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THREE-DIMENSIONAL VENICE MASKS

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To the reason of my life, my dear family.

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Abstract

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Let M be a compact 3-manifold and let X be a vector field C^r, $r \geq 1$ on M. The flow generated by X is denoted by X_t, $t \in I\!R$. An *attracting set* is a set to which all nearby positive orbits converge. An subset $\Lambda \subset M$ is *transitive* if $\Lambda = \omega_X(x)$ for some $x \in \Lambda$. A *closed orbit* is a compact orbit (singularity or periodic orbit).

A compact invariant set Λ is *sectional-hyperbolic* for X_t if Λ is partially-hyperbolic with area expanding on each two-dimensional space in the central subbundle, and each singularity in Λ is hyperbolic.

The flow X_t is *sectional-Anosov*, if the *maximal invariant* set of X defined by $M(X) = \bigcap_{t\geq 0} X_t(M)$ is sectional-hyperbolic. A sectional-Anosov flow is called *Venice mask* if it is not transitive but has dense periodic orbits.

In this work we prove the following results:

1. Three-dimensional Venice masks with two equilibria do exist. Indeed, we present different types depending on the intersection of the homoclinic classes composing the corresponding maximal invariant set.
2. For each $n \in \mathbb{N}$ there are three-dimensional Venice masks containing exactly n equilibria. These examples are characterized by the maximal invariant set which is finite union of homoclinic classes. Here, the intersection of two different homoclinic classes is contained in the closure of the union of unstable manifolds of the singularities.
3. For every three-dimensional Venice mask the omega-limit set of every non-recurrent point in the unstable manifold of some singularity is a closed orbit.
4. The intersection of two different homoclinic classes of a sectional-Anosov flow decomposes as the disjoint union of singular points, non-singular hyperbolic sets, and regular points whose alpha and omega-limit sets are either singular points or non-singular hyperbolic sets.

Resumo

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Seja M uma 3-variedade compacta e seja X um campo vetorial $C^r, r \geq 1$ em M. O fluxo gerado por X é denotado por X_t, $t \in I\!R$. Um sumidouro é um conjunto compacto tal que a órbita positiva de cada ponto perto dele converge ao conjunto. Um subconjunto $\Lambda \subset M$ é transitivo se $\Lambda = \omega_X(x)$ para algum $x \in \Lambda$. Uma *órbita fechada* é uma *singularidade* ou *órbita periódica*.

Um conjunto compacto invariante Λ é *seccional-hiperbólico* para X_t se Λ é parcialmente-hiperbólico, expande volume em cada espaço 2-dimensional do subfibrado central, e cada singularidade em Λ é hiperbólica.

O fluxo X_t é dito *seccional-Anosov*, se o *conjunto maximal invariante* de X definido por $M(X) = \bigcap_{t \geq 0} X_t(M)$ é seccional-hiperbólico. Um fluxo seccional-Anosov não transitivo é dito *máscara de Veneza* se este possui órbitas periódicas densas.

Neste trabalho vamos provar os seguintes resultados:

1. A existência de duas máscaras de Veneza diferentes, cada uma contendo duas singularidades sobre alguma 3-variedade compacta. Com efeito, são apresentados dois tipos de exemplos nos quais as classes homoclínicas compondo o seu conjunto maximal invariante têm interseção de um modo muito diferente.

2. Para cada $n \in \mathbb{N}$, se mostra a existência de uma máscara de Veneza com n singularidades suportada em alguma 3-variedade compacta. Os exemplos são caracterizados devido a que o conjunto maximal invariante é união de finitas classes homoclínicas. Aqui, a interseção entre duas classes homoclínicas diferentes é contida no fecho da união das variedades instáveis das singularidades de X.

3. Para toda máscara de Veneza definida em uma 3-variedade compacta M, o conjunto omega-limite de todo ponto não recorrente na variedade instável de alguma singularidade, é uma órbita fechada.

4. A interseção de duas classes homoclínicas diferentes de um fluxo seccional-Anosov é obtida como a união disjunta de pontos singulares, conjuntos hiperbólicos não singulares, e pontos regulares cujo α-limite e ω-limite são pontos singulares ou conjuntos hiperbólicos não singulares.

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In the beginning there was π:

$$e^{\pi i} + 1 = 0 \tag{1}$$

CHAPTER

ONE

INTRODUCTION

The theory of dynamical systems deals with the asymptotic behavior of the orbits of a given EDO or map. Because of the impossibility of finding explicit solutions in most of these systems it was proposed proposed to study the qualitative behaviour of the solution without actually finding them. An example of this study is the phenomenom of transverse homoclinic points. Birkhoff proved that any tranverse homoclinic orbit is accumulated by periodic points. The introduction of the *uniformly hyperbolic dynamical systems* by Smale [39] allowed to develop a study of robust models containing infinitely many periodic motions. However, the uniform hyperbolicity was soon proved to be less universal as initially thought. In fact, many classes nonuniformly hyperbolic systems coming from specific models in applications appeared. This motivated the formulation of weaker forms of hyperbolicity as existence of dominated splitting including the partial and sectional hyperbolicities.

Of particular interest are the sectional-hyperbolic sets and sectional-Anosov flows introduced in [28] and [22] to generalize the hyperbolic sets and Anosov flows. Their importance rely on the robustly transitive property in dimension three of certain sectional-Anosov flows [34], and on important examples such as the saddle type hyperbolic attracting sets, the singular horseshoe, the geometric and multidimensional Lorenz attractors [1], [12], [15].

With respect to robustly transitive property, we can mention a clue fact in the scenario of sectional-Anosov flows. As a consequence of the main result in [3] and *Theorem 32* in [6] it follows that every three-dimensional sectional-Anosov flow with a unique singularity is C^r robustly periodic if and only if is C^r robustly transitive. Recall that a C^r vector field is C^r robustly transitive or C^r robustly periodic depending on whether every C^r vector field

C^r-close to it is transitive or has dense periodic orbits.

But unlike the Anosov flows there is no equivalence between transitivity and density of periodic orbits for sectional-Anosov flows. Indeed, there are sectional-Anosov flows which are not transitive but with dense periodic orbits. Another important property related to hyperbolic sets which is not satisfied by general sectional-hyperbolic sets is the Smale's spectral decomposition [39]. It says that any attracting hyperbolic set with dense periodic orbits splits into finitely many disjoint homoclinic classes. The examples in [10] and [32] show that this spectral decomposition is false for general sectional-Anosov flows.

In this context the definition of Venice mask comes in a natural way: A *Venice mask* is a sectional-Anosov flow which is not transitive but has dense periodic orbits. Such flows are necessarily non-Anosov or, equivalently, with at least one singularity. An example with just one singularity was exhibited in [10], and one with three singularities was provided in [32]. Such examples are characterized by the fact that their maximal invariant sets are the union of two (of course different) homoclinic classes intersecting along the unstable manifold of a singularity [32], [31], [10]. Recall that the unstable manifold of a hyperbolic singularity σ is formed by points whose negative orbit converges to σ.

It is natural to ask if there are Venice maks with more singularities and precisely if for each positive integer n there is one with exactly n singularities. It is one of the objectives of this thesis to exhibit examples of three-dimensional Venice masks with two singularities. Namely following the ideas given in [10] we will construct in Chapter 3 two types of Venice masks containing two singularities.

In Chapter 4 we show how generate new examples. First of all, we will briefly described some properties of the examples in [10], [32], [21]. Afterwards, we derive Venice maks with an even number of singulairites from the examples in Chapter 3. Also, from the example in [32] will be constructed Venice masks containing an odd number of singularities. An important conclusioin from these constructions will be that in general the maximal invariant set need not be the union of just two homoclinic classes. Indeed, for every $n \in \mathbb{N}$ we will construct a Venice mask whose maximal invariant set is precisely the union of n homoclinic classes. Moreover, for these flows the intersection of two different homoclinic classes is contained in the closure of the union of the unstable manifold of the singularities.

To finish, in Chapter 5 we will study some properties associated to the dynamics of Venice masks and homoclinic classes of sectional-Anosov flows. Specifically we show for Venice masks that every non-recurrent point in the unstable manifold of a singularity is either a singular point or a hyperbolic periodic orbit. This result can be seen as an extension of the *sectional-connecting lemma* given in [8]. Moreover, we will describe the intersection of two different homoclinic class of any three-dimensional sectional-Anosov flow.

CHAPTER

TWO

PRELIMINARIES

2.1 Definitions and notation

Consider a Riemannian compact manifold M of dimension three (a *compact* 3-*manifold* for short). M is endowed with a Riemannian metric $\langle \cdot, \cdot \rangle$ and an induced norm $\|\cdot\|$. We denote by ∂M the boundary of M. Let $\mathcal{X}^1(M)$ be the space of C^1 vector fields in M endowed with the C^1 topology. Fix $X \in \mathcal{X}^1(M)$, inwardly transverse to the boundary ∂M and denotes by X_t the flow of X, $t \in \mathbb{R}$.

The *omega-limit set* of $p \in M$ is the set $\omega_X(p)$ formed by those $q \in M$ such that $q = \lim_{n\to\infty} X_{t_n}(p)$ for some sequence $t_n \to \infty$. The *alpha-limit set* of $p \in M$ is the set $\alpha_X(p)$ formed by those $q \in M$ such that $q = \lim_{n\to\infty} X_{t_n}(p)$ for some sequence $t_n \to -\infty$. Given $\Lambda \in M$ compact, we say that Λ is *invariant* if $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. We also say that Λ is *transitive* if $\Lambda = \omega_X(p)$ for some $p \in \Lambda$; *singular* if it contains a singularity and *attracting* if $\Lambda = \cap_{t>0} X_t(U)$ for some compact neighborhood U of it. This neighborhood is often called *isolating block*. It is well known that the isolating block U can be chosen to be positively invariant, i.e., $X_t(U) \subset U$ for all $t > 0$. An *attractor* is a transitive attracting set. An attractor is *nontrivial* if it is not a closed orbit.

The *maximal invariant set* of X is defined by $M(X) = \bigcap_{t\geq 0} X_t(M)$.

Definition 2.1.1. *A compact invariant set Λ of X is* hyperbolic *if there are a continuous tangent bundle invariant decomposition $T_\Lambda M = E^s \oplus E^X \oplus E^u$ and positive constants C, λ such that*

- E^X *is the vector field's direction over* Λ.
- E^s *is* contracting, *i.e.,* $\|DX_t(x)\big|_{E^s_x}\| \le Ce^{-\lambda t}$, *for all* $x \in \Lambda$ *and* $t > 0$.
- E^u *is* expanding, *i.e.,* $\|DX_{-t}(x)\big|_{E^u_x}\| \le Ce^{-\lambda t}$, *for all* $x \in \Lambda$ *and* $t > 0$.

A compact invariant set Λ has a *dominated splitting* with respect to the tangent flow if there are an invariant splitting $T_\Lambda M = E \oplus F$ and positive numbers K, λ such that

$$\|DX_t(x)e_x\| \cdot \|f_x\| \le Ke^{-\lambda t}\|DX_t(x)f_x\| \cdot \|e_x\|, \qquad \forall x \in \Lambda, t \ge 0, (e_x, f_x) \in E_x \times F_x.$$

Notice that this definition allows every compact invariant set Λ to have a dominated splitting with respect to the tangent flow (See [9]): Just take $E_x = T_xM$ and $F_x = 0$, for every $x \in \Lambda$ (or $E_x = 0$ and $F_x = T_xM$ for every $x \in \Lambda$).

A compact invariant set Λ is *partially hyperbolic* if it has a *partially hyperbolic splitting*, i.e., a dominated splitting $T_\Lambda M = E \oplus F$ with respect to the tangent flow whose dominated subbundle E is contracting in the sense of Definition 2.1.1.

The Riemannian metric $\langle \cdot, \cdot \rangle$ of M induces a 2-Riemannian metric [35],

$$\langle u, v/w \rangle_p = \langle u, v \rangle_p \cdot \langle w, w \rangle_p - \langle u, w \rangle_p \cdot \langle v, w \rangle_p, \quad \forall p \in M, \forall u, v, w \in T_pM.$$

This in turns induces a 2-norm [14] (or areal metric [20]) defined by

$$\|u, v\| = \sqrt{\langle u, u/v \rangle_p} \qquad \forall p \in M, \forall u, v \in T_pM.$$

Geometrically, $\|u, v\|$ represents the area of the paralellogram generated by u and v in T_pM.

If a compact invariant set Λ has a dominated splitting $T_\Lambda M = F^s \oplus F^c$ with respect to the tangent flow, then we say that its central subbundle F^c is *sectionally expanding* if

$$\|DX_t(x)u, DX_t(x)v\| \ge K^{-1}e^{\lambda t}\|u, v\|, \quad \forall x \in \Lambda, u, v \in F^c_x, t \ge 0.$$

Recall that a singularity of a vector field is hyperbolic if the eigenvalues of its linear part have non zero real part.

By a *sectional-hyperbolic splitting* for X over Λ we mean a partially hyperbolic splitting $T_\Lambda M = F^s \oplus F^c$ whose central subbundle F^c is sectionally expanding.

Definition 2.1.2. *A compact invariant set* Λ *is* sectional-hyperbolic *for* X *if its singularities are hyperbolic and if there is a sectional-hyperbolic splitting for* X *over* Λ*.*

Definition 2.1.3. *We say that* X *is a* sectional-Anosov flow *if* $M(X)$ *is a sectional-hyperbolic set.*

The Invariant Manifold Theorem [18] asserts that if x belongs to a hyperbolic set H of X, then the sets

$$W_X^{ss}(p) = \{x \in M : d(X_t(x), X_t(p)) \to 0, t \to \infty\} \qquad and$$
$$W_X^{uu}(p) = \{x \in M : d(X_t(x), X_t(p)) \to 0, t \to -\infty\},$$

are C^1 immersed submanifolds of M which are tangent at p to the subspaces E_p^s and E_p^u of T_pM respectively.

$$W_X^s(p) = \bigcup_{t\in\mathbb{R}} W_X^{ss}(X_t(p)) \qquad and \qquad W_X^u(p) = \bigcup_{t\in\mathbb{R}} W_X^{uu}(X_t(p))$$

are also C^1 immersed submanifolds tangent to $E_p^s \oplus E_p^X$ and $E_p^X \oplus E_p^u$ at p respectively.

We denote by $Sing(X)$ to the set of singularities of X.

Definition 2.1.4. *We say that a singularity* σ *of a sectional-Anosov flow* X *is* Lorenz-like *if it has three real eigenvalues* $\lambda^{ss}, \lambda^s, \lambda^u$ *with* $\lambda^{ss} < \lambda^s < 0 < -\lambda^s < \lambda^u$*. The strong stable foliation associated to* σ *and denoted by* $\mathcal{F}_X^{ss}(\sigma)$*, is the foliation contained in* $W^s(\sigma)$ *which is tangent to space generated by the eigenvalue* λ^{ss}*.*

We denote as $W^s(Sing(X))$ to $\bigcup_{\sigma\in Sing(X)} W^s(\sigma)$.

Respectively, $W^u(Sing(X)) = \bigcup_{\sigma\in Sing(X)} W^u(\sigma)$.

Definition 2.1.5. *A periodic orbit of* X *is the orbit of some* p *for which there is a minimal* $t > 0$ *(called the period) such that* $X_t(p) = p$*.*

γ is a *transverse homoclinic orbit* of a hyperbolic periodic orbit O if $\gamma \subset W^s(O) \cap W^u(O)$, and $T_qM = T_qW^s(O) + T_qW^u(O)$ for some (and hence all) point $q \in \gamma$. The homoclinic class $H(O)$ of a hyperbolic periodic orbit O is the closure of the union of the transverse homoclinic orbits of O. We say that a set Λ is a homoclinic class if $\Lambda = H(O)$ for some hyperbolic periodic orbit O.

Definition 2.1.6. *A Venice mask is a sectional-Anosov flow with dense periodic orbits which is not transitive.*

$Cl(A)$ denotes the closure of A.

2.2 Manifolds supporting sectional-Anosov flows

We briefly describe some aspects related to *handlebodies*. These will be the starting point for the manifolds supporting the examples exhibited in this work.

D^n denotes the unit ball in $\mathbb{R}^n$ and ∂D^n the boundary of D^n.

An n-cell is a manifold homemorphic to the open ball $D^n \setminus \partial D^n$.

The following definition appears in [17].

Definition 2.2.1. *A handlebody of genus $n \in \mathbb{N}$ (or a cube with n-handles) is a compact 3-manifold with boundary HB_n such that*

- *HB_n contains a disjoint collection of n properly embedded 2-cells.*
- *A 3-cell is obtain of cutting HB_n along the boundary of these 2-cells.*

Observe that a 3-ball is a handlebody of genus 0, whereas a solid torus is a handlebody of genus 1.

In [25] was proved that every orientable handlebody HB_n of genus $n \geq 2$ supports a transitive sectional-Anosov flow. An example is the geometric Lorenz attractor which is supported on a solid bitorus. In particular these flows have $n-1$ singularities.

2.2.1 Punctured 3-handlebodies

To continue, we mention a series of results which were developed in [40]. For more details see also [41].

A *k-handle* of dimension m is a manifold $H_{k,m} := D^k \times D^{m-k}$ with corners. The boundary of the handle is $\partial H_{k,m} = \partial_- H_{k,m} \cup \partial_+ H_{k,m}$, where

$$\partial_- H_{k,m} := \partial D^k \times D^{m-k}, \quad \partial_+ H_{k,m} := D^k \times \partial D^{m-k}.$$

Given a compact m-manifold M_0, we attach a k-handle $H_{k,m}$ to ∂M_0 through an embedding $\phi_k : \partial_- H_{k,m} \to \partial M_0$. We identify each $x \in \partial_- H_{k,m}$ with $\phi_k(x) \in \phi_k(\partial_- H_{k,m})$.

The resulting manifold M_1 will be denoted by $M_1 = M_0 \cup_{\phi_k} H_{k,m}$. Although M_1 has corner points, it is possible to obtain a smooth manifold applying a procedure called *straightening the angle* [23],[38].

In this way is given the following definition.

- A disc D^m is an *m-dimensional generalized handlebody.*
- The manifold $D^m \cup_{\phi^1_{k_1}} H_{k_1,m}$ is an *m-dimensional generalized handlebody*, denoted by $\mathcal{H}(D^m; \phi^1_{k_1})$.
- If $M = \mathcal{H}(D^m; \phi^1_{k_1}, \cdots, \phi^{i-1}_{k_{i-1}})$ is an *m-dimensional generalized handlebody*, then the manifold

$$M \cup_{\phi^i_{k_i}} H_{k_i,m}$$

  obtained from M by attaching a k_i-handle along $\phi^i_{k_i}$, is an *m-dimensional generalized handlebody*, denoted by $\mathcal{H}(D^m; \phi^1_{k_1}, \cdots, \phi^{i-1}_{k_{i-1}}, \phi^i_{k_i})$.

Defined an adequate flow in the handle $H_{k,m}$, was proved in [40] that if a manifold M supports a sectional-Anosov flow, and if we attach the handle in a specifically way, then the resulting manifold supports a sectional-Anosov flow too.

Definition 2.2.2. *A (g, k)-*punctured handlebody *is a handlebody M of genus g with k 2-handles attached to it, so that the attaching spheres of these 2-handles $S \to M$ are null homotopics on ∂M.*

From *Remark 8.0.3* and *Lemma 8.0.4* in [41] follows this remark.

Remark 2.2.3. *A (g, k)-punctured handlebody can be seen as a classical handlebody of genus g with k open balls removed from its interior.*

Now, it is possible to announce the main theorem in [40].

Theorem 2.2.4. *Every punctured 3-handlebody supports a sectional-Anosov flow.*

The idea consists in taking a solid bitorus endowed with the geometric Lorenz attractor. First of all, a $(1, k)$-punctured handlebody is built. For this, are taken $k + 1$ copies of a 2-handle $H^{2,3}_l$, $l = 1, \ldots, k + 1$ endowed with the flow $X^l_t(xe^{\lambda_1 t}, ye^{\lambda_2 t}, ze^{-\lambda_3 t})$. The values $\lambda_1, \lambda_2, \lambda_3$ being taken such as the geometric Lorenz attractor. Then each 2-handle is conveniently attached, one after another at the solid bitorus. After that, it is proved that the resulting flow is sectional-Anosov. Finally, the cases $g = 0$ and $g \geq 2$ are considered through modifications in $\mathcal{H}(D^3; 1, k)$ and the geometric Lorenz attractor in $\mathcal{H}(D^3; 2, 0)$ respectively.

Observe that each X_t^l has a hyperbolic singularity σ_l which is saddle-type. In addition, the stable direction $E^s_{\sigma_l}$ is one-dimensional. This means that just one associated eigenvalue to each σ_l is negative. Unfortunately, the sectional-Anosov flow obtained through this process is not a Venice mask. Indeed, as will be mentioned in Chapter 5, every singular point σ in a Venice mask X shall be Lorenz-like.

This motivates the exploration of another techniques to construct the examples of Venice masks.

2.2.2 Vector fields and the Euler characteristic

The Poincaré-Hopf theorem establishes a connection between the topology of M and the isolated zeroes of a smooth vector field X defined on M (See [24]).

Consider first $X : U \subset \mathbb{R}^n \to \mathbb{R}^n$, where U is an open set containing an isolated singularity σ of X. Define the index $i(\Sigma)$ of X at σ as the degree of the map $\tilde{X}$ given by

$$\tilde{X} = \frac{X(x)}{\|X(x)\|}.$$

Let σ be an isolated singularity σ of X. If $g : U \to M$ is a parametrization of a neighborhood of σ in M , then the index $i(\sigma)$ of X at σ is defined to be equal to the index of the corresponding vector field $dg^{-1} \circ X \circ g$ on U at the zero $g^{-1}(\sigma)$.

On the other hand, given a compact n-manifold M is defined the Euler characteristic $\chi(M)$ as

$$\chi(M) = \Sigma_{k=0}^{n}(-1)^k H_k(M),$$

where each $H_k(M)$ denotes the k-th homology group of M.

Theorem 2.2.5. ***(Poincaré-Hopf)***

Let X be a smooth vector field defined on a compact n-manifold M with isolated singularities. If X inwardly transverse to the boundary ∂M then

$$\chi(M) = \sum_{\sigma \in Sing(X)} i(\sigma).$$

Lemma 2.2.6. *The index $i(\sigma)$ of a non-degenerate singularity σ is equal to signal of $det(DX(\sigma))$.*

From [33] follows that if a sectional-Anosov flow is transitive or has dense periodic orbits, then all singularities are Lorenz-like. So, the following proposition is a direct consequence of the Poincaré Hopf theorem.

Proposition 2.2.7. *The number of singularities of a sectional-Anosov flow with dense periodice orbits supported on a compact* 3*-manifold* M *is equal to* $-\chi(M)$*. The same is valid by interchanging density of perdiodic orbits by transitivity.*

Observe that *Proposition 2.2.7* claims that every three-dimensional Venice mask with n singularities shall be defined on a compact 3-manifold M with $-\chi(M) = n$.

CHAPTER

THREE

EXISTENCE OF VENICE MASKS WITH TWO SINGULARITIES

3.1 Motivation

The dynamical systems theory describes different properties about asymptotic behavior, stability, relationships among system's elements and its characteristics. It is well known that the hyperbolic systems own some features and properties that provide very important information about its behavior. With the purpose of extending the notion of hyperbolicity, arise definitions and a new theory, such as partial hyperbolicity, singular hyperbolicity and sectional hyperbolicity. Thus, we begin by considering the relationship between the hyperbolic and sectional hyperbolic theory. Recall, the sectional hyperbolic sets and sectional Anosov flows were introduced in [28] and [22] respectively as a generalization of the hyperbolic sets and Anosov flows. They contain important examples such as the saddle-type hyperbolic attracting sets, the geometric and multidimensional Lorenz attractors [1], [12], [15].

A natural way is to observe the properties that are preserved or which are not in the new scenario. Particularly, we mention two important properties related to hyperbolic sets which are not satisfied by all sectional hyperbolic sets. The first is the spectral decomposition theorem [39]. It says that an attracting hyperbolic set $\Lambda = Cl(Per(X))$ is a finite disjoint union of homoclinic classes, where $Per(X)$ is the set of periodic points of X. The second says that an Anosov flow on a closed manifold is transitive if and only if it has dense periodic orbits. This results are false for sectional Anosov flows, i.e., sets whose maximal invariant is a sectional-hyperbolic set [31]. Specifically, it is proved that there exists a sectional Anosov flow such that it is supported on a compact 3-manifold, it has dense periodic orbits, is the non-disjoint union of two homoclinic classes but is not transitive. So, a sectional Anosov flow is said a *Venice mask* if it has dense periodic orbits but is not transitive.

The only known examples of Venice masks have one or three singularities, and they are characterized by having two properties: are the union non disjoint of two homoclinic classes and the intersection of its homoclinic classes is the closure of the unstable manifold of a singularity [32], [31], [10]. Particularly, was proved in [32], [31] that every Venice mask with a unique singularity has these properties. Naturally, we can ask whether these two properties are satisfied for every Venice mask. Here, we give a negative answer to this one. Indeed, we provide two examples of Venice masks with two singularities, but with different features. In particular, each one is the union of two different homoclinic classes. However, for the first, the intersection of homoclinic classes is the closure of the unstable manifold of two singularities. Whereas for the second, the intersection of homoclinic classes is just a hyperbolic periodic orbit.

We can state the main results in this chapter.

Theorem A. *There exists a Venice mask X with two singularities supported on a 3-manifold M, such that:*

- *$M(X)$ is the union of two homoclinic classes $\mathcal{H}_X^1, \mathcal{H}_X^2$.*
- *$\mathcal{H}_X^1 \cap \mathcal{H}_X^2 = O$, where O is a hyperbolic periodic orbit.*

Theorem B. *There exists a Venice mask Y with two singularities supported on a 3-manifold N, such that:*

- *$N(Y)$ is the union of two homoclinic classes $\mathcal{H}_Y^1, \mathcal{H}_Y^2$.*
- *$\mathcal{H}_Y^1 \cap \mathcal{H}_Y^2 = Cl(W^u(\sigma_1) \cup W^u(\sigma_2))$, where σ_1, σ_2 are the singularities of Y.*

This is a joint work with Andrés M. López Barragán. See [21].

In section 3.3.2, we shall be described briefly this construction by using one-dimensional and two-dimensional maps. In section 3.4.1, from modifications on the previous maps in Section 3.3.2 and by considering a plug, we shall prove the Theorem A. In the same way, in Section 3.4.2, by using the venice mask with a unique singularity, the Theorem B will be obtained by gluing a particular plug preserving the original flow.

3.2 Preliminaries

3.2.1 Original plugs

In order to obtain the three-dimensional vector field of our example, we begin by considering the well known *Plykin attractor* and the *Cherry flow* (See [37], [36]).

We give a sketch of the flow construction. It will be constructed through three steps, firstly by modifying the Cherry flow. In fact, we consider a vector field in the square whose flow is described in Figure 3.1 a). Note that this vector field has two equilibria: a saddle σ and a sink p. For σ one has that its eigenvalues $\{\lambda_s, \lambda_u\}$ of σ satisfy the relation

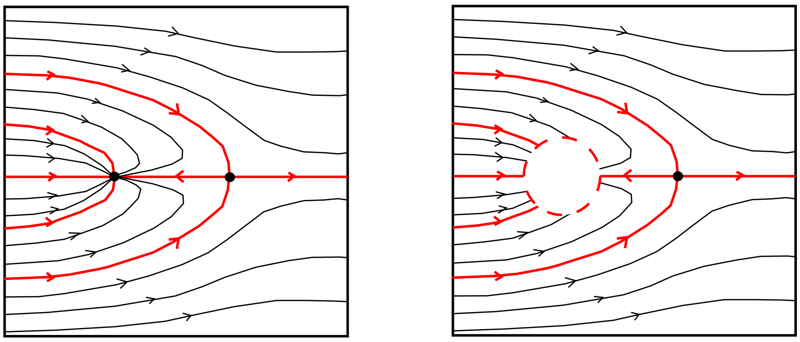
$$\lambda_s < 0 < -\lambda_s < \lambda_u.$$

We have depicted a small disk D centered at the attracting equilibrium p Figure 3.1 b). Note that the flow is pointing inward the edge of the disk. This finishes the first step for the construction.

For the second step we multiply the above vector field by a strong contraction λ_{ss} in order to obtain the vector field described in Figure 3.2 a). We can choose λ_{ss} such that $-\lambda_{ss}$ be large, so the resulting vector field will have a Lorenz-like singularity and this new eigenvalue will be associated with the strong manifold of the singularity. This yields a Cherry flow box and finishes the second step for the construction.

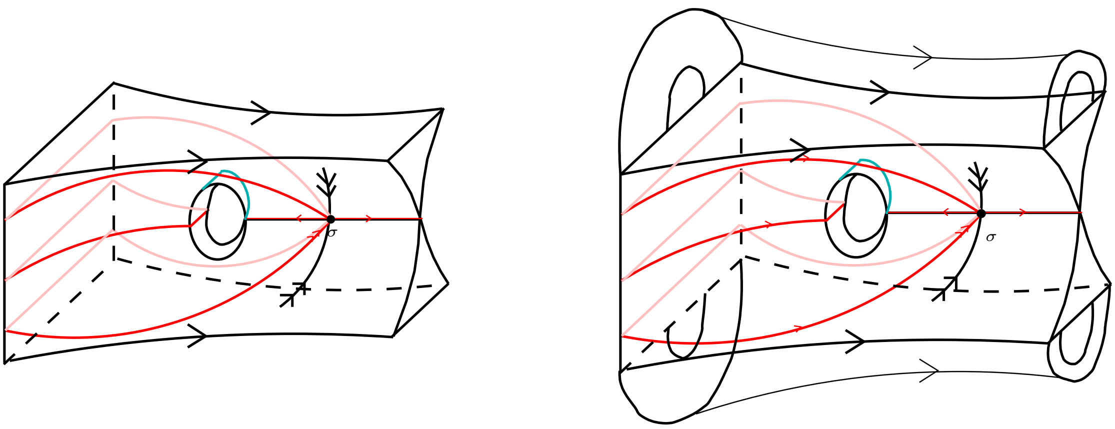
From Plykin attractor follows that the construction must have at least two holes inasmuch as we will use certain return map. Then, the final step is to glue two handles that provides two holes and the three dimensional vector field above in order to obtain the vector field whose flow is given in Figure 3.2 b). Hereafter the resulting vector field will be called of Plug 3.2.

The hole indicated in this Figure 3.2 is nothing but the disk D times a compact interval I_1. Again, note that the flow is pointing inward the edge of the hole by construction. For this reason, we take a solid 3-ball and we define a flow on this one. Indeed, it flow has no singularities, it acts as in Figure 3.3 and will be used for to glue the hole's bound with this one. Hereafter the resulting vector field will be called of Plug 3.3.



$a)$ $b)$

Figure 3.1: Cherry flow.



$a)$ $b)$

Figure 3.2: Cherry flow box and Plug 3.2.

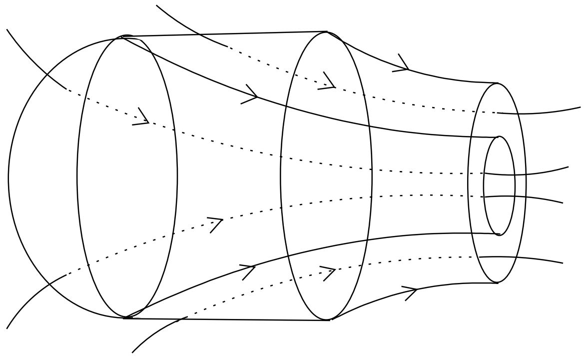


Figure 3.3: Plug 3.3.

3.3 Modified maps

We begin by considering the construction made in [10] like model in order to obtain the vector fields X and Y of the main theorems. Recall that the original model provides tools for a three dimensional example with a unique singularity. The main aim is to modify the original maps, in order to make a suspension of the modified maps via the new plugs. For this purpose, we will do such modifications followed by its original maps.

3.3.1 One-dimensional map

Thus, in the same way of [10], we consider the branched 1-manifold $\mathcal{B}$ consisting of a compact interval and a circle with branch point b. We cut $\mathcal{B}$ open along b to obtain a compact interval which we assume to be $[0, 1]$ for simplicity. In $[0, 1]$ we consider three points $0 < d_1 < d_* < d_2 < 1$, where d_* is depicted also in the Figure 3.4. These will be the discontinuity points of f as a map of $[0, 1]$. The set $\mathcal{B} \setminus \{d_*\}$ will be the domain of f. We define $f : \mathcal{B} \setminus \{d_*\} \to \mathcal{B}$ in a way that its graph in $[0, 1]$ is the one in Figure 3.4.

By construction one has that f satisfies the following hypotheses:

(H1): $Dom(f) = [0, 1] \setminus \{d_*\}$.

(H2): $f(0) = 0$; $f(d_1) = f(d_2) = 1$; $f(1) = f(b) \in (0, d_1)$.

(H3): $f(d_1+) = f(d_2+) = b$; $f(d_1-) = f(d_2-) = 1$; $f(d_*+) = f(d_*-) = 0$.

(H4): $f([0, d_1]) = [0, 1]$; $f((d_1, d_*)) = (0, b)$; $f((d_*, d_2]) = (0, 1]$; $f((d_2, 1]) = [f(b), b)$.

(H5): f is expanding, i.e., f is C^1 in $Dom(f)$ and there is $\lambda > 1$ such that $|f'(x)| \geq \lambda$, for each $x \in Dom(f)$.

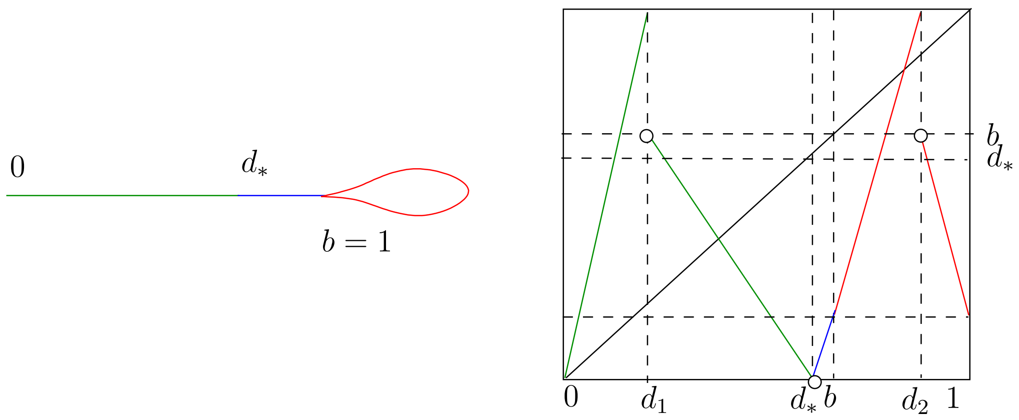


Figure 3.4: The quotient space and one-dimensional map.

3.3.2 Modified one-dimensional map

We realize a modification of the above map f . Denote $d_* = d^+$ and let $f^+ : \mathcal{B}^+ \setminus \{d^+\} \to \mathcal{B}^+$ be in a way that its graph in $[0, 1]$ is the one in Figure 3.5. More specifically, we consider a map f_1^+ and a map f_2^+ such that, $f_1^+|_{(0,d_1)}$ is contained in a small neighborhood of $f|_{(0,d_1)}$ in the topology $C^1(0, d_1)$, and $f_2^+|_{(d^+,b)}$ is contained in a small neighborhood of $f|_{(d^+,b)}$ in the topology $C^1(d^+, b)$. In this way, we take $\delta_1 > 0$ and we define $f_1^+(x) = \alpha x^2 + \beta x$ with $f_1^+(0) = 0$, $f_1^+(\frac{d_1}{2}) = \frac{1}{2} + \delta_1$ and $f_1^+(d_1) = 1$. Therefore we have $\alpha = \frac{-4\delta_1}{d_1^2}$ and $\beta = \frac{1+4\delta_1}{d_1^2}$. Moreover $(f_1^+)'(x) > 1$ for all $x \in (0, d_1)$ if δ_1 is small.

In the second case, we need a map f_2^+ which satisfies $(f_2^+)'(b-) = f'(b-)$. For this purpose, we use the fact that $\bar{f}(x) = exp\left[\frac{1}{x^2-1}\right]$ is such that $\bar{f}'(1-) = \bar{f}'(-1+) = 0$. Now, let $\delta_2 > 0$ be small and we define $f_2^+(x) = -\delta_2 exp\left[\frac{1}{(x-d^+)(x-b)}\right] + \frac{f(b)}{b-d^+}(x - d^+)$. So,

$$f^+(x) = \begin{cases} f_1^+(x), & x \in [0, d_1] \\ f(x), & x \in (d_1, d^+) \cup [b, 1] \\ f_2^+(x), & x \in (d^+, b) \end{cases}.$$

Here, there exist $\varepsilon > 0$ small such that $\int_0^{d_1} \sqrt{[(f)'(x)]^2 + 1}dx < \int_0^{d_1} \sqrt{[(f^+)'(x)]^2 + 1}dx < \int_0^{d_1} \sqrt{[(f)'(x)]^2 + 1}dx + \varepsilon$.

Also $\int_{d_*}^{b} \sqrt{[(f)'(x)]^2 + 1}dx < \int_{d_*}^{b} \sqrt{[(f^+)'(x)]^2 + 1}dx < \int_{d_*}^{b} \sqrt{[(f)'(x)]^2 + 1}dx + \varepsilon$. Moreover f^+ satisfies **(H1)-(H5)**. We define $f^-(x) = f(-x)$ and denote $-d^+ = d^-$. $f^- : \mathcal{B}^- \setminus \{d^-\} \to \mathcal{B}^-$.

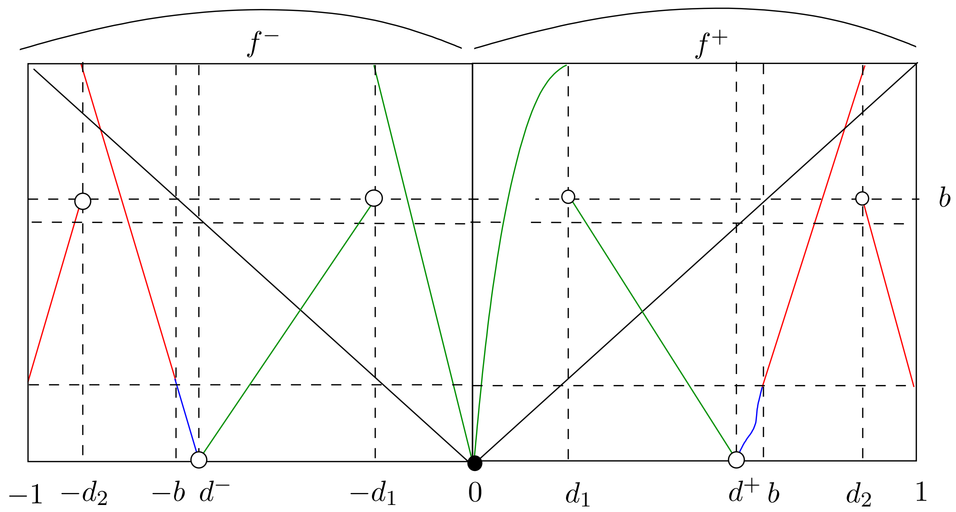


Figure 3.5: Modified one-dimensional map.

The following results proceed to examining the properties of f and appears in [10]. This in turns through a structure closely related to [10] and by construction we obtain the same properties for the f^+ map.

Definition 3.3.1. *We say that* f *is* locally eventually onto (leo for short) *if given any open interval* $I \subset [0, 1]$ *there is* $m \geq 0$ *such that* $f^m(I) = [0, 1]$.

Theorem 3.3.2. f^+ *is* leo.

Corollary 3.3.3. *The periodic points of* f^+ *are dense in* $\mathcal{B}$. *If* $x \in \mathcal{B}$, *then*

$$\mathcal{B} = Cl\left(\bigcup_{n\geq 0}(f^+)^{-n}(x)\right).$$

3.3.3 Two-dimensional map

We consider the twice punctured planar region R depicted in Figure 3.6. It is formed by: two half-annuli A, F, and four rectangles B, C, D, E. There is a middle vertical line denoted by l. Note that l defines a plane reflexion throughout denoted by θ. We assume $\theta(D) = C$, $\theta(E) = B$ and $\theta(F) = A$. In particular, $\theta(R) = R$ and $\theta(d^+) = d^-$, where the vertical segments d^-, d^+ correspond to the right-hand and left-hand boundary curves of B and D respectively. We define $H^- = A \cup B \cup C$ and $H^+ = D \cup E \cup F$. We take the same foliation $\mathcal{F}$ of R given in [10]. It is formed by vertical segments in the rectangular components B, C, D, E and radial segments in the annuli components A, F.

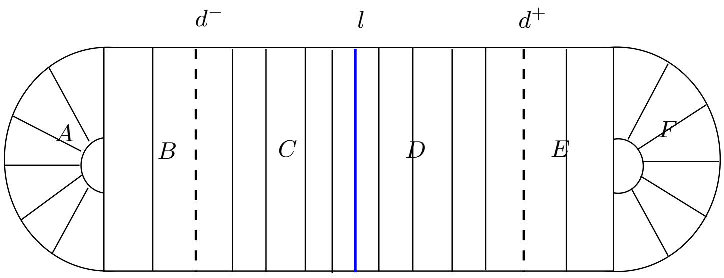


Figure 3.6: Region R.

In [10] was defined the C^∞ map $G : R \setminus \{d^-, d^+\} \to Int(R)$. It satisfies the following hypotheses:

(G1): G and θ commute, i.e., $G \circ \theta = \theta \circ G$.

(G2): G preserves and contracts the foliation $\mathcal{F}$.

(G3): Let $g : K \setminus \{d^-, d^+\} \to K$ be the map induced by G in the leaf space K.

Then, the map f^+ defined by $f^+ = g|_{\mathcal{B}^+}$ satisfies the hypotheses **(H1)-(H5)**, with $f = f^+$, $\mathcal{B} = \mathcal{B}^+$ and $d_* = d^+$.

Properties of G

- By **(G1)**, H^+ and H^- are invariant under G.
- Since G contracts $\mathcal{F}$ (**(G2)**) we have that $W^s(x, G)$ is union of leaves of $\mathcal{F}$. It follows from **(G2)**, **(G3)** and the expansiveness in **(H5)** that all periodic points of G are hyperbolic saddles.
- By **(G1)** we have that $G(l) \subset l$ and so G has a fixed point P in l. Clearly one has $\pi(P) = 0$.

Define

$$A_G^- = Cl\left(\bigcap_{n\geq 1} G^n(H^-)\right), \qquad A_G^+ = Cl\left(\bigcap_{n\geq 1} G^n(H^+)\right).$$

Theorem 3.3.4. *A_G^- and A_G^+ are homoclinic classes and $P \in A_G^+ \cap A_G^-$.*

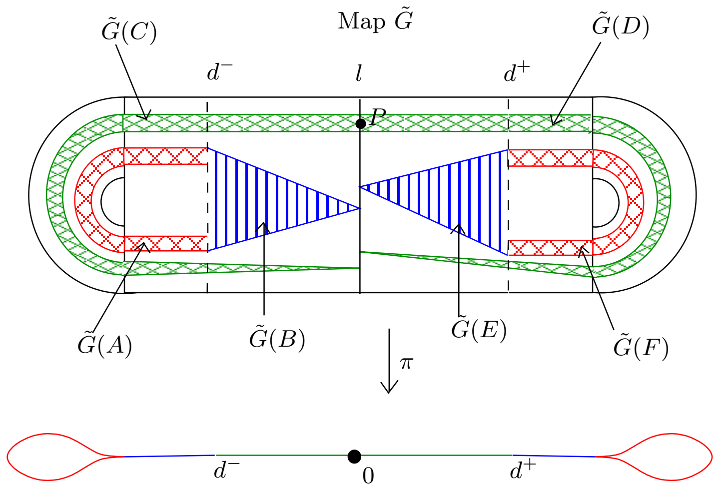


Figure 3.7: The quotient space and modified two-dimensional map.

3.3.4 Modified two-dimensional map

For the region R in Figure 3.6, we define the C^∞ map $\tilde{G} : R \setminus \{d^-, d^+\} \to Int(R)$ in a way that its image is as indicated in Figure 3.7. We require the following hypotheses:

(L1): H^-, H^+ are invariant under $\tilde{G}$. $\tilde{G}(H^- \setminus \{d^-\}) \subset H^-$ and $\tilde{G}(H^+ \setminus \{d^+\}) \subset H^+$.

(L2): $\tilde{G}$ preserves and contracts the foliation $\mathcal{F}$.

(L3): Let $\tilde{g} : K \setminus \{d^-, d^+\} \to K$ be the map induced by $\tilde{G}$ in the leaf space K.

Then, the map $f^{+(-)}$ defined by $f^{+(-)} = \tilde{g}|_{\mathcal{B}^{+(-)}}$ satisfies the hypotheses **(H1)-(H5)**, $\mathcal{B} = \mathcal{B}^{+(-)}$ and $d_* = d^{+(-)}$.

We observe that **(L1)** implies $\tilde{G}(l) \subset l$ and by contraction, $\tilde{G}$ has a fixed point $P \in l$. Again, for

$$A^-_{\tilde{G}} = Cl\left(\bigcap_{n\geq 1} \tilde{G}^n(H^-)\right), \qquad A^+_{\tilde{G}} = Cl\left(\bigcap_{n\geq 1} \tilde{G}^n(H^+)\right)$$

we have that $A^+_{\tilde{G}}$ and $A^-_{\tilde{G}}$ are homoclinic classes and $\{P\} = A^+_{\tilde{G}} \cap A^-_{\tilde{G}}$.

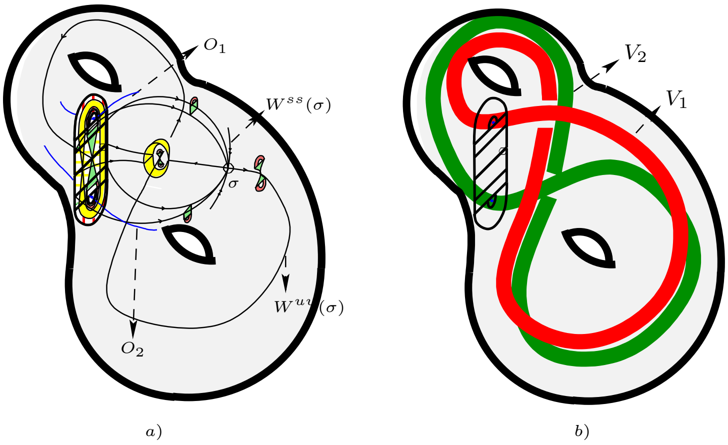


Figure 3.8: Venice mask with one singularity

3.3.5 Venice mask with one singularity

Recall, by considering the original maps (Subsection 3.3.1, 3.3.3), and by using the plugs 3.2, 3.3, in [10] was construct the venice mask example with one singularity. Here, we provides a graphic idea in order to compare it with the new examples.

The Figure 3.8 a) shows the flow, whereas the Figure 3.8 b) shows the ambient manifold that supports this one. The ambient manifold is a solid bi-torus excluding two tori neighborhoods V_1, V_2 associated to two repelling periodic orbits O_1, O_2 respectively.

3.4 Venice mask's examples with two-singularities

3.4.1 Vector field X and Example 1.

In this section, we construct a vector field X which will satisfy the properties in the Theorem A by using the subsection 3.3.2 and 3.3.4.

We begin by considering a vector field as the Cherry flow described in Figure 3.1, with the same conditions of subsection 3.3.2.

We called this flow of A and we proceed to perturbe it, following the ideas of the well known DA-Attractor introduced by Smale (see [37]). Let U be a neighborhood (relatively

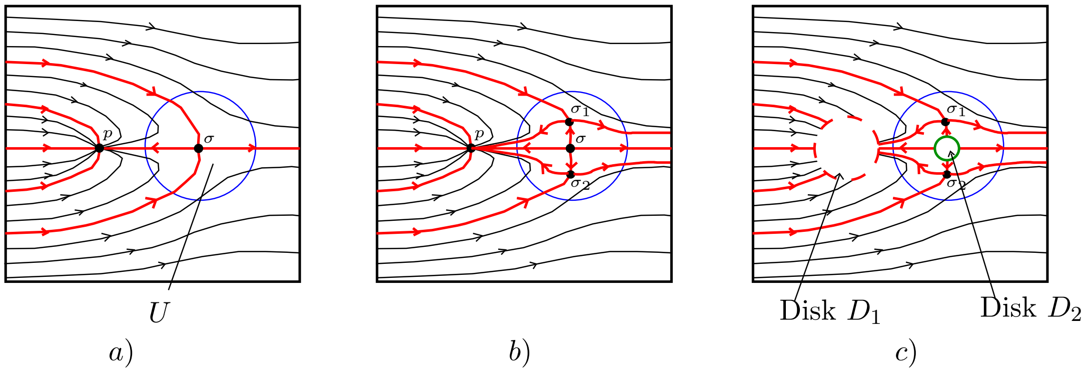


Figure 3.9: Perturbed Cherry flow

small) of σ. We can obtain a flow φ^t such that $supp(\varphi^t - id) \subset U$ (Figure 3.9 *a*)). Also, the derivative of the flow at σ with respect to canonical basis in $T_\sigma Q$ is

$$D\varphi^t_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix}.$$

We deform such a flow in order to obtain a one-parameter family of flows $B^t = \varphi^t \circ A$. Let $\tau > 0$ be such that $e^\tau \lambda_s > 1$, so σ is a source for B^τ. Moreover, the new map has three fixed points on $W^s_X(\sigma)$, σ a source and σ_1, σ_2 saddles. Moreover, there exists a neighborhood V of σ (not containing σ_1 and σ_2) contained in U such that $B^\tau_s(V) \supset V$ for all $s > 0$ (Figure 3.9 *b*)). Thus, we obtain a vector field as the square Q whose flow A is described in Figure 3.9.

Now, we remove two small disks D_1, $D_2 = V$ centered at the attracting equilibrium p and at the repelling equilibrium σ respectively (Figure 3.9 *c*)).

In the next step, we multiply the above vector field by a strong contraction λ_{ss} in order to obtain the similar vector field described in Figure 3.2 *b*). We choose λ_{ss} such that σ_1 and σ_2 are Lorenz-like.

Now, we consider an interval $I_0 = I_1 \times \{p_0\}$, where p_0 is the point of intersection between $W^u_X(\sigma)$ and the disk D_1. We realize a modification in the flow such that a branched of $W^u_X(\sigma_1)$ intersects a connected component of $I_0 \setminus \{p_0\}$ and a branched of $W^u_X(\sigma_2)$ intersects the other connected component of $I_0 \setminus \{p_0\}$ (See 3.10).

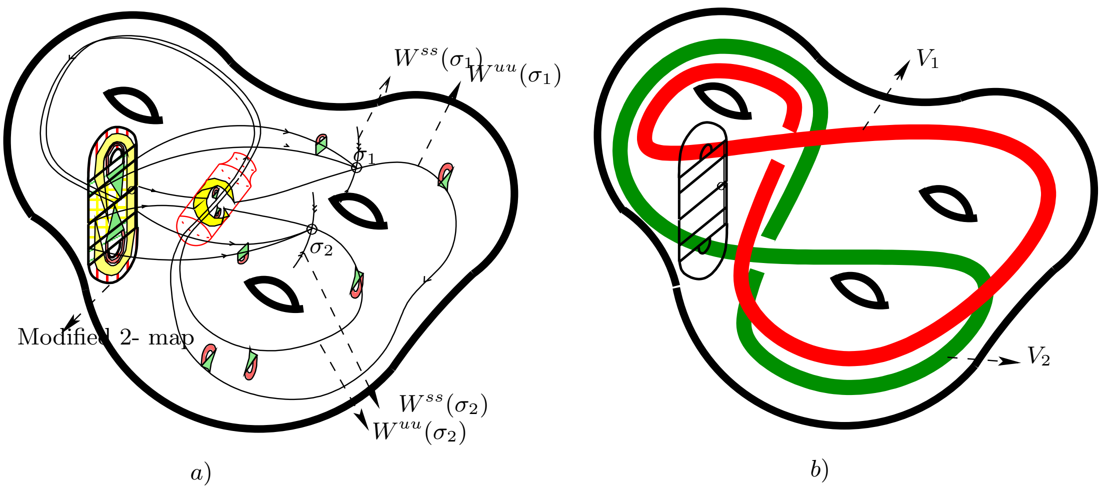


Figure 3.10: Plug X and its associated manifold.

The final step is to glue two handles on the 3-dimensional vector field above in order to obtain the vector field whose flow is given in Figure 3.10 a). The resulting vector field is what we shall call Plug X.

In the same way as in Figure 3.2, in this case, by multiplying the above vector field by a strong contraction generate two holes and it is nothing but the disks D_1 times a compact interval I_1, and D_2 times a compact interval I_2. Also, let us to use the Plug 3.3 and apply on the hole associated to D_1. Note that the interval I_2 is chosen such that $D_2 \times I_2$ produces the third hole on the ambient manifold. It generates a solid tritorus (see Figure 3.10 b)).

Then, we construct a vector field X on a solid tritorus ST_1 in a way that $X_t(ST_1) \subset Int(ST_1)$ for all $t > 0$ and X is transverse to the boundary of the solid tritorus. The flow is obtained gluing plugs X and 3.3 as indicated in Figure 3.10 a).

We require the following hypotheses:

(X1): There are two repelling periodic orbits O_1, O_2 in $Int(ST_1)$ crossing the holes of R.

(X2): There are two solid tori neighborhoods $V_1, V_2 \subset Int(ST_1)$ of O_1, O_2 with boundaries transverse to X_t such that if $M = ST_1 \setminus (V_1 \cup V_2)$, then M is a compact neighborhood with smooth boundary transverse to X_t and $X_t(M) \subset M$ for $t > 0$. As M is a solid tritorus with two solid tori removed, we have that M is connected as indicated in Figure 3.10 b).

(X3): $R \subset M$ and the return map $\tilde{G}$ induced by X in R satisfies the properties

(L1)-(L3) in Section 3.3.4. Moreover,

$$\{q \in M : X_t(q) \notin R, \forall t \in \mathbb{R}\} = \{\sigma_1, \sigma_2\}.$$

Now, define

$$A^+ = Cl\left(\bigcup_{t\in\mathbb{R}} X_t(A^+_{\tilde{G}})\right) \qquad \text{and} \qquad A^- = Cl\left(\bigcup_{t\in\mathbb{R}} X_t(A^-_{\tilde{G}})\right).$$

Proposition 3.4.1. *$W^u_X(\sigma_1) \subset A^+$ and $W^u_X(\sigma_2) \subset A^-$.*

Proof. If $x \in H^+$ is a periodic point of $\tilde{G}$, then $\tilde{G}^n(x) \in R$ for all $n \leq 0$ and so $x \in A^+_{\tilde{G}} = Cl(\bigcap_{n\geq 1} \tilde{G}^n(H^+))$. Therefore $x \in A^+$ (for $A_{\tilde{G}^+} \subset A^+$) and by invariance of A^+, the full orbit of x is contained in A^+.

Second, the periodic points of f^+ in **(L3)** are dense in $\mathcal{B}$ by Corollary 3.3.3. Then, the periodic points of $\tilde{G}$ accumulate on d^+ in both connected components of $H^+ \setminus d^+$. Since d^+ is contained in $W^s_X(\sigma_1)$, the full X_t-orbit of the periodic points of $\tilde{G}$ accumulating d^+ also accumulate on $W^u_X(\sigma_1)$. Then $W^u_X(\sigma_1) \subset A^+$ because A^+ is closed. Analogously, we have $W^u_X(\sigma_2) \subset A^-$. □

Define $A_{\tilde{G}} = A^+_{\tilde{G}} \cup A^-_{\tilde{G}}$ and

$$A = Cl\left(\bigcup_{t\in\mathbb{R}} X_t(A_{\tilde{G}})\right),$$

Lemma 3.4.2. *A^+ and A^- are homoclinic classes of X and $A = A^+ \cup A^-$.*

Proof. See [10]. □

Proposition 3.4.3. *X is a sectional Anosov flow.*

Proof. In the same way of [10], we will prove that A is a sectional-hyperbolic set and $M(X) = A$. Indeed, how $A = A_1 \cup A_2$ is union of homoclinic classes then A has dense periodic orbits (Birkhoff-Smale Theorem). Moreover, of the hypotheses **(L2)** and **(L3)** follows that every periodic orbit of X contained in A has a hyperbolic splitting $T_O M = E^s_O \oplus E^X_O \oplus E^u_O$. Here, E^s_O is due to **(L2)**, E^u_O by **(L3)** and E^X_O is the one-dimensional subbundle over O induced by X. Let $Per(A)$ be the union of the periodic orbits of X contained in A. Define the splitting

$$T_{Per(A)}M = F^s_{Per(A)} \oplus F^c_{Per(A)},$$

where $F^s_x = E^s_x$ and $F^c_x = E^X_x \oplus E^u_x$ for $x \in Per(A)$. As every periodic orbit in M of every vector field C^1 close to X is hyperbolic of saddle type, we can use the arguments in [33] to prove that the splitting $T_{Per(A)}M = F^s_{Per(A)} \oplus F^c_{Per(A)}$ over $Per(A)$ extends to a sectional-hyperbolic splitting $T_A M = F^s_A \oplus F^c_A$ over the whole $A = Cl(Per(A))$.

We conclude that X is a sectional Anosov flow on M.

□

Proof of Theorem A.

By using the Lemma 3.4.2 and the Proposition 3.4.3 we have that X is a sectional Anosov flow and $M(X)$ is the union of two homoclinic classes H^1_X, H^2_X, where $H^1_X = A^+$ and $H^2_X = A^-$. Since $\{P\} = A^+_{\tilde{G}} \cap A^-_{\tilde{G}}$, it implies that $H^1_X \cap H^2_X = O$, with O the orbit associated to P. In particular X is a Venice mask, and by construction it has two singularities.

3.4.2 Vector field Y and Example 2.

In this section, we construct a vector field Y which will satisfy the properties in the Theorem B by using the results from [10].

Firstly, in order to obtain the vector field Y, we begin by considering the venice mask with one singularity. Unlike the previous section, in this case we will not perturb the flow. Moreover, we will change the flow by preserving the plugs 3.2, 3.3 and we will remove a connected component of the flow and its ambient manifold.

The main aim of removing a connected component will be to glue a new plug with different features, properties and that provides other singularity. This process is done in

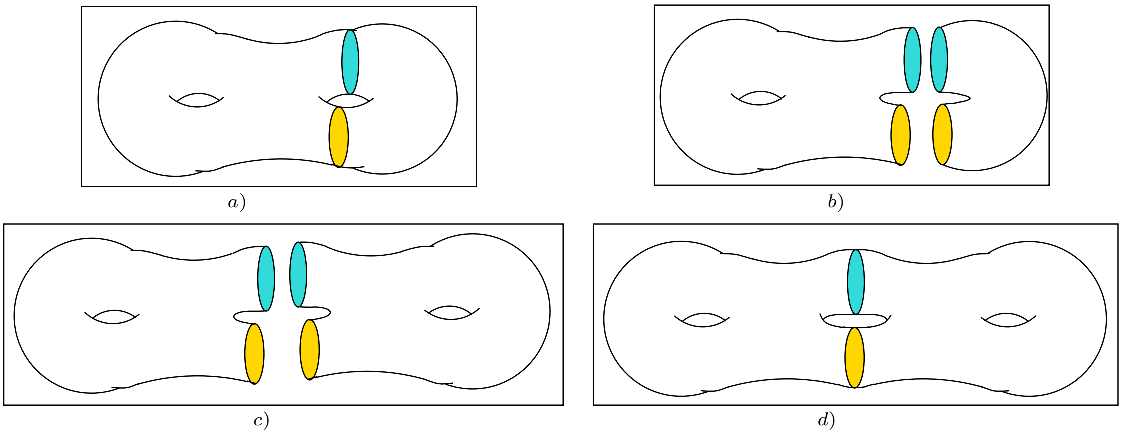


Figure 3.11: Steps by gluing the new plug.

some simple steps (see Figure 3.11). Indeed, the important steps are Figure 3.11 c), d) and since we want a plug by containing a singularity, we will see that the this one has a hole, which is produced by the singularity.

3.4.3 Flow through of the faces

We begin by considering the plug 3.2 described in Figure 3.2 with the same conditions of subsections 3.3.1, 3.3.3.

For this purpose we need to observe with detail the flow behavior through of the faces removed. Indeed, we observe the vector field in the square whose flow is described in Figure 3.2.

Thus, it will be constructed the new plug through two steps. Firstly, we will be depicted a circle that represents the face 1 on the Cherry flow and let us to study the flow behavior. It should be noted that this vector field exhibits two leaves which belong to the region R and converge to the singularity, i.e., the region R exhibits two singular leaves. Note that these leaves are crossing outward to the face 1. In addition, note that there are trajectories crossing inward to the face 1 too, such as the branch unstable manifold of the singularity. This shows that extensive analysis is necessary for understand the flow behavior to the face 1.

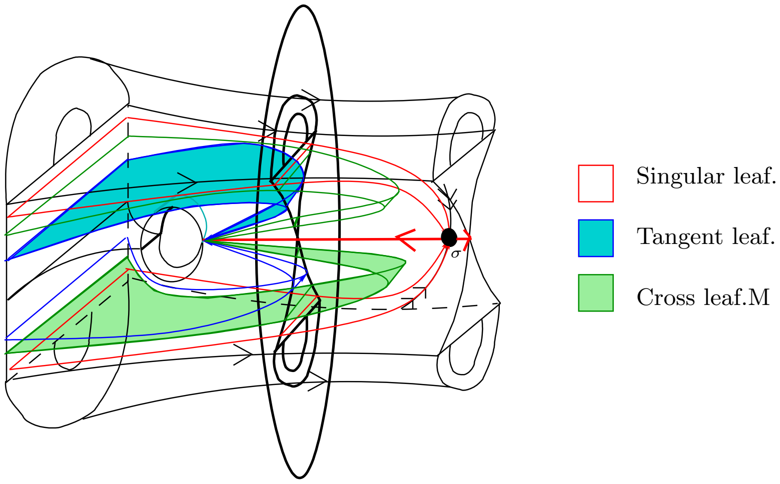


Figure 3.12: Flow through of the face 1.

We can observe that the top and bottom region of the singular leaves saturated by the flow are crossing through the face 1, i.e., the flow is pointing outward of the face 1.

By studying the complement of these regions, we have that the behavior of the leaves is depicted as Figure 3.12. Here, this region exhibits two tangent leaves, whereas the other leaves intersect the region twice, i.e., the other leaves cross and return.

Also, we must research the flow behavior inside to the face 1, but in the complement of Cherry box flow. However, we can to observe that the behavior flow is extended to the whole circle. This finishes the first step.

We must observe the flow behavior on the face 2. In this case, is easy to verify that all trajectories are crossing inward to the face 2. Thus, the flow through the two faces is depicted in Figure 3.13.

Now, we construct a plug Y containing a singularity σ_2. Consequently, the dynamical system can be transferred by means of plug Y surgery from one bitorus onto another manifold exporting some of its properties. This singularity generates a hole and this in turns generates a solid tritorus ST_2 in a way that $Y_t(ST_2) \subset Int(ST_2)$ for all $t > 0$ and Y is transverse to the boundary tritorus. The flow is obtained gluing the plugs 3.2, 3.3 with the plug Y as indicated in Figure 3.14. Indeed, the third hole is generated by the unstable manifold of the singularity σ_2.

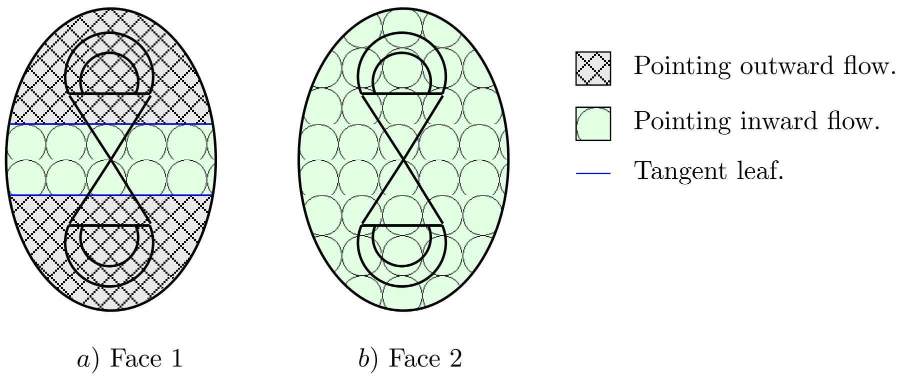


Figure 3.13: Direction of flow through the faces.

In the same way of previous subsection, we require some hypotheses for the ambient manifold (after of gluing).

($\hat{X}$1): There are two repelling periodic orbits O_1, O_2 in $Int(ST_2)$ crossing the holes of R.

($\hat{X}$2): There are two solid tori neighborhoods $V_1, V_2 \subset Int(ST_2)$ of O_1, O_2 with boundaries transverse to Y_t such that if $N = ST_2 \setminus (V_1 \cup V_2)$, then N is a compact neighborhood with smooth boundary transverse to Y_t and $Y_t(N) \subset N$ for $t > 0$. As N is a solid tritorus with two solid tori removed, we have that N is connected.

($\hat{X}$3): $R \subset N$ and the return map G induced by Y in R satisfies the properties **(G1)-(G3)** in Section 3.3.3. Moreover,

$$\{q \in N : Y_t(q) \notin R, \forall t \in \mathbb{R}\} = Cl(W^{uu}_Y(\sigma_2)).$$

Now, we define

$$\hat{A}^+ = Cl\left(\bigcup_{t\in\mathbb{R}} Y_t(A^+_G)\right) \qquad \text{and} \qquad \hat{A}^- = Cl\left(\bigcup_{t\in\mathbb{R}} Y_t(A^-_G)\right).$$

By using the Propositions 3.4.1, 3.4.3 and Lemma 3.4.2 we can obtain that the intersection of homoclinic classes is the closure of the unstable manifold of two singularities.

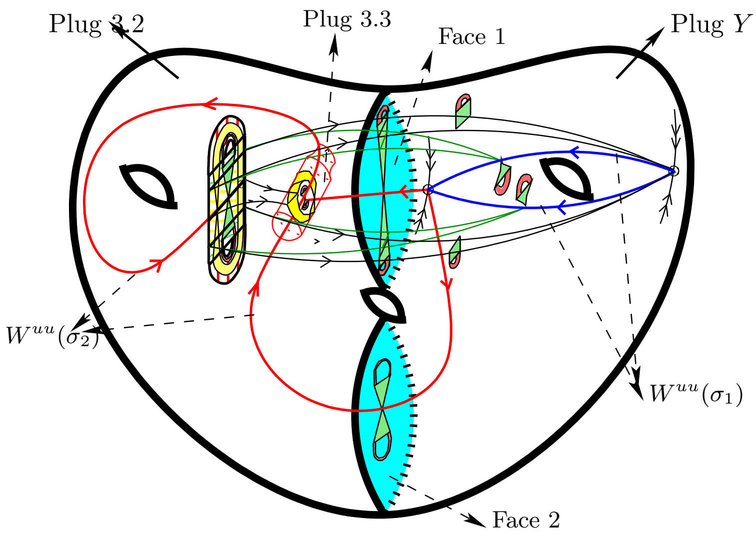


Figure 3.14: Plug Y

Proof of Theorem B.

By using the Lemma 3.4.2 and the Proposition 3.4.3 we have that Y is a sectional Anosov flow and $N(Y)$ is the union of two homoclinic classes $\mathcal{H}_Y^1, \mathcal{H}_Y^2$, where $\mathcal{H}_Y^1 = \hat{A}^+$ and $\mathcal{H}_Y^2 = \hat{A}^-$. It implies that $\mathcal{H}_Y^1 \cap \mathcal{H}_Y^2 = Cl(W_Y^u(\sigma_1) \cup W_Y^u(\sigma_2))$. In particular Y is a Venice mask, and by construction it has two singularities.

CHAPTER

FOUR

GENERATING NEW EXAMPLES SUPPORTED ON 3-MANIFOLDS

For each $n \in \mathbb{N}$, we show the existence of Venice masks containing n equilibria on certain compact 3-manifolds. These examples are characterized because of the maximal invariant set is finite union of homoclinic classes. Here, the intersection between two different homoclinic classes is contained in the closure of the union of unstable manifolds of the singularities. This is a joint work with Andrés M. López Barragán.

4.1 Introduction

As we already mention, there are examples of sectional-Anosov flows non-transitive with dense periodic orbits supported on compact three dimensional manifolds. An example of Venice mask with a unique singularity was given in [10], and for three singularities was provided in [32]. Recently, [21] showed the construction the examples with two equilibria, which were the exhibited in Chapter 2.

All these flows have the common property that the maximal invariant set is union non disjoint of two homoclinic classes, and the intersection between their classes is contained in the closure of the union of unstable manifolds of the singularities.

The above observations motivate the following questions,

1. It is possible to obtain Venice masks with more singularities ?

2. The maximal invariant set of every Venice mask is union of two homoclinic classes?

3. How is the intersection of these homoclinic classes?

The answer to the first question is positive. We use the ideas developed in [21] and [32] for the construction of these examples, which provide more tools and clues for a general theory of Venice masks. In particular, we construct an example with five singularities which is non-disjoint union of three homoclinic classes. So, the answer to the second question is false.

Theorem C. *For each $n \in \mathbb{N}$ there exists a Venice mask X with n singularities supported on a compact 3-manifold M, such that:*

- *$M(X)$ can be decomposed as finite union of homoclinic classes .*
- *The intersection of two different homoclinic classes of $M(X)$ is contained in $Cl(W^u(Sing(X)))$.*

In section 4.2, we describe briefly the construction and some important properties for the known examples with two and three singularities. In section 4.3, using the techniques of the Venice masks with two singularities, we provide an example with four singularities. In the same way, in Section 4.4, by using the Venice mask with three singularities, the example with five equlibria will be obtained. Theorems 4.3.2 and 4.4.2 will be consequence of a inductive process. Finally, *Theorem* C will be a direct consequence of *Theorem* 4.3.2 and *Theorem* 4.4.2.

4.2 Preliminaries

We make a brief description about the known Venice masks.

An example with a unique singularity was given in [10], and in [32] was proved that every Venice mask $X_{(1)}$ with one equilibrium satisfies the following properties:

- $M(X_{(1)})$ is union of two homoclinic classes $H^1_{X_{(1)}}$, $H^2_{X_{(1)}}$.
- $H^1_{X_{(1)}} \cap H^2_{X_{(1)}} = Cl(W^u_{X_{(1)}}(\sigma))$ where σ is the singularity of $X_{(1)}$.

In [21] were exhibited two Venice masks containing two equilibria σ_1, σ_2.

For the first example we have a vector field X verifying:

- $M(X)$ is the union of two homoclinic clases H^1_X, H^2_X.
- $H^1_X \cap H^2_X = O$, where O is a hyperbolic periodic orbit.
- $O = \omega_X(q)$, for all $q \in W^u_X(\sigma_1) \cup W^u_X(\sigma_2) \setminus \{\sigma_1, \sigma_2\}$.

The vector field Y that determines the second example with two singularities σ_1, σ_2 satisfies:

- $M(Y)$ is the union of two homoclinic clases H^1_Y, H^2_Y.
- $H^1_Y \cap H^2_Y = Cl(W^u_Y(\sigma_1) \cup W^u_Y(\sigma_2))$.

An essential element to obtain the examples with two singularities is the existence of a return map defined in a cross section R. A foliation $\mathcal{F}$ is defined on R, which has vertical segments in the rectangular components B, C, D, E and radial segments in the annuli components A, F.

We are interested to take the C^∞ two-dimensional map $\tilde{G} : R \setminus \{d^-, d^+\} \to Int(R)$ given in Section 3.3.4, satisfying the hypotheses **(L1)-(L3)** established there. In particular, **(L1)** and **(L2)** imply the contraction and the invariance of the leaf l by $\tilde{G}$. So, the map $\tilde{G}$ has a fixed point $P \in l$. We define $H^+ = A \cup B \cup C$ and $H^- = D \cup E \cup F$. For

$$A^-_{\tilde{G}} = Cl\left(\bigcap_{n\geq 1} \tilde{G}^n(H^-)\right), \qquad A^+_{\tilde{G}} = Cl\left(\bigcap_{n\geq 1} \tilde{G}^n(H^+)\right)$$

follow that $A^+_{\tilde{G}}$ and $A^-_{\tilde{G}}$ are homoclinic classes and $\{P\} = A^+_{\tilde{G}} \cap A^-_{\tilde{G}}$.

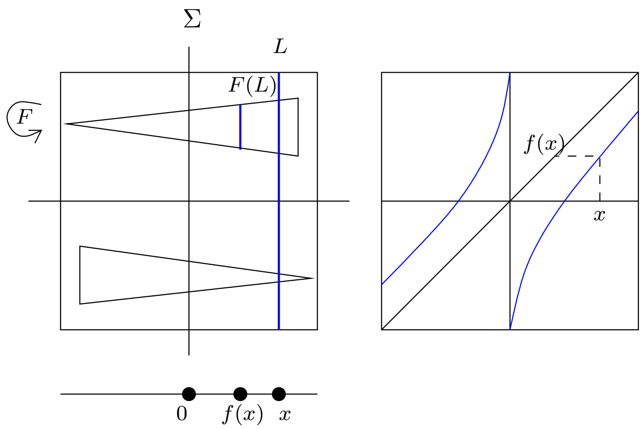


Figure 4.1: Map F.

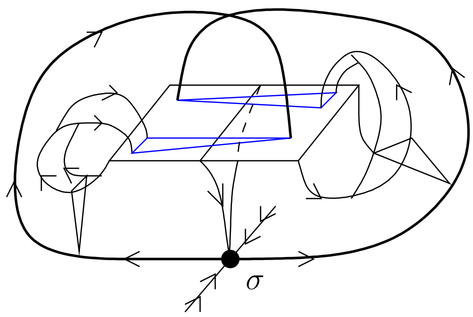


Figure 4.2: Geometric Lorenz Attractor.

The mode to obtain the example with three singularities described in [32] is easier. First of all, is important to know some properties about the dynamic of the Geometric Lorenz Attractor (GLA for short) [15].

In [4] was proved that this attractor is a homoclinic class. The result is obtained due to the existence of a return map F for the flow, defined on a cross section Σ. This map preserves the stable foliation $\mathcal{F}^s$, where the leaves are vertical lines. The induced map f in the leaf space is differentiable and expansive.

The GLA is modified in [32] by adding two singularities to the flow located at $W^u(\sigma)$. We called this modification as GLA_{mod}. We glue together in a C^∞ fashion two copies of this flow along the unstable manifold of the singularity σ, thus generating the flow depicted in Figure 4.3. In this way is obtained a sectional-Anosov flow $X_{(3)}$ with dense periodic orbits

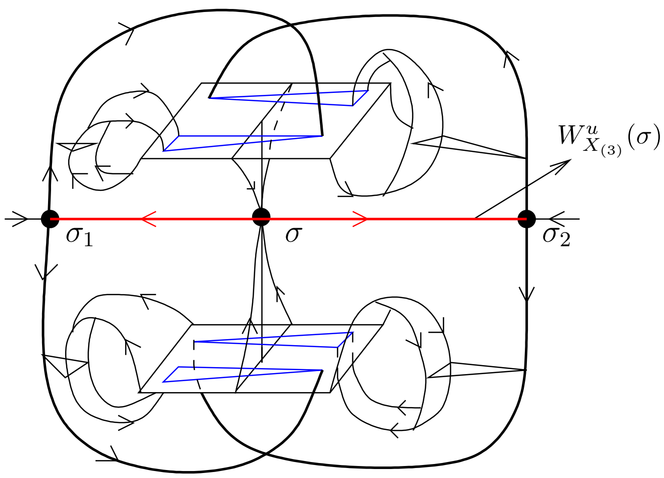


Figure 4.3: Example with three singularities.

and three equilibria whose maximal invariant set is non-disjoint union of two homoclinic classes. In this case, the intersection between the homoclinic classes is $Cl(W^u_{X_{(3)}}(\sigma))$.

Observe that this flow is supported on a handlebody of genus 4.

4.3 Venice mask's examples with an even number of singularities

4.3.1 Vector field Z

We provide an example with four singularities. We start with the vector field X associated to the Venice mask with two singularities. Then, we construct a plug Z containing two addittional equilibria σ_3, σ_4. In this way, the flow is obtained through plug Z surgery from one solid tritorus onto another manifold exporting some of its properties.

The vector field X is supported on a solid tritorus ST_1. Now, we remove a connected component B of ST_1 as in Figure 4.4.

The behavior across the faces removed is similar with respect to observed in the example given by the vector field Y in Section 3.4.2.

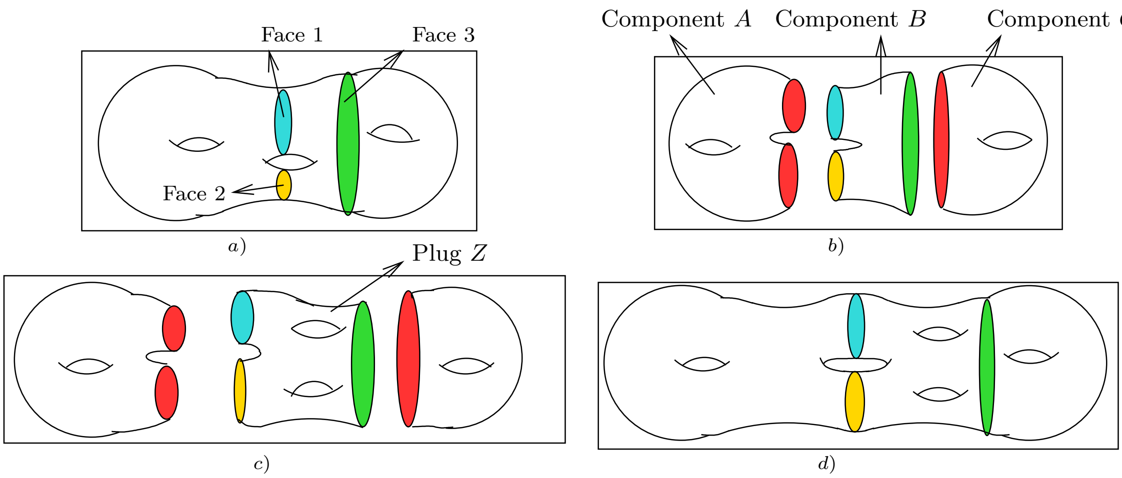


Figure 4.4: Steps by gluing the new plug.

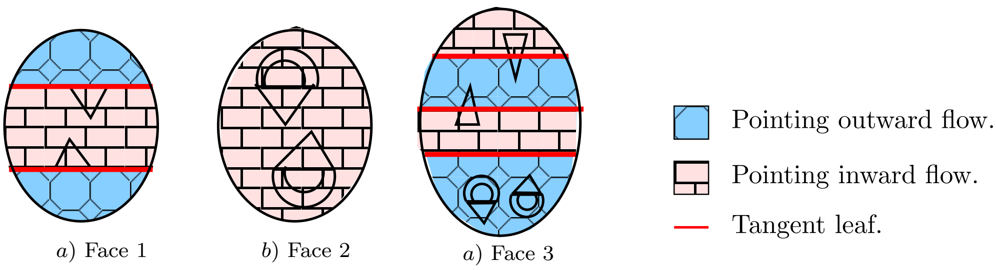


Figure 4.5: Faces.

On face 1, we identify three regions determinated by the singular leaves saturated by the flow. In the middle region on face 1, the trajectories crossing inward to ∂A, such as the branch unstable manifold of the two initial singularities. On face 3 there are three singular leaves which generate four regions such as is exhibited in Figure 4.5. There, the flow crossing inward and outward to ∂C. All trajectories are crossing inward to face 2 as ∂A.

As we before mention, will be constructed an adequate plug Z to include the additional equilibria. We ask the singularities to be Lorenz-like . On the other hand, two holes are generated by the unstable manifold of the singularities σ_1, σ_2 respectively. Therefore, we obtain a handlebody HB_5 of genus five. So, the vector field Z produced by gluing plug Z instead the removed connected component B, satisfies $Z_t(HB_5) \subset Int(HB_5)$ for all $t > 0$. Moreover Z is transverse to the boundary handlebody.

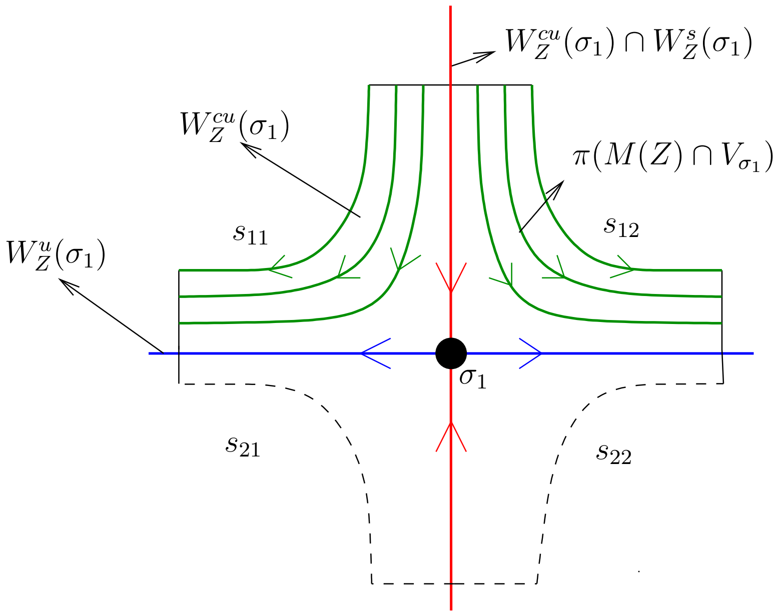


Figure 4.6:

We exhibit with details the behavior near to the singularities. For that, we mention some facts that appear in [26]. As every singularity is Lorenz-like, there exists a center unstable manifold $W^{cu}_Z(\sigma_i)$ associated to σ_i $(i = 1, 2, 3, 4)$. It is divided by $W^u_Z(\sigma_i)$ and $W^s_Z(\sigma_i) \cap W^{cu}_Z(\sigma_i)$ in the four sectors $s_{11}, s_{12}, s_{21}, s_{22}$. There is also a projection $\pi : V_{\sigma_i} \to W^{cu}_Z(\sigma_i)$ defined in a neighborhood V_{σ_i} of σ_i via the strong stable foliation of the maximal invariant set associated to flow.

For $\sigma \in Sing(Z)$, we define the matrix

$$A(\sigma) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$a_{ij} = \begin{cases} 1 & \textit{if } \sigma \in Cl(\pi(M(Z) \cap V_\sigma)) \cap s_{ij} \\ 0 & \textit{if } \sigma \notin Cl(\pi(M(Z) \cap V_\sigma)) \cap s_{ij}. \end{cases}$$

$A(\sigma)$ does not depend on the chosen center unstable manifold $W^{cu}_Z(\sigma)$.

Figure 4.6 shows the case for the singularity σ_1 of the example.

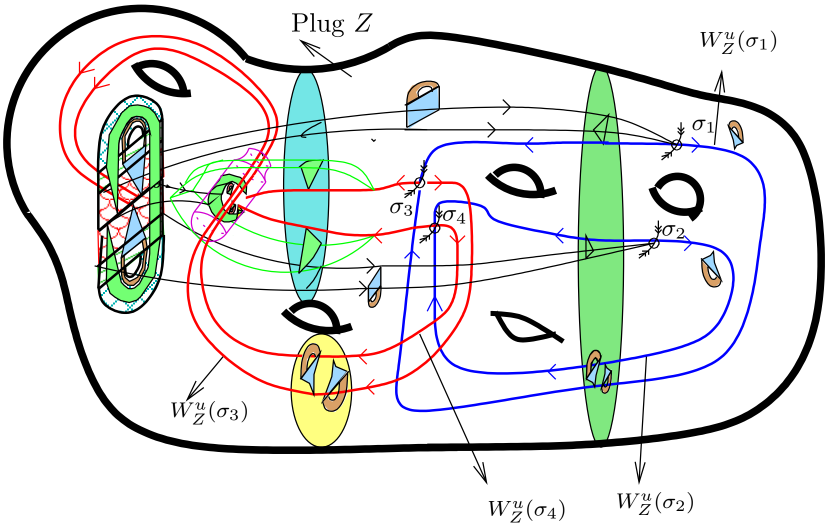


Figure 4.7: Venice mask with four singularities.

These are the associated matrices to the singularities of our vector field Z.

$$A_{\sigma_1} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_{\sigma_2} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_{\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{\sigma_4} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, we consider the following hypotheses.

(Z1): There are two repelling periodic orbits O_1, O_2 in $Int(HB_5)$ crossing the holes of R.

(Z2): There are two solid tori neighborhoods $V_1, V_2 \subset Int(HB_5)$ of O_1, O_2 with boundaries transverse to Z_t such that if $N_4 = HB_5 \setminus (V_1 \cup V_2)$, then N_4 is a compact neighborhood with smooth boundary transverse to Z_t and $Z_t(N_4) \subset N_4$ for $t > 0$. N_4 is a handlebody of genus five with two solid tori removed.

(Z3): $R \subset N_4$ and the return map $\tilde{G}$ induced by Z in R satisfies the properties **(L1)-(L3)** given in Section 3.3.4. Moreover,

$$\{q \in N : Z_t(q) \notin R, \forall t \in \mathbb{R}\} = Cl(W^u_Z(\sigma_1) \cup W^u_Z(\sigma_2)).$$

We define

$$A^+_Z = Cl\left(\bigcup_{t\in\mathbb{R}} Z_t(A^+_{\tilde{G}})\right) \quad \text{and} \quad A^-_Z = Cl\left(\bigcup_{t\in\mathbb{R}} Z_t(A^-_{\tilde{G}})\right).$$

Proposition 4.3.1. *Z is a Venice mask with four singularities supported on the compact 3-manifold N_4. $N_4(Z)$ is the union of two homoclinic classes A^+_Z, A^-_Z. The intersection between A^+_Z and A^-_Z is a hyperbolic periodic orbit O contained in $Cl(W^u_Z(\sigma_3) \cup W^u_Z(\sigma_4))$.*

Proof. By construction Z has four singularities. The proof to be A^+_Z, A^-_Z homoclinic classes is the same given in [21]. Also the fact to be Z a Venice mask. The intersection between the homoclinic classes is reduced to a hyperbolic periodic orbit O because of $\{P\} = A^+_{\tilde{G}} \cap A^-_{\tilde{G}}$ and by hypotheses **(Z3)**. Here, $O = O_Z(P)$. We observe that the branches of the unstable manifolds of σ_3 and σ_4 intersect the leaf l of the foliation $\mathcal{F}$ in R. Then the hypotheses **(L1)**, **(L2)** of the map $\tilde{G}$, and the invariance of the flow imply $O \subset \omega_Z(q)$ for all regular point $q \in W^u_Z(\sigma_3) \cup W^u_Z(\sigma_4)$. As $W^u_Z(\sigma_3) \subset A^+_Z$ and $W^u_Z(\sigma_4) \subset A^-_Z$ (see *Proposition 4.1* [21]) we conclude $A^+_Z \cap A^-_Z \subset Cl(W^u_Z(\sigma_3) \cup W^u_Z(\sigma_4))$. □

4.3.2 General case

We expose a general result. More specifically the following theorem holds.

Theorem 4.3.2. *For every n even, there exists a Venice mask $X_{(n)}$ with n singularities supported on a handlebody N_n of genus $n+1$ with two solid tori removed. $N_n(X_{(n)})$ is the non-disjoint union of two homoclinic classes, and the intersection between them is a hyperbolic periodic orbit contained in $Cl(W^u(Sing(X_{(n)})))$.*

Proof. Consider $n = 2k$ with $k \geq 3$. Again, we remove the same connected component B to the manifold that supports the Venice mask X with two equilibria. We glue a plug Z_n containing $n-2 = 2k-2$ Lorenz-like singularities. For each singularity in Z_n, we have a connection of saddle-type between $W^u_{X_{(n)}}(\sigma_i)$ and $W^s_{X_{(n)}}(\sigma_{i+2})$, $i = 1, \ldots, n-2$. Figure 4.8 exhibits the particular case for Plug Z_6.

So, the new manifold is a handlebody HB_{n+1} of genus $n+1$ and supports a flow $X_{(n)_t}$ with $n = 2k$ equilibria. The flow is obtained by gluing plug Z_n instead the connected component B. In this way, the vector field $X_{(n)}$ on HB_{n+1} satisfies $X_{(n)_t}(HB_{n+1}) \subset Int(HB_{n+1})$ for all $t > 0$. In addition, $X_{(n)}$ is transverse to the boundary handlebody.

Here,

$$A_{\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{\sigma_4} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{\sigma_{2k-1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad A_{\sigma_{2k}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$k = 3, \ldots, n/2$.

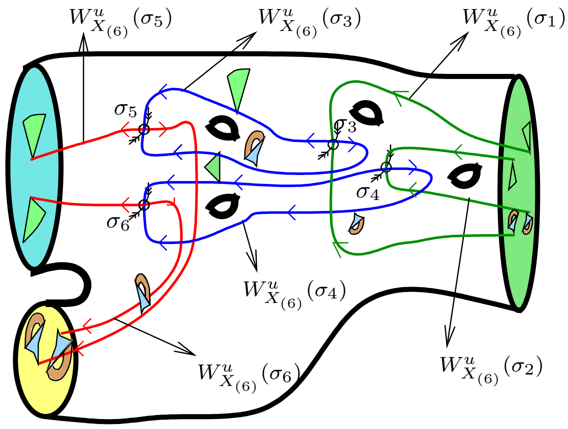


Figure 4.8: Plug Z_6.

We assume $X_{(n)}$ satisfying the hypotheses:

(Z_n1): There are two repelling periodic orbits O_1, O_2 in $Int(HB_{n+1})$ crossing the holes of R.

(Z_n2): There are two solid tori neighborhoods $V_1, V_2 \subset Int(HB_{n+1})$ of O_1, O_2 with boundaries transverse to $X_{(n)_t}$ such that if $N_n = HB_{n+1} \setminus (V_1 \cup V_2)$, then N_n is a compact neighborhood with smooth boundary transverse to $X_{(n)_t}$ and $X_{(n)_t}(N_n) \subset N_n$ for $t > 0$. N_n is a handlebody of genus $n+1$ with two solid tori removed.

(Z_n3): $R \subset N_n$ and the return map $\tilde{G}$ induced by $X_{(n)}$ in R satisfies the properties **(L1)-(L3)** given in Section 3.3.4. Moreover,

$$\{q \in N_n : X_{(n)_t}(q) \notin R, \forall t \in \mathbb{R}\} = Cl\left(\bigcup_{m=1}^{n-2} W^u_{X_{(n)}}(\sigma_m)\right).$$

We define

$$A^+_{X_{(n)}} = Cl\left(\bigcup_{t\in\mathbb{R}} X_{(n)_t}(A^+_{\tilde{G}})\right) \qquad \text{and} \qquad A^-_{X_{(n)}} = Cl\left(\bigcup_{t\in\mathbb{R}} X_{(n)_t}(A^-_{\tilde{G}})\right).$$

$A^+_{X_{(n)}}$ and $A^-_{X_{(n)}}$ are homoclinic classes for $X_{(n)}$. Moreover $A^+_{X_{(n)}} \cup A^-_{X_{(n)}} = N_n(X_{(n)})$ and $A^+_{X_{(n)}} \cap A^-_{X_{(n)}} = O$, where $O = O_{X_{(n)}}(P)$ with P the fix point associated to map $\tilde{G}$ defined in R.

The proof follows the same ideas to construct the example with four singularities. □

4.4 Venice mask's examples with an odd number of singularities

As was observed in Section 4.2, Venice masks containing one or three equilibria have already been developed. To continue, we provide an example with five singularities. The idea is very simple. We just proceed such as the process made to obtain the vector field $X_{(3)}$.

First of all, the GLA as sectional-Anosov flow, is supported on a solid bitorus (see [4]). The holes on the manifold are produced because of the branches of the unstable manifold of the saddle-type singularity. Therefore, $X_{(3)}$ is a Venice mask defined on a handlebody of genus 4. The holes are generated by the branches of the unstable manifolds of σ_1 and σ_2.

Now, for the vector field $X_{(3)}$, we add two Lorenz-like singularities located at the branches of $W^u_{X_{(3)}}(\sigma_2)$. We glue together in a C^∞ fashion one copy of GLA_{mod} along the unstable manifold of the singularity σ_2. Thus is obtained the vector field $X_{(5)}$ whose flow is depicted in Figure 4.9.

For each $i = 1, 2, 3$, there is a cross section Σ_i and return map F_i such that

$$\Lambda_i = Cl\left(\bigcap_{n\geq 0} F_i^n(\Sigma_i)\right)$$

is a homoclinic class for F_i. Therefore

$$H_i = Cl\left(\bigcup_{t\in\mathbb{R}} X_{(5)}(\Lambda_i)\right)$$

is a homoclinic class for flow $X_{(5)}$. Moreover, $H_1 \cap H_2 \subset Cl(W^u_{X_{(5)}}(\sigma))$, $H_1 \cap H_3 \subset Cl(W^u_{X_{(5)}}(\sigma_2))$ and $H_2 \cap H_3 \subset Cl(W^u_{X_{(5)}}(\sigma_2))$.

Proposition 4.4.1. *$X_{(5)}$ is a Venice mask supported on a handlebody HB_6 of genus 6. The maximal invariant set $HB_6(X_{(5)})$ is non-disjoint union of three homoclinic classes. The intersection between two different homoclinic classes is contained in $Cl(W^u(Sing(X_{(5)})))$.*

It is possible to continue gluing copies of GLA_{mod} to produce Venice masks with any odd number of equilibria. Each copy is glued along the unstable manifold of some singularity σ_i. The equilibrium σ_i is chosen such that were previously possible to add two Lorenz-like

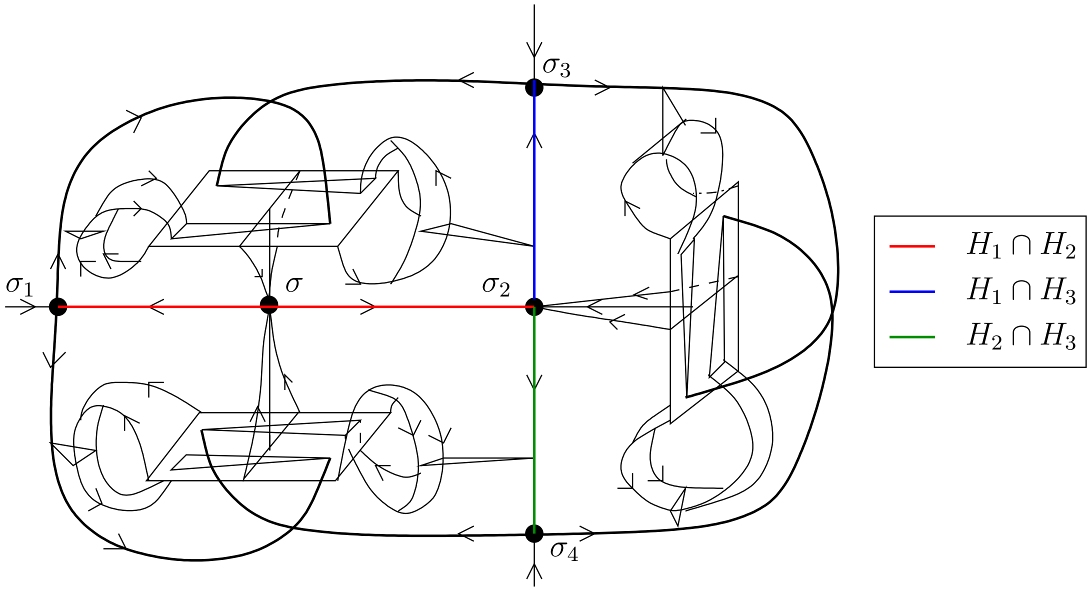


Figure 4.9: Venice mask with five singularities.

singularities in its unstable manifold, one on each branch. More specifically, each σ_i is selected to add two new singular points if previously we have

$$A_{\sigma_i} \neq \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

In this way, the following theorem holds.

Theorem 4.4.2. *For every n odd, there exists a Venice mask $X_{(n)}$ with n singularities supported on a handlebody HB_{n+1} of genus $n+1$. The maximal invariant set $HB_{n+1}(X_{(n)})$ is non-disjoint union of $(n+1)/2$ homoclinic classes. The intersection between two different homoclinic classes is contained in $Cl(W^u(Sing(X_{(n)})))$.*

Theorem C follows from *Theorem* 4.3.2 and *Theorem* 4.4.2.

CHAPTER

FIVE

INTERSECTION OF HOMOCLINIC CLASSES IN VENICE MASKS

5.1 Introduction

In search of properties which allow to characterized the dynamic of Venice masks , will be studied the behavior of homoclinic classes and its relation with the unstable manifolds of the singularities. As was seen in previous chapters, all known examples of Venice masks have the maximal invariant set as finite union of homoclinic classes. Moreover in a Venice mask X, the intersection between two different homoclinic classes is contained in $Cl(W^u(Sing(X)))$. Specifically, this intersection can be decomposed as the disjoint union of, a singularity σ, a closed orbit C, and regular points such that its *alpha-limit set* is σ and the *omega-limit set* is C.

As we mention, the dynamical systems theory is interested to describes the behavior as time goes to infinity for the majority of orbits in a determinated system. An important tool for hyperbolic sets is the known *connecting lemma* [16], [2], [11]. Specifically, the lemma says that if X is an Anosov flow on a compact manifold M and $p, q \in M$ satisfy that for all $\varepsilon > 0$ there is a trajectory from a point $\varepsilon-$close to p to a point ε-close to q, then there is a point $x \in M$ such that $\alpha_X(x) = \alpha_X(p)$ and $\omega_X(x) = \omega_X(q)$. In [8] was proved a similar result for sectional-Anosov flows, which is known as *sectional-connecting lemma.* A fundamental hypothesis in the sectional-hyperbolic case consists in the alpha-limit set of $p \in M(X)$ to be non-singular.

On the other hand, the unstable manifold of every singularity σ of a sectional-Anosov X is contained in the maximal invariant set $M(X)$. Would be interesting to know what is the omega-limit set of a point in $W^u_X(\sigma)$. In fact, it can be seen as a extension of the *sectional-connecting lemma*. Here, we give an answer when the vector field is a Venice mask.

5.2 Main statements

We show that if X is a *Venice mask* supported on a compact 3-manifold, then the omega-limit set of every non-recurrent point in the unstable manifold of some singularity is a closed orbit. In addition, we prove that the intersection of two different homoclinic classes in the maximal invariant set of a sectional-Anosov flow can be decomposed as the disjoint union of, singular points, a non-singular hyperbolic set, and regular points whose *alpha-limit set* and *omega-limit set* is formed by singular points or hyperbolic sets.

Specifically, we have the following statements.

Theorem D. *If X is a Venice mask and σ is a singularity of X, then for all $q \in W^u_X(\sigma)$ such that q is non-recurrent we have the following dichotomy:*

- $\omega_X(q) \in Sing(X)$.
- $\omega_X(q) = O$, *where O is a hyperbolic periodic orbit.*

Theorem E. *The intersection of two different homoclinic classes H_1, H_2 in the maximal invariant set of a sectional-Anosov flow X is the disjoint union of a set S (possibly empty) of singularities, a non-singular hyperbolic set H (possibly empty), and a set R (possibly empty) of regular points such that if $q \in R$ then $\alpha_X(q) \subset H \cup S$ and $\omega_X(q) \subset H \cup S$.*

5.3 Preliminary results

We mention the following results which are essentials to proving the theorems.

Theorem 5.3.1 ([33]). *Let Λ be a sectional-hyperbolic set with dense periodic orbits. Then, every $\sigma \in Sing_X(\Lambda)$ is Lorenz-like and satisfies $\Lambda \cap \mathcal{F}^{ss}_X(\sigma) = \{\sigma\}$.*

We observe that $W^s_X(\sigma) \setminus \mathcal{F}^{ss}_X(\sigma)$ is decomposed by two connected components $W^{s,+}_X(\sigma)$ and $W^{s,-}_X(\sigma)$ (see figure 5.3). Hence for a Venice mask, a regular point in $M(X)$ contained in the stable manifold of some singularity σ, necessarily is contained either $W^{s,+}_X(\sigma)$ or $W^{s,-}_X(\sigma)$.

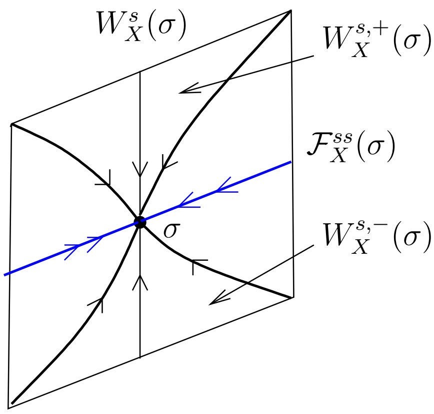


Figure 5.1: Connected components.

Lemma 5.3.2 (Hyperbolic lemma [33]). *A compact invariant set without singularities of a sectional-hyperbolic set is hyperbolic saddle-type.*

Remark 5.3.3. *Theorem 5.3.1 and the Hyperbolic Lemma imply that every Venice mask has singularities, and these are Lorenz-like.*

Definition 5.3.4. *We say that a C^1 vector field X with hyperbolic closed orbits has the Property (P) if for every periodic orbit O there is a singularity σ such that*

$$W^u_X(O) \cap W^s_X(\sigma) \neq \emptyset. \tag{5.1}$$

The above definition is useful by the interesting fact below.

Lemma 5.3.5. *Every point in the closure of the periodic orbits of a vector field with the Property (P) is accumulated by points for which the omega-limit set is a singularity.*

Moreover, we have an important property.

Lemma 5.3.6 ([32]). *Every sectional-Anosov flow with singularities and dense periodic orbits on a compact 3-manifold has the Property (P).*

Remark 5.3.7. *By Lemma 5.3.5 and Lemma 5.3.6 we can assert that every Venice mask X has the Property (P) and $W^s(Sing(X)) \cap M(X)$ is dense in $M(X)$.*

Definition 5.3.8. *Given $\Sigma \subset M$ we say that $q \in M$ satisfies Property $(P)_\Sigma$ if $Cl(O^+(q)) \cap \Sigma = \emptyset$ and there is open arc I in M with $q \in \partial I$ such that $O^+(x) \cap \Sigma \neq \emptyset$ for every $x \in I$.*

We finish to exhibit the preliminar statements with the following characterization.

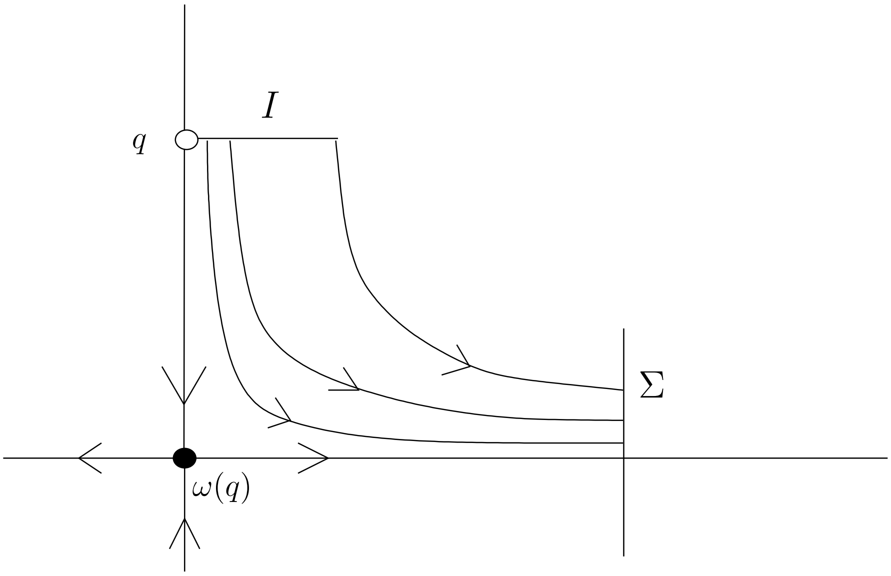


Figure 5.2: Property $(P)_\Sigma$.

Theorem 5.3.9 ([7]). *Let X be a C^1 vector field in a compact 3-manifold M. If $q \in M$ has sectional-hyperbolic omega-limit set $\omega(q)$, then the following properties are equivalent:*

- *$\omega(q)$ is a closed orbit.*
- *q satisfies $(P)_\Sigma$ for some closed subset Σ.*

In Figure 5.2 is exhibited the case when the omega-limit set $\omega(q)$ of the point q is a hyperbolic singularity of saddle-type.

5.4 Characterizing the omega-limit set

In this section we will prove the *Theorem D*. The idea is to consider a sequence of points satisfying the Property $(P)_\Sigma$, which approximates a point q in the unstable manifold of a fixed singularity. We show that q satisfies the Property $(P)_\Sigma$ too. Hereafter in this section, we assume that every regular point $q \in W^u(Sing(X))$ is non-recurrent.

First, we mention some facts of topology. Given a compact metric space (Y, d), define a distance function between any point x of Y and any non-empty set B of Y by:

$$d(x, B) = \inf\{d(x, y) | y \in B\}.$$

Now, consider the collection $\mathcal{C}(Y) = \{C \in Y : C \text{ is a non-empty compact subset of } (Y, d)\}$. For $\mathcal{C}(Y)$, take the Hausdorff metric d_H defined as the distance function between any two non-empty sets A and B of Y by:

$$d_H(A, B) = \sup\{d(x, B) | x \in A\}.$$

Lemma 5.4.1. *Let $\{A_n : n \in \mathbb{N}\}$ be a sequence of closed sets contained in a compact metric space (Y, d), such that $A_n \to A$ in the Hausdorff metric induced by d. Then $\partial A_n \to \partial A$.*

For now and on this section, let M be a riemaniann compact 3-manifold, and let X be a Venice mask on M. So, for a hyperbolic point p of X, $W^s_X(p)$ is just denoted by $W^s(p)$. The same interchanging s by u.

5.4.1 Existence of singular partitions

We introduce the following definition which extends the notion given in [30]. This can also be found in [5] and [6].

A cross section of X is a codimension one submanifold S transverse to X. We denote the interior and the boundary (in topological sense) of S by $Int(S)$ and ∂S respectively. If $\mathcal{R} = \{S_1, \cdots, S_k\}$ is a collection of cross sections we still denote by $\mathcal{R}$ the union of its elements. Moreover

$$\partial\mathcal{R} := \bigcup_{i=1}^{k} \partial S_i \qquad \textit{and} \qquad Int(\mathcal{R}) := \bigcup_{i=1}^{k} Int(S_i)$$

The size of $\mathcal{R}$ will be the sum of the diameters of its elements.

Definition 5.4.2. *A singular partition of an invariant set H of a vector field X is a finite disjoint collection $\mathcal{R}$ of cross sections of X such that $H \cap \partial\mathcal{R} = \emptyset$ and*

$$H \cap Sing(X) = \{y \in H : X_t(y) \notin \mathcal{R}, \forall t \in \mathbb{R}\}.$$

We remember a fact mentioned in *Section* 4.3. For a Lorenz-like singularity σ, the center unstable manifold $W^{cu}_X(\sigma)$ associated is divided by $W^u(\sigma)$ and $W^s(\sigma) \cap W^{cu}(\sigma)$ in the four sectors s_{11}, s_{12}, s_{21}, s_{22} (see Figure 4.6). $\pi : V_\sigma \to W^{cu}(\sigma)$ is the projection defined in a neighborhood V_σ of σ.

Lemma 5.4.3. *Consider σ a Lorenz-like singularity of a Venice mask X, and O a hyperbolic periodic orbit satisfying $Cl(W^u(O)) \cap W^{s,+}(\sigma) \neq \emptyset$ and $Cl(W^u(O)) \cap W^{s,-}(\sigma) \neq \emptyset$. Moreover, $\pi(Cl(W^u(O))) \cap s_{1i} \neq \emptyset$ and $\pi(Cl(W^u(O))) \cap s_{2i} \neq \emptyset$ for some $i \in \{1, 2\}$. If q is a regular point in $W^u(\sigma) \cap Cl(s_{1i}) \cap Cl(s_{2i})$, then $O = \omega_X(q)$.*

Proof. We take $q \in W^u(\sigma)$ a regular point close to σ. We assert that $q \in W^s(O)$. Indeed, if we suppose that is not the case, we will get a contradiction.

So, we assume $q \in W^u(\sigma) \setminus W^s(O)$. Then, there is a sequence $p_n^- \to q$ such that $p_n^- \in W^u(O) \cap W^s(O)$ for all n. In addition, $\{O_X(p_n^-) : n \in \mathbb{N}\}$ accumulates some regular point p^- in $W^{s,-}(\sigma)$ or in $W^{s,+}(\sigma)$. We can suppose the accumulation in some point of $W^{s,-}(\sigma)$. Also, we can take $\{p_n^+ : n \in \mathbb{N}\} \subset W^u(O)$ be a sequence such that $p_n^+ \to q$. Moreover, $\{O_X(p_n^+) : n \in \mathbb{N}\}$ accumulates σ and some point p^+ in $W^{s,+}(\sigma)$. We have $p_n^+, p_n^- \notin W^u(\sigma)$ for all n. On the other hand, $q \in Cl(W^u(O))$ and the invariance of $W^u(\sigma)$ imply $O_X(q) \subset Cl(W^u(O))$. But $Cl(W^u(O))$ is a closed set, therefore $Cl(O_X(q)) \subset Cl(W^u(O))$. Applying the compactness of $Cl(W^u(O))$ and Tubular Flow Box Theorem [36] in a neighborhood of $O^+(q)$ we obtain that $\{O^+(p_n^+) : n \in \mathbb{N}\}$ and $\{O^+(p_n^-) n \in \mathbb{N}\}$ acummulate all point close $\omega_X(q)$.

As O and $\omega_X(q)$ are invariant closed sets, then they are disjoints and $d(x, \omega_X(q)) > 0$ for all $x \in O$. This implies that there exists $\varepsilon > 0$ such that every point y closen to $\omega_X(q)$ satisfies $d(y, O) > \varepsilon$. Moreover $y \notin O_X(q)$ and, $\{O^+(p_n^+) : n \in \mathbb{N}\}$, $\{O_X^+(p_n^-) : n \in \mathbb{N}\}$ acummulate y. The positive orbits of p_n^+ and p_n^- cannot intersect $\omega_X(q)$. So, we have two possibilities, either any orbit intersects $O_X(q)$, or no orbit does it. The first case means that there is a point $w \in W^u(\sigma) \cap W^u(O)$ which is absurd. So, neither orbit intersects $O_X(q)$. Now, q is a non-recurrent point. Then, $\{O_X^+(p_n^+) : n \in \mathbb{N}\}$ does not accumulate on $W^{s,+}(\sigma)$. But this contradicts the choice of the sequences. Therefore $q \in W^s(O)$. So, we conclude $O = \omega_X(q)$.

□

From *Lemma* 5.4.3 we obtain the following corollary.

Corollary 5.4.4. *Consider σ a Lorenz-like singularity of a Venice mask X, and O a hyperbolic periodic orbit satisfying $W^u(O) \cap W^{s,+}(\sigma) \neq \emptyset$ and $W^u(O) \cap W^{s,-}(\sigma) \neq \emptyset$. Let q be a regular point in $W^u(\sigma) \cap Cl(W^u(O))$ such that for $\{p_n : n \in \mathbb{N}\} \subset Cl(W^u(O)) \cap W^s(O)$ and $p_n \to q$. Then $p_n \in O_X(q)$.*

Proof. For this is sufficient to observe that for $p_n \in W^s(O) \cap Cl(W^u(O))$ such that $p_n \to q$, then $p_n \in O_X(q)$ for all n large.

□

Remark 5.4.5. Corollary *5.4.4 says that for $i \in \{1,2\}$ and for every hyperbolic periodic orbit O of X, is not possible $H(O) \cap s_{1i} \neq \emptyset$ and $H(O) \cap s_{2i} \neq \emptyset$ simultaneously.*

Lemma 5.4.6. *Let σ be a singularity of a Venice mask X, and let O be a hyperbolic periodic orbit such that $W^u(O) \cap W^s(\sigma) \neq \emptyset$. Then for $q \in W^u(\sigma) \setminus \{\sigma\}$, $\omega_X(q)$ has singular partitions of arbitrarily small size.*

Proof. We adapt the proof of *Theorem 17* given in [6]. Observe that $\omega_X(q)$ is sectional-hyperbolic. Therefore, if $\omega_X(q)$ is a closed orbit, then *Theorem* 5.3.9 implies that q satisfies the property $(P)_\Sigma$ for some closed subset Σ. Moreover, we can apply Theorem 16 in [6] to conclude that $\omega_X(q)$ has singular partitions of arbitrarily small size.

Hereafter, we assume $\omega_X(q)$ is not a closed orbit. By *Proposition 3* in [6] is sufficient to prove that for all $z \in \omega_X(q)$ there is cross section Σ_z close to z such that $z \in Int(\Sigma_z)$ and $\omega_X(q) \cap \partial\Sigma z = \emptyset$.

We assert that $\omega_X(q)$ cannot contain any local strong stable manifold. Indeed, we first assume that $\omega_X(q)$ has no singularities. By *Hyperbolic lemma*, it is hyperbolic saddle-type. Suppose $\omega_X(q)$ containing a local strong stable manifold. Then, by *Lemma 11* in [6], q would be a recurrent point. Therefore using *Lemma 5.6* in [29], there is $x^* \in Per(X) \cap \omega_X(q)$ such that $q \in W^s_X(x^*)$. This means that $\omega_X(q)$ is a periodic orbit which contradicts our assumption. Now, if $\omega_X(q)$ is a sectional-hyperbolic set with singularities, applying *Main Theorem* in [27], $\omega_X(q)$ cannot contain any local strong stable manifold.

We can fix a foliated rectangle of small diameter R^0_z such that $z \in Int(R^0_z)$ and $\omega_X(q) \cap \partial^h R^z_0 = \emptyset$. By *Theorem* 5.3.1, the intersection of $W^u(O)$ with $W^s(\sigma)$ occurs in some connected component $W^{s,+}(\sigma')$ or $W^{s,-}(\sigma)$ (or both). We initially assume the intersection in $W^{s.+}(\sigma)$.

Since $z \in \omega_X(q)$ and the omega-limit set is not a closed orbit, we have that the positive orbit of q intersects either only one or the two connected components of $R^0_z \setminus \mathcal{F}^s(z, R^0_z)$.

Assume the intersection is occurring in just one component only, we shall consider the following cases:

- $W^{s,-}(\sigma) \cap M(X) = \emptyset$.

Using this and linear coordinates around σ, we can construct an open interval $I^+ = I^+_q \subset W^u(O)$, contained in a suitable cross section throught $q \in W^u(\sigma) \setminus \{\sigma\}$ and $q \in \partial I^+$. As $W^u(O) \cap W^{s,+}(\sigma)$ is dense in $W^u(O)$ we have $I^+ \cap W^{s,+}(\sigma)$ is dense in I^+.

It is possible to assume I^+ is contained in that component of $R^0_z \setminus \mathcal{F}^s(z, R^0_z)$. It is because of the positive orbit of q carries the positive orbit of I^+ into such a component. Furthermore, the stable manifolds throught I^+ form a subrectangle R^+_I in there. So, $W^{s,+}(\sigma) \cap R^+_I$ is dense in R^+_I.

Now, as in *Theorem 17* of [6], we suppose $\omega_X(q) \cap Int(R^+_I) \neq \emptyset$ to obtain a contradiction. By hypothesis, the omega-limit set of q is not a periodic orbit. Then *Lemma 5.6* in [29] implies that the positive orbit of q cannot intersects $\mathcal{F}^s(q, R^0_z)$ infinitely many times. Now, if it intersects R^+_I, then by the density of $W^{s,+}(\sigma) \cap R^+_I$ in R^+_I, we can assert that the positive orbit of a point p in $W^{s,-}(\sigma)$ would intersect R^+_I. Therefore $p \in Cl(W^u(O)) \subset M(X)$ which we get a contradiction. So $\omega_X(q) \cap Int(R^+_I) = \emptyset$.

To continue, we choose a point $z' \in Int(R^+_I)$ and a point z'' in the connected component $R^0_z \setminus \mathcal{F}^s(z, R^0_z)$ not intersected by the positive orbit of q. The desired rectangle Σ_z is a subrectangle of R^0_z bounded by $\mathcal{F}^s(z', R^0_z)$ and $\mathcal{F}^s(z'', R^0_z)$.

- $W^s(\sigma) \cap W^u(O) \subset W^{s,+}(\sigma)$ and $W^s(\sigma) \cap W^u(O') \subset W^{s,-}(\sigma)$ for some hyperbolic periodic orbit $O' \neq O$.

  In this way, we have the hypotheses of *Theorem 17* in [6]. Therefore there exists an interval $I^- \subset W^u(O')$ contained in that component of $R^0_z \setminus \mathcal{F}^s(z, R^0_z)$, such that $q \in \partial I^-$ and $I^- \cap W^{s,-}(\sigma)$ is dense in I^-. The stable manifolds throught $I = I^+ \cup \{q\} \cup I^-$ form a subrectangle R_I in there, with $Int(R_I) \cap \omega_X(q) = \emptyset$. So, the existence of Σ_z is guaranteed such as last item.

- $W^{s,+}(\sigma) \cap W^u(O) \neq \emptyset$ and $W^{s,-}(\sigma) \cap W^u(O) \neq \emptyset$.

  We assert that there are O_1, O_2 hyperbolic periodic orbits such that, $W^s(\sigma) \cap W^u(O_1) \subset W^{s,+}(\sigma)$ and $W^s(\sigma) \cap W^u(O_2) \subset W^{s,-}(\sigma)$. Indeed, we take $q_1 \in W^{s,+}(\sigma) \cap W^u(O)$ and $q_2 \in W^{s,-}(\sigma) \cap W^u(O)$.

  As $M(X)$ is union of homoclinic classes and $W^u(O) \subset M(X)$, there are hyperbolic periodic orbits O_1, O_2 satisfying $q, q_1 \in H(O_1)$ and $q, q_2 \in H(O_2)$. Therefore $O_X(q_1) \subset H(O_1)$ and $O_X(q_2) \subset H(O_2)$. Moreover, since the homoclinic classes are closed set we have that σ and O are in $H(O_1) \cap H(O_2)$. From *Remark* 5.4.5 follows $H(O_1) \cap W^s(\sigma) \subset W^{s,+}(\sigma)$ and $H(O_2) \cap W^s(\sigma) \subset W^{s,-}(\sigma)$. On the other hand, let $W^+(O)$ be the connected component of $W^u(O) \setminus O$ containing q_1, then $W^+(O) \subset H(O_1)$. Analogously, for $W^-(O)$, the connected component of $W^u(O) \setminus O$

containing q_2, we have $W^-(O) \subset H(O_2)$. Therefore $W^u(O_1) \cap W^s(\sigma) \subset W^{s,+}(\sigma)$ and $W^u(O_2) \cap W^s(\sigma) \subset W^{s,-}(\sigma)$. Again we have the hypotheses of *Theorem 17* in [6].

- $W^{s,+}(\sigma) \cap W^u(O) \neq \emptyset$ and $W^{s,-}(\sigma) \cap H(O) \neq \emptyset$.

  It is not possible by *Corollary* 5.4.4.

- $W^{s,+}(\sigma) \cap W^u(O) \neq \emptyset$, $W^{s,-}(\sigma) \cap Cl(W^u(O')) \neq \emptyset$ and $q \in Cl(W^u(O'))$, where O' is a hyperbolic periodic orbit of X.

  From last item $O' \notin H(O)$. As X satisfies the Property (P), there is $\sigma' \in Sing(X)$ such that $W^u(O) \cap W^s(\sigma') \neq \emptyset$. If $\sigma' = \sigma$ then $W^u(O')$ intersects $W^{s,+}(\sigma)$ or $W^{s,-}(\sigma)$. Observe that those alternatives was already analyzed. If $\sigma' \neq \sigma$, then we can obtain a interval J^- such that $J^- \subset W^u(O')$ and $J^- \cap W^s(\sigma')$ is dense in J^-. Moreover we can assume $W^s(\sigma) \cap W^u(O) \subset W^{s,+}(\sigma)$ to obtain a interval I^+ such that $I^+ \subset W^u(O)$ and $I^+ \cap W^{s,+}(\sigma)$ is dense in I^+. Because of $O' \notin H(O)$, follows that $W^u(O') \nsubseteq H(O)$. Therefore $W^u(O')$ cannot intersect $W^{s,+}(\sigma)$. In this way, there is an open arc $I^- \subset \bigcup_{t \geq 0} X_t(J^-)$ such that $q \in \partial I^-$. I^- works such as in second item. The stable manifolds throught $I = I^+ \cup \{q\} \cup I^-$ generates a subrectangle R_I. This acts such as *Theorem* 17 in [6].

Now assume the positive orbit intersect both components of $R_z^0 \setminus \mathcal{F}^s(z, R_z^0)$. Therefore we take I (or I^+ to first case) with the positive orbit as before to obtain two subrectangles R_I^t and R_I^b, like R_I (or R_I^+ to first case), in each component. Then we select two points $z' \in Int(R_I^t)$ and $z'' \in Int(R_I^b)$ and define Σ_z as the rectangle in R_z^0 bounded by $\mathcal{F}^s(z', R_z^0)$ and $\mathcal{F}^s(z'', R_z^0)$.

From *Proposition* 3 in [6] we conclude the result.

□

We remember the concept of *singular cross section* that appears in [31]. For a disjoint collection of rectangles $\mathcal{S} = \{S_1, \cdots, S_l\}$ we denote $\mathcal{S}^o = \mathcal{S} \setminus \partial\mathcal{S}$. and $\partial^*\mathcal{S} = \bigcup_{S \in \mathcal{S}} \partial^* S$ for $* = h, v, o$.

Definition 5.4.7. *A singular cross section of X is a finite disjoint collection $\mathcal{S}$ of foliated rectangles with $M(X) \cap \partial^h S = \emptyset$ such that for every $S \in \mathcal{S}$ there is a leaf l_S of $\mathcal{F}^s$ in S^o such that the return time $t_S(x)$ for $x \in S \cap Dom(\Pi_S)$ goes uniformly to infinity as x approaches l_S.*

We define the singular curve of $\mathcal{S}$ as the union,

$$l_{\mathcal{S}} = \bigcup_{S \in \mathcal{S}} l_S.$$

Proposition 5.4.8. *Let q be a regular point in $W^u(\sigma)$, with σ a singularity of a Venice mask X, and let O be a hyperbolic periodic orbit such that $W^u(O) \cap W^s(\sigma) \neq \emptyset$. Then $\omega_X(q)$ is a closed orbit.*

Proof. If $\omega_X(q)$ is a singularity, then it is done. Hereafter, we assume that $\omega_X(q)$ is not a singularity. From *Lemma 5.4.6* follows that $\omega_X(q)$ has singular partitions or arbitrarily small size. On the other hand, let $T_U M = \hat{F}^s_U \oplus \hat{F}^c_U$ be a continous extension of the sectional-hyperbolic splitting $T_{\omega_X(q)}M = F^s_{\omega_X(q)} \oplus F^c_{\omega_X(q)}$ of $\omega_X(q)$ to a neighborhood U of $\omega_X(q)$. Let I be an arc tangent to $\hat{F}^c_U$, transverse to X, with q as boundary point. *Theorem 18* in [6] guarantees for every singular partition $\mathcal{R} = \{S_1, \cdots S_k\}$ of $\omega_X(q)$, the existence of $S \in \mathcal{R}$, $\delta > 0$, a sequence $q'_1, q'_2, \cdots \in S$ in the positive orbit of q, and a sequence of intervals $J'_1, J'_2 \cdots \subset S$ in the positive orbit of I with q'_j as a boundary point of J'_j for all such that $length(J'_j) \geq \delta$, for all $j = 1, 2, 3, \cdots$.

We can assume $I = J'_1$. As $q, q'_j \in M(X)$ and X is a Venice mask, we can use the *Lemma 5.3.5* to obtain a sequence $\{q_n : n \in \mathbb{N}\} \subset M$ such that $q_n \to q$ and $\omega(q_n)$ is a singularity for any n. As X has just a finite singular points, we can take $\omega(q_n) = \{\sigma'\}$ for all n, and some $\sigma' \in Sing(X)$. If $q_n \in W^u(\sigma)$ for all n, then $\omega(q) = \{\sigma'\}$ which contradicts our assumption. Therefore $q_n \notin W^u(\sigma)$ for any n. We can take q_n such that $q_n \in S$ for all n

On the other hand, for σ' are possible the following two alternatives, either $\sigma' \in \omega_X(q)$, or $\sigma' \notin \omega_X(q)$. We begin to consider $\sigma' \in \omega_X(q)$. *Lemma 14* in [6] asserts $O^+(q) \cap \mathcal{R} = \{\hat{q}_1, \hat{q}_2, \cdots\}$ an infinite sequence ordered in a way that $\Pi(\hat{q}_n) = \hat{q}_{n+1}$, and the existence of a curve $c_n \subset W^s(Sing(X) \cap \omega_X(q)) \cap B_\delta(\hat{q}_n)$ such that

$$B^+_\delta(\hat{q}_n) \subset Dom(\Pi) \qquad \textit{and} \qquad \Pi|_{B^+_\delta(\hat{q}_n)} \quad \textit{is} \quad C^1,$$

where $B^+_\delta(\hat{q}_n)$ denotes the connected component of $B_\delta(\hat{q}_n) \setminus c_n$ containing $\hat{q}_n$.

In particular, we can reduce δ to obtain $\Pi_S = \Pi|_S$ such that

$$(\Pi_S)|_{B^+_\delta(q)} \quad \textit{is } C^1.$$

However $W^s(\sigma')$ accumulates q on S, so we obtain a contradiction.

Therefore the first alternative cannot occur. We conclude $\sigma' \notin \omega_X(q)$.

Hartman-Grobman's Theorem implies the existence of a neighborhood $V_{\sigma'}$ of σ', where the flow is C^0-conjugated to its linear part. Let $\eta > 0$ be such that $V_{\sigma'} \subset B_\eta(\sigma')$ and $O^+(q) \cap V_{\sigma'} = \emptyset$. From *Lemma 2.2* in [31] there are singular cross sections $\Sigma^+, \Sigma^- \subset V_{\sigma'}$ such

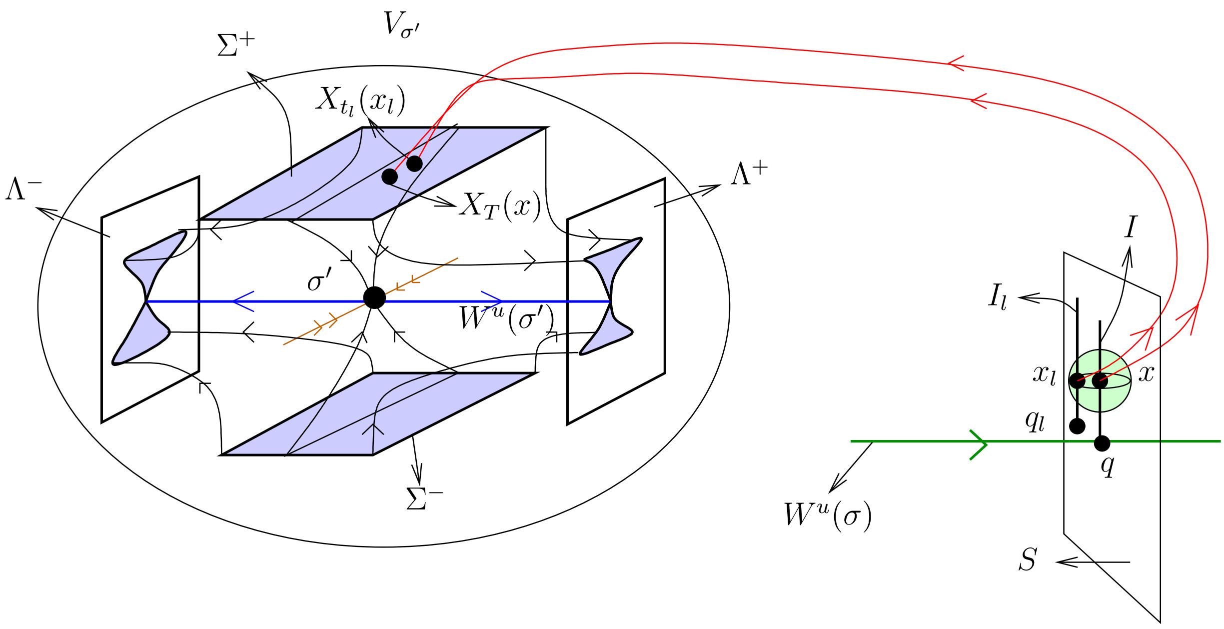


Figure 5.3: Proof *Proposition 5.4.8.*

that every orbit of $M(X)$ passing close to some point in $W^{s,+}(\sigma')$ (respectively $W^{s,-}(\sigma')$) intersects Σ^+(respectively Σ^-). Moreover *Lemma 2.3* in [3] guarantees the existence of two disks $\Lambda^+, \Lambda^- \subset V_{\sigma'}$ transverse to X such that for $B_\varepsilon(\sigma) \subset V_{\sigma'}$, and for any point $x \in B_\varepsilon(\sigma')$, there are two numbers $t_- < 0 < t_+$ with $X_{t_-}(x) \in \Sigma^+ \cup \Sigma^-$ and $X_{t_+}(x) \in \Lambda^+ \cup \Lambda^-$. In addition, $X_t(x) \in V_{\sigma'}$ for all $t \in (t_-, t_+)$. See Figure 5.3.

As $q_n \to q$, we can take a sequence of open arcs $I_1, I_2, \cdots$ with q_n as a boundary point of I_n such that $Cl(I_n)$ converges to $Cl(I)$. In particular, we can assume $\delta \leq length(I_n) < \epsilon$ for all $n = 1, 2, 3, \cdots$ and $diam(S) = \epsilon$. In addition, we can take $I_n \subset S$ for all n. On the other hand, $q_n \in W^s(\sigma')$ implies that $O^+(q_n)$ intersects $\Sigma^+ \cup \Sigma^-$. Assume that the intersection occurs in Σ^+ for all n. As we can choose the singular partition of arbitrarily small size and q is non-recurrent, there is $\varepsilon' > 0$ such that $diam(\mathcal{R}) = \varepsilon'$ and $O^+(s_n) \cap \Sigma^+ \neq \emptyset$ for all $s_n \in I_n$.

We assert that q satisfies the property $(P)_\Sigma$, where $\Sigma = \Sigma^+$. Indeed, from $O^+(q) \cap V_{\sigma'} = \emptyset$ follows $O^+(q) \cap \Sigma^+ = \emptyset$. Now, for $x \in I$ there are $\beta_1, \beta_2 > 0$ such that $B_{\beta_1}(x) \cap \partial I = \emptyset$, $B_{\beta_2}(x) \cap \{q_l\} = \emptyset$ and $B_{\beta_2}(x) \cap I_l \neq \emptyset$ l for all l large. We define $\beta = \min\{\beta_1, \beta_2\}$. Let $\{x_l\}_l$ be a sequence with $x_l \in I_l \cap B_\beta(x)$ such that $x_l \to x$. As in [6], we define the *holonomy map* Π_{S,Σ^+} from S to Σ^+ by

$$Dom(\Pi_{S,\Sigma^+}) = \{y \in S : X_t(y) \in \Sigma^+ \text{ for some } t > 0\}$$

and

$$\Pi_{S,\Sigma^+}(y) = X_{t_{S,\Sigma^+}(y)}(y),$$

where $t_{S,\Sigma^+}(y) = \inf\{t > 0 : X_t(y) \in \Sigma^+\}$.

Therefore $x_l \in Dom(\Pi_{S,\Sigma^+})$ for all n. From *Lemma 19* and *Theorem 22* in [6] follows that $x \in Dom(\Pi_{S,\Sigma^+})$.

Finally, *Theorem 5.3.9* implies that $\omega_X(q)$ is a closed orbit. As we assume $\omega_X(q)$ not being a singularity, then we conclude that the omega-limit set of q is a periodic orbit.

□

5.4.2 Property $(P_{\sigma'})_q^+$

Definition 5.4.9. *Let $\sigma, \sigma' \in Sing(X)$ and q be a regular point in $W^u(\sigma)$. We say that an open arc $I \subset M$ satisfies the property $(P_{\sigma'})_q^+$ if $q \in \partial I$ and $I \cap W^{s,+}(\sigma')$ is dense in I. In a similar way, an open arc $J \subset M$ satisfies the Property $(P_{\sigma'})_q^-$ if $q \in \partial J$ and $J \cap W^{s,-}(\sigma')$ is dense in J.*

Proposition 5.4.10. *Let O be a hyperbolic periodic orbit of a Venice mask X. Assume $\sigma' \in Sing(X)$ satisfying $\emptyset \neq W^u(O) \cap W^s(\sigma') \subset W^{s,+}(\sigma')$. Then, for all singularity σ and all regular point $q \in W^u(\sigma) \cap Cl(W^u(O))$, there is an open arc satisfying the property $(P_{\sigma'})_q^+$. The same interchanging $+$ by $-$.*

Proof. Let $p \in W^u(\sigma')$ be a regular point. We assert that there is an open interval J satisfying the property $(P_{\sigma'})_p^+$. Indeed, σ' and p are contained in $Cl(W^u(O))$. As $W^u(O)$ intersects $W^{s,+}(\sigma')$, then $W^u(O) \cap W^s(\sigma)$ is dense in $W^{s,+}(\sigma')$. Consider an open arc $J \subset W^u(O)$ with $p \in \partial J$. So, the density of $W^u(O) \cap W^{s,+}(\sigma)$ in $W^u(O)$ implies that $J \cap W^{s,+}(\sigma')$ is dense in J.

If $\sigma = \sigma'$, then we obtain the desired result. Now, we consider $\sigma \neq \sigma'$. From *Lemma 5.4.8* follows that the omega-limit set of every point in $W^u(\sigma')$ is a closed orbit. Now, take two point p_1, p_2, one on each branch of $W^u(\sigma') \setminus \{\sigma'\}$. We analize the following cases which are ilustrated in Figure 5.4.

- $\omega_X(p_1)$ is a singularity. Let σ_1 be a singularity with $\omega_X(p_1) = \{\sigma_1\}$. If $\omega_X(p_1) = \{\sigma'\}$, then $\omega_X(p_2) \neq \{\sigma'\}$. Indeed, $\omega_X(p_1) = \{\sigma'\} = \omega_X(p_2)$ implies either $W^u(O) \cap$

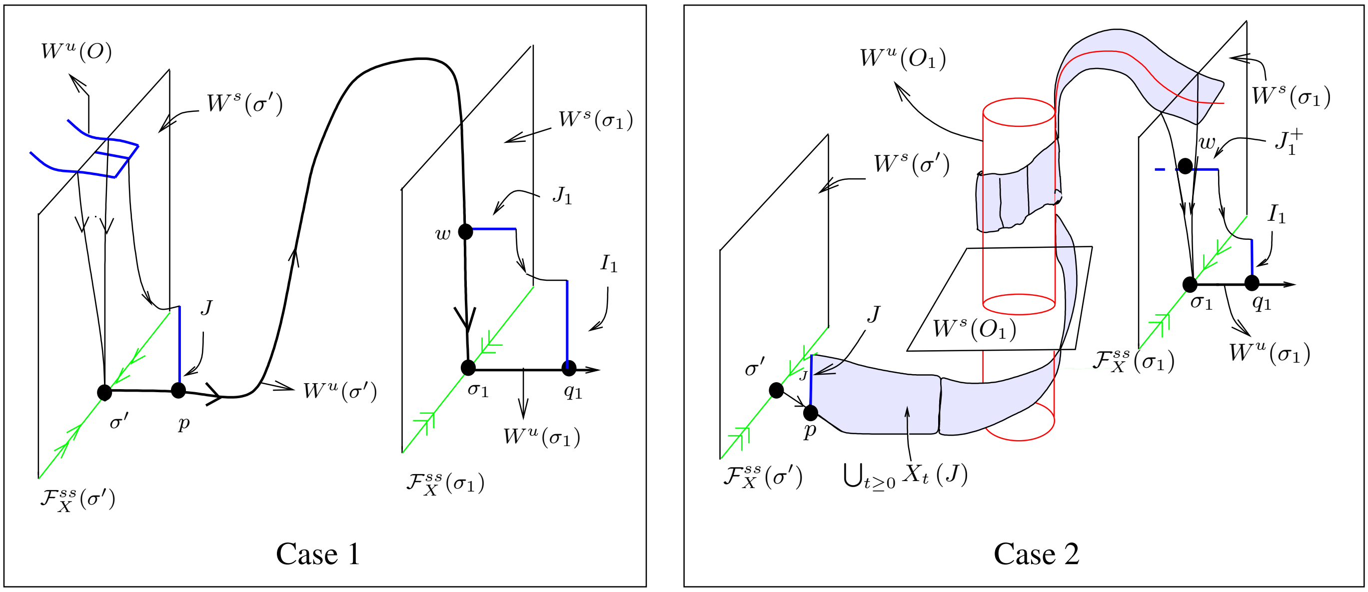


Figure 5.4: Proof Proposition 5.4.10

$W^s(\sigma) \neq \emptyset$ or $Cl(W^u(O)) \cap W^s(\sigma) \neq \emptyset$. But $W^u(O) \cap W^s(\sigma) = \emptyset$ by hypothesis. Moreover $\sigma \in Cl(W^u(O))$. So, $\sigma_1 \neq \sigma'$.

Let $w \in W^u(\sigma') \cap W^s(\sigma_1)$ be a point in $O_X^+(p_1)$ close to σ_1. Using it and linear coordinates around σ_1, we can construct an open interval $J_1 \subset \bigcup_{t\geq 0} X_t(J) \subset W^u(O)$ contained in a suitable cross section throught w, such that $w \in \partial J_1$. From *Inclination lemma* [36], follows that $W^u(O)$ accumulates points in some branch of $W^u(\sigma_1)$. Therefore, for $q_1 \in (W^u(\sigma_1) \cap Cl(W^u(O))) \setminus \{\sigma_1\}$ there is an open arc I_1 such that $I_1 \subset \bigcup_{t\geq 0} X_t(J_1)$ and $q_1 \in \partial I_1$. The density of $W^{s,+}(\sigma') \cap W^u(O)$ in $W^u(O)$ implies the density of $W^{s,+}(\sigma') \cap I_1$ in I_1. Then I_1 satisfies $(P_{\sigma'})^+_{q_1}$.

- When the omega-limit set of p_1 and p_2 are respectively hyperbolic periodic orbits O_1, O_2, we have that $W^u(O_i)$ intersects the stable manifold of some singularity σ_i of X, $i = 1, 2$. We first assume $\sigma_1 = \sigma_2 = \sigma'$. That intersection cannot just only occurs in $W^s(\sigma')$ because of this would imply $\sigma \notin Cl(W^u(O_1) \cup W^u(O_2))$ and $Cl(W^u(O)) \subset Cl(W^u(O_1) \cup W^u(O_2))$. But $\sigma \in Cl(W^u(O))$ which produces a contradiction. Therefore we can assume that $W^u(O_1) \cap W^s(\sigma_1) \neq \emptyset$ with $\sigma_1 \neq \sigma'$.

  Applying *Inclination lemma*, $Cl(W^u(O))$ and $\bigcup_{t\geq 0} X_t(J)$ intersect $W^s(\sigma_1)$ transversally. Again, let $w \in W^u(O) \cap W^s(\sigma)$ be a point in $\bigcup_{t\geq 0} X_t(J)$ close to σ_1. Using it and linear coordinates around σ_1, we can construct an open interval $J_1 \subset W^u(O)$ contained in a suitable cross section throught w. $J_1 \setminus \{w\}$ is formed by two open arcs $J_1^+, J_1^- \subset W^u(O)$. Therefore, for $q_1 \in W^u(\sigma_1) \setminus \{\sigma_1\}$ there is an open arc I_1

such that and $q_1 \in \partial I_1$ and, $I_1 \subset \bigcup_{t\geq 0} X_t(J^+)$, or $I_1 \subset \bigcup_{t\geq 0} X_t(J^-)$. The density of $W^{s,+}(\sigma') \cap W^u(O)$ in $W^{s,+}(\sigma')$ implies the density of $W^{s,+}(\sigma) \cap I_1$ in I_1. Then I_1 satisfies $(P_{\sigma'})^+_{q_1}$.

If $\sigma_1 = \sigma$, then the result is obtained. Otherwise, we apply a similar process to σ_1 to get $\sigma_3 \in Sing(X)$ with $\sigma_3 \notin \{\sigma', \sigma_1\}$, and an open arc $I_3 \subset Cl(W^u(O))$ such that I_3 satisfies the property $(P_{\sigma'})^+_{q_3}$.

As $\sigma \in Cl(W^u(O))$ and X just has finitely many singularities, we conclude the existence of some open arc satisfying the property $(P_{\sigma'})^+_q$ for $q \in W^u(\sigma) \cap Cl(W^u(O))$.

□

5.4.3 Proof of Theorem D

It is sufficient to prove the existence of singular partitions of arbitrarily small size.

Let q be a regular point in $W^u(\sigma)$, where $\sigma \in Sing(X)$.

As $M(X)$ is union of homoclinic classes, there is a hyperbolic periodic orbit O such that σ and q are contained in the homoclinic class associated to O, denoted by $H(O)$. In addition $H(O)$ intersects only one or the two connected components $W^{s,+}(\sigma), W^{s,-}(\sigma)$ of $W^s(\sigma) \setminus \mathcal{F}^{ss}_X(\sigma)$. We begin to analize the intersection in $W^{s,+}(\sigma)$. On the other hand, X satisfies the property (P). This implies that there is a singularity $\sigma' \in Sing(X)$ with $W^u(O) \cap W^s(\sigma') \neq \emptyset$. By *Theorem* 5.3.1, the intersection of $W^u(O)$ with $W^s(\sigma')$ is either only one or the two connected components $W^{s,+}(\sigma'), W^{s,-}(\sigma')$ of $W^s(\sigma') \setminus \mathcal{F}^{ss}_X(\sigma')$. If $\sigma = \sigma'$ then from *Lemma 5.4.6* follows the existence of singular partitions of arbitrarily small size. Hereafter, we assume $\sigma \neq \sigma'$ and $W^{s.+}(\sigma') \cap W^u(O) \neq \emptyset$.

If $Cl(W^u(O)) \cap W^{s,-}(\sigma') \neq \emptyset$, then *Lemma* 5.4.3 and *Proposition* 5.4.8 imply that for some $p \in W^u(\sigma') \cap Cl(W^u(O))$, $O = \omega_X(p)$ and $H(O) \subset Cl(W^u(\sigma'))$. But $q \notin W^u(\sigma')$. This contradicts $q \in H(O)$. So, $Cl(W^u(O)) \cap W^{s,-}(\sigma') = \emptyset$. *Proposition* 5.4.10 guarantees the existence of an open arc $I^+ \subset M$ satisfying the property $(P_{\sigma'})^+_q$.

We suppose $\omega_X(q)$ is not a periodic orbit. Let z be a point in $\omega_X(q)$. In a similar way as *Lemma* 5.4.6, we fix a foliated rectangle of small diameter R^0_z such that $z \in Int(R^0_z)$ and $\omega_X(q) \cap \partial^h R^z_0 = \emptyset$. The positive orbit of q intersects either only one or the two connected components of $R^0_z \setminus \mathcal{F}^s(z, R^0_z)$.

Assume the intersection is occurring in just one component only.

Now, analize the following cases:

- $q \notin H(O')$ for all hyperbolic periodic orbit O' of X such that $H(O') \cap W^{s,-}(\sigma) \neq \emptyset$.

  The existence of the singular partitions of arbitrarily small size is obtained such as the first case in *Lemma* 5.4.6.

- There is a sequence $\{p_n\}_n \subset W^u(O)$ such that $p_n \to p \in W^{s,-}(\sigma)$, and there is a sequence $\{q_n\}$ such that $q_n \in O_X(p_n)$ and $q_n \to q$.

  From *Lemma* 5.4.3 follows that $\omega_X(q) = O$. But this contradicts our assumption that the omega-limit set is not a periodic orbit.

- For some periodic orbit $O' \neq O$, there is a sequence $\{p_n : n \in \mathbb{N}\} \subset W^u(O')$ such that $p_n \to p \in W^{s,-}(\sigma)$, and there is a sequence $\{q_n : n \in \mathbb{N}\}$ satisfying $q_n \in O_X(p_n)$ and $q_n \to q$.

  Again, *Lemma* 5.4.3 implies that $W^u(O')$ does not intersect the open arc I^+. From Property (P), there is $\sigma'' \in Sing(X)$ such that $W^u(O') \cap W^s(\sigma'') \neq \emptyset$. Then for some $r \in W^u(\sigma'')$ there is an interval $J^- \subset W^u(O')$, such that $r \in \partial J$ and $J^- \cap W^s(\sigma'')$ is dense in J^-. Also there is an open arc $I^- \subset \bigcup_{t\geq 0} X_t(J^-)$ satisfying $q \in \partial I^-$. Therefore $I^- \subset W^u(O')$ and $I^- \cap W^s(\sigma'')$ is dense in I^-. In addition, $W^{s,+}(\sigma) \cap I^- = \emptyset$. The stable manifolds throught $I = I^+ \cup \{q\} \cup I^-$ generates a subrectangle R_I. This rectangle acts such as *Lemma* 17 in [6].

The existence of the singular partition of arbitrarily small size is obtain such as *Lemma* 5.4.6.

If the intersection of $O^+_X(q)$ with R^0_z occurs in both connected components of $R^0_z \setminus \mathcal{F}^s(z, R^0_z)$, then we proceed such as *Lemma* 5.4.6 to get a cross section Σ_z with $z \int \Sigma_z$ and $\partial\Sigma_z \cap \omega_X(q) = \emptyset$.

In this way, *Proposition* 3 in [6] implies the existence of the singular partition of arbitrarily small size for $\omega_X(q)$.

Finally, we follow the proof of *Proposition* 5.4.8 to conclude that $\omega_X(q)$ is a closed orbit.

5.5 Intersection of homoclinic classes

In this section we are interested in the study of the intersection of homoclinic classes in a sectional-Anosov flow. We follow some ideas developed in [9] to obtain the theorem E. More specifically we prove that in this context that the intersection can be decomposed in three specific sets. a non-singular hyperbolic set, finitely many singularities and regular orbits joining them. Recall that an invariant set is nontrivial if it does not reduces to a single orbit. The conclusion of *Theorem E* is obvious when H_1 or H_2 is a trivial invariant set. Hereafter, H_1 and H_2 are two non trivial different transitive sets in $M(X)$. Let Λ be the intersection between H_1 and H_2. We start with the following lemma.

Lemma 5.5.1. *Assume that there is a singularity $\sigma \in \Lambda$, then for $\delta > 0$ small, every sequence $\{x_n : n \in \mathbb{N}\} \subset \Lambda \cap B_\delta(\sigma)$ such that $x_n \to \sigma$ is contained in $W^s(\sigma) \cup W^u(\sigma)$.*

Proof. We suppose by contradiction that there is a sequence $\{x_n : n \in \mathbb{N}\} \subset \Lambda \cap B_\delta(\sigma)$ such that $x_n \to \sigma$ and $x_n \notin W^s(\sigma) \cup W^u(\sigma)$ for all n.

So, we obtain two sequences x_n^s and x_n^u, in the orbit of x_n such that $x_n^s \to y^s$ and $x_n^u \to y^u$ for some $y^s \in W^s(\sigma) \setminus \{\sigma\}$ and $y^u \in W^u(\sigma) \setminus \{\sigma\}$ close to σ. Let O_1, O_2 be two orbits such that $H(O_1) = H_1$ and $H(O_2) = H_2$. Then there exist sequences $\{p_n : n \in \mathbb{N}\} \subset (W^u(O_1) \cap W^s(O_1))$ and $\{q_n : n \in \mathbb{N}\} \subset (W^u(O_2) \cap W^s(O_2))$ satisfying $p_n \to x_n^s$ and $q_n \to x_n^s$. We can assume $p_n \notin H_2$ for all n. This means that $p_n \to x^s$ and $q_n \to x^s$ too. The behavior of the orbits of x_n, p_n and q_n nearby σ, are as described in Figure 5.5.

Since homoclinic classes have density of periodic points [19], for each n we have that p_n and q_n are approximated respectively by a sequence of periodic orbits $\{O_1^{mn} : m \in \mathbb{N}\}$ and $\{O_2^{mn} : m \in \mathbb{N}\}$. Define the map $\pi : B_\delta(\sigma) \to W^{cu}(\sigma)$ such as in *Section 4.3.* Observe that $\{\pi(W^u(O_1^{mn})) : m \in \mathbb{N}\}$ and $\{\pi(W^u(O_2^{mn})) : m \in \mathbb{N}\}$ accumulate y^s in the same sector s_{ij} of $W^{cu}(\sigma)$. Follows from *Lemma* 3.1 in [13] that these sequences can be chosen in a way that, for $i = 1, 2$ and for all n, m, $W^s(O_i^{nm})$ is uniformly bounded away from zero. This implies that for m_1, m_2, n_1, n_2 large, $W^u(O_1^{m_1 n_1}) \cap W^s(O_2^{m_2 n_2}) \neq \emptyset$. Consider $x \in W^u(O_1^{n_1 m_1}) \cap W^s(O_2^{m_2 n_2})$. As $O_1^{m_1 n_1} \subset (H_1 \setminus H_2)$ and $O_2^{m_2 n_2} \subset H_2$, then there is $x^* \in O_X(x)$ such that $x^* \in \Lambda$. But Λ is an invariant closed set, then $O_1^{m_1 n_1} \subset Cl(O_X(x^*)) = Cl(O_X(x^*)) \subset \Lambda$. However $O_1^{m_1 n_1} \not\subset H_2$ and $\Lambda \subset H_2$, which is a contradiction.

We conclude $x_n \in W^s(\sigma) \cup W^u(\sigma)$ for all $n \in \mathbb{N}$.

□

5.5.1 Proof theorem E

Theorem E gives a description about the set Λ.

Proof. The idea of the proof is the same given in *Lemma 3.3* by [9]. Follows to *Lemma 5.5.1* that there is $\delta > 0$ such that $\Lambda \cap B_\delta(\sigma) \subset W^s(\sigma) \cup W^u(\sigma)$, and the balls $B_\delta(\sigma)$ are pairwise disjoint for every $\sigma \in \Lambda \cap Sing(X) = S$. Define

$$H = \bigcap_{(t,\sigma) \in \mathbb{R} \times S} X_t(\Lambda \setminus B_\delta(\sigma)).$$

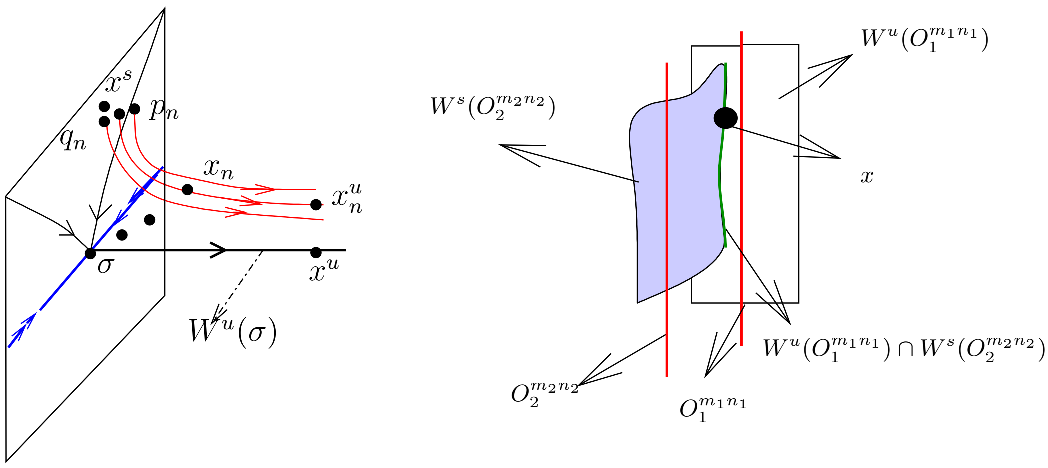


Figure 5.5: *Lemma* 5.5.1

By construction, H is a non-singular, compact invariant sectional-hyperbolic set. So, applying *Lemma 5.3.2* we have that H is hyperbolic. Now define $R = \Lambda \setminus (S \cup H)$. For $x \in R$ there is $(t, \sigma) \in \mathbb{R} \times S$ with $X_t(x) \in B_\delta(\sigma)$, and by *Lemma 5.5.1* $X_t(x) \in W^s(\sigma) \cup W^u(\sigma)$.

If $x \in W^u(\sigma)$ we obtain $\alpha(x) \subset H \cup S$. Assume $X_s(x) \notin \bigcup_{\rho \in S} B_\delta(\rho)$ for all $s \geq 0$, then $\omega(x) \subset H$. Now, if there is $(s, \rho) \in \mathbb{R} \times S$ such that $X_s(x) \in B_\delta(\rho)$ then $x \in W^s(\rho)$, So $\omega(x) \in H \cup S$.

With a similar argument we have $\alpha(x) \subset H \cup S$ and $\omega(x) \subset H \cup S$ for $x \in W^s(\sigma)$. So, we conclude the result.

□

CHAPTER

SIX

CONCLUSIONS AND PERSPECTIVES

From this work we have the following conclusions for sectional-Anosov flows of Venice mask type on compact 3-manifolds:

1. The existence of Venice masks containing any finite number of singularities. These examples are characterized because the associated maximal invariant set is finite union of homoclinic classes. In addition, the intersection between two different homoclinic classes is contained in the closure of the union of unstable manifold of the singular points of the Venice mask.

2. There exist Venice masks such that the maximal invariant set cannot be decomposed as the union of two homoclinic classes.

3. The *omega-limit set* of every non-recurrent point in the unstable manifold of a equilibrium of a Venice mask is a closed orbit.

4. The intersection between two different homoclinic classes in a sectional-Anosov flow can be decomposed as the disjoint union of singular points, a non-singular hyperbolic set H, and regular points whose *alpha-limit set* and *omega-limit set* are contained in the union of singular points and the non-singular hiperbolic set H.

Because of the study developed in this work, different questions have appeared. Such as we mention in Chapter 4, all known examples of Venice mask are characterized because the maximal invariant set is the finite union of homoclinic classes and the intersection between two different homoclinic classes H_1 and H_2 is contained in $Cl(W^u(Sing(X)))$. Moreover, every regular point $q \in W^u(Sing(X)) \cap H_1 \cap H_2$ is non-recurrent.

Consider a Venice mask X supported on a compact 3-manifold M. Let H_1 and H_2 be two different homoclinic classes in $M(X)$ and let Λ be the intersection between H_1 and H_2. Assume the decomposition of Λ given in *Theorem E*, it is $\Lambda = S \cup H \cup R$.

We announce the following conjecture.

Conjecture 6.0.1. *Every regular point $q \in R$ is non-recurrent.*

By *Lemma 5.5.1* we have $x \in W^s(\sigma) \cup W^u(\sigma)$ for some $\sigma \in S$. If $x \in W^u(\sigma)$ then $\alpha(x) = \{\sigma\}$. Now we take $x \in W^s(\sigma) \setminus W^u(\sigma)$, therefore we shall consider two cases, either $\alpha(x) = \{\rho\}$ for some $\rho \in S$ or $\alpha(x) \subset H$. In the first case, we obtain the desired result. If we prove that the second case cannot occur, then the following conjecture would be true.

Conjecture 6.0.2. $\Lambda \subset Cl(W^u(Sing(X)))$.

Let us state direct consequence of the hyperbolic Lemma 5.3.2 that appears in [6].

Corollary 6.0.3. *Every periodic orbit of a sectional-Anosov flow on a compact manifold is hyperbolic. In particular, all such flows have countably many closed orbits.*

This implies that the maximal invariant set of every Venice mask is union of countably many homoclinic classes. So, if *Conjecture 6.0.1* and *Conjecture 6.0.2* are true, then would be possible to realize the following statement.

Conjecture 6.0.4. *The maximal invariant set of every Venice mask is finite union of homoclinic classes.*

Proof. Let X be a Venice mask supported on a compact 3-manifold M. Then X has finite many singularities, we say n. Let H_1, H_2 be two different homoclinic classes associated to $M(X)$. From Conjecture 6.0.1 and 6.0.2 is possible to apply Theorem D to conclude that for each singularity σ of X, $Cl(W^u(\sigma)) = \{\sigma\} \cup W^u(\sigma) \cup C$, it is a disjoint union and C is a closed orbit. On the other hand, the branches of $W^u(\sigma)$ are uni-dimensional. Therefore Theorem 6.0.2 implies $H_1 \cap H_2$ has just only a finite number of possibilities to occur. Moreover, at most three homoclinic classes can contain the branch of the unstable manifold of some singularity.

This finishes the proof.

□

Definition 6.0.5. *We say that a sectional-Anosov flow X supported on a compact manifold M has codimension k if the dimension of the central subbundle is $k+1$*

Observe that all examples developed in this work has codimension 1 and these are defined by three-dimensional vector fields. It is not difficult to construct Venice masks of codimension 1 supported on some compact n-manifold M, where $n \geq 4$. For this, is sufficient to take a Venice mask of dimension 3 and multiply it by a strong stable foliation of dimension $n-3$. Verify the existence or not, of a Venice mask of codimension $k \geq 2$ can be more difficult. So, we have the following question.

Is there a Venice mask of codimension $k \geq 2$?

In case that the answer to be positive, we would like to study the dynamic of this type of flows.

Finally, as was mentioned in Chapter 1, follows from [3], and *Theorem 32* in [6] that every sectional-Anosov flow with a unique singularity on a compact 3-manifold is C^r robustly periodic if and only if is C^r robustly transitive. The hypothesis of a unique singularity is essencial to prove this statement. Therefore we ask:

Every C^r robustly periodic sectional-Anosov flow on a compact 3-manifold is C^r robustly transitive?

BIBLIOGRAPHY

[1] Afraimovich, V. S., Bykov, V. V., and Shilnikov, L. P. On structurally unstable attracting limit sets of lorenz attractor type. *Trudy Moskov. Mat. Obshch. 44*, 2 (1982), 150–212.

[2] Araújo, V., and Pacífico, M. J. *Three-dimensional flows*, vol. 53. Springer Science & Business Media, 2010.

[3] Arroyo, A., and Pujals, E. Dynamical properties of singular-hyperbolic attractors. *Discrete Contin. Dyn. Syst. 19*, 1 (2007), 67–87.

[4] Bautista, S. The geometric lorenz attractor is a homoclinic class. *Bol. Mat.(NS) 11*, 1 (2004), 69–78.

[5] Bautista, S. Sobre conjuntos singulares-hiperbólicoss. *Tese de Doutorado, UFRJ* (2005).

[6] Bautista, S., and Morales, C. A. Lectures on sectional-anosov flows. http://preprint.impa.br/Shadows/SERIE_D/2011/86.html.

[7] Bautista, S., and Morales, C. A. Characterizing omega-limit sets which are closed orbits. *J. Differential Equations 245*, 3 (2008), 637–652.

[8] Bautista, S., and Morales, C. A. A sectional-anosov connecting lemma. *Ergodic Theory Dynam. Systems 30*, 2 (2010), 339–359.

[9] Bautista, S., and Morales, C. A. On the intersection of sectional-hyperbolic sets. *J. of Modern Dynamics 10*, 1 (2016), 1–16.

[10] Bautista, S., Morales, C. A., and Pacifico, M. J. On the intersection of homoclinic classes on singular-hyperbolic sets. *Discrete and continuous Dynamical Systems 19*, 4 (2007), 761–775.

[11] Bonatti, C., Diaz, L., and Viana, M. *Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective,.* Encyclopaedia of Mathematical Sciences, 102. Mathematical Physics, III. Springer-Verlag, Berlin, 2005.

[12] Bonatti, C., Pumariño, A., and Viana, M. Lorenz attractors with arbitrary expanding dimension. *C. R. Acad. Sci. Paris Sér. I Math. 325*, 8 (1997), 883–888.

[13] Carballo, C. M., and Morales, C. A. Omega-limit sets close to singular-hyperbolic attractors. *Illinois Journal of Mathematics 48*, 2 (2004), 645–663.

[14] Gähler, S. Lineare 2-normierte räume. *Mathematische Nachrichten 28*, 1-2 (1964), 1–43.

[15] Guckenheimer, J., and Williams, R. F. Structural stability of lorenz attractors. *Publications Mathématiques de l'IHÉS 50*, 1 (1979), 59–72.

[16] Hayashi, S. Connecting invariant manifolds and the solution of the c^1-stability and ω-stabililty conjectures for flows. *Annals of Math. 145* (1997), 81–137.

[17] Hempel, J. *3-manifolds.* Ann. of Math. Studies, No. 86. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1976.

[18] Hirsch, M. W., Pugh, C. C., and Shub, M. *Invariant manifolds*, vol. 583. Springer Berlin, 1977.

[19] Katok, A., and Hasselblatt, B. *Introduction to the Modern Theory of Dynamical Systems*, vol. 54. Cambridge University Press, 1997.

[20] Kawaguchi, A., and Tandai, K. On areal spaces i. *Tensor NS 1* (1950), 14–45.

[21] López Barragan, A. M., and Sánchez, H. M. S. Sectional anosov flows: Existence of venice masks with two singularities. *Bulletin of the Brazilian Mathematical Society, New Series 48*, 1 (2017), 1–18.

[22] Metzger, R., and Morales, C. A. Sectional-hyperbolic systems. *Ergodic Theory and Dynamical Systems 28*, 05 (2008), 1587–1597.

[23] Milnor, J. *Differentiable manifolds which are homotopy spheres.* Princeton University, 1954.

[24] Milnor, J. *Topology from the differentiable viewpoint, Based on notes by David W. Weaver.* The University Press of Virginia Charlottesville, 1965.

[25] Morales, C. Singular-hyperbolic attractors with handlebody basins. *Dyn. Control Syst. 13*, 1, 15–24.

[26] Morales, C. A. Examples of singular-hyperbolic attracting sets. *Dynamical Systems 22*, 3 (2007), 339–349.

[27] Morales, C. A. Strong stable manifolds for sectional-hyperbolic sets. *Discrete and Continuous Dynamical Systems 17*, 3 (2007), 553–560.

[28] Morales, C. A. Sectional-anosov flows. *Monatshefte für Mathematik 159*, 3 (2010), 253–260.

[29] Morales, C. A., and Pacífico, M. J. Mixing attractors for 3-flows. *Nonlinearity 14*, 2 (2001), 359–378.

[30] Morales, C. A., and Pacífico, M. J. A dichotomy for three-dimensional vector fields. *Ergodic Theory Dynam. Systems 23*, 5 (2003), 1575–1600.

[31] Morales, C. A., and Pacífico, M. J. Sufficient conditions for robustness of attractors. *Pacific journal of mathematics 216*, 2 (2004), 327–342.

[32] Morales, C. A., and Pacífico, M. J. A spectral decomposition for singular-hyperbolic sets. *Pacific Journal of Mathematics 229*, 1 (2007), 223–232.

[33] Morales, C. A., Pacífico, M. J., and Pujals, E. R. Singular hyperbolic systems. *Proceedings of the American Mathematical Society 127*, 11 (1999), 3393–3401.

[34] Morales, C. A., Pacífico, M. J., and Pujals, E. R. Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers. *Annals of mathematics* (2004), 375–432.

[35] Morales, C. A., and Vilches, M. On 2-riemannian manifolds. *SUT J. Math. 46*, 1 (2010), 119–153.

[36] Palis, J., and De Melo, W. *Geometric theory of dynamical systems.* Springer, 1982.

[37] Robinson, C. *Dynamical systems: stability, symbolic dynamics, and chaos.* CRC press, 1995.

[38] Smale, S. Generalized poincares conjecture in dimensions greater than four. *The Annals of Mathematics 74*, 2 (1961), 391–406.

[39] Smale, S. Differentiable dynamical systems. *Bulletin of the American mathematical Society 73*, 6 (1967), 747–817.

[40] Sodero, T. Sectional-anosov flows on certain compact 3-manifolds. *Bull. Braz. Math. Soc. (N.S.) 42*, 3 (2011), 439–454.

[41] Sodero, T. Flows: The specification property and the existence of sectional anosov flows. *Tese de Doutorado, UFRJ* (2014).

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