On the entropy of the continuum hyperspace map

Jennyffer Smith Bohorquez Barrera

Tese de Doutordo apresentada ao Programa de Pós-graduação do Instituto de Matemática, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários á obtenção do título de Doutor em Matemática.

Orientador: Alexander Eduardo Arbieto Mendoza

Rio de Janeiro

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On the entropy of continuum hyperspace map

Jennyffer Smith Bohorquez Barrera

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In the last decades several researchers have shown interest in studying relations between the "individual" dynamics and the "collective" dynamics of a given system. In this thesis, we study mainly the "collective" dynamics of Morse-Smale diffeomorphisms and dendrite homeomorphisms.

In the case of Morse-Smale diffeomorphisms, we prove that the topological entropy of the induced map $C(f)$ is infinite or zero. Besides that, if the base space is S^1 then the induced map $C(f)$ does not have the shadowing property.

In the case of dendrite homeomorphisms, we prove that the topological entropy of the induced map $C(f)$ is zero or infinite.

Finally, we give three sufficient conditions to obtain infinite topological entropy on the hyperspace.

Key-words: Morse-Smale diffeomorphism, homeomorphism, dendrite, hyperspace, topological entropy, shadowing.

Entropia topológica do mapa induzido no hiperespaço dos continuos

Jennyffer Smith Bohorquez Barrera

Orientador: Alexander Eduardo Arbieto Mendoza

Nas últimas décadas varios pesquiçadores tem mostrado interesse em estudar relações entre a dinâmica individual e a dinâmica coletiva de um sistema dado. Neste trabalho, nós estudamos principalmente a dinâmica coletiva dos difeomorpfismos Morse-Smale e dos homeomorfismos definidos sob dendrites.

No caso dos difeomorpfismos Morse-Smale, provamos que a entropia topológica do mapa induzido $C(f)$ é infinita ou zero. Além disso, se o espaço base é S^1 então o mapa induzido $C(f)$ não tem a propriedade de sombreamento.

No caso dos homeomorfismos definidos sob dendrites, provamos que a entropia topológica do mapa induzido $C(f)$ é zero ou infinita.

Finalmente, obtivemos três criterios para obter entropia topológica infinita no hiperespaço.

Palavras-chave: Difeomorfismo Morse-Smale, homeomorfismo, dendrite, hiperespaço, entropia topológica, sombreamento.

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Contents

Chapter 1

Introduction

It is well known that every map $f : X \to X$ on a compact metric space X induces of a natural way, the induced map $2^f : 2^X \to 2^X$ on the hyperspace of all nonempty and closed subsets of X, defined as $2^f(A) = f(A)$ for all $A \in 2^X$. The first work that appears in the literature about the study of relationships between f and its induced map 2^f is the Bauer-Sigmund article: *Topological dynamics of transformations induced on the space of probability measures*, published in 1975 (see [7]). They studied, in particular, the property of having positive topological entropy and proved if a map has positive topological entropy then the induced map 2^f has infinite topological entropy.

Topological entropy is a way of measuring the complexity of a dynamical system. In this context, a dynamical system is said to be *chaotic* if the topological entropy is positive. The original definition was introduced in [4] by Adler, Konheim, and McAndrew in 1965 to continuous maps on a compact topological space. In the 1970s, Bowen and Dinaburg introduced in [11] and [17] an equivalent and useful definition of topological entropy. One advantage of this definition is that the base space does not need to be a compact space provided the map be uniformly continuous. However, in this work we will consider only dynamical systems on compact metric spaces.

After a bit more than 25 years, this study is resumed and currently several articles have been published in different journals (see for example [3] [8] [18] [22] [26] [27]).

When X is a continuum space, f induces the map $C(f) = 2^f|C(X)$ on the hyperspace of all connected elements of 2^X. We call $C(f)$ the induced continuum map. Many techniques used to study the map 2^f can not be applied to $C(f)$ like approximate a compact set by a finite number of points. Actually, as we will see, some results for the continuum map $C(f)$ are different that the ones for 2^f. For instance, if f has positive topological entropy does not imply that the continuum map $C(f)$ has infinite topological entropy. In [27] the authors presented a counterexample.

On the other hand, Koichi Yano [42] showed that generic homeomorphisms on manifolds n-dimensional with $n \geq 2$ has infinite topological entropy, therefore its induced hyperspace map has infinite topological entropy and analogously this holds for the induced continumm map. A natural question is: what happens if f has entropy zero? Lampart-Raith characterized the topological entropy of the induced continumm map of homeomorphisms defined on the unit circle S^1 or on the interval I, see [27]. They showed the topological entropy of the induced hyperspace map of homeomorphisms on the unit circle (or interval) can be zero or infinite, while the topological entropy of the induced continuum map is zero, that is, topological entropy zero can be generate infinite topological entropy in its induced hyperspace map but not in its induced continuum map. In particular, the topological entropy of the induced hyperspace map of all Morse-Smale diffeomorphism on S^1 (or M a compact connected manifold without boundary) is infinity while the topological entropy of $C(f)$ is zero when $M = S^1$.

Another class of dynamical systems with zero topological entropy are the dendrite homeomoprhisms. A continuum D is said to be a dendrite if is a locally connected and contains no simple closed curves. In 2015, P. Hernández and H. Méndez [22], generalized the result of Lampart-Raith to the induced dendrite homeomorphisms. The authors proved that if $f : D \to D$ is a dendrite homeomorphism then the topological entropy is infinity if and only if the set of recurrent points of f is different from D. However, unlike the homeomorphisms on the interval, there exist examples in [1] and [3] of dendrite homeomorphism such that the continuum map has infinite topological entropy.

In this thesis we are interesting in to study the induced continuum map of two classes of dynamical systems with entropy zero: the Morse-Smale diffeomorphisms and Dendrite homeomorphisms. Mainly, we study shadowing and entropy in $C(X)$.

In Chapter 2 we give the definitions and preliminary results that will be needed for the rest of the work.

In Chapter 3 we study the nonwandering set of the induced map and continuum map of Morse-Smale diffeomorphisms on S^1. We will see that the dynamics are very different, while the dynamics of 2^f present many interesting properties the dynamics of $C(f)$ remain, in some sense, equal to a Morse-Smale diffeomorphism. Although the dynamics of the continuum map $C(f)$ might be very simple, as seen in Theorem A, they are interesting because they do not have the shadowing property. In fact, in [18] Good and Fernández showed that f has the shadowing property if and only if the induced map 2^f has the shadowing property. Besides if the continuum map $C(f)$ has the shadowing property then f does too. The Morse-Smale diffeomorphisms are examples of dynamical systems with the shadowing property, see [35]. Thus the main results in this chapter are

Theorem A. Let $f : S^1 \to S^1$ be a Morse-Smale diffeomorphism then

i. If f preserves orientation, then $\Omega(C(f)) = Per_N(C(f)) \cup \{S^1\}$ for some $N \geq 1$,

ii. If f reverse orientation, then $\Omega(C(f)) = Per_2(C(f)) \cup Fix(C(f))$.

Theorem B. Let $f : S^1 \to S^1$ be a Morse-Smale diffeomorhism. Then the continuum map $C(f)$ does not have the shadowing property.

In Chapter 4 we give three sufficient conditions to obtain infinite topological entropy on the hyperspace. In section 4.1 we give a criteria to obtain infinity topological entropy on 2^X proving the following:

Theorem C. Let $f : X \to X$ be a surjective map and let X be a continuum space. If there exists an infinite countable set $A = \{a_1, a_2, ...\} \subset X$ such that

i. $L = \bigcup_{i\geq 1} \alpha(\{x^i_{-n}\}_{n\in\mathbb{Z}_+}, f) \cup \omega(a_i, f)$ and $M = \bigcup_{i\geq 1} \{x^i_{-n}\}_{n\in\mathbb{Z}_+}$ are disjoint,

ii. For every pair $i \neq j$, $i \geq 1$, $j \geq 1$,

$$Orb(\{x^i_{-n}\}_{n\in\mathbb{Z}_+}, f) \cap Orb(\{x^j_{-n}\}_{n\in\mathbb{Z}_+}, f) = \emptyset,$$

then $h(2^f) = \infty$.

There exist several dynamical systems with zero topological entropy which satisfy the hypotheses of Theorem C. In particular, we have that the induced map 2^f of every surjective finite map has infinity topological entropy. So, not only the induced map of Morse-Smale diffeomorphism has infinite entropy but also the induced map of surjective Morse-Smale endomorphisms, as we can see in the following:

Theorem D. Let $f : X \to X$ be a surjective finite map and let X be a continuum space. If the positively recurrent points are not dense in X then $h(2^f) = \infty$.

In Section 4.2 we give two criteria to obtain infinity topological entropy on $C(X)$. In the first criteria, we introduce one special dynamics that guarantees infinite entropy on the hyperspace as follows:

Theorem E. Let $f : X \to X$ be homeomorphism on a continuum metric space. If f admits a special dendrite, then $h(C(f)) = \infty$ and therefore $h(2^f) = \infty$.

In the second criterion, we introduce the notion of curves self-accumulated such that together with some extra-hypothesis guarantee infinite entropy on $C(X)$ as follows:

Theorem F. Let M^n be a compact, connected n-dimensional space with $n \geq 2$ and let $f : M^n \to M^n$ be a homeomorphism with three fixed points p, q and σ. If there exists an infinite countable set $A = \{a_0, a_1, a_2, ...\} \subset M^n$ such that

i. for every $i \geq 0$, $\alpha(a_i, f) = \{p\}$ and $\omega(a_i, f) = \{q\}$,

ii. for every $i \geq 0$, $a_i \neq p$ and $a_i \neq q$,

iii. for every pair $i \neq j$, $i \geq 0$, $j \geq 0$,

$$\{f^k(a_i) : k \in \mathbb{Z}\} \cap \{f^k(a_j) : k \in \mathbb{Z}\} = \emptyset,$$

iv. for every $r \geq 1$ and $i \in \{0, 1, ..., r-1\}$, there exist arcs γ_i from a_i to σ, such that the sequence $\{f^k(\gamma_i)\}_{k\in\mathbb{Z}}^{i\in\{0,...,r-1\}}$ is not self-accumulated.

Then $h(C(f)) = \infty$.

In Section 4.3 we prove using the definition of topological entropy by separated sets that the continuum map of every Morse-Smale diffeomorphisms has infinity topological entropy as we can see in the following:

Theorem G. Let $f : M \to M$ be a Morse-Smale diffeomorphism then the topological entropy of its induced map $C(f)$ is infinite.

Finally, Chapter 5 is devoted to study the continuum map of dendrite homeomorphisms. Our conjecture is: The continuum map of all dendrite homeomorphism has zero or infinite topological entropy. We prove the conjecture but for a subclass of dendrite homeomorphisms. In fact, we show that the existence of no-recurrent branch points could generate infinite topological entropy on the continuum hyperspace. A point $x \in D$ is said to be a branch point of D if the number of all components of $D \setminus \{x\}$ is greater or equal than 3. The main result in Chapter 5 is the following:

Theorem H. Let $f : D \to D$ be a dendrite homeomorphism. Then

i. If there is a no-recurrent branch point in D, then the topological entropy of its induced map $C(f)$ is ∞.

ii. If all point in D is a recurrent point, then the topological entropy of its induced map $C(f)$ is 0.

Chapter 2

Preliminaries

Our purpose in this chapter is to present definitions and some of the well-known or not so well-known facts about Morse-Smale diffeomorphisms, Dendrite homemorphisms, hyperspaces and induced maps, recurrence and topological entropy that will be used throughout the whole work.

For general background see for instance [4], [9], [23], [28], [35], [36], [40], [16].

2.1 Hyperspaces and induced maps

Hyperspaces

Let (X, d) be a compact metric space. A *hyperspace* of X is a specified collection of subsets of X endowed with some metric. For convenience, we exclude the empty set $\emptyset$ from being a point of hyperspace. We restrict our attention to the following hyperspaces:

$$2^X = \{A \subset X : A \text{ is nonempty and closed in } X\}$$

if X is a compact metric space and

$$C(X) = \{A \in 2^X : A \text{ is connected}\}$$

if X is a *continuum*, that is, a compact connected metric space. Throughout this text we called *hyperspace* and *continuum hyperspace* respectively. Notice that $C(X) \subset 2^X$.

We consider the Hausdorff Metric d_H on 2^X (or $C(X)$), which was first considered by F. Hausdorff in [20]. Let (X, d) be a compact metric space. For any $x \in X$ and $A \in 2^X$ (or $C(X)$), let

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

For any $r > 0$ and any $A \in 2^X$ let

$$N(x, A) = \{x \in X : d(x, A) < r\}.$$

We say that $N(x, A)$ is the *generalized open d-ball in X about A of radius r*. Thus, *The Hausdorff metric* for 2^X, induced by d, which is denoted by d_H, is defined as follows: for any $A, B \in 2^X$ (or $C(X)$),

$$d_H(A, B) = \inf\{r > 0 : A \subset N(B, r) \text{ and } B \subset N(A, r)\}.$$

Theorem 2.1.1. [*See*[*23*], *p.*11] *If (X, d) is a compact metric space, then d_H is a metric on 2^X (or $C(X)$), that is, for all A, B and C in 2^X (or $C(X)$)*

- $d_H(A, B) \geq 0,$
- $d_H(A, B) = 0$ *if and only if* $A = B,$
- $d_H(A, B) = d_H(B, A)$ *and*
- $d_H(A, B) \leq d_H(A, C) + d_H(C, B).$

Intuitively, A and B are close together with respect to d_H provided that each point of A is close to a point of B and A and B are approximately the same. The interested reader can be find in the books [23] and [28] a detailed study about properties of Hausdorff metric and the hyperspaces 2^X and $C(X)$. However, we give the main properties of them without proof. The fundamental theorem about the topological invariance of 2^X and $C(X)$ is the following:

Theorem 2.1.2. [*See*[*23*], *p.7*] *If X and Y are homeomorphic, then 2^X and 2^Y ($C(X)$ and $C(Y)$) are homeomorphic.*

Theorem 2.1.3. [*See*[*23*], *p.16*] *If (X, d) is a compact metric space, then $(2^X, d_H)$ is a compact metric space. And if (X, d) is a continuum then $C(X)$ is a continuum space.*

There are other hyperspaces as $F_n(X)$ the collection of subsets of X with at most n elements, $C_n(X)$ the collection of subsets of X (supposing that X is a continuum) with at most n connected components of X, among others. Nevertheless we do not study them.

Induced maps

Let (X, d) and (Y, ρ) be compact metric spaces and let $f : X \to Y$ be a continuous map. The map f induces in a natural way a map on 2^X,

$$2^f : 2^X \to 2^Y$$

given by $2^f(A) = f(A) = \{y \in Y : y = f(x) \text{ for some } x \in A\}$ for all $A \in 2^X$. Since f is a continuous map, 2^f is well defined.

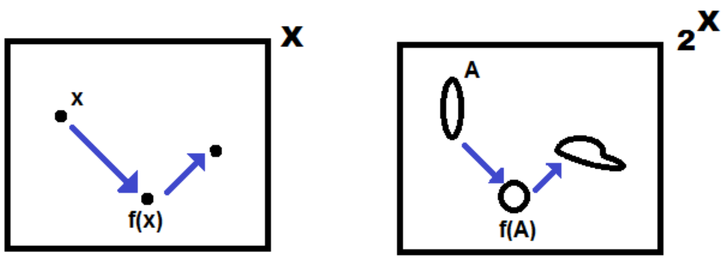

Figure 2.1: Individual dynamics vs Collective dynamics.

Proposition 2.1.4. [*See [30], p. 17*] *Let (X, d) and (Y, ρ) be a compact metric spaces. If $f : X \to Y$ is a continuous map (homeomorphism) then $2^f : 2^X \to 2^Y$ is a continuous map (homeomorphism).*

Throughout this text, we called 2^f *the induced hyperspace map* and $C(f) = 2^f|_{C(X)}$ *the induced continuum map*. Thus, if (X, d) is a continuum, then the dynamical system (X, f) induces the dynamical systems $(2^X, 2^f)$ and $(C(X), C(f))$. Therefore, it is natural and interesting to study the relationships between f and its induced maps 2^f and $C(f)$. When a dynamical property of f remains valid for its induced maps, and conversely. A number of well-studied subspaces (such as the collections $C_n(X)$ of closed sets with at most n components, $F(X)$ of finite subsets, or $F_n(X)$ of subsets sets with at most n points) are invariant under 2^f and therefore form dynamical systems in their own right, see [23] and [28]. Nevertheless, we do not study them here since they are not our main object of study.

2.2 Recurrence and orbit perturbation

Recurrence

Let $f : X \to X$ be a homeomorphism of X a compact metric space. For a point $x \in X$ we define the following sets:

- The *orbit* of x is the $\mathcal{O}(x) = \{f^n(x) : n \in \mathbb{Z}\}$. We can also define the future (resp. past) orbit of x by $\mathcal{O}^+(x) = \{f^n(x) : n \geq 0\}$ (resp. $\mathcal{O}^-(x) = \{f^n(x) : n \leq 0\}$).
- The *omega-limit set* (resp. *alpha-limit set*) is the set $\omega(x, f) = \{y \in X : \exists n_j \to +\infty \text{ such that } f^{n_j}(x) \to y\}$ (resp. $\alpha(x, f) = \omega(x, f^{-1})$). In general, when f is understood, we shall omit it from the notation.

We can classify the points according to the behavior of their orbit:

- $Fix(f) = \{x \in X : f(x) = x\}$ is the set of *fixed points*.
- $Per(f) = \{x \in X : \exists n \in \mathbb{N} \text{ such that } f^n(x) = x\}$ is the set of *periodic points*.
- We say that a point x is *recurrent* if $x \in \omega(x)$.
- $R(f) = \{x \in X : x \in \omega(x, f)\}$ is the set of *recurrent points*.
- $\Omega(f) = \{x \in X : \forall \varepsilon > 0, \exists n > 0; f^n(B(x, \varepsilon)) \cap B(x, \varepsilon) \neq \emptyset\}$ is the nonwandering set of f.

We refer the reader to [24], [36] and [39] for examples showing the strict inclusions in the following chain of closed sets which is easy to verify:

$$Fix(f) \subset Per(f) \subset R(f) \subset \Omega(f)$$

We say that f is *transitive* if there exists $x \in X$ such that $\mathcal{O}(x)$ is dense in X. It is an easy exercise to show the following equivalences (see for example [40], p.127):

Proposition 2.2.1. *The homeomorphism $f : X \to X$ is transitive if and only if for every U, V open sets there exists $n \in \mathbb{Z}$ such that $f^n(U) \cap V \neq \emptyset$ if and only if there is a residual subset of points whose orbit is dense.*

We say that f is *minimal* if every orbit is dense. Many of the dynamical properties one obtains are invariant under what is called *conjugacy*. We say that two dynamical systems (X, f) and (Y, g) are (topologically) *conjugated* if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$.

Shadowing

Let $\delta > 0$. We say that a sequence $(x_n)_{n\in\mathbb{Z}}$ is a *δ-pseudo orbit of f* if

$$d(f(x_n), x_{n+1}) \leq \delta \text{ for all } n \in \mathbb{Z}.$$

The homeomorphism f is said to *have the shadowing property* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo orbit $(x_n)_{n\in\mathbb{Z}}$ of f is ε-shadowed by a real orbit of f, i.e, there exists $x \in X$ such that

$$d(x_n, f^n(x)) < \varepsilon \text{ for all } n \in \mathbb{Z}.$$

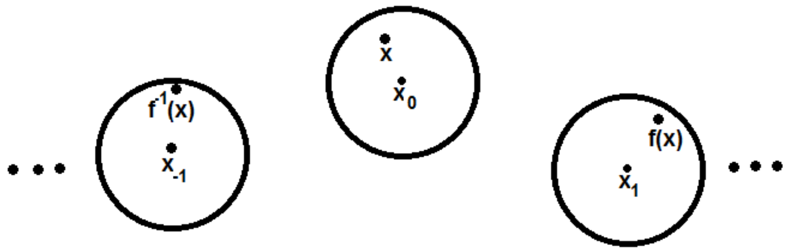

Figure 2.2: Shadowing Property.

2.3 Topological Entropy

The most important numerical invariant related to the orbit growth is topological entropy. It represents the exponential growth rate for the number of orbit segments distinguishable with arbitrarily fine but finite precision. In a sense, the topological entropy describes in a crude but suggestive way the total exponential complexity of the orbit structure with a single number. In this section, first, we give the definition of topological entropy introduced by Bowen-Dinaburg in 1970. Then we present some important properties and examples of how to calculate entropy.

Definition(Bowen-Dinaburg)

Let $f : X \to X$ be a continuous map on X a compact metric space with a metric d. We define an increasing sequence of metrics d_n, $n \in \mathbb{N}$ as follows:

$$d_n(x, y) = \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y)).$$

In other words, d_n measures the distance between the orbit segments $I_x^n = \{x, ..., f^{n-1}(x)\}$ and I_y^n. We denote the open ball $\{y \in X : d_n(x, y) < \varepsilon\}$ by $B(x, \varepsilon, n)$. By definition, we have that

$$B(x, \varepsilon, n) = \bigcap_{i=0}^{n-1} f^{-i}(B(f^i(x), \varepsilon))$$

We say that $F \subset X$ is (n, ε)-spanning if for each $x \in X$ there exists $y \in F$ such that $d_n(x, y) < \varepsilon$, that is,

$$X \subset \bigcup_{x \in F} B(x, \varepsilon, n).$$

Let $r(n, \varepsilon)$ be the minimal cardinality of an (n, ε)-spanning set, or equivalently the cardinality of a minimal (n, ε)-spanning set. The topological entropy of f is given by

$$h^{sp}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \varepsilon).$$

One more way to define topological entropy is via separated sets. We say that $E \subset X$ is (n, ε)-separated set if for every pair of different points x, y in E, $d_n(x, y) > \varepsilon$, that is, the set $\bigcap_{i=0}^{i=n-1} f^{-i}(B(f^i(x), \varepsilon))$ does not contain another point of E. Let $s(n, \varepsilon)$ be the maximal cardinality of and (n, ε)-separated set, or equivalent the cardinality of a maximal (n, ε)-separated set. The topological entropy of f is given by

$$h^{se}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon).$$

Proposition 2.3.1. [*See [40], p. 169*] *Let $f : X \to X$ be a continuous map on a compact metric space X, with metric d. Then $h^{sp}(f) = h^{se}(f)$.*

Thus, we denote the topological entropy of f by $h(f)$ (definition given by Bowen-Dinaburg). Therefore, it is possible to calculate the topological entropy using spanning sets or separated sets.

Properties

The topological entropy is a topological invariant. This property is very important because it provides a method for calculating topological entropy. Notice that topological entropy is not easy to calculate, though it is possible to do it in some cases. Therefore, if we want to calculate the topological entropy of a dynamical system using the following theorem, we must first calculate the topological entropy of a simpler conjugated dynamical system. The proof of the following properties can be found in [9] and [40].

Theorem 2.3.2. *Let $f : X \to X$ and $g : Y \to Y$ be continuous map. If f and g are conjugated then $h(f) = h(g)$.*

Proposition 2.3.3. *Let $f : X \to X$ be a homeomorphism. Then $h(f) = h(f^{-1})$.*

Proposition 2.3.4. *Let $f : X \to X$ be a continuous map on a compact metric space and let $Y \subset X$ be a closed subset such that $f(Y) = Y$. Then $h(f|_Y) \leq h(f)$.*

Proposition 2.3.5. *Let $f : X \to X$ be a continuous map on a compact metric space and let $m \in \mathbb{N}$. Then $h(f^m) = mh(f)$.*

Proposition 2.3.6. *Let f_i be a continuous map on a compact metric space (X_i, d_i) for $i = 1, 2$. Let d be a metric defined on $X_1 \times X_2$ as follows:*

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

Then $(X_1 \times X_2, d)$ is a compact metric space, $f_1 \times f_2 : X_1 \times X_2 \to X_1 \times X_2$ is a continuous map and

$$h(f_1 \times f_2) = h(f_1) + h(f_2).$$

The following result says that all entropy is contained in the nonwandering set, that is, the orbits of wandering points do not contribute to topological entropy.

Proposition 2.3.7. *Let $f : X \to X$ be a continuous map on a compact metric space. Then $h(f) = h(f|_{\Omega(f)})$.*

2.4 Morse-Smale diffeomorphisms

Let M be a compact connected manifold without boundary, let $1 \leq r < \infty$ and let $f : M \to M$ be a C^r-diffeomorphism. Given $p \in Per(f)$ we have the following linear map:

$$D_p f^n : T_p M \to T_p M$$

where $f^n(p) = p$. We say that $p \in Per(f)$ is a *hyperbolic periodic point* if $D_p f^n$ has no eigenvalues of modulus 1. For a hyperbolic periodic point p we have that the tangent bundle admits a splitting into two vector subbundles $T_p M = E^s(p) \bigoplus E^u(p)$ where $E^s(p)$ (resp. $E^u(p)$) corresponds to the eigenspace of $D_p f^n$ associated to the eigenvalues of modulus smaller than 1 (resp. larger than 1) which satisfy $D_p f^n(E^\sigma(p)) = E^\sigma(p)$ by $\sigma = s, u$.

A hyperbolic periodic point p of period n is called a *sink* or *attracting point* provided all the eigenvalues of $D_p f^n$ are less than one in absolute value, that is, $E^u(p) = \{0\}$. In the same way, a hyperbolic periodic point is called a *source* or *repelling point* provided all the eigenvalues are greater than one in absolute value. Finally, a hyperbolic periodic point with $E^u(p) \neq \{0\}$ and $E^s(p) \neq \{0\}$ is a *saddle*.

A hyperbolic fixed point gives rise to the so-called stable manifold and the unstable manifold. The stable manifold consists of all the points in the space that converge under the iterates of the map to the fixed point, while the unstable manifold consists of all those points that converge to the fixed point under the iterates of the inverse map.

Definition 2.4.1. [See [33], $p.73$] The stable, respectively unstable, invariant manifold of a hyperbolic fixed point p of the diffeomorphism f are the sets

$$W^s(p) = \{x \in M : f^j(x) \to p, j \to \infty\},$$

$$W^u(p) = \{x \in M : f^{-j}(x) \to p, j \to \infty\},$$

These sets contain the fixed point p and are clearly invariant under the map f,

$$f(W^\sigma(p)) = W^\sigma(p), \sigma = s, u.$$

We say that the submanifolds $W^s(p)$ and $W^u(q)$ in M are *transverse* (in M) provided for any point $x \in W^s(p) \cap W^u(q)$, we have that $T_x W^s(p) + T_x W^u(q) = T_x M$. (This allows for the possibility that $W^s(p) \cap W^u(q) = \emptyset$).

In this section we consider a class of diffeomorphisms with only finitely many periodic orbits and no other (or no other wandering points):

Definition 2.4.2. A C^r-diffeomorphims f on a connected compact manifold M without boundary is called *Morse-Smale* provided:

i. The nonwandering set is a finite set of periodic points, each of which is hyperbolic, and

ii. each pair of stable and unstable manifolds of periodic points is transverse, that is, if $p, q \in Per(f)$ then $W^s(p)$ is transverse to $W^u(q)$.

The follow important facts about Morse-Smale diffeomorphisms can be found in [33], [36], [39].

- Every Morse-Smale diffeomorphism $f : M \to M$ admits a filtration, i.e., there exists a sequence $\emptyset = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_k = M$ of compact submanifolds with boundary of M, such that $f(M_i) \subset IntM_i$ for each $0 < i < k$, and in $M_{i+1} \setminus M_i$, the maximal invariant set of f, is just one critical element (fixed point) p_{i+1}, that is,

$$\bigcap_{n \in \mathbb{Z}} f(M_{i+1} \setminus IntM_i) = p_{i+1}.$$

- Every Morse-Smale has an attractor periodic point and a repeller periodic point.

- If p is an attractor periodic point then there exists a repeller periodic point q such that the stable and unstable manifolds intersect transversaly.

- $M = \bigcup W^s(p_i) = \bigcup W^u(p_i)$ where $p_i \in \Omega(f)$.

- The set of Morse-Smale diffeomorphisms is open (and nonempty) in $Diff^r(M^n)$ for any manifold M^n and any $r \geq 1$.

- A Morse-Smale diffeomorphism $f : M \to M$ is structurally stable, i.e., if there is a neighborhood U of f in the set of C^1-diffeomorphisms such that f is conjugate to every g in U.
- The set of Morse-Smale diffeomorphisms is dense in $Diff^r(S^1)$, $r \geq 1$.
- The set of Morse-Smale diffeomorphisms is not dense in $Diff^r(M^n)$, $n \geq 2$.

2.5 Dendrite homeomorphisms

In this section we present some basic properties of dendrites and of maps defined on dendrites.

Dendrites

Throughout this section D denotes a nondegenerate dendrite, that is, D is a continuum such that it is a locally connected and contains no simple closed curves. We say that D is *locally connected* provided that for every point $x \in D$ and each neighborhood of x contains a connected neighborhood of x which is open in D.

Proposition 2.5.1. [*See [25], p. 132*] *If C is connected and $C \subset W \subset \bar{C}$, then W is connected.*

Proposition 2.5.2. [*See [31], p. 83*] *A topological space is locally connected if and only if each component of each open set is open.*

Let $x \in D$. The point x is said to be an *end point* of D provided that $D \setminus \{x\}$ is connected; the set of all end points of D is denoted by $E(D)$. The point x is said to be a *cut point* of D if $D \setminus \{x\}$ is not connected. The *order* of x, $ord(x)$, is the cardinality of the set of all components of $D \setminus \{x\}$. If $ord(x) = 2$, the point x is called *ordinary point* of D. If $ord(x) \geq 3$, the point x is called *branch point* of D.

The following structural characterizations of dendrites are known.

Theorem 2.5.3. *For a continuum D the following conditions are equivalent:*

i. D is a dendrite;

ii. each point of D is either a cut point or an end point of D;

iii. each nondegenerate subcontinua of D contains uncountable many cut points of D;

iv. the intersection of every two connected subsets of D is connected.

Proof. See [41], (1.1), p. 88. □

The reader is referred to [15] for a complete structural characterizations of dendrites. Besides the characterization, we recall some important properties of these continua.

Theorem 2.5.4. *The following conditions hold*

i. Each subcontinuum of D is a dendrite.

ii. Every connected subset of D is arcwise connected.

Proof. See [41], (1.3) (i) and ($ii.$), p. 89. □

Theorem 2.5.5. *The set of all branch points of D is countable.*

Proof. See [31], (10.23), p. 174. □

Corolary 2.5.6. *Each nondegenrate subcontinuum of D contains cut points of order* 2.

Propositions 2.5.7 and 2.5.8 are proved in [29].

Proposition 2.5.7. *Let $\{A_n\}$ be a sequence of nonempty connected subsets of D such that for each pair $n \neq m$, $A_n \cap A_m = \emptyset$. Then*

$$\lim_{n\to\infty} diam(A_n) = 0.$$

Given two distinct points a and b in D, there is only one arc from a to b contained in D. We denote such an arc by $[a, b]$. Also we use the following notation: $(a, b] = [a, b] \setminus \{a\}$, $[a, b) = [a, b] \setminus \{b\}$, and $(a, b) = [a, b] \setminus \{a, b\}$.

Proposition 2.5.8. *For every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for any pair of points* $a, b \in D$*,* $d(a,b) < \delta$ *implies* $diam([a,b]) < \varepsilon$*.*

Corolary 2.5.9. *Let* $\{a_n\}$ *be a sequence in* $D \setminus \{a\}$ *such that* $\lim_{n\to\infty} a_n = a$*. Then*

$$\lim_{n\to\infty} diam([a_n, a]) = 0$$

Dendrite Homeomorphisms

We present some basic properties of dendrite homeomorphisms. Recall D represents a nondegenerate dendrite.

Proof of Proposition 2.5.10 can be found in [32].

Proposition 2.5.10. *Let* $f : D \to D$ *be a homeomorphism. Then for each arc* $[a,b]$ *contained in* D*,* $f([a,b]) = [f(a), f(b)]$*.*

Theorem 2.5.11 is one of the main results in [2].

Theorem 2.5.11. *Let* $f : D \to D$ *be a homeomorphism and* $x \in D$*. Then* $\omega(x,f)$ *is either a periodic orbit or a Cantor set. Moreover, if* $\omega(x,f)$ *is a Cantor set,* f *restricted to* $\omega(x,f)$ *is an adding machine.*

The following proposition is well known and the proof can be found in [[41], p. 243].

Proposition 2.5.12. *Any dendrite homeomorphism has the fixed point property.*

In [32] the author proved the following two interesting and useful results.

Proposition 2.5.13. *Let* $f : D \to D$ *be a homeomorphism. Then*

$$R(f) = cl(Per(f))$$

where $R(f)$ *denote the set of recurrent points of* f*.*

Proposition 2.5.14. *Let* $f : D \to D$ *be a homeomorphism. If* $R(f) = D$*, then every cut point of* D *is a periodic point of* f*.*

Remark 2.5.15. *By Proposition 2.5.13 and Proposition 2.5.2, if there is* $x \in D \setminus R(f)$ *then the connected component of* $D \setminus R(f)$ *that contains* x *is an open set. Therefore,* $U \neq \{x\}$.

A proof of Lemma 2.5.16 can be found in [2] and a proof of lemmas 2.5.17 and 2.5.18 can be found in [22].

Lemma 2.5.16. *Let* $f : D \to D$ *be a homeomorphism. If* x_0 *is an end point of* D *such that* $f(x_0) = x_0$*, then* $|Fix(f)| \geq 2$.

Lemma 2.5.17. *Let* $f : D \to D$ *be a homeomorphism. Let* a, b, c *be three distinct end points of* D*. If* $\{a, b, c\} \subset Fix(f)$*, then there exists a cut point of* D*, say* u*, such that* $u \in Fix(f)$.

Lemma 2.5.18. *Let* $f : D \to D$ *be a homeomorphism. Let* $a, b \in Fix(f)$*,* $a \neq b$*. If* a *and* b *are end points of* D *and* $|Fix(f)| = 2$*, then one of the following two conditions holds:*

i. For every $x \in D \setminus \{a, b\}$*,* $\alpha(x, f) = \{a\}$ *and* $\omega(x, f) = \{b\}$.

ii. For every $x \in D \setminus \{a, b\}$*,* $\alpha(x, f) = \{b\}$ *and* $\omega(x, f) = \{a\}$.

The dynamics of dendrite homeomorphisms can be more complicated than Morse-Smale diffeomorphism, however they also have zero topological entropy. The proof can be found in [2].

Theorem 2.5.19. *Let* $f : D \to D$ *be a dendrite homeomorphim, then the topological entropy is zero.*

Chapter 3

Recurrence and shadowing of induced Morse-Smale diffeomorphisms

As mentioned in section 1.4, the dynamics of Morse-Smale diffeomorphisms is very simple since the nonwandering set consists only of finitely many hyperbolic periodic points. This implies, in particular, that Morse-Smale diffeomorphisms are not topologically transitive and have topological entropy zero. However, Morse-Smale diffeomorphisms could generate interesting topological properties in its induced hyperspace 2^f, as seen in Proposition 3.1.1 of this chapter. Some of this properties has been well studied in the last years for continuous maps, see for example [7] [8] [26], but we will provide them here.

Our main interest is to study the dynamics of the continuum map $C(f)$ of Morse-Smale diffeormorphisms, mainly, recurrence and shadowing. We notice that some properties of $C(f)$ depend on the dimension of the base space. For instance, when the base space is S^1 we prove in Section 3.1 that the non-recurrent set of $C(f)$ is a finite number of periodic points as the following theorem state:

Theorem A. *Let $f : S^1 \to S^1$ be a Morse-Smale diffeomorphism then*

i. *If f preserves orientation, then $\Omega(C(f)) = Per_N(C(f)) \cup \{S^1\}$ for some $N \geq 1$,*

ii. *If f reverse orientation, then* $\Omega(C(f)) = Per_2(C(f)) \cup Fix(C(f))$.

Consequently, the continuum map $C(f)$ is not transitive and has zero topological entropy. The dynamics of $C(f)$ seems not to be very interesting however the Morse-Smale diffeomorphisms provide us counterexamples of dynamical systems with the following property:

Theorem B. *Let $f : S^1 \to S^1$ be a Morse-Smale diffeomorphism. Then the continuum map $C(f)$ does not have the shadowing property.*

The proof of Theorem B can be found in Section 3.2. It is well known that Morse-Smale diffeomorphisms have the shadowing property, see Theorem 3.1.1 in [35]. On the other hand, in [18] Good and Fernández showed that a continuous map $f : X \to X$ on a compact metric space has the shadowing property if and only if its induced map 2^f has the shadowing property. Also, with the same arguments, they showed that if $C(f)$ has shadowing then f has shadowing. Thus, we obtained a C^1-open and dense set in $Diff^1(S^1)$ with the shadowing property such that the continuum map $C(f)$ does not have shadowing, for any $f \in Diff^r(S^1)$.

3.1 Recurrence of induced Morse-Smale diffeomorphisms

This section is devoted to study only the induced maps 2^f and $C(f)$ of Morse-Smale diffeomorphisms. For this, first we study the subclass of Morse-Smale diffeomorphisms defined on the circle S^1 since they are well known, as we have seen in Section 2.4. Indeed, the expanding and contracting periodic points alternate and if it is orientation preserving, then all periodic points have the same period, and if it is orientation reversing all periodic points have period one or two. We follow the techniques used in [8] by N. Bernardes and R. Vermersch about *dumbbells*.

Proposition 3.1.1. *Let $f : S^1 \to S^1$ be a Morse-Smale diffeomorphism then*

i. 2^f has uncountably many periodic points of each period $n \geq 1$;

ii. $R(2^f) \neq \Omega(2^f) = \overline{Per(2^f)} = CR(2^f)$;

iii. The topological entropy of 2^f *is infinity;*

iv. The induced map 2^f *is not transitive;*

v. 2^f *has the shadowing property;*

where $R(2^f)$ *and* $CR(2^f)$ *denote, respectively, the set of recurrent and chain recurrent points of* 2^f.

Proof. Suppose that f preserves orientation. Without loss of generality, we can assume that all periodic points of the diffeomorphism f are fixed (otherwise we pass to some iteration of f).

i. Fix $n \in \mathbb{N}$. For all $x \in S^1 \backslash Fix(f)$, we have the set $\Lambda_n = \overline{\cup_{i\in\mathbb{Z}}\{f^{ni}(x)\}}$ is a n-periodic point of 2^f.

ii. Let $Fix(f) = \{p_1, p_2, ..., p_{2k}\}$ and let $x \in S^1 \setminus Fix(f)$. Define

$$K = \{p_1, p_2, ..., p_{2k}\} \cup \{x\} \in 2^{S^1}.$$

Since

$$d_H(K, 2^{f^n}(K)) \geq \min\{d(x, f(x)), d(x, f^{-1}(x))\}$$

for all $n \in \mathbb{N}$, we have that K is not a recurrent point of 2^f. On the other hand, for each $n \in \mathbb{N}$, let

$$K_n = K \cup \{f^{-n}(x)\} \in 2^{S^1}.$$

Then, we obtain

$$K_n \to K \text{ and } f^n(K_n) \to K,$$

that is, K is a nonwandering point of 2^f. Thus, we obtain a compact subset $K \in S^1$ such that $K \in \Omega(2^f)$ but $K \notin R(2^f)$.

Now, let $K \in \Omega(2^f)$ and $\varepsilon > 0$ small enough. Consider the set of fixed points $Fix(f) = \{p_1, p_2, ..., p_{2k}\}$ such that p_{i+1} is an attractor point and p_{2i} is a repeller point. We claim if $K \cap (p_i, p_{i+1}) \neq \emptyset$ then $p_i, p_{i+1} \in K$. In fact, let $\hat{K}_i = K \cap [p_i, p_{i+1}]$

and $\hat{a} = \min\{x \in S^1 : x \in \hat{K}_i\}$. Since $\hat{K}_i$ is compact then $p_i < \hat{a}$. Analogously, if $\hat{b} = \max\{x \in S^1 : x \in \hat{K}_i\}$ then $\hat{b} < p_{i+1}$. So, if $\hat{\varepsilon} < \min\{d(\hat{a}, f(\hat{a})), d(\hat{b}, f(\hat{b}))\}/2$ we have that

$$B_H(K, \hat{\varepsilon}) \cap 2^{f^k}(B_H(K, \hat{\varepsilon})) = \emptyset,$$

for all $k \geq 1$. That is a contradiction and therefore $p_i, p_{i+1} \in K$. In this case we consider $B(p_i, \varepsilon)$ for each $p_i \in K$. Besides that, for each $\hat{K}_i$ there exist N_i points such that $\hat{K}_i \subset \cup_{r \leq N_i} B(x_r^i, \varepsilon)$ and $\hat{K}_i \cap B(x_r^i, \varepsilon) \neq \emptyset$. Using the same argument to $i = 1, ..., 2k-1$ we obtain a finite set

$$X = \{x_1^1, ..., x_{N_1}^1\} \cup ... \cup \{x_1^{2k-1}, ..., x_{N_{2k-1}}^{2k-1}\}$$

and N such that $f^N(x) \in \bigcup_{i \leq 2k} B(p_i, \varepsilon)$ for all $x \in X$. Finally, we consider $Z = \overline{\cup_{j \in \mathbb{Z}} f^{jN}(X)}$. Thus

$$f^N(Z) = Z \text{ and } d_H(Z, K) < \varepsilon.$$

Thus, we have the first equality $\Omega(2^f) = \overline{Per(2^f)}$. The second equality follow of v, see Theorem 3.1.2 in [5].

iii. In [27], the authors obtain sufficient conditions to obtain infinite topological entropy for the induced hyperspace map. It can quickly be checked that these conditions hold for Morse-Smale diffeomorphisms on the circle.

iv Follows from Theorem 2.1 in [34].

v. Follows from Theorem 6 in [18] and Theorem 3.1.1 in [35].

□

Now we will prove Theorem A and notice that the dynamics of $C(f)$ is very different from the dynamics of the induced map 2^f, but some properties remain valid, for example, the non transitivity.

Theorem A. *Let $f : S^1 \to S^1$ be a Morse-Smale diffeomorphism then*

i. *If f preserves orientation, then $\Omega(C(f)) = Per_N(C(f)) \cup \{S^1\}$ for some $N \geq 1$;*

ii. *If f reverses orientation, then* $\Omega(C(f)) = Per_2(C(f)) \cup Fix(C(f))$;

where $Per_j(C(f))$ denote the set of periodic points of $C(f)$ of period j.

Proof. i. Let $f : S^1 \to S^1$ be orientation preserving Morse-Smale diffeomorphism such that every point in the non-wandering set is a fixed point, i.e, $\Omega(f) = Fix(f) = \{p_1, \dots, p_k\}$. We say that a subset $J \subset S^1$ is a D-arc if it is homeomorphic to an interval (possibly degenerated) and the boundary of J is contained in the set of fixed points, which we denote by $\partial J \subset Fix(f)$. Note that, if f preserves orientation and $J \in C(S^1)$ then $J = [a, b]$ for some $a, b \in S^1$ and $f([a, b]) = [f(a), f(b)]$. Therefore, we have that $Fix(C(f)) = \{J : J \text{ is a } D\text{- arc}\} \cup \{S^1\}$.

We will show that $\Omega(C(f)) = Fix(C(f))$, i.e, the non-wandering set consists of finitely many fixed points. Indeed, suppose that there is an interval $I = [x, y]$ in $\Omega(C(f))$ such that I is not a fixed point of $C(f)$. Without loss of generality we can suppose that x is not a fixed point of f and therefore there is $p_i \in Fix(f)$ such that $x \in W^s(p_i)$, see definition in Section 2.4. Since that $I \in \Omega(C(f))$, we have that for every $n \in \mathbb{N}$, there are $N_n \geq 1$ and an interval $J_n = [a_n, b_n]$ such that

$$d_H(I, J_n) < \frac{1}{n} \quad \text{and} \quad d_H(I, C(f)^{N_n}(J_n)) < \frac{1}{n}.$$

This implies that $\lim_{n\to\infty} a_n = x$ and $\lim_{n\to\infty} f^{N_n}(a_n) = x$. Since f is a continuous map, for $\varepsilon = \min\{d(x, f(x)), d(f(x), p_i)\}/2$, there is $\delta > 0$ small enough such that $f(B(x, \delta)) \subset B(f(x), \varepsilon)$ and $B(x, \delta) \cap B(f(x), \varepsilon) = \emptyset$. Besides that, there is $n_0 \geq 1$ such that a_{n_0} and $f^{N_{n_0}}(a_{n_0})$ belong to $B(x, \delta)$. As f preserves orientation we have that $f^n(a_{n_0}) \in [p_i, f(x)] \cup B(f(x), \varepsilon)$ for every $n \geq 1$ which is a contradiction. This contradiction yields the proof.

Now suppose that f is orientation preserving and the non-wandering set consists of a finite number of periodic points with the same period $N > 1$. If f is orientation preserving then f^N is orientation preserving too. Thus, if $J = [x, y]$, for some $x, y \in S^1$, then $f^N([x, y]) = [f^N(x), f^N(y)]$. We say that, J is a D-arc of f^N if and only if J is homeomorphic to a interval (posibly degenerated) and $\partial J \subset Fix(f^N)$. Therefore, we have that $Per_N(C(f)) = \{J : J \text{ is a } D\text{-arc of } f^N\}$.

We will show that $\Omega(C(f)) = \{S^1\} \cup \{J : J \text{ is a } D\text{-arc of } f^N\}$, i.e., the set of non-wandering points consists of finitely many periodic points and a fixed point. Indeed, suppose that there is an interval $I = [x, y]$ in $\Omega(C(f))$ such that I is not a periodic point of $C(f)$. We can assume without loss of generality that x is not a periodic point of f and therefore there is $p_i \in Per_N(f)$ such that $x \in W^s(f^j(p_i))$. Since that $I \in \Omega(C(f))$, we have that for every $n \in \mathbb{N}$, there are $N_n \geq 1$ and an interval $J_n = [a_n, b_n]$ such that

$$d_H(I, J_n) < \frac{1}{n} \text{ and } d_H(I, C(f)^{N_n}(J_n)) < \frac{1}{n}.$$

This implies that $\lim_{n\to\infty} a_n = x$ and $\lim_{n\to\infty} f^{N_n}(a_n) = x$. Since f is a continuous map, for $\varepsilon = \{d(x, f^N(x)), d(f^N(x), f^j(p_i))\}/2$ there is $\delta > 0$ small enough such that $f^N(B(x, \delta)) \subset B(f^N(x), \varepsilon)$. Besides that, there is $n_0 \geq 1$ such that a_{n_0} and $f^{N_{n_0}}(a_{n_0})$ belong to $B(x, \delta)$. Since f preserves orientation, we have that $f^{N.n}(a_{n_0}) \in [f^j(p_i), f^N(x)] \cup B(f^N(x), \varepsilon)$ for every $n \geq 1$. Note that N_{n_0} is a multiple of N which contradicts that $f^{N_{n_0}}(a_{n_0}) \in B(x, \delta)$. This contradiction yields the proof.

ii. Suppose that f is orientation reversing, then all periodic points have period one or two. We say that a set $J \subset S^1$ is a D-mix-arc if it is homeomorphic to an interval (possibly degenerated) and at least one point in ∂J belongs to $Per_2(f)$. Note that, if f is orientation reversing, then f^2 preserves orientation. Therefore, we have that $Fix(C(f)) = \{J : J \text{ is a D-arc}\} \cup \{S^1\}$ and $Per_2(C(f)) = \{J : J \text{ is a D-mix-arc}\}$.

    We will show that $\Omega(C(f)) = Fix(C(f)) \bigcup Per_2(C(f))$. Indeed, suppose that there is an interval $I = [x, y] \in \Omega(C(f))$ such that $I \notin Fix(C(f)) \cup Per_2(C(f))$. We can assume without loss of generality that x is not a fixed point of f nor a periodic point of f and, therefore, there is $p_i \in Fix(f) \cup Per_2(f)$ such that $x \in W^s(f^j(p_i))$ and repeat the argument in item i.

□

Since, the non-wandering set consists only of a finite number of fixed points, we have that the topological entropy is zero and there is not a point with dense orbit on $C(S^1)$ as follows:

Corolary 3.1.2. *Let $f : S^1 \to S^1$ be a Mose-Smale diffeomorphism then*

i. $R(C(f)) = \Omega(C(f)) = Per(C(f))$*;*

ii. The topological entropy of $C(f)$ *is zero;*

iii. The continuum map $C(f)$ *is not transitive.*

Note that if $f : X \to X$ is a continuous map on a compact metric space, then for all $n \in \mathbb{N}$ the induced map $C_n(f)$ is a subsystem of 2^f which we denoted by $C_n(f) \leq 2^f$. So

$$C(f) \leq C_2(f) \leq C_3(f) \leq ... \leq 2^f.$$

Therefore, if f is a Morse-Smale diffeomorphisms on the circle we have that

$$h(C(f)) = 0 \leq h(C_2(f)) \leq h(C_3(f)) \leq ... \leq h(2^f) = \infty.$$

Nevertheless, following the proof of Theorem A we can show $h(C_n(f)) = 0$ for all $n \in \mathbb{N}$. Note that if $\Omega(f) = Fix(f)$, with r fixed points, then $\Omega(C_n(f)) = \Omega(C_r(f))$ for all $n \geq r$. Thus, we have the following question.

Question. There exists a dynamical system (X, f) such that

$$h(C(f)) < h(C_n(f)) < h(2^f)$$

for some $n \geq 2$?

The following result claim that every point in $C(S^1) \setminus \Omega(C(f))$ is a heteroclinic point.

Proposition 3.1.3. *Let $f : S^1 \to S^1$ be a Morse Smale diffeomorphism. If J is a wandering point of $C(f)$ then J is a heteroclinic point.*

Proof. Suppose that f preserves orientation. Without loss of generality, we can assume that all periodic points of the diffeomorphism f are fixed (otherwise we pass to some iteration of f). By Theorem A, we have that J is not a fixed point of $C(f)$, then $J \neq S^1$ and J is not a D-arc. Therefore, there is $x \in \partial J$ such that x is not a fixed point of f. We can suppose that $J = [x, y]$. Since f is a Morse-Smale diffeomorphism, $x \in$

$W^s(p_i) \cap W^u(p_j)$ and $y \in W^s(\hat{p}_i) \cap W^u(\hat{p}_j)$ for some fixed points $p_i \neq p_j$, $\hat{p}_i \neq \hat{p}_j$ of f. For this reason, we have that

$$\lim_{n\to\infty} C(f)^n([x,y]) = [p_i, \hat{p}_i]$$

and

$$\lim_{n\to\infty} C(f)^{-n}([x,y]) = [p_j, \hat{p}_j].$$

□

3.2 Shadowing of induced Morse-Smale diffeomorphisms

Good and Fernández, in [18], showed that a map $f : X \to X$ on a compact metric space has the shadowing property if and only if its induced map 2^f has the shadowing property. Besides that, with the same arguments they showed that if $C(f)$ has the shadowing property then f has shadowing. Nevertheless, it was not known if there is a dynamical system with the shadowing property such that the continuum map does not have shadowing. Despite the dynamics of Morse-Smale diffeomorphisms is very simple, in the sense that the non-recurrent points are only a finite number of periodic points, they have an important topological property which is the shadowing property. This section is devoted to prove:

Theorem B. *Let $f : S^1 \to S^1$ be a Morse-Smale diffeomorhism. Then the continuum map $C(f)$ does not have the shadowing property.*

Proof. Without loss of generality, we can assume that all periodic points of f are fixed (otherwise we pass to some iteration of f). First, suppose that f has only two fixed points, an attractor p and a repeller q. Furthermore, suppose that $C(f)$ has the shadowing property. Then for $\varepsilon > 0$ small enough, there is $\delta > 0$ such that every δ-pseudo orbit $\{x_n\}$ of $C(f)$ is ε-shadowed by real orbit of $C(f)$. For each $\delta > 0$, we can consider the following δ-pseudo orbit:

$$x_0 = \{S^1\}, x_1 = \{J_p^*\}, x_{-1} = \{J_q^*\}$$

$$x_i = C(f)^i(x_1), \text{ and } x_{-i} = C(f)^{-i}(x_{-1}) \text{ if } i \geq 2$$

where J_p^* is an arc which contains the point p but not the point q and $d_H(J_p^*, S^1) < \delta$. In the same way, J_q^* is an arc that contains the point q but not the point p and such that $d_H(J_q^*, S^1) < \delta$. Observe that

$$\lim_{i \to \infty} C(f)^i(x_1) = \{p\} \text{ and } \lim_{i \to -\infty} C(f)^i(x_{-1}) = \{q\}.$$

Fix $\varepsilon > 0$ small enough and $\delta > 0$ of the shadowing property. We claim that the δ-pseudo orbit $\{x_i\}$ defined above is not shadowed. Indeed, suppose that there is $A \in C(S^1)$ such that $d_H(C(f)^i(A), x_i) < \varepsilon$ for all $i \in \mathbb{Z}$. Then we have two possibilities, A contains q or A does not contain q. If A contains q, then there is $i_0 > 1$ big enough such that $d_H(C(f)^{i_0}(A), x_{i_0}) > \varepsilon$ and it is a contradiction. If A does not contain q then there is $j_0 > 1$ big enough such that $d_H(C(f)^{-j_0}(A), x_{-j_0}) > \varepsilon$ and it is a contradiction. Therefore, $C(f)$ does not have the shadowing property.

If f has more than two fixed points, then we consider one attractor point and one repeller point and define δ-pseudo orbit as above. The difference is that $\lim_{i \to \infty} C(f)^i(x_1)$ and $\lim_{i \to -\infty} C(f)^i(x_{-1})$ are D-arcs. □

Chapter 4

Sufficient conditions to obtain infinite topological entropy on the hyperspace

In this section we will present three sufficient conditions to obtain infinite topological entropy on the hyperspace.

4.1 Infinite topological entropy in 2^X

The first case, is a generalization of Theorem 20 in [26]. First we need some preliminary lemmas and notations. Let $r \geq 2$ and let Σ_r be the space of all infinite sequences of elements of $\{1, 2, ..., r\}$ with the product topology. Let $S \subset \Sigma_r$ be the set of all sequence for which the symbol $i \in \{1, ..., r\}$ occurs at most once. For $n \in \mathbb{Z}$ we denote by b_n^i the point in S whose nth coordinate is the symbol i, and by a we denote the sequence $(..., 0, 0, 0, ...) \in S$. It is easy to see that S is a subshift of the one-side shift Σ_r. We will write shortly σ_S for $\sigma|_S$.

The proof we present to Proposition 4.1.1 follows, with slight changes, the proof given for Theorem 13 in [26].

Proposition 4.1.1. *For the subshift (S, σ_S) defined above: $h(\sigma_S) = 0$, but $h(2^{\sigma_S}) = \log r$*

Proof. Since that $\Omega(\sigma_S) = \{a\}$, σ_S has the topological entropy equal to zero. We will show that the induced map 2^{σ_S} is semiconjugated to the full shift $\{\{0,1\}^r\}^{\mathbb{Z}}$. For the proof we define the map $\phi : 2^S \to \{\{0,1\}^r\}^{\mathbb{Z}}$ by

$$\phi(A) = \{(y_n^1, y_n^2, ..., y_n^r)\}_{n\in\mathbb{Z}}, \text{ where } y_n^i = \begin{cases} 1, & \text{if } \; b_n^i \in A; \\ 0, & \text{otherwise}; \end{cases}$$

for any $A \in 2^S$. It is easy to check that ϕ is an uniformly finite-to-one semiconjugacy from 2^{σ_S} to σ. Hence $h(2^{\sigma_S}) = r \log 2$. □

Taking $r \geq 2$ large enough, we can produce an arbitrary gap between the topological entropy of the base map and its induced map to the hyperspace.

Corolary 4.1.2. *For all $M > 0$ there exists dynamical system (X, f) such that $h(f) = 0$ and $h(2^f) > M$.*

Now, we will give an example of a compact metric space and homeomorphism such that the topological entropy of the map is equal to zero, but the induced map has infinite topological entropy. We also generalize Theorem 17 in [26].

Example 4.1.3. *Let Q be the Hilbert Cube, see [3], and S_Q be the set of all bi-infinite sequences for which symbol $1/p$ occurs at most once, where $p \in \{1, 2, ...\}$. For $n \in \mathbb{Z}$ we denote by b_n^p the point in S_Q whose nth coordinate is the symbol $1/p$, and by a, we denote the sequence $(..., 0, 0, 0, ...) \in S_Q$. It is easy to see that S_Q is a subshift of the full two-sided shift Q. We will write shortly σ_{S_Q} for $\sigma|_{S_Q}$.*

Lemma 4.1.4. *The subshift (S_Q, σ_{S_Q}) defined above: $h(\sigma_{S_Q}) = 0$ but $h(2^{\sigma_{S_Q}}) = \infty$.*

Proof. Since that $\Omega(\sigma_{S_Q}) = \{a\}$, σ_{S_Q} has the topological entropy equal zero. We will show that for any $r \geq 1$, the induced map $2^{\sigma_{S_Q}}$ is semiconjugated to the full shift $\{\{0,1\}^r\}^{\mathbb{Z}}$. For the proof, we define the map $\phi : 2^{S_Q} \to \{\{0,1\}^r\}^{\mathbb{Z}}$ by

$$\phi(A) = \{(y_n^1, y_n^2, ..., y_n^r)\}_{n\in\mathbb{Z}}, \text{ where } y_n^i = \begin{cases} 1, & \text{if } \ p \le r \ \text{ and } \ b_n^p \in A; \\ 0, & \text{otherwise}; \end{cases}$$

for any $A \in 2^{S_Q}$. It is easy to check that, for any $r \ge 1$, the map Φ is a semiconjugacy from $2^{\sigma_{S_Q}}$ to σ. Hence, $h(2^{\sigma_{S_Q}}) = \infty$. □

Note that the above lemma remains valid if one replaces two-sided subshift by its one-sided analog. We generalize further the above lemma and prove

Theorem C. *Let $f : X \to X$ be a surjective map and let X be a continuum space. If there exists an infinite countable set $A = \{a_1, a_2, ...\} \subset X$ such that*

i. $L = \bigcup_{i\ge1} \alpha(\{x_{-n}^i\}_{n\in\mathbb{Z}_+}, f) \cup \omega(a_i, f)$ *and* $M = \bigcup_{i\ge1} \{x_{-n}^i\}_{n\in\mathbb{Z}_+}$ *are disjoint,*

ii. *For every pair $i \ne j$, $i \ge 1$, $j \ge 1$,*

$$Orb(\{x_{-n}^i\}_{n\in\mathbb{Z}_+}, f) \cap Orb(\{x_{-n}^j\}_{n\in\mathbb{Z}_+}, f) = \emptyset,$$

then $h(2^f) = \infty$.

Proof. We will construct a semiconjugacy from some subsystem of (X, f) to the one-side version of subshift of Proposition 4.1.1. First, fix an $r \in \mathbb{N}$. For each $i \in \{1, ..., r\}$ extend the given negative orbit through a_i to the full $Orb(\{x_n^i\}_{n\in\mathbb{Z}_+}, f)$. Then define the closed f-invariant set as

$$\Lambda = \bigcup_{i=0}^{r-1} \overline{Orb(\{x_n^i\}_{n\in\mathbb{Z}_+}, f)}.$$

With the notation introduced in Proposition 4.1.1, we define a map $\phi : \Lambda \to S_Q$ by

$$\phi(x) = \begin{cases} b_n^i, & \text{if } \ x = x_{-n}^i \text{ for some } n \ge 0 \text{ and } 1 \le i \le r; \\ a, & \text{otherwise.} \end{cases}$$

Clearly ϕ is the desired semiconjugacy. This implies that 2^Φ is a semiconjugacy from 2^f to $2^{\sigma_{S_Q}}$. Therefore, $h(2^f) \ge \log r$ for all $r \in \mathbb{N}$. □

As corollary we obtain Theorem 5.7 in [22]. This result was given before by Lampart-Raith in [27].

Theorem D. *Let $f : X \to X$ be a surjective finite map and let X be a continuum space. If the positively recurrent points are not dense in X, then $h(2^f) = \infty$.*

Proof. In [26] Theorem 20, the authors showed that, for every point $x_0 \in X \setminus c(f)$ and every negative orbit $\{x_{-n}\}_{n\in\mathbb{Z}_+}$, we have that $\alpha(\{x_{-n}\}_{n\in\mathbb{Z}_+}, f) \cup \omega(x_0, f)$ and $\{x_{-n}\}_{n\in\mathbb{Z}_+}$ are disjoint. Thus, we have that for all every negative orbit $\{x_{-n}\}_{n\in\mathbb{Z}_+}$ there exist $\lambda > 0$ such that $d(x_0, \alpha(\{x_{-n}\}_{n\in\mathbb{Z}_+}, f) > \lambda$. We will construct by induction a set A which verifies the hypotheses of Lemma 4.1.4. Fix $a_1 \in X \setminus c(f)$, a negative orbit $\{x^1_{-n}\}_{n\in\mathbb{Z}_+}$ through a_1 and the set $\Lambda_{a_1} = \bigcup_{n\in\mathbb{Z}_+} f^{-n}(\{a_1\})$ of all negative orbits through a_1. We claim

$$c(f) \cup \overline{Orb(\{x^1_{-n}\}_{n\in\mathbb{Z}_+}, f)} \cup \overline{\Lambda}_{a_1} \neq X.$$

Indeed, suppose that

$$c(f) \cup \overline{Orb(\{x^1_{-n}\}_{n\in\mathbb{Z}_+}, f)} \cup \overline{\Lambda}_{a_1} = X.$$

Since by hypothesis we have that a_1 is not a recurrent point, $c(f)$ is a closed set and $d(\alpha(\{x^1_{-n}\}_{n\in\mathbb{Z}_+}, f), a_1) > 0$, then there exists $\varepsilon_1 > 0$ small enough such that

$$B(a_1, \varepsilon_1) \subset X \setminus c(f) \text{ and } B(a_1, \varepsilon_1) \cap \overline{Orb(\{x^1_{-n}\}_{n\in\mathbb{Z}_+}, f)} = \{a_1\}.$$

Therefore,

$$B(a_1, \varepsilon) \cap \overline{\Lambda}_{a_1} = B(a_1, \varepsilon_1).$$

Since Λ_{a_0} is a countable set and X is a continuum space, we have that

$$a_1 \in \alpha(\{x^1_{-n}\}_{n\in\mathbb{Z}_+}, f)$$

which is a contradiction. Therefore,

$$c(f) \cup \overline{Orb(\{x^1_{-n}\}_{n\in\mathbb{Z}_+}, f)} \cup \overline{\Lambda}_{a_1} \neq X.$$

Thus, we can consider

$$a_2 \in X \setminus c(f) \cup \overline{Orb(\{x^1_{-n}\}_{n\in\mathbb{Z}_+}, f)} \cup \overline{\Lambda}_{a_1},$$

and repeat the argument. Thus we obtain a set $A = \{a_1, a_2, ...\} \subset X$ that verifies the hypotheses of Lemma 4.1.4 and therefore $h(2^f) = \infty$.

□

Corolary 4.1.5. *The hyperspace map of every Morse-Smale diffeomorphism has infinite topological entropy.*

4.2 Infinite topological entropy on $C(X)$

In this section we present two cases to obtain infinite topological entropy on $C(X)$.

Case 1.

In [3] the authors present two examples of dynamical systems defined on dendrites such that the continuum induced map has infinite topological entropy. We are interested in one of these examples and we will present it here.

First consider, in $\mathbb{R}^2$, the points $p = (-1, 0)$, $q = (1, 0)$ and the sequence $((a_n, 0))_{n\in\mathbb{Z}}$ such that $a_0 = 0$ and, for each $n \in \mathbb{N}$, $a_n = 1 - \frac{1}{n+1}$ and $a_{-n} = -a_n$. Note that $a_n < a_{n+1}$ for each $n \in \mathbb{Z}$. Moreover:

$$\lim_{n\to\infty} a_n = 1 \quad \text{and} \quad \lim_{n\to-\infty} a_n = -1.$$

Now, given $n \in \mathbb{Z}$, let L_n be the straight line segment $L_n = \{a_n\} \times [0, \frac{1}{|n|+1}]$. We denote the segment $[-1, 1] \times \{0\}$ by pq. Let

$$X = pq \cup \left(\bigcup_{n\in\mathbb{Z}} L_n \right).$$

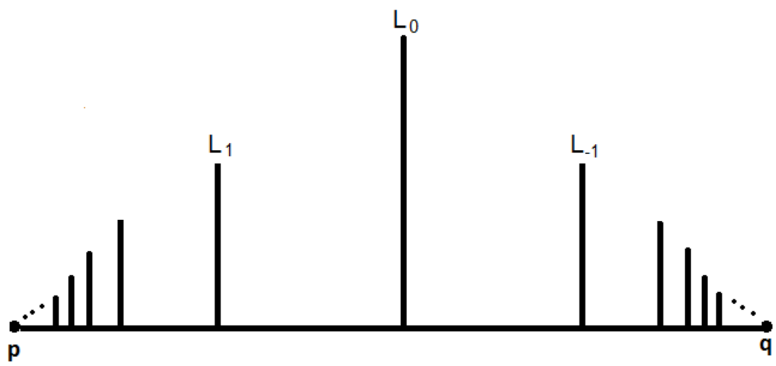

Figure 4.1: Dendrite X.

Note that X is a dendrite with free arcs. Let $F : X \to X$ be a homeomorphism with the following properties:

i. $F|_{pq}$ is a homeomorphism from pq onto itself such that $F(p) = p$, $F(q) = q$ and

$F((a_n,0)) = (a_{n+1},0)$ for each $n \in \mathbb{Z}$ (so the image under F of the arc from $(a_n,0)$ to $(a_{n+1},0)$ is the arc from $(a_{n+1},0)$ to $(a_{n+2},0)$;

ii. for each $n \in \mathbb{Z}$, $F|_{L_n} : L_n \to L_{n+1}$ is a linear homeomorphism.

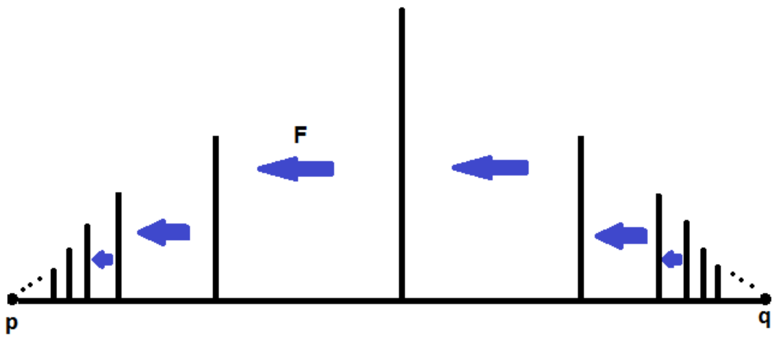

Figure 4.2: Dynamics of $F : X \to X$.

Note that $Fix(F) = \{p,q\}$. Moreover: $\omega(p,F) = \{p\}$ and $\omega(x,F) = \{q\}$, for each $x \in X - \{p\}$. Thus F is not transitive. Since X is a dendrite and F is a homeomorphism, by [2] the topological entropy of F is zero. Now, since F is a homeomorphism, the induced map $C(F) : C(X) \to C(X)$ is also a homeomorphism, see [23] . By either Theorem 4.5 or Theorem 6.2 in [3], $C(F)$ is not transitive. Note that, $\omega(A, C(F)) = \{\{q\}\}$, for each $A \in C(X)$ such that $p \notin A$.

The authors in [3] constructed a closed subset Λ of $C(X)$ such that $C(F)_{|\Lambda}$ is topologically conjugate to *the shift map* defined on the Hilbert cube. For this reason, $C(F)|_\Lambda$ has the following properties

i. The homeomorphism $C(F)|_\Lambda : \Lambda \to \Lambda$ is *Devaney chaotic*, i.e., is transitive and periodically dense.

ii. The topological entropy of $C(F)|_\Lambda$ is infinite, so $C(F)|_\Lambda$ and $C(F)$ are topologically chaotic.

iii. $C(F)|_\Lambda$ has uncountable periodic points of each period.

We observed that there are hyperbolic dynamical systems with this dendrite. Therefore, we will introduce the definition of special dendrite for homeomorphisms.

Definition 4.2.1. Let $f : X \to X$ be a homeomorphism on a compact metric space X. We say that a closed subset $\Lambda \subset X$ is a *Special Dendrite* if there is $k \in \mathbb{N}$ such that Λ is f^k-invariant and $f^k|_\Lambda$ is conjugated to F (homeomorphism defined above). In this case, we say that f *admit a Special Dendrite.*

Let us present some examples of dynamical systems that admit a special dendrite, and note that the base space is a n-dimensional manifold with $n \geq 2$:

i. North Pole-South Pole diffeomorphism on S^2: the north pole p_N is a repeller fixed point, the south pole p_S is an attractor fixed point and for all $x \in S^2 - \{p_N, p_S\}$, $\omega(x) = \{p_S\}$ and $\alpha(x) = \{p_N\}$. This diffeomorphism admits a special dendrite, see Proposition 4.2.2.

ii. Consider the torus $T^2 \subset \mathbb{R}^3$ and let $X = grad(t)$ where t is the height function of points of T^2 above the horizontal plane. This vector field has four singularities p_1, p_2, p_3, p_4 where p_1 is a sink, p_2 and p_3 are saddles and p_4 is a source. The stable manifold of p_2 intersects the unstable manifold of p_3 nontransversally. The diffeomorphism time-one map of X (see definition in [33]) admits a special dendrite.

ii. In the example above we can destroy the intersection nontransversally with a small perturbation of the field X. The resulting field Y is a Morse-Smale field and will therefore not be equivalent to X. The diffeomorphism time-one map of Y admits a special dendrite, see Proposition 4.2.6 or Theorem G.

Theorem E. *Let $f : X \to X$ be homeomorphism on a continuum metric space X. If f admits a special dendrite, then $h(C(f)) = \infty$ and therefore $h(2^f) = \infty$.*

Proof. Since that f admits a special dendrite, there exist a closed subset $\Lambda \subset X$ and $k \in \mathbb{N}$ such that Λ is f^k-invariant and $f^k|_\Lambda$ is conjugate to F. So the induced continuum systems $C(f^k|_\Lambda)$ and $C(F)$ are conjugate too (see [38], Theorem 4). Therefore, we have that $h(f) \geq h(C(F)) = \infty$. □

Morse-Smale diffeomorphisms with only two hyperbolic fixed points admit a special dendrite.

The set $MS_2(M^n)$ of diffeomorphisms Morse-Smale on a compact manifold without boundary and with only two fixed points is non-empty, when the manifold is an n-dimensional sphere. In this case, the dynamics is very simple: all non-fixed points move from the source to the sink. On a sphere of a given dimension, any two such diffeomorphisms are topologically conjugate. We will show that every Morse-Smale diffeomorphism with only two hyperbolic fixed points admits a special dendrite and therefore the topological entropy of the continuum map is infinite.

Proposition 4.2.2. *Let M be a compact, connected and orientable n-dimensional manifold without boundary and let $n \geq 2$. If $f \in MS_2(M^n)$ then f admits a special dendrite.*

Proof. Let $f \in MS_2(M^n)$. Since that, all non-fixed points move from the source to the sink, we have $W^s(p) \bigcap W^u(q) = M - \{p, q\}$ is an open and connected n-dimensional set. Fix a point $x \in W^s(p) \bigcap W^u(q)$ and let $\varepsilon > 0$ be sufficiently small such that

- $B(p, \varepsilon) \bigcap B(q, \varepsilon) \bigcap B(x, \varepsilon) = \emptyset$.
- $f(\overline{B(p, \varepsilon)}) \subset B(p, \varepsilon)$ and
- $f^{-1}(\overline{B(q, \varepsilon)}) \subset B(q, \varepsilon)$.

Then, there exists $k_0 \in \mathbb{N}$ such that $f^k(x) \in B(p, \varepsilon)$ and $f^{-k}(x) \in B(q, \varepsilon)$ for all $k \geq k_0$. The following lemma will help us to construct the dendrite. We denote γ_x^y an arc joining the points x and y.

Lemma 4.2.3. *There are $y \in M - \{p, q\}$ and arcs $\gamma_x^{f(x)}$ and β_x^y with the following properties:*

i. There is $t_0 \in \mathbb{N}$ such that $f^{-t_0}(\gamma_x^{f(x)} \cup \beta_x^y) \subset B(q, \varepsilon)$ and $f^{t_0}(\gamma_x^{f(x)} \cup \beta_x^y) \subset B(p, \varepsilon)$.

ii. For all $n \neq m \in \mathbb{Z}$, $f^n(\beta_x^y) \cap f^m(\beta_x^y) = \emptyset$.

Proof. First we will show the item $i.$. Since that M is connected manifold then there is $\gamma_x^{f(x)}$, which we denote it by γ. It is sufficient to prove that there is $t_1 \in \mathbb{N}$ such that $f^{-t_1}(\gamma) \subset B(q,\varepsilon)$. Suppose that, for all $k \geq k_0$ there is y_k in γ such that $f^{-k}(y_k)$ is not contained in $B(q,\varepsilon)$. Then we obtain two sequence $\{y_k\}_{k\geq k_0}$ in γ and $\{f^{-k}(y_k)\}_{k\geq k_0}$ in $B(q,\varepsilon)^c$. Taking a subsequence, if it is necessary, we have $\lim_{k\to\infty} y_k = x_0 \in \gamma$. For x_0, there exists $m_0 \in \mathbb{N}$ such that $f^{-m_0}(x_0) \in B(q,\varepsilon)$. Let $\varepsilon_1 > 0$ be such that $B(f^{-m_0}(x_0),\varepsilon_1) \subset B(q,\varepsilon)$, so since f is continuous, there exists, $\delta > 0$ such that

$$f^{-m_0}(B(x_0,\delta)) \subset B(f^{-m_0}(x_0),\varepsilon_1).$$

Now, let $n_0 > m_0$ be such that $y_{n_0} \in B(x_0,\delta)$ and $f^{-m_0}(y_{n_0}) \in B(f^{-m_0}(x_0),\varepsilon_1)$. Recall that $B(f^{-m_0}(x_0),\varepsilon_1) \subset B(q,\varepsilon)$. Thus, we have $f^{-(m_0+t)}(y_{n_0}) \in B(q,\varepsilon)$ for all $t \geq 1$, which is a contradiction. With the same argument we prove that there exist $t_2 \in \mathbb{N}$ such that $f^{t_2}(\gamma) \subset B(p,\varepsilon)$. Thus, it is sufficient to consider $t_0 = \max\{t_1,t_2\}$. Again by continuity of f, there is a neighborhood $V(\gamma)$ of γ such that

$$f^{t_0}(V(\gamma)) \subset B(p,\varepsilon) \text{ and } f^{-t_0}(V(\gamma)) \subset B(q,\varepsilon)$$

Thus, we can consider a point $\bar{y} \in V(\gamma) - \gamma$ and an arc $\beta_x^{\bar{y}} \subset V(\gamma)$ joining the points x and $\bar{y}$ such that $\beta_x^{\bar{y}} \cap \gamma = \{x\}$.

To show the item $ii.$ we choose β (in item $i.$) such that $f^n(\beta) \bigcap f^m(\beta) = \emptyset$ for $|m|,|n| \leq k_0$. Indeed, let $\delta > 0$ sufficiently small such that

$$\delta < \min_{|i|,|j|\leq k_0-1}\{d(f^i(x),f^j(x))\}.$$

Since that f and f^{-1} are continuous, there is η_1, with $\eta_1 < \delta/2$ and

$$f(B(f^{-1}(x),\eta_1)), f^{-1}(B(f(x),\eta_1)) \subset B\left(x,\frac{\delta}{2}\right).$$

Also, there is $\eta_2 > 0$, with $\eta_2 < \eta_1$ and

$$f(B(f^{-2}(x),\eta_2)) \subset B(f^{-1}(x),\eta_1) \text{ and } f^{-1}(B(f^2(x),\eta_2)) \subset B(f^{-1}(x),\eta_1).$$

Continuing this process, there is $\eta_{k_0-1} > 0$, with $\eta_{k_0-1} < \eta_{k_0-2}$ such that

$$f(B(f^{-(k_0-1)}(x),\eta_{k_0-1}) \subset B(f^{-(-k_0-2)}(x),\eta_{k_0-2}),$$

$$f^{-1}(B(f^{k_0-1}(x),\eta_{k_0-1}) \subset B(f^{k_0-2}(x),\eta_{k_0-2}).$$

Finally, we choose an arc β contained in the open set

$$\bigcap_{|j|\leq k_0-1} f^j(B(f^j(x),\eta_{|j|})) \subset B\left(x,\frac{\delta}{2}\right)$$

where $\eta_0 = \frac{\delta}{2}$. By construction, for all $n,m \in \{-k_0-1,...,k_0-1\}$ we have that $f^n(\beta)\bigcap f^m(\beta)=\emptyset$. This implies that $f^n(\beta)\bigcap f^m(\beta)=\emptyset$ for all $n\in\mathbb{Z}$. □

Now, we consider the arc $\alpha=\gamma\cup\beta$ and we will construct a special dendrite as following:

Let $\Lambda=\overline{\bigcup_{n\in\mathbb{Z}} f^n(\alpha)} = \bigcup_{n\in\mathbb{Z}} f^n(\alpha)\bigcup\{p,q\}$ be a compact and f-invariant set. We claim that $f|_\Lambda$ is conjugate to F (see section 4.2, Case 1). Indeed, we know that there is a homeomorphism $G:\alpha\to L_0\bigcup\left([0,\frac{1}{2}]\times\{0\}\right)$ and we are going to construct the homeomorphism $H:\Lambda\to pq\cup\left(\bigcup_{n\in\mathbb{Z}} L_n\right)$ by induction. Let $y\in\Lambda$ then there is $n\in\mathbb{Z}$ such that $f^n(y)\in\alpha$, by definition $G(f^n(y))\in L_0\bigcup\left([0,\frac{1}{2}\times\{0\}]\right)$. We defined $H(y):=F^n\circ G\circ f^n(x)$. By construction H is a homeomorphism and $H\circ f=F\circ H$. Therefore, f admits a special dendrite. □

Corolary 4.2.4. *The induced continuum map of every Morse-Smale with only two hyperbolic fixed points has infinite topological entropy.*

Morse-Smale diffeomorphisms with an attractor, a repeller and a saddle fixed point admit a special dendrite.

Let $MS_3^*(M^n)$ be the subclass of $MS_3(M^n)$ of diffeomorphisms with only three periodic points. The first question about this subclass of diffeomorphisms is if this set is non-empty. It is well known that there are no Morse-Smale diffeomorphisms on 2-manifolds with exactly three periodic points. In [19], the authors showed that, there are no Morse-Smale diffeomorphisms on 3-manifolds whose set of non-wandering points consists of exactly three periodic points. In [43], the existence of closed n-manifolds with $n\geq 4$ admitting Morse functions with precisely three critical points was proved, and such manifolds were studied. Thus, in the case $n\geq 4$, there exists Morse-Smale diffeomorphisms with precisely three periodic points. In [43], the authors showed that any such diffeomorphism has precisely one saddle, one sink and one source. Also, they showed that if n is an even number and $n \geq 4$, then the unstable and the stable separatrix are $n/2$-dimensional spheres. Thus we obtain the following result.

Proposition 4.2.5. *Let M^n be a compact and connected, orientable n-dimensional manifold with $n \geq 4$. Let $f : M^n \to M^n$ be a Morse-Smale diffeomorphism with only three periodic points, an attractor p, a repeller q and a saddle σ. Then f admits a special dendrite and, therefore, $h(C(f)) = \infty$.*

Proof. Without loss of generality, we can assume that all periodic points of the diffeomorphism f are fixed (otherwise we pass to some iteration of f). By the Main Theorem in [43], we have that

$$W^u(\sigma) \cup \{p\} = S_p \quad \text{and} \quad W^s(\sigma) \cup \{q\} = S_q$$

are $n/2$-dimensional spheres. In this case the dynamic is very simple: all non-fixed points move from the source to the saddle or from the saddle to the sink. Therefore, by Proposition 4.2.2, we have that f admits a special dendrite. □

Proposition 4.2.6. *Let M^n be a compact and connected, orientable n-dimensional manifold with $n \geq 4$. Let $f : M^n \to M^n$ be a Morse-Smale diffeomorphism with an attractor point p and a repeller point q such that $W^s(p) \cap W^u(q) \neq \emptyset$ has a f^k-invariant connected component with dimension at least 2. Then f admits a special dendrite and therefore $h(C(f)) = \infty$.*

Proof. Since that the set $W^s(p) \cap W^u(q) \neq \emptyset$ has dimension at least 2 and it is f^k-invariant, we can construct a special dendrite with the same arguments in Proposition 4.2.2. □

Case 2.

The authors, in [26], also provided an example of a zero entropy map f on a connected space X such that the topological entropy of $C(f)$ is positive. We include it here and we will show the topological entropy of $C(f)$ is infinite.

Example 4.2.7. *Let (S, σ_S) be the shift system from Proposition 4.1.1. By Cone(S) we denote the cone over the space S that is the space obtained from $S \times [0,1]$ by collapsing a subset $S \times \{1\}$ to a single point. We can extend σ_S to the map σ_c defined on the whole Cone(S). Then the set of cones over compact subsets of S is a closed invariant for $C(\sigma_c)$. With the same arguments given by the authors in [26], we have that $h(\sigma_c) = 0$ and $h(C(\sigma_c)) \geq \log r$.*

Proposition 4.2.8. *Let $(Cone(S), \sigma_c)$ be the system in example above. Then the topological entropy of $C(\sigma_c)$ is infinite.*

Proof. Let $S_1 = \{a, b_n^1\}_{n\in\mathbb{Z}}$. Given $n \in \mathbb{Z}$, let $L_n = \{b_n^1\} \times [0,1]$. Note that, $Cone(S_1)$ is a compact and connected σ_c-invariant set. We denote $[1] = S \times \{1\}$, the fixed point in $Cone(S_1)$. Now, consider the following set:

$$\Lambda = \{A \in C(Cone(S_1)) : [1] \in A\}$$

Note that Λ is a closed subset of $C(Cone(S))$. Since $\sigma_c([1]) = [1]$, Λ is strongly invariant under $C(\sigma_c)$. Consider $\sigma : Q \to Q$ *the shift map* defined on a Hilbert Cube, $Q = \Pi_{n\in\mathbb{Z}}[0,1]$, see [3] section 7. We will show the induced map $C(\sigma_c)|_\Lambda$ is semi-conjugated to *the shift map* σ defined on the Hilbert cube. For $A \in \Lambda$, let $\phi(A) = \hat{(t)} = (t_n)_{n\in\mathbb{Z}}$, where $t_n = 1 - \min(\pi_2(L_n \cap A))$, for every $n \in \mathbb{Z}$. In this way we have a function $\phi : \Lambda \to Q$. Note that, ϕ is continuous and onto. Moreover $\phi \circ C(\sigma_c)|_\Lambda = \sigma^{-1} \circ \phi$ since $\sigma_c(b_n^i, a) = (b_{n+1}^i, a)$ for any (b_n^i, a) in $Cone(S)$. Thus, the topological entropy of $C(\sigma_c)$ is greater than or equal to the topological entropy of σ. Since the topological entropy of σ is infinite, by Theorem 7.6 in [3], we conclude that the topological entropy of $C(\sigma_c)$ is infinite. □

Now, Let $f : M \to M$ be a homeomorphism on a continuum metric space M, with three fixed points p, q and σ. Fix $r \in \mathbb{N}$ and suppose that there exist $a_0, a_1, ..., a_{r-1} \in M$ with the following property: $\lim_{n\to\infty} f^n(a_i) = p$ and $\lim_{n\to-\infty} f^n(a_i) = q$ for every $i \in \{0, ..., r-1\}$. Also, suppose there exist arcs γ_i joining the points a_i and σ such that

$$f^{n_1}(\gamma_{i_1}) \cap f^{n_2}(\gamma_{i_2}) = \{\sigma\} \text{ for all } (i_1, n_1) \neq (i_2, n_2).$$

In this section, we will denote that $f^n(\gamma_i) = \gamma_{i,n}$ and $\hat{\gamma}_{i,n} := \gamma_{i,n} \backslash \{\sigma\}$.

Definition 4.2.9. We say that the sequence $\{f^n(\gamma_i)\}_{n\in\mathbb{Z}}^{i\in\{0,\ldots,r-1\}}$ *is Self-Accumulated* if there exist $x \in \bigcup_{n\in\mathbb{Z}} f^n(\gamma_0 \cup \gamma_1 \cup \ldots \cup \gamma_{r-1})$, a sequence of points $\{x_j\}_{j\in\mathbb{N}} \subset \gamma_0 \cup \ldots \cup \gamma_{r-1}$ and $\{k_j\}_{j\in\mathbb{N}} \subset \mathbb{Z}$, $|k_j| \to \infty$ when $j \to \infty$, such that $f^{k_j}(x_j) \to x$ when $j \to \infty$.

Remark 4.2.10. *If the sequence $\{f^n(\gamma_i)\}_{n\in\mathbb{Z}}^{i\in\{0,\ldots,r-1\}}$ is not self-accumulated then, for any $(i_0, n_0) \in \{0, ..., r-1\} \times \mathbb{Z}$ and each $x \in \hat{\gamma}_{i_0,n_0}$, there exists $\varepsilon_0 > 0$ such that*

$$B(x, \varepsilon_0) \cap \gamma_{i,n} = \emptyset \textit{ for all } (i, n) \neq (i_0, n_0).$$

From now, in this section, we will assume that M is a continuum metric space and $f : M \to M$ is a homeomorphism with three fixed points p, q and σ. Besides that, we suppose that there exist an infinite countable set $A = \{a_0, a_1, ...\} \subset M$ with $\alpha(a_i) = \{q\}$ and $\omega(a_i) = \{p\}$ for all $i \geq 0$, and a sequence of arcs $\{\gamma_i\}_{i\geq 0}$, where γ_i joins the points a_i and σ such that for every $r \in \mathbb{N}$ the sequence $\{f^n(\gamma_i)\}_{n\in\mathbb{Z}}^{i\in\{0,\ldots,r-1\}}$ is not self-accumulated. For all $i \in \{0, ..., r-1\}$ we set

$$\Lambda_i = \{x \in M : \exists\ n_j \to \infty \text{ and } x_{i,j} \in \gamma_{i,n_j} \text{ such that } x_{i,j} \to x \text{ when } j \to \infty\}$$

and

$$\Gamma_i = \{x \in M : \exists\ n_j \to -\infty \text{ and } x_{i,j} \in \gamma_{i,n_j} \text{ such that } x_{i,j} \to x \text{ when } j \to \infty\}$$

Note that, these sets are compact subsets of M by definition.

Lemma 4.2.11. *For all $i \in \{0, ..., r-1\}$, we have that*

$$\Lambda_i = \bigcup_{A_\lambda \in \omega(\gamma_{i,0})} A_\lambda \qquad \textit{and} \qquad \Gamma_i = \bigcup_{B_\lambda \in \alpha(\gamma_{i,0})} B_\lambda$$

where $\omega(\gamma_{i,0}) = \omega(\gamma_{i,0}, C(f))$ and $\alpha(\gamma_{i,0}) = \alpha(\gamma_{i,0}, C(f))$. Therefore, Λ_i and Γ_i are compact and connected subsets of M such that $q, \sigma \in \Lambda_i$ and $p, \sigma \in \Gamma_i$.

Proof. First, suppose that $x \in A_\lambda$ with $A_\lambda \in \omega(\gamma_{i,0}, C(f))$. Then there is a subsequence γ_{i,n_j} such that $d_H(\gamma_{i,n_j}, A_\lambda) \to 0$. From the definition of Hausdorff metric, we have that there is a subsequence $y_{i,n_j} \in \gamma_{i,n_j}$ such that $d(x, y_{i,n_j}) \to 0$. Therefore, $x \in \Lambda_i$. Now, suppose that $x \in \Lambda_i$, then there are $n_j \to \infty$ and $x_{i,j} \in \gamma_{i,n_j}$ such that $x_{i,j} \to x$ when $j \to \infty$. Since $C(M)$ is a compact metric space (see [23]), there are $A_{\lambda_0} \in \omega(\gamma_{i,0}, C(f))$

and a subsequence of $\{\gamma_{i,n_j}\}_{j\geq 1}$ such that $d_H(\gamma_{i,n_j}, A_{\lambda_0}) \to 0$. We claim that $x \in A_{\lambda_0}$. Indeed, by definition of Hausdorff metric, we have that $d(x_j, A_{\lambda_0}) \to 0$ so by triangle inequality we have $d(x, A_{\lambda_0}) = 0$. Therefore $x \in A_{\lambda_0}$. We use the same argument to show the second equality. □

Note that, for all $i \in \{0, ..., r-1\}$ the set $\Lambda_i \cup \Gamma_i$ is a fixed point of $C(f)$ since that $\omega(\gamma_{i,0}, C(f))$ is strongly invariant set of $C(f)$. We set

$$\Delta_r = \bigcup_{i=0}^{r-1} \Lambda_i \cup \Gamma_i$$

and

$$L_r = \bigcup_{n\in\mathbb{Z}} \left(\bigcup_{i=o}^{r-1} \gamma_{i,n} \cup \Delta_r \right).$$

Note that, both Δ_r and L_r are a compact, connected and f-invariant subsets of M. Therefore, $C(L_r)$ is a subset of $C(M)$. We denote $\hat{\gamma}_{i,n} = \gamma_{i,n} \backslash \{\sigma\}$ and we say that a set $K \in C(L_r)$ is a *Full Cone* if it satisfies the following properties:

i. If $K \cap \hat{\gamma}_{i,n} \neq \emptyset$ then $\gamma_{i,n} \subset K$,

ii. $\Delta_r \subset K$.

We denote by H_r the set of all *Full Cones* in $C(L_r)$. Note that, there are only two types of full cones in H_r. We say that $K \in H_r$ is a *Finite Full Cone* if the set $\Sigma = \{(i, n) \in \{0, ..., r-1\} \times \mathbb{Z} : K \cap \hat{\gamma}_{i,n} \neq \emptyset\}$ is finite or *Infinite Full Cone* if Σ is infinite. Therefore, if K is a full cone we can write this as

$$K = \bigcup_{(i,n)\in\Sigma} \gamma_{i,n} \cup \Delta_r.$$

Lemma 4.2.12. *The subset H_r is a closed subset of $C(L_r)$ and $C(f)$-invariant.*

Proof. First, we will prove that H_r is a closed subset of $C(L_r)$. Indeed, let $\{K_m\}_{m\in\mathbb{N}}$ be a sequence in H_r such that $d_H(K_m, K_0) \to 0$ if $m \to \infty$. First, note that for all $m \in \mathbb{N}$, $\Delta_r \subset K_m$, therefore $\Delta_r \subset K_0$. Now, suppose that for some $i_0 \in \{0, ..., r-1\}$ and some $n_0 \in \mathbb{Z}$, $\hat{\gamma}_{i,n_0} \cap K_0 \neq \emptyset$. We claim that $\hat{\gamma}_{i_0,n_0} \subset K_0$. Indeed, let $x \in \hat{\gamma}_{i_0,n_0} \cap K_0$. So there exist

$\varepsilon_0 > 0$ such that $B(x,\varepsilon_0)\cap\gamma_{i,n} = \emptyset$ for any $i \neq i_0$ and $n \neq n_0$. Note that, $B(x,\varepsilon_0)\cap\Delta_r = \emptyset$. Then for all $\varepsilon < \varepsilon_0$, there exists $m_0 \in \mathbb{N}$ such that $d_H(K_m, K_0) < \varepsilon$ for each $m \geq m_0$. Thus, for x there exists $y_m \in K_m$ with $d(x, y_m) < \varepsilon$, but this implies $y_m \in \gamma_{i_0,n_0}$ and, therefore, $\gamma_{i_0,n_0} \subset K_m$. This implies that for every $\varepsilon > 0$, $\gamma_{i_0,n_0} \subset V(K_0,\varepsilon)$. So we conclude that $\gamma_{i_0,n_0} \subset K$, i.e, K_0 is a full cone.

Now, we will prove that H_r is a $C(f)$ invariant set. Let K be a full cone, then

$$K = \bigcup_{(i,n)\in\Sigma} \gamma_{i,n} \cup \Delta_r.$$

Thus, we have that

$$C(f)(K) = f(\bigcup_{(i,n)\in\Sigma} \gamma_{i,n} \cup \Delta_r) = \bigcup_{(i,n)\in\Sigma} f(\gamma_{i,n}) \cup \Delta_r = \bigcup_{(i,n)\in\Sigma} \gamma_{i,n+1} \cup \Delta_r.$$

Therefore, $C(f)(K)$ is a full cone. Using the same argument above for $C(f^{-1})$, we obtain that H_r is $C(f)$ invariant. □

By Lemma 4.2.12, we have that $(C(f), H_r)$ is a subsystem of $(C(f), C(M))$. Now, we will show that $C(f^{-1})$ contain a full shift of 2^r symbols.

Lemma 4.2.13. *The induced map $C(f^{-1}) : H_r \to H_r$ is conjugate to the full shift $\sigma : (\{0,1\}^r)^{\mathbb{Z}} \to (\{0,1\}^r)^{\mathbb{Z}}$.*

Proof. Let $\phi : H_r \to (\{0,1\}^r)^{\mathbb{Z}}$ be a map defined by

$$\phi(K) = ((w_{1,n}, w_{2,n}, ..., w_{r,n}))_{n\in\mathbb{Z}} \text{ where } w_{i,n} = \begin{cases} 1, & \text{if } \hat{\gamma}_{i,n} \cap K \neq \emptyset; \\ 0, & \text{otherwise}; \end{cases}$$

for every $K \in H_r$.

It is clear that ϕ is onto and injective. We claim that ϕ is continuous. In fact, let $\{K_j\}_{j\in\mathbb{N}}$ be a sequence of full cones such that $d_H(K_j, K) \to 0$ when $j \to \infty$. Since the sequence $\{\gamma_{i,n}\}_{n\in\mathbb{Z}}^{i\in\{0,\dots,r-1\}}$ is not self-accumulated, for any $\varepsilon > 0$, small enough, there exist $j_0, n_0 \in \mathbb{N}$ such that if $|n| \leq n_0$, $i \in \{0, ..., r-1\}$ and $\hat{\gamma}_{i,n} \subset K$ then $\hat{\gamma}_{i,n} \subset K_j$ for every $j \geq j_0$. Therefore, $d(\phi(K_j), \phi(K)) \to 0$ when $j \to \infty$ and thus ϕ is continuous. The fact that $\sigma \circ \Phi = \Phi \circ C(f^{-1})|_{H_r}$ follows from:

$$\hat{\gamma}_{i,n} \cap K \neq \emptyset \Leftrightarrow f^{-1}(\hat{\gamma}_{i,n}) \cap f^{-1}(K) \neq \emptyset \Leftrightarrow \hat{\gamma}_{i,n-1} \cap f^{-1}(K) \neq \emptyset.$$

□

We are ready to show the main result.

Theorem F. *Let M^n be a compact, connected n-dimensional space with $n \geq 2$ and let $f : M^n \to M^n$ be a homeomorphism with three fixed points p, q and σ. If there exists an infinite countable set $A = \{a_0, a_1, a_2 ...\} \subset M^n$ such that*

i. *for every $i \geq 0$, $\alpha(a_i, f) = \{p\}$ and $\omega(a_i, f) = \{q\}$,*

ii. *for every $i \geq 0$, $a_i \neq p$ and $a_i \neq q$,*

iii. *for every pair $i \neq j$, $i \geq 0$, $j \geq 0$,*

$$\{f^k(a_i) : k \in \mathbb{Z}\} \cap \{f^k(a_j) : k \in \mathbb{Z}\} = \emptyset,$$

iv. *for every $r \geq 1$ and $i \in \{0, 1, ..., r-1\}$, there exist arcs γ_i from a_i to σ, such that the sequence $\{f^k(\gamma_i)\}_{k\in\mathbb{Z}}^{i\in\{0,...,r-1\}}$ is not self-accumulated.*

Then $h(C(f)) = \infty$.

Proof. Fix $r \in \mathbb{N}$. We consider $H_r \subset C(M^n)$ the set of full arcs and its induced map $C(f^{-1}) : H_r \to H_r$. By Lemma 4.2.13 we have $C(f^{-1})$ is semiconjugate to σ defined on $(\{0, 1\}^r)^{\mathbb{Z}}$. Therefore,

$$h(C(f)) \geq h(C(f^{-1})|_{H_r}) = r \log 2.$$

Since that r is fix but arbitrary, we obtain $h(C(f)) = \infty$. □

4.3 The induced continuum Morse-Smale diffeomorphism has infinite topological entropy

In section 4.2, we proved that some Morse-Smale diffeomorphisms admit a special dendrite. However, it is not known if every Morse-Smale admits special dendrite. The next theorem proves that the continuum map of every Morse-Smale diffeomorphism has infinite topological entropy. The technical issue that we still do not know how to deal with it is the non self-accumulation of curves transversal to stable manifolds of saddles. Indeed, this could happen due to the lambda-lemma. However, it is still possible, but we do not know, that the special dendrite exists in a place where there are no such transversal curves. We use the definition of topological entropy with separated sets, see section 2.3 in Preliminaries.

Theorem G. *If $f : M \to M$ is a Morse-Smale diffeomorphism, then the topological entropy of its induced map $C(f)$ is infinite.*

Proof. Without loss of generality, we can assume that f has only fixed points (otherwise we pass to some iteration of f). It is known there are an attractor p and a repeller q such that $W^s(p) \cap W^u(q) \neq \emptyset$. Let B^s be an open ball with center at p such that $f(\partial B^s) \subset B^s$ where $\partial B^s = \overline{B}^s \setminus B^s$ is the boundary of B^s. Let $Q^s = B^s \setminus f(\overline{B}^s)$ be the interior of a fundamental domain for the stable manifold of p. Similarly, we consider B^u an open ball with center at q such that $f^{-1}(\partial B^u) \subset B^u$ where $\partial B^u = \overline{B}^u \setminus B^u$ is the boundary of B^u and $Q^u = B^u \setminus f(\overline{B}^u)$ is the interior of a fundamental domain for the unstable manifold of q. Since that $W^u(q)$ is a connected set we have $\partial B^s \cap W^u(q) \neq \emptyset$. Let $x \in \partial B^s \cap W^u(q) \neq \emptyset$ and consider the connected component of $Q^s \cap W^u(q)$ with x in its boundary and we denote this component by N. Notice that N is an open set of M.

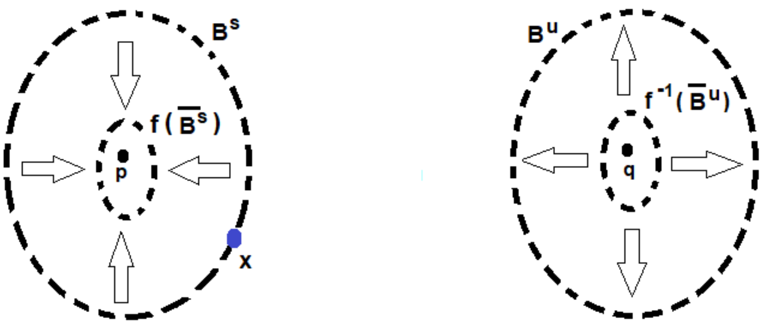

Figure 4.3: Construction of the set N.

Consider $y \in N$, an arc γ with end points x and $f(x)$ such that $\gamma \setminus \{x, f(x)\} \subset Q^s$, $y \in \gamma$ and the subarc $\hat{\gamma} \subset \gamma$ with end points y and x is contained in $N \cup \{x\}$. Let $m_0 = \min\{n : f^{-n}(y) \in Q^u\}$ and $\varepsilon > 0$ small enough such that $B(f^i(y), \varepsilon) \cap B(f^j(y), \varepsilon) = \emptyset$

for $i \neq j$ and $i, j \in \{0, -1, -2, ..., -m_0\}$. By the continuity of f, we can consider an arc $\beta \subset N$ such that y is an end point of β, $\gamma \cap \beta = \{y\}$ and $f^j(\beta) \subset B(f^j(y), \varepsilon)$ for $j \in \{0, -1, ..., -m_0\}$. Therefore, $f^n(\beta) \cap f^m(\beta) = \emptyset$ for all $n \neq m \in \mathbb{Z}$.

Lemma 4.3.1. *There is $\eta > 0$ such that $V(\beta, \eta) \cap f^n(\gamma) = \emptyset$ for all $n \in \mathbb{Z} \setminus \{0\}$.*

Proof. Suppose that for every $k \in \mathbb{N}$ there is $n_k \in \mathbb{Z} \setminus \{0\}$ such that $f^{n_k}(\alpha) \cap V(\beta, \frac{1}{k}) \neq \emptyset$. Then we have that there exists a subsequence of $\{f^{n_k}(x_k)\}$, where $x_k \in \alpha$, such that

$$\lim_{k \to \infty} f^{n_k}(x_k) \in \beta \text{ or } \lim_{k \to -\infty} f^{n_k}(x_k) \in \beta.$$

In the first case, we have there is n_k big enough such that $f^{n_k}(x_k) \in B(y, \varepsilon) \subset N$. Then $x_k = f^{-n_k}(f^{n_k}(x_k)) \in W^u(q) \setminus \overline{B}^s$ that is a contradiction. In the second case, we have there is n_k big enough such that $f^{-n_k}(x_k) \in B(y, \varepsilon) \subset N$. Therefore, the point $x_k = f^{n_k}(f^{-n_k}(x_k)) \in f(B^s)$ and that is a contradiction. Thus we obtain the desired result. □

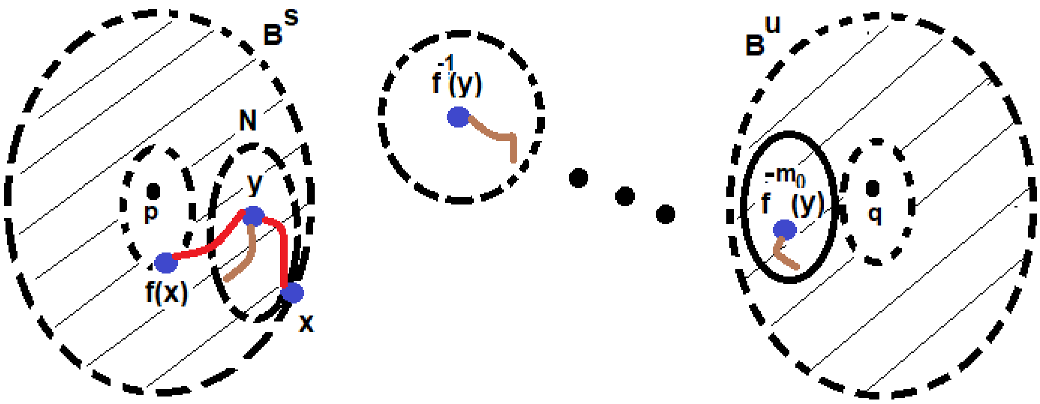

Figure 4.4: Construction of γ, β and η arcs.

Lemma 4.3.2. *If $k \in \mathbb{N}$, there is $\delta > 0$ such that $s(n+1, C(f)^{-1}, \delta) \geq k^n$ for all $n \in \mathbb{N}$.*

Proof. Let y and z be the end points of the arc β. Fix $\bar{y} \in \beta$ close to the point y and consider δ_0 small enough such that $V([\bar{y}, z], \delta_0)$ does not contain points of γ. Since $[\bar{y}, z]$ is an arc, it is homeomorphic to interval $[0,1]$. Let $H : [\bar{y}, z] \to [0,1]$ be a homeomorphism such that $H(\bar{y}) = 0$ and $H(z) = 1$. Let $\eta > 0$ be given by Lemma 4.3.1 and let $k \in \mathbb{N}$. There exists $\delta < \min\{\frac{1}{k}, \eta, \delta_0\}$ such that

$$d(x, y) < \delta \text{ implies } |H(x) - H(y)| < \frac{1}{k}.$$

Let $a_i = H^{-1}(i/k)$ for $i = 1, ..., k-1$. Let $n \in \mathbb{N}$, let $\sigma = (\sigma_0, \sigma_1 ..., \sigma_{n-1}) \in \{1, 2, ..., k\}^n$ and let C_σ be the subtree of $\Lambda = \overline{\bigcup_{j \in \mathbb{Z}} f^j(\gamma \cup \beta)}$ defined as follow:

$$C_\sigma = \bigcup_{j=0}^{n-1} f^j([a_{\sigma_j}, y]) \cup \bigcup_{j=0}^{n-2} f^j(\gamma).$$

If $\sigma \neq \sigma' \in \{1, 2, ..., k\}^n$, there is $j_0 \in \{0, 1, ..., n-1\}$ such that $\sigma_{j_0} \neq \sigma'_{j_0}$. Without loss of generality, we can assume $\sigma_{j_0} < \sigma'_{j_0}$. Then

$$\Big|\frac{\sigma'_{j_0}}{k} - t\Big| > \frac{1}{k} \text{ for all } t \in \big[0, \frac{\sigma_{j_0}}{k}\big].$$

By the continuity of H we have that

$$d(a_{\sigma'_{j_0}}, H^{-1}(t)) > \delta \text{ for all } t \in \big[0, \frac{\sigma_{j_0}}{k}\big].$$

If it is necessary, we consider a smaller $\delta > 0$ such that

$$a_{\sigma'_{j_0}} \in f^{-j_0}(C_{\sigma'}) \text{ and } a_{\sigma'_{j_0}} \notin V(f^{-j_0}(C_\sigma), \delta).$$

Therefore $d_H(f^{-j_0}(C_\sigma), f^{-j_0}(C_{\sigma'})) \geq \delta$. Thus, the set $\{C_\sigma : \sigma \in \{1, ..., k\}^n\} \subset C(\Lambda')$ is $(n, C(f)^{-1}, \delta)$-separated set and $s(n, C(f)^{-1}, \delta) \geq k^n$. □

As a consequence, $h(C(f^{-1})) \geq \ln(k)$ for all $k \in \mathbb{N}$. By definition of topological entropy, we have $h(C(f)) = h(C(f)^{-1}) = \infty$. □

Chapter 5

Entropy of induced continuum dendrite homeomorphisms

Chapters 2 and 3 were devoted to study the topological entropy of the induced maps 2^f and $C(f)$ of Morse-Smale diffeomorphisms. Our motivation was the fact that the set of topological entropy of Morse-Smale diffeomorphisms is zero. Another class of dynamical systems with this property are the dendrite homeomorphisms, see [2]. In particular, homeomorphisms on the interval belong to this class. The dynamics of dendrite homeomorphisms can be more complicated than Morse-Smale diffeomorphisms. In Section 2.5, we presented a folklore example of a dendrite homeomorphism such that the periodic points is dense in the dendrite.

In 2010, M. Lampart and P. Raith [27], proved that if $f : I \to I$ is a homeomorphism on the interval, then the topological entropy of the induced map 2^f is zero or infinite. Moreover, the topological entropy of the induced continuum map $C(f)$ is zero.

In 2015, P. Hernández and H. Méndez [22], generalized the above result to the induced dendrites homeomorphisms. The authors proved that if $f : D \to D$ is a dendrite homeomorphism, then the topological entropy of 2^f is infinity if and only if the set of recurrent points of f is different from D. We can ask if the topological entropy of the induced continuum map is still zero. Nevertheless, there are examples in [3] and [1] of dendrite homeomorphisms $f : D \to D$ such that its induced continuum map $C(f)$ has infinite topological entropy. Thus, it is possible to formulate the following dichotomy:

Conjecture. Let $f : D \to D$ be a dendrite homeomorphism. Then the topological entropy of $C(f) : C(D) \to C(D)$ has only two values: 0 or ∞.

In this chapter we prove the Conjecture but for a subclass of dendrite homeomorphisms. We will see that the existence of no-recurrent branch points could generate infinite topological entropy in the continuum hyperspace. The main result of this chapter is the following

Theorem H. *Let $f : D \to D$ be a dendrite homeomorphism. Then*

i. *If there is a no-recurrent branch point in D, then the topological entropy of its induced map $C(f)$ is ∞.*

ii. *If each point in D is a recurrent point, then the topological entropy of its induced map $C(f)$ is 0.*

In order to show the item *i.* of Main Theorem, we divided the proof in two cases. The first case, Theorem I stated below, is when the connected component of $D \setminus R(f)$ which contains the non-recurrent points is f^n-invariant for some $n \in \mathbb{N}$. Some dynamical properties of f restricted to the closure of the connected component allow us to construct a special chaotic dendrite (see Definition 4.2.1) and, therefore, to obtain infinite topological entropy of its induced continuum dendrite homeomorphism as follows:

Theorem I. *Let $f : D \to D$ be a homeomorphism such that $R(f) \neq D$. Let $x_0 \in D \setminus R(f)$ be a branch point and let U be the component of $D \setminus R(f)$ that contains x_0. If there is an $n \in \mathbb{N}$ such that U is f^n-invariant, then f admits a special dendrite. In particular $h(C(f)) = \infty$.*

The second case, Theorem J stated below, is the opposite case of Theorem I, i.e., it is when the connected component of $D \setminus R(f)$, which contains the non-recurrent points, is not f^n-invariant for any $n \in \mathbb{N}$. We will see that despite we do not have sufficient tools to construct a special dendrite, it is possible to prove that the topological entropy of $C(f)$ is infinite. We will use the definition of topological entropy with the notion of separated sets given by Bowen-Dinaburg. For interested readers see the definition of topolgical entropy in Chapter 1.

Theorem J *Let $f : D \to D$ be a homeomorphism such that $R(f) \neq D$. Let $x_0 \in D \setminus R(f)$ be a branch point and let U be the component of $D \setminus R(f)$ that contains x_0 and suppose it is not f^n-invariant for any $n \in \mathbb{N}$. Then $h(C(f)) = \infty$.*

The item *ii.* is a direct consequence of Theorem 6.6 of P. Hernández and H. Méndez in [22]

The reader is advised to read Section 1.5 of Chapter 1 before to begin the reading of the following sections, since basic properties of dendrite homeomorphisms will be used.

This chapter is organized as follows: In Section 5.1, we present a proof of Theorem I. In Section 5.2, we give a proof of Theorem J. Finally, in Section 4.3, we present a proof of the Main Theorem.

5.1 Proof of Theorem I

As we mentioned before, this section is devoted to show:

Theorem I. *Let $f : D \to D$ be a homeomorphism such that $R(f) \neq D$. Let $x_0 \in D \setminus R(f)$ a branch point and U be the component of $D \setminus R(f)$ that contains x_0. If there is $n \in \mathbb{N}$ such that U is f^n-invariant then f admit a special dendrite. In particular $h(C(f)) = \infty$.*

To prove Theorem I, we will construct a special chaotic dendrite in the closure of the connected component of the non-recurrent branch points as can be seen in 5.1.4. But to show this result we need an auxiliary lemmas: Lemmas 5.1.1, 5.1.2 and 5.1.3.

In Lemma 5.1.1, we show that the dynamics of some iterate of f restricted to the closure of the connected component of the non-recurrent branch points is simple in the following sense: there exist only two fixed points, one attractor and one repeller. Also, this fixed points are end points and the closure of the orbit of any other point in the connected component is the orbit together with the fixed points. The proof of this Lemma was given by P. Hernández and H. Méndez in [22], Proposition 6.1, however we include it here.

Lemma 5.1.2 and 5.1.3 are dedicated to give sufficient conditions to construct the special chaotic dendrite. We follow the ideas of P. Hernández and H. Méndez in [22]. And finally, we prove in 5.1.4 that f admits a special chaotic dendrite.

Proof of Theorem I. Let $x_0 \in D \setminus R(f)$ be a branch point and let U be the connected component of $D \setminus R(f)$ that contains the point x_0. We assume for simplicity that $ord(x_0) = 3$. Notice that the sets $R(f)$ and $D \setminus R(f)$ are strongly invariant under f. Let $N = \min\{n \in \mathbb{N} : f^n(U) = U\}$ and let $g : W \to W$ be the dendrite homeomorphism given by $g = f^N$ and $W = cl(U)$. Since $g(U) = U$, $g(W) = W$.

Lemma 5.1.1. *Let $g : W \to W$ be a dendrite homeomorphism defined above. Then g has exactly two fixed points in dendrite W and both of them are end points of W. Furthermore, $Per(g) = Fix(g)$.*

Proof. By Proposition 2.5.12, there exists a fixed point $w_0 \in W$ of g. Since w_0 is not in U, $w_0 \in W \setminus U$. By Proposition 2.5.1, $W \setminus \{w_0\}$ is a connected set and, by Theorem 2.5.3 item *ii.*, w_0 is an end point of W. Lemma 2.5.16 implies that there exists another fixed point of g in W. Let $w_1 \in Fix(g) \cap W$, $w_0 \neq w_1$. Note that w_1 is an end point of W. Observe that, g can not have a third fixed point in W. In fact, if g had a third fixed point, by Lemma 2.5.17, there would exist a cut point w of W such that $g(w) = w$, a contradiction. Therefore g has exactly two fixed points in W and both of them are end points of W.

Now, suppose that there exists $u \in Per(g) \setminus Fix(g)$, where $Fix(g) = \{w_0, w_1\}$. So, $g^k(u) = u$ for some $k \in \mathbb{N}$. Note that u is an end point of W. If we consider the dendrite homeomorphism $G = g^k : W \to W$, then $\{w_0, w_1, u\} \subset Fix(G)$ and Lemma 2.5.17 implies there exists a cut point $\bar{u}$ of W such that $\bar{u} \in Fix(G)$, a contradiction. □

Let w_0 and w_1 be the two fixed points in W. By Lemma 2.5.18, we can assume, without loss of generality, w_0 is an attractor point, w_1 is a repeller point and for all $x \in W \setminus \{w_0, w_1\}$

$$\omega(x, g) = \{w_0\} \text{ and } \alpha(x, g) = \{w_1\}.$$

On the other hand notice that the arc $[w_0, w_1]$ is strongly invariant under g.

Lemma 5.1.2. *There exist a point $u \in W \setminus [w_0, w_1]$ and a point $x \in (w_0, w_1)$ such that $[u, x] \cap [w_0, w_1] = \{x\}$.*

Proof. By hypothesis of Theorem I., $x_0 \in D \setminus R(f)$ and $ord(x_0) = 3$, then $x_0 \in W \setminus E(W)$, where $E(W)$ denote the set of end points of W. Then we have two possibilities: either $x_0 \in (w_0, w_1)$ or $x_0 \notin (w_0, w_1)$.

First suppose that $x_0 \in (w_0, w_1)$. Since $ord(x_0) = 3$, we can consider a point $u \in W \setminus [w_0, w_1]$ such that $[u, x_0] \cap [w_0, w_1] = \{x_0\}$.

If $x_0 \notin (w_0, w_1)$ then there exists $x \in (w_0, w_1)$ such that $[x_0, x] \cap [w_0, w_1] = \{x\}$. Thus, we consider $u = x_0$. □

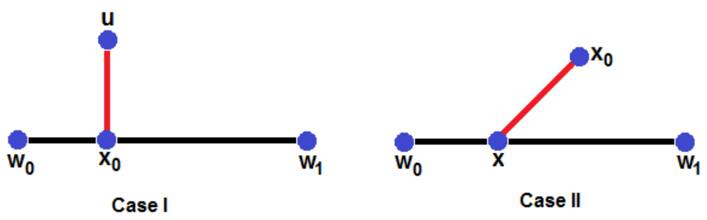

Figure 5.1: Construction of special dendrite.

Lemma 5.1.3. *Let $x \in (w_0, w_1)$. Then $[x, g(x)] \cap [g(x), g^2(x)] = \{g(x)\}$ and, therefore, it is an arc.*

Proof. Since the arc $[w_0, w_1]$ is strongly invariant under g, we have that $g^j(x) \in (w_0, w_1)$ for all $x \in (w_0, w_1)$ and each $j \in \mathbb{Z}$. Thus, $[x, g(x)]$ is a subarc of (w_0, w_1). We shall show that $[x, g(x)] \cap [g(x), g^2(x)] = \{g(x)\}$. Suppose, on the contrary, that either $[x, g(x)] \cap [g(x), g^2(x)] = [g(x), g^2(x)]$ or $[x, g(x)] \cap [g(x), g^2(x)] = [x, g(x)]$

If $[x, g(x)] \cap [g(x), g^2(x)] = [g(x), g^2(x)]$ then $g^2(x) \in (x, g(x))$. This implies that $g^3(x) \in (g(x), g^2(x))$ but $(g(x), g^2(x)) \subset [x, g(x)]$ therefore $g^3(x) \in [x, g(x)]$. By induction, we have that $g^j(x) \in [x, g(x)]$ for all $j \geq 0$. As a consequence $\omega(x) \subset [x, g(x)]$ which is a contradiction.

If $[x, g(x)] \cap [g^2(x), g(x)] = [x, g(x)]$ then $[x, g(x)] \subset [g^2(x), g(x)]$. Thus, we have that $g([x, g(x)]) = [g^2(x), g(x)]$ and $x \in [g^2(x), g(x)]$. So iterating we obtain

$$[g^2(x), g(x)] \cap [g^2(x), g^3(x)]) = [g^2(x), g(x)].$$

Then $[g^2(x), g(x)] \subset [g^2(x), g^3(x)]$ and $g(x) \in [g^2(x), g^3(x)]$. Continuing with the same argument, we obtain a subsequence $\{g^{n_k}(x)\}$ such that $\lim_{n_k \to \infty} g^{n_k}(x) \neq w_0$, a contradiction. □

Lemma 5.1.4. *Let $g : W \to W$ be the homeomorphism defined above. Then g admits a special dendrite.*

Proof. Let u, x be points in W given by Lemma 5.1.2. Since the dendrite W does not have simple closed curves, for each $n \in \mathbb{N}$ the arc $g^n([u, x]) = [g^n(u), g^n(x)]$ is disjoint from the arc $[u, x]$. This implies that for every pair $n, m \in \mathbb{Z}$, with $n \neq m$, $g^n([u, x]) \cap g^m([u, x]) = \emptyset$. Therefore, by Proposition 2.5.7, we have

$$\lim_{n \to \infty} diam(g^n([u, x])) = 0 \text{ and } \lim_{n \to -\infty} diam(g^n([u, x])) = 0.$$

Now, consider the set $\Lambda = \bigcup_{j \in \mathbb{Z}} g^j([u, x]) \cup [w_0, w_1]$ and note that, by construction, it is a connected compact and g-invariant set. We claim $g|_\Lambda$ is conjugated to F defined in Section 4.2, Case 1. Indeed, it is known (see [31], p. 3) that there is a homeomorphism $G : [u, x] \cup [x, g(x)] \to L_0 \cup ([0, \frac{1}{2}] \times \{0\})$ which preserves the end points, i.e., $G(u) = (0, 1)$ and $G(g(x)) = (\frac{1}{2}, 0)$. Besides, we can consider G such that $G(x) = (0, 0)$. Define the conjugation $H : \Lambda \to pq \bigcup (\cup_{n \in \mathbb{Z}} L_n)$ by induction as follows:

- $H(w_0) := p$ and $H(w_1) := q$,
- If $y \in [u, x] \cup [x, g(x)]$, then $H(y) := G(y)$,
- If $y \in \bigcup_{j \in \mathbb{Z} \setminus \{0\}} g^j([u, x] \cup [x, g(x)])$, then there exists $j_0 \in \mathbb{Z} \setminus \{0\}$ such that $g^{-j_0}(y) \in [u, x] \cup [x, g(x)]$. Thus, we let $H(y) := F^{j_0} \circ G \circ g^{-j_0}(y)$.

By definition, H is a homeomorphism. Now, we claim that $H \circ g|_\Lambda = F \circ H$. In fact, this is clear if $y \in Fix(g|_\Lambda)$. If y is not a fixed point, there exists $j_0 \in \mathbb{Z}$ such that

$g^{-j_0}(y) \in [u,x] \cup [x, g(x)]$. So

$$H(g(y)) = F^{(j_0+1)} \circ G \circ g^{-j_0-1}(g(y)) = F \circ F^{j_0} \circ G \circ g^{-j_0}(y) = F \circ H(y).$$

□

Finally, by Lemma 5.1.4, Proposition 2.3.4 and Theorem E., we have that

$$h(C(f)) \geq h(C(f^N|_\Lambda)) = h(C(g|_\Lambda)) = \infty.$$

□

5.2 Proof of Theorem J

This section is devoted to show

Theorem J. *Let $f : D \to D$ be a homeomorphism such that $R(f) \neq D$. Let $x_0 \in D \setminus R(f)$ be a branch point and let U be the component of $D \setminus R(f)$ that contains x_0 and it is not f^n-invariant for any $n \in \mathbb{N}$. Then $h(C(f)) = \infty$.*

This theorem is the opposite case of Theorem I. Although we do not have sufficient tools to construct a special chaotic dendrite, it is possible to prove that the topological entropy of the induced continuum map $C(f)$ is infinite. By definition of topological entropy using separated sets, we prove that the topological entropy of the continuum map $C(f)$ is as big as we want.

Proof of Theorem J. To say that U is not f^n-invariant for any $n \in \mathbb{N}$ is equivalent to say that $f^n(U) \cap U = \emptyset$ for all $n \in \mathbb{N}$. So, for each pair $n, m \in \mathbb{Z}$, with $n \neq m$, $f^n(U) \cap f^m(U) = \emptyset$. Therefore, according to Proposition 2.5.7

$$\lim_{n\to\infty} diam(f^n(U)) = 0 \text{ and } \lim_{n\to-\infty} diam(f^n(U)) = 0.$$

As in the proof of Theorem I., we assume for simplicity that $ord(x_0) = 3$. Therefore, we can consider three points $a, b, c \in U$ such that each one of them belongs to one component of $(D \setminus \{x_0\}) \cap U$ and the arcs $[a, x_0]$, $[b, x_0]$ and $[c, x_0]$ are contained in U.

We set $W_n = f^n([a, x_0] \cup [b, x_0] \cup [c, x_0])$ for all $n \in \mathbb{N}$. Moreover, there exist $y_0 \in W_0$ and $y_1 \in W_1$ such that $[y_0, y_1] \cap W_0 = \{y_0\}$ and $[y_0, y_1] \cap W_1 = \{y_1\}$. For instance, y_0 and y_1 are such that $[a, f(c)] \cap W_0 = [a, y_0]$ and $[a, f(c)] \cap W_1 = [y_1, f(c)]$.

Notice that the arc $[y_0, y_1]$ can intersect at most a finite number of W_i. Therefore, consider $N = \max\{|n| : [y_0, y_1] \cap W_n \neq \emptyset\}$ and the points $\bar{y_0} \in W_0$ and $\bar{y_1} \in W_N$ such that the arc $[\bar{y_0}, \bar{y_1}] \cap W_0 = \{\bar{y_0}\}$ and $[\bar{y_0}, \bar{y_1}] \cap W_N = \{\bar{y_1}\}$.

In addition, let $\Lambda = \overline{\bigcup_{j \in \mathbb{Z}} f^{Nj}(W_0 \cup [\bar{y_0}, \bar{y_1}])}$ be a connected compact f^N-invariant subset of D and let β be the branch of W_0 such that $\beta \cap \{\bar{y_0}, f^{-N}(\bar{y_1})\} = \emptyset$.

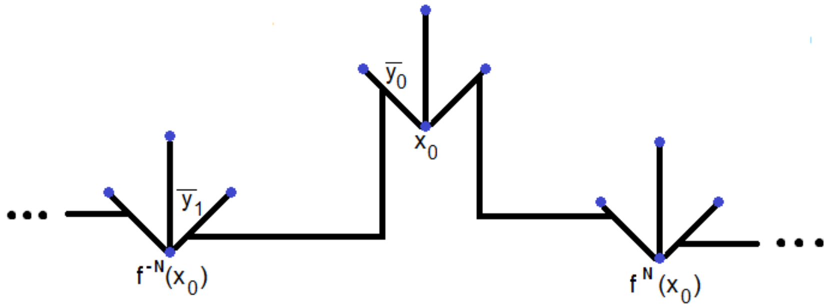

Figure 5.2: Construction of Λ if $[\bar{y_0}, \bar{y_1}] \cap Fix(f^N) = \emptyset$.

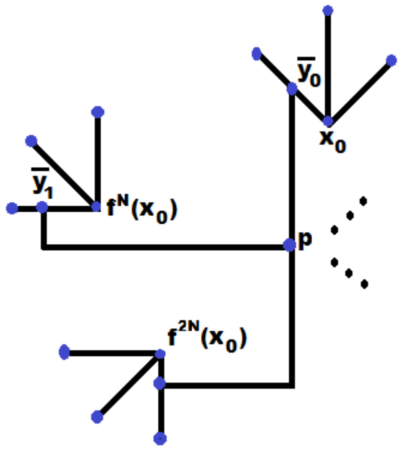

Figure 5.3: Construction of Λ if $[\bar{y_0}, \bar{y_1}] \cap Fix(f^N) = \{p\}$.

Now, we need the following lemma to have more information about the dynamics of $f^N|_\Lambda$ in a neighborhood of β and, thus, we can find a sufficient number of separate sets.

Let $g : \Lambda \to \Lambda$ be the dendrite homeomorphism given by $g = f^N$.

Lemma 5.2.1. *There exists* $\eta > 0$ *such that*

i. For every $n \neq 0$, $V(\beta,\eta)\bigcap g^n(W_0) = \emptyset$,

ii. For every $n \neq 0,-1$, $V(\beta,\eta)\bigcap g^n([\bar{y_0},\bar{y_1}]) = \emptyset$,

where $V(\beta,\eta) = \bigcup_{x\in\beta} B(x,\eta)$ *is a neighborhood of* β.

Proof. First, we show that there exists $\eta_1 > 0$ that satisfies i. Then we show that there exists $\eta_2 > 0$ that satisfies ii. and finally we consider $\eta = \min\{\eta_1, \eta_2\}$.
i. Suppose (arguing by contradiction) that for all $k \in \mathbb{N}$, there exists $n_k \neq 0$ such that

$$V(\beta, 1/k)\bigcap g^{n_k}(W_0) \neq \emptyset.$$

We have two cases. Assume first that the set

$$A = \{n_k : n_k \in \mathbb{Z}\setminus\{0\} \text{ and } V(\beta,1/k)\cap g^{n_k}(W_0) \neq \emptyset\}$$

is finite: i.e., $A = \{n_1, ..., n_{k_0}\}$. Since both β and $g^{n_j}(W_0)$ are disjoint compact sets, there exists $k \in \mathbb{N}$ big enough such that

$$V(\beta,1/k)\cap V(g^{n_j}(W_0),1/k) = \emptyset$$

for $j = 1, ..., k_0$ which is a contradiction.

If the set A is infinite, then we can assume, without loss of generality, that $n_k > 0$ for every k. So, there exist $x \in W_0$ and $y \in \beta$ such that $g^{n_k}(x_k) \to y$ when $k \to \infty$ and $x_k \to x$ when $k \to \infty$. We claim $y \in \omega(y, g)$. In fact, let $\varepsilon > 0$. Then there is $k_0 \in \mathbb{N}$ such that $d(g^{n_k}(x_k), y) < \varepsilon$ for all $k \geq k_0$. We can assume that

$$d(g^{n_k}(x_k), y) > d(g^{n_{k+1}}(x_{k+1}), y) \text{ for all } k \geq k_0$$

Let $\delta = \min\{d(g^{n_{k_0}}(x_{n_{k_0}}), y), \varepsilon - d(g^{n_{k_0}}(x_{n_{k_0}}), y)\}$. So we have that

$$B(g^{n_k}(x_{n_k}),\delta) \subset B(y,\varepsilon) \text{ for all } k \geq k_0.$$

On the other hand, since $\lim_{n\to\infty} diam(g^n(U)) = 0$, there exists $N \in \mathbb{N}$ such that $diam(g^n(U)) < 2\delta$ and, therefore:

$$g^{n_k}(x_k) \in g^{n_k}(U) \subset B(g^{n_k}(x_k),\delta) \text{ for all } n_k \geq N.$$

Thus, if $n_k \geq \max\{N, n_{k_0}\}$, then $g^{n_k}(y) \in B(y, \varepsilon)$ which concludes.

ii. Suppose (arguing by contradiction) that for all $k \in \mathbb{N}$, there exists $n_k \neq 0, -1$ such that

$$V(\beta, 1/k) \cap g^{n_k}([\bar{y_0}, \bar{y_1}]) \neq \emptyset.$$

As above, we consider the set

$$A = \{n_k : n_k \in \mathbb{Z} \setminus \{0, -1\} \text{ and } V(\beta, 1/k) \cap g^{n_k}(([\bar{y_0}, \bar{y_1}]) \neq \emptyset\}.$$

Therefore, if A is a finite set with the same argument as above, we have a contradiction. If A is an infinite set, we can assume, without loss of generality, that $n_k > 0$ for every k. So, there are a sequence $\{x_k\}_{k\in\mathbb{N}}$ in $[\bar{y_0}, \bar{y_1}]$ and $\{n_k\}_{k\in\mathbb{N}}$ in A such that $g^{n_k}(x_k) \to y \in \beta$ and $x_k \to x \in [\bar{y_0}, \bar{y_1}]$ when $k \to \infty$. Let $\varepsilon < \eta_1$, there exists $\delta < \varepsilon$ such that satisfies Proposition 2.5.8, that is, for any pair of points $a, b \in D$ with $d(a, b) < \delta$ implies $diam([a, b]) < \varepsilon$. Also, there exists $k_0 \in \mathbb{N}$ such that $d(y, g^{n_{k_0}}(x_{k_0})) < \delta$ and $d(x, x_{k_0}) < \delta$. Thus, according Proposition 2.5.8,

$$diam([y, g^{n_{k_0}}(x_{k_0})]) < \varepsilon$$

and, therefore, $[y, g^{n_{k_0}}(x_{k_0})] \subset B(y, \varepsilon)$. Meanwhile, by the construction of set Λ we have

$$[y, g^{n_{k_0}}(x_{k_0})] \cap g^j(W_0) \neq \emptyset$$

for $j = 1, ..., n_{k_0}$ which contradicts item i. □

Lemma 5.2.2. *If $k \in \mathbb{N}$, there is $\delta > 0$ such that $s(n+1, C(g)^{-1}, \delta) \geq k^n$ for all $n \in \mathbb{N}$.*

Proof. Without loss of generality, we can assume that $\beta = [x_0, b]$. Fix $\bar{a} \in (x_0, a)$ close to the point x_0 and consider δ_0 small enough such that $V([\bar{a}, a], \delta_0)$ does not contain points of $([\bar{y_0}, \bar{y_1}] \cup W_0) \bigcup f^{-N}([\bar{y_0}, \bar{y_1}] \cup W_0) \setminus (x_0, a]$. Since $[\bar{a}, a]$ is an arc, it is homeomorphic to interval $[0, 1]$. Let $H : [\bar{a}, a] \to [0, 1]$ be a homeomorphism such that $H(\bar{a}) = 0$ and $H(a) = 1$. Let $\eta > 0$ be given by Lemma 5.2.1 and let $k \in \mathbb{N}$. There exists $\delta < \min\{\frac{1}{k}, \eta, \delta_0\}$ such that

$$d(x, y) < \delta \text{ implies } |H(x) - H(y)| < \frac{1}{k}.$$

Set $a_i = H^{-1}(i/k)$ for $i = 1, ..., k-1$. Let $n \in \mathbb{N}$, let $\sigma = (\sigma_0, \sigma_1 ..., \sigma_{n-1}) \in \{1, 2, ..., k\}^n$ and let C_σ the subtree of $\Lambda = \overline{\bigcup_{j\in\mathbb{Z}} g^j(W_0 \cup [\bar{y_0}, \bar{y_1}])}$ defined as follows:

$$C_\sigma = \bigcup_{j=0}^{n-1} g^j([a_{\sigma_j}, x_0]) \cup \bigcup_{j=0}^{n-2} g^j([\bar{y_0}, \bar{y_1}] \cup [a, x_0] \cup [b, x_0]).$$

If $\sigma \neq \sigma' \in \{1, 2, ..., k\}^n$, there is $j_0 \in \{0, 1, ..., n-1\}$ such that $\sigma_{j_0} \neq \sigma'_{j_0}$. Without loss of generality, we can assume that $\sigma_{j_0} < \sigma'_{j_0}$. Then

$$\left|\frac{\sigma'_{j_0}}{k} - t\right| > \frac{1}{k} \text{ for all } t \in \left[0, \frac{\sigma_{j_0}}{k}\right].$$

By the continuity of H, we have that

$$d(a_{\sigma'_{j_0}}, H^{-1}(t)) > \delta \text{ for all } t \in \left[0, \frac{\sigma_{j_0}}{k}\right].$$

If it is necessary, we consider a smaller $\delta > 0$ such that

$$a_{\sigma'_{j_0}} \in g^{-j_0}(C_{\sigma'}) \text{ and } a_{\sigma'_{j_0}} \notin V(g^{-j_0}(C_\sigma), \delta).$$

Therefore, $d_H(g^{-j_0}(C_\sigma), g^{-j_0}(C_{\sigma'}) \geq \delta$. Thus, the set $\{C_\sigma : \sigma \in \{1, ..., k\}^n\} \subset C(\Lambda)$ is $(n, C(g)^{-1}, \delta)$-separated set and $s(n, C(f)^{-1}, \delta) \geq k^n$. $\square$

According to Lemma 5.2.2, if $k \in \mathbb{N}$, then there is $\delta > 0$ such that for all $n \in \mathbb{N}$ $s(n, C(g)^{-1}, \delta) \geq k^n$. So, for all $n \in \mathbb{N}$

$$\frac{1}{n} \log s(n, C(g)^{-1}, \delta) \geq \log k$$

and the definition of topological entropy gives us

$$h(C(g)^{-1}) = \lim_{\delta\to 0} \limsup \frac{1}{n} \log s(n, C(g)^{-1}, \delta) \geq \log k.$$

Thus, we conclude that $h(C(f)) = h(C(g)) = h(C(g)^{-1}) = \infty$. $\square$

5.3 Proof of Theorem H

Now we are ready to prove:

Theorem H *Let $f : D \to D$ be a dendrite homeomorphism. Then*

i. *if there exists a branch point x_0 in $D \setminus R(f)$, then $h(C(f)) = \infty$, and*

ii. if $R(f) = D$, then $h(C(f)) = 0$.

Proof. i. Let $x_0 \in D \setminus R(f)$ and let U be a component of $D \setminus R(f)$ that contains the point x_0. We assume that $ord(x_0) = 3$ for simplicity. Note that the sets $R(f)$ and $D \setminus R(f)$ are strongly invariant under f. Since f is a homeomorphism, for all $n \in \mathbb{Z}$, $f^n(U)$ is a component of $D \setminus R(f)$. We consider two cases. First, there exists $n \in \mathbb{N}$ such that $f^n(U) = U$. Then by Theorem I, we have that $h(C(f)) = \infty$. The other case is if for all $n \in \mathbb{N}$, $f^n(U) \cap U = \emptyset$. Then by Theorem J we have that $h(C(f)) = \infty$.

ii. The proof follows directly from Theorem 6.6 in [22]. □

Chapter 6

Appendix

6.1 Questions

- Does there exist a dynamical system (X, f) such that

$$h(C(f)) < h(C_n(f)) < h(2^f)$$

for some $n \geq 2$?

- If X is a continuum and $f : X \to X$ is an expansive homeomorphism with positive entropy then is $h(C(f)) = \infty$?

- If X is a continuum with $dim(X) \geq 2$ and $f : X \to X$ is a homeomorphism with positive entropy, then is $h(C(f)) = \infty$?

- Let $f : D \to D$ be a dendrite homeomorphism such that for every $x \in D \setminus R(f)$, $ord(x) \leq 2$. Then is $h(C(f)) \in \{0, \infty\}$?

- If f is a partial hyperbolic diffeomorphism, then is $C(f)$ transitive?

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