# Shadowable, topologically stable and distal points for flows

by

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# Shadowable, topologically stable and distal points for flows.

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*For Iván Aponte, in memoriam.*

**Abstract**

A shadowable point for a flow is a point where the shadowing lemma holds for pseudo-orbits passing through it. We prove that this concept satisfies the following properties: the set of shadowable points is invariant and a $G_\delta$ set. A flow has the pseudo-orbit tracing property if and only if every point is shadowable. The chain recurrent and nonwandering sets coincide when every chain recurrent point is shadowable. The chain recurrent points which are shadowable are exactly those that can be are approximated by periodic points when the flow is expansive. These results extends those presented in [42]. We study the relations between shadowable points of a homeomorphism and the shadowable points of its suspension flow. We characterize the set of forward shadowable points for transitive flows and chain transitive flows. We prove that the geometric Lorenz attractor does not have shadowable points. We show that in the presence of shadowable points chain transitive flows are transitive and that transitivity is a necessary condition for chain recurrent flows with shadowable points whenever the phase space is connected. Finally, as an application these results we give concise proofs of some well known theorems establishing that flows with POTP admitting some kind of recurrence are minimal.

We introduce the concept of topological stable point for flows. We see that this set is invariant under the flow and under topological conjugacy. We prove that if the chain recurrent set is contained in the set of topologically stable points, then it coincides with the closure of the periodic orbits. Finally we show that if an orbit of the suspension flow of a map is topologically stable then its base point is also topologically stable. These results extends to the flow context some of those given in [34].

We also study the variation of distality for flows $\phi$ obtained by making the proximal cell in [10] to depend on a given subset $\mathcal{F}$ of the full set of reparametrizations $\mathcal{C}$. We consider first the case when $\mathcal{F}$ reduces to a single continuous map $s:\mathbb{R}\to\mathbb{R}$ fixing the origin different from the identity. In such a case if the $s$-dependent proximal cells are trivial, then the flow is uniformly closed (or trivial if $s$ is bounded). Next we show that the flow $\phi$ is closed if and only if the $s$-depending proximal cells reduce to the corresponding orbit for every (or some) $s$ bounded. Furthermore, nonsingular flows admits points whose proximal cell (with $s$ being the identity) does not reduce to the orbit. Afterwards, we consider the case when $\mathcal{F}$ is either $\mathcal{H}$ (the set of homeomorphisms $s:\mathbb{R}\to\mathbb{R}$ fixing 0) or the whole $\mathcal{C}$. From this we obtain a characterization of the classical pointwise almost periodicity.

Finally, we begin the study of the two-limit shadowable property for flows $LmSP$. We show that the geometric Lorenz attractor does not have $LmSP$. We show that if a map has the two-side limit shadowing property then the suspension has $LmSP$ and show an example where the reciprocal is not true.

## Resumo

Um ponto sombreável para um fluxo é um ponto em que o lema de sombremento é válido para pseudo-órbitas que passam por ele. Provamos que este conceito satisfaz as seguintes propriedades: o conjunto de pontos sombreáveis é invariante e um conjunto $G_\delta$. Um fluxo tem a propriedade de propriedade de traçado de pseudo-órbitas se e somente se cada ponto for sombreável. Os conjunto recurrente por cadeias e n ao errantes coincidem quando cada ponto recorrente por cadeia é sombreável. Os pontos recorrentes por cadeia que s ao sombréaveis s ao exatamente aqueles que podem ser aproximados por pontos periódicos quando o fluxo é expansivo. Estes resultados estendem os apresentados em [42]. Estudamos as relações entre pontos sombréaveis de um homeomorfismo e os pontos sombréaveis do seu fluxo de suspensão. Caracterizamos o conjunto de pontos sombréaveis para os fluxos transitivos e os fluxos transitivos por cadeia. Provamos que o atractor geométrico de Lorenz não tem pontos sombréaveis. Mostramos que, na presença de pontos sombréaveis, os fluxos transitivos por cadeia são transitivos e que a transitividade é uma condição necessária para fluxos recorrentes por cadeia com pontos sombreáveis sempre que o espaço de fase é conexo. Finalmente, como uma aplicação desses resultados damos provas concisas de alguns teoremas bem conhecidos que estabelecem que fluxos com POTP admitindo algum tipo de recorrência são mínimais.

Apresentamos o conceito de ponto topologicamente estável para fluxos. Vemos que este conjunto é invariante sob o fluxo e sob conjugação topológica. Provamos que, se o conjunto recorrente da cadeia estiver contido no conjunto de pontos topologicamente estáveis, então ele coincide com o fecho das órbitas periódicas. Finalmente, mostramos que se uma órbita do fluxo suspensão de um mapa é topologicamente estável, então seu ponto base também é topologicamente estável. Esses resultados estendem ao contexto de fluxo alguns dos dados em [34].

Também estudamos a variação de distalidade para fluxos $\phi$ obtidos fazendo células proximais em [10] depender de um subconjunto dado $\mathcal{F}$ de funções contínuas. Primeiro consideramos o caso quando $\mathcal{F}$ se reduz a uma única função contínua $s : \mathbb{R} \to \mathbb{R}$. Em tal caso se a células proximais dependentes de $s$ são triviais, então o fluxo é uniformemente fechado (ou trivial se $s$ é limitada). Logo mostramos que o fluxo $\phi$ é fechado se e somente se as células $s$ proximais se reduzem a órbita correspondente para cada (ou alguma) $s$ limitada. Mais ainda, fluxos não singulares admitem pontos cuja célula proximal (com $s$ sendo a identidade) não se reduz à órbita. Logo, consideramos o caso quando $\mathcal{F}$ é ou $\mathcal{H}$ (o conjunto de homeomorphismos $s : \mathbb{R} \to \mathbb{R}$ fixando 0) ou $\mathcal{C}$ tudo. Disto obtemos uma caracterização da clássica quase periodicidade pontual.

Finalmente, começamos o estudo da propriedade de sombreamento bilateral no limite para fluxos $LmSP$. Mostramos que o atractor geométrico de Lorenz não tem $LmSP$. Mostramos que, se um mapa tiver a propriedade de sombreamento bilateral no limite, então a suspensão tem $LmSP$ e mostramos um exemplo em que o recíproco não é verdadeiro.

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# ONE

# INTRODUCTION AND BASIC DEFINTIONS

In this work we study the general problem of redefining global dynamical properties in terms of local ones. When can we recover global properties in terms of local ones? What are the advantages or disadvantages of studying dynamical systems from this viewpoint and it how help us to understand it better. Recently this problem has attracted many researchers. Indeed, there are now several concepts in topological dynamics admitting pointwise counterparts. That is the case of equicontinuous, expansive, distal and persistence homeomorphisms that admit concepts of *equicontinuous*, *expansive* [22,46], *distal* [6] and *persitence* [38] points as counterparts, respectively, and its study constitute what is called today *pointwise dynamical systems*. A more recent example are the entropy points [56]. There are also pointwise concepts admitting global counterparts as its the case of transitive points admitting the general context of transitive homeomorphisms [54]. In [42], the concept of *shadowable points* which are points such every pseudo-orbits passing through them can be shadowed by a real orbit. Later, this was generalized in [31] and several interesting questions were answered. On the other hands, similar ideas involving the pointwise counterparts of dynamical properties, were used in [41] and [37] to define distal and expansive measures, respectively.

Our main object of study will be the so-called continuous dynamical systems, which are actions of the additive group $(\mathbb{R}, +)$ over certain set $X$ called the phase space of the system. Our phase spaces will be compact topological metric spaces. We make some contributions to the theory of pointwise dynamical systems in this context.

## 1.1 Continuous flows and equivalence

We begin this section by reminding the definition of a continuous flow on a general topological space.

**Definition 1.1.1.** Let $X$ a compact topological space. A *flow* on $X$ is a function $\phi\colon X \times \mathbb{R} \to X$ satisfying the following properties:

i) $\phi(x, 0) = x$ for all $x \in X$.

ii) $\phi(x, t+s) = \phi(\phi(x, t), s)$.

Every flow considered in this work will be continuous with respect to the usual product topology on $X \times \mathbb{R}$ and we will always refer one simply as a flow. The definition of a flow $\phi$ implies immediately that the family of maps $\{\phi_t \colon X \to X\}_{t\in\mathbb{R}}$ given by $\phi_t(x) = \phi(x, t)$ is a family of homeomorphisms of $X$ satisfying $\phi_0 = id_X$, the identity map of $X$, and $\phi_t \circ \phi_s = \phi_{t+s}$. From these two properties follows that $(\phi_t)^{-1} = \phi_{-t}$. A subset $A \subset X$ is *invariant under* $\phi$, or simply is *an invariant set of* $\phi$ if $\phi_t(A) = A$ for all $t \in \mathbb{R}$. Given $I \subset \mathbb{R}$ we set $\phi_I(A) = \{\phi_t(x) : (x, t) \in A \times I\}$. If $A$ consists of a single point $x$, then we write $\phi_I(x)$ instead of $\phi_I(\{x\})$. If $A$ is an invariant subset of $X$ then we defined the *restriction flow of* $\phi$ *to* $A$ by the flow $\phi|_A \colon A \times \mathbb{R} \to A$ defined by $\phi|_A(x, t) = \phi(x, t)$.

There are mainly two ways in which two flows may be topologically similar. The first of these is conjugacy.

**Definition 1.1.2.** We say that the flows $\phi\colon X \times \mathbb{R} \to X$ and $\psi\colon Y \times \mathbb{R} \to Y$ are *topologically conjugate* if there exists a homeomorphism $h\colon X \to Y$ such that for each real number $t$ the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\phi_t} & X \\ h\downarrow & & \downarrow h \\ Y & \xrightarrow{\psi_t} & Y \end{array}$$

That is, for all $x \in X$ and all $t \in \mathbb{R}$ we have $\psi_t(h(x)) = h(\phi_t(x))$.

A weaker form of topological similarity is that of topological equivalence.

**Definition 1.1.3.** Let $\phi\colon X \times \mathbb{R} \to X$ and $\psi\colon Y \times \mathbb{R} \to Y$ be two flows. We say that $\phi$ is *equivalent* to $\psi$, denoted by $\phi \sim_X \psi$, if there exists a homeomorphism $h\colon X \to Y$ and a function $\sigma\colon X \times \mathbb{R} \to \mathbb{R}$ with the following properties:

i) $\sigma$ is a reparametrizarion, i.e, $\sigma(x, \cdot)\colon \mathbb{R} \to \mathbb{R}$ is strictly increasing and onto, for all $x \in X$;

ii) $h(\phi_{\sigma(x,t)}(x)) = \psi_t(h(x))$.

Properties i) and ii) imply that and $h$ transform orbits of $\phi$ onto orbits of $\psi$ preserving their orientation. The pair $(h, \sigma)$ is called an *equivalence from* $\phi$ *to* $\psi$. Given an equivalence, the function $\Phi(x, t) : X \times \mathbb{R} \to X$ defined by $\Phi(x, t) = \phi_{\sigma(x,t)}(x)$ is a flow whose phase portrait is exactly that of $\phi$. So $\Phi$ is a so-called *time change* of $\phi$. In fact, a flow $\phi$ is equivalent to $\psi$ if and only if there is a time change of $\phi$ which is topologically conjugate to $\psi$ . Note that given an equivalence $(h, \sigma)$ from $\phi$ to $\psi$ we can find a new one such that its reparametrization fixes the origin of $\mathbb{R}$. Indeed, it is enough to define the new reparametrization as $\sigma(x, \cdot) - \sigma(x, 0)$ for all $x \in X$. From this observation we can always suppose that our reparametrizations fix 0. Note also that we do not require the function $\sigma\colon X \times \mathbb{R} \to \mathbb{R}$ to be continuous either. However, under certain condition, continuity can be guaranteed as is the case of continuous flows without singularities [48]. In this case we say that the flows are *continuously equivalent* and the corresponding equivalence $(h, \sigma)$ will be called continuous. Clearly a conjugacy defines a continuous equivalence if we define $\sigma(x, t) = t$ for all $t \in \mathbb{R}$.

We have that $\sim_X$ is indeed an equivalence relation on the set on continuous flows on a topological space $X$. In fact, by taking $f = id_X$ and $\sigma(x, t) = t$ we can show that $\phi$ is equivalent to itself, and if $\phi$ is equivalent to $\psi$ under the equivalence $(h, \sigma)$, the definition of a reparametrization implies that $\sigma(\cdot, x) : \mathbb{R} \to \mathbb{R}$ is invertible for each $x \in X$. Therefore, for each $s \in \mathbb{R}$ and $y \in Y$ there exists a unique $t \in \mathbb{R}$ such that $s = \rho(t, h^{-1}(y))$. If we define $\tilde{\sigma} \colon \mathbb{R} \times Y \to \mathbb{R}$ by $\tilde{\sigma}(s, y) = t$, then it is easy to prove that $(h^{-1}, \tilde{\sigma})$ is an equivalence from $\psi$ to $\phi$. Finally, if $(h, \sigma)$ is an equivalence from $\phi$ to $\psi$ and $(g, \theta)$ is equivalent to $\psi$ to $\rho$ then the the pair $(g \circ h, \gamma)$ where $\gamma \colon \mathbb{R} \times X \to \mathbb{R}$ is defined by $\gamma(x, t) = \sigma(x, \theta(x, h(x)))$ is an equivalence from $\phi$ to $\rho$ and the transitivity of $\sim_X$ follows.

## 1.2 Recurrence

The simplest notion of recurrence of a flow is that of periodic behavior. We begin by giving some standard notation. The *orbit*, *forward orbit* and *backward orbit* of a point $x$ under the flow $\phi$ are, respectively, the sets

$$\mathcal{O}_\phi(x) := \{\phi_t(x)\}_{t\in\mathbb{R}},\ \mathcal{O}^+_\phi(x) := \{\phi_t(x)\}_{t\geq 0},\ \mathcal{O}^-_\phi(x) := \{\phi_t(x)\}_{t\leq 0}.$$

When the flow is implicit or obvious from the context, we usually omit the subscript $\phi$. A *singularity* of $\phi$ is a point $x \in X$ such that $\phi_t(x) = x$ for all $t \in \mathbb{R}$. If $x$ is not a singularity, then $x$ is a *regular* point of $\phi$. The set of singular points will be denoted by $Sing(\phi)$. An *orbit* of $\phi$ is any subset of $X$ equal to $\mathcal{O}_\phi(x)$ for some $x \in X$.

**Definition 1.2.1.** A point $x$ is *periodic*, if there exists $t > 0$ with $\phi^t(x) = x$.

If $x$ is a periodic point, then number $\pi(x) := \inf\{t > 0\colon\ \phi_t(x) = x\}$ exists and is greater or equal than zero. We say that $\pi(x)$ is the *period* of $x$. In compact metric space the period zero periodic points are exactly the singularities of $\phi$. A *periodic orbit* is any subset of the space equal to an orbit of a periodic point. The set of periodic points of $\phi$ will be denoted by $Per(\phi)$ and it is clearly the union of all its periodic orbits.

Next we introduce the notion of recurrence points. Given a flow on the topological space $X$, the *omega limit set* and *alpha limit set* of a point $x \in X$ are, respectively, the sets

$$\omega_\phi(x) = \omega(x) := \{y \in X\colon\ d(\phi_{t_i}(x), y) \to 0,\ \text{for some } t_i \to +\infty\},$$

and

$$\alpha_\phi(x) = \alpha(x) := \{y \in X\colon\ d(\phi_{t_i}(x), y) \to 0,\ \text{for some } t_i \to -\infty\}.$$

If $X$ is compact, then $\omega_\phi(x)$ and $\alpha_\phi(x)$ are nonempty compact invariant sets of $\phi$.

**Definition 1.2.2.** A point $x \in X$ is *recurrent* if $x \in \omega(x)$ and $\phi$ is a *recurrent flow* if every point is recurrent.

The set of recurrent points of $\phi$ is denoted by $R(\phi)$, so $\phi$ is recurrent if and only if $R(\phi) = X$.

**Definition 1.2.3.** We say that a point $x \in X$ is *non-wandering* if for every neighborhood $U$ of $x$ and every $T \in \mathbb{R}$ there is $t \geq T$ such that $\phi_t(U) \cap U \neq \emptyset$, and we say that $\phi$ is a *non-wandering flow* if every point point is non-wandering.

The set of non-wandering points of $\phi$ is denoted by $\Omega(\phi)$. So a flow is non-wandering if and only if $\Omega(\phi) = X$. It can be also seen that is closed non-empty subset of $X$.

The next notion of recurrence is chain recurrence.

Given $\delta, T > 0$, $a \in \mathbb{Z} \cup \{-\infty\}$, $b \in \mathbb{Z} \cup \{+\infty\}$ with $a \leq b$, we say that a sequence of pairs $(x_i, t_i)_{i=a}^{b}$ in $X \times \mathbb{R}$ is a $(\delta, T)$-*pseudo-orbit* of $\phi$ if for all integer indexes $i$ such that $a \leq i \leq b-1$ we have that $t_i \geq T$ and $d(\phi_{t_i}(x_i), x_{i+1}) \leq \delta$. If $a, b \in \mathbb{Z}$ and $ab \leq 0$, we say that it is a *finite* $(\delta, T)$-*pseudo-orbit.* If $a = 0$ and $b = \infty$ we say that it is a *forward* $(\delta, T)$-*pseudo-orbit* and if $a = 0$ and $b < \infty$ we say that it is a $(\delta, T)$-*chain.* (see [32, 47]).

Two points $x$ and $y$ are $(\delta, T)$-related if there are two $(\delta, T)$-chains $(x_i, t_i)_{i=0}^{m}$ and $(y_i, s_i)_{i=0}^{n}$ such that $p = x_0 = y_n$ and $y = y_0 = x_m$. We say that $p$ and $q$ are related (written $p \sim q$) if they are $(\delta, T)$-related for every $\delta, T > 0$.

**Definition 1.2.4.** A point $x \in X$ is *chain recurrent* if $x \sim x$, and $\phi$ is a *chain recurrent flow* if every point is chain recurrent.

The set chain recurrent points is denoted by $CR(\phi)$. So a flow a chain recurrent if and only if $CR(\phi) = X$. As in the case of nonwandering points it is a closed non-empty subset of $X$. The relation $\sim$ defines an equivalence relation on $CR(\phi)$ and so it can be divided in disjoint and invariant equivalence classes called *chain components.* As a matter of fact, this classes are exacty the connected components of $CR(\phi)$.

Clearly $\Omega(\phi) \subseteq CR(\phi)$ and the inclusion may be proper [1]. In fact, we have the following chain of inclusions:

$$Sing(\phi) \subseteq Per(\phi) \subseteq R(\phi) \subseteq \Omega(\phi) \subseteq CR(\phi)$$

## 1.3 Suspension flows

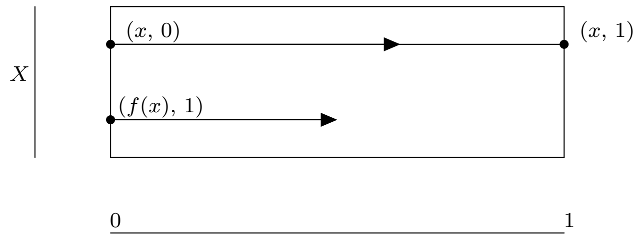
An important source of flows comes from homeomorphisms on a compact metric space through a construction known as the suspension of a map. Let $(X, d)$ be a compact metric space and $f : X \to X$ be a homeomorphism and $\tau \colon X \to (0, +\infty)$ be a continuous function. Consider the set

$$X^{\tau,f} := \{(x,t) : \ 0 \leq t \leq \tau(x), \ x \in X\}/\sim,$$

where $(x, \tau(x)) \sim (f(x), 0)$ for all $x \in X$ and give it the usual quotient topology. The *suspension flow over* $f$ *with height function* $\tau$ is the flow on $Y^{\tau,f}$ defined by

$$\phi_t^{\tau,f}(x,s) := (f^n(x), s+t-n),$$

whenever $s + t \in [n, \tau(x)(n+1))$ for some $n \in \mathbb{Z}$.




Every suspension of $f$ is conjugate to the suspension of $f$ under the constant function 1. A homeomorphism from $Y^{1,f}$ to $Y^{\tau,f}$ that conjugates the flows is given by the map $(x,t) \mapsto (x, t\tau(x))$. Since all the properties studied in this work are invariant under conjugacy, we will concentrate only in suspensions flows over $f \equiv 1$. In this case we denote the flow $\phi^{1,\,f}$ simply by $\phi^f$. We can replace $d$ by the equivalent metric $\hat{d}\colon X \times X \to [0,\,\infty)$ defined by $\hat{d}(x,\,y) = \frac{d(x,y)}{\operatorname{diam}(X)}$, for all $x,\,y \in X$, if necessary, and assume that $\operatorname{diam}(X) \leq 1$. We show that $X^{\tau,f}$ is metrizable by giving it a metric, known as the *Bowen-Walters metric* [14], as follows: Consider the subset $X \times \{t\}$ of $X$ and give it the metric $d_t$ defined by $d_t((x,\,t),\,(y,\,t)) = (1-t)d(x,\,y) + td(f(x),\,f(y))$ for all $x$, $y \in X$. Note that $d_0((x,\,0),\,(y,\,0)) = d(x,\,y)$ and $d_1((x,\,1),\,(y,\,1)) = d(f(x),\,f(y))$. Let $(x,\,t),\,(y,\,s) \in X^{1,f}$ and consider all the finite sequences $(z_i,\,t_i)_{i=1}^n$ of elements of $X^{1,f}$ such that $(z_1,\,t_1) = (x,\,t)$ , $(z_2,\,t_2) = (y,\,s)$ and for each $1 \leq i \leq n-1$ either $(x_i,\,t_i)$ and $(x_{i+1},\,t_{i+1})$ belongs to $X \times \{t\}$ (in which case we call $[(x_i,\,t_i),\,(x_{i+1},\,t_{i+1})]$ a *horizontal segment*) or $(x_i,\,t_i)$ and $(x_{i+1},\,t_{i+1})$ belongs to the same orbit of the suspension flow (and then we call $[(x_i,\,t_i),\,(x_{i+1},\,t_{i+1})]$ a *vertical segment*). The length of a horizontal segment will be given by the distance $d_t$ and the length of a vertical segment will be the shortest distance between $(x_i,\,t_i)$ and $(x_{i+1},\,t_{i+1})$ along the orbit, using the usual distance of $\mathbb{R}$. In case $(x_i,\,t_i) \neq (x_{i+1},\,t_{i+1})$ and they are in the same horizontal and vertical segment, we take the length of $[(x_i,\,t_i),\,(x_{i+1},\,t_{i+1})]$ as the length given by $d_t$ since this is always less than 1. The length of a chain will be the sum of the length of its horizontal and vertical segments. We now define a metric $d^f$ as follows

$$d^f((x,\,t),\,(y,\,s)) = \inf\{\text{length of all chains between } (x,\,t) \text{ and } (y,\,s)\}\,.$$

If $d'$ is the metric on $X$ defined by $d'(x,\,y) = \min\{d(x,\,y),\,d(f(x),\,f(y))\}$, then $d'$ is equivalent to $d$ and clearly $d_t((x,\,t),\,(y,\,t)) \geq d'(x,\,y)$. So $d^f((x,\,t),\,(y,\,s)) = 0$ if and only if $(x,\,t) = (y,\,s)$. It can be seen easily that $d^f$ is symmetric and satisfies the triangle inequality. Therefore $d^f$ defines a metric on $X^{1,f}$ under which the suspension flow is continuous. Even more, this metric generates the quotient topology [14].

CHAPTER

# TWO

# SHADOWABLE POINTS FOR FLOWS

The theory of shadowing in dynamical systems has been largely studied by many researchers and is well documented (see for instance [44]). It refers to the general problem of approximating orbits obtained in the presence of noise or round-off error (for instance solutions obtained by numerical computations). There are several ways of defining the *shadowing property* for flows, see for instance [45] and references therein. In essence, the central idea among the majority of definitions of shadowing for flows is the existence of time reparametrizations.

Recently, in [42] the definition of shadowing for homeomorphisms in a compact metric space was generalized by introducing the notion of shadowable points, which are points where the shadowing property holds for pseudo-orbits passing through them.

Given these results, it is natural to consider a notion of shadowable points for flows. We prove that this concept satisfies the following properties: the set of shadowable points is invariant and a $G_\delta$ set. A flow has the pseudo-orbit tracing property if and only if every point is shadowable. The chain recurrent and nonwandering sets coincide when every chain recurrent point is shadowable. The chain recurrent points which are shadowable are exactly those that can be are approximated by periodic points when the flow is expansive. These results extends those presented in [42]. We study the relations between shadowable points of a homeomorphism and the shadowable points of its suspension flow. We characterize the set of forward shadowable points for transitive flows and chain transitive flows. We prove that the geometric Lorenz attractor does not have shadowable points. We show that in the presence of shadowable points chain transitive flows are transitive and that transitivity is a necessary condition for chain recurrent flows with shadowable points whenever the phase space is connected. Finally, as an application these results we give concise proofs of some well known theorems establishing that flows with POTP admitting some kind of recurrence are minimal.

## 2.1 Shadowable points

In this section we overview the work entitled *Shadowable Points* [42].

Let $X$ be a metric space. If $f : X \to X$ is a homeomorphism and $\delta > 0$, we say that a bi-infinite sequence $\xi = (\xi_n)_{n\in\mathbb{Z}}$ of $X$ is a *$\delta$-pseudo-orbit* if $d(f(\xi_n), \xi_{n+1}) \leq \delta$ for all $n \in \mathbb{Z}$. Given $\varepsilon > 0$ we say that $\xi$ *can be $\varepsilon$-shadowed* if there is $x \in X$ such that $d(f^n(x), \xi_n) \leq \varepsilon$ for all $n \in \mathbb{Z}$. We say that $f$ has the *pseudo-orbit tracing property* (abbrev. *POTP*) if for every $\varepsilon > 0$ there is $\delta > 0$ such that every $\delta$-pseudo-orbit can be $\varepsilon$-shadowed. Homeomorphisms with the POTP have been widely studied [3], [45].

Now we introduce the following concept closely related to that of absolutely nonshadowable points [57]. It splits the POTP into individual shadowings.

**Definition 2.1.1.** A point $x \in X$ is *shadowable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that every $\delta$-pseudo-orbit $\xi$ *with* $\xi_0 = x$ can be $\varepsilon$-shadowed. We denote by $Sh(f)$ the set of shadowable points of $f$.

Let us present some related examples.

**Example 1.** Clearly if $f$ has the POTP, then $Sh(f) = X$ (i.e. every point is shadowable). The converse is true on compact metric spaces by Theorem 2.1.2 below. As we shall see, the identity of the circle has no shadowable points. Examples where $Sh(f)$ is a proper nonempty set will be given later on.

We present some properties of $Sh(f)$ through the following standard definitions. We say that a point $x \in X$ is *nonwandering* if for every neighborhood $U$ of $x$ there is $k \in \mathbb{N}^+$ such that $f^n(U) \cap U \neq \emptyset$. We say that $x$ is *chain recurrent* if for every $\rho > 0$ there is a *$\rho$-chain* from $x$ to itself, i.e., a finite sequence $\{x_i : 0 \leq i \leq n\}$ satisfying $x_0 = x$, $x_n = x$ and $d(f(x_i), x_{i+1}) \leq \rho$ for all $i$ with $0 \leq i \leq n-1$. Denote by $\Omega(f)$ and $CR(f)$ the set of nonwandering and chain recurrent points of $f$ respectively. Clearly $\Omega(f) \subset CR(f)$ and the inclusion may be proper. We say that $\Lambda \subset X$ is *invariant* if $f(\Lambda) = \Lambda$.

With these definitions we can state our first result.

**Theorem 2.1.2.** *If $f : X \to X$ is a homeomorphism of a compact metric space $X$, then*

*(1) $Sh(f)$ is an invariant set (empty or nonempty, possibly noncompact);*

*(2) $f$ has the POTP if and only if $Sh(f) = X$;*

*(3) if $CR(f) \subset Sh(f)$, then $CR(f) = \Omega(f)$.*

*Remark* 1. A shorter elegant proof of item 2 of the previous theorem was presented in [22].

The following example is related to Item (3) of the above theorem.

**Example 2.** It is easy to find examples where $CR(f)$ is a proper subset of $Sh(f)$: just take a homeomorphism with the POTP (so $Sh(f) = X$) with $CR(f)$ being a proper subset of $X$.

Given a homeomorphism $f : X \to X$ and $p \in X$ we define the *omega-limit set*,

$$\omega(p) = \{q \in X : q = \lim_{l\to\infty} f^{n_l}(p) \text{ for some sequence } n_l \to \infty\}.$$

We say that $p$ is *recurrent* if $p \in \omega(p)$. Denote by $R(f)$ the set of recurrent points of $f$. A homeomorphism $f : X \to X$ is *pointwise-recurrent* if $R(f) = X$.

On the other hand, the space $X$ is *totally disconnected at* $p \in X$ if the connected component of $X$ containing $p$ is $\{p\}$. As in [8] we define

$$X^{deg} = \{p \in X : X \text{ is totally disconnected at } p\}.$$

With these notations we obtain the following result.

**Theorem 2.1.3.** *If $f : X \to X$ is a pointwise-recurrent homeomorphism of a compact metric space $X$, then $Sh(f) \subset X^{deg}$.*

Let us state some consequences of Theorem 2.1.3. A *continuum* is a compact connected metric space. We say that it is *nondegenerated* if it does not reduce to a single point. Clearly a nondegenerated continuum $X$ satisfies $X^{deg} = \emptyset$ and so the following result holds by Theorem 2.1.3.

**Corollary 2.1.4.** *Pointwise-recurrent homeomorphisms on nondegenerated continua have no shadowable points.*

In particular, a pointwise recurrent homeomorphism of a nondegenerated continuum does not have the POTP (see the remark after the Main Theorem in [35]).

Examples were the above results apply are as follows: A homeomorphism $f : X \to X$ is *minimal* if the orbit $\{f^n(x) : n \in \mathbb{Z}\}$ of every point $x \in X$ is dense in $X$. We say that $f$ is *semisimple* if there is a collection $\{E_\alpha : \alpha \in I\}$ of compact invariant subsets of $X$ such that $X = \cup_{\alpha \in I} E_\alpha$ and $f|_{E_\alpha}$ is minimal, for all $\alpha \in I$. We say that $f$ is *distal* if $\inf_{n \in \mathbb{Z}} d(f^n(x), f^n(y)) > 0$ for distinct points $x, y \in X$. Every minimal homeomorphism is semisimple and the same property holds for distal homeomorphism (see Theorem 11.5.9 in [3] or Corollary 4 in p. 68 of [10]). Every semisimple homeomorphism is clearly pointwise-recurrent. Consequently, distal or minimal homeomorphisms of nondegenerated continua have no the POTP (this fact was proved earlier [2], [43]).

The second corollary deals with compact metric spaces exhibiting pointwise-recurrent homeomorphisms with the POTP. Recall that $X$ is *totally disconnected* if it is totally disconnected at any point (i.e. $X = X^{deg}$).

**Corollary 2.1.5.** *A compact metric space admits a pointwise-recurrent homeomorphism with the POTP if and only if it is totally disconnected.*

*Proof.* Every totally disconnected compact metric space exhibits a pointwise-recurrent homeomorphism with the POTP (e.g. the identity, see Theorem 2.3.2 p. 79 in [3]). On the other hand, if there is a pointwise-recurrent homeomorphism with the POTP, then every point is shadowable and so the space is totally disconnected by Theorem 2.1.3. □

*Remark* 2. Theorem 2.1.3 motivates the question whether every pointwise-recurrent homeomorphism $f : X \to X$ of a compact metric space $X$ satisfies $Sh(f) = X^{deg}$. However this was proved to be false [31].

The answer for the above question is clearly positive on nondegenerated continua (by Corollary 2.1.4). Another partial positive answer can be obtained as follows. We say that a homeomorphism $f : X \to X$ is *equicontinuous* if for every $\alpha > 0$ there is $\beta > 0$ such that $x, y \in X$ and $d(x, y) \leq \beta$ imply $d(f^n(x), f^n(y)) \leq \alpha$ for all $n \in \mathbb{Z}$. It is easy to see that every equicontinuous homeomorphism of a compact metric space is distal (hence pointwise-recurrent). For such homeomorphisms we have the following result.

**Theorem 2.1.6.** *If $f : X \to X$ is an equicontinuous homeomorphism of a compact metric space $X$, then $Sh(f) = X^{deg}$.*

Since $X^{deg}$ is always a $G_\delta$ subset of $X$ (see p. 746 in [8]), we conclude from the above theorem that the set of shadowable points of an equicontinuous homeomorphism is a $G_\delta$ too.

Theorem 2.1.6 also implies the following corollary extending the conclusion of Theorem 4 in [36] to distal homeomorphisms.

**Corollary 2.1.7.** *Let $X$ be a compact metric space and $f : X \to X$ be a distal homeomorphism. Then, $f$ has the POTP if and only if $X$ is totally disconnected.*

*Proof.* As already said, every distal homeomorphism $f$ is pointwise-recurrent. If $f$ has the POTP, then $X$ is totally disconnected by Corollary 2.1.5. Conversely, if $X$ is totally disconnected, then $f$ is equicontinuous (e.g. Corollary 1.9 in [7]) so $Sh(f) = X$ (by Theorem 2.1.6) thus $f$ has the POTP (by Theorem 2.1.2). □

Recall that a homeomorphism $f : X \to X$ is *transitive* if the orbit $\{f^n(x) : n \in \mathbb{Z}\}$ of some point $x \in X$ is dense in $X$.

*Remark* 3. One can ask if there is a transitive homeomorphism $f$ for which $Sh(f)$ is a non-empty non-compact subset. This was answered to be negative in [31].

**Example 3.** There are a compact metric space $X$ and a homeomorphism $f : X \to X$ such that $Sh(f)$ is a nonempty noncompact subset of $X$.

*Proof.* Define $X = C \cup [1, 2]$ with the topology induced from $\mathbb{R}$, where $C$ be the ternary Cantor set of $[0, 1]$. Clearly $X^{deg} = C \setminus \{1\}$. Now take $f : X \to X$ as the identity of $X$. Since the identity is an equicontinuous homeomorphism, we obtain $Sh(f) = X^{deg}$ by Theorem 2.1.6. Then $Sh(f) = C \setminus \{1\}$. Since $C \setminus \{1\}$ is nonempty and noncompact, we are done. □

## 2.2 Shadowable points for flows

For any sequence of real numbers $(t_j)_{j\in\mathbb{Z}}$ we write

$$s_i = \begin{cases} \displaystyle\sum_{j=0}^{i-1} t_j & i > 0, \\ 0 & i = 0, \\ \displaystyle -\sum_{j=i}^{-1} t_j & i < 0. \end{cases}$$

and we will say that $(s_i)_{i\in\mathbb{Z}}$ is the *sequence of sums of* $(t_i)_{i\in\mathbb{Z}}$.

Let $(x_i, t_i)_{i\in\mathbb{Z}}$ be a $(\delta, T)$-pseudo-orbit of $\phi$ and let $t \in \mathbb{R}$, we denote by $x_0 \star t$ a point in the $(\delta, T)$-pseudo-orbit $t$ *units from* $x_0$ [32]. More precisely,

$$x_0 \star t = \phi_{t-s_i}(x_i) \text{ whenever } s_i \leq t < s_{i+1}.$$

We denote by Rep the set of strictly increasing surjective functions $h\colon \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$.

Given $\varepsilon, \delta > 0$ and $T > 0$, a $(\delta, T)$-pseudo-orbit $(x_i, t_i)_{i\in\mathbb{Z}}$ is $\varepsilon$-shadowed by an orbit $(\phi_t(y))_{t\in\mathbb{R}}$ if there exists $h \in$ Rep such that

$$d(x_0 \star t, \phi_{h(t)}(y)) \leq \varepsilon, \text{ for every } t \in \mathbb{R}.$$

We recall the definition of pseudo orbit tracing property for flows [47].

**Definition 2.2.1.** A flow $\phi$ on X is said to have the pseudo orbit tracing property with respect to the parameter $T > 0$, if for all $\varepsilon > 0$ there exists $\delta > 0$ such that every $(\delta, T)$-pseudo-orbit is $\varepsilon$-shadowed by an orbit of $\phi$. A flow has the pseudo-orbit tracing property, *POTP*, if has it with respect to the parameter $T = 1$.

By following [42] we introduce the main definition of this chapter [4].

**Definitions 2.2.2.** Given positive numbers $\delta$, $T$ and $\varepsilon$, we say that a $(\delta, T)$-pseudo-orbit $(x_i, t_i)_{i\in\mathbb{Z}}$ of $\phi$ passes through $p$ if $x_0 = p$, and we say it is $\varepsilon$-*shadowed* if there are a point $y \in X$ and a function $h \in$ Rep such that

$$d(x_0 \star t, \phi_{h(t)}(y)) \leq \varepsilon, \text{ for each } t \in \mathbb{R}.$$

A point $p \in X$ is *shadowable with respect to the parameter* $T > 0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every $(\delta, T)$-pseudo-orbit passing through $p$ can be $\varepsilon$-shadowed. Finally, we say that $p$ is *shadowable* if it is shadowable with respect to the parameter $T = 1$.

We denote by $Sh(\phi)$ the set of shadowable points of $\phi$ in $X$.

**Example 4.** If a flow $\phi$ on $X$ has the POTP then $Sh(\phi) = X$. The converse is also true for compact metric spaces as we shall see shortly.

Next, we are going to prove the basic properties of $Sh(\phi)$

**Theorem 2.2.3.** *The set $Sh(\phi)$ is an invariant set of $\phi$. So, if not empty, is union of orbits of $\phi$.*

*Proof.* Let $x$ be a shadowable point of $X$. Let $\varepsilon > 0$ and $s \in \mathbb{R}$ given. Since $\phi_s$ is uniformly continuous, we can choose $0 < \varepsilon' < \varepsilon$ such that whenever $d(x, y) < \varepsilon'$ we have $d(\phi_s(x), \phi_s(y)) < \varepsilon$. For $\varepsilon'$, let $\delta > 0$ such that any $(\delta, 1)$-pseudo-orbit passing through $x$ can be $\varepsilon'$-shadowed. Similarly, $\phi_{-s}$ is uniformly continuous so we can choose $\delta' > 0$ with the property that $d(\phi_{-s}(x), \phi_{-s}(y)) \leq \delta$ whenever $d(x, y) < \delta'$. Now let $(x_i, t_i)_{i\in\mathbb{Z}}$ be a $(\delta', 1)$-pseudo-orbit passing through $\phi_s(x)$. Because $d(\phi_{t_i}(x_i), x_{i+1}) \leq \delta'$ we have by the choice of $\delta'$ that

$$d(\phi_{-s}(\phi_{t_i}(x_i)), \phi_{-s}(x_{i+1})) = d(\phi_{t_i}(\phi_{-s}(x_i)), \phi_{-s}(x_{i+1})) \leq \delta$$

and hence $(\phi_{-s}(x_i), t_i)_{i\in\mathbb{Z}}$ is a $(\delta, 1)$-pseudo-orbit passing through $x$. By definition, there are $h \in \text{Rep}$ and $y \in X$ such that

$$d(x \star t, \phi_{h(t)}(y)) \leq \varepsilon', \text{ for every } t \in \mathbb{R}.$$

Then, if $s_i \leq t < s_{i+1}$, it follows that $d(\phi_{t-s_i}(\phi_{-s}(x_i)), \phi_{h(t)}(y)) \leq \varepsilon'$ for every $t \in \mathbb{R}$ which implies $d(\phi_{t-s_i}(x_i), \phi_{h(t)}(\phi_s(y))) \leq \varepsilon$. Therefore, $d(x_0 \star t, \phi_{h(t)}(\phi_s(y))) \leq \varepsilon$ for each $t \in \mathbb{R}$. Thus, every $(\delta', 1)$-orbit passing through $\phi_s(x)$ can be $\varepsilon$-shadowed by a point in $X$. This completes the proof. □

Let $X$ be a compact metric space. We say that a sequence $(x_n, t_n)_{n\in\mathbb{Z}}$ of $X \times \mathbb{R}$ *is through some subset* $K \subseteq X$ if $x_0 \in K$ (see [42]). Now we introduce the following auxiliary definition.

**Definition 2.2.4.** We say that a flow $\phi \colon X \times \mathbb{R} \to X$ has the *POTP through a subset $K$ with respect to the parameter* $T > 0$, if given $\varepsilon > 0$, there exists $\delta > 0$ such that every $(\delta, T)$-pseudo-orbit passing through $K$ is $\varepsilon$-shadowed by an orbit of $\phi$. We say that $\phi$ has the *POTP through a subset $K$* if it has it with respect to the parameter $T = 1$.

When $K = X$ this definition coincides with the classical POTP. Note that we do not require the entire $(\delta, 1)$-pseudo-orbit to be contained in $K$, therefore the definition 2.2.4 is stronger than the POTP *on* $K$ [45].

A sequence of pairs $(x_i, t_i)_{i\in\mathbb{Z}}$ is a $(\delta, T_1, T_2)$-pseudo-orbit of $\phi$ if it is a $(\delta, T_1)$-pseudo-orbit of $\phi$ and satisfies $t_i \leq T_2$, for all $i \in \mathbb{Z}$.

In [47], Thomas proved that a flow satisfies the POTP with respect to the parameter $T$ if and only if for every $\varepsilon > 0$ we can find $\delta > 0$ such that every $(\delta, T, 2T)$-pseudo-orbit can be $\varepsilon$-shadowed. He also showed that if $T > 0$ then a flow has the POTP if and only has the POTP with respect to the parameter $T$. We present pointwise versions of these results:

**Lemma 2.2.5.** *Let $a > 0$, $K \subseteq X$ and $\phi$ be a flow on a compact metric space $X$. Then, the following statements are equivalent:*

*(1) For all $\varepsilon > 0$ there exists $\delta > 0$ such that every $(\delta, a, 2a)$-pseudo-orbit passing through $K$ is $\varepsilon$-shadowed by an orbit of $\phi$.*

*(2) $\phi$ has the POTP through $K$ with respect to the parameter $a$.*

*(3) $\phi$ has the POTP through $K$.*

*Proof.* Clearly (2) $\Longrightarrow$ (1). To see that (1) $\Longrightarrow$ (2), suppose that for all $\varepsilon > 0$ there exists $\delta > 0$ such that every $(\delta, a, 2a)$-pseudo-orbit passing through $K$ is $\varepsilon$-shadowed by an orbit of $\phi$. Let $(x_i, t_i)_{i\in\mathbb{Z}}$ be any $(\delta, a)$-pseudo-orbit of $\phi$ passing through $K$. For each $n \in \mathbb{Z}$, there exists $m_n \in \mathbb{N}$ such that $t_n = m_n a + r_n$ with $a \leq r_n < 2a$. Let $(s_n^m)_{n\in\mathbb{Z}}$ the sequence of sums associated to $m = (m_n)_{n\in\mathbb{Z}}$. Denote $A_n = s_n^m + n$ for all $n \in \mathbb{Z}$ and define the sequence $(y_i)_{i\in\mathbb{Z}}$ on $X$ such that $y_i = \phi_{a(i-A_n)}(x_n)$ if $A_n \leq i < A_{n+1}$. In addition, we define a sequence $\lambda = (\lambda_i)_{i\in\mathbb{Z}}$ of real numbers in the following way, for each $i \in \mathbb{Z}$, we set

$$\lambda_i = \begin{cases} a & \text{if } A_n \leq i < A_{n+1} - 1, \\ r_n & \text{if } i = A_{n+1} - 1. \end{cases}$$

Given $i \in \mathbb{Z}$ note that $a \leq \lambda_i < 2a$ and let $n \in \mathbb{Z}$ be such that $A_n \leq i < A_{n+1}$. We have two cases:

*Case 1:* if $i < A_{n+1} - 1$, then

$$d(\phi_{\lambda_i}(y_i), y_{i+1}) = d(\phi_a(\phi_{a(i-A_n)}(x_n)), \phi_{a(i+1-A_n)}(x_n)) = 0.$$

*Case 2:* if $i = A_{n+1} - 1$, bearing in mind that $A_{n+1} - A_n = s^m_{n+1} - s^m_n + 1 = m_n + 1$ we obtain

$$\begin{aligned} d(\phi_{\lambda_i}(y_i), y_{i+1}) = d(\phi_{r_n}(\phi_{a(A_{n+1}-1-A_n)}(x_n)), x_{n+1}) &= d(\phi_{r_n}(\phi_{am_n}(x_n)), x_{n+1}) \\ &= d(\phi_{t_n}(x_n), x_{n+1}) \leq \delta. \end{aligned}$$

That is, $(y_i, \lambda_i)_{i\in\mathbb{Z}}$ is a $(\delta, a, 2a)$-pseudo-orbit of $\phi$ passing through $K$. Then, there are $z \in X$ and $h \in \text{Rep}$ such that $d(\phi_{r-s^\lambda_n}(y_n), \phi_{h(r)}(z)) \leq \varepsilon$ where $s^\lambda_n \leq r < s^\lambda_{n+1}$ and $(s^\lambda_i)$ is the sequence of sums associated to $\lambda = (\lambda_i)_{i\in\mathbb{Z}}$. Let $w \in \mathbb{R}$ and $n \in \mathbb{Z}$ such that $s^t_n \leq w < s^t_{n+1}$, where $(s^t_n)$ is associated to $t = (t_i)_{i\in\mathbb{Z}}$. Since $s^t_n = s^\lambda_{A_n}$, then $s^\lambda_{A_n} \leq w < s^\lambda_{A_{n+1}} = s^\lambda_{A_n+m_n+1}$. Hence, there is $0 \leq j \leq m_n$ such that $s^\lambda_{A_n+j} \leq w < s^\lambda_{A_n+j+1}$ and then

$$\begin{aligned} \varepsilon \geq d(\phi_{w-s^\lambda_{A_n+j}}(y_{A_n+j}), \phi_{h(w)}(z)) &= d(\phi_{w-s^t_n}(\phi_{s^t_n - s^\lambda_{A_n+j}}(y_{A_n+j})), \phi_{h(w)}(z)) \\ &= d(\phi_{w-s^t_n}(\phi_{s^t_n - s^\lambda_{A_n+j}}(\phi_{aj}(x_n))), \phi_{h(w)}(z)) \\ &= d(\phi_{w-s^t_n}(x_n), \phi_{h(w)}(z)). \end{aligned}$$

It follows that $\phi$ has the POTP through $K$ with respect to the parameter $a$. Now we prove that (2) $\implies$ (3). We can assume that $a > 1$. Fix $m \in \mathbb{N}$ such that $m \geq a$. Given $\varepsilon > 0$ choose $\delta > 0$ satisfying the following conditions:

(1) Every $(\delta, a)$-pseudo-orbit passing through $K$ is $\frac{\varepsilon}{2}$-shadowable.

(2) For each $0 \leq t \leq 2m$ we have $d(\phi_t(x), \phi_t(y)) < \frac{\varepsilon}{2}$, whenever $d(x, y) < \delta$.

Let $0 < \delta' < \delta/m$ and take $0 < \beta < \delta'$ so that $d(x, y) < \beta$ implies that $d(\phi_t(x), \phi_t(y)) < \delta'$ for $0 \leq t \leq 2m$. Let $(x_n, t_n)_{n\in\mathbb{Z}}$ be a $(\beta, 1)$-pseudo-orbit for $\phi$ passing through $K$ with $1 \leq t_n \leq 2$ for all $n \in \mathbb{Z}$. Consider the sequence of pairs $(x_{im}, \lambda_i)_{i\in\mathbb{Z}}$ where $\lambda_i = \sum_{j=0}^{m-1} t_{j+im}$ for every $i \in \mathbb{Z}$. We denote $\lambda_i(k) = \sum_{j=k}^{m-1} t_{j+im}$ with $0 \leq k < m$. Then

$$d(\phi_{\lambda_i}(x_{im}), x_{(i+1)m}) \leq \sum_{r=1}^{m} d(\phi_{\lambda_i(r)}(\phi_{t_{im+r-1}}(x_{im+r-1})), \phi_{\lambda_i(r)}(x_{im+r})) \leq m\delta' < \delta,$$

because $a \leq \lambda_i \leq 2m$. So, $(x_{im}, \lambda_i)_{i\in\mathbb{Z}}$ is a $(\delta, a)$-pseudo-orbit for $\phi$ passing through $K$. Hence, there are $z \in X$ and $h \in \text{Rep}$ such that $d(\phi_{t-s^\lambda_n}(x_{nm}), \phi_{h(t)}(z)) \leq \frac{\varepsilon}{2}$ where $s^\lambda_n \leq t < s^\lambda_{n+1}$. Now, for $0 \leq k < m$ denote $s^t_k(r) = \sum_{j=r}^{k-1} t_j$ we have

$$d(\phi_{s^t_k}(x_0), x_k) \leq \sum_{r=1}^{k} d(\phi_{s^t_k(r)}(\phi_{t_{r-1}}(x_{r-1})), \phi_{s^t_k(r)}(x_r)) < k\delta' < \delta.$$

Then for $s^t_k \leq t < s^t_{k+1}$

$$d(\phi_{t-s^t_k}(x_k), \phi_{h(t)}(z)) \leq d(\phi_{t-s^t_k}(x_k), \phi_{t-s^t_k}(\phi_{s^t_k}(x_0))) + d(\phi_t(x_0), \phi_{h(t)}(z)) \leq \varepsilon.$$

For $m \leq k < 2m$, we follow in the same manner. So we will have that the orbit $(\phi_t(z))_{t\in\mathbb{R}}$ $\varepsilon$-shadows the $(\beta, 1, 2)$-pseudo-orbit of $\phi$ passing through $K$. Applying (1) $\Longleftrightarrow$ (2) we obtain that $(\phi_t(z))_{t\in\mathrm{R}}$ can be $\varepsilon$-shadows by a $(\beta, 1)$-pseudo-orbit of $\phi$ passing through $K$. By using similar ideas we obtain (3) $\Longrightarrow$ (2). This completes the proof. □

Hence a flow $\phi$ in a compact metric space $X$ has POTP trough $K$ if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that every $(\delta, 1, 2)$-pseudo-orbit passing through $K$ is $\varepsilon$-shadowed by an orbit of $\phi$.

Clearly, if a flow has the POTP through a set $K$ then every point is $K$ is shadowable respectively. The reciprocal is also true if $K$ is compact as shown in the following lemma.

**Lemma 2.2.6.** *Let $\phi$ be a flow on a compact metric space $X$. If every point of a compact subset $K$ of $X$ is shadowable, then $\phi$ has the POTP through the set $K$.*

*Proof.* Assume by contradiction that there exists a nonempty compact subset $K$ such that every point in $K$ is shadowable but does not have the POTP through $K$. Then there is $\varepsilon > 0$ and a sequence $(\xi^k)_{k\in\mathbb{N}} = (\xi^k_n,\, t^k_n)_{n\in\mathbb{Z}}$ of $(\frac{1}{k},\, 1,\, 2)$-pseudo-orbits passing through $K$ which cannot be $2\varepsilon$-shadowed. Since $K$ and $[1,\, 2]$ are compact, we can assume that $\xi^k_0 \to p$ for some $p \in K$ and $t^k_0 \to t_0$ for some time $t_0 \in [1,\, 2]$. We have that $p$ is shadowable, so for $\varepsilon$ as above, we choose $\delta > 0$ from the shadowableness of $p$ with $\delta < \frac{\varepsilon}{3}$. Since $X \times [0,\, 2]$ is compact, $\phi|_{X\times[1,\,2]}$ is uniformly continuous and so our $\delta$ can also be chosen so that if $d((x,\, t),\, (y,\, s)) \leq \delta$ with $0 \leq s,\, t \leq 2$ then $d(\phi_t(x),\, \phi_s(y)) \leq \frac{\varepsilon}{3}$. We set a sequence $\hat{\xi}^k = (\hat{\xi}^k_n,\, \hat{t}^k_n)_{n\in\mathbb{Z}}$ as follows,

$$\hat{\xi}^k = \begin{cases} (\xi^k_n,\, t^k_n), & \text{if } n \neq 0, \\ (p,\, t_0), & \text{if } n = 0. \end{cases}$$

Clearly all such sequences are passing through $p$. Moreover,

$$d(\phi_{\hat{t}^k_n}(\hat{\xi}^k_n),\, \hat{\xi}^k_{n+1}) = \begin{cases} d(\phi_{t^k_n}(\xi^k_n),\, \xi^k_{n+1}), & \text{if } n \neq 0,\, -1, \\ d(\phi_{t_0}(p),\, \xi^k_1), & \text{if } n = 0, \\ d(\phi_{t^k_{-1}}(\xi^k_1),\, p), & \text{if } n = -1, \end{cases}$$

so

$$d(\phi_{\hat{t}^k_n}(\hat{\xi}^k_n),\, \hat{\xi}^k_{n+1}) \leq \begin{cases} \frac{1}{k}, & \text{if } n \neq 0,\, -1, \\ d(\phi_{t_0}(p),\, \phi_{t^k_0}(\xi^k_0)) + \frac{1}{k}, & \text{if } n = 0, \\ d(\xi^k_0,\, p) + \frac{1}{k}, & \text{if } n = -1. \end{cases}$$

As $\phi$ is continuous and $(\xi^k_0,\, t^k_0) \to (\xi_0,\, p)$ we obtain that $(\hat{\xi}^k_n)$ is a $(\delta,\, 1,\, 2)$-pseudo-orbit for $k$ large. Then for such $k$ it follows that there are $x_k \in X$ and $h \in \mathrm{Rep}$ such that $d(p \star t,\, \phi_{h(t)}(x_k)) \leq \varepsilon$ for all $t \in \mathbb{R}$. For the sequences $(\hat{t}^k_i)_{k\in\mathbb{Z}}$ and $(t^k_i)_{k\in\mathbb{Z}}$ we write

$$\hat{s}^k_i = \begin{cases} \displaystyle\sum_{j=0}^{i-1} \hat{t}^k_j & i > 0, \\ 0 & i = 0, \\ \displaystyle -\sum_{j=i}^{-1} \hat{t}^k_j & i < 0, \end{cases}$$

and

$$s_i^k = \begin{cases} \sum_{j=0}^{i-1} t_j^k & i > 0, \\ 0 & i = 0, \\ -\sum_{j=i}^{-1} t_j^k & i < 0. \end{cases}$$

We will consider the three possible cases: $t_0^k < \hat{t}_0^k$ , $t_0^k = \hat{t}_0^k$ and $t_0^k > \hat{t}_0^k$. Note that in every case we have $|s_i^k - \hat{s}_i^k| = |t_0^k - \hat{t}_0^k|$ for all $i \in \mathbb{Z}$. We consider only the $t_0^k < \hat{t}_0^k$ case being the other two cases analogous. Then $s_i^k < \hat{s}_i^k$ for all $i \in \mathbb{Z}$. Let $t \in \mathbb{R}$ and let $i \in \mathbb{Z}$ such that $s_i^k \leq t < s_{i+1}^k$. We have two cases:

*Case 1:* if $s_i^k \leq t < s_{i+1}^k$, then in particular $\hat{s}_i^k \leq t < \hat{s}_{i+1}^k$, so

$$\begin{aligned} d(\phi_{t-s_i^k}(\xi_i^k),\, \phi_{h(t)}(x_k)) &\leq d(\phi_{t-s_i^k}(\xi_i^k),\, \phi_{t-\hat{s}_i^k}(\hat{\xi}_i^k)) + d(\phi_{t-\hat{s}_i^k}(\hat{\xi}_i^k),\, \phi_{h(t)}(x_k)) \\ &\leq \frac{\varepsilon}{3} + \varepsilon < 2\varepsilon. \end{aligned}$$

*Case 2:* if $s_i^k \leq t < \hat{s}_i^k$, again in particular, $\hat{s}_{i-1}^k \leq t < \hat{s}_i^k$, so

$$\begin{aligned} d(\phi_{t-s_i^k}(\xi_i^k),\, \phi_{h(t)}(x_k)) &\leq d(\phi_{t-s_i^k}(\xi_i^k),\, \phi_{t-\hat{s}_{i-1}^k}(\hat{\xi}_{i-1}^k)) + d(\phi_{t-\hat{s}_{i-1}^k}(\hat{\xi}_{i-1}^k),\, \phi_{h(t)}(x_k)) \\ &\leq d(\phi_{t-s_i^k}(\xi_i^k),\, \hat{\xi}_i^k) + d(\hat{\xi}_i^k,\, \phi_{\hat{t}_{i-1}^k}(\hat{\xi}_{i-1}^k)) + d(\phi_{\hat{t}_{i-1}^k}(\hat{\xi}_{i-1}^k),\, \phi_{t-\hat{s}_i^{k-1}}(\hat{\xi}_{i-1}^k)) + \varepsilon \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \varepsilon = 2\varepsilon, \end{aligned}$$

thus $d(\phi_{t-s_i^k}(\xi_i^k),\, \phi_{h(t)}(x_k)) \leq 2\varepsilon$ for all $s_i^k \leq t < s_{i+1}^k$. It follows that $\xi^k$ can be $2\varepsilon$-shadowed, which is a contradiction. This proves the result. □

**Theorem 2.2.7.** *Let $(X,\, d)$ be a compact metric space and $\phi$ be a flow on $X$. Then $\phi$ has the POTP if and only if $Sh(\phi) = X$*

*Proof.* Simply take $K = X$ in Lemma 2.2.6. □

Now we proof that that set of shadowable points is a dynamical invariant under continuous equivalences.

**Theorem 2.2.8.** *Let $\phi$ and $\psi$ be two flows on the compact metric spaces $(X,\, d_x)$ and $(Y,\, d_y)$, respectively. If $(h,\, \sigma)$ is a continuous equivalence from $\psi$ to $\phi$, then $h(Sh(\phi)) = Sh(\psi)$*

*Proof.* Let $(h^{-1},\, \tilde{\sigma})$ the corresponding inverse equivalence from $\psi$ to $\phi$. Let $a = \min\{\tilde{\sigma}(h(x),\, 1) : x \in X\}$. By compactness of $Y$ such $a$ exists and indeed is positive. Now, given $\varepsilon > 0$, choose $\varepsilon' > 0$ such that $d_y(y_1, y_2) < \varepsilon'$ implies $d_x(h^{-1}(y_1), h^{-1}(y_2)) < \varepsilon$ for every $y_1, y_2 \in Y$. Suppose $p \in h^{-1}(Sh(\phi))$. By Lemma 2.2.5, there exists $\delta' > 0$ such that each $(\delta', a)$-pseudo-orbit passing through $h(p)$ can be $\varepsilon'$-shadowed by an orbit of $\psi$. Also choose $\delta > 0$ so that $d_y(h(x_1), h(x_2)) < \delta'$ whenever $d_x(x_1, x_2) < \delta$ for all $x_1, x_2 \in X$. Now let $(x_n,\, t_n)_{n\in\mathbb{Z}}$ be a $(\delta,\, 1)$-pseudo-orbit for $\phi$ passing through $p$. Then $d_y(h(\phi_{t_n}(x_n)),\, h(x_{n+1})) < \delta'$. By definition of equivalence we have

$$d_x(\psi_{\tilde{\sigma}(h(x_n),\, t_n)}(h(x_n)),\, h(x_{n+1})) \leq \delta'.$$

Consider the sequence $(h(x_n),\, \tilde{\sigma}(h(x_n),\, t_n))_{n\in\mathbb{Z}}$. Since $t_n \geq 1$ it follows that $\tilde{\sigma}(x_n,\, t_n) \geq a$ for all $n \in \mathbb{Z}$. So $(h(x_n),\, \tilde{\sigma}(h(x_n),\, t_n))_{n\in\mathbb{Z}}$ is a $(\delta',\, a)$-pseudo-orbit for $\psi$ passing through $h(p)$. Then there are $y = h(z)$ in $Y$ and $\alpha \in \mathrm{Rep}$ such that

$$d_y(h(p) \star t,\, \psi_{\alpha(t)}(y)) < \varepsilon', \text{ for all } t \in \mathbb{R}$$

It follows that

$$d_x(h^{-1}(h(p) \star t),\, h^{-1}(\psi_{\alpha(t)}(h(z)))) < \varepsilon, \quad \text{for all } t \in \mathbb{R}. \tag{2.1}$$

Fix $t \in \mathbb{R}$ and $n \in \mathbb{Z}$ such that $s_n \leq t < s_{n+1}$ and let $(\widehat{s}_n)_{n\in\mathbb{Z}}$ the sequence of sums associated to $(\tilde{\sigma}(h(x_n),\, t_n))_{n\in\mathbb{Z}}$. Since $\tilde{\sigma}(h(x_n),\, \cdot) \in \mathrm{Rep}$ we have $0 \leq \tilde{\sigma}(h(x_n),\, t - s_n) < \tilde{\sigma}(h(x_n),\, t_n)$. So, $\widehat{s}_n \leq \tilde{\sigma}(h(x_n),\, t - s_n) + \widehat{s}_n < \widehat{s}_{n+1}$. Set $\widehat{t} = \tilde{\sigma}(h(x_n),\, t - s_n) + \widehat{s}_n$, so $h(p) \star \widehat{t} = \psi_{\tilde{\sigma}(h(x_n),\, t-s_n)}(h(x_n))$. By (2.1) it follows that

$$d_x(\phi_{t-s_n}(x_n), \phi_{\sigma(z,\, \alpha(\widehat{t}))}(z)) = d_x(h^{-1}(\psi_{\widehat{t}-\widehat{s}_n}(h(x_n))), h^{-1}(\psi_{\alpha(\widehat{t}\,)}(h(z)))) \leq \varepsilon.$$

Let $\widehat{\alpha}(t) = \sigma(z,\, \alpha(\widehat{t}))$ for all $t \in \mathbb{R}$. Since $t \mapsto \tilde{\sigma}(h(x_n),\, t - s_n) + \widehat{s}_n$ is increasing, then $\widehat{\alpha} \in \mathrm{Rep}$. It follows that

$$d_x(\phi_{t-s_n}(x_n), \phi_{\widehat{\alpha}(t)}(z)) \leq \varepsilon, \quad \text{for every } s_n \leq t < s_{n+1}, \text{ and } t \in \mathbb{R}.$$

This proves that $(x_i,\, t_i)_{i\in\mathbb{Z}}$ is $\varepsilon$-shadowed by $\mathcal{O}(z)$. Therefore, $h^{-1}(Sh(\psi)) \subseteq Sh(\phi)$. The inclusion $h(Sh(\phi)) \subseteq Sh(\psi)$ is obtained analogously considering the equivalence $(h,\, \sigma)$ from $\phi$ to $\psi$. This completes the proof. □

Now we introduce another auxiliary definition.

**Definition 2.2.9.** Let $\varepsilon > 0$ and $K$ be a subset of $X$. We say that a flow $\phi$ has the $\varepsilon$-POTP through a subset $K$ if there exists $\delta > 0$ such that every $(\delta,\, 1)$-pseudo-orbit passing through $K$ can be $\varepsilon$-shadowed.

We denote by $B[\,\cdot,\, \delta]$ the close ball operation on $X$.

**Lemma 2.2.10.** *Let $\phi$ be a flow on the compact metric space $X$ and let $\varepsilon > 0$. If the flow $\phi$ has the $\varepsilon$-POTP through a compact subset $K$, then there is $\delta > 0$ so that $\phi$ has the $2\varepsilon$-POTP through $B[K,\, \delta]$.*

*Proof.* Suppose by contradiction that a flow $\phi$ has the $\varepsilon$-POTP through a subset $K$ but for every $\delta > 0$, we can find a $(\delta,\, 1)$-pseudo-orbit passing through $B[K,\, \delta]$ that cannot be $2\varepsilon$-shadowed.

Take a $\delta > 0$ from the $\varepsilon$-POTP through $K$ with $\delta < \varepsilon$, and let $(\xi^k)_{k\in\mathbb{N}}$ be a sequence of $(\frac{1}{k},\, 1)$-pseudo-orbits passing through $B[K,\, \frac{1}{k}]$ which cannot be $2\varepsilon$-shadowed. For every $k \in \mathbb{N}$ we write $\xi^k = (\xi^k_n,\, t^k_n)_{n\in\mathbb{Z}}$. It follows from the definition that there is a sequence $x_k \in K$ such that $d(\xi^k_0, x_k) \leq \frac{1}{k}$ for all $k \in \mathbb{N}$. Since $X$ is compact, the flow $\phi$ is uniformly continuous in $X \times [-t^1_0,\, t^1_0]$, so we can choose $k$ with the property that

$$\max\{\max_{-t^1_0 \leq t \leq t^1_0}\{d(\phi_t(\xi^k_0),\, \phi_t(x_k))\},\, \tfrac{1}{k}\} \leq \frac{\delta}{2}.$$

Fix $k$ and define a sequence $\xi = (\xi_n, t_n)_{n\in\mathbb{Z}}$ by

$$(\xi_n, t_n) = \begin{cases} (\xi_n^k, t_n^k) & \text{if } n \neq 0, \\ (x_k, t_0^k) & \text{otherwise.} \end{cases}$$

Clearly, $d(\phi_{t_n}(\xi_n), \xi_{n+1}) \leq \frac{1}{k} < \delta$ for $n \neq -1, 0$. Since

$$d(\phi_{t_{-1}}(\xi_{-1}), \xi_0) = d(\phi_{t_{-1}^k}(\xi_{-1}^k), x_k) \leq d(\phi_{t_{-1}^k}(\xi_{-1}^k), \xi_0^k) + d(x_k, \xi_0^k) \leq \tfrac{1}{k} + \tfrac{1}{k} = \frac{2}{k} \leq \delta$$

and

$$d(\phi_{t_0}(\xi_0), \xi_1) = d(\phi_{t_0^1}(x_k), \xi_1^k) \leq d(\phi_{t_0^1}(x_k), \phi_{t_0^1}(\xi_0^k)) + d(\phi_{t_0^1}(\xi_0^k), \xi_1^k) \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

we see that $\xi$ is a $(\delta, 1)$-pseudo-orbit. Since $\xi_0 = x_k \in K$ by definition, we obtain that $\xi$ can be $\varepsilon$-shadowed by a point $y \in X$. Thus, there exists $h \in \text{Rep}$ such that

$$d(\xi_0 \star t, \phi_{h(t)}(y)) < \varepsilon, \quad \text{for each } t \in \mathbb{R}.$$

Note that for $i \neq 0$ and $t$ such that $s_i \leq t < s_{i+1}$, we have

$$\xi_0 \star t = \phi_{t-s_i}(\xi_i) = \phi_{t-s_i}(\xi_i^k) = \xi_0^k \star t.$$

Hence $\xi_0 \star t = \xi_0^k \star t$ for $t \notin [s_0, s_1)$. Furthermore, for $t \in [s_0, s_1)$,

$$\begin{aligned} d(\xi_0^k \star t, \phi_{h(t)}(y)) &\leq d(\xi_0^k \star t, \xi_0 \star t) + d(\xi_0 \star t), \phi_{h(t)}(y)) \\ &\leq d(\phi_t(\xi_0^k), \phi_t(\xi_0)) + d(\xi_0 \star t, \phi_{h(t)}(y)) \\ &\leq \frac{\delta}{2} + \varepsilon \\ &\leq 2\varepsilon. \end{aligned}$$

Thus, $d(\xi_0^k \star t, \phi_{h(t)}(y)) \leq 2\varepsilon$ for all $t \in \mathbb{R}$. It follows that $\xi^k$ is $2\varepsilon$-shadowed, which is a contradiction. This proves the result □

A subset of $X$ is a $G_\delta$ set if it is a countable intersection of open sets of $X$. In [42], examples of homeomorphisms where the set of shadowable points is a $G_\delta$ sets are given. On the other hand, in [31] the author proved that the set of shadowable points of a homeomorphism is a Borel set. The following theorem extends this result in the flow case.

**Theorem 2.2.11.** *The set of shadowable points of $\phi$ is a $G_\delta$ set of $X$. In particular, it is a Borel set.*

*Proof.* Given $\varepsilon > 0$ we denote by $Sh(\phi, \varepsilon)$ the set of points $p \in X$ such that the flow has the $\varepsilon$-POTP through a subset $\{p\}$ (see Definition 2.2.9). Note that

$$Sh(\phi) = \bigcap_{\varepsilon>0} Sh(\phi, \varepsilon). \tag{2.2}$$

Let $\varepsilon_0 > 0$. We can suppose $Sh(\phi) \neq \emptyset$. Given $x \in Sh(\phi)$, since $x \in Sh(\phi, \frac{\varepsilon_0}{2})$, by Lemma 2.2.10 there is $\delta_{x,\varepsilon_0} > 0$ such that every $(\delta_{x,\varepsilon_0}, 1)$-pseudo-orbit passing through $B[x,\, \delta_{x,\,\varepsilon_0}]$ can be $\varepsilon_0$-shadowed. It follows that $B(x,\, \delta_{x,\,\varepsilon_0}) \subset Sh(\phi,\, \varepsilon_0)$. So, for every $\varepsilon_0 > 0$

$$Sh(\phi,\, \varepsilon_0) = A(\varepsilon_0) \cup B(\varepsilon_0),$$

where $A(\varepsilon_0) = \bigcup_{x \in Sh(\phi)} B(x,\, \delta_{x,\,\varepsilon_0})$ and $B(\varepsilon_0) = Sh(\phi,\, \varepsilon_0) \setminus A(\varepsilon_0)$. Moreover, note that $A(\varepsilon_0)$ is open and $B(\varepsilon_0) \subset Sh(\phi,\, \varepsilon_0) \setminus Sh(\phi)$. By (2.2) we have $\bigcap_{\varepsilon_0 > 0} B(\varepsilon_0) = \emptyset$ and

$$Sh(\phi) = \bigcap_{n \in \mathbb{N}} Sh(\phi,\, \tfrac{1}{n}) = \bigcap_{n \in \mathbb{N}} A(\tfrac{1}{n}) \cup B(\tfrac{1}{n}) = \bigcap_{n \in \mathbb{N}} A(\tfrac{1}{n}).$$

That is, $Sh(\phi)$ is a $G_\delta$ set of $X$. □

*Remark* 4. The same ideas behind the proof of Theorem 2.2.11 implies that the set of shadowable points of a homeomorphism is a $G_\delta$ set too.

A flow $\phi$ on $X$ said to be is *minimal* if all of its orbits are dense in $X$. It is said to be an *isometric flow* if $d(\phi_t(x),\, \phi_t(y)) = d(x,\, y)$ for every $x,\, y \in X$ and $t \in \mathbb{R}$.

**Example 5.** If $\phi$ and $\psi$ are continuous flows on $X$ and then is not always true that $Sh(\phi) \times Sh(\psi) \subset Sh(\phi \times \psi)$. Indeed, if we consider $\phi(z,\, t) = e^{2t\pi i} z$, defined in the unit circle $S^1$, this flow has the POTP. Then $Sh(\phi) = S^1$ by Theorem 2.2.7. If the inclusion holded then $Sh(\phi \times \phi) = S^1 \times S^1$. Again, by Theorem 2.2.7, $\phi \times \phi$ would have POTP. However this is not possible because this flow is isometric and is not minimal [32].

## 2.3 Shadowable points of suspension flows

In this section we give the relation between the set of shadowable points of a homeomorphism and the shadowable points of its suspension flows.

**Theorem 2.3.1.** *If $\phi^{f,\tau}$ is the suspension flow of a homeomorphism $f$ on the compact metric space $(X, d)$ under a continuous map $\tau : X \to \mathbb{R}_+$, then*

$$Sh(\phi^f) = (Sh(f) \times [0, 1]) / \sim .$$

*Proof.* We can assume without loss of generality that $\tau \equiv 1$. Given $(z, t) \in Sh(\phi^f)$ since $Sh(\phi^f)$ is invariant by $\phi^f$, then $(z, \frac{1}{2}) \in Sh(\phi^f)$. Let $\varepsilon > 0$ be given. Choose $\varepsilon' > 0$ with $\varepsilon' < \min\{\varepsilon, \frac{1}{4}\}$ so that $d(f^i(x), f^i(y)) < \varepsilon$ for $i = -1, 0, 1$, whenever $d(x, y) < \varepsilon'$. Choose $\delta > 0$ from the definition of shadowable point for $\phi^f$ respect to $\varepsilon'$. Also take $0 < \delta' < \delta$ so that $d(x, y) < \delta'$ implies $d(f(x), f(y)) < \delta$. Let $\{x_n\}_{n \in \mathbb{Z}}$ be any $\delta'$-pseudo-orbit of $f$ with $x_0 = z$. Consider the pair of sequences $(x_n, \frac{1}{2})_{n \in \mathbb{Z}}$ and $(t_n)_{n \in \mathbb{Z}}$ such that $t_n = 1$ for each $n \in \mathbb{Z}$. Then

$$\begin{aligned} d^f(\phi^f_{t_n}(x_n, \tfrac{1}{2}), (x_{n+1}, \tfrac{1}{2})) &= d^f((f(x_n), \tfrac{1}{2}), (x_{n+1}, \tfrac{1}{2})) \\ &= \tfrac{1}{2} d(f(x_n), x_{n+1}) + \tfrac{1}{2} d(f^2(x_n), f(x_{n+1})) \le \delta. \end{aligned}$$

That is $((x_n, \frac{1}{2}), t_n)_{n \in \mathbb{Z}}$ is a $(\delta, 1)$-pseudo-orbit of $\phi^f$ with $x_0 = z$. So, there are $(x, s) \in X^{1,f}$ and $\alpha \in \mathrm{Rep}$ such that

$$d^f(\phi^f_{\alpha(t)}(x, s), \phi^f_{t-n}(x_n, \tfrac{1}{2})) < \varepsilon', \quad \text{for } n \le t < n + 1 \quad (n \in \mathbb{Z}).$$

Now as $t = 0$, we have

$$d^f((x,s),(z,\tfrac{1}{2})) < \frac{1}{4},$$

so $|s - \frac{1}{2}| < \frac{1}{4}$. Moreover, since $d^f(\phi^f_{\alpha(t)}(x,s), \phi^f_t(z,\frac{1}{2})) < \varepsilon' < \frac{1}{4}$ for all $0 \le t < 1$, it follows that $d^f((x, s+\alpha(1)), (z, \frac{3}{2})) < \frac{1}{4}$. Thus we obtain

$$|\tfrac{3}{2} - s - \alpha(1)| < \tfrac{1}{4}.$$

Then $1 \le s + \alpha(1) < 2$ and so $\phi^f_{\alpha(1)}(x,s)$ should be represented as $(f(x), s^{(1)})$ where $0 \le s^{(1)} < 1$. Also we have $d^f(\phi^f_{\alpha(t)}(x,s), \phi^f_{t-1}(x_1,\frac{1}{2})) < \varepsilon' < \frac{1}{4}$ for all $1 \le t < 2$. Thus $d^f((f(x), s^{(1)} + \alpha(2) - \alpha(1)), (x_1, \frac{3}{2})) < \frac{1}{4}$, therefore $\phi^f_{\alpha(2)}(x,s)$ should be represented as $(f^2(x), s^{(2)})$ where $0 \le s^{(2)} < 1$. If we carry on in the same manner we will have that $\phi^f_{\alpha(n)}(x,s)$ should be represented as $(f^n(x), s^{(n)})$ where $0 \le s^{(n)} < 1$ for each $n \in \mathbb{Z}$. For $t = n$, we have

$$d^f(\phi^f_n(x,s), (x_n, \tfrac{1}{2})) = d^f((f^n(x), s^{(n)}), (x_n, \tfrac{1}{2})) < \varepsilon'.$$

If $f^n(x) = x_n$, $d(f^n(x), x_n) < \varepsilon$ is trivial. If $f^n(x) \ne x_n$, it follows that

$$\tfrac{1}{2}d(f^n(x), x_n) + \tfrac{1}{2}d(f^{n+1}(x), f(x_n)) \le d^f((f^n(x), s^{(n)}), (x_n, \tfrac{1}{2})) < \varepsilon'.$$

Hence $d(f^n(x), x_n) < \varepsilon'$ or $d(f^{n+1}(x), f(x_n)) < \varepsilon'$. From the way we chose $\varepsilon'$ this implies that $d(f^n(x), x_n) < \varepsilon$ for every $n \in \mathbb{Z}$. Therefore $z \in Sh(f)$.

Conversely, let $z \in Sh(f)$ and $r \in [0,1]$. Given $\varepsilon > 0$, take $0 < \varepsilon' < \varepsilon$ so that $d(x,y) < \varepsilon'$ implies $d(f^i(x), f^i(y)) < \frac{1}{2}\varepsilon$ for $i = 0,1,2$. Take $\delta$ with $0 < \delta < \frac{1}{2}\varepsilon'$ from the shadowableness of $z$ with respect to $\varepsilon'$. Take $0 < \delta' < \min\{\frac{1}{4}, \delta\}$ as in Lemma 2.5 in [47] and let $((x_k, s_k), t_k)_{k\in\mathbb{Z}}$ be a $(\delta', 2, 4)$-pseudo orbit of the suspension flow $\phi^f$ of $f$ on $X^{1,f}$ passing through $(z, r)$. Let $w_k = \lfloor s_k + t_k \rfloor$ denote the integer part of $s_k + t_k$. We have

$$d^f((f^{w_k}(x_k),\ s_k + t_k - w_k),\ (x_{k+1}, s_{k+1})) < \delta' \quad \text{for all } k \in \mathbb{Z}.$$

Since $\delta' < \frac{1}{4}$, by Lemma 2.4 in [47], we have that $|s_k + t_k - w_k - s_{k+1}| < \delta'$ or $|1 + s_k + t_k - w_k - s_{k+1}| < \delta'$ or $|1 + s_{k+1} + w_k - t_k - s_k| < \delta'$. Now, let $n_k$ be a positive integer defined as follows

$$n_k = \begin{cases} w_k & \text{if } |s_k + t_k - w_k - s_{k+1}| < \delta', \\ w_k - 1 & \text{if } |1 + s_k + t_k - w_k - s_{k+1}| < \delta', \\ w_k + 1 & \text{if } |1 + s_{k+1} + w_k - t_k - s_k| < \delta'. \end{cases}$$

Then by Lemma 2.5 in [47] we obtain that $d(f^{n_k}(x_k),\ x_{k+1}) < \delta$ for all $k \in \mathbb{Z}$. Define a sequence $(y_i)_{i\in\mathbb{Z}}$ in $X$ as follows:

$$y_i = f^{i-N_k}(x_k) \quad \text{for } N_k \le i < N_{k+1},$$

where $(N_k)_{k\in\mathbb{Z}}$ is the sequence of sums associated to $(n_k)_{k\in\mathbb{Z}}$. Obviously this sequence is a $\delta$-pseudo-orbit of $f$ passing through $z$. Hence, there exists $x \in X$ such that $d(f^i(x),\ y_i) < \varepsilon'$ for every $i \in \mathbb{Z}$. In particular one has

$$d(f^{j+N_k}(x),\ f^j(x_k)) < \varepsilon' \quad \text{for } 0 \le j < n_k \ (k \in \mathbb{Z}). \tag{2.3}$$

Now, take the point $(x,\, t) \in X^{1,f}$ and define $\alpha : \mathbb{R} \to \mathbb{R}$ in the following way:

$$\alpha(t) = \frac{s_{k+1} + n_k - s_k}{t_k}(t - T_k) + s_k + N_k - s_0 \quad \text{for } T_k \le t < T_{k+1},$$

where $(T_k)_{k\in\mathbb{Z}}$ is the sequence of sums associated to $(t_k)_{k\in\mathbb{Z}}$. It is clear that $\alpha$ is continuous with $\alpha(0) = 0$. Moreover, since $n_k \ge 1$ then $\alpha \in \text{Rep}$. We claim that $\mathcal{O}(x, r)$ is an orbit of $\phi^f$ which $\varepsilon$-traces $((x_k,\, s_k),\, t_k)_{k\in\mathbb{Z}}$. Indeed, let $t \in \mathbb{R}$ and let $k \in \mathbb{Z}$ such that $T_k \le t < T_{k+1}$. We get

$$\begin{aligned} |\alpha(t) - s_k - N_k + s_0 - (t - T_k)| &= \left| \frac{s_{k+1} + n_k - s_k - t_k}{t_k}(t - T_k) \right| \\ &= |s_{k+1} + n_k - s_k - t_k| \left| \frac{t - T_k}{t_k} \right|. \end{aligned}$$

Since $|s_k + t_k - n_k - s_{k+1}| < \delta'$ and $0 \le t - T_k < t_k$, we have

$$|\alpha(t) - s_k - N_k + s_0 - (t - T_k)| < \delta'. \tag{2.4}$$

Now if $j$ is a positive integer which makes $0 \le s_k + t - T_k - j < 1$, then $0 \le j \le s_k + t_k \le n_k + 2$. So by (2.3) and the choice of $\varepsilon'$ we get $d(f^{j+N_k}(x), f^j(x_k)) < \frac{1}{2}\varepsilon$ for $0 \le j \le n_k + 2$. Finally,

$$d^f(\phi^f_{\alpha(t)}(x,\, s_0),\, \phi^f_{t-T_k}(x_k,\, s_k)) \;=\; d^f\left((f^{N_k}(x),\, s_0 + \alpha(t) - N_k),\, (x_k,\, s_k + t - T_k)\right)$$

$$\begin{aligned} &= d^f\left((f^{j+N_k}(x),\, s_0 + \alpha(t) - N_k - j),\, (f^j(x_k),\, s_k + t - T_k - j)\right) \\ &\le d^f\left((f^{j+N_k}(x),\, s_0 + \alpha(t) - N_k - j),\, (f^{j+N_k}(x),\, s_k + t - T_k - j)\right) \\ &\quad + d^f\left((f^{j+N_k}(x),\, s_k + t - T_k - j), (f^j(x_k),\, s_k + t - T_k - j)\right) \\ &\le |s_0 + \alpha(t) - N_k - j - (s_k + t - T_k - j)| + (s_k + t - T_k - j)d(f^{j+N_k+1}(x),\, f^{j+1}(x_k)) \\ &\quad + (1 - s_k - t + T_k + j)d(f^{j+N_k}(x),\, f^j(x_k)) \\ &< \delta' + \tfrac{1}{2}(1 - (s_k + t - T_k - j))\varepsilon + \tfrac{1}{2}(s_k + t - T_k - j)\varepsilon \le \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Hence $(x,\, s_0) \in Sh(\phi^f)$. □

With this theorem we have the following example of a flow on a compact metric space whose shadowable set is a nonclosed subset of $X$.

**Example 6.** Let $f \colon X \to X$ be the map of Example 3. Then, by Theorem 2.3.1, $Sh(\phi^f) = (C \setminus \{1\})^{1,f}$ which is a nonclosed proper subset of $X^{1,f}$.

This example motivates the question of whether there exists a flow with a nonclosed dense subset of shadowable points.

**Definition 2.3.2.** A flow $\phi$ on a compact metric space $X$ has the *almost POPT* if $Sh(\phi)$ is dense in $X$.

Obviously, a flow having the POTP has the almost POTP. The following theorem shows that the converse may not be true.

**Theorem 2.3.3.** *There exists a flow with the almost POPT but without the POPT.*

*Proof.* Take an almost totally disconnected compact metric space $X$ (that is a space $X$ where $X^{deg}$ is dense in $X$) which is not totally disconnected (e.g. [23] or a non-totally disconnected cantoroid as in Definition 2 p. 740 of [8]). Then, take any equicontinous map $f : X \to X$, so that $Sh(f) = X^{deg}$, but it does no have the POTP (by Corollary 2.1.7). Suspend this map to obtain a flow with the desired properties. □

**Example 7.** The above Theorem also provides an example of a flow $\phi$ for which $Sh(\phi)$ is a proper subset of $CR(\phi)$.

Another application of Theorem 2.3.1 is that the set of shadowable points of a homeomorphism is a $G_\delta$ set of its phase space. In order to see it we first state two elementary lemmas.

**Lemma 2.3.4.** *Let $X$ be topological space and consider the space $X \times [0, 1]$ with the usual product topology. If $A \subseteq X$ and $A \times [0, 1]$ is $G_\delta$ set of $X \times [0, 1]$ then $A$ is a $G_\delta$ set of $X$.*

*Proof.* By considering $X$ as a subset of $X \times [0, 1]$ under a canonical inclusion, we note that if $A \times [0, 1] \subseteq U$ where $U$ is open, then $U \cap X$ is a nonempty open set of $X$ containing $A$. Indeed, $A \times \{0\} \subseteq U$ and if $(x, 0) \in U$ there exists open sets $V \subseteq X$ containing $x$ and $W \subseteq [0, 1]$ containing 0 such that $(x, 0) \in V \times W \subseteq U$. In particular, we have $V \subseteq U \cap (X \times \{0\})$ showing that $U \cap X$ is open in $X$. Hence, if $A \times [0, 1] = \bigcap_{n=1}^{\infty} A_n$ for some sequence $(A_n)_{n=1}^{\infty}$ of open sets of $X \times [0, 1]$, then $A \times [0, 1] \subseteq A_n$ for each $n \in \mathbb{N}$ and so $A \subseteq A_n \cap X$ with each $A_n \cap X$ an open set of $X$. On the other hand,

$$\bigcap_{n=1}^{\infty}(A_n \cap X) = X \cap \left(\bigcap_{n=1}^{\infty} A_n\right) = X \cap (A \cap [0, 1]) = A.$$

Therefore $A$ is a $G_\delta$ set of $X$. □

Remind that for a given map $p\colon X \to Y$, where $X$ and $Y$ are any sets, a subset $U$ of $X$ is *saturated* with respect to $p$ if $U = p^{-1}(p(U))$. It is easy to see that $U$ is saturated if and only if for every $x \in U$ and $y \in X$ the equality $p(x) = p(y)$ implies that $y \in U$.

**Lemma 2.3.5.** *Let $f$ be a homeomorphism on a compact metric space $X$ and let $\phi^f$ the suspension flow of $f$ on the space $X^{1,f}$. Let $p\colon X \times [0, 1] \to X^{1,f}$ be the quotient map of the topology of $X^{1,f}$. If $A \subseteq X$ is invariant with respect to $f$, then $A \times [0, 1]$ is saturated with respect to $p$.*

*Proof.* Let $(x, t) \in A \times [0, 1]$ and $(y, s) \in X \times [0, 1]$ such that $p(x, t) = p(y, s)$. Then $(y, s) = (x, t)$ if $0 < t < 1$, $(y, s) = (f^{-1}(x), 1)$ if $t = 0$ and $(y, s) = (f(x), 0)$ if $t = 1$. In any case $y \in A$ because $A$ is invariant. This proves the lemma. □

**Theorem 2.3.6.** *The set $Sh(f)$ of shadowable points of a homeomorphism $f\colon X \to X$ defined on compact metric space $(X, d)$ is a $G_\delta$ set of $X$.*

*Proof.* By theorem 2.2.11, if $\phi^f$ is the suspension of $f$ under the constant function 1, then $Sh(\phi^f)$ is a $G_\delta$ set of $X^{1,f}$. So, there exists a sequence $(A_n)_{n=1}^{\infty}$ of open set of $X^{1,f}$ such that $Sh(\phi^f) = \bigcap_{n=1}^{\infty} A_n$ On the other hand, by Theorem 2.3.1 we have $Sh(\phi^f) = Sh(f) \times [0, 1]/\sim$. Let $p\colon X \times [0, 1] \to X^{1,f}$ be the quotient map associated to the quotient topology. We have that $p(Sh(f) \times [0, 1]) = Sh(\phi^f)$. Since $Sh(f)$ is invariant with respect to $f$, then $Sh(f) \times [0, 1]$ is saturated with respect to $p$ by Lemma 2.3.5. So $Sh(f) \times [0, 1] = p^{-1}(p(Sh(\phi) \times [0, 1])) = p^{-1}(Sh(\phi^f))$. By definition of quotient maps, $p^{-1}(A_n)$ is an open set of $X \times [0, 1]$, hence

$$Sh(f) \times [0, 1] = p^{-1}\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} p^{-1}(A_n)$$

and so $Sh(f) \times [0, 1]$ is a $G_\delta$ set of. Finally, $Sh(f)$ is a $G_\delta$ set of $X$ by Lemma 2.3.4. This concludes the proof. □

*Remark* 5. In [31], the author proved that the set of shadowable points of a homeomorphism is a Borel set. But what he proved indeed is that such a set is a $F_{\sigma\delta}$ set of phase space, i.e., a countable intersection of countable union of closed sets. So Theorem 2.3.6 is improvement of Kawaguchi's result.

## 2.4 Shadowable points and recurrence

In this section we study the recurrence behavior of a flow under the presence of shadowable points.

**Lemma 2.4.1.** *If $\phi$ is a flow on a compact metric space $X$, then $Sh(\phi) \cap CR(\phi) \subseteq \Omega(\phi)$.*

*Proof.* Let $p \in Sh(\phi) \cap CR(\phi)$ and $\varepsilon > 0$ be given. Then there exists $\delta > 0$ from the shadowableness of $p$. Since $p$ is a chain recurrent point, there exists a $(\delta, 1)$-chain $(x_i, t_i)_{i=0}^{k}$ with $p = x_0 = x_k$ and $t_0 = t_k$. For every integer number $n$ we put $x_{kn+i} = x_i$, $t_{kn+i} = t_i$ for $0 \leq i < k$. So, $(x_i, t_i)_{i\in\mathbb{Z}}$ is a $(\delta, 1)$-pseudo-orbit passing through $p$ for $\phi$ and therefore there are $y \in X$ and $g \in \text{Rep}$ such that $d(p \star t, \phi_{g(t)}(y)) \leq \varepsilon$ for every $t \in \mathbb{R}$. It follows that $y \in B[p, \varepsilon]$ because $g(0) = 0$ by definition. For every $j \geq 0$ make $m_j = j\sum_{i=0}^{k-1} t_i$. Then,

$$\begin{aligned} d(x, \phi_{g(m_j)}(y)) &= d(x \star m_j, \phi_{g(m_j)}(y)) \\ &\leq \varepsilon, \ \forall j \geq 0 \end{aligned}$$

and $m_j \geq jk$ for all $j \geq 0$. So $m_j \to \infty$ $(j \to \infty)$. Since $g \in \text{Rep}$, $g(m_j) \to \infty$. Therefore $x \in \Omega(\phi)$, and the lemma follows. □

**Theorem 2.4.2.** *If $CR(\phi) \subseteq Sh(\phi)$ then $CR(\phi) = \Omega(\phi)$.*

*Proof.* Since $\Omega(\phi) \subset CR(\phi)$, if $CR(\phi) \subseteq Sh(\phi)$ then $\Omega(\phi) = CR(\phi)$ by Lemma 2.4.1. □

In some especial case, we can improve Theorem 2.4.2. We first remind the definition of expansive flow [14]:

**Definition 2.4.3.** A flow $\phi$ on the metric space $X$ is *expansive* if given $\varepsilon > 0$ there exists $\delta > 0$ with the property that if $d(\phi_t(x), \phi_{h(t)}(y)) \leq \delta$ for every pair of points $x$ and $y$, for all $t \in \mathbb{R}$ and some $h\colon \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$, then $y = \phi_t(x)$ with $|t| < \varepsilon$.

This notion is invariant under flow equivalences and it can be seen that if a flow is expansive, then every singularity is isolated. Even more, in the case where there are not singularities, we can take the function $h$ belonging to Rep. (see [14]).

The following Lemma can be found is Lemma 3.10 in [53].

**Lemma 2.4.4.** *Let $\phi$ be a continuous flow without singularities on a compact metric space $X$. For each $\lambda > 0$ small enough, there exists $\varepsilon > 0$ such that for every $x, y \in X$, for every interval $[T_1, T_2]$ containing the origin and for every $\alpha \in$ Rep, the following holds: if $d(\phi_t(x), \phi_{\alpha(t)}(y)) \leq \varepsilon$ for all $t \in [T_1, T_2]$, then $|\alpha(t) - t| < \lambda$ for $|t| \leq 2$ in $[T_1, T_2]$ and $|\alpha(t) - t| < |t|\lambda$ for $|t| > 2$ in $[T_1, T_2]$.*

**Lemma 2.4.5.** *If $\phi$ is a expansive flow on a compact metric space $X$, then $CR(\phi) \cap Sh(\phi) \subset \overline{Per(\phi)}$.*

*Proof.* Without loss of generality, by expansiveness, we can suppose that the flow has no singularities. Let $x \in CR(\phi) \cap Sh(\phi)$ and $\varepsilon \in (0, 1)$. We can take $\varepsilon$ satisfying Lemma 2.4.4 with respect to $\lambda = \frac{1}{2}$. Take $\delta > 0$ satisfying the definition of shadowing with respect to $\varepsilon$. Since $x \in CR(\phi)$, there is a $(\frac{\delta}{2}, 3)$-chain $(x_i, t_i)_{i=0}^k$ where $x_0 = x_k = x$ and $t_i \geq 3$. Assume also that $\delta$ satisfies the expansiveness of $\phi$ with respect to $\varepsilon$. Extend the $(\frac{\delta}{2}, 3)$-chain $(x_i, t_i)_{i=0}^k$ to a $(\frac{\delta}{2}, 3)$-pseudo-orbit $(x_i, t_i)_{i\in\mathbb{Z}}$ by putting for every integer number $n$, $x_{kn+i} = x_i$, and $t_{kn+i} = t_i$ provided $0 \leq i < k$. Thus, there are $z \in X$ and $\alpha \in$ Rep such that $d(\phi_{\alpha(t)}(z), \phi_{t-s_i}(x_i)) \leq \frac{\delta}{2}$ for $s_i \leq t < s_{i+1}$. If $L = t_0 + \ldots + t_{k-1}$, for $d(\phi_{\alpha(t+L)}(z), \phi_{t-s_i}(x_i)) \leq \frac{\delta}{2}$ whenever $s_i \leq t < s_{i+1}$. Therefore

$$d(\phi_{\alpha(t+L)}(z), \phi_{\alpha(t)}(z)) \leq \delta \text{ for every } t \in \mathbb{R}.$$

Take $u = \alpha(t)$, then

$$d(\phi_{\alpha(\alpha^{-1}(u)+L)}(z), \phi_u(z)) = d(\phi_{\alpha(\alpha^{-1}(u)+L)-\alpha(L)}(\phi_{\alpha(L)}(z)), \phi_u(z)) \leq \delta$$

for every $u \in \mathbb{R}$, where $u \mapsto \alpha(\alpha^{-1}(u) + L) - \alpha(L) \in$ Rep. Hence, by expansiveness, $\phi_{\alpha(L)}(z) \in \phi_{(-\varepsilon,\varepsilon)}(z)$. Moreover since $d(\phi_{\alpha(t)}(z), \phi_t(x)) \leq \frac{\varepsilon}{2}$ for $0 \leq t < t_0$, then $\frac{1}{2}s \leq \alpha(s)$ for some $2 \leq s \leq t_0$, by Lemma 2.4.4. Then $\varepsilon \leq \alpha(s) \leq \alpha(L)$ since $s \leq L$. Therefore $z \in Per(\phi)$. ☐

So in the case that $\phi$ is an expansive flow, we obtain the following improvement of Theorem 2.4.2:

**Theorem 2.4.6.** *If $\phi$ is expansive and $CR(\phi) \subseteq Sh(\phi)$, then $CR(\phi) = \overline{Per(\phi)}$.*

*Proof.* It follows directly from the fact that $\overline{Per(\phi)} \subseteq CR(\phi)$. Then $CR(\phi) = \overline{Per(\phi)}$ by Lemma 2.4.5. ☐

We denote by $Sh^+(\phi)$ the set of points such that given $\varepsilon > 0$ there exists $\delta > 0$ such that every forward $(\delta, 1)$-pseudo-orbit $(x_i, t_i)_{i=0}^{\infty}$ passing through $p$ can be $\varepsilon$-shadowed. Each element of $Sh^+(\phi)$ is said *forward shadowable point*. All theorems proven until now about shadowable points are equally valid for forward shadowable points. From now on, we will use this observation without explicit mention of it.

We recall that a chain transitive flow $\phi$ is one where $X$ is a chain transitive set. That is, for every $p, q \in X$ we have $p \sim q$ [1].

**Theorem 2.4.7.** *If the flow $\phi$ is chain transitive, then $Sh^+(\phi) = X$ or $Sh^+(\phi) = \emptyset$.*

*Proof.* Suppose that $Sh^+(\phi) \neq \emptyset$. Let $p \in Sh^+(\phi)$ and $q \in X$. Let $\varepsilon > 0$ and take $\delta > 0$ from the forward shadowableness of $p$. Let $(x_i,\, t_i)_{i=0}^{\infty}$ a forward $(\delta,\, 1)$-pseudo-orbit passing through $q$ and let $(y_i,\, s_i)_{i=0}^{m}$ a $(\delta,\, 1)$-chain such that $y_0 = p$ and $y_m = q$. We have that the sequence of pairs $(z_j,\, r_j)_{j=0}^{\infty}$ given by

$$(z_j,\, r_j) = \begin{cases} (y_j,\, s_j) & \text{if } 0 \le j < m, \\ (x_{j-m},\, t_{j-m}) & \text{if } j \ge m. \end{cases}$$

is a forward $(\delta,\, 1)$-pseudo-orbit passing through $p$. We set

$$\hat{r}_i = \begin{cases} \sum_{j=0}^{i-1} r_j & i > 0, \\ 0 & i = 0, \end{cases}$$

and

$$\hat{t}_i = \begin{cases} \sum_{j=0}^{i-1} t_j & i > 0, \\ 0 & i = 0. \end{cases}$$

Hence there are $h \in \mathrm{Rep}$ and a point $y \in X$ such that $d(p \star t,\, \phi_{h(t)}(y)) \le \varepsilon$, for $t \in [0,\, \infty)$. Let $g(t) = h(t + \hat{r}_m) - h(\hat{r}_m)$. Clearly $g \in \mathrm{Rep}$.

Note that

$$\hat{r}_{m+k} - \hat{r}_m = \sum_{j=0}^{m+k-1} r_j - \sum_{j=0}^{k-1} r_j = \sum_{j=m}^{m+k-1} t_j = \hat{t}_k.$$

So, if $\hat{t_k} \le t < \hat{t}_{k+1}$ with $k \ge 0$, then $\hat{r}_{m+k} \le t + \hat{r}_m < \hat{r}_{k+m+1}$ and therefore

$$\begin{aligned} d(q \star t,\, \phi_{g(t)}(\phi_{h(\hat{r}_m)}(y))) &= d(\phi_{t-\hat{t}_k}(x_k),\, \phi_{h(t+\hat{r}_m)-h(\hat{r}_m)}(\phi_{h(\hat{r}_m)}(y))) \\ &= d(\phi_{t+\hat{r}_m-\hat{r}_{m+k}}(z_{m+k}),\, \phi_{h(t+\hat{r}_m)}(y)) \\ &\le \varepsilon. \end{aligned}$$

We have shown that the given forward $(\delta,\, 1)$-pseudo-orbit passing through $q$ can be $\varepsilon$-shadowed by the point $\phi_{h(\hat{t}_m)}(y)$ and as this $\varepsilon$ was arbitrary we conclude that $q$ forward shadowable. That is $q \in Sh^+(\phi)$. This completes the proof. □

Recall that a transitive flow $\phi$ [9] is one for which there exists a point $x \in X$ such that $\omega(x) = X$ where

$$\omega(x) = \{y \in X : y = \lim_{t_n \to +\infty} \phi_{t_n}(x) = y \text{ for some sequence } t_n \to +\infty\}.$$

A well known result in topological dynamics states that every transitive flow is chain transitive [1]. The following corollary, which is an immediate consequence of Theorem 2.4.7, gives a partial negative answer to this question in the flow case.

**Corollary 2.4.8.** *If $\phi$ is a transitive flow $Sh^+(\phi) = X$ or $Sh^+(\phi) = \emptyset$.*

*Remark* 6. There is a weaker notion of transitivity, which is original Birkhoff's definition (see [13]), for which Corollary 2.4.8 is not valid. We say that a flow is *transitive in Birkhoff's sense* if it has a dense orbit. Clearly, transitivity as described in this work, implies transitivity in Birkhoff's sense. In [31], an example of a transitive in Birkhoff's sense homeomorphism $f\colon X \to X$ such that $Sh(f) = \emptyset$ but $Sh^+(f)$ is a proper nonempty subset of $X$ is given. So, by suspending $f$ we obtain a flow $\phi^f$ on $X^{1,f}$ without shadowable points and such that $Sh^+(\phi^f)$ is a nonempty proper subset of $X^{1,f}$. This also says that in general $Sh(\phi) \subsetneq Sh^+(\phi)$.

In [33], it is proved that if $\phi$ is a geometric Lorenz flow, then it does not have the forward POTP provided that its return map $f$ satisfies that $f(0) \neq 0$ or $f(1) \neq 1$. In this case we have $Sh^+(\phi) \neq X$. It follows that $Sh^+(\phi) = \emptyset$. We have proved the following result.

**Corollary 2.4.9.** *A geometric Lorenz flows whose return map is such that $f(0) \neq 0$ or $f(1) \neq 1$ does not have shadowable points.*

We state a known theorem about transitive flows on compact metric spaces (and more general on second countable Baire topological spaces) which will be useful.

**Theorem 2.4.10.** *On a Baire, Second countable topological space $X$ a flow is transitive if, and only if, for every pair of open nonempty sets $U$ and $V$, there exists a nonnegative time $T$ such that $\phi_T(U) \cap V \neq \emptyset$.*

We do not know if Theorem 2.4.7 remains valid if we substitute forward shadowable points for shadowable points. We have, however a related result:

**Theorem 2.4.11.** *Let $X$ compact metric space and $\phi$ be a chain transitive flow on $X$. If $\phi$ admits shadowable points then $\phi$ is transitive.*

*Proof.* As $X$ is a Baire Space, it is enough to prove that for any pairs of open sets $U$ and $V$ of $X$, there exists a non-negative $T$ with $\phi_T(U) \cap V \neq \emptyset$. By hypothesis there exists at least one shadowable point $x$. Choose two points $p \in U$ and $q \in V$ and let $\varepsilon > 0$ such that $B(p, \varepsilon) \subseteq U$ and $B(q, \varepsilon) \subseteq V$. Let $\delta > 0$ from the shadowableness of $x$ with respect to $\varepsilon$. By chain transitivity there exists a $(\delta, 1)$-chain $(x_i, s_i)_{i=0}^m$ from $p$ to $x$ and a $(\delta, 1)$-chain $(y_j, r_j)_{j=0}^n$ from $x$ to $q$. Define a $(\delta, 1)$-pseudo-orbit $(z_i, t_i)_{i\in\mathbb{Z}}$ passing through $x$ as follows:

$$(z_k, t_k) = \begin{cases} (\phi_{k+m}(x_0), 1) & \text{if } k < -m \\ (x_{m+k}, s_{m+k}) & \text{if } -m \leq k < 0 \\ (y_k, r_k) & \text{if } 0 \leq k \leq n-1 \\ (\phi_{k-n}(y_n), 1) & \text{if } k \geq n \end{cases}$$

Then there is $y \in X$ and $h \in \mathrm{Rep}$ such that

$$d(x \star t, \phi_{h(t)}(y)) \leq \varepsilon, \ \forall t \in \mathbb{R},$$

in particular, $d(p, \phi_{s_{-m}}(y)) \leq \varepsilon$ and $d(p, \phi_{s_n}(y)) \leq \varepsilon$. Set $T = -s_{-m} + s_n$ which is nonnegative. Then $\phi_{s_n}(y) \in \phi_T(U) \cap V$. This concludes the proof. □

**Corollary 2.4.12.** *On a compact metric space, a flow admitting shadowable points is chain transitive if, and only if, it is transitive.*

*Proof.* We have already observed that transitive flows are always chain transitive. The converse follows from Theorem 2.4.11. □

So, nontransitive chain transitive flows on compact metric spaces does not have shadowable points. For a concrete example take $X = S^1$ and $\phi$ the flow associated to the differential equation

$$\dot{\theta} = \sin^2 \theta$$

in angular coordinates. In this case $Sh(\phi) = \emptyset$. A more interesting family of examples are the so-called Venice Masks which are sectional-Anosov flows with dense periodic orbits which are not transitive [39]. This flows are chain transitive when the manifold is connected for $\overline{Per(\phi)} = CR(\phi) = M$. Hence they cannot have shadowable points. In light of this, we believe that the following statement may be true:

**Conjecture 2.4.13.** *A sectional Anosov flow with shadowable points is an Anosov flow.*

The following example shows that the conclusion of Theorem 2.4.11 cannot be guaranteed if we drop the connectedness hypothesis.

**Example 8.** In [36], it is proved that an equicontinuous homeomorphism on a compact metric space $X$ $f : X \to X$ has the POTP if and only if $X$ is totally disconnected So if $C$ is the usual ternary Cantor set in the interval $[0, 1]$, then identity map $id\colon C \to C$ has the POTP. The suspension flow of this map is then an equicontinuous flow that has POTP [47]. But this flow is not transitive for its phase space is not connected.

In what follows, we show some applications of shadowable points for flows. Remind that $\phi$ is *minimal* if for every $x \in X$ the orbit $\mathcal{O}(x)$ is dense in $X$. Komuro showed that isometric flows with the pseudo orbit tracing property are minimal flows [32]. Meanwhile Kato proved that equicontinuous flows with the pseudo orbit tracing property are also minimal flows [29]. Later, He and Wang proved that distal flows with the pseudo-orbit tracing property are minimal too [27].

In light of these results, is natural to ask if we still can conclude minimality if the pointwise recurrence hypothesis is weakened to suppose the flow to be chain recurrent. The answer is negative as there are nonminimal chain recurrent flows with the pseudo orbit tracing property, for instance the suspension of the usual linear Anosov map on the torus. However, there are not known examples of nontransitive chain recurrent flows with the pseudo-orbit tracing property on connected spaces. The following corollary shows that transitivity is a necessary condition is the phase space is assumed to be connected.

**Corollary 2.4.14.** *If $\phi$ is a chain recurrent flow on a compact connected metric space $X$ then $\phi$ is transitive.*

*Proof.* Since $X$ is connected, the only chain component of $CR(\phi)$ is $X$ itself. So $\phi$ is chain transitive. The corollary then follows from theorem 2.4.11. □

A flow $\phi$ is distal if whenever $\inf_{t\in\mathbb{R}} d(\phi_t(x), \phi_t(y)) = 0$ implies $x = y$. Every distal flow is chain recurrent and every transitive distal flow is minimal. It is equicontinuous if the family of $t$-time maps $\{\phi_t\}_{t\in\mathbb{R}}$ is an equicontinuous family of homeomorphisms in $X$. Equicontinuous and isometric flows are distal flows. The following corollaries are then immediate consequences of Theorem 2.4.11.

**Corollary 2.4.15** (He, L., Wang, M., [27]). *Every distal flow with POTP on a connected compact metric is minimal.*

**Corollary 2.4.16** (Kato, K., [29]). *Let $M$ a Riemannian manifold and $\phi$ a equicontinuous flow with respect to the Riemannian metric of $M$. If $\phi$ has the finite POTP then $\phi$ is minimal.*

**Corollary 2.4.17** (Komuro, M., [32]). *Let $M$ a Riemannian manifold and $\phi$ a isometric flow with respect to the Riemannian metric of $M$. If $\phi$ has the finite POTP then $\phi$ is minimal.*

CHAPTER

# THREE

# $\mathcal{F}$-SHADOWABLE POINTS

## 3.1 $\mathcal{F}$-POTP and $\mathcal{F}$-shadowable points

There several ways to define the *shadowing property* for flows, see for instance [45] and references therein. In essence, the central idea among the majority of definitions of shadowing for flows is the following: even if small errors occur at each iteration, one can track the resulting pseudo-orbit by a true orbit with a time reparametrization.

Below we will give a brief idea about how this dependence originates with the time reparametrizations. Bowen and Walters [14] introduced the definition of expansive flow by using reparametrizations. Afterward, Keynes and Sears [25, 26] restricted the reparametrizations in the definition of expansive flow to subsets $\mathcal{F}$ giving rise to the concept of *$\mathcal{F}$-expansive transformation group*. Thomas [47] relates expansiveness with the shadowing property and stability for flows in metric spaces. Later, in [49], he proves that by adding canonical coordinates the expansive flow verifies the shadowing property. Thus the expansivity and in particular the use of reparameterizations is strongly related to the property of shadowing [47, 49, 51, 52].

In [53] the restriction of the reparametrizations $\mathcal{C}$ allows us to generalize known results about expansive measures for homeomorphism to flows. Moreover when $\mathcal{C}$ is endowed with the $\infty$-metric [15], in the sense that it allows infinite distances between certain points, important generalizations are obtained for the expansiviness [53].

Given these results, it is natural to consider a notion of shadowable points for flows where we restrict the reparametrizations as in [25] or [53]. Hence, we obtain the concepts of *$\mathcal{F}$-shadowable points* and *$\mathcal{F}$-orbit shadowing property for flows* in which $\mathcal{F}$ is a given subset of the set of reparametrizations.

Denote by $\mathcal{C}$ the set of continuous maps $h : \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$ which will be called the set of reparameterizations. We endow $\mathcal{C}$ with the supremum metric

$$\widehat{d}(f, g) = \sup\{|f(x) - g(x)| : x \in \mathbb{R}\}.$$

Under this distance, we obtain that $(\mathcal{C}, \widehat{d})$ is a so-called $\infty$-*metric space* in the sense that it allows infinite distances between certain points (see [15, 53]).

The following is a straightforward reformulation of the notion of shadowing for flows [47]. Given $\varepsilon, \delta > 0$ and $T > 0$, a $(\delta, T)$-pseudo-orbit $(x_i, t_i)_{i\in\mathbb{Z}}$ is $\varepsilon$-shadowed by an orbit $(\phi_t(y))_{t\in\mathbb{R}}$ if there exists $h \in \text{Rep}$ such that

$$d(x_0 \star t, \phi_{h(t)}(y)) \le \varepsilon, \text{ for every } t \in \mathbb{R}.$$

To motivate our main definition we recall the following generalization of expansive flow introduced by Keynes and Sears in [25]. Given a subset $\mathcal{F} \subset \mathcal{C}$ we say that a flow $\phi$ is *$\mathcal{F}$-expansive* if for every $\varepsilon > 0$ there exists $\delta > 0$ with the property that if $d(\phi_t(x), \phi_{h(t)}(y)) \le \delta$ for all $t \in \mathbb{R}$ for some pair of points $x, y \in X$ and some $h \in \mathcal{F}$, then $y = \phi_s(x)$ for some $s \in (-\varepsilon, \varepsilon)$.

Hence, by combining the definitions of shadowable points and $\mathcal{F}$-expansiveness we obtain the following objects that generalize the notion of shadowing for flows.

**Definitions 3.1.1.** Given $\mathcal{F}$ a subset of $\mathcal{C}$ and positive numbers $\delta$, $T$ and $\varepsilon$, we say that a $(\delta, T)$-pseudo-orbit $(x_i, t_i)_{i\in\mathbb{Z}}$ of $\phi$ passes through $p$ if $x_0 = p$, and we say that is *$(\mathcal{F}, \varepsilon)$-shadowed* if there are a point $y \in X$ and a function $h \in \mathcal{F}$ such that

$$d(x_0 \star t, \phi_{h(t)}(y)) \le \varepsilon, \text{ for each } t \in \mathbb{R}.$$

A flow $\phi$ has the $\mathcal{F}$-POTP with respect to the parameter $T > 0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every $(\delta, T)$-pseudo-orbit can be $(\mathcal{F}, \varepsilon)$-shadowed. Finally, we say that a flow has the *$\mathcal{F}$-POTP* if it has the *$\mathcal{F}$-POTP* with respect to the parameter $T = 1$.

Now we introduce the main objects of study.

**Definition 3.1.2.** A point $p \in X$ is *$\mathcal{F}$-shadowable with respect to the parameter* $T > 0$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that every $(\delta, T)$-pseudo-orbit of $\phi$ passing through $p$ can be $(\mathcal{F}, \varepsilon)$-shadowed. When $p$ is $\mathcal{F}$-shadowable with respect to the parameter $T = 1$ we say that $p$ is *$\mathcal{F}$-shadowable.*

We denote by $Sh(\phi, \mathcal{F})$ the set of $\mathcal{F}$-shadowable points of $\phi$ in $X$. In what follows we will give some examples of $\mathcal{F}$-shadowable points.

**Example 9.** If a flow $\phi$ on $X$ has the $\mathcal{F}$-POTP then $Sh(\phi, \mathcal{F}) = X$. The converse is also true as we shall see shortly.

We denote by $\mathcal{B}_0$ the subsets of $\mathcal{C}$ consisting of bounded functions, and by $Sing(\phi)$ the set of singularities of the flow $\phi$, that is the set of points $p \in X$ where $\phi_t(p) = p$ for all $t \in \mathbb{R}$.

**Example 10.** If a flow $\phi$ has no singularities on $X$ and $\mathcal{F} \subset \mathcal{B}_0$ then $Sh(\phi, \mathcal{F}) = \emptyset$.

*Proof.* By Theorem 3 in [14] for $\lambda > 0$ small enough, there are $\varepsilon_\lambda > 0$ and $\tau_\lambda > 0$ such that if $(x_i, t_i)_{i\in\mathbb{Z}}$ is a $(\delta, 1)$-pseudo-orbit for $\phi$, where $\delta > 0$ corresponds to $\varepsilon_\lambda$, $x_{i+1} = \phi_{t_i}(x_i)$, and there are $h \in \mathcal{F}$ and $z \in X$ with $d(\phi_t(x), \phi_{h(t)}(z)) \le \varepsilon$ for each $t \in \mathbb{R}$, then $h(t + \lambda) - h(t) \ge \tau_\lambda$ for every $t \in \mathbb{R}$. Thus $h \notin \mathcal{B}_0$. □

Hereafter, the closure operation will be denoted by $\overline{(\cdot)}$. The following example shows that the set of $\mathcal{F}$-shadowable points can be finite.

**Example 11.** If $\phi$ is a $\mathcal{F}$-expansive flow on $X$ where $\mathcal{F} \subset \mathcal{C}$ such that $\mathcal{F} \cap \mathcal{B}_0 \neq \emptyset$, then $Sh(\phi, \mathcal{F}) = Sing(\phi)$ whenever $id_{\mathbb{R}} \notin \overline{\mathcal{F}}$.

*Proof.* Suppose $id_{\mathbb{R}} \notin \overline{\mathcal{F}}$. By Lemma 2 in [14] for $\lambda > 0$ small enough such that $\lambda < \widehat{d}(id_{\mathbb{R}}, \mathcal{F})$ there exists $\gamma > 0$ with the property that $d(\phi_{\pm\lambda}(x), y) > \gamma$ provided that $x, y \in X \setminus Sing(\phi)$ and $d(x, y) < \gamma$. Also, there exists $\varepsilon > 0$ such that

$$\text{if } y \in \phi_{(-\varepsilon,\varepsilon)}(x), \text{ then } d(\phi_t(x), \phi_t(y)) < \frac{\gamma}{2} \text{ for all } (x, t) \in (X \setminus Sing(\phi)) \times \mathbb{R}. \quad (3.1)$$

Fix $x \in Sh(\phi, \mathcal{F}) \setminus Sing(\phi)$ and take $\varepsilon_0 > 0$ such that $\varepsilon_0 < \min\{\frac{\gamma}{2}, \delta_0\}$ where $\delta_0$ is the $\mathcal{F}$-expansivity constant of $\phi$ for $\varepsilon > 0$. There exists $\delta > 0$ such that every $(\delta, 1)$-pseudo-orbit passing through $x$ can be $(\mathcal{F}, \varepsilon_0)$-shadowable. Let $(x_i, t_i)_{i\in\mathbb{Z}}$ be a $(\delta, 1)$-pseudo-orbit for $\phi$ where $x_{i+1} = \phi_{t_i}(x_i)$ and $x_0 = x$. Then, there are $h \in \mathcal{F}$ and $z \in X \setminus Sing(\phi)$ such that

$$d(x \star t, \phi_{h(t)}(z)) = d(\phi_t(x), \phi_{h(t)}(z)) \leq \varepsilon_0, \quad \text{for every } t \in \mathbb{R}.$$

So, there is $s \in (-\varepsilon, \varepsilon)$ such that $z = \phi_s(x)$. Then from (3.1) for each $t \in \mathbb{R}$ we have

$$d(\phi_t(z), \phi_{h(t)}(z)) \leq d(\phi_t(z), \phi_t(x)) + d(\phi_t(x), \phi_{h(t)}(z)) < \gamma.$$

It follows that $d(\phi_t(z), \phi_{h(t)-t}(\phi_t(z))) < \gamma$ for all $t \in \mathbb{R}$. Then $|h(t) - t| < \lambda$ for every $t \in \mathbb{R}$. That is, $\widehat{d}(id_{\mathbb{R}}, h) < \lambda$ and we obtain a contradiction. Moreover, since $\mathcal{F} \cap \mathcal{B}_0 \neq \emptyset$, then $Sing(\phi) \subset Sh(\phi, \mathcal{F})$. Therefore $Sing(\phi) = Sh(\phi, \mathcal{F})$. □

The following result can be proved as per the above example.

**Example 12.** Let $\phi$ be a flow without singularities on $X$. If $\mathcal{F}_i \subset \mathcal{C}$ with $i = 1, 2$ are such that $\widehat{d}(\mathcal{F}_1, \mathcal{F}_2) > 0$, then $Sh(\phi, \mathcal{F}_1) \cap Sh(\phi, \mathcal{F}_2) = \emptyset$.

Given a subset $\mathcal{F} \subset \mathcal{C}$ we write $\mathrm{Rep} \circ \mathcal{F} \circ \mathrm{Rep} \subset \mathcal{F}$ if $g \circ f \circ h \in \mathcal{F}$ whenever $g, h \in \mathrm{Rep}$ and $f \in \mathcal{F}$ [53].

## 3.2 General Properties of $\mathcal{F}$-shadowable points

The main result of this chapter are the basic properties of $\mathcal{F}$-shadowable points.

**Theorem 3.2.1.** *Given a subset $\mathcal{F} \subset \mathcal{C}$ and a flow $\phi$ in a compact metric space $(X, d)$, the set of $\mathcal{F}$-shadowable points satisfies the following properties:*

*(a) $Sh(\phi, \mathcal{F})$ is invariant.*

*(b) $Sh(\phi, \overline{\mathcal{F}}) = Sh(\phi, \mathcal{F})$.*

*(c) The flow has the $\mathcal{F}$-POTP if and only if $Sh(\phi, \mathcal{F}) = X$.*

*(d) If $\mathcal{F} = \{id_{\mathbb{R}}\}$ and the flow has the $\mathcal{F}$-POTP, then $\phi_t$ has the POTP for every $t \neq 0$.*

*(e) If $\mathcal{F} \subset \mathrm{Rep}$ and $CR(\phi) \subseteq Sh(\phi, \mathcal{F})$ then $CR(\phi) = \Omega(\phi)$.*

*(f) If $\mathrm{Rep} \circ \mathcal{F} \circ \mathrm{Rep} \subset \mathcal{F}$ and $f$ is a conjugacy between $\phi$ and $\psi$, then either both have $\mathcal{F}$-POTP or neither of them does.*

*Proof.* Item (a), is proved as Theorem 2.2.3. To prove Item (b), let $x \in Sh(\phi, \overline{\mathcal{F}})$ and $\varepsilon > 0$. There exists $\delta > 0$ such that every $(\delta, 1)$-pseudo-orbit of $\phi$ passing through $x$ can be $(\overline{\mathcal{F}}, \varepsilon)$-shadowable. Let $(x_i, t_i)_{i\in\mathbb{Z}}$ be a $(\delta, 1)$-pseudo-orbit of $\phi$ passing through $x$. Then there are $h \in \overline{\mathcal{F}}$ and $z \in X$ such that $d(x \star t, \phi_{h(t)}(z)) \leq \frac{\varepsilon}{2}$, for every $t \in \mathbb{R}$. By a compactness argument we can show that there exists $\varepsilon_0 > 0$ such that $\phi_{(-\varepsilon_0,\varepsilon_0)}(x) \subset B(x, \frac{\varepsilon}{2})$ for every $x \in X$. Take $g \in \mathcal{F}$ such that $\widehat{d}(h, g) < \varepsilon_0$ and fix $t \in \mathbb{R}$. Then,

$$d(x \star t, \phi_{g(t)}(z)) \leq \frac{\varepsilon}{2} + d(\phi_{h(t)}(z), \phi_{g(t)}(z)).$$

Since $\phi_{(-\varepsilon_0,\varepsilon_0)}(\phi_{h(t)}(z)) \subset B(\phi_{h(t)}(z), \frac{\varepsilon}{2})$ and $\widehat{d}(h, g) < \varepsilon_0$, we have

$$d(\phi_{h(t)}(z), \phi_{g(t)}(z)) = d(\phi_{h(t)}(z), \phi_{g(t)-h(t)}(\phi_{h(t)}(z))) < \frac{\varepsilon}{2}.$$

So, $d(x \star t, \phi_{g(t)}(z)) \leq \varepsilon$. It follows that $x \in Sh(\phi, \mathcal{F})$.

Item (c) is proved as Theorem 2.2.7

To prove Item (d), first we take $t > 0$. By Theorem 1.1 in [42] is sufficient prove that $X \subset Sh(\phi_t)$. Let $x \in X$ and let $\varepsilon > 0$ be given. Since $X = Sh(\phi, \{id_\mathbb{R}\})$, by Lemma 2.2.5 there exists $\delta > 0$ such that each $(\delta, t)$-pseudo-orbit can be $(\{id_\mathbb{R}\}, \varepsilon)$-shadowable. Given $(x_n)_{n\in\mathbb{Z}}$ a $\delta$-pseudo-orbit of $\phi_t$ passing through the point $x \in X$, that is $x_0 = x$. Then, for every integer $n$ we have that $d(\phi_t(x_n), x_{n+1}) \leq \delta$. That is $(x_n, t_n)_{n\in\mathbb{Z}}$, where $t_n = t$, is a $(\delta, t)$-pseudo orbit of $\phi$. Therefore, there is $z \in X$ such that $d(\phi_{r-s_n}(x), \phi_r(z)) \leq \varepsilon$ where $s_n \leq r < s_{n+1}$. Since $s_n = nt$ for every $n \in \mathbb{Z}$, then for $r = nt$ we have $d((\phi_t)^n(z), x_n) = d(x_n, \phi_{nt}(z)) \leq \varepsilon$ for each $n \in \mathbb{Z}$. It follows that $x \in Sh(\phi_t)$. The proof for $t < 0$ is analogous.

Item (e) and (f) are proved as Theorems 2.4.2 and 2.2.8 respectively.

□

Next, we give some examples related to Theorem 3.2.1.

We say that $X$ is *totally disconnected at* $p \in X$ if the connected component of $X$ containing $p$ is $\{p\}$. Denote by $X^{deg}$ the set of totally disconnected points of $X$ [3, 11].

**Example 13.** The result of Item (d) of Theorem 3.2.1 can not be extended to all $t$-maps in the arbitrary compact space case. Indeed, if the flow has the $\{id_\mathbb{R}\}$-POTP, then $\phi_t$ has the POTP for every $t \in \mathbb{R}$ if and only if $X = X^{deg}$ [3].

As a consequence of the above theorem we obtain the following corollary.

**Corollary 3.2.2.** *If $g \in \mathcal{F}$ is such that for every $x \in X$ and for every $\delta > 0$ there exists $f \in \mathcal{F}$ with $d(\phi_{g(t)}(x), \phi_{f(t)}(x)) \leq \delta$ for all $t \in \mathbb{R}$, then $Sh(\phi, \mathcal{F}) = Sh(\phi, \mathcal{F} \setminus \{g\})$.*

**Theorem 3.2.3.** *Let $\mathcal{F}$ be a subset of $\mathcal{C}$. Then the $\mathcal{F}$-shadowable points is a $G_\delta$ set of $X$.*

*Proof.* Same proof as Theorem 2.2.11. □

To state our next result we will need the following finite versions of the notions of shadowing for flows.

**Definitions 3.2.4.** Given $\mathcal{F}$ a subset of $\mathcal{C}$, positive numbers $\delta$, $T$ and $\varepsilon$, and integers $a$ and $b$ such that $ab \leq 0$ we say that a finite $(\delta, T)$-pseudo-orbit $(x_i, t_i)_{i=a}^{b}$, of $\phi$ passes through $p$ if $x_0 = p$, and we say that is finitely *$(\mathcal{F}, \varepsilon)$-shadowed* if there are a point $y \in X$ and a function $h \in \mathcal{F}$ such that

$$d(x_0 \star t, \phi_{h(t)}(y)) \leq \varepsilon, \text{ for all } t \in [s_a, s_{b+1}).$$

A flow $\phi$ has the *finite $\mathcal{F}$-POTP* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every finite $(\delta, 1)$-pseudo-orbit can be $(\mathcal{F}, \varepsilon)$-shadowed.

**Definition 3.2.5.** A point $p \in X$ is *finitely $\mathcal{F}$-shadowable*, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that every finite $(\delta, 1)$-pseudo-orbit of $\phi$ passing through $p$ can be finitely $(\mathcal{F}, \varepsilon)$-shadowed.

Given $f \in \mathcal{C}$, we define $\mathcal{B}_f = \{h \in \mathcal{C} : \widehat{d}(f, h) < \infty\}$ and $d_f = \widehat{d}_{|\mathcal{B}_f}$. It follows that $\mathcal{C}$ can be written as a union of metric spaces $(\mathcal{B}_f, d_f)$. Note that in a $\infty$-metric space a subset of $\mathcal{C}$ is compact if and only if is a union of a finite number of compact subsets each one belonging to some $(\mathcal{B}_f, d_f)$ (p. 15 in [15]).

We denote by $Sh_f(\phi, \mathcal{F})$ the set of finitely $\mathcal{F}$-shadowable points of $\phi$ in $X$. In [47], it is proved that if $\phi$ has no singularities we have $Sh(\phi, \text{Rep}) = Sh_f(\phi, \text{Rep})$. The following theorem shows that for a compact family of funtions $\mathcal{F}$ we always have an equality.

**Theorem 3.2.6.** *If $\mathcal{F}$ is a compact subset of $\mathcal{C}$, then the $\mathcal{F}$-shadowable points of $\phi$ coincide with the finitely $\mathcal{F}$-shadowable points of $\phi$.*

*Proof.* It is enough to show that $Sh_f(\phi, \mathcal{F}) \subseteq Sh(\phi, \mathcal{F})$. Let $p \in Sh_f(\phi, \mathcal{F})$ and $\varepsilon > 0$. Choose $\delta > 0$ from the finitely $\mathcal{F}$-shadowableness of $p$ and let $(x_i, t_i)$ be a $(\delta, 1)$-pseudo-orbit passing through $p$. For each $n \in \mathbb{N}$, the sequence of pairs $(x_i, t_i)_{i=-n}^{n}$ is a $(\delta, 1)$-chain passing through $p$. So there are $q_n \in X$ and $h_n \in \mathcal{F}$ such that $d(p \star t, \phi_{h_n(t)}(q_n)) \leq \varepsilon$ for all $t \in [s_{-n}, s_n)$. By compactness we can suppose that there exists some $q \in X$ such that $q_n \to q$ when $n \to \infty$. Since for each natural number $n$ we have $h_n \in \mathcal{F}$ and $\mathcal{F}$ is compact, then there are $h \in \mathcal{F}$ and a subsequence $(h_{n_j})_{j\in\mathbb{N}}$ of $(h_n)_{n\in\mathbb{N}}$ in $\mathcal{B}_h$ converging to $h$.

Fix any $t_0 \in \mathbb{R}$ and choose $N \in \mathbb{N}$ large enough so that $t \in [s_{-N}, s_N)$. Note that for $n \geq N$ we always have that $d(p \star t_0, \phi_{h_n(t_0)}(q_n)) \leq \varepsilon$ because $[s_{-N}, s_N) \subseteq [s_{-n}, s_n)$. So, we have

$$d(p \star t_0, \phi_{h(t_0)}(q)) = \lim_{j\to\infty} d(p \star t_0, \phi_{h_{n_j}(t_0)}(q_{n_j})) \leq \varepsilon.$$

Because $t_0$ was arbitrary we conclude that $p \in Sh(\phi, \mathcal{F})$. This completes the proof of the theorem. □

Now, we consider an important class of family of functions of $\mathcal{C}$.

**Definition 3.2.7.** Let $\mathcal{F} \subseteq \mathcal{C}$ be any family of functions. We say that $\mathcal{F}$ is invariant by translations if for every $c \in \mathbb{R}$ and $f \in \mathcal{F}$ the function $f_c$ defined by $f_c(t) = f(t + c) - f(c)$ for all $t \in \mathbb{R}$ also belongs to $\mathcal{F}$.

**Example 14.** The following families of functions in $\mathcal{C}$ are invariant by translations: $\mathcal{C}$, Rep, $\mathcal{B}_0$, $\{\text{id}_{\mathbb{R}}\}$, the homogeneous dilations $\{h : h(t) = at,\ a \in \mathbb{R}\}$ and the polynomials fixing zero.

Now we present some results related to this family of reparametrizations. To do this, we need to introduce some definitions and notations.

We denote by $Sh^+(\phi, \mathcal{F})$ the set of points such that given $\varepsilon > 0$ there exists $\delta > 0$ such that every forward $(\delta, 1)$-pseudo-orbit $(x_i, t_i)_{i=0}^{\infty}$ passing through $p$ can be $(\mathcal{F}, \varepsilon)$-shadowed. Each element of $Sh^+(\phi, \mathcal{F})$ is said forward $\mathcal{F}$-shadowable point.

We recall that a chain transitive flow $\phi$ is one where $X$ is a chain transitive set. That is, for every $p, q \in X$ we have $p \sim q$ [1].

The following result and its corollary characterize the set of forward $\mathcal{F}$-shadowable points for chain transitive flows and transitive flows and their proofs are identical to those of Theorem 2.4.7 and Corollary 2.4.8

**Theorem 3.2.8.** *If the flow $\phi$ is chain transitive and $\mathcal{F}$ is a family of functions invariant by translations, then $Sh^+(\phi, \mathcal{F}) = X$ or $Sh^+(\phi, \mathcal{F}) = \emptyset$.*

**Corollary 3.2.9.** *If $\phi$ is a transitive flow and $\mathcal{F}$ is a family of functions invariant by translations, then $Sh^+(\phi, \mathcal{F}) = X$ or $Sh^+(\phi, \mathcal{F}) = \emptyset$.*

Corollary 3.2.9 can be used to obtain information about the geometric Lorenz attractor [33, 55]. In [33], it is proved that if $\phi$ is the geometric Lorenz attractor, then it does not have the finite forward Rep-POTP provided that its return map $f$ satisfies that $f(0) \neq 0$ or $f(1) \neq 1$. In this case we have $Sh^+(\phi, \mathcal{F}) \neq X$ whenever $\mathcal{F} \subset \mathrm{Rep}$. Moreover, since Rep is invariant by translations, then by Corollary 2.4.8 it follows that $Sh^+(\phi, \mathrm{Rep}) = \emptyset$. So we obtain the following result.

**Corollary 3.2.10.** *If $\mathcal{F} \subset \mathrm{Rep}$, then the geometric Lorenz attractor does not have $\mathcal{F}$-shadowable points.*

Finally, using invariant by translations families we extend to flows the results for homeomorphisms given in [30].

The following is a generalization of the limit shadowing property for flows.

**Definition 3.2.11.** A point $p \in X$ is limit $\mathcal{F}$-shadowable for $\phi$ if for each asymptotic 1-pseudo-orbit $(x_i, t_i)_{i \in \mathbb{N}}$ of $\phi$ passing through $p$ there are $h \in \mathcal{F}$ and $z \in X$ such that

$$\lim_{t \to +\infty} d(\phi_{t-s_j}(x_j), \phi_{h(t)}(z)) = 0, \quad \text{where} \quad s_j \leq t < s_{j+1}.$$

We says that $\phi$ has the limit $\mathcal{F}$-shadowing property when every point of $X$ is limit $\mathcal{F}$-shadowable.

Let $M(X)$ be the space of Borel probability measures on $X$. We consider in $M(X)$ the *Prohorov metric* $\widetilde{d}$. Thus $M(X)$ it turns into a compact space. From the Riesz representation theorem we may regard $M(X)$ as a subset of the unit sphere in $C(X)^*$ (the dual space of $C(X)$) and write $\mu(f) = \int f d\mu$ for $f \in C(X)$.

Each flow $(X, \phi)$, induces a map $\widetilde{\phi}\colon M(X) \times \mathbb{R} \to M(X)$ defined by

$$\widetilde{\phi}(\mu, t)(f) = \mu(f \circ \phi_t) \quad \text{where} \quad (\mu, t) \in M(X) \times \mathbb{R}.$$

This map is a flow on $M(X)$ (see [10]). Taking into account these notions we can state the following result that extends to flows Theorem 2 of [30].

**Theorem 3.2.12.** *If $\mathcal{F}$ is invariant by translation, then the limit $\mathcal{F}$-shadowable points for $\widetilde{\phi}$ are finitely $\mathcal{F}$-shadowable points for $\widetilde{\phi}$.*

*Proof of Theorem 3.2.12.* Suppose the Theorem is false. Then there exists a point $\mu \in M(X)$ limit $\mathcal{F}$-shadowable but is not finite $\mathcal{F}$-shadowable. Thus, there is $\varepsilon > 0$ such that for every $k \geq 1$ there is a $(\frac{1}{k}, 1)$-pseudo-orbit $(\mu_i^k, t_i^k)_{i=a_k}^{N_k}$ of $\widetilde{\phi}$ passing through $\mu$ such that is not $(\mathcal{F}, \varepsilon)$-shadowed. Without loss of generality we can assume that $a_k = 0$ for each $k \geq 1$. Since $\widetilde{\phi}$ is chain transitive by Lemma **??**, there is a $(\frac{1}{k}, 1)$-pseudo-orbit $(\nu_i^k, \lambda_i^k)_{i=0}^{L_k}$ with $\nu_0^k = \mu_{N_k}^k$ and $\nu_{L_k}^k = \mu_0^{k+1}$. Renewing the indices of the sequences

$$\{\mu_0^1, \mu_1^1, \cdots, \mu_{N_1}^1, \nu_1^1, \nu_2^1, \cdots, \nu_{L_1-1}^1, \mu_0^2, \mu_1^2, \cdots, \mu_{N_2}^2, \nu_1^2, \nu_2^2, \cdots, \nu_{L_2-1}^2, \cdots\},$$

and

$$\{t_0^1, t_1^1, \cdots, t_{N_1}^1, \lambda_1^1, \lambda_2^1, \cdots, \lambda_{L_1-1}^1, t_0^2, t_1^2, \cdots, t_{N_2}^2, \lambda_1^2, \lambda_2^2, \cdots, \lambda_{L_2-1}^2, \cdots\}$$

we obtain a sequence of pairs $(\mu_i, t_i)_{i\in\mathbb{N}}$ of $(M(X), \widetilde{\phi})$ with the following property: Given $i \in \mathbb{N}$, if $\sum_{j=1}^r (N_j + L_j) \leq i \leq N_{r+1} + \sum_{j=1}^r (N_j + L_j)$ for some $r \geq 1$, then

$$(\mu_i, t_i) = \left(\mu^{r+1}_{i-\sum_{j=1}^r (N_j+L_j)}, t^{r+1}_{i-\sum_{j=1}^r (N_j+L_j)}\right),$$

and if $N_{r+1} + \sum_{j=1}^r (N_j + L_j) < i < \sum_{j=1}^{r+1} (N_j + L_j)$ we consider

$$(\mu_i, t_i) = \left(\nu^{r+1}_{i-N_{r+1}-\sum_{j=1}^r (N_j+L_j)}, \lambda^{r+1}_{i-N_{r+1}-\sum_{j=1}^r (N_j+L_j)}\right).$$

Moreover, the sequence $(\mu_i, t_i)_{i\in\mathbb{N}}$ satisfies $\mu^{r+1}_{\sum_{j=1}^r (N_j+L_j)} = \mu$ for every $r \geq 1$ and

$$\lim_{i\to+\infty} \widetilde{d}(\widetilde{\phi}_{t_i}(\mu_i), \mu_{i+1}) = 0.$$

Since $(M(X), \widetilde{\phi})$ has the limit $\mathcal{F}$-shadowing property, there are $\eta \in M(X)$ and $h \in \mathcal{F}$ such that

$$\lim_{w\to+\infty} \widetilde{d}(\widetilde{\phi}_{w-s_j}(\mu_j), \phi_{h(w)}(\eta)) = 0, \quad \text{where} \quad s_j \leq w < s_{j+1}.$$

Therefore for some $T_0 > 0$ we have

$$\widetilde{d}(\widetilde{\phi}_{w-s_j}(\mu_j), \phi_{h(w)}(\eta)) < \varepsilon, \quad \text{for every} \quad s_j \leq w < s_{j+1} \text{ where } j \geq T_0.$$

Let $k_0 \in \mathbb{N}$ such that $T_0 \leq \sum_{j=1}^{k_0} (N_j + L_j)$. For each $l \in \{1, \cdots, N_{k_0+1}\}$ we denote $\widetilde{s}_l = \sum_{j=0}^{l-1} t_j^{k_0+1}$ and $\widetilde{s}_0 = 0$. Fix $t \in \mathbb{R}$ and $0 \leq m < N_{k_0+1}$ such that $\widetilde{s}_m \leq t < \widetilde{s}_{m+1}$. We denote $c(m) = \sum_{j=1}^{k_0} (N_j + L_j) + m$. Since $t_{c(m)} = t_m^{k_0+1}$, then for each $0 \leq w - s_{c(m)} < t_m^{k_0+1}$ we have

$$\widetilde{d}\left(\widetilde{\phi}_{w-s_{c(m)}}\left(\mu_{c(m)}\right), \phi_{h(w)}(\eta)\right) < \varepsilon.$$

Moreover from $\mu_{c(m)} = \mu_m^{k_0+1}$, it follows that

$$\widetilde{d}\left(\widetilde{\phi}_{t-\widetilde{s}_m}(\mu_m^{k_0+1}), \widetilde{\phi}_{\widetilde{h}(t)}(\gamma)\right) < \varepsilon \quad \text{for every } \widetilde{s}_m \leq t < \widetilde{s}_{m+1},$$

where $\widetilde{h}(t) = h(t + s_{c(m)}) - h(s_{c(m)}) \in \mathcal{F}$ and $\gamma = \widetilde{\phi}(\eta, h(s_{c(m)})) \in M(X)$. Therefore the $(\frac{1}{k_0+1}, 1)$-pseudo-orbit $(\mu_i^{k_0+1}, t_i^{k_0+1})_{i=0}^{N_{k_0+1}}$ of $\widetilde{\phi}$ passing through $\mu$ is $(\mathcal{F}, \varepsilon)$-shadowed. This is a contradiction. □

From Theorem 3.2.6, the following corollary is immediate.

**Corollary 3.2.13.** *If $\mathcal{F}$ is a compact invariant by translations family of functions and $\widetilde{\phi}$ has the limit $\mathcal{F}$-shadowing property, then $\widetilde{\phi}$ has the $\mathcal{F}$-POTP.*

CHAPTER

# FOUR

# TOPOLOGICALLY STABLE POINTS

In this chapter we present a pointwise counterpart of topologically stability for flows [28,47,49], by extending a recent work of Koo, Lee and Morales [34] to the flow context. We show that set of topologically stable points is invariant and preserved by continuous equivalences.

## 4.1 Topologically stable points

Let $(X,\, d)$ be a metric space and $A \subseteq X$ Given two functions $f,\, g\colon A \subseteq X \to X$. The $C^0$ *distance on* $A$ between $f$ and $g$ is defined by

$$d_{C^0}(f,\, g) = \sup_{x \in A} d(f(x),\, g(x)).$$

The following definition is a weaker version of the notion of equivalence.

**Definition 4.1.1.** Let $\phi\colon X \times \mathbb{R} \to X$ and $\psi\colon Y \times \mathbb{R} \to Y$ be two flows. We say that $\psi$ is *semiconjugate* to $\phi$ if there are a continuous function $h\colon X \to Y$ and a surjective continuous function $\sigma\colon X \times \mathbb{R} \to \mathbb{R}$ with the following properties:

i) $\sigma$ is a reparametrizarion fixing 0, i.e, $\sigma(x,\, \cdot)\colon \mathbb{R} \to \mathbb{R}$ is strictly increasing, onto and $\sigma(x,\, 0) = 0$ for all $x \in X$

ii) $h(\psi_{\sigma(x,\, t)}(x)) = \phi_t(h(x))$.

The pair $(h,\, \sigma)$ is called a *semiconjugacy* from $\psi$ to $\phi$, and it is easy to see that a semiconjugacy sends orbits or periodic orbits of $\psi$ to orbits or periodics orbits of $\phi$, respectively.

Now we remind the definition of topological stability [28,47,49].

**Definition 4.1.2.** Let $X$ be a metric space. A continuous flow $\phi$ is *topologically stable* if given $\varepsilon > 0$ there is $\delta > 0$ such that for every continuous flow $\psi$ satisfying $d_{C^0}(\phi^t,\, \psi^t) \le \delta$ for every $t \in [0,\, 1]$ there is a semiconjugacy $(h,\, \sigma)$ from $\psi$ to $\phi$ such that $d_{C^0}(h,\, id_X) \le \varepsilon$.

Our definition of topologically stable point will be based on the following elementary remark:

*Remark* 7. A necessary condition for a given flow $\phi$ being topologically stable is that for every $\varepsilon > 0$ and every $x \in X$ there is $\delta_x > 0$ such that for every continuous flow $\psi$ satisfying $d_{C^0}(\phi^t, \psi^t) \le \varepsilon$, for every $t \in [0, 1]$ there is a semiconjugacy $(k, \tau)$ from $\psi|_{\overline{\mathcal{O}_\psi(x)}}$ to $\phi$ such that $d_{C^0}(k, i_{\overline{\mathcal{O}_\psi(x)}}) \le \varepsilon$. Indeed, for $\varepsilon > 0$ then we can take $\delta_x = \delta$, $h = k|_{\overline{\mathcal{O}_\psi}}$ and $\tau = \sigma|_{\overline{\mathcal{O}_\psi(x)}\times\mathbb{R}}$ where $\delta$ and $k\colon X \to X$ come from the topological stability of $\phi$.

This remark motivates the content of the main definition of this chapter.

**Definition 4.1.3.** Let $\phi$ be a flow on a metric space $X$ and let $x \in X$. It is said that $x$ *is topologically stable* if given $\varepsilon > 0$ there is $\delta_x > 0$ such that for every flow $\psi$ on $X$ such that $d_{C^0}(\phi_t, \psi_t) \le \delta_x$ for every $t \in [0, 1]$, there is a semiconjugacy $(k, \tau)$ from $\psi|_{\overline{\mathcal{O}_\psi(x)}}$ to $\phi$ such that $d_{C^0}(k, i_{\overline{\mathcal{O}_\psi(x)}}) \le \varepsilon$.

We denote by $T(\phi)$ the subset of $X$ of topologically stable points of the flow $\phi$. In what follows we give some examples.

**Example 15.** If a flow $\phi$ is topologically stable then by the previous remark $T(\phi) = X$. It is unknown, however, if the converse is also true.

**Example 16.** An isolated point of a flow $\phi$ is topological stable. Indeed, if $x$ is isolated then $\phi_t(x) = x$ for all $t \in \mathbb{R}$ and we can choose $\delta_x > 0$ small enough so that $B(x, \delta_x) = \{x\}$. If $\psi$ is any flow satisfying $d_{C^0}(\psi_t, \phi_t) \le \delta_x$ for all $t \in [0, 1]$ then $\psi_t(x) = x$. So if $\tau(x, t) = t$, then $(id_{\{x\}}, \tau)$ is a semiconjugacy from $\psi|_{\overline{\mathcal{O}_\psi(x)}}$ to $\phi$.

The following Lemma is from [48].

**Lemma 4.1.4.** *Let $\phi$ and $\psi$ be two flows on a compact metric space $X$ and let $a > 0$. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that $d_{C^0}(\phi_t, \psi_t) \le \delta$ for all $t \in [0, 1]$, implies $d_{C^0}(\phi_t, \psi_t) \le \varepsilon$ for all $t \in [0, a]$.*

*Proof.* If $a \ge 1$ there take $\delta = \varepsilon$. Assume $a < 1$ and fix $k \in \mathbb{N}$ greater than $\frac{1}{a}$. Let us define a sequence $(\delta_n)_{n\in\mathbb{N}}$ of real numbers as follows: $\delta_1 = \delta$ and for $n > 1$ set $\delta_n$ inductively as having the following properties

1. $0 < \delta_n < \frac{\delta_{n-1}}{2}$
2. $d(x, y) \le \delta_n \Longrightarrow d(\phi_t(x), \phi_t(y)) \le \frac{\delta_{n-1}}{2}$, $\forall\, x, y \in X$ and $\forall\, t \in [0, a]$.

Now take $\delta = \delta_k$ and assume $d_{C^0}(\phi_t, \psi_t) \le \delta$ for $t \in [0, a]$. Then

$$d_{C^0}(\phi_{t'} \circ \phi_t, \phi_{t'} \circ \psi_t) \le \tfrac{\delta_{n-1}}{2}, \quad \forall\, t' \in [0, a] \text{ and } \forall\, t \in [0, 1].$$

Therefore,

$$d_{C_0}(\phi_{t'+t}, \psi_{t'+t}) \le d_{C_0}(\phi_{t'} \circ \phi_t, \phi_{t'} \circ \psi_t) + d_{C_0}(\phi_{t'} \circ \phi_t, \psi_{t'} \circ \psi_t).$$

That is $d_{C_0}(\phi_t, \psi_t) \le \delta_{k-1}$ for all $t \in [0, 2a]$. Continuous in this manner, we will have that $d_{C_0}(\phi_t, \psi_t) \le \delta$ for all $t \in [0, ka]$. This completes the proof. □

The next lemma says that we do not need to restrict ourselves to perturbations $\psi$ which are $C^0$-closed to $\psi$ in the interval $[0, 1]$. We can choose any interval $[0, a]$ with $a > 0$.

**Lemma 4.1.5.** *Let $\phi$ be a flow on the compact metric space $X$ and $a > 0$. A point $x_0 \in X$ is topologically stable if and only if given $\varepsilon > 0$ there exists $\delta_{x_0}$ such that for every flow $\psi$ on $X$ such that $d_{C^0}(\phi_t, \psi_t) \leq \delta_{x_0}$ for every $t \in [0, a]$ there exists a semiconjugacy $(k, \tau)$ from $\psi|_{\overline{\mathcal{O}_\psi(x)}}$ to $\phi$*

*Proof.* Suppose that $x_0$ is topologically stable. Given $\epsilon > 0$ take $\delta'_{x_0}$ from the topological stability of $x$. If $a \geq 1$ simply take $\delta_{x_0} = \delta'_{x_0}$ . Assume that $a < 1$. By Lemma 4.1.4 there is $\delta > 0$ such that if $d_{C^0}(\phi_t, \psi_t) \leq \delta$ for all $t \in [0, a]$ then $d_{C^0}(\phi_t, \psi_t) \leq \delta'_{x_0}$ for all $t \in [0, 1]$. Take $\delta_{x_0} = \delta$. □

We are ready to show that under continuous equivalences, the set of topological stable points is preserved. In fact we proof a stronger result.

**Theorem 4.1.6.** *Let $\phi$ and $\psi$ be two flows on the compact metric spaces $X$ and $Y$, respectively. Suppose that $\phi$ and $\psi$ are equivalent under a homeomorphism $h\colon X \to Y$ with a continuous reparametrization $\sigma\colon X \times \mathbb{R} \to \mathbb{R}$ fixing zero. Then $h(T(\phi)) = T(\psi)$.*

*Proof.* Let $y_0 \in T(\psi)$, $\varepsilon > 0$ and $x_0$ the only point in $X$ with $h(x_0) = y_0$. We'll see that $x_0 \in T(\phi)$. By compacity, we can choose $0 < \varepsilon' < \varepsilon$ so that $d(y_1, y_2) < \varepsilon'$ implies that $d(h^{-1}(y_1), h^{-1}(y_2)) < \varepsilon'$ for every $y_1, y_2$ in $Y$. Let $\delta'_{y_0}$ from the topological stability of $y_0$ with respect to $\varepsilon'$. Choose $0 < \delta < \delta'_{y_0}$ such that $d(h(x_1), h(x_2)) \leq \delta'_{y_0}$ whenever $d(x_1, x_2) < \delta$ for all $x_1, x_2 \in X$. Let $a = \sup_{x\in X} \sigma(x, 1)$ which exists and is greater than zero in virtue of the compacity of $X$, the continuity of the reparametrization $\sigma$ and its zero fixing property. Let $\phi'$ any other flow on $Y$ with $d_{C^0}(\phi'_t, \phi_t) < \delta$ for all $t \in [0, a]$. Define a flow $\psi'$ on $Y$ by

$$\psi'_t(h(x)) = h(\phi'_{\sigma(x,t)}(x)).$$

Then, for every $t \in [0, 1]$ and $x \in X$ we have

$$d(\psi'_t(h(x)), \psi_t(h(x))) = d(h(\phi_{\sigma(x,t)}(x)), h(\phi'_{\sigma(x,t)}(x))) \leq \delta'_{y_0}.$$

Using the topological stability of $y_0$, there exists a semiconjugacy $(k, \tau)$ from $\psi'|_{\overline{\mathcal{O}(y_0)}}$ to $\psi$ such that $d_{C^0}(k, id_{\overline{\mathcal{O}_{\psi'}(y_0)}}) \leq \varepsilon'$. Define a function $l := h^{-1} \circ k \circ h\colon \overline{\mathcal{O}_{\phi'}(x_0)} \to X$. Note that $l$ is well defined because $h(\overline{\mathcal{O}_{\phi'}(x_0)}) = \overline{\mathcal{O}_{\psi'}(y_0)}$. Since $d_{C^0}(k \circ h|_{\overline{\mathcal{O}_{\phi'}(x_0)}}, h|_{\overline{\mathcal{O}_{\phi'}(x_0)}}) < \varepsilon'$, then

$$d_{C^0}\left(h^{-1}|_{\overline{\mathcal{O}_{\psi'}(y_0)}} \circ k \circ h|_{\overline{\mathcal{O}_{\phi'}(x_0)}}, id|_{\overline{\mathcal{O}_{\phi'}(x_0)}}\right) < \varepsilon,$$

this is $d_{C^0}(l, id|_{\overline{\mathcal{O}_{\phi'}(x_0)}}) < \varepsilon$. Also define a reparametrization $\rho\colon \overline{\mathcal{O}_{\phi'}(x_0)} \times \mathbb{R} \to \mathbb{R}$ by $\rho(x, t) = \sigma(x, \tau(h(x), \tilde{\sigma}(h(x), t)))$ where $\tilde{\sigma}$ is the inverse reparametrization of $\sigma$. We have

$$
\begin{aligned}
l(\phi'_{\rho(x,t)}(x)) &= h^{-1}\circ k\circ h\left(\phi'_{\sigma(x,\,\tau(h(x),\tilde{\sigma}(h(x),\,t)))}(x)\right)\\
&= h^{-1}\circ k\left(\psi'_{\tau(h(x),\tilde{\sigma}(h(x),\,t))}(h(x))\right)\\
&= h^{-1}(\psi_{\tilde{\sigma}(h(x),\,t)})(k\circ h(x))\\
&= \phi_t\left(h^{-1}\circ k\circ h(x)\right)\\
&= \phi_t(l(x)).
\end{aligned}
$$

Then, $(l,\,\rho)$ is a semiconjugacy from $\phi'_{\overline{\mathcal{O}(x_0)}}$ to $\phi$ and therefore $x_0\in T(\phi)$. We have shown that $h^{-1}(T(\psi))\subseteq T(\phi)$. The contrary inclusion can be seen analogously. This completes the proof. □

Since every continuous change velocity of $\phi$ satisfy the previous theorem $\phi$, the following corollary is immediate.

**Corollary 4.1.7.** *Let $\phi$ be a flow on a metric space $X$. Then the set of topologically stable points remains invariant under continuous velocity changes.*

From topological equivalence we can obtain invariance.

**Corollary 4.1.8.** *The set $T(\phi)$ of topologically stable points of $\phi$ is invariant under $\phi$. In particular, $T(\phi)$ is union of orbit of $\phi$.*

*Proof.* Let $s$ be any real number. The function $\phi_s$ is a conjugacy between $\phi$ and itseld, because $\phi_t=\phi_s\circ\phi_t\circ\phi_{-s}$. As pointed before, every conjugacy between two flows is an equivalence. Therefore $\phi_s(T(\phi))=T(\phi)$, this is, $T(\phi)$ is invariant under $\phi$. □

So, given $x\in T(f)$ we have $\mathcal{O}(x)\subseteq T(f)$. An orbit will be called a *topologically stable orbit* if is the orbit of some topologically stable point.

Remind that a *closed manifold* is a compact one without boundary.

**Theorem 4.1.9.** *Let $\phi$ be a $C^1$ flow on a closed smooth manifold $M$ of dimension different from 2. Then $T(\phi)\cap CR(\phi)\subseteq\overline{Per(\phi)}$. In particular if $CR(\phi)\subseteq T(\phi)$, then $CR(\phi)=\overline{Per(\phi)}$.*

*Proof.* Let $x\in T(\phi)\cap CR(\phi)$ and let $\varepsilon>0$. First suppose that $\dim M\geq 3$. Then there exists $\delta>0$ such that for every $(\delta,\,1)$-chain from $x$ to $x$ there is a $C^1$ flow $\psi$ on $M$ such that $d_{C^0}(\phi_t,\,\psi_t)\leq\varepsilon$ for all $t\in[0,\,1]$ and such that if $(x_i,\,t_i)_{i=0}^n$ a $(\delta,\,1)$-chain from $x$ to itself then $\mathcal{O}_\psi(x)$ is a periodic orbit for $\psi$. (see [28]). By decreasing $\delta>0$ if necessary we can suppose it satisfies the topological stability of $x$. By assumption, there is some $(\delta,\,1)$-chain $(x_i,\,t_i)_{i=0}^n$ from $x$ to itself. Take $\psi$ as above and let $(k,\,\tau)$ be a semiconjugacy from $\psi|_{\overline{\mathcal{O}_\psi(x)}}$ to $\phi$. Since $\mathcal{O}_\psi(x)$ is a closed orbit of a $C^1$ flow we have that $\mathcal{O}_\psi(x)=\overline{\mathcal{O}_\psi(x)}$ and $k(\mathcal{O}_\psi(x))$ is a closed orbit of $\phi$ satisfying $d(k(\mathcal{O}_\psi(x)),\,x)\leq d(k(x),\,x)\leq d(k,\,id_{\mathcal{O}_\psi(x)})\leq\varepsilon$. Since $\varepsilon$ is arbitrary we conclude that $x\in\overline{Per(\phi)}$. If $\dim M=1$ or $\dim M=0$ the conclusion is immediate. In particular, if $CR(\phi)\subseteq T(\phi)$ then $CR(\phi)\subseteq\overline{Per(\phi)}$ and therefore $CR(\phi)=\overline{Per(\phi)}$ and the claim follows. □

**Example 17.** By Theorem 4.1.9 if $\phi$ is a minimal flow on a smooth manifold $M$ of dimension greater than 2, then $T(f) = \emptyset$. Indeed $CR(\phi) = M$ and $\overline{Per(\phi)} = \emptyset$. In particular distal, equicontinuous or isometric flows admitting shadowable points on a closed manifold of dimension greater than 3 do not have topologically stable points.

**Theorem 4.1.10.** *If a flow $\phi$ on a compact metric space $X$ can be arbitrarily $C^0$-approximated by minimal flows then either* $\text{int}(Per(\phi)) \cap T(f) = \emptyset$ *or $X$ is a periodic orbit.*

*Proof.* Let $x \in \text{Int}(Per(\phi))$ and $\varepsilon > 0$ be such that $d(x, y) \leq \varepsilon$ implies that $y \in Per(\phi)$. Suppose that $x \in T(\phi)$ and let $\delta > 0$ satisfying the topological stability of $\phi$. Let $\psi$ be a minimal flow which is $\delta$-closed to $\phi$ and $(k, \tau)$ a semiconjugacy from $\psi|_{\overline{\mathcal{O}_\psi(x)}}$ to $\phi$ such that $d_{C^0}(k, id_{\overline{\mathcal{O}_\psi(x)}}) \leq \varepsilon$. Then $\overline{\mathcal{O}_\phi(x)} = X = k(\overline{\mathcal{O}_\psi(x)})$. However, $d(k(x), x) \leq \varepsilon$ and so $\mathcal{O}_\phi(k(x))$ is a periodic orbit of $\phi$. This is only possible if $\mathcal{O}_\phi(k(x)) = X$ and the theorem follows. □

## 4.2 Topologically stable points and suspensions

In this section we study the relation between the topological stable points of a homeomorphism and of its suspended flow.

We begin with a technical lemma whose proof can be found in [47].

**Lemma 4.2.1.** *Let $\phi$ be a flow on a compact metric space $X$ and suppose that $Sing(\phi) = \emptyset$. Then there exists $T_0 > 0$ such that if $0 < t < T_0$, there is $\lambda > 0$ such that $d(x, y) < \lambda$ implies that $d(\phi_t(x), y) > \lambda$ for all $x, y \in X$.*

Next we proof that topological points can be "lowered" from the suspension flow.

**Theorem 4.2.2.** *Let $f: X \to X$ be a homeomorphism on a compact metric space $X$, $\phi^{\tau, f}$ be the suspension flow of $f$ on the suspended space $X^{\tau, f}$ with height function $\tau: X \to (0, \infty)$ and $(x, r) \in X^{\tau, f}$. If $\mathcal{O}_{\phi^{\tau, f}}(x, r) \subseteq T(\phi^{\tau, f})$, then $x \in T(f)$.*

*Proof.* Since the set of topological points of a flow is invariant under equivalences, we only consider the case where $\tau \equiv 1$. Moreover, since $\phi^f$ does not have singularities, by making a velocity change we can suppose $\phi^f$ satisfies Lemma 4.2.1 with $T_0 \geq 1$. Fix $\lambda > 0$ as in Lemma 4.2.1 for $t = \frac{1}{4}$. Now, let $\varepsilon > 0$ and $(x, r)$ be a topologically stable point of $\phi^f$. We can assume without loss of generality that $\varepsilon < \min\{\lambda, \frac{1}{4}\}$. Take $0 < \varepsilon_0 < \varepsilon$ with the following properties:

1. If $d((x, r), (y, s)) \leq \varepsilon_0$ then $d(\pi(x, r), \pi(y, s)) \leq \varepsilon$, where $\pi: X \times [0, 1] \to X$ is the projection map on $X$,
2. $d^f((x, r), (y, s)) \leq \varepsilon_0$ implies $d^f(\phi^f_t(x, r), \phi^f_t(y, s)) \leq \varepsilon$.

Next choose $0 < \delta_0 < \varepsilon_0$ with $\delta_0$ satisfying the topological stability of $(x, \frac{1}{2})$ with respect to $\varepsilon_0$ and let $0 < \delta < \delta_0$ such that $d(f(x), f(y)) \leq \delta_0$ whenever $d(x, y) \leq \delta$ for every $x, y \in X$. We claim that $x$ is a topological stable point of $f$. Indeed, let

$g\colon X \to X$ be another homeomorphism of $X$ such that $d_{C^0}(f,\, g) \leq \delta$ and let $\phi^g$ its corresponding suspended flow. The for $t \in [0,\, 1]$ and $s \in [0,\, 1)$ we have that

$$d^f(\phi^f_t(x,\, s),\, \phi^g_t(x,\, s)) = d^f((x,\, s+t),\, (x,\, s+t)) = 0, \text{ if } s+t \in [0,\, 1),$$

and

$$\begin{aligned} d^f(\phi^f_t(x,\, s),\, \phi^g_t(x,\, s)) &= (2-s-t)d(f(x),\, g(x)) + (s+t-1)d(f(f(x)),\, f(g(x))) \\ &< (2-s-t)\delta_0 + (s+t-1)\delta < \delta_0 \text{ if } s+t \in [1,\, 2). \end{aligned}$$

So, $d^f_{C^0}(\phi^f,\, \phi^g) < \delta_0$. Hence there is a semiconjugacy $(k,\, \sigma)$ from $\phi^g|_{\overline{\mathcal{O}_{\phi^g}(x,\, \frac{1}{2})}}$ to $\phi^f$ with $d_{C^0}(k,\, id_{\overline{\mathcal{O}_{\phi^g}(x,\, \frac{1}{2})}}) \leq \varepsilon_0$. For $(y,\, s) \in \overline{\mathcal{O}_{\phi^g}(x,\, \frac{1}{2})}$ we have

$$\begin{aligned} & d^f(\phi^f_{t-\sigma(y,\, s,\, t)}(\phi_{\sigma(y,\, s,\, t)}(k(y,\, s))),\, \phi^f_{\sigma(y,\, s,\, t)}(y,\, s)) \\ & \leq d^f(k(\phi^g_{\sigma(y,\, s,\, t)}(y,\, s)),\, \phi^g_{\sigma(y,\, s,\, t)}(y,\, s)) + d^f(\phi^g_{\sigma(y,\, s,\, t)}(y,\, s),\, \phi^f_{\sigma(y,\, s,\, t)}(y,\, s)) \\ & < \varepsilon_0 + \delta_0 < \delta, \text{ for } t \in \mathbb{R}. \end{aligned}$$

By Lemma 2.4 in [47] and the continuity of $\sigma(y,\, s,\, \cdot)$ we obtain that $|\sigma(y,\, s,\, t) - t| \leq \frac{1}{4}$ for all $t \in \mathbb{R}$ and every point $(y,\, s)$ in $\overline{\mathcal{O}_{\phi^g}(x,\, \frac{1}{2})}$. Let us define $h\colon \overline{\mathcal{O}_g(x)} \to X$ by $h(y) = \pi(k(x,\, \frac{1}{2}))$ which is clearly continuous. Since $d_{C^0}(k,\, id_{\overline{\mathcal{O}_{\phi^g}(x,\, \frac{1}{2})}}) \leq \varepsilon_0$ we have

$$d(h(y),\, y) = d(\pi(k(y,\, \tfrac{1}{2})),\, \pi(y,\, \tfrac{1}{2})) \leq \varepsilon,\ \forall\, y \in \overline{\mathcal{O}_g(x)}$$

and therefore $d_{C^0}(h,\, id_{\overline{\mathcal{O}_g(x)}}) \leq \varepsilon$. Finally to see the conjugacy $h \circ g = f \circ h$ note that if we write $k(y,\, \frac{1}{2}) = (y',\, u)$ and $1 = \sigma(y,\, \frac{1}{2},\, r)$ then $|w - \frac{1}{2}| \leq \frac{1}{4}$ and $|r-1| \leq \frac{1}{4}$ by Lemma 2.4 in [47] and so $1 \leq u + r \leq 2$. Hence

$$\begin{aligned} & h(g(y)) = \pi(k(g(y),\, \tfrac{1}{2})) = \pi(k(\phi^g_1(y,\, \tfrac{1}{2}))) = \pi(\phi^f_r(k(y,\, \tfrac{1}{2}))) = \pi((y',\, u+r)) = \\ & = \pi((f(y'),\, u+r-1) = f(y') = f(\pi(k(y,\, \tfrac{1}{2}))) = f(h(y)), \end{aligned}$$

as desired. This conclude the proof. □

CHAPTER

# FIVE

# $\mathcal{F}$-DISTAL FLOWS

In this chapter we study a variation of distality for flows $\phi$ obtained by making the proximal cell in [10] to depend on a given subset $\mathcal{F}$ of the full set of reparametrizations $\mathcal{C}$. We consider first the case when $\mathcal{F}$ reduces to a single continuous map $s : \mathbb{R} \to \mathbb{R}$ fixing the origin different from the identity. In such a case if the $s$-dependent proximal cells are trivial, then the flow is uniformly closed (or trivial if $s$ is bounded). Next we show that the flow $\phi$ is closed if and only if the $s$-depending proximal cells reduce to the corresponding orbit for every (or some) $s$ bounded. Furthermore, nonsingular flows admits points whose proximal cell (with $s$ being the identity) does not reduce to the orbit. Afterwards, we consider the case when $\mathcal{F}$ is either $\mathcal{H}$ (the set of homeomorphisms $s : \mathbb{R} \to \mathbb{R}$ fixing 0) or the whole $\mathcal{C}$. From this we obtain a characterization of the classical pointwise almost periodicity.

## 5.1 $s$-distal flows

In this section we make a first attempt to generalize the concept of distal flows.

Recall that a flow $\phi$ is *distal* if $x = y$ whenever $x, y \in X$ satisfy

$$\inf_{t\in\mathbb{R}} d(\phi_t(x), \phi_t(y)) = 0.$$

The distal flows have been widely study in topological dynamics [10], [21], [23]. By imitating the approach made in [37], we can restate the definition of distal flow as follows. A flow $\phi$ is distal if $\mathcal{P}(x) = \{x\}$ for every $x \in X$ where $\mathcal{P}(x)$ is the *proximal cell* defined by

$$\mathcal{P}(x) = \left\{ y \in X : \inf_{t\in\mathbb{R}} d(\phi_t(x), \phi_t(y)) = 0 \right\}.$$

This definition suggests a similar one depending on a continuous map $s : \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$:

**Definition 5.1.1.** Given a flow $\phi$ of $X$ and a continuous map $s\colon \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$, the *$s$-dependent proximal cell of $x$ with respect to $\phi$* is the set

$$\mathcal{P}_s(x) = \left\{ y \in X : \inf_{t\in\mathbb{R}} d(\phi_t(x), \phi_{s(t)}(y)) = 0 \right\}.$$

Clearly, $\mathcal{P}(x) = \mathcal{P}_{id}(x)$ where $id\colon \mathbb{R} \to \mathbb{R}$ is the identity. Therefore, $\phi$ is distal if and only if

$$\mathcal{P}_{id}(x) = \{x\}, \qquad \forall x \in X.$$

Recall that $\mathcal{C}$ denote the space of continuous maps $s : \mathbb{R} \to \mathbb{R}$ fixing 0. The definition bellow is motivated by the previous observation:

**Definition 5.1.2.** Let $\phi$ be flow on the compact metric space $X$ and $s \in \mathcal{C}$. It is said that $\phi$ is *s-distal* if

$$\mathcal{P}_s(x) = \{x\}, \qquad \forall x \in X. \tag{5.1}$$

So a *id*-distal flow is a distal flow. The natural question is: for an arbitrary $s\colon \mathbb{R} \to \mathbb{R}$, which flows are $s$-distal flows? In this section we give an answer for this question.

We will use the following notations. Given a flow $\phi$ of a metric space $X$, $x \in X$ and $s : \mathbb{R} \to \mathbb{R}$ we define

$$\omega_s(x) = \left\{ y \in X : \lim_{n\to\infty} d(\phi_{t_n}(x), \phi_{s(t_n)}(y)) = 0 \text{ for some sequence } t_n \to \infty \right\},$$

$$\alpha_s(x) = \left\{ y \in X : \lim_{n\to\infty} d(\phi_{t_n}(x), \phi_{s(t_n)}(y)) = 0 \text{ for some sequence } t_n \to -\infty \right\}$$

and

$$\mathcal{O}_s(x) = \{\phi_{t-s(t)}(x) : t \in \mathbb{R}\}.$$

If $s = 0$ is the zero map, then $\omega_0(x) = \omega(x)$, $\mathcal{O}_0(x) = \mathcal{O}(x)$ and $\alpha_0(x) = \alpha(x)$.

Remind that both the alpha and the omega-limit sets sets are closed, invariant and satisfy the formula

$$\overline{\mathcal{O}(x)} = \alpha(x) \cup \mathcal{O}(x) \cup \omega(x), \qquad \forall\, x \in X. \tag{5.2}$$

For general maps $s \in \mathcal{C}$ we obtain a similar identity:

**Lemma 5.1.3.** *For every flow $\phi$ of a metric space $X$ one has*

$$\mathcal{P}_s(x) = \alpha_s(x) \cup \mathcal{O}_s(x) \cup \omega_s(x), \qquad \forall\, x \in X,\ \forall\, s \in \mathcal{C}.$$

*Proof.* It follows easily from the definitions that $\alpha_s(x) \cup \omega_s(x) \subseteq \mathcal{P}_s(x)$. If $t \in \mathbb{R}$, the constant sequence $t_n = t$ satisfies

$$\lim_{n\to\infty} d(\phi_{t_n}(x), \phi_{s(t_n)}(\phi_{t-s(t)}(x))) = \lim_{n\to\infty} d(\phi_t(x), \phi_t(x)) = 0.$$

Hence $\phi_{t-s(t)}(x) \in \mathcal{P}_s(x)$ and so

$$\alpha_s(x) \cup \mathcal{O}_s(x) \cup \omega_s(x) \subseteq \mathcal{P}_s(x).$$

Now take $y \in \mathcal{P}_s(x)$. Then, there is a sequence $t_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} d(\phi_{t_n}(x), \phi_{s(t_n)}(y)) = 0. \tag{5.3}$$

If $t_n$ is unbounded, we can assume by passing to a subsequence if necessary that $t_n \to \infty$ or $t_n \to -\infty$. Hence $y \in \alpha_s(x) \cup \omega_s(x)$. Otherwise, $t_n$ is bounded and so we can assume that $t_n \to t$ for some $t \in \mathbb{R}$ by passing to a subsequence if necessary. Then, (5.3) implies $d(\phi_t(x), \phi_{s(t)}(y)) = 0$ i.e. $y = \phi_{t-s(t)}(x) \in \mathcal{O}_s(x)$. All together imply

$$\mathcal{P}_s(x) \subseteq \alpha_s(x) \cup \mathcal{O}_s(x) \cup \omega_s(x)$$

proving the result. □

For bounded functions $s \in \mathcal{B}_0$ we have a stronger lemma:

**Lemma 5.1.4.** *For every flow $\phi$ of a compact metric space $X$ and every $s \in \mathcal{B}_0$ one has*

$$\mathcal{O}(x) \subseteq \mathcal{P}_s(x) \subseteq \overline{\mathcal{O}(x)}, \qquad \forall\, x \in X.$$

*Proof.* Fix $x \in X$. Obviously $\mathcal{O}_s(x) \subseteq \mathcal{O}(x)$. Now take $y \in \omega_s(x)$. Then, there is a sequence $t_n \to \infty$ such that $d(\phi_{t_n}(x), \phi_{s(t_n)}(y)) \to 0$ as $n \to \infty$. Since $s$ is bounded, we can assume by passing to a subsequence if necessary that $s(t_n) \to a$ for some $a \in \mathbb{R}$. Since $X$ is compact, we can also assume that $\phi_{t_n}(x) \to z$ for some $z \in X$. Clearly $z \in \omega(x)$. Since

$$d(z, \phi_a(y)) \leq d(z, \phi_{t_n}(x)) + d(\phi_{t_n}(x), \phi_{s(t_n)}(y)) + d(\phi_{s(t_n)}(y), \phi_a(y)),$$

$d(z, \phi_{t_n}(x)) \to 0$, $d(\phi_{t_n}(x), \phi_{s(t_n)}(y)) \to 0$ and

$$d(\phi_{s(t_n)}(y), \phi_a(y)) \to d(\phi_a(y)), \phi_a(y)) = 0$$

as $n \to \infty$, we obtain $d(z, \phi_a(y)) = 0$, i.e., $\phi_a(y) = z \in \omega(x)$. Therefore, $y \in \omega(x)$ proving $\omega_s(x) \subseteq \omega(x)$. Similarly we prove $\alpha_s(x) \subseteq \alpha(x)$. By Lemma 5.1.3 and (5.2) we conclude that

$$\mathcal{P}_s(x) \subseteq \overline{\mathcal{O}(x)}.$$

On the other hand, since $s$ is bounded continuous, the map $t \in \mathbb{R} \to t - s(t)$ is onto. Take $y \in \mathcal{O}(x)$. Then, $y = \phi_r(x)$ for some $r \in \mathbb{R}$. Taking $t \in \mathbb{R}$ such that $r = t - s(t)$ we get $y = \phi_r(x) = \phi_{t-s(t)}(x) \in \mathcal{O}_s(x)$. Then, Lemma 5.1.3 implies

$$\mathcal{O}(x) \subseteq \mathcal{P}_s(x)$$

and the proof follows. □

*Remark* 8. Lemma 5.1.4 is false if $s : \mathbb{R} \to \mathbb{R}$ were unbounded. Take for instance a minimal distal flow $\phi$ of a compact metric space with more than one point and $s = id$.

**Definition 5.1.5.** We say that a flow $\phi$ is *trivial* (resp. *closed*) if $\phi_t(x) = x$ for every $x \in X$ (resp. every orbit $\mathcal{O}(x)$ is closed). We say that $\phi$ is *uniformly closed* if there is $T \neq 0$ such that $\phi_T(x) = x$ for every $x \in X$.

The following theorem answer our last question:

**Theorem 5.1.6.** *Let $\phi$ be a flow of a metric space $X$. If $\phi$ is $s$-distal for some $s \in \mathcal{C} \setminus \{id\}$, then $\phi$ is uniformly closed. If additionally $s \in \mathcal{B}_0$, then $\phi$ is trivial.*

*Proof.* Let $\phi$ be a flow of a metric space $X$. Suppose that there is $s \in \mathcal{C} \setminus \{id\}$ satisfying (5.1). Then, Lemma 5.1.3 implies $\mathcal{O}_s(x) \subseteq \{x\}$ and so $\phi_{t-s(t)}(x) = x$ for every $x \in X$ and every $t \in \mathbb{R}$. Since $s \neq id$, $T = t - s(t) \neq 0$ for some $t \in \mathbb{R}$. For this $T$ we obtain $\phi_T(x) = x$ for every $x \in X$ and then $\phi$ is uniformly closed.

Finally, if $s$ is bounded, then $\mathcal{O}(x) = \{x\}$ for every $x \in X$ by Lemma 5.1.4 and so $\phi$ is trivial. □

## 5.2 $\mathcal{F}$-distal flows

Theorem 5.1.6 says that, for general $s \in \mathcal{C}$, $s$-distal flows are quite trivial . We can improve the definition by considering the equation (5.1) but replacing $\{x\}$ by another set depending on $x$. Of course, natural candidates for such a replacement are the orbit $\mathcal{O}(x)$ or the *orbit closure* $\overline{\mathcal{O}(x)}$. The first of these candidates yields to flows satisfying:

$$\mathcal{P}_s(x) = \mathcal{O}(x), \qquad \forall\, x \in X. \tag{5.4}$$

Related examples are as follows.

**Example 18.** Clearly $\mathcal{P}_0(x) = \overline{\mathcal{O}(x)}$ for every $x \in X$ and every flow $\phi$ of a metric space $X$, where $0$ here is the zero map ($t \in \mathbb{R} \mapsto 0$). Then, the sole flows $\phi$ satisfying (5.4) with $s = 0$ are the closed ones.

**Example 19.** For every uniformly closed flow $\phi$ there is $s \in \mathcal{B}_0$ satisfying (5.4) (take for instance $s = 0$).

**Lemma 5.2.1.** *If $\phi$ is a flow of a metric space $X$ and $s : \mathbb{R} \to \mathbb{R}$ is bounded, then*

$$\omega(x) \subseteq \bigcup_{a\in\mathbb{R}} \phi_a(\omega_s(x)) \quad \text{and} \quad \alpha(x) \subseteq \bigcup_{a\in\mathbb{R}} \phi_a(\alpha_s(x)) \quad \text{for every} \quad x \in X.$$

*Proof.* Fix $x \in X$. If $y \in \omega(x)$, there is a sequence $t_n \to \infty$ such that $d(\phi_{t_n}(x), y) \to 0$ as $n \to \infty$. Since $s$ is bounded, we can assume up to passing to a subsequence if necessary that $s(t_n) \to a$ for some $a \in \mathbb{R}$. Since

$$d(\phi_{t_n}(x), \phi_{s(t_n)}(\phi_{-a}(y))) \leq d(\phi_{t_n}(x), y) + d(y, \phi_{s(t_n)}(\phi_{-a}(y))),$$

$d(\phi_{t_n}(x), y) \to 0$, and

$$d(y, \phi_{s(t_n)}(\phi_{-a}(y))) \to d(y, \phi_a(\phi_{-a}(y))) = d(y, y) = 0$$

as $n \to \infty$, we obtain

$$\lim_{n\to\infty} d(\phi_{t_n}(x), \phi_{s(t_n)}(\phi_{-a}(y))) = 0$$

and so

$$\phi_{-a}(y) \in \omega_s(x).$$

Hence $y \in \phi_a(\omega_s(x))$ proving the first inclusion. The proof of the second inclusion is analogous. □

We have the following corollary.

**Corollary 5.2.2.** *Let $\phi$ be a flow of a compact metric space $X$. If $x \in X$ satisfies $\mathcal{P}_s(x) = \mathcal{O}(x)$ for some $s \in \mathcal{B}_0$, then $\mathcal{O}(x)$ is closed.*

*Proof.* By Lemma 5.1.3 one has $\omega_s(x) \cup \alpha_s(x) \subseteq \mathcal{O}(x)$. Then, Lemma 5.2.1 implies

$$\omega(x) \cup \alpha(x) \subseteq \bigcup_{a\in\mathbb{R}} \phi_a(\mathcal{O}(x)) = \mathcal{O}(x)$$

and so $\mathcal{O}(x)$ is closed by (5.2). □

The theorem below is motivated by Examples 18 and 19.

**Theorem 5.2.3.** *The following properties are equivalent for every flow $\phi$ of a compact metric space $X$:*

*(1) $\phi$ is closed.*

*(2) $\phi$ satisfies (5.4) for every $s \in \mathcal{B}_0$.*

*(3) $\phi$ satisfies (5.4) for some $s \in \mathcal{B}_0$.*

*Proof.* Let $\phi$ be a flow of a compact metric space $X$. If $\phi$ is closed, $\overline{\mathcal{O}(x)} = \mathcal{O}(x)$ and then $\mathcal{P}_s(x) = \mathcal{O}(x)$ for every $x \in X$ and every $s \in \mathcal{C}$ bounded by Lemma 5.1.4. Therefore, Item (1) implies Item (2). Obviously Item (2) implies Item (3) and Item (3) implies Item (1) by Corollary 5.2.2. □

On the other hand, if the function is not bounded, then the flow is forced to have singularities:

**Theorem 5.2.4.** *If a flow $\phi$ of a compact metric space $X$ satisfies (5.4) with $s = id$, then $Sing(\phi) \neq \emptyset$.*

*Proof.* Suppose by contradiction that there is a flow without singularities $\phi$ on a compact metric space $X$ satisfying (5.4) with $s = id$. Since $\phi$ has no singularities, and $X$ is compact, there is $r > 0$ such that $X_r(x) \neq x$ for every $x \in X$. Now, take any $x \in X$. It follows from (5.4) with $s = id$ that there is a sequence $t_n \in \mathbb{R}$ such that $d(\phi_{t_n}(x), \phi_{t_n+r}(x)) \to 0$ as $n \to \infty$. By compactness we can assume that $\phi_{t_n}(x) \to z$ for some $z \in X$. Then, $\phi_r(z) = z$ which is impossible. □

The next step would be to consider flows $\phi$ satisfying (5.1) but with the orbit closure $\overline{\mathcal{O}(x)}$ instead of $\{x\}$, namely, satisfying the equation

$$\mathcal{P}_s(x) = \overline{\mathcal{O}(x)}, \qquad \forall x \in X. \tag{5.5}$$

However, we have the following example.

**Example 20.** Since $\mathcal{P}_0(x) = \overline{\mathcal{O}(x)}$ for $x \in X$ and every flow $\phi$ of $X$ (e.g. Example 18), every flow $\phi$ satisfies (5.5) with $s = 0$.

This example suggests a different approach to obtain concrete results from the equation (5.5).

We are going to consider proximal cells depending not on a single map $s : \mathbb{R} \to \mathbb{R}$ but rather on a subset $\mathcal{F} \subseteq \mathcal{C}$ as we did with $\mathcal{F}$-shadowable points:

**Definition 5.2.5.** Given a flow $\phi$, $x \in X$ and $\mathcal{F} \subseteq \mathcal{C}$ we define the *$\mathcal{F}$-proximal cell of $x$* as the set

$$\mathcal{P}_{\mathcal{F}}(x) = \left\{ y \in X : \inf_{t \in \mathbb{R}} d(\phi_t(x), \phi_{s(t)}(y)) = 0, \text{ for some } s \in \mathcal{F} \right\}.$$

Precisely $\mathcal{P}_s(x) = \mathcal{P}_{\{s\}}(x)$ when $\mathcal{F}$ consists of a single map $s$. Moreover,

$$\mathcal{P}_{\mathcal{F}}(x) = \bigcup_{s\in\mathcal{F}} \mathcal{P}_s(x), \qquad \forall x \in X.$$

Natural candidates for $\mathcal{F}$ are $\mathcal{C}$ itself or else $\mathcal{H}$, the set of homeomorphisms of $\mathbb{R}$ fixing 0. We have a first technical lemma.

**Lemma 5.2.6.** *The following properties hold for every flow $\phi$ of a compact metric space $X$ and every $x \in X$:*

*(1) If $y \in \mathcal{P}_{\mathcal{H}}(x)$, then $x \in \mathcal{P}_{\mathcal{H}}(y)$.*

*(2) $\overline{\mathcal{O}(x)} \subseteq \mathcal{P}_{\mathcal{H}}(x)$.*

*Proof.* To prove Item (1), take $x \in X$ and $y \in P_{\mathcal{H}}(x)$. Then, there are a homeomorphism $s : \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$ and a sequence $t_n \in \mathbb{R}$ such that $d(\phi_{t_n}(x), \phi_{s(t_n)}(y)) \to 0$ as $n \to \infty$. The homeomorphism $s^{-1} : \mathbb{R} \to \mathbb{R}$ and the sequence $t'_n = s(t_n)$ satisfy $s^{-1}(0) = 0$ and $d(\phi_{t'_n}(y), \phi_{s^{-1}(t'_n)}(x)) \to 0$ as $n \to \infty$ so $x \in \mathcal{P}_{\mathcal{H}}(y)$. This proves Item (1).

To prove Item (2), we first observe that $x \in \mathcal{P}_{\mathcal{H}}(x)$ (just take $s(t) = t$ in the definition of $\mathcal{P}_{\mathcal{H}}(x)$). Now, take $y \in \mathcal{O}(x)$ thus $y = \phi_r(x)$ for some $r \in \mathbb{R}$. By the previous observation we can assume that $r \neq 0$. First suppose $r > 0$ and define

$$s(t) = \begin{cases} t & \text{if } \ t \leq r; \\ \frac{1}{2}(t + r) & \text{if } \ r \leq t \leq 3r; \\ t - r & \text{if } \ 3r \leq t. \end{cases}$$

Clearly $s : \mathbb{R} \to \mathbb{R}$ is a homeomorphism and $s(0) = 0$. Since

$$d(\phi_t(x), \phi_{s(t)}(y)) = d(\phi_t(x), \phi_{t-r}(\phi_r(x))) = d(\phi_t(x), \phi_t(x)) = 0, \quad \forall t \geq 3r,$$

we get $\inf_{t\in\mathbb{R}} d(\phi_t(x), \phi_{s(t)}(y)) = 0$ proving $y \in \mathcal{P}_{\mathcal{H}}(x)$. Similarly, $y \in \mathcal{P}_{\mathcal{H}}(x)$ when $r < 0$ hence $\mathcal{O}(x) \subseteq \mathcal{P}_{\mathcal{H}}(x)$. Now take $y \in \omega(x)$. Since $X$ is compact, we can choose $q \in \omega(y)$. Hence $q \in \omega(x)$ and so there is a sequence $t_n \to \infty$ such that $d(\phi_{t_n}(x), q) \to 0$ as $n \to \infty$. But $q \in \omega(y)$ so there is another sequence $s_n \to \infty$ such that $d(\phi_{s_n}(y), q) \to 0$ as $n \to \infty$. Then, $d(\phi_{t_n}(x), \phi_{s_n}(y)) \to 0$ as $n \to \infty$ by the triangle inequality. Clearly, we can assume that both $s_n$ and $t_n$ converge monotonically to infinity as $n \to \infty$. Hence, by defining $s(t_n) = s_n$ and extending linearly to all $\mathbb{R}$ we obtain a homeomorphism $s \colon \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$. This homeomorphism satisfies $d(\phi_{t_n}(x), \phi_{s(t_n)}(y)) \to \infty$ as $n \to \infty$ and so $y \in \mathcal{P}_{\mathcal{H}}(x)$. Therefore, $\omega(x) \subseteq \mathcal{P}_{\mathcal{H}}(x)$. Similarly, $\alpha(x) \subseteq \mathcal{P}_{\mathcal{H}}(x)$ and then Item (2) holds by (5.2). □

**Theorem 5.2.7.** *A flow $\phi$ of a compact metric space $X$ is trivial if and only if $\mathcal{P}_{\mathcal{H}}(x) = \{x\}$ for every $x \in X$.*

*Proof.* Clearly $\mathcal{P}_{\mathcal{H}}(x) = \{x\}$ for every $x \in X$ whenever $\phi$ is trivial. Conversely, if $\mathcal{P}_{\mathcal{H}}(x) = \{x\}$ for every $x \in X$, then $\overline{\mathcal{O}(x)} = \{x\}$ for every $x \in X$ by Lemma 5.2.6 and so $\phi$ is trivial. □

Finally we will consider Equation (5.5) but for general subsets of reparametrizations $\mathcal{F}$ (instead of $s$):

**Definition 5.2.8.** Let $\phi$ be a flow on a compact metric space $X$ and $\mathcal{F} \subseteq \mathcal{C}$. It is said that $\phi$ is a *$\mathcal{F}$-distal* if

$$\mathcal{P}_{\mathcal{F}}(x) = \overline{\mathcal{O}(x)}, \qquad \forall x \in X. \tag{5.6}$$

To state a concrete result we introduce the following basic concepts:

**Definitions 5.2.9.** A subset $A \subseteq \mathbb{R}$ is *syndetic* if there is $K \subseteq \mathbb{R}$ compact such that $\mathbb{R} = \{a + k : (a, k) \in A \times K\}$. We say that $x \in X$ is an *almost periodic point* of $\phi$ if $\{t \in \mathbb{R} : \phi_t(x) \in U\}$ is syndetic for every neighborhood $U$ of $x$. A flow $\phi$ is *pointwise almost periodic* (also called *semisimple flow* [50]) if every $x \in X$ is an almost periodic point of $\phi$.

Every distal flow is pointwise almost periodic but not conversely (c.f. Remark 2.24 in [54]). A subset $A \subseteq X$ is a *minimal subset* of $X$ with respect to $\phi$ if it is nonempty closed and invariant, with property that if $B \subseteq A$ is also nonempty, closed and invariant, then $A = B$.

**Lemma 5.2.10** (Theorem 3.2(c) in [54])**.** *A flow $\phi$ is almost pointwise periodic if and only if $\overline{\mathcal{O}(x)}$ is compact minimal subset of $X$ for all $x \in X$.*

With this result, we can state the main theorem of this chapter:

**Theorem 5.2.11.** *The following properties are equivalent for every flow $\phi$ of a compact metric space $X$:*

*(1) $\phi$ is pointwise almost periodic.*

*(2) $\phi$ is $\mathcal{C}$-distal.*

*(3) $\phi$ is $\mathcal{H}$-distal.*

*Proof.* First we prove that Item (1) implies Item (2). Take $x \in X$ and $y \in \mathcal{P}_{\mathcal{C}}(x)$. Then, there are $s : \mathbb{R} \to \mathbb{R}$ continuous with $s(0) = 0$ and a sequence $t_n \in \mathbb{R}$ such that $d(\phi_{t_n}(x), \phi_{s(t_n)}(y)) \to 0$ as $n \to \infty$. Since $X$ is compact, we can assume by taking a subsequence if necessary that $\phi_{t_n}(x) \to z$ as $n \to \infty$ for some $z \in X$. It follows that $\phi_{s(t_n)}(y) \to z$ as $n \to \infty$ thus $z \in \overline{\mathcal{O}(x)} \cap \overline{O(y)}$. Then, $\overline{\mathcal{O}(x)} \cap \overline{O(y)} \neq \emptyset$ and so $\overline{\mathcal{O}(x)} = \overline{O(y)}$ since both $\overline{\mathcal{O}(x)}$ and $\overline{O(y)}$ are minimal sets. In particular, $y \in \overline{\mathcal{O}(x)}$ and so $\mathcal{P}_{\mathcal{C}}(x) \subseteq \overline{\mathcal{O}(x)}$. Since $\overline{\mathcal{O}(x)} \subseteq \mathcal{P}_{\mathcal{H}}(x)$ (by Lemma 5.2.6) and $\mathcal{P}_{\mathcal{H}}(x) \subseteq \mathcal{P}_{\mathcal{C}}(x)$ (by definition because $\mathcal{H} \subseteq \mathcal{C}$), we obtain $\overline{\mathcal{O}(x)} \subseteq \mathcal{P}_{\mathcal{C}}(x)$. Then, $\overline{\mathcal{O}(x)} = \mathcal{P}_{\mathcal{C}}(x)$ for every $x \in X$ proving Item (2).

By Lemma 5.2.6 we have that Item (2) implies Item (3).

Finally we prove that Item (3) implies Item (1). Take $y \in \overline{\mathcal{O}(x)}$ for some $x \in X$. Then, $y \in \mathcal{P}_{\mathcal{H}}(x)$ and so $x \in \mathcal{P}_{\mathcal{H}}(y) = \overline{\mathcal{O}(x)}$ by Lemma 5.2.6. It follows that $\overline{\mathcal{O}(x)} \subseteq \overline{O(y)}$ hence $\overline{\mathcal{O}(x)}$ is minimal thus $x$ is almost periodic. Since $x$ is arbitrary, Item (1) holds. □

Finally, we can see what happens with $\mathcal{F}$-distal flows under the presence of shadowable points. The following result resembles Corollary 2.4.15:

**Corollary 5.2.12.** *Let $\phi$ be a flow satisfying one (and hence any) of the properties stated in items 1, 2 or 3 in Theorem 5.2.11, and $X$ be a connected compact metric space. If $\phi$ has shadowable points, then $\phi$ is minimal.*

*Proof.* By Lemma 5.2.10, $\overline{\mathcal{O}(x)}$ is a minimal compact subset of $X$ for every $x \in X$. Since $Sh(\phi) \neq \emptyset$ and $X$ is connected, by Corollary 2.4.14 $\phi$ is transitive. So, there exists at least one $x \in X$ such that $\overline{\mathcal{O}(x)} = X$. Therefore $X$ is itself a minimal subset of $X$ and we can see easily that this is equivalent to say the $\phi$ is minimal. □

CHAPTER

# SIX

# THE LIMIT SHADOWING PROPERTY FOR FLOWS

In this final chapter we study flows with limit shadowing properties and explain why pointwise counterparts of them are helpless. The theory of limit shadowing for flows is largely unexplored and here we only scratch the surface. A deep study of this property for flows is expected for future work.

We begin reminding the notion of limit shadowing for homeomorphism [20]. Let $f\colon \mathbb{R} \to \mathbb{R}$, a homeomorphism on a compact metric space $(X,\, d)$. A sequence $(x_n)_{n\geq 0}$ of points of $X$ is said to be a *limit pseudo-orbit* of $f$ if

$$\lim_{n\to+\infty} d(f(x_n),\, x_{n+1}) = 0,$$

and we say it is *limit-shadowed* if there exists a point $y \in X$ such that

$$\lim_{n\to+\infty} d(f^n(y),\, x_n) = 0.$$

As explained in [20], from the numerical viewpoint this property means the following: if we apply a numerical method that approximates $f$ with 'improving accuracy' so that one step errors tend to zero as time goes to infinity, then the numerically obtained orbits tend to real ones.

Similarly, a negative sequence $(x_n)_{n\leq 0}$ (resp. two-sided$(x_n)_{n\in\mathbb{Z}}$) of points of $X$ is said to be a *negative limit pseudo-orbit* (resp. *two-sided limit pseudo-orbit*) of $f$ if

$$\lim_{n\to-\infty} d(f(x_n),\, x_{n+1}) = 0 \text{ (resp. } \lim_{|n|\to\infty} d(f(x_n),\, x_{n+1}) = 0),$$

and we say it is *negatively limit-shadowed* (resp. *two-sided limit-shadowed*) if there exists a point $y \in X$ such that

$$\lim_{n\to-\infty} d(f^n(y),\, x_n) = 0, \text{ (resp. } \lim_{|n|\to\infty} d(f^n(y),\, x_n) = 0).$$

The two-side limit shadowing property for homeomorphisms has been studied recently in [16–19]. This is among the strongest notions of shadowing in existence, and an important class of maps, including transitive Anosov diffeomorphisms and shift maps, has it.

A definition of limit shadowable point was also given [22]. A limit-shadowable point is one such that every limit pseudo-orbit passing through it is limit shadowed by a true orbit. As noted in [22], this definition is somewhat trivial because the important behavior of a limit pseudo-orbit only happens in the limit. Even more, if a point is limit-shadowable, then any other point would be because redefining the initial point of a limit pseudo-orbit does not affect its asymptotic behavior. So pointwise counterparts of limit shadowing properties does not help us to understand them better.

## 6.1 The limit shadowing property

We begin by giving flow counterparts of limit shadowing properties for flows [58] which are stronger to those given in [24].

**Definition 6.1.1.** Let $\phi$ be a flow on a metric space $X$. Given $T > 0$, a sequence of pairs $(x_i,\, t_i)_{i\in\mathbb{Z}}$ is a *limit $T$-pseudo-orbit* of $\phi$ if $t_i \geq T$ for all $i \in \mathbb{Z}$ and

$$\lim_{|i|\to\infty} d(\phi_{t_i}(x_i),\, x_{i+1}) = 0.$$

**Definition 6.1.2.** Let $\phi$ be a flow on compact metric space $X$. It is said that $\phi$ has the *limit shadowing property with respect the parameter* $T > 0$ if for every limit $T$-pseudo-orbit $(x_i,\, t_i)_{i\in\mathbb{Z}}$ of $\phi$ there are $h \in \mathrm{Rep}$ and $y \in X$ such that

$$\lim_{|t|\to\infty} d(x_0 \star t,\, \phi_{h(t)}(y)) = 0.$$

In this case we say that $y$ *limit-shadows* the limit pseudo-orbit or that it *is limit-shadowed by the orbit* $\mathcal{O}(y)$. We say that $\phi$ has the *limit shadowing property LmSP* if has the limit shadowing property with respect to $T = 1$.

Definitions of positive and negative limit pseudo-orbits, and of positive and negative limit shadowing property, $LmSP_+$ and $LmSP_-$, are analogous.

Our first lemma deals with the fact that we can control the time length of limits pseudo-orbits. A sequence of pairs $(x_i,\, t_i)_{i\in\mathbb{Z}}$ is an *limit $(T_1,\, T_2)$-pseudo-orbit* of $\phi$ if it is a limit $T_1$-pseudo-orbit of $\phi$ and $t_i \leq T_2$, for all $i \in \mathbb{Z}$. Again analogous definitions for positive and negative limit $(T_1,\, T_2)$-pseudo-orbit are possible.

**Lemma 6.1.3.** *Let $\phi$ be a flow on $X$ and $T > 0$. Then $\phi$ has the limit shadowing property with respect to the parameter $T$ if and only if every limit $(T,\, 2T)$-pseudo-orbit $(x_i,\, t_i)_{i\in\mathbb{Z}}$ of $\phi$ is limit shadowed by an orbit of $\phi$.*

*Proof.* If $\phi$ has the limit shadowing property with respect the parameter $T$, then obviously every limit $(T,\, 2T)$-pseudo-orbit passing can be limit shadowable by an orbit of $\phi$. To prove the converse, suppose $T > 1$ and that every limit $(T,\, 2T)$-pseudo-orbit is limit shadowed by an orbit of $\phi$. Let $(x_i,\, t_i)_{i\in\mathbb{Z}}$ be any limit $a$-pseudo-orbit of$\phi$. For each $n \in \mathbb{Z}$, there exists $m_n \in \mathbb{N}$ such that $t_n = m_n T + r_n$ with $T \leq r_n < 2T$. Let $(s^m_n)_{n\in\mathbb{Z}}$ the sequence of sums associated to $m = (m_n)_{n\in\mathbb{Z}}$. Denote $A_n = s^m_n + n$ for all $n \in \mathbb{Z}$ and define the sequence $(y_i)_{i\in\mathbb{Z}}$ on $X$ such that $y_i = \phi_{T(i-A_n)}(x_n)$ if $A_n \leq i < A_{n+1}$. In addition, we define a sequence $\lambda = (\lambda_i)_{i\in\mathbb{Z}}$ of real numbers in the following way, for each $i \in \mathbb{Z}$, we set

$$\lambda_i = \begin{cases} T & \text{if } A_n \leq i < A_{n+1} - 1, \\ r_n & \text{if } i = A_{n+1} - 1. \end{cases}$$

Given $i \in \mathbb{Z}$ note that $T \leq \lambda_i < 2T$ and let $n \in \mathbb{Z}$ be such that $A_n \leq i < A_{n+1}$. We have two cases.

*Case 1:* if $i < A_{n+1} - 1$, then

$$d(\phi_{\lambda_i}(y_i), y_{i+1}) = d(\phi_T(\phi_{T(i-A_n)}(x_n)), \phi_{T(i+1-A_n)}(x_n)) = 0.$$

*Case 2:* if $i = A_{n+1} - 1$, bearing in mind that $A_{n+1} - A_n = s^m_{n+1} - s^m_n + 1 = m_n + 1$ we obtain

$$\begin{aligned} d(\phi_{\lambda_i}(y_i), y_{i+1}) = d(\phi_{r_n}(\phi_{T(A_{n+1}-1-A_n)}(x_n)), x_{n+1}) &= d(\phi_{r_n}(\phi_{Tm_n}(x_n)), x_{n+1}) \\ &= d(\phi_{t_n}(x_n), x_{n+1}) \end{aligned}$$

Let $\varepsilon > 0$, and $N \in \mathbb{N}$ such that $|n| \geq N$ implies $d(\phi_{t_n}(x_n), x_{n+1}) \leq \varepsilon$. If $i \geq A_N$ or $i \leq A_{-N}$, then $A_n \leq i < A_{n+1}$ for some $n \geq N$ or for some for some $n$ with $n \leq -N$ respectively. In either case, if $i < A_{n+1} - 1$, then by Case 1 $d(\phi_{\lambda_i}(y_i), y_{i+1}) = 0$. If $i = A_{n+1} - 1$, then by Case 2 $d(\phi_{\lambda_i}(y_i), y_{i+1}) \leq \varepsilon$. So if $|i| \geq \max\{A_N, -A_{-N}\}$ then $d(\phi_{\lambda_i}(y_i), y_{i+1}) \leq \varepsilon$. That is $(y_i, \lambda_i)_{i\in\mathbb{Z}}$ is a limit $(T, 2T)$-pseudo-orbit of $\phi$. Then, there are $z \in X$ and $h \in \text{Rep}$ such that $\lim_{|r|\to\infty} d(\phi_{r-s^\lambda_i}(y_i), \phi_{h(r)}(z)) = 0$, where $(s^\lambda_i)$ is the sequence of sums associated to $\lambda = (\lambda_i)_{i\in\mathbb{Z}}$, and $s^\lambda_i \leq r < s^\lambda_{i+1}$, for $i \in \mathbb{Z}$. Let $r \in \mathbb{R}$ and $n \in \mathbb{Z}$ such that $s^t_n \leq r < s^t_{n+1}$, where $(s^t_n)$ is associated to $t = (t_i)_{i\in\mathbb{Z}}$. Since $s^t_n = s^\lambda_{A_n}$, then $s^\lambda_{A_n} \leq r < s^\lambda_{A_{n+1}} = s^\lambda_{A_n+m_n+1}$. Hence, there is $0 \leq j \leq m_n$ such that $s^\lambda_{A_n+j} \leq r < s^\lambda_{A_n+j+1}$, and then

$$\begin{aligned} d(\phi_{r-s^\lambda_{A_n+j}}(y_{A_n+j}), \phi_{h(r)}(z)) &= d(\phi_{r-s^t_n}(\phi_{s^t_n - s^\lambda_{A_n+j}}(y_{A_n+j})), \phi_{h(r)}(z)) \\ &= d(\phi_{w-s^t_n}(\phi_{s^t_n - s^\lambda_{A_n+j}}(\phi_{aj}(x_n))), \phi_{h(r)}(z)) \\ &= d(\phi_{r-s^t_n}(x_n), \phi_{h(r)}(z)). \end{aligned}$$

Let $\varepsilon > 0$ and $K \in \mathbb{R}$ such that $|r| \geq K$ implies that $d(\phi_{r-s^\lambda_i}(y_i), \phi_{h(r)}(z)) \leq \varepsilon$ with $s^\lambda_i \leq r < s^\lambda_{i+1}$. Then, by the previous identity if $|r| \geq K$ and $s^t_n \leq r < s^t_{n+1}$ then $d(\phi_{r-s^t_n}(x_n), \phi_{h(r)}(z)) \leq \varepsilon$. It follows that $(x_i, t_i)$ is limit-shadowed by $\mathcal{O}(z)$. This proves the lemma. □

As an application of this lemma we see that limit shadowing property with respect to the parameter $T > 0$ does not depend on the choice of $T$.

**Theorem 6.1.4.** *Let $\phi$ be a flow on $X$, $p \in X$ and $T > 0$. Then $\phi$ has LmSP if and only if, it the limit shadowing property with respect to the parameter $T$.*

*Proof.* Suppose that $\phi$ has the limit shadowing property with respect to the parameter $T > 0$, with $T > 1$. By Lemma 6.1.3, to conclude that $\phi$ has $LmSP$, it is enough to show that every limit $(1, 2)$-pseudo-orbit is limit shadowed by an orbit of $\phi$. Let $(x_i, t_i)$ be a limit $(1, 2)$-pseudo-orbit of $\phi$. Fix $m \in \mathbb{N}$ such that $m \geq T$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ and $\delta < \frac{\varepsilon}{m}$ satisfying the following conditions:

1. If $|i| \geq N$ then $d(\phi_{t_i}(x_i), x_{i+1}) \leq \frac{\varepsilon}{2}$
2. For each $0 \leq t \leq 2m$ we have $d(\phi_t(x), \phi_t(y)) \leq \delta$, whenever $d(x, y) \leq \varepsilon$.

Consider the sequence of pairs $(x_{im},\, \lambda_i)_{i\in\mathbb{Z}}$ where $\lambda_i = \sum_{j=0}^{m-1} t_{j+im}$ for every $i \in \mathbb{Z}$. We denote $\lambda_i(k) = \sum_{j=k}^{m-1} t_{j+im}$ with $0 \le k < m$. If $|i| \ge N$, then

$$d(\phi_{\lambda_i}(x_{im}),\, x_{(i+1)m}) \le \sum_{r=1}^{m} d(\phi_{\lambda_i(r)}(\phi_{t_{im+r-1}}(x_{im+r-1})),\, \phi_{\lambda_i(r)}(x_{im+r})) \le m\delta < \varepsilon,$$

because $T \le \lambda_i(k) \le 2m$. So, $(x_{im},\, \lambda_i)_{i\in\mathbb{Z}}$ is a limit $T$-pseudo-orbit of $\phi$ passing through $p$. Hence, there are $z \in X$ and $h \in \text{Rep}$ such that $\lim_{|t|\to\infty} d(\phi_{t-s_n^\lambda}(x_{nm}), \phi_{h(t)}(z)) = 0$ where $s_n^\lambda \le t < s_{n+1}^\lambda$. Take $K > 0$ such that $d(\phi_{t-s_n^\lambda}(x_{nm}),\, \phi_{h(t)}(z)) \le \varepsilon$ whenever $|t| \ge K$. For $0 \le k < m$ and $|t| \ge K$ denote $s_k^t(r) = \sum_{j=r}^{k-1} t_j$ we have

$$d(\phi_{s_k^t}(x_0),\, x_k) \le \sum_{r=1}^{k} d(\phi_{s_k^t(r)}(\phi_{t_{r-1}}(x_{r-1})),\, \phi_{s_k^t(r)}(x_r)) < k\delta < \varepsilon.$$

Then for $s_k^t \le t < s_{k+1}^t$

$$d(\phi_{t-s_k^t}(x_k),\, \phi_{h(t)}(z)) \le d(\phi_{t-s_k^t}(x_k), \phi_{t-s_k^t}(\phi_{s_k^t}(x_0))) + d(\phi_t(x_0), \phi_{h(t)}(z)) \le \varepsilon.$$

For $m \le k < 2m$, we follow in the same manner. So we have that the orbit $\mathcal{O}(z)$ limit-shadows the $(1, 2)$-pseudo-orbit of $\phi$. This conclude the proof of the theorem. □

Now we prove that the limit shadowing property is invariance under continuous equivalences.

**Theorem 6.1.5.** *Let $\phi$ and $\psi$ be two flows on the compact metric spaces $(X,\, d_x)$ and $(Y,\, d_y)$, respectively. If $\phi$ is continuously equivalent to $\psi$ then either both have $LmSP$ or neither of them has.*

*Proof.* Let $(h,\, \sigma)$ a continuous equivalence from $\phi$ to $\psi$ and let $(h^{-1},\, \tilde{\sigma})$ be the corresponding inverse equivalence from $\psi$ to $\phi$. Suppose first that $\psi$ has $LmSP$. For each $x \in X$, we have a function $\sigma(x,\, \cdot) \in \text{Rep}$ such that

$$h^{-1}(\psi_{\tilde{\sigma}(h(x),\, t)}(h(x))) = \phi_t(x), \text{ for each } t \in \mathbb{R}.$$

Let $a = \min\{\tilde{\sigma}(h(x),\, 1) : x \in X\}$. By compactness of $Y$, such $a$ exists and is positive. Now, given $\varepsilon > 0$, choose $0 < \varepsilon' < \varepsilon$ such that $d_y(y_1, y_2) \le \varepsilon'$ implies $d_x(h^{-1}(y_1), h^{-1}(y_2)) \le \varepsilon$ for every $y_1, y_2 \in Y$. By Theorem 6.1.4, every limit $a$-pseudo-orbit of $\psi$ can be limit-shadowed by an orbit of $\psi$. Suppose $\varepsilon'$ is small enough so that $d_y(h(x_1), h(x_2)) \le \varepsilon$ whenever $d_x(x_1, x_2) \le \varepsilon'$ for all $x_1, x_2 \in X$. Now let $(x_n,\, t_n)_{n\in\mathbb{Z}}$ be a limit pseudo-orbit of $\phi$ and let $N \in \mathbb{N}$ be such that $|n| \ge N$ implies $d_x(\phi_{t_n}(x_n),\, x_{n+1}) \le \varepsilon'$. By definition of equivalence, we have $d_x(h^{-1}(\psi_{\tilde{\sigma}(h(x_n),\, t_n)}(h(x_n))),\, x_{n+1}) \le \varepsilon'$. Then $d(\psi_{\tilde{\sigma}(h(x_n),\, t_n)}(h(x_n)),\, h(x_{n+1})) \le \varepsilon$ whenever $|n| \ge N$. Since $t_n \ge 1$ and $\tilde{\sigma}$ is strictly increasing, then $\tilde{\sigma}(h(x_n),\, t_n) \ge a$ for all $n \in \mathbb{Z}$, so $(h(x_n), \tilde{\sigma}(h(x_n),\, t_n))_{n\in\mathbb{Z}}$ is a limit $a$-pseudo-orbit of $\psi$. Then there are $y = h(z)$ in $Y$ and $\alpha \in \text{Rep}$ such that $\lim_{|t|\to\infty} d_y(\psi_{t-\widehat{s}_i}(h(x_n)),\, \psi_{\alpha(t)}(y)) = 0$, with $\widehat{s}_i \le t < \widehat{s}_{i+1}$, where $\widehat{s}_n = \sum_{j=0}^{n-1} \tilde{\sigma}(h(x_j),\, t_j)$ if $n > 0$, $\widehat{s}_0 = 0$ and $\widehat{s}_n = -\sum_{j=n}^{-1} \tilde{\sigma}(h(x_j),\, t_j)$ if $n < 0$.

Let $K > 0$ such that, $|t| \geq K$ and $\widehat{s}_i \leq t < \widehat{s}_{i+1}$, imply $d_y(\psi_{t-\widehat{s}_i}(h(x_n)),\, \psi_{\alpha(t)}(y)) \leq \varepsilon'$. It follows that

$$d_x(h^{-1}(\psi_{t-\widehat{s}_i}(h(x_n))),\, h^{-1}(\psi_{\alpha(t)}(h(z)))) \leq \varepsilon, \tag{6.1}$$

for all $|t| \geq K$ with $\widehat{s}_i \leq t < \widehat{s}_{i+1}$. Fix $t \in \mathbb{R}$ and $n \in \mathbb{Z}$ such that $s_n \leq t < s_{n+1}$. Since $\tilde{\sigma}(h(x_n), \cdot) \in \text{Rep}$ we have $0 \leq \tilde{\sigma}(h(x_n), t - s_n) < \tilde{\sigma}(h(x_n), t_n)$. So, $\widehat{s}_n \leq \tilde{\sigma}(h(x_n), t - s_n) + \widehat{s}_n < \widehat{s}_{n+1}$. Set $\widehat{t} = \tilde{\sigma}(h(x_n), t - s_n) + \widehat{s}_n$, so $h(p) \star \widehat{t} = \psi_{\tilde{\sigma}(h(x_n), t-s_n)}(h(x_n))$. It follows that

$$d_x(\phi_{t-s_n}(x_n), \phi_{\sigma(z,\alpha(\widehat{t}))}(z)) = d_x(h^{-1}(\psi_{\widehat{t}-\widehat{s}_n}(h(x_n))), h^{-1}(\psi_{\alpha(\widehat{t}\,)}(h(z)))) \text{ for } t \in \mathbb{R}.$$

Let $\widehat{\alpha}(t) = \sigma(z, \alpha(\widehat{t}))$ for all $t \in \mathbb{R}$. Since $t \mapsto \tilde{\sigma}(h(x_n), t - s_n) + \widehat{s}_n$ is increasing, then $\widehat{\alpha} \in \text{Rep}$. Choose $K' > 0$ such that $|t| \geq K'$ implies that $|\hat{t}| \geq K$. It follows from (6.1) that

$$d_x(\phi_{t-s_n}(x_n), \phi_{\widehat{\alpha}(t)}(z)) \leq \varepsilon, \quad \text{for every } s_n \leq t < s_{n+1}, \text{ and } |t| \geq K'.$$

This proves that $(x_i,\, t_i)_{i\in\mathbb{Z}}$ is limit-shadowed by $\mathcal{O}(z)$. Hence $\phi$ has *LmSP* if $\psi$ has. Conversely, if $\phi$ has *LmSP* we can conclude that $\psi$ also has analogously by considering the corresponding equivalence $(h,\, \sigma)$ from $\phi$ to $\psi$. This completes the proof. □

We introduce a finite version of the POPT and of shadowable points.

**Definitions 6.1.6** ( [32]). Given positive numbers $\delta$, $T$ and $\varepsilon$, and the nonnegative integer $a$, we say that a finite $(\delta,\, T)$-chain $(x_i,\, t_i)_{i=0}^{a}$ of $\phi$ passes through $p$ if $x_0 = p$, and we say that is finitely $\varepsilon$-shadowed if there are a point $y \in X$ and a function $h \in \text{Rep}$ such that

$$d(x_0 \star t,\, \phi_{h(t)}(y)) \leq \varepsilon, \text{ for all } t \in [0,\, s_b).$$

A flow $\phi$ has the *finite POTP* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every finite $(\delta,\, 1)$-chain can be $\varepsilon$-shadowed.

**Definition 6.1.7.** A point $p \in X$ is *finitely shadowable*, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that every finite $(\delta,\, 1)$-pseudo-orbit of $\phi$ passing through $p$ can be finitely $\varepsilon$-shadowed.

We denote by $Sh_f(\phi)$ the set of finitely shadowable points. It is clear that $Sh(\phi) \subseteq Sh^+(\phi) \subseteq Sh_f(\phi)$.

The proofs of the following theorems are analogous to the proofs of the corresponding theorem for shadowable and forward shadowable points and we omit their proofs.

**Theorem 6.1.8.** *A flow $\phi$ has the finite POPT if and only every point is finitely shadowable.*

**Theorem 6.1.9.** *If the flow $\phi$ is chain transitive, then $Sh_f(\phi) = X$ or $Sh_f(\phi) = \emptyset$.*

For chain transitive flows, the forward limit shadowing property implies that the flow has the finite POTP.

**Theorem 6.1.10.** *Let $\phi$ a chain transitive flow on the metric space $X$. If $\phi$ has $LmSP_+$, then every point is finitely shadowable. In particular, $\phi$ has the finite POTP.*

*Proof.* Suppose by the contrary that this is not the case. Then for some $p \in X$ there is $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there is a finite forward $(\frac{1}{n}, 1)$-pseudo-orbit $\alpha_n$ passing through $p$ that is not $\varepsilon$-shadowed by any orbit of $\phi$. By chain transitivity, for each $n$ there is a $(\frac{1}{n}, 1)$-chain $\beta_n$ such that the concatenated sequence $\alpha_n\beta_n\alpha_{n+1}$ forms a finite forward $(\frac{1}{n}, 1)$-pseudo-orbit of $\phi$. Then the sequence

$$\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3\cdots$$

is a forward asymptotic pseudo-orbit passing through $p$. Denote this pseudo-orbit by $(x_i, t_i)_{i\in\mathbb{N}}$. There are $h \in \text{Rep}$ and $q \in X$ such that

$$\lim_{t\to+\infty} d(p \star t, \phi_{h(t)}(q)) = 0.$$

Let $N \in \mathbb{N}$ big enough so that $s_n \le t < s_{n+1}$ implies that $d(\phi_{t-s_n}(x_n), \phi_{h(t)}(q)) \le \varepsilon$ whenever $n \ge N$. For $n \in \mathbb{N}$ let $g \in \text{Rep}$ be the function $g_n(t) = h(t + s_n) - h(s_n)$. Then the $(\frac{1}{n}, 1)$-pseudo-orbit $(x_i, t_i)_{i=n}^{\infty}$ can be $\varepsilon$-shadowed by $\phi_{h(s_n)}(q)$. Indeed,

$$d(x_n \star t, \phi_{g_n(t)}(\phi_{h(s_n)}(q))) = d(x_0 \star (t + s_n), \phi_{h(t+s_n)}(q)) \le \varepsilon.$$

But this would imply that some $\alpha_n$ can be finitely $\varepsilon$-shadowed by some orbit of $\phi$. This is a contradiction and therefore the proof is completed.

□

As a Corollary we obtain a result already established in [5] (although with a weaker notion of limit shadowing).

**Corollary 6.1.11.** *The Geometric Lorenz flows whose return map satisfies $f(0) \neq 0$ or $f(1) \neq 1$ do not have LmSP.*

*Proof.* If it had $LmSP$, it would have finite $POTP$ which is a contradiction. □

## 6.2 The limit shadowing property and suspension

The following theorem establish the connection between the two-sided limit shadowing property for homeomorphisms and the limit shadowing property for flows.

**Theorem 6.2.1.** *Let $\phi^f$ be the suspension flow of a homeomorphism $f$ on $(X, d)$ under the continuous map $\tau : X \to \mathbb{R}_+$. If $f$ has the two-sided limit shadowing property, then $\phi^{f,\tau}$ has $LmSP$.*

*Proof.* By Theorem 6.1.5, it is enough to prove the result for the suspension of $f$ under the map $\tau \equiv 1$. Suppose that $f$ has the two-sided limit shadowing property and let $((x_k, s_k), t_k)_{k\in\mathbb{Z}}$ a limit 2-pseudo orbit of $\phi^f$. Take $0 < \delta' < \frac{1}{4}$. Let $w_k = \lfloor s_k + t_k \rfloor$ denote the integer part of $s_k + t_k$. Hence $t_k \ge 2$ and

$$\lim_{|k|\to\infty} d^f((f^{w_k}(x_k), s_k + t_k - w_k), (x_{k+1}, s_{k+1})) = 0.$$

Set $\delta_k = d^f((f^{w_k}(x_k), s_k + t_k - w_k), (x_{k+1}, s_{k+1}))$ for $k \in \mathbb{Z}$, so that $\delta_k \to 0$ as $|k| \to \infty$. Let $\varepsilon > 0$. Take $0 < \varepsilon' < \varepsilon$ and let $\delta > 0$ satisfying Lemma 2.5 in [47] with respect to $\varepsilon'$. Also make $\varepsilon'$ satisfies

$$d(x, y) < \varepsilon' \text{ then } d(f^i(x), f^i(y)) \le \tfrac{1}{2}\varepsilon \text{ for } i = 1, 2, 3.$$

Let $\delta' \le \min\{\varepsilon', \frac{1}{4}, \delta\}$. Take $M \in \mathbb{N}$ such that $|k| \ge M$ implies $\delta_k \le \delta'$. Since $\delta' < \frac{1}{4}$, then by Lemma 2.4 in [47] we have that, for $|k| \ge M$, $|s_k + t_k - w_k - s_{k+1}| < \delta'$ or $|1 + s_k + t_k - w_k - s_{k+1}| < \delta'$ or $|1 + s_{k+1} + w_k - t_k - s_k| < \delta'$. Now, let $n_k$ be a positive integer defined, for $|k| \ge M$, as follows

$$n_k = \begin{cases} w_k & \text{if } |s_k + t_k - w_k - s_{k+1}| < \delta', \\ w_k - 1 & \text{if } |1 + s_k + t_k - w_k - s_{k+1}| < \delta', \\ w_k + 1 & \text{if } |1 + s_{k+1} + w_k - t_k - s_k| < \delta'. \end{cases}$$

and for $|k| < M$ we can define $n_k$ as any arbitrary integer. Then, by Lemma 2.5 in [47], if $|k| \ge M$ then $d(f^{n_k}(x_k), x_{k+1}) \le \varepsilon'$. Take an arbitrary point $p \in X$ and define a sequence $(y_i)_{i\in\mathbb{Z}}$ in $X$ as follows:

$$y_i = \begin{cases} f^{i-N_k}(x_k) & \text{for } \; N_k \le i < N_{k+1} \text{ and } |k| \ge M \\ p & \text{if } |k| < M, \end{cases}$$

where $(N_k)_{k\in\mathbb{Z}}$ is the sequence of sums associated to $(n_k)_{k\in\mathbb{Z}}$. If $|i| > \max\{N_M, N_{-M}\}$, then $d(f(y_i), y_{i+1}) \le \varepsilon'$, so $(y_i)_{i\in\mathbb{Z}}$ is a two-sided limit pseudo-orbit of $f$. Then there exists $x \in X$ such that $\lim_{|i|\to\infty} d(f^i(x), y_i) = 0$. Therefore there is $N \in \mathbb{N}$ such that if $|i| \ge N$ then $d(f^i(x), y_i) \le \varepsilon'$. In particular if $|k| \ge M$ is such that $|N_k| \ge N$, then

$$d(f^{j+N_k}(x), f^j(x_k)) \le \varepsilon', \quad \text{for } 0 \le j < n_k. \tag{6.2}$$

Take $(x, s_0) \in X^{1,f}$. We claim that $(x, s_0)$ limit shadows $((x_k, s_k), t_k)_{k\in\mathbb{Z}}$. Indeed, define $\alpha\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$\alpha(t) = \begin{cases} \frac{t - s_M - N_M - s_0}{s_{-M} + N_{-M} - s_M - T_M} & \text{if } T_{-M} \le t \le T_M, \\ \frac{s_{k+1} + n_k - s_k}{t_k}(t - T_k) + s_k + N_k - s_0 & \text{if } T_k \le t < T_{k+1} \text{ and } k \notin [-M, M+1], \end{cases}$$

where $(T_k)_{k\in\mathbb{Z}}$ is the sequence of sums associated to $(t_k)_{k\in\mathbb{Z}}$. It is clear that $\alpha$ is continuous with $\alpha(0) = 0$. Moreover, since $n_k \ge 1$ then $\alpha \in \mathrm{Rep}$. Let $t \in \mathbb{R}$ and let $k \in \mathbb{Z}$ be such that $T_k \le t < T_{k+1}$ and $|k| \ge M$ we get

$$\begin{aligned} |\alpha(t) - s_k - N_k + s_0 - (t - T_k)| &= \left| \frac{s_{k+1} + n_k - s_k - t_k}{t_k}(t - T_k) \right| \\ &= |s_{k+1} + n_k - s_k - t_k| \left| \frac{t - T_k}{t_k} \right|. \end{aligned}$$

Since $|s_k + t_k - n_k - s_{k+1}| < \delta'$ and $0 \le t - T_k < t_k$, we have

$$|\alpha(t) - s_k - N_k + s_0 - (t - T_k)| < \delta'. \tag{6.3}$$

Now if $j$ is a positive integer which makes $0 \le s_k + t - T_k - j < 1$, then $0 \le j \le s_k + t_k \le n_k + 2$ for $|k| \ge M$. So, by (6.2) and the choice of $\varepsilon'$, if $|N_k| \ge N$ we have $d(f^{j+N_k}(x), f^j(x_k)) < \frac{1}{2}\varepsilon$ for $0 \le j \le n_k + 2$. Finally, for $|k| > M$ such that $N_k > N$, if $T_k \le t < T_{k+1}$ one has

$$d^f(\phi^f_{\alpha(t)}(x,\, s_0), \phi^f_{t-T_k}(x_k, s_k)) \quad = \quad d^f\left((f^{N_k}(x), s_0 + \alpha(t) - N_k), (x_k, s_k + t - T_k)\right)$$

$$\begin{aligned}
&= \quad d^f\left((f^{j+N_k}(x), s_0 + \alpha(t) - N_k - j),\, (f^j(x_k), s_k + t - T_k - j)\right)\\
&\leq \quad d^f\left((f^{j+N_k}(x),\, s_0 + \alpha(t) - N_k - j),\, (f^{j+N_k}(x_k), s_k + t - T_k - j)\right)\\
&\quad\quad + d^f\left((f^{j+N_k}(x), s_k + t - T_k - j),\, (f^j(x_k),\, s_k + t - T_k - j)\right)\\
&\leq \quad |s_0 + \alpha(t) - N_k - j - (s_k + t - T_k - j)| + (s_k + t - T_k - j)d(f^{j+N_k+1}(x),\, f^{j+1}(x_k))\\
&\quad\quad + (1 - s_k - t + T_k + j)d(f^{j+N_k}(x), f^j(x_k))\\
&< \quad \delta' + \tfrac{1}{2}(1 - (s_k + t - T_k - j))\varepsilon + \tfrac{1}{2}(s_k + t - T_k - j)\varepsilon \leq \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,
\end{aligned}$$

that is to say

$$\lim_{|t|\to\infty} d^f(\phi^f_{\alpha(t)}(x,\, s_0), \phi^f_{t-T_k}(x_k, s_k)) = 0.$$

Hence $\mathcal{O}_{\phi^f}(x,\, s_0)$ limit shadows $((x_k,\, s_k),\, t_k)_{k\in\mathbb{Z}}$ □

Theorem 6.2.1 can be used to give examples of flows with $LmSP$.

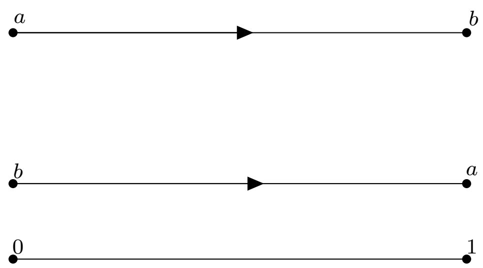
**Example 21.** It is clear that the identity map defined on an space consisting of sole point has the two-sided limit shadowing property. So, by Theorem 6.2.1, the suspension of this map, which is conjugate to the so-called rotation flow $\phi_t(z) = e^{2\pi it}z$ defined in $S^1$, has $LmSP$.

It would be tempting to believe that the the converse of Theorem 6.2.1 is also true. Sadly this is not the case as the following example shows.

**Example 22.** Let $X = \{a,\, b\}$ be a set with two different points and give it the discrete metric $d$, this is

$$d(x,\, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f\colon X \to X$ define by $f(a) = b$ and $f(b) = a$. In [16] it is shown that this map does not have the two-sided limit shadowing property. The suspension space of $X$ is the space $X^{1,f} = (\{a\} \times [0,\, 1] \cup \{b\} \times [0,\, 1])/\sim$ with $(b,\, 1) \sim (a,\, 0)$ and $(a,\, 1) \sim (b,\, 0)$. It can be seen easily that $X^{1,f}$ is homeomorphic to $S^1$ and that $\phi^f$ is the rotation flow with a velocity change of factor 2. So this flow is conjugate to $\psi_t(z) = e^{\pi it}z$ and equivalent to $\phi_t(z) = e^{2\pi it}z$ under the equivalence $(id_{S^1}, \sigma)$ where $\sigma(z,\, t) = 2t$ and therefore has $LmSP$ by Example 21 and Theorem 6.1.5.



*Remark* 9. If consider positive shadowing only then it can be establish that a homeomorphism has the limit shadowing property if and on if its suspension flows have $LmSP_+$ [58].

There is also a notion a expansiveness for homeomorphisms: a homeomorphism $f\colon X \to X$ is expansive if there is $\delta > 0$, called expansiveness constant, such that if $d(f^n(x),\, f^n(y)) \leq \delta$ then $x = y$. In [14], it was proved that a homeomorphism $f$ is expansive if and only if its suspension flow $\phi^f$ on $X^{1,f}$ is expansive.

**Corollary 6.2.2.** *There exists a non-expansive flow having $LmSP$.*

*Proof.* Let $X = [0,\, 1]^{\mathbb{Z}}$ and $\sigma\colon X \to X$ the usual shift map. Then $\phi$ has the two-sided limit shadowing property and is not expansive [18]. Then the suspension flow $\phi^\sigma$ has the two-sided limit shadowing property and is not expansive. □

In [18], it was shown that transitive Anosov diffeomosphisms are exactly those which satisfy the two-side limit shadowing property. The following corollary is a consequence of the fact that transitive suspension flows are exactly those whose base homeomorphisms are also transitive.

**Corollary 6.2.3.** *A Anosov flow which is the suspension of a Anosov diffeomorphism is transitive if, and only if, has $LmSP$.*

*Remark* 10. An interesiting question is if the above Corollary is true for every transitive Anosov flow.

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