called canonical heights, are sometimes of great use. The theory of canonical heights has had profound applications throughout the field of Arithmetic geometry.

Over abelian varieties A defined over a number field K, Néron and Tate constructed canonical height functions $\hat{h}_L : A(\bar{K}) \to \mathbb{R}$ with respect to symmetric ample line bundles L which enjoy nice properties, and can be used to prove Mordell-Weil theorem for the rational points of the variety. More generally, in [9], Call and Silverman constructed canonical height functions on projective varieties X defined over a number field which admit a morphism $f : X \to X$ with $f^*(L) \cong L^{\otimes d}$ for some line bundle L and some $d > 1$. In another direction, Silverman [31] constructed canonical height functions on certain $K3$ surfaces S with two involutions σ_1, σ_2 (called Wheler's $K3$ surfaces) and developed an arithmetic theory analogous to the arithmetic theory on abelian varieties.

It was an idea of Kawaguchi [16] to consider polarized dynamical systems of several maps, namely, given X/K a projective variety, $f_1, ...f_k : X \to X$ morphisms on defined over K, $\mathcal{L}$ an invertible sheaf on X and a real number $d > k$ so that $f_1^*\mathcal{L} \otimes ... \otimes f_k^*\mathcal{L} \cong \mathcal{L}^{\otimes d}$, he constructed a canonical height function associated to the polarized dynamical system $(X, f_1, ..., f_k, \mathcal{L})$ that generalizes the earlier constructions mentioned above. In the Wheler's K3 surfaces' case above, for example, the canonical height defined by Silverman arises from the system formed by (σ_1, σ_2) by Kawagushi's method.

This context provides a short glimpse of a connection between two areas of mathematics, Number Theory and Dynamical Systems. Many of the motivating theorems in the new subject of Arithmetic Dynamics may be viewed as the transposition of classical results in the theory of Diophantine equations to the setting of discrete dynamical systems. We can start associating rational and integral points on varieties with rational and integral points on orbits, in particular associating torsion points on abelian varieties with periodic and preperiodic points of rational maps. The works listed in the last paragraphs above deal mainly with dynamics in dimension greater than one, where there is an abundant variety of varieties that can admit self-maps of infinite order even in dimension 2, imperfectly understood. On the other side, the only self-maps of a curve of genus greater than 1 are automorphisms of finite order. This thesis is concerned with dynamics in dimension greater than one, and dynamics associated to algebraic groups, which can provide important examples and testing grounds for general results in arithmetic dynamics.

Given $X/\mathbb{C}$ smooth projective variety, $f : X \dashrightarrow X$ dominant rational map inducing f^* :NS$(X)_\mathbb{R}$ $\to$NS$(X)_\mathbb{R}$ on the Néron-Severi group, the dynamical degree is defined as $\delta_f := \lim_{n\to\infty} \rho((f^n)^*)^{\frac{1}{n}}$, where ρ denotes the spectral radius of a given linear map, or the biggest number among the absolute values of its eigenvalues. This limit converges and is a birational invariant that has been much studied over the past couple of decades. In [18] we find a list of references.

In [18], Kawaguchi and Silverman studied an analogous arithmetic degree for X and f defined over $\bar{\mathbb{Q}}$ on points with well defined foward orbit over $\bar{\mathbb{Q}}$. Namely, $\alpha_f(P) := \lim_{n\to\infty} h_X^+(f^n(P))^{\frac{1}{n}}$, where h_X is a Weil height relative to an ample divisor and $h_X^+ = \max\{1, h_X\}$. Such degree measures the arithmetic complexity of the orbit of P by f, and $\log \alpha_f(P)$ has been interpreted as a measure of the arithmetic entropy of the orbit $\mathcal{O}_f(P)$. It is showed in [18] that the arithmetic degree determines the height counting function for points in orbits, and that the arithmetic complexity of the f-orbit of an algebraic point never exceeds the geometrical-dynamical complexity of the map f, as well as more arithmetic consequences. We could ask if this kind of research could be done in the setting of general dynamical systems as treated by Kawagushi, with several maps, as in the case of Wheler's K3 surfaces. This is the first subject found in this thesis.

Given X/K be a projective variety, $f_1, ..., f_k : X \dashrightarrow X$ rational maps, $\mathcal{F}_n = \{f_{i_1} \circ ... \circ f_{i_n}; i_j = 1, ..., k\}$, we define a more general dynamical degree of a system of maps as $\delta_{\mathcal{F}} = \limsup_{n\to\infty} \max_{f\in\mathcal{F}_n} \rho(f^*)^{\frac{1}{n}}$, and extend the definition of arithmetic degree for $\alpha_{\mathcal{F}}(P) = \frac{1}{k}\lim_{n\to\infty}\{\sum_{f\in\mathcal{F}_n} h_X^+(f(P))\}^{\frac{1}{n}}$, obtaining also the convergence of $\delta_{\mathcal{F}}$, and that $\alpha_{\mathcal{F}}(P) \leq \delta_{\mathcal{F}}$ when $\alpha_{\mathcal{F}}(P)$ exists, all done in the first chapter. Still in it, motivated by [18], we give an elementary proof that our new arithmetic degree is related with height counting functions in orbits, when $\alpha_{\mathcal{F}}(P)$ exists, by:

$$\lim_{B\to\infty} \frac{\#\{n \geq 0; \sum_{f\in\mathcal{F}_n} h_X(f(P)) \leq B\}}{\log B} = \frac{1}{\log(k.\alpha_{\mathcal{F}}(P))},$$

$$\liminf_{B\to\infty}(\#\{Q \in \mathcal{O}_{\mathcal{F}}(P); h_X(Q) \leq B\})^{\frac{1}{\log B}} \geq k^{\frac{1}{\log(k.\alpha_{\mathcal{F}}(P))}}.$$

We are able to extend theorem 1 of [18], showing explicitely how the dynamical degree of a system with several maps can offer an uniform upper bound for heights on iterates of points in orbits, when K is a number field or an one variable function field. Precisely, for every $\epsilon > 0$, there exists a positive constant $C = C(X, h_X, f, \epsilon)$ such that for all $P \in X_{\mathcal{F}}(\bar{K})$ and all $n \geq 0$,

$$\sum_{f\in\mathcal{F}_n} h_X^+(f(P)) \leq C.k^n.(\delta_{\mathcal{F}} + \epsilon)^n.h_X^+(P).$$

In particular, $h_X^+(f(P)) \leq C.k^n.(\delta_{\mathcal{F}} + \epsilon)^n.h_X^+(P)$ for all $f \in \mathcal{F}_n$.

This theorem becomes a tool to show the second very important theorem of the first chapter. As we have seen, for a pair $(X/K, f_1, ..., f_k, L)$ with k self-morphisms on X over K, and L a divisor satisfying a linear equivalence $\otimes_{i=1}^k f_i^*(L) \sim L^{\otimes d}$ for $d > k$, there is a well known theory of canonical heights developed by Kawaguchi in [16]. Now we are partially able to generalize this to cover the case that the relation $\otimes_{i=1}^k f_i^*(L) \equiv L^{\otimes d}$ is only an algebraic relation. Hence the limit

$$\hat{h}_{L,\mathcal{F}}(P) = \lim_{n\to\infty} \frac{1}{d^n} \sum_{f\in\mathcal{F}_n} h_L(f(P)).$$

converges for certain eigendivisor classes relative to algebraic relation. For L ample and K a number field, we obtain that :

$$\hat{h}_{L,\mathcal{F}}(P) = 0 \iff P \text{ has finite } \mathcal{F}\text{-orbit}.$$

These kind of generalization was firstly done for just one morphism by Silverman in [18], extending his own theory of canonical heights in [9], and we work out for several maps in the present work.

After this, one can ask when it is possible to define a general canonical height function on a projective variety X for a dynamical system with several rational maps that eventually are not all morphisms, and when that can be done without an algebraic equivalence identity previously exposed. Inspired by the work [29] of Silverman, we propose two ways to define a canonical height using a factor of correction in chapter 2, both generalizing Silverman's ideas in slightly different forms from each other, on points with well defined orbit. The first one is

$$\hat{\mathbf{h}}_{D,\mathcal{F}}(P) = \limsup_{n\to\infty} \frac{1}{n^{(l_{\mathcal{F}})} k^n \delta_{\mathcal{F}}^n} \sum_{f\in\mathcal{F}_n} h_D(f(P)),$$

where $l_{\mathcal{F}} := \inf\{l \geq 0 : \sup_{n\geq 1} \frac{\rho(\mathcal{F}_n)}{n^l \delta_{\mathcal{F}}^n} < \infty\}$ is the factor of correction, $\rho(\mathcal{F}_n) := \max_{f\in\mathcal{F}_n} \rho(f^*)$, and D is the divisor defining the Weil height h_D. We conjecture the existence of $l_{\mathcal{F}}$, as was done in [29] with just one map.

Conjecture A: *$l_{\mathcal{F}}$ exists.*

For the above height, we easily have that $\hat{\mathbf{h}}_{D,\mathcal{F}}(P) = 0$ for $P \in \text{Preper}(\mathcal{F})$. However the reciprocal is already false for $k = 1$. From the other side, if $\hat{\mathbf{h}}_{D,\mathcal{F}}(P) > 0$, we prove $\delta_{\mathcal{F}} = \alpha_{\mathcal{F}}(P)$. In various cases, $\delta_{\mathcal{F}}$ appears as root of a characteristic polynomial, which takes us to generalize a conjecture made for just one map in [29], which is

Conjecture B: $\{\alpha_{\mathcal{F}}(P) | P \in X_{\mathcal{F}}(\bar{K})\}$ *is a finite subset of the ring of algebraic numbers* $\mathcal{O}_K$.

The second proposed height function is

$$\hat{h}_{D,\mathcal{F}}(P) = \limsup_{n\to\infty} \frac{1}{n^{(l_{\mathcal{F}})} \delta_{\mathcal{F}}^n} \sum_{f\in\mathcal{F}_n} h_D(f(P)).$$

We study this second height for rational maps on the projective space arising from endomorphisms of the multiplicative group. We define and consider a class of dynamical systems of this kind whose dynamical degree can be calculated as if we had just one map in the system. Doing so, we can show, as it was done by Silverman in [29] for just one monomial map, that the points with canonical height zero lie in a set that is not Zariski dense. We show that the orbit of a point with canonical height zero is not Zariski dense in this case, and that, under some conditions on the characteristic polynomials of the matrices inducing the monomial maps, it is true now that

$$\hat{h}_{D,\mathcal{F}}(P) = 0 \textit{ if and only if the orbit } \mathcal{O}_{\mathcal{F}}(P) \textit{ is finite.}$$

According to all this, it follows that a point with Zariski dense orbit might have positive canonical height. A subsequent natural question then, made by Silverman [29] (remark 30) for the case with one map, is to ask if one can find explicitely a positive lower bound for the canonical height of a point that has Zariski dense orbit, where the bound would depend only on the naive height of the point and on the matrix which induces the endomorphism. Silverman's suggestion is to try to use some effective form of the famous theorem of Alan Baker for such, because he already had used it to prove his previous related results that had risen the question. In the end of the second chapter we find such bounds answering this question for the case where the initial matrix has real eigenvalues. For this, we use an algorithm to find Jordan normal forms in [25], and an effective form of Baker's theorem due to Philippon and Waldschmidt. As Silverman predicted, we see that the bigger the naive height of the point is, the smaller the calculated constants will be.

As a canonical height, it is known that Kawaguchi's height is, up to a constant, equal to a Weil height. In the third chapter we start studying how such bounds can vary more explicitely in families of varieties, when for each fiber we associate one canonical height. This kind of research was made by Silverman and Tate in [30] for families of abelian varieties, and afterwards by Silverman in [9] for a family of general varieties and the canonical height developed there. We therefore generalize their results for the Kawaguchi situation. Namely, for a family $\pi : \mathcal{V} \to T$ of varieties with a system $\mathcal{Q}$ of maps $\phi_i : \mathcal{V}/T \dashrightarrow \mathcal{V}/T$ and a divisor η satisfying $\pi(|(\bigotimes_{i=1}^k \phi_i^* \eta) - \alpha\eta|) \neq T$. Then on all fibers $\mathcal{V}_t$ for t in a certain T^0, there is a canonical height $\hat{h}_{\mathcal{V}_t,\eta_t,\mathcal{Q}_t}$, and we ask to bound the difference between this height and a given Weil height $h_{\mathcal{V},\eta}$ in terms of the parameter t. We show that there exist constants c_1 and c_2 such that

$$|\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t}(x) - h_{\mathcal{V},\eta}(x)| \leq c_1 h_T(t) + c_2 \text{ for all } t \in T^0 \text{ and all } x \in \mathcal{V}_t.$$

We can find also a kind of local version for the above result. In fact, in theorem 4.2.1 of [16] Kawaguchi showed that his canonical height can be seen as a sum of local canonical heights as in the case of abelian varieties. Therefore, we show an estimate for the difference between the canonical local height and a given Weil height.

When T is a curve, h_T is a Weil height associated to a divisor of degree one, $P : T \to \mathcal{V}$ is a section, V the generic fiber of $\mathcal{V}$ is a variety over the global function field $V(\bar{K}(T))$, $P_t := P(t)$, and the section P corresponds to a rational point $P_V \in V(\bar{K}(T))$, we show that

$$\lim_{h_T(t)\to\infty, t\in T^0(\bar{K})} \frac{\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t}(P_t)}{h_T(t)} = \hat{h}_{V,\eta_V,(\mathcal{Q})_V}(P_V),$$

generalizing a result of Silverman in [9].

The third and last chapter is also place for an analysis of canonical local heights for non-archimedean places in the context of intersection theory. Inspired in the final section of [9], this theme was already discussed for abelian varieties. We work out with system of several maps again. We prove more generally that if V has a model $\mathcal{V}$ over a complete ring $\mathcal{O}_v$ such that every rational point extends to a section and such that k morphisms $\phi_i : V \to V$ extend to finite morphisms $\Phi_i : \mathcal{V} \to \mathcal{V}$, then the canonical local height of Kawagushi is given by an intersection multiplicity on $\mathcal{V}$. For such generalization, we use Frobenius-Perron theory on eigenvalues of matrices.

In the appendix, we point out that the admissible metric, defined by Kawaguchi in [16] for a dynamical system of maps associated to a bundle, does not change if we start with another dynamical system that has maps commuting with the maps of the first system, and that are associated with the same divisor by a similar divisorial relation. The canonical measures risen by both systems would then be the same as well, and the main results of [26] are now in a more general setting.

Chapter 1

The dynamical and arithmetical degrees for eigensystems of rational self-maps

We study the existence of a canonical height function of points in projective varieties for several morphisms, defined with motivation in the height constructed by S. Kawaguchi in [16], but now with algebraic equivalence hypothesis, instead of linear equivalence. This also generalizes Kawaguchi's work with J. Silverman for systems with only one morphism, which was done in [18]. With this purpose, we define the arithmetical and dynamical degrees of eigensystems of self-rational maps, study some properties and connections of these degrees with orbits of points, and find a bound for summed heights of iterates of a point depending on such dynamical degree.

1.1 Notation, and first definitions

Throughout this chapter, K will be either a number field or a one-dimensional function field of characteristic 0 . We let $\bar{K}$ be an algebraic closure of K. The uple $(X, f_1, ..., f_k)$ is called a dynamical system, where either X is a smooth projective variety and $f_i : X \dashrightarrow X$ are dominant rational maps all defined over K, or X is a normal projective variety and $f_i : X \dashrightarrow X$ are dominant morphisms.

We denote by $h_X : X(\bar{K}) \to [0, \infty)$ the absolute logarithmic Weil height function relative to an ample divisor A of X, and for convenience we set $h^+_X(P)$ to be $\max\{1, h_X(P)\}$.

The sets of iterates of the maps in the system are denoted by $\mathcal{F}_0 = \{\mathrm{Id}\}, \mathcal{F}_1 = \mathcal{F} = \{f_1, ..., f_k\}$, and $\mathcal{F}_n = \{f_{i_1} \circ ... \circ f_{i_n}; i_j = 1, ..., k\}$, inducing what we call $\mathcal{O}_{\mathcal{F}}(P)$ the forward $\mathcal{F}$-orbit of P=$\{f(P); f \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_n\}$. A point P is said preperiodic when its $\mathcal{F}$-orbit is a finite set.

We write I_{f_i} for the indeterminacy locus of f_i, i.e., the set of points which f_i is not well-defined, and $I_{\mathcal{F}}$ for $\bigcup_{i=1}^k I_{f_i}$. Also we define $X_{\mathcal{F}}(\bar{K})$ as the set of points $P \in X(\bar{K})$ whose forward orbit is well-defined, in other words, $\mathcal{O}_{\mathcal{F}}(P) \cap I_{\mathcal{F}} = \emptyset$.

The set of Cartier divisors on X is denoted by $\mathrm{Div}(X)$, while $\mathrm{Pic}(X)$ denotes The Picard group of X, and $\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is called the Neron-Severi Group of X. The equality in this group is denoted by the symbol $\equiv$, which is called algebraic equivalence.

Given a rational map $f : X \dashrightarrow X$, the linear map induced on the tensorized Néron-Severi Group $\mathrm{NS}(X)_{\mathbb{R}} = \mathrm{NS}(X) \otimes \mathbb{R}$ is denoted by f^*. So, when looking for a dynamical system $(X, \mathcal{F})$, it is convenient for us to use the notation $\rho(\mathcal{F}_n) := \max_{f\in\mathcal{F}_n} \rho(f^*,\mathrm{NS}(X)_{\mathbb{R}})$.

For definitions and properties about Weil height functions, we refer to [14].

Next, we define the dynamical degree of a set of rational maps on a complex variety, which is a measure of the geometric complexity of the iterates of the maps in the set, when it exists. This is a generalization for several morphisms of the dynamical degree appearing as the first definition of [18].

Definition 1.1: *Let $X/\mathbb{C}$ be a (smooth) projective variety and let $\mathcal{F}$ be as above. The dynamical degree of $\mathcal{F}$, when it exists, is defined by*

$$\delta_{\mathcal{F}} = \limsup_{n\to\infty} \rho(\mathcal{F}_n)^{\frac{1}{n}}$$

In this sense, we also generalize the second definition in the introduction of [18], introducing now the arithmetic degree of a system of maps $\mathcal{F}$ at a point P. This degree measures the growth rate of the heights of n-iterates of the point by maps of the system as n grows, and so it is a measure of the arithmetic complexity of $\mathcal{O}_{\mathcal{F}}(P)$.

Definition 1.2: *Let $P \in X_{\mathcal{F}}(\bar{K})$. The arithmetic degree of $\mathcal{F}$ at P is the quantity*

$$\alpha_{\mathcal{F}}(P) = \frac{1}{k}\lim_{n\to\infty}\{\textstyle\sum_{f\in\mathcal{F}_n} h_X^+(f(P))\}^{\frac{1}{n}}$$

assuming that the limit exists.

Definition 1.3: *In the lack of the convergence, we define the upper and the lower arithmetic degrees as*

$$\bar{\alpha}_{\mathcal{F}}(P) = \frac{1}{k}\limsup_{n\to\infty}\{\textstyle\sum_{f\in\mathcal{F}_n} h_X^+(f(P))\}^{\frac{1}{n}}$$
$$\underline{\alpha}_{\mathcal{F}}(P) = \frac{1}{k}\liminf_{n\to\infty}\{\textstyle\sum_{f\in\mathcal{F}_n} h_X^+(f(P))\}^{\frac{1}{n}}$$

Remark 1.4: Let X be a projective variety and D a Cartier divisor. If $f : X \to X$ is a surjective morphism, then f^*D is a Cartier divisor. In the case where X is smooth, and $f : X \dashrightarrow X$ a merely rational map, we take a smooth projective variety $\tilde{X}$ and a birational morphism $\pi : \tilde{X} \to X$ such that $\tilde{f} := f \circ \pi : \tilde{X} \to X$ is a morphism. And we define $f^*D := \pi_*(\tilde{f}^*D)$. It is not hard to verify that this definition is independent of the choice of X and π. This is done in section 1 of [18] for example.

1.2 Basic properties of the arithmetic degree

In this section we check that the upper and lower degrees defined in the end of the section above are independent of the Weil height function chosen for X, and so they are well defined. Some examples of these degrees are computed is this section as well. We also present and prove our first counting result for points in orbits for several maps, and state an elementary and useful linear algebra's lemma.

Proposition 1.5: *The upper and lower arithmetic degrees $\bar{\alpha}_{\mathcal{F}}(P)$ and $\underline{\alpha}_{\mathcal{F}}(P)$ are independent of the choice of the height function h_X.*

Proof. If the $\mathcal{F}$-orbit of P is finite, then the limit $\alpha_{\mathcal{F}}(P)$ exists and is equal to 1, by definition of such limit, whatever the choice of h_X is. So we consider the case when P is not preperiodic, which allows us to replace h_X^+ with h_X when taking limits.

Let h and h' be the heights induced on X by ample divisors D and D' respectively, and let the respective arithmetic degrees denoted by $\bar{\alpha}_{\mathcal{F}}(P)$, $\underline{\alpha}_{\mathcal{F}}(P)$, $\bar{\alpha}'_{\mathcal{F}}(P)$, $\underline{\alpha}'_{\mathcal{F}}(P)$. By the definition of ampleness, there is an integer m such that $mD - D'$ is ample, and thus the functorial properties of height functions imply the existence of a non-negative constant C such that:

$$mh(Q) \geq h'(Q) - C \text{ for all } Q \in X(\bar{K}).$$

We can choose a sequence of indices $\mathcal{N} \subset \mathbb{N}$ such that:

$$\bar{\alpha}'_{\mathcal{F}}(P) = \frac{1}{k} \limsup_{n\to\infty} \{\sum_{f\in\mathcal{F}_n} h'(f(P))\}^{\frac{1}{n}} = \frac{1}{k} \lim_{n\in\mathcal{N}} \{\sum_{f\in\mathcal{F}_n} h'(f(P))\}^{\frac{1}{n}}$$

Then

$$\bar{\alpha}'_{\mathcal{F}}(P) = \frac{1}{k} \lim_{n\in\mathcal{N}} \{\sum_{f\in\mathcal{F}_n} h'(f(P))\}^{\frac{1}{n}}$$

$$\leq \frac{1}{k} \lim_{n\in\mathcal{N}} \{\sum_{f\in\mathcal{F}_n} mh(f(P)) + C\}^{\frac{1}{n}}$$

$$\leq \frac{1}{k} \limsup_{n\to\infty} \{\sum_{f\in\mathcal{F}_n} mh(f(P)) + C\}^{\frac{1}{n}}$$

$$= \frac{1}{k} \limsup_{n\to\infty} \{m(\sum_{f\in\mathcal{F}_n} h(f(P))) + Ck^n\}^{\frac{1}{n}}$$

$$= \frac{1}{k} \limsup_{n\to\infty} \{\sum_{f\in\mathcal{F}_n} h(f(P))\}^{\frac{1}{n}}$$

$$= \bar{\alpha}_{\mathcal{F}}(P)$$

This proves the inequality for the upper arithmetic degrees. Reversing the roles of h and h' in the calculation above we also prove the opposite inequality, which demonstrates that $\bar{\alpha}_{\mathcal{F}}(P) = \bar{\alpha}'_{\mathcal{F}}(P)$. In the same way we prove that $\underline{\alpha}_{\mathcal{F}}(P) = \underline{\alpha}'_{\mathcal{F}}(P)$. □

Our next lemma says that points belonging to a fixed orbit have their upper and lower arithmetic degrees bounded from above by the respective arithmetic degrees of the given orbit generator point.

Lemma 1.6: *Let $\mathcal{F} = \{f_1, ..., f_k\}$ be a set of self-rational maps on X defined over $\bar{K}$. Then, for all $P \in X_{\mathcal{F}}(\bar{K})$, all $l \geq 0$, and all $g \in \mathcal{F}_l$,*

$$\bar{\alpha}_{\mathcal{F}}(g(P)) \leq \bar{\alpha}_{\mathcal{F}}(P) \text{ and } \underline{\alpha}_{\mathcal{F}}(g(P)) \leq \underline{\alpha}_{\mathcal{F}}(P)$$

Proof. We calculate

$$\bar{\alpha}_{\mathcal{F}}(g(P)) = \frac{1}{k} \limsup_{n\to\infty} \{\sum_{f\in\mathcal{F}_n} h_X^+(f(g(P)))\}^{\frac{1}{n}} =$$

$$\frac{1}{k} \limsup_{n\to\infty} \{\sum_{f\in\mathcal{F}_n, g'\in\mathcal{F}_l} h_X^+(f(g'(P))) - \sum_{f\in\mathcal{F}_n, g'\in\mathcal{F}_l-\{g\}} h_X^+(f(g'(P)))\}^{\frac{1}{n}}$$

$$\leq \frac{1}{k} \limsup_{n\to\infty} \{[\sum_{f\in\mathcal{F}_{n+l}} h_X^+(f(P))] + O(1).k^{n+l}\}^{\frac{1}{n}}$$

$$= \frac{1}{k} \limsup_{n\to\infty} \{\sum_{f\in\mathcal{F}_{n+l}} h_X^+(f(P))\}^{\frac{1}{n}}$$

$$= \frac{1}{k} \limsup_{n\to\infty} \{\sum_{f\in\mathcal{F}_{n+l}} h_X^+(f(P))\}^{\frac{1}{n+l}.(1+\frac{l}{n})}$$

$$= \frac{1}{k} \limsup_{n\to\infty} \{\sum_{f\in\mathcal{F}_{n+l}} h_X^+(f(P))\}^{\frac{1}{n+l}}$$

$$= \bar{\alpha}_{\mathcal{F}}(P)$$

The proof for $\underline{\alpha}_{\mathcal{F}}(P)$ is similar.

□

Here are some examples:

Example: Let S be a $K3$ surface in $\mathbb{P}^2 \times \mathbb{P}^2$ given by the intersection of two hypersurfaces of bidegrees (1,1) and (2,2) over $\overline{\mathbb{Q}}$, and assume that $\mathrm{NS}(S) \cong \mathbb{Z}^2$, generated by $L_i := p_i^* O_{\mathbb{P}^2}(1), i = 1, 2$, where $p_i : S \to \mathbb{P}^2$ is the projection to the i-factor for $i = 1, 2$. These induce noncommuting involutions $\sigma_1, \sigma_2 \in \mathrm{Aut}(S)$. By [31, Lemma 2.1], we have

$$\sigma_i^* L_i \cong L_i, \sigma_i^* L_j \cong 4L_i - L_j, \text{ for } i \neq j.$$

The line bundle $L := L_1 + L_2$ is ample on S and satisfies $\sigma_1^* L + \sigma_2^* L \cong 4L$, and thus $h := \hat{h}_{L,\{\sigma_1,\sigma_2\}}$ exists on $S(\overline{\mathbb{Q}})$ by [16, theorem 1.2.1]. Noting that

$$\sigma_1^* \sim \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}, \sigma_2^* \sim \begin{bmatrix} -1 & 0 \\ 4 & 1 \end{bmatrix}, (\sigma_1 \circ \sigma_2)^* \sim \begin{bmatrix} -1 & -4 \\ 4 & 15 \end{bmatrix}, (\sigma_2 \circ \sigma_1)^* \sim \begin{bmatrix} 15 & 4 \\ -4 & -1 \end{bmatrix},$$

$$(\sigma_1 \circ \sigma_2 \circ \sigma_1)^* \sim \begin{bmatrix} 15 & 56 \\ -4 & -15 \end{bmatrix}, (\sigma_2 \circ \sigma_1 \circ \sigma_2)^* \sim \begin{bmatrix} -15 & -4 \\ 56 & 15 \end{bmatrix},$$

we calculate that

$$\rho(\sigma_1^*) = 2 + \sqrt{3}, \rho(\sigma_2^*) = 2 + \sqrt{3}, \rho((\sigma_1 \circ \sigma_2)^*) = 7 + 4\sqrt{3},$$
$$\rho((\sigma_2 \circ \sigma_1)^*) = 7 + 4\sqrt{3}, \rho((\sigma_1 \circ \sigma_2 \circ \sigma_1)^*) = 1, \rho((\sigma_2 \circ \sigma_1 \circ \sigma_2)^*) = 1.$$

This gives that $\delta_{\{\sigma_1,\sigma_2\}} = 2 + \sqrt{3}$. Furthermore, since h is a Weil Height with respect to an ample divisor,

$$\alpha_{\{\sigma_1,\sigma_2\}}(P) = (1/2). \lim_{n\to\infty} [\textstyle\sum_{f \in \{\sigma_1,\sigma_2\}_n} h(f(P))]^{\frac{1}{n}} = 1/2.[4^n.h(P)]^{\frac{1}{n}} = 2$$

for all $P \in S(\bar{\mathbb{Q}})$ non-preperiodic, i.e, P such that $h(P) \neq 0$.

Observe that in this case $\bar{\alpha}_{\{\sigma_1,\sigma_2\}}(P) = 2 \leq 2 + \sqrt{3} = \delta_{\{\sigma_1,\sigma_2\}}$, which we will prove in Corollary 1.16 to be true in our general conditions.

Example: Let S be a $K3$ surface in $\mathbb{P}^2 \times \mathbb{P}^2$, as in the example 1.4.5 of [16], given by the intersection of two hypersurfaces of bidegrees (1,2) and (2,1) over $\overline{\mathbb{Q}}$, and assume that $\mathrm{NS}(S) \cong \mathbb{Z}^2$, generated by $L_i := p_i^* O_{\mathbb{P}^2}(1), i = 1, 2$, where $p_i : S \to \mathbb{P}^2$ is the projection to the i-factor for $i = 1, 2$. These induce noncommuting involutions $\sigma_1, \sigma_2 \in \mathrm{Aut}(S)$. By similar computations we have $\sigma_i^* L_i \cong L_i, \sigma_i^* L_j \cong 5L_i - L_j$, for $i \neq j$. The ample line bundle $L := L_1 + L_2$ exists on S and satisfies $\sigma_1^* L + \sigma_2^* L \cong 5L$, and thus $h := \hat{h}_{L,\{\sigma_1,\sigma_2\}}$ exists on $S(\overline{\mathbb{Q}})$ by [16, theorem 1.2.1]. Proceeding in the same way as in the previous example, we have that

$$\bar{\alpha}_{\{\sigma_1,\sigma_2\}}(P) = 5/2 \leq \sqrt{\frac{23 + 5\sqrt{21}}{2}} = \delta_{\{\sigma_1,\sigma_2\}}.$$

Example: Let S be a hypersurface of tridegree (2,2,2) in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ over $\overline{\mathbb{Q}}$, as in the example 1.4.6 of [16]. For $i = 1, 2, 3$, let $p_i : S \to \mathbb{P}^1 \times \mathbb{P}^1$ be the projection to the $(j, k)-$th factor with $\{i, j, k\} = \{1, 2, 3\}$. Since p_i is a double cover, it gives an involution $\sigma_i \in \mathrm{Aut}(S)$. Let also, $q_i : S \to \mathbb{P}^1$ be the projection to the $i-$th factor, and set $L_i := q_i^* O_{\mathbb{P}^1}, L := L_1 + L_2 + L_3$ ample, and we assume that $\mathrm{NS}(S) =< L_1, L_2, L_3 >\cong \mathbb{Z}^3$. By similar computations as above we have

$$\sigma_i^*(L_i) \cong -L_i + 2L_j + 2L_k \text{ for } \{i, j, k\} = \{1, 2, 3\}$$
$$\sigma_j^*(L_i) \cong L_i \text{ for } i \neq j.$$

Then $\sigma_1^*L+\sigma_2^*L+\sigma_3^*L\cong 5L$, which gives us the existence of $h:=\hat{h}_{L,\{\sigma_1,\sigma_2,\sigma_3\}}$ by [16, theorem 1.2.1]. We note that if $h(P)\neq 0$, then a similar computation as in the previous examples yields $\alpha_{\{\sigma_1,\sigma_2,\sigma_3\}}(P)=5/3$. While we can also calculate that:

$$(\sigma_3\circ\sigma_2\circ\sigma_1)^*\sim\begin{bmatrix}1&-2&-2\\2&3&10\\2&6&15\end{bmatrix}$$

with its big eigenvalue being aproximatelly $\rho((\sigma_3\circ\sigma_2\circ\sigma_1)^*)\sim 18,3808$. As $(18,3808)^{1/3}\sim 2,639$, we have that $\delta_{\{\sigma_1,\sigma_2,\sigma_3\}}\geq 2,63>5/3=\alpha_{\{\sigma_1,\sigma_2,\sigma_3\}}(P)$

Example: Let A be an abelian variety over $\bar{\mathbb{Q}}$, L a symmetric ample line bundle on A. Let $f=(F_0:...:F_N):\mathbb{P}^N\to\mathbb{P}^N$ be a morphism defined by the homogeneous polynomials $F_0,...,F_N$ of same degree $d>1$ such that 0 is the only common zero of $F_0,...,F_N$. Set $X=A\times\mathbb{P}^N, g_1=[2]\times\mathrm{id}_{\mathbb{P}^N}$, and $g_2=\mathrm{id}_A\times f$. Put $M:=p_1^*L\otimes p_2^*O_{\mathbb{P}^N}(1)$, where p_1 and p_2 are the obvious projections. Then

$$\overbrace{g_1^*(M)\otimes...\otimes g_1^*(M)}^{(d-1)\ \text{times}}\otimes g_2^*(M)\otimes g_2^*(M)\otimes g_2^*(M)\cong M^{\otimes(4d-1)}.$$

This gives us that a canonical height $h:=\hat{h}_{\{g_1,...,g_1,g_2,g_2,g_2\}}$ exists by [16, theorem 1.2.1]. Again, if $h(P)\neq 0$, then $\alpha_{\{g_1,...,g_1,g_2,g_2,g_2\}}(P)=\dfrac{4d-1}{d+2}$, and we can also see that $\delta_{\{g_1,...,g_1,g_2,g_2,g_2\}}=\max\{\delta_F,\delta_{[2]}\}=\max\{d,4\}$, which leads also to the same as the previous examples, since $\dfrac{4d-1}{d+2}<\max\{d,4\}$.

The next proposition is a counting orbit points result in the case of a system possibly with several maps. This result describes some information about the growth of the height counting function of the orbit of P as given below.

Proposition 1.7: *Let $P\in X_{\mathcal{F}}(\bar{K})$ whose $\mathcal{F}$-orbit is infinite, and such that the arithmetic degree $\alpha_{\mathcal{F}}(P)$ exists. Then*

$$\lim_{B\to\infty}\frac{\#\{n\geq 0;\sum_{f\in\mathcal{F}_n}h_X(f(P))\leq B\}}{\log B}=\frac{1}{\log(k.\alpha_{\mathcal{F}}(P))}$$

and in particular,

$$\liminf_{B\to\infty}(\#\{Q\in\mathcal{O}_{\mathcal{F}}(P);h_X(Q)\leq B\})^{\frac{1}{\log B}}\geq k^{\frac{1}{\log(k.\alpha_{\mathcal{F}}(P))}}$$

Proof. Since $\mathcal{O}_{\mathcal{F}}(P)=\infty$, it is only necessary to prove the same claim with h_X^+ in place of h_X. For each $\epsilon>0$, there exists an $n_0(\epsilon)$ such that

$$(1-\epsilon)\alpha_{\mathcal{F}}(P)\leq\frac{1}{k}(\textstyle\sum_{f\in\mathcal{F}_n}h_X^+(f(P)))^{\frac{1}{n}}\leq(1+\epsilon)\alpha_{\mathcal{F}}(P)\text{ for all }n\geq n_0(\epsilon).$$

It follows that

$$\{n\geq n_0(\epsilon):(1+\epsilon)\alpha_{\mathcal{F}}(P)\leq\frac{B^{\frac{1}{n}}}{k}\}\subset\{n\geq n_0(\epsilon):\textstyle\sum_{f\in\mathcal{F}_n}h_X^+(f(P))\leq B\}$$

and

$$\{n\geq n_0(\epsilon):\textstyle\sum_{f\in\mathcal{F}_n}h_X^+(f(P))\leq B\}\subset\{n\geq n_0(\epsilon):(1-\epsilon)\alpha_{\mathcal{F}}(P)\leq\frac{B^{\frac{1}{n}}}{k}\}$$

Counting the number of elements in these sets yields

$$\frac{\log B}{\log(k(1+\epsilon)\alpha_{\mathcal{F}}(P))} - n_0(\epsilon) - 1 \le \#\{n \ge 0 : \sum_{f\in\mathcal{F}_n} h_X^+(f(P)) \le B\}$$

and

$$\#\{n \ge 0 : \sum_{f\in\mathcal{F}_n} h_X^+(f(P)) \le B\} \le \frac{\log B}{\log(k(1-\epsilon)\alpha_{\mathcal{F}}(P))} + n_0(\epsilon) + 1$$

Dividing by $\log B$ and letting $B \to \infty$ gives

$$\frac{1}{\log(k(1+\epsilon)\alpha_{\mathcal{F}}(P))} \le \liminf_{B\to\infty} \frac{\#\{n \ge 0 : \sum_{f\in\mathcal{F}_n} h_X^+(f(P)) \le B\}}{\log B}$$

and

$$\limsup_{B\to\infty} \frac{\#\{n \ge 0 : \sum_{f\in\mathcal{F}_n} h_X^+(f(P)) \le B\}}{\log B} \le \frac{1}{\log(k(1-\epsilon)\alpha_{\mathcal{F}}(P))}$$

Since the choice for ϵ is arbitrary, and the lim inf is less or equal to the lim sup, this finishes the proof that

$$\lim_{B\to\infty} \frac{\#\{n \ge 0 : \sum_{f\in\mathcal{F}_n} h_X^+(f(P)) \le B\}}{\log B} = \frac{1}{\log(k.\alpha_{\mathcal{F}}(P))}$$

Moreover, we also have that

$$\{n \ge 0 : \sum_{f\in\mathcal{F}_n} h_X^+(f(P)) \le B\} \subset \{n \ge 0 : h_X^+(f(P)) \le B \text{ for all } f \in \mathcal{F}_n\}$$

and thus

$$\frac{\log B}{\log(k(1+\epsilon)\alpha_{\mathcal{F}}(P))} - n_0(\epsilon) - 1 \le \#\{n \ge 0 : h_X^+(f(P)) \le B \text{ for all } f \in \mathcal{F}_n\}$$

This implies that

$$\frac{k^{\frac{\log B}{\log(k(1+\epsilon)\alpha_{\mathcal{F}}(P))} - n_0(\epsilon)} - 1}{k-1} \le \#\{Q \in \mathcal{O}_{\mathcal{F}}(P); h_X^+(Q) \le B\}$$

Taking $\frac{1}{\log B}$-roots and letting $B \to \infty$ gives

$$k^{\frac{1}{\log(k.\alpha_{\mathcal{F}}(P))}} \le \liminf_{B\to\infty} (\#\{Q \in \mathcal{O}_{\mathcal{F}}(P); h_X^+(Q) \le B\})^{\frac{1}{\log B}}.$$

□

We finish this section by stating the following elementary lemma from linear algebra. This lemma will be useful in the following sections.

Lemma 1.8: *Let $A = (a_{ij}) \in M_r(\mathbb{C})$ be an r-by-r matrix. Let $||A|| = \max |a_{ij}|$, and let $\rho(A)$ denote the spectral radius of A. Then there are constants c_1 and c_2, depending on A, such that*

$$c_1\rho(A)^n \le ||A^n|| \le c_2 n^r \rho(A)^n \text{ for all } n \ge 0.$$

In particular, we have $\rho(A) = \lim_{n\to\infty} ||A^n||^{\frac{1}{n}}$.

Proof. See [18, lemma 14] □

1.3 Some divisor and height inequalities for rational maps

We let $h, g : X \dashrightarrow X$ be rational maps, and $f \in \mathcal{F}_n$ for $\mathcal{F} = \{f_1, ..., f_k\}$ a dynamical system of self-rational maps on X. The aim of this section is mainly to prove the next result below. It states that the action of $f \in \mathcal{F}_n$ on the vector space $\mathrm{NS}(X)_{\mathbb{R}}$ is related with the actions of the maps $f_1, ..., f_k$ by the existence of certain inequalities. This result guarantees, for instance, that the dynamical degree converges, and afterwards will also be important in order to prove that $h_X^+(f(P)) \leq O(1).k^n.(\delta_{\mathcal{F}} + \epsilon)^n h_X^+(P)$ for all $f \in \mathcal{F}_n$. In order to achieve this goal, we state without proof, at the end of the section, an arithmetic inequality relating the Weil height functions $h_{X,D} \circ f$ and h_{Y,f^*D}, with $X/\bar{K}$ and $Y/\bar{K}$ smooth projective varieties, $f : Y \dashrightarrow X$ a dominant rational map defined over $\bar{K}$, and $D \in \mathrm{Div}(X)$ be an ample divisor.

Proposition 1.9: *Let X be a smooth projective variety, and fix a basis $D_1, ..., D_r$ for the vector space $NS(X)_{\mathbb{R}}$. A dominant rational map $h : X \dashrightarrow X$ induces a linear map on $NS(X)_{\mathbb{R}}$, and we write*

$$h^*D_j \equiv \textstyle\sum_{i=1}^r a_{ij}(h)D_i \text{ and } A(h) = (a_{ij}(h)) \in M_r(\mathbb{R}).$$

We let $||.||$ denote the sup norm on $M_r(\mathbb{R})$. Then there is a constant $C \geq 1$ depending on $D_1, ..., D_r$ such that for any dominant rational maps $h, g : X \dashrightarrow X$, any $n \geq 1$, and any $f \in \mathcal{F}_n$ we have

$$||A(g \circ h)|| \leq C||A(g)||.||A(h)||$$
$$||A(f)|| \leq C.(r. \max_{i=1,...,k.} ||A(f_i)||)^n.$$

The proof of this result will be made in the sequel. An immediate corollary of this is the convergence of the limit defining the dynamical degree.

Corollary 1.10: *The limit $\delta_{\mathcal{F}} = \limsup_{n\to\infty} \rho(\mathcal{F}_n)^{\frac{1}{n}}$ exists.*

Proof. With notation as in the statement of proposition 1.9, we have

$$\rho(\mathcal{F}_n) = \max_{f\in\mathcal{F}_n} \rho(f^*,\mathrm{NS}(X)_{\mathbb{R}}) = \max_{f\in\mathcal{F}_n} \rho(A(f))$$

Denoting $||A(\mathcal{G})|| = \max_{g\in\mathcal{G}} ||A(g)||$, where $\rho(\mathcal{G}) := \rho(g)$ for $\mathcal{G}$ dynamical system and $g \in \mathcal{G}$, proposition 1.9 give us that

$$\log ||A(\mathcal{F}_{n+m})|| \leq \log ||A(\mathcal{F}_m)|| + \log ||A(\mathcal{F}_n)|| + O(1)$$

Using this convexity estimate, we can see that $\dfrac{1}{n} \log ||A(\mathcal{F}_n)||$ converges. Indeed, if a sequence $(d_n)_{n\in\mathbb{N}}$ of nonnegative real numbers satisfies $d_{i+j} \leq d_i + d_j$, then after fixing a integer m and writing $n = mq + r$ with $0 \leq r \leq m - 1$, we have

$$\frac{d_n}{n} = \frac{d_{mq+r}}{n} \leq \frac{(qd_m + d_r)}{n} = \frac{d_m}{m}\frac{1}{(1 + r/mq)} + \frac{d_r}{n} \leq \frac{d_m}{m} + \frac{d_r}{n}.$$

Now take the limsup as $n \to \infty$, keeping in mind that m is fixed and $r \leq m - 1$, so d_r is bounded. This gives

$$\limsup_{n\to\infty} \frac{d_n}{n} \leq \frac{d_m}{m}.$$

taking the infimum over m shows that

$$\limsup_{n\to\infty} \frac{d_n}{n} \leq \inf_{m\geq 1} \frac{d_m}{m} \leq \liminf_{m\to\infty} \frac{d_m}{m},$$

and hence all three quantities must be equal.

As the sequence $(||A(\mathcal{F}_n)||^{1/n})_{n\in\mathbb{N}}$ is convergent and therefore bounded, lemma 1.8 guarantees that the sequence $(\rho(\mathcal{F}_n)^{1/n})_{n\in\mathbb{N}}$ is bounded as well. □

We also conjecture that the limit $\lim_{n\to\infty}\rho(\mathcal{F}_n)^{\frac{1}{n}}$ exists and is a birational invariant. The proof for dynamical degrees of systems with only one map given in [12, prop. 1.2] should be extented naturally for our present definition of degree with several maps. In the mentioned article, the dynamical degree is firstly defined using currents, and afterwards such definition is proved to coincide with the one using the limit of roots of spectral radius. Such result can be worked out in some future paper. Thus, from now on, we assume that

$$\delta_{\mathcal{F}} := \lim_{n\to\infty}\rho(\mathcal{F}_n)^{\frac{1}{n}},$$

and that it exists.

We start the proof of proposition 1.9 stating the following proposition and lemmas whose proofs can be found in [18]:

Proposition 1.11: *Let $X^{(0)}, X^{(1)}, ..., X^{(m)}$ be smooth projective varieties of the same dimension N, and let $f^{(i)} : X^{(i)} \dashrightarrow X^{(i-1)}$ be dominant rational maps for $1 \le i \le m$. Let D be a nef divisor on $X^{(0)}$. Then for any nef divisor H on $X^{(m)}$, we have*

$$(f^{(1)}\circ f^{(2)}\circ ...\circ f^{(m)})^*D.H^{N-1} \le (f^{(m)})^*...(f^{(2)})^*(f^{(1)})^*D.H^{N-1}.$$

Proof. See [18, Prop. 17] □

For the lemmas, we need to set the following notation:

- N : The dimension of X, wich we assume is at least 2.
- $\mathrm{Amp}(X)$: The ample cone in $\mathrm{NS}(X)_{\mathbb{R}}$ of all ample $\mathbb{R}-$divisors.
- $\mathrm{Nef}(X)$: The nef cone in $\mathrm{NS}(X)_{\mathbb{R}}$ of all nef $\mathbb{R}-$divisors.
- $\mathrm{Eff}(X)$: The effective cone in $\mathrm{NS}(X)_{\mathbb{R}}$ of all effective $\mathbb{R}-$divisors.
- $\overline{\mathrm{Eff}}(X)$: The $\mathbb{R}-$closure of $\mathrm{Eff}(X)$.

As explained in [11, section 1.4], we have the facts

$$\mathrm{Nef}(X) = \overline{\mathrm{Amp}}(X) \text{ and } \mathrm{Amp}(X) = \mathrm{int}(\mathrm{Nef}(X)).$$

In particular, since $\mathrm{Amp}(X) \subset \mathrm{Eff}(X)$, it follows that $\mathrm{Nef}(X) \subset \overline{\mathrm{Eff}}(X)$.

Lemma 1.12: *With notation as above, let $D \in \overline{Eff}(X) - \{0\}$ and $H \in Amp(X)$. Then*

$$D.H^{N-1} > 0.$$

Proof. See [18, lemma 18] □

Lemma 1.13: *Let $H \in Amp(X)$, and fix some norm $|.|$ on the $\mathbb{R}-$vector space $NS(X)_{\mathbb{R}}$. Then there are constants $C_1, C_2 > 0$ such that*

$$C_1|v| \leq v.H^{N-1} \leq C_2|v| \text{ for all } v \in \overline{\text{Eff}}(X).$$

Proof. See [18, lemma 19] □

Now we start the proof of theorem 1.9. We fix a norm $|.|$ on the $\mathbb{R}-$vector space $\text{NS}(X)_{\mathbb{R}}$ as before. Additionally, for any $A : \text{NS}(X)_{\mathbb{R}} \to \text{NS}(X)_{\mathbb{R}}$ linear transformation, we set

$$||A||' = \sup_{v \in \text{Nef}-\{0\}} \frac{|Av|}{|v|},$$

which exists because the set $\overline{\text{Eff}}(X) \cap \{w \in \text{NS}(X)_{\mathbb{R}} : |w| = 1\}$ is compact.
We note that for linear maps $A, B \in \text{End}(\text{NS}(X)_{\mathbb{R}})$ and $c \in \mathbb{R}$ we have

$$||A + B||' \leq ||A||' + ||B||' \text{ and } ||cA||' = |c|||A||'.$$

Further, since $\text{Nef}(X)$ generates $\text{NS}(X)_{\mathbb{R}}$ as an $\mathbb{R}-$vector space, we have $||A||' = 0$ if and only if $A = 0$. Thus $||.||'$ is an $\mathbb{R}-$norm on $\text{End}(\text{NS}(X)_{\mathbb{R}})$.
Similarly, for any linear map $A : \text{NS}(X)_{\mathbb{R}} \to \text{NS}(X)_{\mathbb{R}}$, we set

$$||A||'' = \sup_{v \in \text{Eff}-\{0\}} \frac{|Aw|}{|w|},$$

then $||.||''$ is an $\mathbb{R}-$norm on $\text{End}(\text{NS}(X)_{\mathbb{R}})$.

We note that $\overline{\text{Eff}}(X)$ is preserved by f^* for f self-rational map on X, and that $\text{Nef}(X) \subset \overline{\text{Eff}}(X)$. Thus if $v \in \text{Nef}(X)$, then g^*v and h^*v belong to $\overline{\text{Eff}}(X)$. This allows us to compute

$$\begin{aligned}
&||(g \circ h)^*||' = \sup_{v \in \text{Nef}(X)-\{0\}} \frac{|(g \circ h)^*v|}{|v|} \\
&\leq C_1^{-1} \sup_{v \in \text{Nef}(X)-\{0\}} \frac{(g \circ h)^*v.H^{N-1}}{|v|} \text{ from lemma 1.13} \\
&\leq C_1^{-1} \sup_{v \in \text{Nef}(X)-\{0\}} \frac{(h^*g^*v).H^{N-1}}{|v|} \text{ from proposition 1.11} \\
&= C_1^{-1} \sup_{v \in \text{Nef}(X)-\{0\}, g^*v \neq 0} \frac{(h^*g^*v).H^{N-1}}{|v|} \\
&= C_1^{-1} (\sup_{v \in \text{Nef}(X)-\{0\}, g^*v \neq 0} \frac{(h^*g^*v).H^{N-1}}{|g^*v|}.\frac{|g^*v|}{|v|}) \\
&\leq C_1^{-1} (\sup_{v \in \text{Nef}(X)-\{0\}, g^*v \neq 0} \frac{(h^*g^*v).H^{N-1}}{|g^*v|}).(\sup_{v \in \text{Nef}-\{0\}} \frac{|g^*v|}{|v|}) \\
&= C_1^{-1} (\sup_{v \in \text{Nef}(X)-\{0\}, g^*v \neq 0} \frac{(h^*g^*v).H^{N-1}}{|g^*v|}).||g^*||' \\
&\leq C_1^{-1} (\sup_{w \in \overline{\text{Eff}}(X)-\{0\}} \frac{(h^*w).H^{N-1}}{|w|}).||g^*||' \text{ since } g^*v \in \overline{\text{Eff}}(X) \\
&\leq C_1^{-1} C_2 (\sup_{w \in \overline{\text{Eff}}(X)-\{0\}} \frac{|h^*w|}{|w|}).||g^*||' \text{ from lemma 1.13} \\
&= C_1^{-1} C_2 ||h^*||''.||g^*||'.
\end{aligned}$$

We remember that we defined $||.||$ to be the sup norm on $M_r(\mathbb{R}) = \text{End}(\text{NS}(X)_{\mathbb{R}}$, where the identification is via the given basis $D_1, ..., D_r$ of $\text{NS}(X)_{\mathbb{R}}$. We thus have three norms $||.||, ||.||'$ and $||.||''$ on $\text{End}(\text{NS}(X)_{\mathbb{R}}$, so there are positive constants C_3', C_4', C_3'' and C_4'' such that

$$C_3'||\gamma|| \le ||\gamma||' \le C_4'||\gamma|| \text{ and } C_3''||\gamma|| \le ||\gamma||'' \le C_4''||\gamma|| \ 1\ \forall \gamma \in \mathrm{End}(\mathrm{NS}(X)_{\mathbb{R}}.$$

Hence

$$||A(g \circ h)|| = ||(g \circ h)^*|| \le C_3'^{-1}||(g \circ h)^*||'$$

$$\le C_3'^{-1}C_1^{-1}C_2||h^*||''.||g^*||'$$

$$\le C_3'^{-1}C_1^{-1}C_2C_4'C_4''||h^*||.||g^*||$$

$$= C_3'^{-1}C_1^{-1}C_2C_4'C_4''||A(h)||.||A(g)||.$$

Similarly, if $v \in \mathrm{Nef}(X)$, $f := f_{i_1} \circ ... \circ f_{i_n} \in \mathcal{F}_n$, then $f^*v \in \overline{\mathrm{Eff}}(X)$. A similar calculation gives

$$||f^*||' = \sup_{v\in \mathrm{Nef}(X)-\{0\}} \frac{|f^*v|}{|v|}$$

$$\le C_1^{-1} \sup_{v\in \mathrm{Nef}(X)-\{0\}} \frac{(f^*v).H^{N-1}}{|v|} \text{ from lemma 1.13}$$

$$= C_1^{-1} \sup_{v\in \mathrm{Nef}(X)-\{0\}} \frac{(f_{i_1} \circ ... \circ f_{i_n})^*v.H^{N-1}}{|v|}$$

$$\le C_1^{-1} \sup_{v\in \mathrm{Nef}(X)-\{0\}} \frac{((f_{i_n})^*...(f_{i_1})^*v).H^{N-1}}{|v|} \text{ from proposition 1.11}$$

$$\le C_1^{-1}C_2(\sup_{v\in \mathrm{Nef}(X)-\{0\}} \frac{|(f_{i_n})^*...(f_{i_1})^*v|}{|v|}) \text{ from lemma 1.13}$$

$$= C_1^{-1}C_2.||(f_{i_n})^*...(f_{i_1})^*||'.$$

Hence

$$||A(f)|| = ||f^*|| \le C_3'^{-1}||f^*||'$$

$$\le C_3'^{-1}C_1^{-1}C_2||(f_{i_n})^*...(f_{i_1})^*||'$$

$$\le C_3'^{-1}C_1^{-1}C_2C_4'C_4''||(f_{i_n})^*...(f_{i_1})^*||$$

$$\le C_3'^{-1}C_1^{-1}C_2C_4'C_4''r^n||(f_{i_n})^*||...||(f_{i_1})^*||$$

$$\le C_3'^{-1}C_1^{-1}C_2C_4'C_4''.[r. \max_{i=1,...,k.} ||A(f_i)||]^n,$$

As we wanted to show.

As it was said in the beginning of this section, the next proposition is a height inequality for rational maps, with eyes towards future applications.

Proposition 1.14: *Let $X/\bar{K}$ and $Y/\bar{K}$ be smooth projective varieties, let $f : Y \dashrightarrow X$ be a dominant rational map defined over $\bar{K}$, let $D \in Div(X)$ be an ample divisor, and fix Weil height functions $h_{X,D}$ and $h_{Y,f^*D}(P)$ associated to D and f^*D. Then*

$$h_{X,D} \circ f(P) \le h_{Y,f^*D}(P) + O(1) \text{ for all } P \in (Y - I_{\mathcal{F}})(\bar{K}),$$

where the $O(1)$ bound depends on X, Y, f, and the choice of height functions, but is independent of P.

Proof. See [18, Prop. 21]. □

1.4 A bound for the sum of heights on iterates

This section is devoted for the proof of a quantitative upper bound for $\sum_{f\in\mathcal{F}_n} h_X^+(f(P))$ in terms of the dynamical degree $\delta_{\mathcal{F}}$ of the system.This is one of the main results of this chapter, and is stated below. As a corollary, we see that the arithmetic degree of any point is upper bounded by the dynamical degree of the system.

Theorem 1.15: *Let K be a number field or a one variable function field of characteristic 0 , let $\mathcal{F} = \{f_1, ..., f_k\}$ be a set of dominant self rational maps on X defined over K as stated before, let h_X be a Weil height on $X(\bar{K})$ relative to an ample divisor, let $h_X^+ = \max\{h_X, 1\}$, and let $\epsilon > 0$. Then there exists a positive constant $C = C(X, h_X, f, \epsilon)$ such that for all $P \in X_{\mathcal{F}}(\bar{K})$ and all $n \geq 0$,*

$$\textstyle\sum_{f\in\mathcal{F}_n} h_X^+(f(P)) \leq C.k^n.(\delta_{\mathcal{F}} + \epsilon)^n.h_X^+(P).$$

In particular, $h_X^+(f(P)) \leq C.k^n.(\delta_{\mathcal{F}} + \epsilon)^n.h_X^+(P)$ for all $f \in \mathcal{F}_n$.

Before proving the theorem, we note that it implies the fundamental inequality $\bar{\alpha}_{\mathcal{F}}(P) \leq \delta_{\mathcal{F}}$.

Corollary 1.16: *Let $P \in X_{\mathcal{F}}(\bar{K})$. Then*

$$\bar{\alpha}_{\mathcal{F}}(P) \leq \delta_{\mathcal{F}}.$$

Proof. Let $\epsilon > 0$. Then

$\bar{\alpha}_{\mathcal{F}}(P) = \frac{1}{k} \limsup_{n\to\infty} \{\sum_{f\in\mathcal{F}_n} h_X^+(f(P))\}^{\frac{1}{n}}$ by definition of $\bar{\alpha}_{\mathcal{F}}$

$\leq \limsup_{n\to\infty} (C.(\delta_{\mathcal{F}} + \epsilon)^n.h_X^+(P))^{\frac{1}{n}}$ from theorem 1.15

$= \delta_{\mathcal{F}} + \epsilon.$

This holds for all $\epsilon > 0$, which proves that $\bar{\alpha}_{\mathcal{F}}(P) \leq \delta_{\mathcal{F}}$. □

Now we prove theorem 1.15. If P has a finite orbit, then $\bar{\alpha}_{\mathcal{F}}(P) = 1$ and $1 \leq \delta_{\mathcal{F}}$, so there is nothing to prove. We assume henceforth that $\#\mathcal{O}_{\mathcal{F}}(P)$ is not finite. We let m and l be positive integers to be chosen later, and we set

$$\mathcal{G} := \mathcal{F}_{ml}.$$

We note that $X_{\mathcal{F}}(\bar{K}) \subset X_{\mathcal{G}}(\bar{K})$. We choose ample divisors $D_1, ..., D_r$ in $\mathrm{Div}(X)$ whose algebraic equivalence classes form a basis for $\mathrm{NS}(X)_{\mathbb{Q}}$, and we fix functions $h_{D_1}, ..., h_{D_r}$ associated to the divisors $D_1, ..., D_r$. We note that any two ample heights are commensurate with one another, in other words, $h_X >< h'_X$, so we may take h_X to be

$$h_X(Q) = \max_{1\leq i\leq r} h_{D_i}(Q).$$

We further can assume that $h_{D_i} \geq 1$, so $h_X = h_X^+$.

Applying g^* to the divisors in our basis of $\mathrm{NS}(X)_{\mathbb{Q}}$, where g is any self-rational map on X/K, we have algebraic relations

$$(*) \qquad g^*D_t \equiv \textstyle\sum_{i=1}^{r} a_{it}(g)D_i \quad \text{for some } a_{it}(g) \in \mathbb{Q}.$$

We set the notation

$$A(g) = (a_{it}(g)), ||A(g)|| = \max_{i,t} |a_{it}(g)|, ||A(\mathcal{G})|| = \max_{g\in\mathcal{G}} ||A(g)||.$$

Algebraic equivalence of divisors implies a height relation as in the following result.

Lemma 1.17: *Let $E \in Div(X)_{\mathbb{R}}$ be a divisor that is algebraic equivalent to 0, and fix a height function h_E associated to E. Then there is a constant $C = C(h_X, h_E)$ such that*

$$|h_E(P)| \leq C\sqrt{h_X^+(P)} \text{ for all } P \in X(\bar{K}).$$

Proof. See for example the book of Diophantine Geometry of Hindry-Silverman[14, Theorem B.5.9]. □

Applying lemma 1.17 to (*) and using additivity of height functions, we find a positive constant $C_t(\epsilon, g)$ such that

$$|h_{g^*D_t}(Q) - \textstyle\sum_{i=1}^r a_{it}(g)h_{D_i}(Q)| \leq C_t(\epsilon, g)\sqrt{h_X(Q)} \text{ for all } Q \in X(\bar{K}).$$

Making $A := \max_{g\in\mathcal{G},t} C_t(\epsilon, g)$, we have, for all points $Q \in X(\bar{K})$ and $g \in \mathcal{G}$ that

$h_X(g(Q)) = \max_{i\leq t\leq r} h_{D_t}(g(Q))$

$\leq \max_{i\leq t\leq r}(h_{g^*D_t}(Q) + O(1))$ from proposition 1.14

$\leq \max_{i\leq t\leq r}(\sum_{i=1}^r a_{it}(g)h_{D_i}(Q)) + O(1) + A\sqrt{h_X(P)}$ from lemma 1.17

$\leq (r \max_{i,t,g\in\mathcal{G}} |a_{it}(g)|)h_X(Q) + O(\sqrt{h_X(P)})$

$= r||A(\mathcal{G})||h_X(Q) + O(\sqrt{h_X(P)})$.

An elementary lemma will be used.

Lemma 1.18: *Let S be a set, $\mathcal{G} = \{g_1, ..., g_s\}$ self maps on S, and a map $h : S \to [1, \infty)$. Let $a, b \geq 1$ be constants. Suppose that for all $x \in S$ we have*

$$h(g_i(x)) \leq ah(x) + c\sqrt{h(x)}.$$

Then for all $x \in S, n \geq 0, g^{(n)} \in \mathcal{G}_n$,

$$h(g^{(n)}(x)) \leq a^n(h(x) + (2\sqrt{2}c)^n\sqrt{h(x)}).$$

Proof. We set $\gamma = 2\sqrt{2}$ and proceed by induction on n. The inequality is true for $n = 0, 1$ by hypothesis, then we suppose that it is true for a given $n \in \mathbb{N}_{\geq 1}$. Let $g^{(n+1)} := g_{i_n+1} \circ ... \circ g_{i_2} \circ g_{i_1} := g^{(n)} \circ g_{i_1} \in \mathcal{G}_{n+1}, g^{(n)} \in \mathcal{G}_n$.

Then

$h(g^{(n+1)}(x)) = h(g^{(n)}(g_{i_1}(x)))$
$\leq a^n(h(g_{i_1}(x)) + (\gamma c)^n\sqrt{h(g_{i_1}(x))})$ from the induction hypothesis
$\leq a^n(ah(x) + c\sqrt{h(x)} + (\gamma c)^n\sqrt{ah(x) + c\sqrt{h(x)}})$
$\leq a^n(ah(x) + c\sqrt{h(x)} + (\gamma c)^n\sqrt{2ach(x)})$
$= a^{n+1}h(x) + (a^nc + (\gamma ac)^n\sqrt{2ac})\sqrt{h(x)}$.

Hence

$a^{n+1}(h(x) + (\gamma c)^{n+1}\sqrt{h(x)}) - h(g^{(n+1)}(x))$
$\geq (a^{n+1}(h(x) + (\gamma ac)^{n+1}\sqrt{h(x)})) - (a^{n+1}h(x) + (a^nc + (\gamma ac)^n\sqrt{2ac}) \sqrt{h(x)})$
$= \sqrt{h(x)}a^nc(\gamma^{n+1}ac^n - 1 - \gamma^na^{1/2}c^{n-1/2}\sqrt{2})$
$\geq \sqrt{h(x)}a^nc(\gamma^{n+1}ac^n - 1 - \gamma^nac^n\sqrt{2})$

$= \sqrt{h(x)}a^n c(\gamma^n ac^n(\gamma - \sqrt{2}) - 1)$
$= \sqrt{h(x)}a^n c(\gamma^n ac^n\sqrt{2} - 1)$
$> 0.$

□

We apply lemma 1.18 to the inequality above the previous lemma to obtain

$$\begin{aligned} h_X(g^{(n)}(Q)) &\leq (r||A(\mathcal{G})||)^n(h_X(Q) + O(1)^n\sqrt{h_X(Q)}) \\ &\leq (C_5 r||A(\mathcal{G})||)^n h_X(Q), \end{aligned}$$

for all $g^{(n)} \in \mathcal{G}_n, n \in \mathbb{N}, Q \in X_{\mathcal{F}}(\bar{K})$. And then

$$\textstyle\sum_{g\in\mathcal{G}_n} h_X(g(Q)) \leq (C_5.r.k.||A(\mathcal{G})||)^n h_X(Q)$$

for all $Q \in X_{\mathcal{F}}(\bar{K})$.

We recall that $\mathcal{G} = \mathcal{F}_{ml}$, which lets us estimate

$$\begin{aligned} &||A(\mathcal{G})|| = ||A((\mathcal{F}_l)_m)|| \\ &= \sup_{f\in(\mathcal{F}_l)_m} ||A(f)|| \text{ by definition} \\ &\leq (O(1).r.||A(\mathcal{F}_l)||)^m \text{ by Proposition} \qquad 1.9 \\ &\leq (C_6.r.\rho(\mathcal{F}_l))^m \text{ by lemma 1.8,} \end{aligned}$$

By definition, the dynamical degree is the limit of $\rho(\mathcal{F}_l)^{1/l}$ as $l \to \infty$. So we now fix an $l = l(\epsilon, \mathcal{F})$ such that

$$\rho(\mathcal{F}_l) \leq (\delta_{\mathcal{F}} + \epsilon)^l \text{ and } (C_6.r)^{1/l} < (1+\epsilon)^{1/3}.$$

For this choice of l, we have

$$||A(\mathcal{G})|| \leq [C_6.r.(\delta_{\mathcal{F}} + \epsilon)^l]^m.$$

Using $\mathcal{G} = \mathcal{F}_{ml}$ gives

$$\begin{aligned} &\textstyle\sum_{f\in\mathcal{F}_{mln}} h_X(f(P)) = \sum_{f\in\mathcal{G}_n} h_X(f(P)) \\ &\leq (C_5.r.k.||A(\mathcal{G})||)^n h_X(P) \\ &\leq (C_5.r.k.[C_6.r.(\delta_{\mathcal{F}} + \epsilon)^l]^m)^n h_X(P) \\ &\leq [C_7.C_6^m.r^{m+1}(\delta_{\mathcal{F}} + \epsilon)^{ml}]^n h_X(P). \end{aligned}$$

where $C_7 := C_5.k$. We now take $P \in X_{\mathcal{F}}(\bar{K})$ as in the statement of the theorem, and we apply the inequality right above to each of the sets $\mathcal{F}_0, ..., \mathcal{F}_{ml-1}$ to obtain, for $i = 0, ..., ml-1$, that

$$\begin{aligned} &\textstyle\sum_{f\in\mathcal{F}_{mln+i}} h_X(f(P)) = \\ &= \textstyle\sum_{f\in\mathcal{F}_{mln}, g\in\mathcal{F}_i} h_X(f(g(P))) \\ &\leq [C_7.C_6^m.r^{m+1}(\delta_{\mathcal{F}} + \epsilon)^{ml}]^n (\textstyle\sum_{g\in\mathcal{G}_i} h_X(g(P))) \\ &\leq C_{8,i}.k^i[C_7.C_6^m.r^{m+1}(\delta_{\mathcal{F}} + \epsilon)^{ml}]^n h_X(P). \end{aligned}$$

Where the last inequality follows from the fact that the ample height h_X dominates any other height h_D. Then

$\max_{0\leq i<ml} \sum_{f\in\mathcal{F}_{mln+i}} h_X(f(P))$

$\leq \max_{1\leq i<ml} C_{8,i}.k^{ml}[C_7.C_6^m.r^{m+1}(\delta_{\mathcal{F}}+\epsilon)^{ml}]^n h_X(P)$

$\leq C_8 k^{ml}[C_7.C_6^m.r^{m+1}(\delta_{\mathcal{F}}+\epsilon)^{ml}]^n h_X(P).$

Now let $q \geq 1$ be any integer and write

$$q = mln + i \text{ with } 0 \leq i < ml.$$

Then using $mln \leq q$ and $n \leq \dfrac{q}{ml}$ and the inequality quite above, we have

$$\sum_{f\in\mathcal{F}_q} h_X(f(P)) \leq C_8.k^{ml}.C_7^{q/ml}.C_6^{q/l}.r^{q/l+q/ml}(\delta_{\mathcal{F}}+\epsilon)^q.h_X(P).$$

The quantity $(C_7)^{1/ml}$ is independent of q and goes to 1 as $m \to \infty$. So we now fix a value of m such that $(C_7)^{1/ml} < (1+\epsilon)^{1/3}$. This value of m depend on $\epsilon, X, \mathcal{F}$, but not on q nor P, and the same is true for the constants. We finally get that

$$\sum_{f\in\mathcal{F}_q} h_X(f(P)) \leq C_8.k^q.(1+\epsilon)^q(\delta_{\mathcal{F}}+\epsilon)^q h_X(P).$$

After adjusting ϵ, this inequality above is the desired result, which completes the proof of theorem 1.15.

1.5 Application to canonical heights

In this final section of chapter 1, we show that the canonical height limit, proposed and constructed by S. Kawaguchi in [16, theorem 1.2.1], is convergent for certain eigendivisor classes relative to algebraic equivalence, instead of linear equivalence case worked by Kawaguchi in the source [3]. The theorem is also an extension of theorem 5 of [18], where the eigensystem of the hypothesis has just one morphism.

Theorem 1.19: *Assume that $\mathcal{F} = f_1, ..., f_k : X \to X$ are morphisms, and let $D \in Div(X)_{\mathbb{R}}$ that satisfies the algebraic relation*

$$\sum_{i=1}^k f_i^* D \equiv \beta D \text{ for some real number } \beta > \sqrt{\delta_{\mathcal{F}}}k,$$

where $\equiv$ denotes algebraic equivalence in $NS(X)_{\mathbb{R}}$. Then

(a) For all $P \in X(\bar{K})$, the following limit converges:

$$\hat{h}_{D,\mathcal{F}}(P) = \lim_{n\to\infty} \frac{1}{\beta^n} \sum_{f\in\mathcal{F}_n} h_D(f(P)).$$

(b) The canonical height in (a) satisfies

$$\sum_{i=1}^k \hat{h}_{D,\mathcal{F}}(f_i(P)) = \beta \hat{h}_{D,\mathcal{F}}(P) \text{ and } \hat{h}_{D,\mathcal{F}}(P) = h_D(P) + O(\sqrt{h_X^+(P)}).$$

(c) If $\hat{h}_{D,\mathcal{F}}(P) \neq 0$, then $\underline{\alpha}_{\mathcal{F}}(P) \geq \beta/k$.

(d) If $\hat{h}_{D,\mathcal{F}}(P) \neq 0$ and $\beta = \delta_{\mathcal{F}}k$, then $\alpha_{\mathcal{F}}(P) = \delta_{\mathcal{F}}$.

(e) Assume that D is ample and that K is a number field. Then

$$\hat{h}_{D,\mathcal{F}}(P) = 0 \iff P \text{ is preperiodic, i.e, has finite } \mathcal{F}\text{-orbit.}$$

Proof. (a) Theorem 1.15 says that for every $\epsilon > 0$ there is a constant $C_1 = C_1(X, h_X, \mathcal{F}, \epsilon)$ such that

$$\textstyle\sum_{f\in\mathcal{F}_n} h_X^+(f(P)) \leq C_1.k^n.(\delta_{\mathcal{F}}+\epsilon)^n.h_X^+(P) \text{ for all } n \geq 0.$$

We are given that $\sum_{i=1}^k f_i^* D \equiv \beta D$. Applying lemma 1.17 with $E = \sum_{i=1}^k f_i^* D - \beta D$, we find a positive constant $C_2 = C_2(D, \mathcal{F}, h_X)$ such that

$$|h_{\sum_{i=1}^k f_i^* D}(Q) - \beta h_D(Q)| \leq C_2\sqrt{h_X^+(Q)} \text{ for all } Q \in X(\bar{K}).$$

Since we assumed that the f_i are morphisms, standard functoriality of Weil height states that

$$h_{\sum_{i=1}^k f_i^* D} = \textstyle\sum_{i=1}^k h_D \circ f_i + O(1),$$

so the above inequality is reformulated as follows

$$(^{**})\ |\textstyle\sum_{i=1}^k h_D(f_i(Q)) - \beta h_D(Q)| \leq C_3\sqrt{h_X^+(Q)} \text{ for all } Q \in X(\bar{K}).$$

For $N \geq M \geq 0$ we estimate a telescopic sum,

$$\begin{aligned}
&|\beta^{-N}\textstyle\sum_{f\in\mathcal{F}_N} h_D(f(P)) - \beta^{-M}\sum_{f\in\mathcal{F}_M} h_D(f(P))| \\
&= |\textstyle\sum_{n=M+1}^N \beta^{-n}[\sum_{f\in\mathcal{F}_n} h_D(f(P)) - \beta\sum_{f\in\mathcal{F}_{n-1}} h_D(f(P))]| \\
&\leq \textstyle\sum_{n=M+1}^N \beta^{-n}|\sum_{f\in\mathcal{F}_n} h_D(f(P)) - \beta\sum_{f\in\mathcal{F}_{n-1}} h_D(f(P))| \\
&\leq \textstyle\sum_{n=M+1}^N \beta^{-n}[\sum_{f\in\mathcal{F}_{n-1}} |\sum_{i=1}^k h_D(f_i(f(P))) - \beta h_D(f(P))|] \\
&\leq \textstyle\sum_{n=M+1}^N \beta^{-n}(\sum_{f\in\mathcal{F}_{n-1}} C_3\sqrt{h_X^+(f(P))}) \text{ by } (^{**}) \\
&\leq \textstyle\sum_{n=M+1}^N \beta^{-n}.k^{(n-1)/2}.C_3.\sqrt{\sum_{f\in\mathcal{F}_{n-1}} h_X^+(f(P))} \text{ by Cauchy-Schwarz} \\
&\leq \textstyle\sum_{n=M+1}^N \beta^{-n}.k^{n-1}.C_3.C.(\delta_{\mathcal{F}}+\epsilon)^{(n-1)/2}.\sqrt{h_X^+(P)} \text{ by Thm. 1.15} \\
&\leq CC_3\sqrt{h_X^+(P)}\textstyle\sum_{n=M+1}^\infty [\dfrac{k^2(\delta_{\mathcal{F}}+\epsilon)}{\beta^2}]^{n/2}.
\end{aligned}$$

And

$$\textstyle\sum_{n=M+1}^\infty [\dfrac{k^2(\delta_{\mathcal{F}}+\epsilon)}{\beta^2}]^{n/2} < \infty \iff \dfrac{k^2(\delta_{\mathcal{F}}+\epsilon)}{\beta^2} < 1.$$

Since $\beta > \sqrt{\delta_{\mathcal{F}} k^2}$, we can choose $0 < \epsilon < \frac{\beta^2}{k^2} - \delta_{\mathcal{F}}$, which implies $\frac{k^2(\delta_{\mathcal{F}}+\epsilon)}{\beta^2} < 1$ and the desired convergence. Also we obtain the following estimate (***):

$$\begin{aligned}
&|\beta^{-N}\textstyle\sum_{f\in\mathcal{F}_N} h_D(f(P)) - \beta^{-M}\sum_{f\in\mathcal{F}_M} h_D(f(P))| \\
&\leq CC_3[\dfrac{k^2(\delta_{\mathcal{F}}+\epsilon)}{\beta^2}]^{M/2}\sqrt{h_X^+(P)}.
\end{aligned}$$

(b) The formula

$$\textstyle\sum_{i=1}^k \hat{h}_{D,\mathcal{F}}(f_i(P)) = \beta\hat{h}_{D,\mathcal{F}}(P)$$

follows immediately from the limit defining $\hat{h}_{D,\mathcal{F}}$ in part (a). Next, letting $N \to \infty$ and setting $M = 0$ in (***) gives

$$|\hat{h}_{D,\mathcal{F}}(P) - h_D(P)| = O(\sqrt{h_X^+(P)}),$$

which completes the proof of (b).

(c) We are assuming that $\hat{h}_{D,\mathcal{F}}(P) \neq 0$. If $\hat{h}_{D,\mathcal{F}}(P) < 0$, we change D to $-D$, so we may assume $\hat{h}_{D,\mathcal{F}}(P) > 0$. Let $H \in \mathrm{Div}(X)$ be an ample divisor such that $H + D$ is also ample (this can always be arranged by replacing H with mH for a sufficiently large m). Since H is ample, we may assume that the height function h_H is non-negative. We compute

$$\sum_{f\in\mathcal{F}_n} h_{D+H}(f(P))$$

$$= \sum_{f\in\mathcal{F}_n} h_D(f(P)) + \sum_{f\in\mathcal{F}_n} h_H(f(P)) + O(k^n)$$

$$\geq \sum_{f\in\mathcal{F}_n} h_D(f(P)) + O(k^n) \text{ since } h_H \geq 0$$

$$= \sum_{f\in\mathcal{F}_n} \hat{h}_{\mathcal{F},D}(f(P)) + O(\sum_{f\in\mathcal{F}_n} \sqrt{h_X^+(f(P))}) \text{ from (b)}$$

$$= \beta^n \hat{h}_{\mathcal{F},D}(P) + O(\sum_{f\in\mathcal{F}_n} \sqrt{h_X^+(f(P))}) \text{ from (b)}$$

$$\geq \beta^n \hat{h}_{\mathcal{F},D}(P) + O(\sqrt{\sum_{f\in\mathcal{F}_n} h_X^+(f(P))}) \text{ since } (x \to \sqrt{x}) \text{ is convex}$$

$$= \beta^n \hat{h}_{\mathcal{F},D}(P) + O(\sqrt{Ck^n(\delta_{\mathcal{F}}+\epsilon)^n h_X^+(P)}) \text{ from Theorem 1.15.}$$

This estimate is true for every $\epsilon > 0$, where C depends on ϵ. Using the assumption that $\beta > \sqrt{k.\delta_{\mathcal{F}}}$ we can choose $\epsilon > 0$ such that
$k.(\delta_{\mathcal{F}} + \epsilon) < \beta^2$. This gives

$$\sum_{f\in\mathcal{F}_n} h_{D+H}(f(P)) \geq \beta^n \hat{h}_{\mathcal{F},D}(P) + o(\beta^n),$$

so taking n^{th}-roots, using the assumption that $\hat{h}_{\mathcal{F},D}(P) > 0$, and letting $n \to \infty$ yields

$$\underline{\alpha}_{\mathcal{F}}(P) = \liminf_{n\to\infty} \frac{1}{k}\{\sum_{f\in\mathcal{F}_n} h_{D+H}(f(P)\}^{1/n} \geq \frac{\beta}{k}.$$

(d) From (c) we get that $\underline{\alpha}_{\mathcal{F}}(P) \geq \dfrac{\beta}{k} = \dfrac{\delta_{\mathcal{F}}.k}{k} = \delta_{\mathcal{F}}$, while corollary 1.16 gives $\bar{\alpha}_{\mathcal{F}}(P) \leq \delta_{\mathcal{F}}$. Hence the limit defining $\alpha_{\mathcal{F}}(P)$ exists and is equal to $\delta_{\mathcal{F}}$.

(e) First suppose that $\#\mathcal{O}_{\mathcal{F}}(P) < +\infty$. Since D is ample and the orbit of P is finite, we have that $h_D \geq 0, \hat{h}_{\mathcal{F},D}(P) \geq 0$, and there is a constant $C > 0$ such that $h_D(f(P)) \leq C$ for all $f \in \cup_{l\geq 0}\mathcal{F}_l$. This gives

$$|\hat{h}_{\mathcal{F},D}(P)| \leq \lim_{n\to\infty} \frac{1}{\beta^n} \sum_{f\in\mathcal{F}_n} |h_D(f(P))| \leq \lim_{n\to\infty} C.\frac{k^n}{\beta^n} = 0$$

Since $\beta > k$.

For the other direction, suppose that $\hat{h}_{\mathcal{F},D}(P) = 0$. Then for any $n \geq 0$ and $g \in \mathcal{F}_n$, we apply part (b) to obtain

$$\begin{aligned} 0 = \beta^n \hat{h}_{\mathcal{F},D}(P) &= \sum_{f\in\mathcal{F}_n} \hat{h}_{\mathcal{F},D}(f(P)) \geq \hat{h}_{\mathcal{F},D}(g(P)) \\ &\geq h_D(g(P)) - c\sqrt{h_D(g(P))}. \end{aligned}$$

This gives $h_D(g(P)) \leq c^2$, where c does not depend on P or n. This shows that $\mathcal{O}_{\mathcal{F}}(P)$ is a set of bounded height with respect to an ample height. Since $\mathcal{O}_{\mathcal{F}}(P)$ is contained in $X(K(P))$ and since we have assumed that K is a number field, we conclude that $\mathcal{O}_{\mathcal{F}}(P)$ is finite. □

Remark 1.20: In the same way as pointed in remark 29 of [18], when $f_1, ..., f_k$ are morphisms, there is always one divisor class $D \in \mathrm{NS}(X)_{\mathbb{R}}$ such that $\sum_{i=1}^{k} f_i^* D \equiv \beta D$, where β is the spectral radius of the linear map $\sum_{i \leq k} A(f_i)$ on $\mathrm{NS}(X)_{\mathbb{R}}$. It would remain to check whether it satisfies $\beta > k.\sqrt{\delta_{\mathcal{F}}}$, so that we can use theorem 1.19. In negative case, one should be able to achieve such condition by replacing the morphisms f_i by iterates $f_i^{\circ n_i}$, obtaining a new system $\mathcal{G}$ whose maps are iterates from the initial system. This would make it possible to study the arithmetic of $\mathcal{O}_{\mathcal{F}}$ with the new height associated to $\mathcal{G}$ arising from our theorem, since the orbits $\mathcal{O}_{\mathcal{G}}$ are contained in the orbits $\mathcal{O}_{\mathcal{F}}$ by construction.

Chapter 2

Canonical Heights Induced by Monomial Maps

Let $\Phi = \{\phi_1, ..., \phi_k\}$ be a set of dominant self-rational maps over $\overline{\mathbb{Q}}$ on the n-dimensional projective space. According to the definitions of the last chapter, such dynamical system has dynamical degree $\lim_{n\to\infty} \rho(\Phi_n)^{\frac{1}{n}} = \limsup_{n\to\infty}(\max_{i_j\leq k}(\deg(\phi_{i_1} \circ ... \circ \phi_{i_n})))^{\frac{1}{n}}$. Motivated by the work [29], we propose a definition of canonical height limit induced by the system Φ on projective spaces, make new definitions, and show some properties of such limit, when it exists, but now with several maps. We study the case when the maps are induced by monomial maps on multiplicative groups, generalizing some results of [29] about non-Zariski density of points with canonical height equal to zero. We prove the existence of an effective lower bound for the canonical height of points with infinite orbit as proposed by Silverman also in [29], remark 30.

2.1 Two proposals of dynamical canonical height for dominant rational maps

With the last chapter hypothesis, we defined the dynamical degree of a dynamical system $(X, \mathcal{F} = \{f_1, ..., f_k\})$ as $\delta_{\mathcal{F}} = \lim_{n\to\infty} \rho(\mathcal{F}_n)^{\frac{1}{n}}$, the upper arithmetic degree of a point P as $\bar{\alpha}_{\mathcal{F}}(P) = \dfrac{1}{k} \limsup_{n\to\infty}\{\sum_{f\in\mathcal{F}_n} h_D^+(f(P))\}^{\frac{1}{n}}$, and we showed, in the case where the maps of the system are morphisms, the existence of a canonical height function over the hypothesis that

$$\textstyle\sum_{i=1}^k f_i^* D \equiv \beta D \text{ for some real number } \beta > \sqrt{\delta_{\mathcal{F}}}k.$$

Such height function, on $P \in X(\overline{\mathbb{Q}})$, is given by

$$\hat{h}_{D,\mathcal{F}}(P) = \lim_{n\to\infty} \frac{1}{\beta^n} \textstyle\sum_{f\in\mathcal{F}_n} h_D(f(P)).$$

When $X = \mathbb{P}^N$, and $k = 1, f := f_1$, we have that $\delta_f = \lim_{n\to\infty}(\deg(f^{\circ n}))^{\frac{1}{n}}$. If f is a morphism with degree at least 2, then $(\deg f)^n = \deg(f^{\circ n})$ and $\delta_f = \deg f$ with $f^*\mathcal{O}(1) \equiv (\deg f)\mathcal{O}(1)$, and so $\hat{h}_{\mathcal{O}(1),f}(P) = \lim_{n\to\infty} \dfrac{h_D(f^{\circ n}(P))}{(\deg f)^n}$. But this is not necessarily true when f is not a morphism, since $(\deg f)^n$ may be different from $\deg f^{\circ n}$ in this case. In such situation, since $\deg f^{\circ n}$ grows in the same speed as δ_f^n roughly speaking, Silverman defined in [29], for $P \in \mathbb{P}_f^N$, the height function

$$\hat{h}_{\mathcal{O}(1),f}(P) = \limsup_{n\to\infty} \frac{h(f^{\circ n}(P))}{n^{l_f}(\delta_f)^n}$$

with some correction exponent l_f [29, conjecture 2], defined, when it exists, as $l_{\mathcal{F}} := \inf\{l \geq 0 : \sup_{n\geq 1} \dfrac{\deg(f^{\circ n})}{n^l \delta_f^n} < \infty\}$. Its existence is conjectured by Silverman in the above mentioned reference, as well as that is conjectured there that this exponent is less or equal than N. For X more general smooth projective variety and $k > 1$, we define the generalization of the above correction exponent as below, which existence we conjecture also.

Definition 2.1: $l_{\mathcal{F}} := \inf\{l \geq 0 : \sup_{n\geq 1} \frac{\rho(\mathcal{F}_n)}{n^l \delta_{\mathcal{F}}^n} < \infty\}$, *if it exists.*

So we are in conditions to generalize, now for smooth projective varieties and a system with several rational maps, the above Silverman's definition for canonical heights in projective spaces for a system with just one self-rational map. For instance, we can make this in two ways. The first one is the following, which says in particular that points with nonzero canonical height have arithmetic degree equal to the dynamical degree of the system. In the lack of convergence for the arithmetic degree, we assume from now on that $\alpha_{\mathcal{F}}(P) := \bar{\alpha}_{\mathcal{F}}(P)$.

Definition 2.2: *Assuming the existence of $l_{\mathcal{F}}$, the canonical height of $P \in X_{\mathcal{F}}(\bar{\mathbb{Q}})$ with respect to $\mathcal{F}$ and $D \in Pic(X)$ ample is*

$$\hat{\mathbf{h}}_{D,\mathcal{F}}(P) = \limsup_{n\to\infty} \frac{1}{n^{(l_{\mathcal{F}})} k^n \delta_{\mathcal{F}}^n} \sum_{f\in\mathcal{F}_n} h_D(f(P)).$$

Proposition 2.3: *The canonical height $\hat{h}_{D,\mathcal{F}}$ satisfies the following properties:*
(a) If $h_D \geq 0$ then $\hat{\mathbf{h}}_{D,\mathcal{F}} \geq 0$
(b) $\sum_i \hat{\mathbf{h}}_{D,\mathcal{F}}(f_i(P)) = k\delta_{\mathcal{F}}\hat{\mathbf{h}}_{D,\mathcal{F}}(P)$
(c) If $P \in Preper(\mathcal{F})$, then $\hat{\mathbf{h}}_{D,\mathcal{F}}(P) = 0$
(d) If $\hat{\mathbf{h}}_{D,\mathcal{F}}(P) > 0$, then $\delta_{\mathcal{F}} = \alpha_{\mathcal{F}}(P)$.

Proof. (a) This is obvious, since h_D is a non-negative function.
(b) We compute

$$\begin{aligned}
&\textstyle\sum_{i\leq k} \hat{\mathbf{h}}_{D,\mathcal{F}}(f_i(P)) = \limsup_{n\to\infty} \dfrac{1}{n^{(l_{\mathcal{F}})} k^n \delta_{\mathcal{F}}^n} \sum_{f\in\mathcal{F}_n} \sum_{i\leq k} h_D(f(f_i(P)))\\
&= \limsup_{n\to\infty} \dfrac{1}{n^{(l_{\mathcal{F}})} k^n \delta_{\mathcal{F}}^n} \textstyle\sum_{f\in\mathcal{F}_{n+1}} h_D(f(P))\\
&= \limsup_{n\to\infty} \dfrac{1}{(n-1)^{(l_{\mathcal{F}})} k^{n-1} \delta_{\mathcal{F}}^{n-1}} \textstyle\sum_{f\in\mathcal{F}_n} h_D(f(P))\\
&= k\delta_{\mathcal{F}} \limsup_{n\to\infty} (n/(n-1))^{(l_{\mathcal{F}})} \dfrac{1}{n^{(l_{\mathcal{F}})} k^n \delta_{\mathcal{F}}^n} \textstyle\sum_{f\in\mathcal{F}_n} h_D(f(P))\\
&= k\delta_{\mathcal{F}}\hat{\mathbf{h}}_{D,\mathcal{F}}(P).
\end{aligned}$$

(c) If P is preperiodic, then $\mathcal{O}_{\mathcal{F}}(P)$ is a finite set, lets say, bounded by $C > 0$. Then

$$\begin{aligned}
&|\hat{\mathbf{h}}_{D,\mathcal{F}}(P)| = |\limsup_{n\to\infty} \dfrac{1}{n^{(l_{\mathcal{F}})} k^n \delta_{\mathcal{F}}^n} \textstyle\sum_{f\in\mathcal{F}_n} h_D(f(P))|\\
&\leq \limsup_{n\to\infty} C(\dfrac{k}{k\delta_{\mathcal{F}}})^n.(\dfrac{1}{n^{(l_{\mathcal{F}})}})\\
&= 0
\end{aligned}$$

(d) We are assuming that $\hat{\mathbf{h}}_{D,\mathcal{F}}(P) > 0$, and by definition, $\hat{\mathbf{h}}_{D,\mathcal{F}}(P)$ is the limsup of $n^{-(l_{\mathcal{F}})} k^{-n} \delta_{\mathcal{F}}^{-n} \sum_{f\in\mathcal{F}_n} h_D(f(P))$, so we can find an infinite sequence $\mathcal{N}$ of positive integers such that

$$n^{-(l_{\mathcal{F}})}k^{-n}\delta_{\mathcal{F}}^{-n}\sum_{f\in\mathcal{F}_n} h_D(f(P)) \geq (1/2)\hat{\mathbf{h}}_{D,\mathcal{F}}(P) \text{ for all } n\in\mathcal{N}.$$

It follows that

$$\begin{aligned}\alpha_{\mathcal{F}}(P) &= \frac{1}{k}\limsup_{n\to\infty}\{\textstyle\sum_{f\in\mathcal{F}_n} h_D^+(f(P))\}^{\frac{1}{n}}\\ &\geq (1/k).\limsup_{n\in\mathcal{N}}[n^{(l_{\mathcal{F}})}k^n\delta_{\mathcal{F}}^n(1/2)\hat{\mathbf{h}}_{D,\mathcal{F}}(P)]^{(1/n)}\\ &= \delta_{\mathcal{F}}k/k = \delta_{\mathcal{F}},\end{aligned}$$

since $\hat{\mathbf{h}}_{D,\mathcal{F}}(P) > 0$. The result follows by gathering the above inequality with the fact that $\delta_{\mathcal{F}} \geq \bar{\alpha}_{\mathcal{F}}(P) \geq \underline{\alpha}_{\mathcal{F}}(P)$. □

We know that $\alpha_{\mathcal{F}}(P) = 1$ if P is preperiodic, and that $\alpha_{\mathcal{F}}(P) = \delta_F$ if the above height is positive on P, but we don't know when the converses of (c) and (d) of last proposition are true. So we also ask if eventual Zariski density of $\mathcal{O}_{\mathcal{F}}(P)$ would imply that $\hat{\mathbf{h}}_{D,\mathcal{F}}(P) > 0$, and then that $\alpha_{\mathcal{F}}(P) = \delta_{\mathcal{F}}$. If $k = 1$ and f_1 is morphism, or when $X = \mathbb{P}^N$ and all the maps in the given system are morphisms, then the dynamical degree is root of some charateristic polynomial. We will mention afterwards in this chapter that this also happens when $X = \mathbb{P}^N$, $k = 1$, and f_1 is a rational map induced by a monomial map, which is a kind of map that we will define next section. These informations, together with conjecture 1 of [29], motivate the following generalization for letter (a) of this conjecture.

Question 2.4: Under the hypothesis of section 1.1, is $\{\alpha_{\mathcal{F}}(P) | P \in X_{\mathcal{F}}(\bar{K})\}$ a finite subset of O_K, the ring of algebraic integers?

A survey for the above discussion for $k = 1$ can be found in [28].

It seems also interesting to let the limit in the height definition grow more fastly, omitting powers of k, obtaining another canonical height definition, as a generalization with some more similarities with Silverman's definition, to be seen later. After defining it, we have an analogous version of proposition 2.3 for this new height.

Definition 2.5: *Assuming the existence of $l_{\mathcal{F}}$, the canonical height of $P \in X_{\mathcal{F}}(\bar{\mathbb{Q}})$ with respect to $\mathcal{F}$ and $D \in Pic(X)$ ample is*

$$\hat{h}_{D,\mathcal{F}}(P) = \limsup_{n\to\infty}\frac{1}{n^{(l_{\mathcal{F}})}\delta_{\mathcal{F}}^n}\sum_{f\in\mathcal{F}_n} h_D(f(P)).$$

Proposition 2.6: *The canonical height $\hat{h}_{D,\mathcal{F}}$ satisfies the following properties:*

(a) If $h_D \geq 0$ then $\hat{h}_{D,\mathcal{F}} \geq 0$

(b) $\sum_i \hat{h}_{D,\mathcal{F}}(f_i(P)) = \delta_{\mathcal{F}}\hat{h}_{D,\mathcal{F}}(P)$

(c) If $P \in Preper(\mathcal{F})$, then $\hat{h}_{D,\mathcal{F}}(P) = 0$

(d) If $\hat{h}_{D,\mathcal{F}}(P) > 0$, then $\delta_{\mathcal{F}} \geq \alpha_{\mathcal{F}}(P) \geq \delta_{\mathcal{F}}/k$.

Proof. (a) This is obvious, since h_D is a non-negative function.

(b) We compute

$$\begin{aligned}&\textstyle\sum_{i\leq k}\hat{h}_{D,\mathcal{F}}(f_i(P)) = \limsup_{n\to\infty}\frac{1}{n^{(l_{\mathcal{F}})}\delta_{\mathcal{F}}^n}\sum_{f\in\mathcal{F}_n}\sum_{i\leq k} h_D(f(f_i(P)))\\ &= \limsup_{n\to\infty}\frac{1}{n^{(l_{\mathcal{F}})}\delta_{\mathcal{F}}^n}\textstyle\sum_{f\in\mathcal{F}_{n+1}} h_D(f(P))\\ &= \limsup_{n\to\infty}\frac{1}{(n-1)^{(l_{\mathcal{F}})}\delta_{\mathcal{F}}^{n-1}}\textstyle\sum_{f\in\mathcal{F}_n} h_D(f(P))\\ &= \delta_{\mathcal{F}}\limsup_{n\to\infty}(n/(n-1))^{(l_{\mathcal{F}})}\frac{1}{n^{(l_{\mathcal{F}})}\delta_{\mathcal{F}}^n}\textstyle\sum_{f\in\mathcal{F}_n} h_D(f(P))\\ &= \delta_{\mathcal{F}}\hat{h}_{D,\mathcal{F}}(P).\end{aligned}$$

(c) If P is preperiodic, then $\mathcal{O}_{\mathcal{F}}(P)$ is a finite set, let us say, whose cardinal is bounded from above by $C > 0$. Then

$$|\hat{h}_{D,\mathcal{F}}(P)| = |\limsup_{n\to\infty} \frac{1}{n^{(l_{\mathcal{F}})}\delta_{\mathcal{F}}^{n}} \sum_{f\in\mathcal{F}_n} h_D(f(P))|$$
$$\leq \limsup_{n\to\infty} C(\frac{k}{\delta_{\mathcal{F}}})^n.(\frac{1}{n^{(l_{\mathcal{F}})}})$$
$$= 0$$

(d) We are assuming that $\hat{h}_{D,\mathcal{F}}(P) > 0$, and by definition, $\hat{h}_{D,\mathcal{F}}(P)$ is the limsup of $n^{-(l_{\mathcal{F}})}\delta_{\mathcal{F}}^{-n} \sum_{f\in\mathcal{F}_n} h_D(f(P))$, so we can find an infinite sequence $\mathcal{N}$ of positive integers such that

$$n^{-(l_{\mathcal{F}})}\delta_{\mathcal{F}}^{-n} \sum_{f\in\mathcal{F}_n} h_D(f(P)) \geq (1/2)\hat{h}_{D,\mathcal{F}}(P) \text{ for all } n \in \mathcal{N}.$$

It follows that

$$\alpha_{\mathcal{F}}(P) = \frac{1}{k} \limsup_{n\to\infty}\{\sum_{f\in\mathcal{F}_n} h_D^{+}(f(P))\}^{\frac{1}{n}}$$
$$\geq (1/k). \limsup_{n\in\mathcal{N}}[n^{(l_{\mathcal{F}})}\delta_{\mathcal{F}}^{n}(1/2)\hat{h}_{D,\mathcal{F}}(P)]^{(1/n)}$$
$$= \delta_{\mathcal{F}}/k,$$

since $\hat{h}_{D,\mathcal{F}}(P) > 0$. The result follows by gathering the above inequality with the fact that $\delta_{\mathcal{F}} \geq \alpha_{\mathcal{F}}(P)$. □

2.2 Points of canonical height zero for monomial maps

Monomial maps are endomorphisms of the torus $\mathbb{G}_m^N$. They naturally induce self-rational maps of $\mathbb{P}^N$, by embedding $\mathbb{G}_m^N$ in $\mathbb{P}^N$. In this section we will show, for a class with infinite number of dynamical systems with several self-rational maps of $\mathbb{P}^N$, a generalization for theorem 27 of [29], for the height function of definition 2.5. We will reduce our situation to the case already treated of just one rational map, and this will be possible when the sequence giving the limit defining the dynamical degree is not so unstable, in the sense that it can be calculated using just one of the rational maps in the given system. Before proving results, we state some definitions of sections 6 and 7 of [29], and define our required kind of dynamical system.

Definition 2.7: *We write $Mat_{\mathbb{Z}}^{+}$ for the set of N-by-N matrices with integer coefficients and nonzero determinant. To each matrix $A \in Mat_{\mathbb{Z}}^{+}$ we associate the monomial map $\phi_A : \mathbb{G}_m^N \to \mathbb{G}_m^N$ given by the formula*

$$\phi_A(X_1, ..., X_N) = (X_1^{a_{11}}X_2^{a_{12}}...X_N^{a_{1N}}, X_1^{a_{21}}X_2^{a_{22}}...X_N^{a_{2N}}, ..., X_1^{a_{N1}}X_2^{a_{N2}}...X_N^{a_{NN}}).$$

We call ϕ_A the monomial map associated to A, that induces a rational map $\phi_A : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$. Again, we denote the spectral radius of A by

$$\rho(A) = \max\{|\lambda| : \lambda \in \mathbb{C} \text{ is an eigenvalue for} A\}.$$

It is immediate from the definition that if $A, B \in \mathrm{Mat}_{\mathbb{Z}}^{+}$ are matrices with associated monomial maps ϕ_A and ϕ_B, then

$$\phi_{AB}(P) = (\phi_A \circ \phi_B)(P) \text{ and } \phi_{A+B}(P) = \phi_A(P).\phi_B(P).$$

Definition 2.8: *Let $A \in GL_N(\mathbb{Q})$. A Jordan subspace for A is an A-invariant subspace of $\bar{\mathbb{Q}}^N$ corresponding to a single Jordan block of A. A Jordan subspace $V \subset \bar{\mathbb{Q}}^N$ with associated eigenvalue λ is called a maximal Jordan subspace if $|\lambda| = \rho(A)$ and if the dimension of V is maximal among the Jordan subspaces whose eigenvalue has magnitude equal to $\rho(A)$. We set*

$r(A) =$ *number of maximal Jordan subspaces,*
$\bar{r}(A) = \#\{\sigma(V) : V$ *is a maximal Jordan subspace for* A *and* $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})\}$,
$l(A) = \dim($*any maximal Jordan subspace*$) - 1$.

Thus, $\bar{r}(A)$ is the number of distinct $\bar{\mathbb{Q}}$-subspaces of $\bar{\mathbb{Q}}^N$ that are Galois conjugate to a maximal Jordan subspace of A, and so $\bar{r}(A) \geq r(A) \geq 1$, since A always has at least one maximal Jordan subspace.

Definition 2.9: *Let G be an algebraic subgroup of $\mathbb{G}_m^N$. We write $G(\bar{\mathbb{Q}})^{div}$ for the divisible hull of $G(\bar{\mathbb{Q}})$,*

$$G(\bar{\mathbb{Q}})^{div} = \{(\alpha_1, ..., \alpha_N) \in \mathbb{G}_m^N(\bar{\mathbb{Q}}) : (\alpha_1^n, ..., \alpha_N^n) \in G(\bar{\mathbb{Q}}) \text{ for some } n \geq 1\}.$$

Equivalently, $G(\bar{\mathbb{Q}})^{div}$ is the set of translates of $G(\bar{\mathbb{Q}})$ by points in $\mathbb{G}_m^N(\bar{\mathbb{Q}})_{tors}$.

From now on we denote $\mathcal{A} := \{A_1, ..., A_k\} \subset \mathrm{Mat}_{\mathbb{Z}}^+$, which induces $\Phi := \{\phi_1, ..., \phi_k\} \subset \mathrm{End}(\mathbb{G}_m^N)$, and also self-rational maps of $\mathbb{P}^N$ as described above. We set the simple notation $\hat{h}_\Phi := \hat{h}_{\mathcal{O}(1),\Phi}$ and $\hat{\mathbf{h}}_\Phi := \hat{\mathbf{h}}_{\mathcal{O}(1),\Phi}$.

We know that δ_Φ exists easily because $\deg(\phi \circ \psi) \leq \deg(\phi).\deg(\psi)$ for all ϕ, ψ rational maps on projective spaces, and this implies the convexity estimate resulting the convergence of the limit in the definition. So, throghout this section, we will also say that Φ has the property (*) if this limit can be found using just one map that satisfies certain inequalities. Technically, when it satisfies the following:

$$\exists\ \psi := \phi_{i_1} \circ ... \circ \phi_{i_t} \in \Phi_t \text{ such that } \delta_\Phi = \lim_{n\to\infty} \rho(\Phi_n)^{\frac{1}{n}} \text{ is equal to } \lim_{s\to\infty} \rho(\psi^s)^{\frac{1}{ts}} \text{ and}$$
$$\sup_{n\geq 1} \frac{\rho(\Phi_n)}{n^l \delta_\Phi^n} \leq \sup_{s\geq 1} \frac{\rho(\psi^s)}{(ts)^l \delta_\Phi^{ts}}\ \forall l \geq 0. \qquad (*)$$

This property is satisfied immediately when $k = 1$. For any k, sets of k monomial maps induced by diagonal matrices are examples that satisfy this property above. Effectivelly, proposition 21(c) of [29] says that $\delta_{\phi_A} = \rho(A)$, which yields $\rho(\Phi_n) = \max_{\phi\in\Phi_n} \rho(\phi) = (\max_{1\leq i\leq n} \rho(A_i))^n$, from where the property follows. Another easy way to obtain an infinite number of systems with such property, for matrices not necessarily diagonal, is to consider A_1 not diagonal, $g_2, .., g_k$ any polynomials in $\mathbb{Z}[X]$, and $A_i := g_i(A_1)$ for $i = 2, ..., k$.

The sets with such property satisfy some useful relations expressed in the following proposition.

Lemma 2.10: *Let $\mathcal{A}, \Phi, \psi \in \Phi_t$ satisfying the property (*) as above. Then:*

(a) $\delta_\Phi = \delta_\psi^{\frac{1}{t}}$.
(b) $l_\Phi = l_\psi$.
(c) $\{P \in \mathbb{P}^N(\bar{\mathbb{Q}}) | \hat{h}_\Phi(P) = 0\} \subset \{P \in \mathbb{P}^N(\bar{\mathbb{Q}}) | \hat{h}_\psi(P) = 0\}$.

Proof. (a) By property (*) follows that:

$$\delta_\Phi = \lim_{n\to\infty} \rho(\Phi_n)^{\frac{1}{n}} = \lim_{s\to\infty} \rho(\psi^s)^{\frac{1}{ts}} = \lim_{s\to\infty} (\rho(\psi^s)^{\frac{1}{s}})^{\frac{1}{t}} = \delta_\psi^{\frac{1}{t}}.$$

(b) By theorem 24 of [29] we know that l_ψ exists, and by property (*) again we have

$$l_\Phi = \inf\{l \geq 0 : \sup_{n\geq 1} \frac{\rho(\Phi_n)}{n^l \delta_\Phi^n} < +\infty\}$$
$$\leq \inf\{l \geq 0 : \sup_{s\geq 1} \frac{\rho(\psi^s)}{(ts)^l \delta_\Phi^{ts}} < +\infty\}$$

$$= \inf\{l \geq 0 : \sup_{s\geq 1} \frac{\rho(\psi^s)}{(ts)^l \delta_\psi^{\frac{ts}{t}}} < +\infty\}$$
$$= \inf\{l \geq 0 : \frac{1}{t^l} \sup_{s\geq 1} \frac{\rho(\psi^s)}{s^l \delta_\psi^s} < +\infty\}$$
$$= \inf\{l \geq 0 : \sup_{s\geq 1} \frac{\rho(\psi^s)}{s^l \delta_\psi^s} < +\infty\}$$
$$= l_\psi.$$

The inclusion $\{l \geq 0 : \sup_{n\geq 1} \frac{\rho(\Phi_n)}{n^l \delta_\Phi^n} < +\infty\} \subset \{l \geq 0 : \sup_{s\geq 1} \frac{\rho(\psi^s)}{(ts)^l \delta_\Phi^{ts}} < +\infty\}$ guarantees that $l_\Phi \geq l_\psi$ and hence $l_\Phi = l_\psi$.
(c) $P \in \mathbb{P}^N(\bar{\mathbb{Q}}), D := \mathcal{O}(1)$ and $\hat{h}_\Phi(P) = 0$, so by (a) and (b)

$$0 = \hat{h}_\Phi(P) = \limsup_{n\to\infty} \frac{1}{n^{(l_\Phi)}\delta_\Phi^n} \sum_{\phi\in\Phi_n} h_D(\phi(P))$$
$$= \limsup_{n\to\infty} \frac{1}{n^{(l_\psi)}\delta_\psi^{\frac{n}{t}}} \sum_{\phi\in\Phi_n} h_D(\phi(P))$$
$$\geq \limsup_{s\to\infty} \frac{1}{(ts)^{(l_\psi)}\delta_\psi^s} \sum_{\phi\in\Phi_{ts}} h_D(\phi(P))$$
$$= \frac{1}{t^{(l_\psi)}} \limsup_{s\to\infty} \frac{1}{s^{(l_\psi)}\delta_\psi^s} \sum_{\phi\in\Phi_{ts}} h_D(\phi(P))$$
$$\geq \frac{1}{t^{(l_\psi)}} \limsup_{s\to\infty} \frac{1}{s^{(l_\psi)}\delta_\psi^s} h_D(\psi^s(P))$$
$$= \frac{1}{t^{(l_\psi)}} \hat{h}_\psi(P)$$
$$\geq 0,$$

and thus $\hat{h}_\psi(P) = 0$ □

We are now able to generalize theorem 27, corollary 29 and corollary 31 of [29], now for systems with several rational maps.

Theorem 2.11: *Let $\mathcal{A} := \{A_1, ..., A_k\} \subset Mat_{\mathbb{Z}}^+$ be a set matrices whose set of respective associated monomial maps $\Phi := \{\phi_1, ..., \phi_k\}$ and dynamical degree $\delta_\Phi > 1$ satisfies the property (*) with $\psi \in \Phi_t$ induced by $B \in \mathcal{A}_t$. There is an algebraic subgroup $G \subset \mathbb{G}_m^N$ of dimension*

$$\dim G \geq N - \bar{r}(B)$$

such that

$$\{P \in \mathbb{G}_m^N(\bar{\mathbb{Q}}); \hat{h}_\Phi(P) = 0\} \subset G(\bar{\mathbb{Q}})^{\text{div}}.$$

Proof. By proposition 2.10, $\delta_\Phi = \delta_\psi^{\frac{1}{t}} > 1, \delta_\psi > 1$, and

$$\{P \in \mathbb{P}^N(\bar{\mathbb{Q}}) | \hat{h}_\Phi(P) = 0\} \subset \{P \in \mathbb{P}^N(\bar{\mathbb{Q}}) | \hat{h}_\psi(P) = 0\}.$$

So the theorem is true by theorem 27 of [29] for ψ. □

Corollary 2.12: *Let $\Phi := \{\phi_1, ..., \phi_k\} \subset End(\mathbb{G}_m^N)$ as above with $\delta_\Phi > 1$ satisfying (*), and let P be a point with $\hat{h}_\Phi(P) = 0$. Then there is a proper algebraic subgroup G of $\mathbb{G}_m^N$ with $\mathcal{O}_\Phi(P) \subset G$. In particular, the orbit $\mathcal{O}_\Phi(P)$ is not Zariski dense in $\mathbb{G}_m^N$.*

Proof. We are under the hypothesis of theorem 2.11, so such result is valid. As in the proof of theorem 27 of [29], exists an integral lattice $L \subset \mathbb{Z}^N$ such that the group G of theorem 2.11 is given by

$$G = G_L := \bigcap_{(e_1,...,e_N)\in L}\{X_1^{e_1}...X_N^{e_N} = 1\} \subsetneq \mathbb{G}_m^N.$$

Denote $Q^d = (y_1^d, ..., y_N^d)$ for $Q = (y_1, ..., y_d) \in \mathbb{G}_m^N(\bar{\mathbb{Q}})$. The assumption that $\hat{h}_\Phi(P) = 0$ implies that $\sum_i \hat{h}_\Phi(\phi_i(P)) = \delta_\Phi \hat{h}_\Phi(P) = 0$, which shows that $\mathcal{O}_\Phi(P) \subset G(\bar{\mathbb{Q}})^{\text{div}}$ from theorem 2.11. In particular,

$$\mathcal{O}_\Phi(P) \subset G(\bar{\mathbb{Q}})^{\text{div}} \cap \mathbb{G}_m^N(\mathbb{Q}(P)).$$

Let d be the number of roots of unity in $\mathbb{Q}(P)$ and $Q = (y_1, .., y_N)$ belonging to $G(\bar{\mathbb{Q}})^{\text{div}} \cap \mathbb{G}_m^N(\mathbb{Q}(P))$. This means that there is an $m \geq 1$ such that

$$y_1^{me_1}...y_N^{me_N} = 1 \text{ for all } (e_1, ..., e_N) \in L$$

So $y_1^{e_1}...y_N^{e_N}$ is an m-th root of unity, thus $m|d$ and $Q^d \in G = G_L$, and then $Q \in G_{dL} \subsetneq \mathbb{G}_m^N$. And we just proved that $\mathcal{O}_\Phi(P) \subset G_{dL} \subsetneq \mathbb{G}_m^N$. □

Corollary 2.13: *Let $\Phi := \{\phi_1, ..., \phi_k\} \subset End(\mathbb{G}_m^N)$ be monomial maps induced by $\mathcal{A} := \{A_1, ..., A_k\} \subset Mat_{\mathbb{Z}}^+$ with $\delta_\Phi > k$ and satisfying the property (*) with $\psi \in \Phi_t$ induced by $B \in \mathcal{A}_t$ whose characteristic polynomial is irreducible over $\mathbb{Q}$. Let $P \in \mathbb{G}_m^N(\bar{\mathbb{Q}})$. Then*

$$\hat{h}_\Phi(P) = 0 \iff \#\mathcal{O}_\Phi(P) < +\infty.$$

Proof. If $\#\mathcal{O}_\Phi(P) < +\infty$ then its straightfoward that $\hat{h}_\Phi(P) = 0$ because $\delta_\Phi > k$. Now we suppose $\hat{h}_\Phi(P) = 0$. So

$$\mathcal{O}_\Phi(P) \subset \{Q \in \mathbb{G}_m^N(\bar{\mathbb{Q}}) | \hat{h}_\Phi(Q) = 0\} \subset G(\bar{\mathbb{Q}})^{\text{div}} \cap \mathbb{G}_m^N(\mathbb{Q}(P))$$

from theorem 2.11, where $\dim G = N - \bar{r}(B) = 0$, since the characteristic polynomial is irreducible and all its roots are disjoints.

So $G(\bar{\mathbb{Q}})^{\text{div}} \cap \mathbb{G}_m^N(\mathbb{Q}(P)) = \mathbb{G}_m^N(\mathbb{Q}(P))_{\text{tors}}$, the set of points whose coordinates are roots of unity in the field $\mathbb{Q}(P)$. Such set is finite, and so is $\mathcal{O}_\Phi(P)$. □

2.3 Effective bounds for the canonical height of non-periodic points

For $k = 1, A_1 = A$, Silverman observes in [29, remark 30] that it should be possible to use an effective form of Baker's theorem to prove effective versions of the results above, with an effective computable constant
$C = C(A, h(P)) > 0$ such that

$$\mathcal{O}_{\phi_A}(P) \text{ Zariski dense } \implies \hat{h}_{\phi_A}(P) > C.$$

In this section we work out the cases for that A has real Jordan form, i.e, all of its eigenvalues are real. For this we will make use of an improvement of effective classical Baker's theorem that is due to P. Philippon and M. Waldschmidt, and can be seen for example in [4, chapter 18, theorem 1.1]. This is stated as follows

Theorem 2.14: *Let $\alpha_1, ..., \alpha_n$ be non-zero algebraic numbers that are different from 1, and let $\beta_1, ..., \beta_n$ algebraic numbers not all zero, and $\log \alpha_1, ..., \log \alpha_n$ logarithmic representatives such that $\pi i, \log \alpha_1, ..., \log \alpha_n$ are linearly independent over $\bar{\mathbb{Q}}$ and $\Lambda := \beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 + ... + \beta_n \log \alpha_n$ does not vanish.*

Let D be a positive integer, $A, A_1, A_2, ..., A_n$ be positive real numbers, and B satisfying

$$D \geq [\mathbb{Q}(\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n) : \mathbb{Q}]$$

$$A_j \geq \max\{H(\alpha_j), \exp|\log \alpha_j|, e^n\}, \qquad 1 \leq j \leq n$$

$$A := \max\{A_1, ..., A_n, e^e\}$$

$$B := \max\{H(\beta_j); 1 \leq j \leq n\}$$

Then

$$|\Lambda| \geq e^{-U},$$

where

$$U = -C_{11}(n).D^{n+2}.\log A_1 ... \log A_n.(\log B + \log\log A),$$

$$C_{11}(n) \geq 2^{8n+53}.n^{2n},$$

and $H(\alpha)$ denotes the maximun of absolute values of the coefficients of the minimum polynomial of α over $\mathbb{Q}$.

Proof. Can be found in [4, chapter 18, section 4].

□

Theorem 2.15: *Let $\phi : \mathbb{G}_m^N \to \mathbb{G}_m^N$ be a monomial map induced by A with real eigenvalues, $l(A) \geq 1$ and $\delta_\phi > 1$, and let $P \in \mathbb{G}_m^N(\bar{\mathbb{Q}})$ be a point with orbit $\mathcal{O}_\phi(P)$ Zariski dense. Then there is an effective computable positive constant C depending on A and $h(P)$ such that $\hat{h}_{\phi_A}(P) > C$.*

Proof. We denote

$$\max{}^+((u_1, ..., u_N)) = \max(0, u_1, ...u_N) \text{ for all } (u_1, ..., u_N) \in \mathbb{R}^N,$$

and call

$$\log ||(y_1, ..., y_N)||_v := (\log ||y_1||_v, ..., \log ||y_N||_v) \text{ for all } (y_1, ..., y_N) \in \overline{\mathbb{Q}}^N,$$

and we call $P = (x_1, ..., x_N)$.

Starting from $A \in \mathrm{Mat}_N^+(\mathbb{Z})$, we can write $\bar{\mathbb{Q}}^N = V_1 + ... + V_r + Z$, where $V_1, ..., V_r$ are the distinct maximal Jordan subspaces and $Z := V_{r+1} + ... + V_t$ is the sum of all the other Jordan subspaces for A, as in the equation right below (25) in the proof of theorem 27 of [29].

In [25], we have an effective algorithm for finding a basis where A has Jordan normal form defined over some algebraic extension of $\mathbb{Q}$. In other words, for each $1 \leq i \leq t$, we can find the basis $\{v_1^{(i)}, ..., v_{t_i}^{(i)}\}$ used to put $A_{|V_i}$ in the Jordan normal form, with the canonical coordinates denoted by $v_j^{(i)} = (a_{1j}^{(i)}, ..., a_{Nj}^{(i)})$ effectively computable defined over some algebraic extension of $\mathbb{Q}$, that we call K.

Denoting again $l := l_{\phi_A} = l(A)$, $r(A) = r$, and $\rho := \rho(A)$, the spectral radius of A, we have by (17) of [29, proof of theorem 27] that there exists an infinite subset $\mathcal{N} \subset \mathbb{N}$ and $B \in \mathrm{Mat}_N(\mathbb{R})$ such that the limit $B = \lim_{n \in \mathcal{N}} \dfrac{A^n}{n^l \rho^n}$ is satisfied.

To suppose $\mathcal{O}_{\phi_A}(P)$ Zariski dense implies that $\hat{h}_{\phi_A}(P) > 0$ by theorem 2.11. Extending K, we can suppose P defined over K. By (22) on the proof of theorem 27 of [29] we have that

$$\hat{h}_{\phi_A}(P) = \sum_{v \in M_K} \max{}^+(B \log ||P||_v) > 0$$

and

$$\log ||P||_v \notin \ker_{\mathbb{C}}(B) \ \forall \ v \text{ with nonzero } v\text{-component in the above sum.}$$

By (26) of [29, theorem 27] we can see that $\log ||P||_v \notin \ker_{\mathbb{C}}(B)$ implies that

$$\log ||P||_v \in (< v_{t_1}^{(1)}, ..., v_{t_r}^{(r)} > \bigcup \ker_{\mathbb{C}}(B)) - \ker_{\mathbb{C}}(B).$$

For any $v \in M_K$, we can make an effective change of basis using Cramer rule to obtain $\log ||P||_v$ in the basis where A is in Jordan form, in other words, to obtain $c_{it_i,v}(P) \in \mathbb{C}, i \leq r, b \in \ker_{\mathbb{C}}(B)$, such that $\log ||P||_v = \sum_{i \leq r} c_{it_i,v}(P) v_{t_i}^{(i)} + b$, and so

$$\log ||P||_v = (\sum_{i \leq r} c_{it_i,v}(P) a_{1t_i}^{(i)}, ..., \sum_{i \leq r} c_{it_i,v}(P) a_{Nt_i}^{(i)}) + b.$$

Effectively, let $J(A)$ be the $(N \times N)$-matrix $(a_{il}^{(j)})$ with lines indexed by $1 \leq i \leq N$ and columns indexed by $(j, l); 1 \leq j \leq t, 1 \leq l \leq t_j$ in lexicographic order. So $c_{it_i,v}(P)$ are coodinates of the vector solution z for the linear system $J(A).z = \log ||P||_v$. Using Cramer Rule we see that these solutions have the form $c_{it_i,v}(P) = \sum_{j \leq N} d_{ij,v}(P) \log ||x_j||_v$, for $d_{ij,v}(P) \in K$ effectively computable depending only on A.

From the equation above (25) in the proof of theorem 27 of [29], we see that $Bv_{t_i}^{(i)} = \frac{\xi_i}{\rho^l l!} v_1^{(i)}$ for $\xi_i \in \mathbb{C}, |\xi_i| = 1$ for $1 \leq i \leq r$, and $\xi_i = \lim_n \dfrac{\lambda^n}{\rho^n}$ for some eigenvalue λ of A. The eigenvalues of A are real by hypothesis, therefore ξ_i is equal to 1 or -1. Then for this case we have that

$$B \log ||P||_v = \sum_{i \leq r} c_{it_i,v}(P) B v_{t_i}^{(i)} = \sum_{i \leq r} c_{it_i,v}(P) \frac{\xi_i}{\rho^l l!} v_1^{(i)}$$
$$= \frac{1}{\rho^l l!} (\sum_{i \leq r} c_{it_i,v}(P) a_{11}^{(i)} \xi_i, ..., \sum_{i \leq r} c_{it_i,v}(P) a_{N1}^{(i)} \xi_i) =$$
$$\frac{1}{\rho^l l!} (\sum_j (\sum_{i \leq r} a_{11}^{(i)} \xi_i d_{ij,v}(P)) \log ||x_j||_v, ..., \sum_j (\sum_{i \leq r} a_{N1}^{(i)} \xi_i d_{ij,v}(P)) \log ||x_j||_v),$$

which yields

$$\hat{h}_{\phi_A}(P) = \sum_{v \in M_K} \max{}^+(B \log ||P||_v)$$
$$= \frac{1}{\rho^l l!} \sum_{v \in M_K} \max\{0, \sum_j (\sum_{i \leq r} a_{l1}^{(i)} \xi_i d_{ij,v}(P)) \log ||x_j||_v; l \leq N\} > 0.$$

Now consider $S \in M_K$ the finite set of places v such that

$$\max\{\sum_j (\sum_{i \leq r} a_{l1}^{(i)} \xi_i d_{ij,v}(P)) \log ||x_j||_v; l \leq N\} > 0,$$

(for example $S \subset T := \{v; ||x_j||_v \neq 1 \text{ for some } j\}$ finite, with $S_0, S_\infty, T_0, T_\infty$ the non-archimedean and archimedean places of each set).

Suppose that in this case the maximum in each v-component of the sum is achieved for $l = l_v$, and denote $D_{ij,v} := a_{l_v 1}^{(i)}.d_{ij,v}(P)$. We use the notation $N(v)$ for the norm of the ideal in K corresponding to the place v.

We aim to make use of the fact that $\{\log N(v); v \in S_0\}$ is linearly independent over $\overline{\mathbb{Q}}$ to apply theorem 2.14. We start computing

$\hat{h}_{\phi_A}(P) = \dfrac{1}{\rho^l l!} \sum_{v\in S} \sum_j (\sum_{i\leq r} a^{(i)}_{l_v 1}\xi_i d_{ij,v}(P)) \log ||x_j||_v$
$= \dfrac{1}{\rho^l l!} \sum_{v\in S} \sum_j (\sum_{i\leq r} D_{ij,v}\xi_i) \log ||x_j||_v$
$= \dfrac{1}{\rho^l l!} \sum_{v\in S_0} [\sum_{i,j} D_{ij,v}\xi_i(-v(x_j)) + \sum_{i,j,u\in S_\infty} D_{ij,u}\xi_i.v_{u,j}]. \log N(v),$

where $v_{u,j} \in \mathbb{Q}$ and $\sum_{u\in S_\infty} v_{u,j} = v(x_j)$ for $v \in S_0$ a non-archimedean place. For instance we multiply the point by a integer constant, so we suppose $P \in \mathbb{P}^N(\mathcal{O}_K)$, so that every $v_{u,j}$ is nonnegative.

Using the facts that $H_{\bar{\mathbb{Q}}}(a_1 + ... + a_n) \leq nH_{\bar{\mathbb{Q}}}(a_1)...H_{\bar{\mathbb{Q}}}(a_n)$, and that $H(\alpha) \leq (2.H_{\bar{\mathbb{Q}}}(\alpha))^{[K:\mathbb{Q}]}$[34, first inequation of page 77 and lemma 3.11], where $H_{\bar{\mathbb{Q}}} := \exp h$ is the Weil multiplicative height, we have that

$H(\sum_{i,j} D_{ij,v}\xi_i(-v(x_j)) + \sum_{i,j,u\in S_\infty} D_{ij,u}\xi_i.v_{u,j})$

$\leq [2.H_{\bar{\mathbb{Q}}}(\sum_{i,j} D_{ij,v}\xi_i(-v(x_j)) + \sum_{i,j,u\in S_\infty} D_{ij,u}\xi_i.v_{u,j})]^{[K:\mathbb{Q}]}$

$\leq [4Nr[K:\mathbb{Q}] \prod_{i,j,u\in S_\infty} H_{\bar{\mathbb{Q}}}(D_{ij,v}\xi_i(-v(x_j))H_{\bar{\mathbb{Q}}}(D_{ij,u}\xi_i.v_{u,j})]^{[K:\mathbb{Q}]}$

$\leq (4Nr.[K:\mathbb{Q}]. \max_{i,j} |v(x_j)|^{2Nr[K:\mathbb{Q}]}. \max_{w\in T} H_{\bar{\mathbb{Q}}}(D_{ij,w})^{2Nr[K:\mathbb{Q}]})^{[K:\mathbb{Q}]}$

$\{4Nr.[K:\mathbb{Q}]. \max_j |v(x_j)|.(N-1)! \max_{i,j,l} H_{\bar{\mathbb{Q}}}(a^{(j)}_{il})H_{\bar{\mathbb{Q}}}(\frac{1}{\det J(A)})\}^{2N^2r[K:\mathbb{Q}]^2}$

and then we have by theorem 2.14 that

$$\hat{h}_{\phi_A}(P) \geq \frac{1}{\rho^l l!} \exp(-E. \prod_{v\in T_0} \log A_v.[\log D + \max_{v\in T_0} \log\log A_v]),$$

where

$A_v ::= \max\{\exp|\log N(v)|, e^{\#T_0}, e^e\} \leq N(v)^{12.\#T_0}$ for every $v \in T_0$,

$E := C_{11}(\#T_0).[K:\mathbb{Q}]^{\#T_0+2}$,

$D := \max_{v\in T_0}\{4Nr.[K:\mathbb{Q}]. \max_j |v(x_j)|.(N-1)! \max_{i,j,l} H_{\bar{\mathbb{Q}}}(a^{(j)}_{il})H_{\bar{\mathbb{Q}}}(\frac{1}{\det J(A)})\}^{2N^2r[K:\mathbb{Q}]^2}$,

for $T_0 = \{v \in M_K; ||x_j||_v \neq 1 \text{ for some } j\}$, $C_{11}(n) = 2^{8n+53}.n^{2n}$.

We can note that

$\#T_0 = \sum_{v\in T_0} 1 \leq 2 + \sum_j \sum_{v\in T_0} \log\max\{1, N(v)^{|v(x_j)|}\} \leq 4 + Nh_K(P)$,

$|v(x_j)| \leq \log 3 + \sum_j \sum_{v\in T_0} \log\max\{1, N(v)^{|v(x_j)|}\} \leq 3 + Nh_K(P)$,

$\log N(v) \leq \log N(v)^{|v(x_j)|} \leq \sum_j \sum_{v\in T_0} \log\max\{1, N(v)^{|v(x_j)|}\} \leq 2 + Nh_K(P)$,

where $h_K(P) = [K:\mathbb{Q}].h(P)$, and we used the product formula to conclude $\sum_{v\in T_0} \log\max\{1, N(v)^{|v(x_j)|}\} \leq 2+\sum_{v\in T_0} \log N(v)^{v(x_j)} = \sum_{v\in T_\infty} \log ||x_j||_v \leq \sum_{v\in T_\infty} \log\max\{1, ||x_j||_v\}$ $\leq h_K(x_j) \leq h_K(P)$.

Making C equal to

$$\frac{1}{2\rho^l l!} \exp(-E'.(\log A')^{(4+Nh_K(P))}.[\log D' + \log\log A']),$$

where

$$A' ::= e^{(2+Nh_K(P))(12.(4+Nh_K(P)))},$$

$$E' := C_{11}(4+Nh_K(P)).[K:\mathbb{Q}]^{(6+Nh_K(P))},$$

$$D' := \{4Nr.[K:\mathbb{Q}].(3+Nh_K(P)).(N-1)!\max_{i,j,l} H_{\bar{\mathbb{Q}}}(a_{il}^{(j)})H_{\bar{\mathbb{Q}}}(\tfrac{1}{\det J(A)})\}^{2N^2r[K:\mathbb{Q}]^2},$$

we have what we wanted, since the above constants are efectively computable depending only A and $h(P)$, and the heights $h(P), \hat{h}_{\phi_A}(P)$ do not change when multiplyed by an integer constant. □

Corollary 2.16: *Let $\phi : \mathbb{G}_m^N \to \mathbb{G}_m^N$ be a monomial map induced by A with real eigenvalues, irreducible characteristic polynomial over $\mathbb{Q}$ and $\delta_\phi > 1$, and let $P \in \mathbb{G}_m^N(\bar{\mathbb{Q}})$ be a point with infinite orbit $\mathcal{O}_\phi(P)$. Then there is an effective computable positive constant C depending on A and $h(P)$ such that $\hat{h}_{\phi_A}(P) > C$.*

Proof. By corollary 2.13, we must have $\hat{h}_{\phi_A}(Q) > 0$. Now we follow the proof of theorem 2.15, with some reductions and changes. Again we have $\bar{\mathbb{Q}}^N = V_1 + ... + V_r + Z$, where $V_1, ..., V_r$ are the distinct maximal Jordan subspaces and $Z := V_{r+1} + ... + V_t$ is the sum of all the other Jordan subspaces for A. We assume $v_i := (a_1^{(i)}, ..., a_N^{(i)})$ puts $A_{|V_i}$ in the Jordan normal form. Again $B := \lim_{n\in\mathcal{N}} \dfrac{A^n}{n^l\rho^n} = \lim_{n\in\mathcal{N}} \dfrac{A^n}{\rho^n}$, but now the binomial equation between (24) and (25) in the proof of theorem 27 of [29] shows that

$$Bv_i = \lim_n \frac{\lambda^n}{\rho^n} v_i = \xi_i v_i, \xi_i \in \{-1, 1\}\ \forall\ 1 \le i \le r.$$

Similarly to theorem 2.15, we have that $\log ||P||_v = \sum_{i\le r} c_{i,v}(P)v_i + b$ for $c_{i,v}(P) \in \mathbb{C}, i \le r, b \in \ker_{\mathbb{C}}(B)$, effectively computable and K a number field such that the Jordan normal form of A and the point P are defined over it. So

$$\log ||P||_v = (\textstyle\sum_{i\le r} c_{i,v}(P)a_1^{(i)}, ..., \sum_{i\le r} c_{i,v}(P)a_N^{(i)}) + b.$$

Again $c_{i,v}(P) = \sum_{j\le N} d_{ij,v}(P)\log ||x_j||_v$, for $d_{ij,v}(P) \in K$ effectively computable by Cramer rule, depending only on A implies that

$$B\log ||P||_v = (\textstyle\sum_j(\sum_{i\le r} a_1^{(i)}\xi_i d_{ij,v}(P))\log ||x_j||_v, ..., \sum_j(\sum_{i\le r} a_N^{(i)}\xi_i d_{ij,v}(P))\log ||x_j||_v).$$

For some $S \subset T := \{v; ||x_j||_v \neq 1 \text{ for some } j\}$ finite, $l_v \in \{1, ..., r\}$, denoting $D_{ij,v} := a_{l_v}^{(i)}.d_{ij,v}(P)$, and

$$A ::= e^{(2+Nh_K(P))(12.(4+Nh_K(P)))}$$

$$E := C_{11}(4+Nh_K(P)).[K:\mathbb{Q}]^{(6+Nh_K(P))},$$

$$D := \{4Nr.[K:\mathbb{Q}].(3+Nh_K(P)).(N-1)!\max_{i,j,l} H_{\bar{\mathbb{Q}}}(a_{il}^{(j)})H_{\bar{\mathbb{Q}}}(\tfrac{1}{\det J(A)})\}^{2N^2r[K:\mathbb{Q}]^2},$$

it follows similarly as before that

$$\hat{h}_{\phi_A}(P) = \frac{1}{\rho^l l!}\textstyle\sum_{v\in S}\sum_j(\sum_{i\le r} a_{l_v}^{(i)}\xi_i d_{ij,v}(P))\log ||x_j||_v$$

$$= \frac{1}{\rho^l l!}\textstyle\sum_{v\in S}\sum_j(\sum_{i\le r} D_{ij,v}\xi_i)\log ||x_j||_v$$

$$= \frac{1}{\rho^l l!} \sum_{v \in S_0} [\sum_{i,j} D_{ij,v} \xi_i (-v(x_j)) + \sum_{i,j,u \in S_\infty} D_{ij,u} \xi_i . v_{u,j}]. \log N(v)$$
$$> \frac{1}{2} \exp(-E.(\log A)^{(4+Nh_K(P))}.[\log D + \log\log A])$$

□

Remark 2.17: As Silverman did presume in [29, remark 30], we see that the explicit constants obtained for theorem 2.15 and corollary 2.16 show that if the height of a point is very large, then the referred constants for this point will be smaller, and can be made even smaller if the coordinates of the initial matrix are big. In fact, working out the proof above, the constants C can have a form similar to

$$\frac{1}{2.\rho^l l! \{(4+Nh_K(P))C(A)4Nr[K:\mathbb{Q}](N-1)!\}^{\{c_1.[K:\mathbb{Q}]^2.16.N^2 r.(4+Nh_K(P))\}^{(10(6+Nh_K(P)))}}},$$

where c_1 does not depend on anything, and $C(A) := \max_{i,j,l} H_{\bar{\mathbb{Q}}}(a_{il}^{(j)}) H_{\bar{\mathbb{Q}}}(\frac{1}{\det J(A)})$ depends on the absolute heights of the coordinates of the vectors which transform A in its Jordan form. More sharp effective versions of Baker theorem should lead to more sharp versions of the above constant.

Chapter 3

Heights, Variation, and Intersection numbers

In this chapter we study families of varieties endowed with polarized canonical eigensystems of several maps, inducing canonical heights on the dominating variety as well as on the "good" fibers of the family. We show explicitely the dependence on the parameter for global and local canonical heights when the fibers change, extending previous works of J. Silverman and others. When the base variety is a curve with a Weil height that corresponds to a divisor of degree one, then we can associate canonical heights on fibers and the Weil height on the base variety with the canonical height on the generic fiber by a limit. Finally, fixing an absolute value $v \in K$ and a variety V/K, we descript the Kawaguchi's canonical local height $\hat{\lambda}_{V,E,\mathcal{Q},}(.,v)$ as an intersection number, provided that the polarized system $(V, \mathcal{Q})$ has a certain weak Néron model over $\mathrm{Spec}(\mathcal{O}_v)$ to be defined. With this we extend Néron's work stregthening Silverman's results for systems having only one map.

3.1 Two variation theorems

The following notation will be used for this section and the next section.

- K : A global field with characteristic 0 and a complete set of proper absolute values satisfying a product formula. We will call such a field a global height field.
- $M := M_{\bar{K}}$: The set of absolute values on $\bar{K}$ extending those on K.
- T/K : a smooth projective variety.
- h_T : A fixed Weil height function on T associated to an ample divisor, chosen to satisfy $h_T \geq 0$.
- $\mathcal{V}/K$: a smooth projective variety.
- π : a morphism $\pi : \mathcal{V} \to T$ defined over K whose generic fiber is smooth and geometrically irreducible.
- ϕ_i : rational maps $\phi_i : \mathcal{V}/T \dashrightarrow \mathcal{V}/T$ defined over K for $i = 1, ..., k$, such that ϕ_i is morphism on the generic fiber of $\mathcal{V}/T$. Our assumption that ϕ_i is on $\mathcal{V}/T$ means that $\pi \circ \phi_i = \pi$.
- η : A divisor class $\eta \in \mathrm{Pic}(\mathcal{V}) \otimes \mathbb{R}$ satisfying $\bigotimes_{i=1}^{k} \phi_i^* \eta = \alpha\eta$ for some real $\alpha > k$.
- T^0 : the subset of T having good fibers in the sense that

$$T^0 = \{t \in T : \mathcal{V}_t \text{ is smooth and } (\phi_i)_t : \mathcal{V}_t \to \mathcal{V}_t \text{ is a morphism } \forall i\}.$$

where $\mathcal{V}_t := \pi^{-1}(t)$.

- $\mathcal{Q}_n$ for $n \in \mathbb{N}$: the sets of iterates of functions defined as $\mathcal{Q}_0 = \{\text{Id}\}$, $\mathcal{Q}_1 = \mathcal{Q} = \{\phi_1, ..., \phi_k\}$, and $\mathcal{Q}_n = \{\phi_{i_1} \circ ... \circ \phi_{i_n}; i_j = 1, ..., k\}$.
- $(\mathcal{Q}_n)_t$ for $n \in \mathbb{N}$: the sets of iterates of functions defined as $(\mathcal{Q}_0)_t = \{\text{Id}\}, (\mathcal{Q}_1)_t = (\mathcal{Q})_t = \{(\phi_1)_t, ..., (\phi_k)_t\}$, and $(\mathcal{Q}_n)_t = \{(\phi_{i_1})_t \circ ... \circ (\phi_{i_n})_t; i_j = 1, ..., k\}$, the restrictions of the ϕ_i's to the fiber $\mathcal{V}_t$.

We also assume that the divisor class $\bigotimes_{i=1}^k \phi_i^*\eta - \alpha\eta$ is fibral, which means that it can be represented by a divisor Δ such that $\pi(|\Delta|) \neq T$, or equivalently, there exists a divisor D in $\text{Div}(T)$ such that $\pi^*D > \Delta > -\pi^*D$.

For any $t \in T^0$ we let $i_t : \mathcal{V}_t \to \mathcal{V}$ be the natural inclusion, and then by definition $i_t^*\eta = \eta_t$. The fiber $\mathcal{V}_t$ is irreducible for each $t \in T^0$. If the support of a fibral divisor includes an irreducible fiber, it is always possible to find a linearly equivalent divisor which does not include that fiber. This implies that

$$\bigotimes_{i=1}^k (\phi_i)_t^*\eta_t = \alpha\eta_t \in \text{ Pic}(\mathcal{V}_t) \otimes \mathbb{R} \text{ for all } t \in T^0.$$

From this and from theorem 1.2.1 of [16], we have that for each $t \in T^0$ there is a canonical height

$$\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t} : \mathcal{V}_t(\bar{K}) \to \mathbb{R}.$$

We now fix a Weil height

$$h_{\mathcal{V},\eta} : \mathcal{V}(\bar{K}) \to \mathbb{R}$$

associated to η. It follows from the properties of the height functions of Kawagushi [16], and functoriality of the Weil height function, that

$$\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t} = h_{\mathcal{V}_t,\eta_t} + O(1) = h_{\mathcal{V},\eta} \circ i_t + O(1).$$

where the $O(1)$ depend on t. For $t \in T$, any two canonical heights $\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t}$ differ from the Weil Height $h_{\mathcal{V},\eta}$ by a bounded amount constant depending on t. For applications, it is important to have an explicit bound for such constant. Let us check how we can see this dependence explicitly in the following theorem, which is an extension of theorem 3.1 of [9], for one map's systems. Such case was a more general form of the work of Silverman and Tate for families of abelian varieties done in [30].

Theorem 3.1: *With notation as above, there exist constants c_1, c_2 depending on the family $\mathcal{V} \to T$, the system $\mathcal{Q}$, the divisor class η, and the choice of Weil height functions $h_{\mathcal{V},\eta}$ and h_T, so that*

$$|\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t}(x) - h_{\mathcal{V},\eta}(x)| \leq c_1 h_T(t) + c_2 \quad \textit{for all } t \in T^0 \textit{ and all } x \in \mathcal{V}_t.$$

Proof. From the definition of T^0, one can conclude that $\mathcal{V}^0 := \pi^{-1}(T^0)$ is smooth, so we are able to apply some resolution of singularities to $\mathcal{V}$ without changing $\mathcal{V}^0$. Moreover, although the maps $\phi_i : \mathcal{V} \dashrightarrow \mathcal{V}$ are merely rational, they are morphisms on $\mathcal{V}^0$. This means we can blow-up $\mathcal{V}$ to produce:

(i) smooth projective varieties $\tilde{\mathcal{V}}_i$,

(ii) birational morphisms $\psi_i : \tilde{\mathcal{V}}_i \to \mathcal{V}$ which are isomorphisms on $\mathcal{V}^0$

(iii) morphisms $\xi_i : \tilde{\mathcal{V}}_i \to \mathcal{V}$ which extend the rational maps $\phi_i \circ \psi_i : \tilde{\mathcal{V}}_i \to \mathcal{V}$.
The existence of $\tilde{\mathcal{V}}_i$ with these properties follows from [13] II.7.17.3 and II.7.16, except that $\tilde{\mathcal{V}}_i$ might be singular. We then use Hironaka's resolution of singularities to make $\tilde{\mathcal{V}}_i$ smooth and we have the desired properties.

Next we choose a divisor $E \in \mathrm{Div}(\mathcal{V}) \otimes \mathbb{R}$ in the divisor class of η, and we let $H \in \mathrm{Div}(T)$ be the ample divisor used to define h_T. Our assumption that $\bigotimes_{i=1}^k \phi_i^* \eta - \alpha\eta$ is fibral guarantees the existence of a divisor $D \in \mathrm{Div}(T) \otimes \mathbb{R}$ with

$$\pi^* D > \sum_{i=1}^k \phi_i^* E - \alpha E > -\pi^* D.$$

We also choose an integer $n > 0$ so that the divisors

$$nH + D \text{ and } nH - D \text{ are both ample on } T.$$

The height function with respect to a positive divisor is bounded below out of the support of the divisor, and for an ample divisor such height is everywhere bounded below. So the last assertions imply that

$$|h_{\mathcal{V}, \bigotimes_{i=1}^k \phi_i^* E - \alpha E}| \leq n h_{\mathcal{V}, \pi^* H} + O(1) = n h_{T,H} \circ \pi + O(1)$$
$$\text{for all points } P \text{ in } \mathcal{V}^0 - |D|.$$

Now let $x \in \mathcal{V}^0$ be any point, and let $\tilde{x}_i \in \tilde{\mathcal{V}}_i$ satisfying $\psi_i(\tilde{x}_i) = x$. In the following computation, we write $O(1)$ for a quantity that is boundable in terms of the family $\mathcal{V} \to T$, the maps ϕ_i, the divisor class η, and the choice of Weil height functions $h_{\mathcal{V},\eta} = h_{\mathcal{V},E}$ and $h_T = h_{T,H}$.The most important here is that $O(1)$ is independent of $x \in \mathcal{V}^0$.

$$
\begin{aligned}
&|\textstyle\sum_i h_{\mathcal{V},E}(\phi_i(x)) - \alpha h_{\mathcal{V},E}(x)| \\
&= |\textstyle\sum_i h_{\mathcal{V},E}((\phi_i \circ \psi_i)(\tilde{x}_i)) - \alpha h_{\mathcal{V},E}(x)| \\
&= |\textstyle\sum_i h_{\mathcal{V},E}(\xi_i(\tilde{x}_i)) - \alpha h_{\mathcal{V},E}(x)| \\
&= |\textstyle\sum_i h_{\mathcal{V},\xi_i^* E}(\tilde{x}_i) - \alpha h_{\mathcal{V},E}(x)| + O(1) \\
&= |\textstyle\sum_i h_{\mathcal{V},\psi_i^* \phi_i^* E}(\tilde{x}_i) - \alpha h_{\mathcal{V},E}(x)| + O(1) \\
&= |\textstyle\sum_i h_{\mathcal{V},\phi_i^* E}(\psi_i(\tilde{x}_i)) - \alpha h_{\mathcal{V},E}(x)| + O(1) \\
&= |\textstyle\sum_i h_{\mathcal{V},\phi_i^* E}(x) - \alpha h_{\mathcal{V},E}(x)| + O(1) \\
&= |h_{\mathcal{V}, \bigotimes_{i=1}^k \phi_i^* E - \alpha E}| + O(1) \\
&\leq n h_{T,H}(\pi(x)) + O(1).
\end{aligned}
$$

This inequality is valid on $\mathcal{V}^0$ out of the suport of D. Choosing different divisors E in the class of η to move D, we then obtain the inequality for all points in $\mathcal{V}^0$, in a similar way of [30], pages 203-204. This proves

(*) $|\sum_i h_{\mathcal{V},E}(\phi_i(x)) - \alpha h_{\mathcal{V},E}(x)| \leq n h_{T,H}(\pi(x)) + O(1)$ for all $x \in \mathcal{V}^0$.

In order to complete the proof, we remember theorem 1.2.1 of [16], which says in similar conditions that

$$(1): \quad |\sum_{i=1}^k h_L(f_i(x)) - d h_L(x)| \leq C \text{ implies } (2): \quad |\hat{h}_{L,\mathcal{F}}(x) - h_L(x)| \leq \frac{C}{d-k}.$$

We use (*) in place (1) and obtain

$$|\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t}(x) - h_{\mathcal{V},\eta}(x)| \leq \frac{nh_{T,H}(\pi(x)) + O(1)}{\alpha - k}.$$

□

By $\lambda_{\mathcal{V},E}$ we denote a Weil local height function

$$\lambda_{\mathcal{V},E} : (\mathcal{V} - |E|) \times M_{\bar{K}} \to \mathbb{R}$$

associated to the divisor E. We can now give a similar estimate for the difference between the canonical local height $\hat{\lambda}_{\mathcal{V}_t,E_t,(\mathcal{Q})_t}$ defined by Kawaguchi in theorem 4.2.1 of [16] and a given Weil local height $\lambda_{\mathcal{V},E}$, generalizing Lang's result [20] for abelian varieties. Here we make extensive use of the notation in theorem 7.3 and corollary 7.4 of [27], and obtain a local version of theorem 3.1. So 3.1 can be again deduced adding the following theorem over all absolute values of K and applying theorem 4.3.1 of [16]. Moreover, we note that one may skip to use resolution of singularities to prove the previous theorem just proving the next and adding up local contributions, even obtaining a stronger result, since avoiding to use resolution of singularities gives us that theorem 3.1 is true in any characteristic, using the machinery of [27]. So theorem 3.1 is valid any global field K. For basic facts about local height functions, M_K-bounded functions and M_K- constants, see [20], chapter 10. We here use freely terminology from [27].

Theorem 3.2: *With notation as above, fix a divisor $E \in \mathrm{Div}(\mathcal{V}) \otimes \mathbb{R}$ in the class of η and a Weil local height function $\lambda_{\mathcal{V},E}$. Let U be defined as the following set*

$$\{t \in T : \mathcal{V}_t \text{ is smooth}, (\phi_i)_t : \mathcal{V}_t \to \mathcal{V}_t \text{ are morphisms }, E_t \text{ is a divisor on } \mathcal{V}_t, \text{ and } \textstyle\sum_i (\phi_i)_t^* E_t \sim \alpha E_t\}.$$

(The condition that E_t be a divisor on $\mathcal{V}_t$ means that $|E|$ contains no component of $\mathcal{V}_t$, see [13], III.9.8.5.)

Let $\partial U := T - U$ be the complement of U, and let $\lambda_{\partial U}$ be a local height function associated to ∂U as described in [27].

It is possible to choose canonical local heights $\hat{\lambda}_{\mathcal{V}_t,E_t,(\mathcal{Q})_t}$ as described in Theorem 4.2.1 of [16], one for each $t \in U$, in such a way that

$$|\hat{\lambda}_{\mathcal{V}_t,E_t,(\mathcal{Q})_t}(x,v) - \lambda_{\mathcal{V},E}(x,v)| \leq c\lambda_{\partial U}(t,v) \quad \text{for all } (x,v) \in (\mathcal{V} - |E|) \times M \text{ with } \pi(x) = t \in U.$$

Proof. We substitute $\mathcal{V}$ by the quasi projective variety $\pi^{-1}(U)$, and substitute E by its restriction to this new $\mathcal{V}$. This does not affect the statement of the theorem because [27] section 5 says that our old $\lambda_{\mathcal{V},E}$ and our new $\lambda_{\mathcal{V},E}$ differ by $O(\lambda_{\partial U})$. From the definition of U we have that $\phi_i : \mathcal{V} \to \mathcal{V}$ are morphisms, and on every fiber it is true that $\sum_i (\phi_i)_t^* E_t \sim \alpha E_t$. Hence there is a function $f \in \bar{K}(\mathcal{V})^* \otimes \mathbb{R}$ and a fibral divisor $F \in \mathrm{Div}(\mathcal{V}) \otimes \mathbb{R}$ such that

$$\textstyle\sum_i \phi_i^* E = \alpha E + \mathrm{div}(f) + F, \text{ where } F := \pi^* D \text{ for some } D.$$

Now standard properties of local heights, for example [27] Theorem 5.4, transforms the divisorial relation above into the height relation

$$\textstyle\sum_i \lambda_{\mathcal{V},E}(\phi_i(x),v) = \alpha\lambda_{\mathcal{V},E}(x,v) + v(f(x)) + \lambda_{U,D}(\pi(x),v) + O(\lambda_{\partial U}(\pi(x),v)).$$

Now we can repeat the same idea of the proof of Theorem 4.2.1 of [16], letting $\gamma(x,v) := \sum_i \lambda_{\mathcal{V},E}(\phi_i(x),v)) - \alpha\lambda_{\mathcal{V},E}(x,v) - v(f(x)) - \lambda_{U,D}(\pi(x),v) - O(\lambda_{\partial U}(\pi(x),v))$, and proceeding in the same way. This yields

$$\hat{\lambda}_{\mathcal{V}_t,E_t,(\mathcal{Q})_t} = \lambda_{\mathcal{V},E} + O(\lambda_{U,D}\circ\pi) + O(\lambda_{\partial U}\circ\pi),$$

which is almost what we want to prove. To conclude, we remember the fact that $\sum_i(\phi_i)_t^* E_t \sim \alpha E_t$ on every fiber, so we can repeat the above argument with functions $f_1, ..., f_n$ and divisors $D_1, ..., D_n$ having the property that $\cap|D_i| = \emptyset$. Then

$$\min\{\lambda_{U,D_i}\} = \lambda_{U,\cap D_i} = \lambda_{U,\emptyset}$$

is M_K-bounded, so

$$\hat{\lambda}_{\mathcal{V}_t,E_t,(\mathcal{Q})_t} = \lambda_{\mathcal{V},E} + \min_i O(\lambda_{U,D_i}\circ\pi) + O(\lambda_{\partial U}\circ\pi) = \lambda_{\mathcal{V},E} + O(\lambda_{\partial U}\circ\pi).$$

□

Corollary 3.3: *Theorem 3.1 is true over global fields in any characteristic.*

Proof. As we have said, we just must add the previous result over all places of K and use theorem 4.3.1 of [16]. □

3.2 Variation of the canonical height along sections

In this section we have a more precise result for a one-parameter algebraic family of points. We keep almost the same notation from the previous section with the following addition:

- T/K : we assume that the base variety T has dimension 1, so T is a smooth projective curve.
- h_T : we assume that the Weil height function on T corresponds to a divisor of degree 1.
- P : a section $P : T \to \mathcal{V}$. We can think of the generic fiber V of $\mathcal{V}$ as a variety over the function field $\bar{K}(T)$, and then the section P corresponds to a point $P_V \in V(\bar{K}(T))$.
- The function field $\bar{K}(T)$ is itself an usual global height field , namely, for each point $t \in T$, there is an absolute value ord_t on $\bar{K}(T)$ such that

  $\mathrm{ord}_t(f) :=$ order of vanishing of f at t.

Further, the rational map $\phi_i : \mathcal{V} \dashrightarrow \mathcal{V}$ induces a morphism on the generic fiber $(\phi_i)_V : V \to V$, and we have $\sum_i(\phi_i)_V^*\eta_V = \alpha\eta_V$, where η_V is the restriction of η to the generic fiber. This, by [16], allows us to construct the canonical height

$$\hat{h}_{V,\eta_V,(\mathcal{Q})_V} : V(\overline{K(T)}) \to \mathbb{R},$$

which can be evaluated at the point P_V. We also make $P_t = P(t)$. There are then three heights $\hat{h}_{V,\eta_V,(\mathcal{Q})_V}$, $\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t}$ and h_T which may be compared, as Silverman did for abelian varieties in [29]. Moreover, the following theorem generalizes theorem 4.1 of [9] for the Kawaguchi canonical heights.

Theorem 3.4: *With notation as above,*

$$\lim_{h_T(t)\to\infty, t\in T^0(\bar{K})} \frac{\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t}(P_t)}{h_T(t)} = \hat{h}_{V,\eta_V,(\mathcal{Q})_V}(P_V).$$

Proof. We start by stating together some results. First of all, from theorem 3.1 we have

$$|\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t}(x) - h_{\mathcal{V},\eta}(x)| \leq c_1 h_T(t) + c_2 \text{ for all } t \in T^0 \text{ and all } x \in \mathcal{V}_t.$$

In particular, this is true for $x = P_t$, and the constants c_1, c_2 are independent of both t and x. Second, we apply functoriality of Weil heights to the morphism $P : T \to \mathcal{V}$. We note that P will be a morphism, because we have assumed that T is a smooth curve, so any rational map from T to a variety is automatically a morphism. This gives

$$|h_{\mathcal{V},\eta}(P_t) - h_{T,P^*\eta}(t)| \leq c_3(P) \text{ for all } t \in T.$$

where $c_3(P)$ depends on the section P, but is independent of t. Third, we use [20], Chapter 3, Proposition 3.2 to describe the Weil height h_{V,η_V} on the generic fiber in terms of intersection theory ,

$$|h_{V,\eta_V}(S_V) - \deg S^*\eta| \leq c_4 \text{ for all sections } S : T \to \mathcal{V}.$$

Fourth, we know that a canonical height is a Weil height up to a constant, and then

$$|\hat{h}_{V,\eta_V,(\mathcal{Q})_V}(Q_V) - h_{V,\eta_V}(Q_V)| \leq c_5 \text{ for all } Q_V \in V(K(\bar{T})).$$

Using these four estimates and the triangle inequality, we compute

$$|\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t}(P_t) - \hat{h}_{V,\eta_V,(\mathcal{Q})_V}(P_V)h_T(t)|$$

$$\leq |\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t}(P_t) - h_{\mathcal{V},\eta}(P_t)| + |h_{\mathcal{V},\eta}(P_t) - h_{T,P^*\eta}(t)|$$

$$+ |h_{T,P^*\eta}(t) - (\deg P^*\eta)h_T(t)|$$

$$+ |(\deg P^*\eta)h_T(t) - h_{V,\eta_V}(P_V)h_T(t)|$$

$$+ |h_{V,\eta_V}(P_V)h_T(t) - \hat{h}_{V,\eta_V,(\mathcal{Q})_V}(P_V)h_T(t)|$$

$$\leq (c_1 h_T(t) + c_2) + c_3(P) + |h_{T,P^*\eta}(t) - (\deg P^*\eta)h_T(t)| + c_4 h_T(t) + c_5 h_T(t).$$

We now divide this inequality by $h_T(t)$ and let $h_T(t) \to \infty$. This gives

$$\limsup_{h_T(t)\to\infty} |\frac{\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t}(P_t)}{h_T(t)} - \hat{h}_{V,\eta_V,(\mathcal{Q})_V}(P_V)|$$
$$\leq c_1 + c_4 + c_5 + \limsup_{h_T(t)\to\infty} |\frac{h_{T,P^*\eta}(t)}{h_T(t)} - (\deg P^*\eta)|.$$

Term $c_3(P)$ has disappeared because it depends on P. Moreover, Corollary 3.5 of Chapter 4 from [20] implies that the heights $h_{T,P^*\eta}$ and $(\deg P^*\eta)h_T(t)$ are quasi-equivalent, and so

$$\lim_{h_T(t)\to\infty} \frac{h_{T,P^*\eta}(t)}{h_T(t)} = (\deg P^*\eta).$$

This gives the fundamental estimate

$$\limsup_{h_T(t)\to\infty} |\frac{\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t}(P_t)}{h_T(t)} - \hat{h}_{V,\eta_V,(\mathcal{Q})_V}(P_V)| \leq c_1 + c_4 + c_5,$$

where the constants c_1, c_4 and c_5 are independent of both the section and the point t, so the inequality above works with $f \circ P$ in place of P for all $f \in \mathcal{Q}_n, n \in \mathbb{N}$. By (ii) of Theorem 1.2.1 from [16], we know that

$\sum_{f\in(\mathcal{Q}_n)_t}\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t}(f(x))=\alpha^n\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t}(x)\quad\sum_{f\in(\mathcal{Q}_n)_V}\hat{h}_{V,\eta_V,(\mathcal{Q})_V}(f(x))=\alpha^n\hat{h}_{V,\eta_V,(\mathcal{Q})_V}(x)$.
So we finally obtain

$$\alpha^n\limsup_{h_T(t)\to\infty}|\frac{\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t}(P_t)}{h_T(t)}-\hat{h}_{V,\eta_V,(\mathcal{Q})_V}(P_V)|=$$
$$\limsup_{h_T(t)\to\infty}|\frac{\sum_{f\in(\mathcal{Q}_n)_t}\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t}(f(P_t))}{h_T(t)}-\sum_{f\in(\mathcal{Q}_n)_V}\hat{h}_{V,\eta_V,(\mathcal{Q})_V}(f(P_V))|$$
$$\leq k^n(c_1+c_4+c_5).$$

The right hand-side of the above inequality does not depend on n, while $\alpha>k$, so letting $n\to\infty$ gives us the inequality that we wanted to show.

$$\limsup_{h_T(t)\to\infty}|\frac{\hat{h}_{\mathcal{V}_t,\eta_t,(\mathcal{Q})_t}(P_t)}{h_T(t)}-\hat{h}_{V,\eta_V,(\mathcal{Q})_V}(P_V)|=0.$$

□

3.3 Canonical local heights as intersection multiplicities

In this section we show that Kawaguchi‘s canonical local height $\hat{\lambda}_{V,E,\mathcal{Q},}$ can be computed as an intersection number. We fix an absolute value v on K and let O_v denote the ring of v-integers in K. We continue with the notation used in the previous sections but we add the assumption that V is a smooth projective variety, and that the morphisms $\phi_i:V\to V$ are finite, correspondent to a dynamical system $(V,\phi_1,...,\phi_k)=(V,\mathcal{Q})$. We assume that E is defined over K, where $E\in\mathrm{Div}(V)\otimes\mathbb{R}$ is a divisor satisfying $\sum_{i=1}^k\phi_i^*E\sim\alpha E,\alpha>k$. Let $S:=\mathrm{Spec}(O_v)$. We will say that a smooth scheme $\mathcal{V}/S$ is a weak Néron Model for $(V/K,\mathcal{Q})$ over S if it satisfies the following axioms:

(1) The generic fiber of $\mathcal{V}/S$, denoted by $\mathcal{V}_K$, is V.

(2) Every point $P\in V(K)$ extends to a section $\mathbf{P}:\mathbf{S}\to\mathcal{V}$.

(3) There exist finite morphisms $\Phi_i:\mathcal{V}/S\to\mathcal{V}/S$ whose restriction to the generic fiber are the ϕ_i.

We note that the Néron Model of an Abelian Variety is a weak Néron Model for $(A/K,[n])$ for all $n\geq 2$. Indeed, for an abelian variety A, Néron first showed that any canonical local height $\hat{\lambda}_{A,D}(.,v)$ can be interpreted as an intersection multiplicity on the special fiber of the Néron model of A over $\mathrm{Spec}(\mathcal{O}_v)$(see [20], chapter 11, section 5).

Henceforth we will assume that $(V/K,\mathcal{Q})$ has a weak Néron Model $\mathcal{V}/S$. Let $\mathcal{V}_s$ denote the special fiber of $\mathcal{V}$ and write

$$\mathcal{V}_s=\sum_{j=1}^n\mathcal{V}_s^j,$$

where $\mathcal{V}_s^1,...,\mathcal{V}_s^n\in\mathrm{Div}(\mathcal{V})$ are the irreducible components of $\mathcal{V}_s$. If W is a prime divisor of V rational over K, then $\overline{W}$, its closure in $\mathcal{V}$, is a prime divisor on $\mathcal{V}$. Extending this process by linearity, we obtain a natural injection

$$\mathrm{Div}(V)_K\to\mathrm{Div}(\mathcal{V}),\qquad D\to\bar{D}$$

Similarly, given a point $P\in V(K)$, we write $\bar{P}=\mathbf{P}(\mathbf{S})$ to denote the image of the section $\mathbf{P}\in\mathcal{V}$. Note that the divisor group on S is a cyclic group generated by the special point (s). Hence, for any $D\in\mathrm{Div}(V)_K$ and any $P\in V(K)$ which does not lie in the support of D, we may define the intersection multiplicity $i(D,P)$ (also denoted by $\bar{P}.\bar{D}$) by

$$\mathbf{P}^*\bar{\mathbf{D}}=i(D,P)(s).$$

With these notations in hand, we can now state the main result of this section, which is a stronger version of theorem 6.1 of [9] due to Call and Silverman. For the proof, we will make use of a more refined theorem in algebra linear, that was not required for the proof of their mentioned earlier result.

Theorem 3.5: *Suppose $\mathcal{V}/S$ is a weak Neron model for $(V/K, \mathcal{Q})$ over O_v. Let $\hat{\lambda}_{V,E,\mathcal{Q},f}$ be a canonical local height as constructed in theorem 4.2.1 of [16].Moreover, suppose that $\alpha > nk$ for n the number of irreducible components of the special fiber. Then there exist real numbers $\gamma_1, ..., \gamma_n$ so that for all $P \in V(K) - |E|$,*

$$\hat{\lambda}_{V,E,\mathcal{Q},f}(P) = \bar{P}.(\bar{E} + \textstyle\sum_{j=1}^{n} \gamma_j \mathcal{V}_s^j).$$

An important point in the proof of this theorem is to describe the action of Φ_i on the set of irreducible components $\{\mathcal{V}_s^1, ..., \mathcal{V}_s^n\}$ of $\mathcal{V}_s$. Since Φ_i is a finite morphism, it maps each irreducible component of $\mathcal{V}_s$ onto another irreducible component (possibly the same component) of $\mathcal{V}_s$. Let $N = \{1, ..., n\}$. Then

$$A_i = A_{\Phi_i} : N \to N \text{ defined by } \Phi_i : \mathcal{V}_s^j \to \mathcal{V}_s^{A_i(j)} \text{ for } j \in N.$$

We can identify A_i with a matrix of the following type.

Definition 3.6: *A square matrix M is a permutation-type matrix if every column of M has exactly one* 1 *and all other entries are* 0.

It is a fact (lemma 6.2(b) of [9]) that every eigenvalue of a permutation matrix is 0 or is a root of unity. Such information is used in the proof of theorem 6.1 of [9]. For our more general situation, we will need the following theorem.

Theorem 3.7: *Let A be an $n-$square nonnegative matrix. Then*

$$\min_{1 \le i \le n} \textstyle\sum_{j=1}^{n} a_{ij} \le \rho(A) \le \max_{1 \le i \le n} \textstyle\sum_{j=1}^{n} a_{ij}.$$

In other words, the spectral radius of a nonnegative square matrix is between the smallest row sum and the largest row sum.

Proof. See theorem 5.24 of [36]. □

Proof. (of Theorem 3.5). Since E is assumed to be rational over K, we may fix a rational function $f \in K(V)^* \otimes \mathbb{R}$ so that

$$\textstyle\sum_{i=1}^{k} \phi_i^* E = \alpha E + \mathrm{div}_V(f). \quad (1)$$

Since $K(V) \cong K(\mathcal{V})$, we may also regard f as an element of $K(\mathcal{V})^* \otimes \mathbb{R}$. Then the divisors of f on V and $\mathcal{V}$ differ by a divisor supported on the special fiber, say

$$\overline{\mathrm{div}_V(f)} = \mathrm{div}_{\mathcal{V}}(f) + Z_f, \text{ where } Z_f = \textstyle\sum_{j=1}^{n} m(j, f)\mathcal{V}_s^j, \quad (2)$$

for some constants $m(j, f) \in \mathbb{R}$.

By Theorem 4.2.1 of [16], there is a unique canonical local height $\hat{\lambda}_E := \hat{\lambda}_{V,E,\mathcal{Q},f}$ which satisfies

$$\textstyle\sum_{i=1}^{k} \hat{\lambda}_E(\phi_i(p), v) = \alpha \hat{\lambda}_E(p, v) + v(f(p)). \quad (3)$$

Consider the map $V(K) - |E| \to \mathbb{R}$ defined by $P \to i(E,P) = \bar{P}.\bar{E}$. Given any $P \in V(K) - |E|$, there is a pair (U,g) representing $\bar{E}$ such that $U \subset \mathcal{V}$ is an open neighborhood of P and $g(P) \neq 0, \infty$. Then, by definition, $i(E,P) = v(g(P))$, independent of the choice of the pair (U,g). Thus, the map $P \mapsto i(E,P)$ is a Weil Local Height function for E on $V(K)$.

Note that $\Phi_i^*\bar{E}$ and $\overline{\phi_i^* E}$ differ by a divisor supported on the special fiber, since Φ_i and ϕ_i are the same on the generic fiber. Combining this fact with (1), we have

$$\textstyle\sum_i \Phi_i^*\bar{E} = \alpha\bar{E} + \overline{\mathrm{div}_V(f)} + \sum_{j=1}^n n_j\mathcal{V}_s^j, \qquad (4)$$

for some constants $n_j \in \mathbb{R}$. Further,

$$(\Phi_i)_*\bar{P} = (\Phi_i)_*\mathbf{P}(\mathbf{S}) = \Phi_i\circ\mathbf{P}(\mathbf{S}) = \phi_\mathbf{i}(\mathbf{P})(S) = \overline{\phi_i(P)},$$

where $\phi_\mathbf{i}(\mathbf{P})$ is the section corresponding to $\phi_i(P)$. Hence,

$$\textstyle\sum_i \bar{P}.\Phi_i^*\bar{E} = \sum_i (\Phi_i)_*\bar{P}.\bar{E} = \sum_i \overline{\phi_i(P)}.\bar{E} = \sum_i i(E,\phi_i(P)). \qquad (5)$$

Intersecting both sides of (2) with $\bar{P}$ yields:

$$\textstyle\bar{P}.\overline{\mathrm{div}_V(f)} = \bar{P}.\mathrm{div}_{\mathcal{V}}(f) + \bar{P}.Z_f = v(f(P)) + \bar{P}.\sum_{j=1}^n m(j,f)\mathcal{V}_s^j. \qquad (6)$$

Now, intersecting both sides of (4) with $\bar{P}$ and applying (5) and (6), we conclude

$$\textstyle\sum_i i(E,\phi_i(P)) = \alpha i(E,P) + v(f(P)) + \bar{P}.\sum_{j=1}^n c_j\mathcal{V}_s^j, \qquad (7)$$

where $c_j = m(j,f) + n_j$ are constants which depend on $E, \mathcal{Q}$ and f, but are independent of P. In particular, we see that (7) holds for all $P \in V(K)$ for which the intersection multiplicities $i(E,\phi_i(P))$ and $i(E,P)$ are defined, i.e, for all $P \notin |E| \cup |\phi_1^*E| \cup ... \cup |\phi_k^*E|$.

Next, we will show that one can choose real numbers $x_1, ..., x_n$ so that the function

$$\textstyle\Lambda_E(P) = i(E,P) + \bar{P}.\sum_{j=1}^n x_j\mathcal{V}_s^j \qquad (8)$$

satisfies

$$\textstyle\sum_i \Lambda_E(\phi(P)) = \alpha\Lambda_E(P) + v(f(P)) \qquad (9)$$

For all $P \in V(K) - (|E| \cup |\phi_1^*E| \cup ... \cup |\phi_k^*E|)$. Using (8) and (7), we compute

$$\textstyle\sum_i \Lambda_E(\phi_i(P)) - \alpha\Lambda_E(P) - v(f(P))$$

$$\textstyle= \sum_{i=1}^k(\overline{\phi_i(P)}.\sum_{j=1}^n x_j\mathcal{V}_s^j) - \alpha\bar{P}.\sum_{j=1}^n x_j\mathcal{V}_s^j + \bar{P}.\sum_{j=1}^n c_j\mathcal{V}_s^j.$$

Recall that Φ_i determines a permutation type matrix $A_i := A_{\Phi_i}$ defined by $\Phi_i : \mathcal{V}_s^j \to \mathcal{V}_s^{A_i(j)}$. Since $\bar{P}$ and $\overline{\phi_i(P)} = \Phi_i(\bar{P})$ intersect the components of $\mathcal{V}_s$ transversally, it follows from the definition of A_i that if $\mathbf{P}(\mathbf{s}) \in \mathcal{V}_\mathbf{s}^\mathbf{t}$, then

$$\textstyle\bar{P}.\sum_{j=1}^n x_j\mathcal{V}_s^j = x_t \text{ and } \overline{\phi_i(P)}.\sum_{j=1}^n x_j\mathcal{V}_s^j = x_{A_i(t)}. \qquad (10)$$

Therefore it suffices to find constants $x_1, ..., x_n$ such that

$$\textstyle\sum_{i=1}^k x_{A_i(t)} - \alpha x_t + c_t = 0 \text{ for } t = 1, ..., n.$$

Writing $x_1, ..., x_n$ and $c_1, ..., c_n$ as column forms, we can combine these n equations into a matrix equation

$$\textstyle(\alpha\mathbf{I} - \sum_{\mathbf{i=1}}^{\mathbf{k}}\mathbf{A_i})\mathbf{x} = \mathbf{c}.$$

The A_i are permutation-type matrices, so Theorem 3.7 says that the absolute value of an eigenvalue of $\sum_i A_i$ is at most nk, but $\alpha > nk$ by hypotheses. So $\det(\alpha\mathbf{I} - \sum_{\mathbf{i=1}}^{\mathbf{k}} \mathbf{A_i}) \neq \mathbf{0}$ and $(\alpha\mathbf{I} - \sum_{\mathbf{i=1}}^{\mathbf{k}} \mathbf{A_i})$ is invertible and we may take $\mathbf{x} = (\alpha\mathbf{I} - \sum_{\mathbf{i=1}}^{\mathbf{k}} \mathbf{A_i})^{-\mathbf{1}}\mathbf{c}$. This finishes the proof that we can choose $x_1, ..., x_n$ so that the function Λ defined by (8) satisfies (9).

To complete the proof, we will show that $\hat{\lambda}_E(P, v) = \Lambda_E(P)$ for all P in $V(K) - |E|$. Since $\hat{\lambda}_E(., v)$ and $i(E, .)$ are both Weil local heights for E, their difference has a unique v-continuous extension to a bounded v-continuous function defined on all of $V(K)$ (see [20], chapter 10, proposition 1.5, and 2.3). Hence, by (8), we see that the map $L_E(P) := \hat{\lambda}_E(P, v) - \Lambda_E(P)$ extends to a bounded function on $V(K)$, namely, by a constant C ≥ 0. Further, since $\hat{\lambda}_E$ and Λ_E satisfy (3) and (9), it follows that

$$\sum_i L_E(\phi_i(P)) = \alpha L_E(P) \text{ for all } P \in V(K).$$

Therefore, for any $P \in V(K)$,

$$|L_E(P)| \leq |\alpha^{-N} \sum_{\phi \in \mathcal{Q}_N} L_E(\phi(P))| \leq \frac{k^N}{\alpha^N}.C \rightarrow_{N \rightarrow \infty} 0.$$

We conclude that $L_E \equiv 0$, so $\hat{\lambda}_E(P, v) = \Lambda_E(P)\ \forall P \in V(K) - |E|$. □

Appendix A

Canonical Metrics of Commuting Systems

With two polarized dynamical systems $(X, \mathcal{F} = \{f_1, ..., f_k\}, \mathcal{L}, \alpha), \alpha > k$ and $(X, \mathcal{G} = \{g_1, ..., g_t\}, \mathcal{L}, \beta), \beta > t$, we can build two canonical metrics, two canonical heights, and two canonical measures for $\mathcal{L} \in \text{Pic}(X) \otimes \mathbb{R}$. We will see that if all the maps of one of the systems commute with all the maps of the other, then the canonical metrics, canonical heights, and canonical measures associated to each system are identical.

A.1 The admissible metric

We consider a projective variety X over a number field K, and $(X; f_1, ..., f_k)$ a dynamical eigensystem of k morphisms $\mathcal{F} := \{f_1, ..., f_k\}$ over K associated with an ample line bundle $\mathcal{L} \in \text{Pic}(X) \otimes \mathbb{R}$ of degree $\alpha > k$ as in section 3 of [16], so we have an isomorphim $\phi : \mathcal{L}^{\otimes^\alpha} \to^{\cong} f_1^*\mathcal{L} \otimes ... \otimes f_k^*\mathcal{L}$. This situation will be called a polarized dynamical eigensystem $(X, f_1, ..., f_k, \mathcal{L}, \alpha)$ on X defined over K. Assume that for every place v of K we have chosen a continuous and bounded metric $||.||_v$ on each fibre of $\mathcal{L}_v := \mathcal{L} \otimes_K K_v$. The following theorem is stated in theorem 3.3.1 of [16] for $||.||_\infty$.

Theorem A.1.1: *The sequence defined recurrently by* $||.||_{v,1} := ||.||_v$ *and*

$$||.||_{v,n} = (\phi^*(f_1^*||.||_{v,n-1} ... f_k^*||.||_{v,n-1}))^{\frac{1}{\alpha}} \text{ for } n > 1$$

converge uniformly on $X(\bar{K}_v)$ *to a metric* $||.||_{v,\mathcal{F}}$ *(independent of the choice of* $||.||_{v,1}$*) on* $\mathcal{L}_v$ *which satisfies the equation*

$$||.||_{v,\mathcal{F}} = (\phi^*(f_1^*||.||_{v,\mathcal{F}} ... f_k^*||.||_{v,\mathcal{F}}))^{\frac{1}{\alpha}}.$$

Proof. The proof is the same as is theorem 3.3.1 of [16] with v in place of ∞. □

Definition A.1.2: *The metric* $||.||_{v,\mathcal{F}}$ *is called the canonical metric on* $\mathcal{L}_v$ *relative to the system of maps* $\mathcal{F}$.

The following proposition relates the canonical metrics associating to commuting maps. It represents the main result of this section, and it is a natural and simple generalization of proposition 2.5 of [25] for the metric defined in [16, theorem 3.1.1].

Proposition A.1.3: *Let* $(X, \mathcal{F} = \{f_1, ..., f_k\}, \mathcal{L}, \alpha)$ *and* $(X, \mathcal{G} = \{g_1, ..., g_t\}, \mathcal{L}, \beta)$ *be two polarized systems with* $\alpha > k, \beta > t$ *on* X *defined over* K*. Suppose that* $f_i \circ g_j = g_j \circ f_i$ *for all* i, j*. Then* $||.||_{v,\mathcal{F}} = ||.||_{v,\mathcal{G}}$.

Proof. The key idea is that the canonical metric does not depend on the metric from which we have started the iteration with. Let $s \in \Gamma(X, \mathcal{L})$ be a non-zero section of $\mathcal{L}$. We are going to consider two metrics $||.||_{v,1} = ||.||_{v,\mathcal{F}}$ and $||.||'_{v,1} = ||.||_{v,\mathcal{G}}$ on the line bundle $\mathcal{L}$. By our definition of canonical metric for $\mathcal{F}$, we can start with $||.||'_{v,1}$ and obtain $||s(x)||_{v,\mathcal{F}} = \lim_r (|| \prod_{f \in \mathcal{F}_r} s(f(x))||'_{v,1})^{\frac{1}{\alpha^r}}$, but also by our definition of canonical metric for $\mathcal{G}$ starting with

$$||.||_{v,1} = ||.||_{v,\mathcal{F}} \text{ we get } ||s(x)||_{v,\mathcal{G}} = \lim_l || \prod_{g \in \mathcal{G}_l} s(g(x))||_{v,\mathcal{F}}^{\frac{1}{\beta^l}}.$$

So using the uniform convergence and the commutativity of the maps,

$$||s(x)||_{v,\mathcal{F}} = \lim_r || \prod_{f \in \mathcal{F}_r} s(f(x))||_{v,\mathcal{G}}^{\frac{1}{\alpha^r}}$$

$$= \lim_{r,l} \prod_{f \in \mathcal{F}_r} \prod_{g \in \mathcal{G}_l} ||s(f(g(x)))||_{v,1}^{\frac{1}{\alpha^r \beta^l}}$$

$$= \lim_{r,l} \prod_{f \in \mathcal{F}_r} \prod_{g \in \mathcal{G}_l} ||s(g(f(x)))||_{v,1}^{\frac{1}{\alpha^r \beta^l}}$$

$$= \lim_l || \prod_{g \in \mathcal{G}_l} s(g(x))||_{v,\mathcal{F}}^{\frac{1}{\beta^l}}$$

$$= ||s(x)||_{v,\mathcal{G}},$$

which was the result we wanted to prove. □

Let X n-dimensional projective variety defined over a number field K and $(X, \mathcal{F} = \{f_1, ..., f_k\}, \mathcal{L}, \alpha)$ a polarized system with $\alpha > k$ defined over K. Let v be a place of K over infinity. We can consider morphisms

$$f_i \otimes v : X_v \to X_v \text{ over the complex variety } X_v.$$

Associated to $\mathcal{F}$ and v we also have the canonical metric $||.||_{v,\mathcal{F}}$ and therefore the distribution $\hat{c}_1(\mathcal{L}, ||.||_{v,\mathcal{F}}) = \frac{1}{\pi i} \partial \bar{\partial} \log ||s||_{v,\mathcal{F}}$, with $s \in \Gamma(X, \mathcal{L}) - \{0\}$, analogous to the first Chern form of $(\mathcal{L}, ||.||_{v,\mathcal{F}})$. It can be proved that this is a positive current in the sense of Lelong, as in section 3.2 of [16], we can define the product

$$\hat{c}_1(\mathcal{L}, ||.||_{v,\mathcal{F}})^n = \hat{c}_1(\mathcal{L}, ||.||_{v,\mathcal{F}}) ... \hat{c}_1(\mathcal{L}, ||.||_{v,\mathcal{F}}),$$

which represents a measure μ on X_v.

Definition A.1.4: *The measure $d\mu_{\mathcal{F}} = \hat{c}_1(\mathcal{L}, ||.||_{v,\mathcal{F}})^n / \mu(X)$, is called the canonical measure associated to $\mathcal{F}$ and v. Once we fix $\mathcal{L}$, it depends only on the metric $||.||_{v,\mathcal{F}}$.*

Now we assume that X is an arithmetic variety of absolute dimension $n+1$, that is, given a number field K, X is flat and of finite type over $\mathrm{Spec}(O_K)$ of relative dimension n. We can define (see section 2 of [16]) the arithmetic intersection number $\hat{c}_1(\mathcal{L}_1)...\hat{c}_1(\mathcal{L}_{n+1})$ of the classes of the hermitian line bundles $(\mathcal{L}_i, ||.||)$ on X, which means that each line bundle $\mathcal{L}_i$ on X is equipped with a hermitian metric $||.||_{v,i}$ over $X_v = X \otimes_K \mathrm{Spec}(\mathcal{O}_K)$, for each place v at infinity. Such line bundles are called adelic line metrized bundles when they can be equipped with semipositive metric for all places v. So we can define adelic intersection numbers $\hat{c}_1(\mathcal{L}_{1|Y})...\hat{c}_1(\mathcal{L}_{n+1|Y})$ over a p-cycle $Y \subset X$. Suppose that we are in the presence of a polarized dynamical eigensystem $(X, \mathcal{F}, \mathcal{L}, \alpha)$. In this case the canonical metric $||.||_{\mathcal{F}}$ represents a semipositive metric on $\mathcal{L}$, and we can define the canonical height associated to $(\mathcal{L}, ||.||_{\mathcal{F}})$.

Definition A.1.5: *The canonical height $\hat{h}_{\mathcal{F}}(Y)$ of a p-cycle Y in X is defined as*

$$\hat{h}_{\mathcal{F}}(Y) = \frac{\hat{c}_1(\mathcal{L}_{|Y})^{p+1}}{(\dim Y + 1) c_1(\mathcal{L}_{|Y})^p}.$$

It depends only on $(\mathcal{L}, ||.||_{\mathcal{F}})$, where $||.||_{\mathcal{F}}$ is actually representing a colection of canonical metrics over all places of K. An inportant particular case of canonical height will be the canonical height $\hat{h}_{\mathcal{F}}(P)$ of a point P of X. Since the canonical measure and the canonical height were defined depending only on the canonical metric of the system, the equality of canonical metrics and measure is a direct consequence of proposition A.1.3, as we just state below.

Proposition A.1.6: *Let* $(X, \mathcal{F} = \{f_1, ..., f_k\}, \mathcal{L}, \alpha)$ *and* $(X, \mathcal{G} = \{g_1, ..., g_t\}, \mathcal{L}, \beta)$ *be two polarized systems with* $\alpha > k, \beta > l$ *on* X *defined over* K*. Suppose that* $f_i \circ g_j = g_j \circ f_i$ *for all* i, j*. Then* $\hat{h}_{\mathcal{F}} = \hat{h}_{\mathcal{G}}$ *and* $d\mu_{\mathcal{F}} = d\mu_{\mathcal{G}}$*.*

Proof. It is a consequence of the last two definitions that the statements depend only on the canonical metric presented in proposition A.1.3. □

References

[1] F. Amoroso and S. David *Distribution des points de petite hauteur dans les groupes multiplicatifs*. Ann. Scuola Norm. Cl. Sci. (5) 3, no. 2, 325-348

[2] F. Amoroso and S. David, *Minoration de la hauteur normaliseée dans un tore*, J. Inst. Math. Jussieu 2 (2003), no. 3, 335-381.

[3] F. Amoroso and E. Viada, *Small points on subvarieties of a torus*, Duke Math. J. 150 (2009), no. 3, 407-442 .

[4] Baker, A., *New Advances in Transcendence Theory*, Cambridge University Press(1988).

[5] A. Baragar, *Canonical vector heights on algebraic K3 surfaces with Picard number two*, Canad. Math. Bull. 46 (2003), 495-508 .

[6] A. Baragar, *Rational points on K3 surfaces in* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$,Math. Ann. 305 (1996),541-558 .

[7] Y. Bilu, *Limit distribution of small points on algebraic tori*, Duke Math. J. 89 (1997), 465-508.

[8] Enrico Bombieri and Walter Gubler, *Heights in Diophantine Geometry*, Number 4 in New Mathematical Monographs. Cambridge University Press, Cambridge, 2006.

[9] G. Call and H. Silverman, *Canonical heights on varieties with morphisms*, Compositio Math. 89 (1993), 163-205.

[10] Chambert-Loir, A., *Mesures et équidistribution sur les espaces de Berkovich*, J. Reine Angew. Math., 595 (2006), 215-325 .

[11] Fulton, W, *Intersection theory, 2nded.,Ergeb.Math.Grenzgeb.(3)2*, Springer-Verlag, Berlin 1998.

[12] V.Guedj, *Ergodic properties of rational mappings with large topological degree*, Ann.of Math.(2) 161 (2005), no. 3, 1589-1607.

[13] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.

[14] Marc Hindry and Joseph H. Silverman, *Diophantine Geometry: An Introduction*, volume 201 of Graduate Texts in Mathematics. Springer-Verlag, New York, .

[15] S. Kawaguchi, *Canonical height functions for affine plane automorphisms*, Math. Ann., 335(2):285-310, 2006.

[16] S. Kawaguchi, *Canonical heights, invariant currents, and dynamical eigensystems of morphisms for line bundles,* preprint, math.NT/0405006. (The revised version is accepted in J. Reine Angew. Math.)

[17] S. Kawaguchi, *Projective surface automorphisms of positive topological entropy from an arithmetic viewpoint*, available at arXiv:math/0510634v2 [math.AG].

[18] S. Kawaguchi and Joseph H. Silverman, *On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties*,2012. arXiv:1208.0815.

[19] S. Kawaguchi and J. H. Silverman, *Dynamical canonical heights for Jordan blocks, arithmetic degrees of orbits, and nef canonical heights on abelian varieties*, preprint 2013, http://arxiv.org/abs/1301.4964

[20] Lang, S, *Fundamentals of Diophantine Geometry*, New York, (1983).

[21] C. G. Lee, *Equidistribution of periodic points of some automorphisms on K3 surfaces*, available at arXiv:1102.4860v1[math.NT].

[22] Jan-Li Lin and Chi-Hao Wang, *Canonical height functions for monomial maps*,2012. arXiv:1205.2020v1.

[23] A. Moriwaki, *Arithmetic height functions over finitely generated fields*, Invent. Math. 140 (2000),101-142 .

[24] A. Moriwaki, *The canonical arithmetic height of subvarieties of an abelian variety over a finitely generated field* , J. reine angrew. Math. 530 (2001), 33-54.

[25] Bernard Parisse and Morgane Vaughan , *Jordan Normal and Rational Normal Form Algorithms*,2004. arXiv:cs/0412005v1.

[26] Pineiro, J., *Canonical metrics of commuting maps*, available at arXiv:0803.3468v1[math.NT].

[27] Silverman, J.H, *Arithmetic distance functions and height function in Diophantine geometry*, Math. Ann. 279(1987) 193-216.

[28] Silverman, J.H, *Examples of dynamical degree equals arithmetic degree*, available at arXiv:1212.3015v1[math.NT].

[29] Silverman, J.H, *Dynamical Degrees, Arithmetic Entropy, and Canonical Heights for Dominant Rational Self-Maps of Projective Space*, available at arXiv:1111.5664[math.NT].

[30] Silverman, J.H, *Heights and the specialization maps for families of abelian varieties*, J. Reine Angew. Math. 342(1983) 197-211.

[31] Silverman, J.H, *Rational points on K3 surfaces: a new canonical height*, Invent. Math. 105 (1991), 347-373.

[32] J.H.Silverman, *The Arithmetic of Dynamical Systems*, volume 241 of Graduate Texts in Mathematics. Springer, New York, 2007.

[33] J.H.Silverman, *The arithmetic of elliptic curves*, 2nd ed., Grad. Texts in Math. 106, Springer-Verlag, Dordrecht 2009.

[34] Waldschmidt, M., *Diophantine approximation on linear algebraic groups. Transcendence properties of the exponential function in several variables*, Springer-Verlag, Berlin, 2000.

[35] L. Wang, *Rational points and canonical heights on K3-surfaces in* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, Recent developments in the inverse Galois problem (Seattle, WA, 1993), Contemp. Math. 186 (1995), 273-289.

[36] Zhang, F, *Matrix Theory: Basic Results and Techniques*, Springer-Verlag, (2011).

[37] S. Zhang, *Small points and adelic metrics*, J. Alg. Geom. 4 (1995),281-300 .