The initial-boundary value problem for a fractional type degenerated heat equation

by

Gerardo Jonatan Huaroto Cardenas

Advisor: Wladimir Neves

Doctoral thesis
presented to the Graduate Program in Mathematics, PGPIM,
of the Federal University of Rio de Janeiro
as part of the requirement for the degree of
Doctor in Science

Rio de Janeiro, Brazil, 2017

CIP - Catalogação na Publicação

H874t

Huaroto Cardenas, Gerardo Jonatan
The initial-boundary value problem for a fractional type degenerated heat equation / Gerardo Jonatan Huaroto Cardenas. -- Rio de Janeiro, 2017.
58 f.

Orientador: Wladimir Augusto das Neves.
Tese (doutorado) - Universidade Federal do Rio de Janeiro, Instituto de Matemática, Programa de Pós Graduação em Matemática, 2017.

1. Fractional type degenerated heat equation. 2. The initial-boundary value problem. 3. Fractional Laplacian in bounded domain. 4. Existence of weak solution. I. Neves, Wladimir Augusto das, orient. II. Título.

Elaborado pelo Sistema de Geração Automática da UFRJ com os dados fornecidos pelo(a) autor(a).

The initial-boundary value problem for a fractional type degenerated heat equation

por

Gerardo Jonatan Huaroto Cardenas

Tese de Doutorado submetida ao Programa de Pós-graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários para a obtenção do titulo de Doutor em Matemática.

Área de concentração: Matemática

Aprovada por:

Wladimir Neves.

Prof. Dr. Wladimir Neves - UFRJ-IM
(Orientador)

Prof. Dr. Cesar Niche - UFRJ-IM

Jean Carlos da Silva

Prof. Dr. Jean Carlos da Silva - UFRJ-IM

Prof. Dr. Xavier Carvajal - UFRJ-IM

Prof. Dr. Boyan Sirakov - PUC-RIO

Edgard Almeida Pimentel

Prof. Dr. Edgard Pimentel - PUC-RIO

Rio de Janeiro

Junho de 2017

To the memory of my grandmother, Rosa Felicita Alvites Pino
(1935-2016).
My dedication to her is a small way of saying
thanks and I miss you very much.
A wonderful human being, smart and generous person.
To my parents, Carmen and Efrain who always support me in difficult
times of my career and also to my sisters Maria and Yanina.
to Leen, my fiancée, whose love, confidence and patience
are a constant source of inspiration and encouragement.

ACKNOWLEDGEMENTS

I wish to express my gratitude to many people and institutions that made it possible for me to pursue a life of studies.

I wish to express my infinite gratitude to Prof. Wladimir Neves. For all his help and knowledge that was given to me along these years. In a time when things seemed to be off course, he helped me to put everything in place. His advices have been considered as extremely valuable presents for me and I have tried to follow them the best way I can. To him, my most sincere respect and gratitude.

There have been a lot of people that I have met along these years and I would like to thank Abraham Muñoz, Andres Ortiz, Claudia Salas, Daniel Reis, Dionicio Moreno, Galindo Taza, Aldo Bazán, Taynara Andrade, Vernny Chávez, Wlater Brito, to mention only those closest, who are always by my side to support and encourage me to overcome the difficult stages in the course of this research as well as my life in Rio.

I acknowledge CAPES and CNPq, for the financial support.

Last but never at least, my thanks would go to my beloved family and my fiancée Leen for their encouragement and confidence in me all through these years.

RESUMO

Problema de Valor inicial e contorno para uma equação de calor degenerada de tipo fracionário

Gerardo Jonatan Huaroto Cardenas

Orientador: Wladimir Augusto das Neves

Em proposta trabalho é o estudo do problema de valor inicial e contorno da equação do calor degenerado tipo fracionário colocado em domínios limitados.

$$\begin{cases} \partial_t u = \operatorname{div}(u\,\nabla \mathcal{K} u) & \text{in } \Omega_T, \\ u|_{t=0} = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases}$$

Propomos uma definição de solução fraca que interpreta melhor a condição de Dirichlet homogênea, o qual foi motivado pelo trabalho de Tese de Otto F. [13]. Além disso, mostramos a existência de solução para uma condição inicial u_0, mensurável, limitado e não negativa. O efeito de difusão não local depende do inverso do operador Laplaciano Fracionário $(-\Delta)^{-s}$ e a existência de solução é provada para $s \in (0,1)$.

Palavras-chaves: Laplaciano Fracionário, Problema de valor inicial e contorno, Condição de Dirichlet homogênea.

Rio de Janeiro,
Junho, 2017

ABSTRACT

The initial-boundary value problem for a fractional type degenerated heat equation

Gerardo Jonatan Huaroto Cardenas

Advisor: Wladimir Augusto das Neves

In this thesis, we study the initial-boundary value problem for a fractional degenerated heat type equation posed in bounded domains.

$$\begin{cases} \partial_t u = \operatorname{div}(u\,\nabla \mathcal{K} u) & \text{in } \Omega_T, \\ u|_{t=0} = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases}$$

We state a definition of weak solution, which interprets in a good way the homogeneous Dirichlet boundary condition. This form of interpretation was motivated by Otto F. [13]. Moreover we prove the existence of solutions for measurable and bounded non-negative initial data, and homogeneous Dirichlet boundary condition. The nonlocal diffusion effect relies on an inverse fractional Laplacian $(-\Delta)^{-s}$ operator, and the solvability is proved for any $s \in (0,1)$.

Key words. Fractional Laplacian, initial-boundary value problem, Dirichlet homogeneous boundary condition.

Rio de Janeiro,
June, 2017

Table of Contents

1 Introduction **1**

2 Notation and background **5**

2.1 Functional spaces 5

2.1.1 The space $W^{s,p}(\Omega)$ 5

2.1.2 The space $H^s(\Omega)$, $H_0^s(\Omega)$ and $H_{00}^{1/2}(\Omega)$ 6

2.2 Fractional Laplacian in bounded domains 7

2.2.1 Heat Semigroup and pointwise formula 17

2.2.2 Dirichlet to Neumann operator 18

3 Solvability of IBVP **21**

3.1 Definition of solution 21

3.2 Equivalence Theorem 28

3.3 Basic estimates 31

4 Existence of Weak Solution **33**

4.1 Parabolic regularization 33

4.2 Limit transition 36

4.3 Existence of weak solution 39

5 Appendix **42**

5.1 Deformation 42

5.2 Interpolation spaces: J-method 44

5.3 Area and Coarea Formulas 46

5.4 Aubin-Lions's Theorem 49

Chapter 1

Introduction

The main purpose of this thesis is to study the existence of solutions for an initial-boundary value problem (IBVP) driven by a degenerated fractional heat type equation, that is, we consider

$$\begin{cases} \partial_t u = \operatorname{div}(u \, \nabla \mathcal{K} u) & \text{in } \Omega_T, \\ u|_{t=0} = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega_T := (0,T) \times \Omega$, for $T > 0$ is any real number, and $\Omega \subset \mathbb{R}^n$ is an open bounded domain having smooth (C^2) boundary $\partial\Omega$. Here, $u(t,x)$ is unknown real function, which can physically be an absolute temperature, or a density, also a concentration, thus non-negative. Moreover, the initial data u_0 is a measurable, bounded non-negative function in Ω, and we consider homogeneous Dirichlet boundary condition, while $\mathcal{K} = (-\Delta)^{-s}$, $0 < s < 1$, is the inverse of the $s - fractional$ Laplacian operator.

The nonlocal, possible degenerated, heat type equation is inspired in a non-local Fourier's law, that is

$$q := -\kappa(x,u) \; \nabla \mathcal{K} u,$$

where u is the temperature, q is the heat flux, and $\kappa(x,u)$ denotes here the (nonnegative definite) thermal conductivity tensor. For instance, if we suppose $\kappa(x,u) = 1$, we obtain

$$\partial_t u = \operatorname{div}(\nabla \mathcal{K} u).$$

This is mathematically the fractional version of the standard Heat equation, (written in the divergence form), that is recovered when $s = 0$.

In this thesis, we focus in the (simplest) isotropic degenerated case, that is, $\kappa(x,u) = u$. If we analyse the extreme cases, that is $s = 0$, we observe that $\mathcal{K}$ is the identity operator and we obtain the standard porous medium equation

$$\partial_t u + \frac{1}{2}(-\Delta)(u^2) = 0,$$

whose behavior is well-known (see [9], [18]), and for the other extremal $s = 1$ we have $\mathcal{K} = (-\Delta)^{-1}$, we get

$$\partial_t u = \nabla u \cdot \nabla p - u^2, \quad (-\Delta)p = u.$$

Particularly, in one dimension we obtain $u_t = u_x p_x - u^2$, $p_{xx} = -u$, which can be written as follows

$$u_t = u_x p_x + u p_{xx} = (u p_x)_x.$$

Moreover, if we define $v := -p_x = \int u$, we arrive at

$$v_t + v v_x = c(t),$$

and for $c(t) = 0$, we get the inviscid Burger's equation $v_t + v v_x = 0$. This is a model for non-linear wave propagation, specially in fluid mechanics.

In another context, equation (1.1) was proposed recently by Caffarelli, Vazquez in [2], where they considered a porous media (degenerated) diffusion model given by a fractional potential pressure law. Under some conditions they show existence of solutions for the Cauchy problem, hence the first equation in (1.1) was posed in $\mathbb{R}^n$, and in this case, the fractional Laplacian operator can be defined using Fourier Transform by

$$\widehat{(-\Delta)^s} f(\xi) = |\xi|^{2s} \hat{f}(\xi),$$

which means that the fractional Laplacian is a pseudo-differential operator with principal symbol $|\xi|^{2s}$. The fractional Laplacian can also be described using singular integrals in the following way:

$$(-\Delta)^s f(x) = C_{n,s}\, \text{P.V.} \int_{\mathbb{R}^n} \frac{f(x) - f(\xi)}{|x - \xi|^{n+2s}} d\xi,$$

where $C_{n,s}$ is a suitable constant. Moreover, its inverse, that is to say, $(-\Delta)^{-s}$, $(0 < s < 1)$, is given by convolution with the Riesz kernel $K_s(x) = C_{n,s}|x|^{2s-n}$, that is

$$\mathcal{K} f = K_s * f.$$

However, there does not exist a unified way to define the fractional Laplacian operator in bounded domains (see [19]). There are many choices, and until now a considerable confusion in the literature about the existence of different options. Because of this, we collect in Chapter 2 the main ingredients concerning the fractional Laplacian in bounded domains used in this thesis. More precisely, we use the so-called spectral fractional Laplacian, SFL for short, which is defined as follow

$$(-\Delta)^s u(x) = \sum_{k=1} \lambda_k^s u_k \varphi_k(x),$$

where $\lambda_k > 0$, $k = 1, 2, \cdots$, are the eigenvalues of the Dirichlet Laplacian on Ω with zero boundary condition, and φ_k are corresponding normalized eigenfunctions. In our case $\partial\Omega$ is C^2, then $\varphi_k \in C^\infty(\Omega) \cap C^2(\bar{\Omega})$, namely

$$u_k = \int_\Omega u(x)\varphi_k(x)dx \quad \|\varphi_k\|_{L^2(\Omega)} = 1.$$

Recall that, the zero boundary conditions are built into the definition of the operator. There is another way of defining the SFL using the Caffarelli-Silvestre extension, which turns out to be equivalent (see [3], [4], [5]).

Chapter 3 represent one of the most important parts of this thesis. Indeed, it is shown how the boundary condition is considered. For that, basically we used the notion of C^1-deformation (see Appendix, [16]). For more details, we refer to the Definition 3.1, where the boundary condition was enunciated as follow

$$\text{ess}\lim_{\tau\to 0^+} \int_0^T \int_{\partial\Omega} u(t, \Psi_\tau(r))\nabla\mathcal{K}u(t, \Psi_\tau(r)) \cdot \nu_\tau(\Psi_\tau(r))\gamma(t, r)\, drdt = 0,$$

for each $\gamma \in L^2((0, T) \times \partial\Omega)$, where Ψ is a C^1-deformation, and ν_τ is the unit outward normal field in $\partial\Omega_\tau$. Another important result obtained in this chapter is Theorem 3.1, which express in convenient way the concept of (weak) solution of the IBVP (1.1) as given by Definition 3.1. More precisely, we show an integral equivalent definition

$$\iint_{\Omega_T} u(t, x)\ (\partial_t\phi - \nabla\mathcal{K}u \cdot \nabla\phi)dxdt + \int_\Omega u_0(x)\,\phi(0, x)\ dx = 0,$$

for each test function $\phi \in C_c^\infty((-\infty, T) \times \mathbb{R}^n)$.

In Chapter 4, we prove the existence of (weak) solution as given by the Definition 3.1. To show that, first, we construct auxiliary problems, that is to say, we add a Laplacian term in the first equation of (1.1), and eliminate the degeneracy by raising the level set $\{u = 0\}$ in the diffusion coefficient. We take small numbers δ, $\mu \in (0, 1)$ and consider

$$\partial_t u_{\mu,\delta} - \delta\Delta u_{\mu,\delta} = \operatorname{div}((u_{\mu,\delta} + \mu)\nabla\mathcal{K}u_{\mu,\delta}) \quad \text{in } \Omega_T, \tag{1.2}$$

which has a classical solution. Once the auxiliary problems are solved, estimates are obtained (see Theorem 4.1), that allow us to pass to the limit in $u_{\mu,\delta}$ of (1.2) as $\mu, \delta \to 0^+$. So it is obtained a (weak) solution of (1.1).

The details of the limit transition are explained in Proposition 4.1 and Theorem 4.2. Moreover, the existence of (weak) solution was proved for an non-negative initial data $u_0 \in L^\infty(\Omega)$, and an open bounded domain with C^2 boundary.

In the Appendix, Section 5.1, gathers some useful definitions and properties. One of the most important is the concept of C^1-deformation, which was used to enunciate the definition of (weak) solution.

Besides the interesting mathematical aspects akin to the IBVP (1.1), one may stress some physical important reasons to consider this type of non-local models. First, we recall that, most of the classical partial differential equations describe some physical phenomena related to simple materials, a concept due to Noll [17]. This type of materials may have a perfect temporal memory, but only a very limited non-local effect. For instance, mixtures and porous medium cannot be modeled accurately under this assumption of simple material. Therefore, it seems that the fractional Laplacian operator, in fact its inverse (once we may consider partial differential equations in divergence form) can be used to take into account the presence of the long range interactions in non-simple materials.

Another very important context where the fractional Laplacian operator comes up is anomalous diffusion. For instance, turbulence is known experimentally (also numerically), to have anomalous diffusion, and the application of Lévy process to describe it, leads to partial differential models involving the fractional Laplacian.

Last but not least, it is well-known that any kind of real process modeled by Continuum Mechanics must have dissipation. Also, they could have, or even develop jumps in many applications, in particular related to velocity vector field. In this way the Laplacian operator is not allowed to be present in any kind of partial differential equation modeling these processes. On the other hand, the fractional Laplacian accept the existence of shocks and also gives an amount of dissipation. Consequently, partial differential equations driven by fractional Laplacian could give the correct regularity of solutions for a wide range of real applications.

Chapter 2

Notation and background

In this chapter we state some results of fractional Sobolev space and fractional Laplacian in bounded domain. One can refer to J. L. Lions and E. Magenes [11] and Luc Tartar [14] for an introduction.

2.1 Functional spaces

Throughout this section, we always consider $\Omega \subset \mathbb{R}^n$ a bounded Lipschitz domain, and we will always consider $s \in (0,1)$, unless explicitly stated.

2.1.1 The space $W^{s,p}(\Omega)$

We denote by dx, $d\xi$, etc. the Lebesgue measure on Ω, and by $L^p(\Omega)$ the set of (real or complex) summable functions w.r.t. the Lebesgue measure. Analogously, we have for the Sobolev spaces $W^{s,p}(\Omega)$, where a real $p \geqslant 1$ is the integrability index and a real $s \geqslant 0$ is the smoothness index. More precisely, for $s \in (0,1)$, $p \in [1, +\infty)$, the (fractional) Sobolev space of order s with Lebesgue exponent p is defined by

$$W^{s,p}(\Omega) := \Big\{ u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dxdy < +\infty \Big\},$$

endowed with norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |u|^p dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}.$$

Moreover, for $s > 1$ we write $s = m + \sigma$, where m is an integer and $\sigma \in (0,1)$. In this case, the space $W^{s,p}(\Omega)$ consists of those equivalence classes of functions $u \in W^{m,p}(\Omega)$ whose distributional derivatives $D^\alpha u$, with $|\alpha| = m$, belong to $W^{\sigma,p}(\Omega)$, that is

$$W^{s,p}(\Omega) = \Big\{ u \in W^{m,p}(\Omega) : \sum_{|\alpha|=m} \|D^\alpha u\|_{W^{\sigma,p}(\Omega)} < \infty \Big\},$$

and this is a Banach space with respect to the norm

$$\|u\|_{W^{s,p}(\Omega)} = \Big(\|u\|^p_{W^{m,p}(\Omega)} + \sum_{|u|=m} \|D^\alpha u\|^p_{W^{\sigma,p}(\Omega)} \Big)^{\frac{1}{p}}.$$

Clearly, if $s = m$ is an integer, the space $W^{s,p}(\Omega)$ coincides with the Sobolev space $W^{m,p}(\Omega)$. Also, it is very interesting the case when $p = 2$, i.e. $W^{s,2}(\Omega)$. In this case, the (fractional) Sobolev space is also a Hilbert space, and we can consider the inner product

$$\langle u, v\rangle_{W^{s,2}(\Omega)} = \langle u, v\rangle + \int_\Omega \int_\Omega \frac{(u(x) - u(y))}{|x-y|^{\frac{n}{2}+s}} \frac{(v(x) - v(y))}{|x-y|^{\frac{n}{2}+s}} \, dxdy,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\Omega)$. On the other hand, we can define the subspace $W^{s,p}(\Omega)$ as

$$W_0^{s,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{s,p}(\Omega)}}.$$

2.1.2 The space $H^s(\Omega)$, $H_0^s(\Omega)$ and $H_{00}^{1/2}(\Omega)$

Following J. L. Lions and E. Magenes [11], we can define the spaces $H^s(\Omega)$ by interpolation between $H^1(\Omega)$ and $L^2(\Omega)$, for $s \in (0, 1)$

$$H^s(\Omega) = [H^1(\Omega), L^2(\Omega)]_{1-s}$$

According to this definition, this space is a Hilbert space with the natural norm given by the interpolation Theorem 5.1. Moreover, we can now define the space $H_0^s(\Omega)$ by

$$H_0^s(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^s(\Omega)}}.$$

There exists an equivalent definition given via interpolation, namely Theorem 11.6 of [11], which states that

$$H_0^s(\Omega) = [H_0^1(\Omega), L^2(\Omega)]_{1-s},$$

for all $s \in (0, 1) - \{1/2\}$. The case $s = 1/2$ is special and generates a new space, called Lions-Magenes space defined by

$$H_{00}^{1/2}(\Omega) = [H_0^1(\Omega), L^2(\Omega)]_{1/2},$$

which has the following characterization

$$H_{00}^{1/2}(\Omega) = \left\{ u \in H^{1/2}(\Omega); \int_\Omega \frac{u^2}{dist(x, \partial\Omega)} dx < \infty \right\},$$

where $dist(x, \partial\Omega)$ is the distance from x to the boundary. Moreover, the interpolation norm is equivalent to

$$\|u\|_{H_{00}^{1/2}(\Omega)} := \left\{ \int_\Omega |u(x)|^2 \, dx + \int_\Omega \frac{u^2}{dist(x, \partial\Omega)} \, dx \right\}^{1/2}.$$

We now state here a theorem, which relate $H^s(\Omega)$ and $H_0^s(\Omega)$ spaces.

Theorem 2.1. *Let Ω a bounded open set with a Lipschitz boundary, then $C_c^\infty(\Omega)$ is dense in $H^s(\Omega)$ if and only if $0 < s \leq 1/2$; in this case we have that $H_0^s(\Omega) = H^s(\Omega)$. If $s > 1/2$, then $H_0^s(\Omega) \subset H^s(\Omega)$ and the inclusion is strict.*

Proof. To prove the Theorem, see J. L. Lions and E. Magenes Theorem 11.1 of [11] and Luc Tartar [14] p. 160. □

Finally, we state a Theorem, which characterize functions in $H_0^s(\Omega)$ when $1/2 < s \leq 1$.

Theorem 2.2 (Trace)**.** *Let Ω a bounded open set with a Lipschitz boundary, Then exist an operator T, which is well defined from*

$$T : H^s(\Omega) \to L^2(\partial\Omega),$$

for $s > 1/2$. In this case, it can be shown $T^{-1}(0) = H_0^s(\Omega)$. Therefore, one can characterizes the functions of $H_0^s(\Omega)$ when $1/2 < s \leq 1$ by

$$H_0^s(\Omega) := \{u \in H^s(\Omega) : u = 0 \text{ on } \partial\Omega \text{ in the sense of trace}\}.$$

Proof. To Prove the first part of the Theorem see J. L. Lions and E. Magenes [11] Theorem 9.4, p. 41-42 and the characterization part see Theorem 11.5, p. 62 of the cited reference. □

Remark 2.1. *For a bounded open set Ω, with Lipschitz boundary, the spaces $W^{s,2}(\Omega)$ and $H^s(\Omega)$ are the same space, with equivalent norms, see Luc Tartar [14] p. 83.*

2.2 Fractional Laplacian in bounded domains

Here and subsequently, $\Omega \subset \mathbb{R}^n$ is a bounded set with C^2-boundary $\partial\Omega$. We are mostly interested in fractional powers of a strictly positive self-adjoint operator defined in a domain, which is dense in a (separable) Hilbert space. We are going to consider hereupon the Laplacian operator $(-\Delta)$ with homogeneous Dirichlet data. Due to well known spectral theory of the $(-\Delta)$ operator in Ω, there exists a complete orthonormal basis $\{\varphi_k\}_{k=1}^\infty$ of $L^2(\Omega)$, $\varphi_k \in H_0^1(\Omega)$ eigenfunction corresponding to λ_k for each $k \geq 1$, where one repeats each eigenvalue λ_k according to its (finite) multiplicity:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \to \infty, \quad \text{as } k \longrightarrow \infty.$$

Consequently, we have

$$D(-\Delta) = \{u \in L^2(\Omega);\ \sum_{k=1}^{\infty} \lambda_k^2 \, |\langle u, \varphi_k\rangle|^2 < \infty\},$$

$$(-\Delta)u = \sum_{k=1}^{\infty} \lambda_k \, \langle u, \varphi_k\rangle \; \varphi_k, \quad \text{for each } u \in D(-\Delta).$$

Applying functional calculus, see J. L. Lions and E. Magenes [11], we define for $s > 0$, the fractional Laplacian $(-\Delta)^s : D\big((-\Delta)^s\big) \subset L^2(\Omega) \to L^2(\Omega)$, given by

$$\begin{aligned} (-\Delta)^s u &:= \sum_{k=1}^{\infty} \lambda_k^s \, \langle u, \varphi_k \rangle \; \varphi_k, \\ D\big((-\Delta)^s\big) &= \Big\{ u \in L^2(\Omega) : \; \sum_{k=1}^{\infty} \lambda_k^{2s} \, |\langle u, \varphi_k \rangle|^2 < +\infty \Big\}. \end{aligned} \tag{2.1}$$

Analogously, we can also define $(-\Delta)^{-s} : D\big((-\Delta)^{-s}\big) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$\begin{aligned} (-\Delta)^{-s} u &:= \sum_{k=1}^{\infty} \lambda_k^{-s} \, \langle u, \varphi_k \rangle \; \varphi_k, \\ D\big((-\Delta)^{-s}\big) &= \Big\{ u \in L^2(\Omega) : \; \sum_{k=1}^{\infty} \lambda_k^{-2s} \, |\langle u, \varphi_k \rangle|^2 < +\infty \Big\}. \end{aligned} \tag{2.2}$$

The next theorem give us the main properties of the fractional Laplacian in bounded domains. In particular, we observe that $D\big((-\Delta)^{-s}\big) = L^2(\Omega)$.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain and $s > 0$, consider $(-\Delta)^s$, and $(-\Delta)^{-s}$ the operators defined respectively by* (2.1)*,* (2.2)*. Then, we have:*

(1) *For $s \in (0,1)$, we have $D\big((-\Delta)\big) \subset D\big((-\Delta)^s\big)$, thus $D\big((-\Delta)^s\big)$ is dense in $L^2(\Omega)$.*

(2) *For all $u \in D\big((-\Delta)^s\big)$, exists $\alpha > 0$ which is the coercivity constant of $(-\Delta)$ and satisfies*

$$\langle (-\Delta)^s u, u \rangle \geq \alpha^s \|u\|^2_{L^2(\Omega)}. \tag{2.3}$$

Moreover, $((-\Delta)^s)^{-1} = (-\Delta)^{-s}$, $(-\Delta)^s$ and $(-\Delta)^{-s}$ are self-adjoint.

(3) *$D\big((-\Delta)^s\big)$ endowed with inner product defined by*

$$\langle u, v \rangle_s := \langle u, v \rangle + \int_\Omega (-\Delta)^s u \, (-\Delta)^s v \, dx \tag{2.4}$$

is a Hilbert space.

(4) *For each $u \in D\big((-\Delta)^{s_2}\big)$ and $0 < s_1 \leq s_2$, we have*

$$\|(-\Delta)^{s_1} u\|^{s_2}_{L^2(\Omega)} \leq \|(-\Delta)^{s_2} u\|^{s_1}_{L^2(\Omega)} \, \|u\|^{s_2 - s_1}_{L^2(\Omega)}. \tag{2.5}$$

(5) *Let* $u \in D\big((-\Delta)\big)$, *then*

$$\lim_{s\to 0^+}(-\Delta)^s u = u, \lim_{s\to 1^-}(-\Delta)^s u = (-\Delta)u \quad \text{in } L^2(\Omega). \tag{2.6}$$

Furthermore, (2.5) *holds for* $s_1 = 0$, *and* $s_2 = 1$.

(6) *For any* $\lambda > 0$, *and* $s > 0$, *the operator* $I + \lambda(-\Delta)^s$ *is bijective. Moreover, for any* $v \in L^2(\Omega)$, *the family* $\{v_\lambda\}$, $v_\lambda \in D((-\Delta)^s)$ *defined by*

$$v_\lambda := (I + \lambda(-\Delta)^s)^{-1}v \tag{2.7}$$

converges to v *in* $L^2(\Omega)$ *as* $\lambda \to 0$.

(7) *If* $0 < s_1 < s_2$, *then* $D\big((-\Delta)^{s_2}\big) \hookrightarrow D\big((-\Delta)^{s_1}\big)$. *Moreover, when* $0 < s_1 < s_2 < 1$ *we have* $D\big((-\Delta)^{s_2}\big)$ *is dense in* $D\big((-\Delta)^{s_1}\big)$.

Proof. 1. To Prove (1), let $u \in D((-\Delta))$, and $k_0 \in \mathbb{N}$ be such that $1 \le \lambda_{k_0}$. Then, we have

$$\sum_{j=k_0}^{\infty} \lambda_j^{2s}|\langle u, \varphi_k\rangle|^2 \le \sum_{j=k_0}^{\infty} \lambda_j^{2}|\langle u, \varphi_k\rangle|^2 < \infty.$$

Therefore, we get

$$\sum_{j=1}^{\infty} \lambda_j^{2s}|\langle u, \varphi_k\rangle|^2 < \infty, \quad u \in D((-\Delta)),$$

which implies $u \in D((-\Delta)^s)$, and since $D((-\Delta))$ is dense in $L^2(\Omega)$, so also is $D((-\Delta)^s)$.

2. To show (2), first note that for all $u \in D\big((-\Delta)^s\big)$, it follows that

$$\langle(-\Delta)^s u, u\rangle = \sum_{k=1}^{\infty} \lambda_k^s|\langle u, \varphi_k\rangle|^2 \ge \lambda_1^s \sum_{k=1}^{\infty} |\langle u, \varphi_k\rangle|^2 = \lambda_1^s\|u\|^2_{L^2(\Omega)}, \tag{2.8}$$

which is (2.3) with $\alpha = \lambda_1$, moreover $(-\Delta)^s$ is injective. Now, we show that $(-\Delta)^s$ is also subjective, therefore $((-\Delta)^s)^{-1}$ exists and belong to $\mathcal{L}(L^2(\Omega))$.

To prove that $(-\Delta)^s$ is subjective consider $u \in L^2(\Omega)$ and let v be defined by

$$v := \sum_{k=1}^{\infty} \lambda_k^{-s}\langle u, \varphi_k\rangle\varphi_k.$$

Then, it is easy to see that $v \in D\big((-\Delta)^s\big)$ and $(-\Delta)^s v = u$. Thus $(-\Delta)^s$ is invertible, therefore $((-\Delta)^s)^{-1}$ exists.

Now we show $((-\Delta)^s)^{-1} = (-\Delta)^{-s}$. Indeed, for each $u \in D\big((-\Delta)^s\big)$, and $v \in L^2(\Omega)$

$$
\begin{aligned}
\langle(-\Delta)^{-s}(-\Delta)^s u, v\rangle &= \sum_{k=1}^{\infty} \lambda_k^{-s} |\langle(-\Delta)^s u, \varphi_k\rangle||\langle v, \varphi_k\rangle| \\
&= \sum_{k=1}^{\infty} \lambda_k^{-s} \lambda_k^{s} |\langle u, \varphi_k\rangle||\langle v, \varphi_k\rangle| = \langle u, v\rangle,
\end{aligned}
$$

and hence $(-\Delta)^{-s}$ is the left inverse of $(-\Delta)^s$, therefore $(-\Delta)^{-s} = ((-\Delta)^s)^{-1}$.

Finally we prove $(-\Delta)^s$ and $(-\Delta)^{-s}$ are self-adjoint, since

$$
\begin{aligned}
\langle(-\Delta)^{-s} u, v\rangle &= \sum_{k=1}^{\infty} \lambda_k^{-s} |\langle u, \varphi_k\rangle||\langle v, \varphi_k\rangle| \\
&= \sum_{k=1}^{\infty} |\langle u, \varphi_k\rangle| \lambda_k^{-s} |\langle v, \varphi_k\rangle| = \langle u, (-\Delta)^{-s} v\rangle.
\end{aligned}
$$

Analogously, we can show that $(-\Delta)^s$ is self-adjoint.

3. To show (3), let $|\cdot|_s$ be the norm associated to the scalar product (2.4):

$$
|u|_s^2 = \|u\|_{L^2(\Omega)}^2 + \|(-\Delta)^s u\|_{L^2(\Omega)}^2. \tag{2.9}
$$

If $\{u_n\}$ is a Cauchy sequence for $|\cdot|_s$, then $\{u_n\}$ and $\{(-\Delta)^s u_n\}$ are Cauchy sequences in $L^2(\Omega)$. Therefore, there exists $u, v \in L^2(\Omega)$ such that, as $n \to \infty$

$$
u_n \to u, \quad (-\Delta)^s u_n \to v,
$$

and since $(-\Delta)^s$ is closed, $v = (-\Delta)^s u$.

4. To prove (4), note that $s_1 = s_2$ is trivial, so let us consider $s_1 < s_2$. For each $u \in D\big((-\Delta)^{s_2}\big)$, applying Holder's inequality with $p = \dfrac{s_2}{s_1}$, and $q = \dfrac{s_2}{s_2 - s_1}$

$$
\begin{aligned}
\sum_{k=1}^{\infty} \lambda_k^{2s_1} |\langle u, \varphi_k\rangle|^2 &= \sum_{k=1}^{\infty} \lambda_k^{2s_1} |\langle u, \varphi_k\rangle|^{2s_1/s_2} |\langle u, \varphi_k\rangle|^{2-2s_1/s_2} \\
&\leq \left\{\sum_{k=1}^{\infty} \left(\lambda_k^{2s_1} |\langle u, \varphi_k\rangle|^{2s_1/s_2}\right)^p\right\}^{1/p} \left\{\sum_{k=1}^{\infty} \left(|\langle u, \varphi_k\rangle|^{2-2s_1/s_2}\right)^q\right\}^{1/q} \\
&\leq \left\{\sum_{k=1}^{\infty} \lambda_k^{2s_2} |\langle u, \varphi_k\rangle|^2\right\}^{1/p} \left\{\sum_{k=1}^{\infty} |\langle u, \varphi_k\rangle|^2\right\}^{1/q},
\end{aligned}
$$

which implies (2.5).

5. It remains to show (5), let $u \in D\big((-\Delta)\big)$, then observe

$$\|(-\Delta)^s u - u\|^2_{L^2(\Omega)} = \|(-\Delta)^s u\|^2_{L^2(\Omega)} - 2\langle(-\Delta)^s u, u\rangle + \|u\|^2_{L^2(\Omega)},$$

from the coercivity (2.3) we get

$$\|(-\Delta)^s u - u\|^2_{L^2(\Omega)} \leq \|(-\Delta)^s u\|^2_{L^2(\Omega)} - 2\alpha^s\|u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)},$$

finally using (2.5) with $s_1 = s$ and $s_2 = 1$, we obtain

$$\|(-\Delta)^s u - u\|^2_{L^2(\Omega)} \leq \|(-\Delta)u\|^{2s}_{L^2(\Omega)}\|u\|^{2(1-s)}_{L^2(\Omega)} - 2\alpha^s\|u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)},$$

then passing the limit as $s \to 0^+$, we have $(-\Delta)^s u$ converge to u in $L^2(\Omega)$ as $s \to 0^+$. Analogously, we show that $(-\Delta)^s u$ converge to $(-\Delta)u$ in $L^2(\Omega)$ as $s \to 1^-$.

6. To prove (6). First, we show that the operator $I + \lambda(-\Delta)^s$ is bijective. From the coercivity of $(-\Delta)^s$, we have

$$\|u + \lambda(-\Delta)^s u\|_{L^2(\Omega)} \geq \|u\|_{L^2(\Omega)}, \tag{2.10}$$

which implies that the linear operator $I + \lambda(-\Delta)^s$ is injective. On the other hand, for all $f \in L^2(\Omega)$ there exists $u \in D((-\Delta)^s)$, such that

$$u + \lambda(-\Delta)^s u = f. \tag{2.11}$$

Indeed, it is enough to take

$$u = \sum_{k=1}^{\infty} \frac{\langle f, \varphi_k\rangle}{1 + \lambda\lambda_k^s}\varphi_k,$$

which satisfies (2.11) and $u \in D((-\Delta)^s)$. Therefore $(I + \lambda(-\Delta)^s)$ is a bijective operator.

To show the converge $v_\lambda \to v$ for all $v \in L^2(\Omega)$ as $\lambda \to 0$, we first assume that $v \in D((-\Delta)^s)$ and from (2.7) we observe that

$$(-\Delta)^s v_\lambda = (-\Delta)^s(I + \lambda(-\Delta)^s)^{-1}v = (I + \lambda(-\Delta)^s)^{-1}(-\Delta)^s v.$$

Then, we have

$$v - v_\lambda = \lambda(-\Delta)^s v_\lambda = \lambda(I + \lambda(-\Delta)^s)^{-1}(-\Delta)^s v, \tag{2.12}$$

then from (2.10) and (2.12) implies $v_\lambda \to v$ in $L^2(\Omega)$ as $\lambda \to 0$. Finally, the result follows for $v \in L^2(\Omega)$ applying a standard density argument.

7. It remains to proof (7). First, we show $D((-\Delta)^{s_2}) \hookrightarrow D((-\Delta)^{s_1})$, for $0 < s_1 < s_2$. Consider $u \in D((-\Delta)^{s_2})$ and observe

$$\sum_{k=1}^{\infty} \lambda_k^{2s_1}|\langle u, \varphi_k\rangle|^2 = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{2s_2-2s_1}}\lambda_k^{2s_2}|\langle u, \varphi_k\rangle|^2 \leq \lambda_1^{-2(s_2-s_1)} \sum_{k=1}^{\infty} \lambda_k^{2s_2}|\langle u, \varphi_k\rangle|^2.$$

Hence for $s_2 > s_1$, and each $u \in D((-\Delta)^{s_2})$

$$|u|_{s_1} \leq C(\lambda_1, s_1, s_2)|u|_{s_2}.$$

To show that $D((-\Delta)^{s_2})$ is dense in $D((-\Delta)^{s_1})$ for $0 < s_1 < s_2 < 1$, it is sufficiently to show that, for all $s \in (0,1)$, $D((-\Delta))$ is dense in $D((-\Delta)^s)$. Indeed, for $u \in D((-\Delta)^s)$ let us define for all $n \geq 1$

$$u_n := (I + \frac{1}{n}(-\Delta))^{-1}u. \tag{2.13}$$

Consequently, $u_n \in D((-\Delta))$ and

$$(-\Delta)^s u_n + \frac{1}{n}(-\Delta)(-\Delta)^s u_n = (-\Delta)^s u. \tag{2.14}$$

Now, defining $z := (I + \frac{1}{n}(-\Delta))^{-1}(-\Delta)^s u$, we obtain

$$z + \frac{1}{n}(-\Delta)z = (-\Delta)^s u. \tag{2.15}$$

From (2.14), (2.15)

$$(z - (-\Delta)^s u_n) + \frac{1}{n}(-\Delta)(z - (-\Delta)^s u_n) = 0,$$

and since $I + \frac{1}{n}(-\Delta)$ is an injective operator, it follows that $(-\Delta)^s u_n = z$. In other words,

$$(-\Delta)^s u_n = (I + \frac{1}{n}(-\Delta))^{-1}(-\Delta)^s u. \tag{2.16}$$

Passing to the limit in (2.13), (2.16) as $n \to \infty$,and using the item (6), we obtain

$$u_n \to u, \quad (-\Delta)^s u_n \to (-\Delta)^s u \quad \text{in } L^2(\Omega),$$

thus $u_n \in D((-\Delta))$, and $u_n \to u$ in $D((-\Delta)^s)$ as $n \to \infty$, which finish the proof. $\square$

We now state a Poincare's type inequality for fractional Laplacian and an equivalence norm for the space $D\big((-\Delta)^s\big)$.

Corollary 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain, then for each $s > 0$, there exists a positive constant C_Ω, such that*

$$\|u\|_{L^2(\Omega)} \leq C_\Omega \|(-\Delta)^s u\|_{L^2(\Omega)}, \quad \text{for all } u \in D\big((-\Delta)^s\big).$$

Moreover, the norm defined in (2.9) *and the norm*

$$\|u\|_s^2 = \int_\Omega |(-\Delta)^s u|^2 dx, \tag{2.17}$$

are equivalent.

Hereupon, we consider the following inner product on $D\big((-\Delta)^s\big)$

$$\langle u, v\rangle_s = \int_\Omega (-\Delta)^s u(x)(-\Delta)^s v(x)dx. \tag{2.18}$$

On the other hand, we obtain an equivalent result for the inverse of fractional Laplacian, more precisely $(-\Delta)^{-s}$ is continuous operator.

Corollary 2.2. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain, for each $s > 0$, the operator $(-\Delta)^{-s}$ is continuous from $L^2(\Omega)$ to itself, i.e., there exists a positive constant C_Ω, such that*

$$\|(-\Delta)^{-s}u\|_{L^2(\Omega)} \leq C_\Omega \|u\|_{L^2(\Omega)},$$

for all $u \in L^2(\Omega)$.

We now state a characterization for $D((-\Delta)^s)$, $s \in (0,1)$ via interpolation. In particular, we identify the space $D((-\Delta)^{s/2})$, $s \in (0,1)$ with the fractional Sobolev space. To begin, let us recall that, for $u \in D\big((-\Delta)\big)$ we have

$$\|(-\Delta)^{1/2}u\|_{L^2(\Omega)} = \|u\|_{H_0^1(\Omega)}. \tag{2.19}$$

Consequently, from the density of $D\big((-\Delta)\big)$ in $D\big((-\Delta)^{1/2}\big)$, and also in $H_0^1(\Omega)$, it follows that $D\big((-\Delta)^{1/2}\big) = H_0^1(\Omega)$. Similarly, we have the two following result.

Proposition 2.2. *Let Ω a bounded open set with a Lipschitz boundary, $s \in (0,1)$, then*

$$D\big((-\Delta)^{s/2}\big) = \begin{cases} H^s(\Omega), & if \quad 0 < s < 1/2, \\ H_{00}^{1/2}(\Omega), & if \quad s = 1/2, \\ H_0^s(\Omega), & if \quad 1/2 < s < 1. \end{cases} \tag{2.20}$$

Proof. First, from Section 2.1.2, it is well known

$$\big[H_0^1(\Omega), L^2(\Omega)\big]_{1-s} = \begin{cases} H^s(\Omega), & \text{if } 0 < s < 1/2, \\ H_{00}^{1/2}(\Omega), & \text{if } s = 1/2, \\ H_0^s(\Omega), & \text{if } 1/2 < s < 1. \end{cases}$$

Thus, it is enough to show that $D\big((-\Delta)^{s/2}\big) = \big[H_0^1(\Omega), L^2(\Omega)\big]_{1-s}$. To prove this identification, we apply the discrete version of the J-Method for interpolation, see Appendix.

We recall the definition $D\big((-\Delta)^{s/2}\big) \subset L^2(\Omega)$, namely

$$D\big((-\Delta)^{s/2}\big) = \Big\{u \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^s \, |\langle u, \varphi_k\rangle|^2 < +\infty\Big\},$$

where $\{\varphi_k\}_{k=1}^{\infty} \subset H_0^1(\Omega)$ is orthonormal basis of $L^2(\Omega)$, therefore

$$u = \sum_{k=1}^{\infty} u_k \varphi_k, \quad u_k = \langle u, \varphi_k \rangle.$$

In order to apply the discrete version of the J-Method (see Theorem 5.2), for $u \in D\big((-\Delta)^{s/2}\big)$ we define the following sequence

$$\begin{aligned} U_k &= \left(\lambda_k^{1/2}\right)^{-\theta} J\left(\lambda_k^{1/2}, u_k\varphi_k\right) \\ &= \lambda_k^{-\theta/2} \max\left\{\|u_k\varphi_k\|_{H_0^1(\Omega)}, \lambda_k^{1/2}\|u_k\varphi_k\|_{L^2(\Omega)}\right\} \\ &= \lambda_k^{(1-\theta)/2}|u_k| \end{aligned}$$

and it is clear that $U_k \in \ell^2(\mathbb{N})$ if and only if $\theta = 1 - s$, and that

$$\|u\|_s = \|U_k\|_{\ell^2(\mathbb{N})}.$$

The discrete version of the J-Method, namely Theorem 5.2, allows to identify $D\big((-\Delta)^{s/2}\big)$ as the interpolation space

$$D\big((-\Delta)^{s/2}\big) = \big[H_0^1(\Omega), L^2(\Omega)\big]_{1-s},$$

and the norms are equivalent. □

Proposition 2.3. *Let Ω be an open bounded set in $\mathbb{R}^n$, $s \in (0,1)$, then*

$$D\big((-\Delta)^{(1+s)/2}\big) = \big[H^2(\Omega) \cap H_0^1(\Omega), H_0^1(\Omega)\big]_{1-s}. \tag{2.21}$$

Moreover,

$$D\big((-\Delta)^{(1+s)/2}\big) \subset H^{1+s}(\Omega) \cap H_0^1(\Omega).$$

Proof. To prove the above identification, we apply the discrete version of the J-Method for interpolation, see Appendix.

We recall the definition $D\big((-\Delta)^{(1+s)/2}\big) \subset L^2(\Omega)$, namely

$$D\big((-\Delta)^{(1+s)/2}\big) = \Big\{u \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{1+s} \, |\langle u, \varphi_k \rangle|^2 < +\infty\Big\},$$

where $\{\varphi_k\}_{k=1}^{\infty} \subset H_0^1(\Omega)$ is orthonormal basis of $L^2(\Omega)$, therefore

$$u = \sum_{k=1}^{\infty} u_k \varphi_k, \quad u_k = \langle u, \varphi_k \rangle.$$

In order to apply the discrete version of the J-Method (see Theorem 5.2), for $u \in D\big((-\Delta)^{(1+s)/2}\big)$ and define the following sequence

$$\begin{aligned} U_k &= \left(\lambda_k^{1/2}\right)^{-\theta} J\left(\lambda_k^{1/2}, u_k\varphi_k\right) \\ &= \lambda_k^{-\theta/2} \max\left\{ \|u_k\varphi_k\|_{H^2(\Omega)\cap H_0^1(\Omega)}, \lambda_k^{1/2}\|u_k\varphi_k\|_{H_0^1(\Omega)} \right\}. \end{aligned}$$

On other hand, we observe that for k enough large we have

$$U_k \leq C\lambda_k^{-\theta/2}\lambda_k|u_k|,$$

where C is a constant, and it is clear that $U_k \in \ell^2(\mathbb{N})$ if and only if $\theta = 1-s$.

The discrete version of the J-Method, namely Theorem 5.2, allows to identify $D\big((-\Delta)^{(1+s)/2}\big)$ as the interpolation space

$$D\big((-\Delta)^{(1+s)/2}\big) = \big[H^2(\Omega) \cap H_0^1(\Omega), H_0^1(\Omega)\big]_{1-s},$$

and the norms are equivalent. Moreover from the definition of interpolation we have

$$\big[H^2(\Omega)\cap H_0^1(\Omega), H_0^1(\Omega)\big]_{1-s} \subset \big[H^2(\Omega), H^1(\Omega)\big]_{1-s}\cap H_0^1(\Omega) = H^{1+s}(\Omega)\cap H_0^1(\Omega)$$

see Lions, Magenes [11]. Thus we get

$$D\big((-\Delta)^{(1+s)/2}\big) \subset H^{1+s}(\Omega) \cap H_0^1(\Omega).$$

□

Now, we define for each $s > 0$, the operator $\mathcal{K} : L^2(\Omega) \longrightarrow D\big((-\Delta)^s\big)$ given by $\mathcal{K}u := (-\Delta)^{-s}u$, and analogously $\mathcal{H}u := (-\Delta)^{-s/2}u$, moreover they are continuous operator due to the Corollary 2.2. Moreover, we have the following

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain.*

(1) *If $u \in H_0^1(\Omega)$, then $\mathcal{K}u \in D\big((-\Delta)^{1/2+s}\big)$ and $\mathcal{H}u \in D\big((-\Delta)^{(1+s)/2}\big)$.*

(2) *There exists a positive constant C_Ω, such that*

$$\int_\Omega |\nabla\mathcal{K}u(x)|^2 dx \leq C_\Omega \int_\Omega |\nabla u(x)|^2\, dx, \tag{2.22}$$

for each $u \in H_0^1(\Omega)$.

(3) *If $u \in H_0^1(\Omega)$, then*

$$\int_\Omega \nabla\mathcal{K}u(x)\cdot\nabla u(x)dx = \int_\Omega |\nabla\mathcal{H}u(x)|^2\ dx. \tag{2.23}$$

Proof. 1. To prove (1). First, recall the definition $D\big((-\Delta)^{1/2+s}\big)$ (see (2.1)),

$$D\big((-\Delta)^{1/2+s})\big) = \Big\{u \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{1+2s} \, |\langle u, \varphi_k\rangle|^2 < +\infty\Big\}, \qquad (2.24)$$

where $\{\varphi_k\}_{k=1}^{\infty} \subset H_0^1(\Omega)$ is orthonormal basis of $L^2(\Omega)$.

Now, we show that $\mathcal{K}u \in D\big((-\Delta)^{1/2+s}\big)$, for all $u \in H_0^1(\Omega)$. Indeed, let $u \in H_0^1(\Omega)$, and observe that $\langle \mathcal{K}u, \varphi_k\rangle = \lambda_k^{-s}\langle u, \varphi_k\rangle$, then we obtain

$$\sum_{k=1}^{\infty} \lambda_k^{1+2s} \, |\langle \mathcal{K}u, \varphi_k\rangle|^2 = \sum_{k=1}^{\infty} \lambda_k^{1+2s} \, |\lambda_k^{-s}\langle u, \varphi_k\rangle|^2 = \sum_{k=1}^{\infty} \lambda_k \, |\langle u, \varphi_k\rangle|^2.$$

On the other hand, since $u \in H_0^1(\Omega) = D((-\Delta)^{1/2})$ (see (2.19)), we have $\sum \lambda_k \, |\langle u, \varphi_k\rangle|^2 < \infty$, which implies that $\mathcal{K}u \in D\big((-\Delta)^{1/2+s}\big)$. In particular, from Proposition 2.1, (item 7) we have that $\mathcal{K}u \in D((-\Delta)^{1/2}) = H_0^1(\Omega)$. Analogously, $\mathcal{H}u \in D\big((-\Delta)^{(1+s)/2}\big)$.

2. To show (2). Let $u \in H_0^1(\Omega)$, then using the equivalence norm between $D((-\Delta)^{1/2})$ and $H_0^1(\Omega)$ (see (2.19)) together with $\langle \mathcal{K}u, \varphi_k\rangle = \lambda_k^{-s}\langle u, \varphi_k\rangle$, we have

$$\begin{aligned}
\int_\Omega |\nabla \mathcal{K}u(x)|^2 dx &= \sum_{k=1}^{\infty} \lambda_k |\langle \mathcal{K}u, \varphi_k\rangle|^2 = \sum_{k=1}^{\infty} \lambda_k |\lambda_k^{-s}\langle u, \varphi_k\rangle|^2 \\
&\leq \lambda_1^{-2s} \sum_{k=1}^{\infty} \lambda_k |\langle u, \varphi_k\rangle|^2 = \lambda_1^{-2s} \int_\Omega |\nabla u(x)|^2 dx,
\end{aligned}$$

which implies (2.22).

3. To show (3). First, we consider $u \in C_0^\infty(\Omega)$, then using integration by part together with the definition of $\mathcal{K}u$, we have

$$\int_\Omega \nabla \mathcal{K}u(x) \cdot \nabla u(x) \, dx = -\int_\Omega \Delta \mathcal{K}u(x) u(x) \, dx = \int_\Omega (-\Delta)^{1-s} u(x) u(x) \, dx,$$

due to the fractional Laplacian being self-adjoint (Proposition 2.1, item 2), we arrive to

$$\int_\Omega \nabla \mathcal{K}u(x) \cdot \nabla u(x) \, dx = \int_\Omega |(-\Delta)^{(1-s)/2} u(x)|^2 \, dx,$$

using the equivalence norm (2.19) together with the definition of $\mathcal{H}u$, consequently we obtain

$$\int_\Omega \nabla \mathcal{K}u(x) \cdot \nabla u(x) \, dx = \int_\Omega |\nabla \mathcal{H}u|^2 \, dx.$$

Finally, in view of the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$, we obtain (2.23). $\square$

Remark 2.2. *We observed that, in the previous proof was not used that ∇ commutes with $\mathcal{K}$. In effect, they do not commute.*

We recall another definitions of fractional Laplacian in bounded domains, which are equivalent to that one given above.

2.2.1 Heat Semigroup and pointwise formula

First, given a function $u = \sum_{k=1}^{\infty} u_k \varphi_k$ in $L^2(\Omega)$, the weak solution $v(x,t)$ to the IBVP

$$\begin{cases} v_t - \Delta v = 0, & \text{in } \Omega \times (0, +\infty), \\ v(x,t) = 0, & \text{on } \partial\Omega \times [0, +\infty), \\ v(x,0) = u(x), & \text{in } \Omega \end{cases}$$

is given by

$$v(x,t) = e^{-t(-\Delta)} u(x) = \sum_{k=1}^{\infty} e^{-t\lambda_k} u_k \varphi_k(x),$$

which satisfies the following properties:

- $v \in L^2((0,\infty); H_0^1(\Omega)) \cap C([0,\infty); L^2(\Omega))$.
- $\partial_t v \in L^2((0,\infty); H^{-1}(\Omega))$.

Then, the next lemma shows another representation for $(-\Delta)^s$ and its inverse, due to heat semigroup.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain in $\mathbb{R}^n$ and $0 < s < 1$.*

1. *If $u \in D\big((-\Delta)^s\big)$, then*

$$(-\Delta)^s u = \frac{1}{\Gamma(-s)} \int_0^{\infty} (e^{-t(-\Delta)} u - u) \frac{dt}{t^{1+s}}, \quad in\, L^2(\Omega).$$

2. *If $u \in L^2(\Omega)$, then*

$$(-\Delta)^{-s} u = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-t(-\Delta)} u \frac{dt}{t^{1-s}}, \quad in\, L^2(\Omega).$$

Proof. Let us give a sketch of the proof. First we recall that, the Gamma function is usually defined respectively, for $x > 0$ and $-k < x < -k+1$ ($k \in \mathbb{N}$) by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt,$$

$$\Gamma(x) = \int_0^{\infty} t^{x-1} \left[e^{-t} - \sum_{j=0}^{k-1} \frac{(-t)^j}{j!} \right] dt.$$

Therefore, for $\lambda > 0$ and $0 < s < 1$

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda} \frac{dt}{t^{1-s}},$$

$$\lambda^{s} = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-t\lambda} - 1\right) \frac{dt}{t^{1+s}},$$

Now, from definitions (2.1), (2.2) and Fubini's Theorem, the proof follows. $\square$

2.2.2 Dirichlet to Neumann operator

As already mentioned, an important feature of the fractional Laplacian is its nonlocal character, which could be seen by realizing it as the boundary operator of a suitable extension in the half-cylinder $(0, +\infty) \times \Omega$, which is usually called Dirichlet to Neumann operator.

We recall that, the first general result in this direction was done by Caffarelli and Silvestre [3]. In that paper, they proved that any fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$ for $s \in (0, 1)$ can be determined from the extension problem. Indeed, given $f : \mathbb{R}^n \to \mathbb{R}$ let u be the solution of the extension problem

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla u) \equiv u_{yy} + \dfrac{1-2s}{y} u_y + \Delta u = 0, & (0, +\infty) \times \mathbb{R}^n \\ \qquad\qquad u(0, x) = f(x) & x \in \mathbb{R}^n, \end{cases} \tag{2.25}$$

where div and ∇ act in all (y, x) variables and Δ only on x variable. Moreover, we assume that $\lim_{y\to\infty} u(y, x) = 0$. Then, the relation between the solution u of (2.25) and f is described by

$$(-\Delta)^s f(x) = -\frac{1}{\kappa_s} \lim_{y\to 0^+} y^{1-2s} \frac{\partial u}{\partial y}(y, x), \tag{2.26}$$

where $\kappa_s = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$ and Γ is the standard gamma function.

We give a sketch of the proof for $s = 1/2$. In this case the equation (2.25) is reduced to

$$\begin{cases} u_{yy} + \Delta u = 0, & (y, x) \in (0, +\infty) \times \mathbb{R}^n, \\ u(0, x) = f(x) & x \in \mathbb{R}^n. \end{cases}$$

Taking Fourier transform in x, we obtain

$$\hat{u}_{yy}(y, \xi) - |\xi|^2 \hat{u}(y, \xi) = 0,$$

and $\hat{u}(0,\xi) = \hat{f}(\xi)$. Therefore $\hat{u}(y,\xi) = \hat{f}(\xi)e^{-|\xi|y}$ is the solution of the above problem. Then deriving and passing to the limit as $y \to 0$, we have

$$-\lim_{y\to 0} \hat{u}_y(y,\xi) = |\xi|\hat{f}(\xi)$$

which is equivalent to

$$(-\Delta)^{1/2} f(x) = -\lim_{y\to 0} u_y(y,x)$$

Thus we conclude that (2.26) follows for $s = 1/2$. This technique could be used for the general case. To get a complete precise description, see Caffarelli and Silvestre [3].

Now, let us consider an analogous result of the above formulation for bounded domains with Dirichlet homogeneous data. To follow, we consider $\Omega \subset \mathbb{R}^n$ be an open, bounded domain and define the cylinder

$$\mathcal{C} := (0,+\infty) \times \Omega,$$
$$\partial_L \mathcal{C} := [0,+\infty) \times \partial\Omega.$$

We write points in the cylinder using the notation $(y,x) \in \mathcal{C}$.

Given $s \in (0,1)$, consider the space $H^1_{0,L}(y^{1-2s})$ of measurable functions $u : \mathcal{C} \to \mathbb{R}$ such that $u \in H^1(\Omega \times (s,t))$ for all $0 < s < t < \infty$, $u = 0$ on $\partial_L\mathcal{C}$ and for which the following norm is finite

$$\|u\|^2_{H^1_{0,L}(y^{1-2s})} = \int_{\mathcal{C}} y^{1-2s}|\nabla u|^2 \, dxdy.$$

Consider $f \in D((-\Delta)^s)$, then the boundary value problem

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla u) \equiv u_{yy} + \dfrac{1-2s}{y}u_y + \Delta u = 0, & \text{in } \mathcal{C}, \\ u(0,x) = f(x) & x \in \Omega, \\ u(y,x) = 0 & \text{on } \partial_L\mathcal{C}, \end{cases} \tag{2.27}$$

has a solution $u \in H^1_{0,L}(y^{1-2s})$. Moreover by standard elliptic theory $u(x,y)$ is smooth for $y > 0$. Here it is interesting to use the vector notation, that is, $u : [0,\infty) \longrightarrow H^1_0(\Omega)$ and $[u(y)](x) := u(y,x)$. Then the relation between $(-\Delta)^s f$ and the solution u of the above (extension) problem is given by

$$(-\Delta)^s f(x) = -\frac{1}{\kappa_s} \lim_{y\to 0^+} y^{1-2s}\frac{\partial u}{\partial y}(y,x), \tag{2.28}$$

where $\kappa_s = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$ and Γ is the standard gamma function.

Again, for simplicity we give a sketch of the proof for $s = 1/2$, and since we are dealing with bounded domains, we use the Galerkin's method. Recall that, $\{\varphi_k\}_{k=1}^{\infty}$ are the eigenfunctions for $(-\Delta)$ in Ω with homogeneous Dirichlet boundary conditions, associated to the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$, further

$$\{\varphi_k\}_{k=1}^{\infty} \text{ is an orthogonal base of } H_0^1(\Omega),$$

and

$$\{\varphi_k\}_{k=1}^{\infty} \text{ is an orthonormal base of } L^2(\Omega).$$

Now fix a positive integer m, and look for a function $u_m : [0;1) \to H_0^1(\Omega)$ of the form

$$u_m(y) = \sum_{k=1}^{\infty} a_m^k(y)\varphi_k,$$

where the coefficients $a_m^k(y)$ $(y \geq 0,\ k = 1, \cdots, m)$ solve

$$\begin{cases} \dfrac{d^2 a_m^k}{dy^2}(y) + \lambda_k a_m^k(y) = 0, \quad in\ (0,\infty), \\ \\ \quad a_m^k(0) = \langle f, \varphi_k\rangle, \quad \lim\limits_{y\to\infty} a_m^k(y) = 0. \end{cases} \tag{2.29}$$

Then, it is not difficult to show that, the solution of the above problem is given by $a_m^k(y) = \langle f, \varphi_k\rangle e^{-\sqrt{\lambda_k}y}$. Therefore, the solution of (2.27) for $s = 1/2$ has the form

$$u(y,x) = \sum_{k=1}^{\infty} \langle f, \varphi_k\rangle e^{-\sqrt{\lambda_k}y}\varphi_k(x). \tag{2.30}$$

Finally, to show (2.28), we derive (2.30) with respect to the variable y to obtain

$$u_y(y,x) = -\sum_{k=1}^{\infty} \sqrt{\lambda_k}\langle f, \varphi_k\rangle e^{-\sqrt{\lambda_k}y}\varphi_k(x),$$

then passing to the limit as $y \to 0$, we get (2.28) for $s = 1/2$. This technique could be extended for $s \in (0,1)$. That result has been obtained by Capella, Dávila, Dupaigne, Sire [4], see also Cabré, Tan [5].

Chapter 3

Solvability of IBVP

3.1 Definition of solution

We want to solve the initial-boundary value problem for the equation

$$\partial_t u = \mathrm{div}(u \nabla \mathcal{K} u), \ \ \mathcal{K} = (-\Delta)^{-s},$$

posed in $\Omega_T = (0,T) \times \Omega$, with parameter $0 < s < 1$. Given an initial data $u_0 \in L^\infty(\Omega)$ and considering homogeneous Dirichlet boundary data, we seek for a suitable (weak) solution $u(t,x)$ defined in Ω_T. The next definition tells us in which sense $u(t,x)$ is a solution to the IBVP (1.1).

Definition 3.1. *For $0 < s < 1$, a measurable and bounded function*

$$u \in L^2\Big((0,T); D\big((-\Delta)^{(1-s)/2}\big)\Big),$$

is called a weak solution of the IBVP (1.1), when $u(t,x)$ satisfies:

1. *The fractional degenerated heat type equation: For each $\phi \in C_c^\infty(\Omega_T)$*

$$\iint_{\Omega_T} u(\partial_t \phi - \nabla \mathcal{K} u \cdot \nabla \phi) \, dxdt = 0, \tag{3.1}$$

2. *The initial condition: For all $\zeta \in L^1(\Omega)$*

$$\operatorname*{ess\,lim}_{t \to 0^+} \int_\Omega u(t,x)\zeta(x)\, dx = \int_\Omega u_0(x)\zeta(x)\, dx. \tag{3.2}$$

3. *The boundary condition: For each $\gamma \in L^2((0,T) \times \partial\Omega)$*

$$\operatorname*{ess\,lim}_{\tau \to 0^+} \int_0^T \int_{\partial\Omega} u(t, \Psi_\tau(r)) \nabla \mathcal{K} u(t, \Psi_\tau(r)) \cdot \nu_\tau(\Psi_\tau(r)) \gamma(t,r)\, drdt = 0, \tag{3.3}$$

   where $\Psi : [0,1] \times \partial\Omega \to \overline{\Omega}$ is a C^1-deformation, see Appendix, and ν_τ is the unit outward normal field in $\partial\Omega_\tau$.

we remarks that, given $u \in L^2\left((0,T); D\big((-\Delta)^{(1-s)/2}\big)\right)$ the limit in the left hand side of (3.3), a priori, does not necessarily exist. Indeed, the existence of trace for u and $\nabla \mathcal{K}u \cdot \nu$ are mutually exclusive. For instance, if $0 < s < 1/2$ then from Proposition 2.2, it follows that $u \in L^2\left((0,T); H_0^{1-s}(\Omega)\right)$, which implies that u has trace on $\partial\Omega$. On the other hand, $\mathcal{K}u \in L^2\left((0,T); H^{1+s}(\Omega) \cap H_0^1(\Omega)\right)$, which means that, $\nabla \mathcal{K}u \cdot \nu$ does not have trace on $\partial\Omega$. Vice versa result for $1/2 < s < 1$.

Albeit, if $u \in L^2\left((0,T); D\big((-\Delta)^{(1-s)/2}\big)\right) \cap L^\infty(\Omega_T)$ and satisfies (3.1), then the boundary condition given by (3.3) makes sense. Analogously, the essential limit in (3.2). To show that, first let us consider the following

Lemma 3.1. *Let $u \in L^2\left((0,T); D\big((-\Delta)^{(1-s)/2}\big)\right) \cap L^\infty(\Omega_T)$, with $s \in (0,1)$. Then, for each function $\gamma \in L^2((0,T) \times \partial\Omega)$ and any C^1-deformation Ψ*

$$\int_0^T \int_{\partial\Omega} u(t, \Psi_\tau(r)) \nabla \mathcal{K}u(t, \Psi_\tau(r)) \cdot \nu_\tau(\Psi_\tau(r)) \gamma(t,r)\, dr dt,$$

exists a.e. $\tau > 0$ small enough.

Proof. 1. First, due to $u \in L^2\left((0,T); D\big((-\Delta)^{(1-s)/2}\big)\right)$, the next integral exist

$$\int_0^T \int_{Im(\Psi)} u(t,x) \nabla \mathcal{K}u(t,x) \cdot \nabla h(x)\ \gamma(t, \Psi_{h(x)}^{-1}(x))\ dx dt,$$

where $h(x)$ is the level set associated with the deformation Ψ, which is defined at the Appendix. Then applying the Coarea Formula for the function h we obtain

$$\begin{aligned} &\int_0^T \int_{Im(\Psi)} u(t,x) \nabla \mathcal{K}u(t,x) \cdot \nabla h(x)\ \gamma(t, \Psi_{h(x)}^{-1}(x))\ dx dt \\ &= \int_0^1 \int_0^T \int_{\partial\Omega_\tau} u(t,r) \nabla \mathcal{K}u(t,r) \cdot \nu_\tau(r) \gamma(t, \Psi_\tau^{-1}(r))\, dr\, dt\, d\tau. \end{aligned} \tag{3.4}$$

Thus, from (3.4), we obtain that

$$\int_0^T \int_{\partial\Omega_\tau} u(t,r) \nabla \mathcal{K}u(t,r) \cdot \nu_\tau(r) \gamma(t, \Psi_\tau^{-1}(r))\, dr\, dt, \tag{3.5}$$

exist, a.e. $\tau \in (0,1)$ and $\gamma \in L^2((0,T) \times \partial\Omega)$.

2. Fix any point $x \in \partial\Omega$ and since Ω has boundary of class C^2, there exists a neighbourhood W of x in $\mathbb{R}^n$, an open set $U \subset \mathbb{R}^{n-1}$ and a C^2 mapping $\zeta : U \to \partial\Omega \cap W$, which is C^1-diffeomorphism. Moreover, we recall that, since Ψ is a C^1-deformation, it satisfies

$$\lim_{\tau \to 0} J[\Psi_\tau \circ \zeta] = J[\zeta] \quad in\ \ C(U).$$

where $J[\cdot]$ represent the Jacobian in U.

3. Now, set $\Gamma = \partial\Omega \cap W$, $\Gamma_\tau = \Psi_\tau(\Gamma)$, and consider $\gamma \in L^2((0,T) \times \Gamma)$. Then, from (3.5) and using the Area Formula for the function Ψ_τ, we have

$$\begin{aligned} &\int_0^T \int_\Gamma u(t, \Psi_\tau(r)) \nabla \mathcal{K} u(t, \Psi_\tau(r)) \cdot \nu_\tau(\Psi_\tau(r)) J[\Psi_\tau](r) \gamma(t,r) \, drdt \\ &\quad = \int_0^T \int_{\Gamma_\tau} u(t,r) \nabla \mathcal{K} u(t,r) \cdot \nu_\tau(r) \gamma(t, \Psi_\tau^{-1}(r)) \, dr \, dt, \end{aligned} \tag{3.6}$$

then (3.5) and (3.6) implies that

$$\int_0^T \int_\Gamma u(t, \Psi_\tau(r)) \nabla \mathcal{K} u(t, \Psi_\tau(r)) \cdot \nu_\tau(\Psi_\tau(r)) J[\Psi_\tau](r) \gamma(t,r) \, drdt \tag{3.7}$$

exist a.e. $\tau \in (0,1)$, where $J[\Psi_\tau]$ is defined by

$$J[\Psi_\tau](r) = \frac{J[\Psi_\tau \circ \zeta](\zeta^{-1}(r))}{J[\zeta](\zeta^{-1}(r))}, \tag{3.8}$$

and satisfies $J[\Psi_\tau] \to 1$ uniformly as $\tau \to 0$. To conclude, observe that exist $c > 0$, such that $1 \leq c \, J[\Psi_\tau]$ for $\tau > 0$ small enough, thus from (3.7), follow

$$\int_0^T \int_\Gamma u(t, \Psi_\tau(r)) \nabla \mathcal{K} u(t, \Psi_\tau(r)) \cdot \nu_\tau(\Psi_\tau(r)) \gamma(t,r) \, drdt$$

exist a.e. $\tau > 0$ small enough.

4. Finally, since $\partial\Omega$ is a compact set, applying a standard partition of unity argument, we may exchange the set Γ by $\partial\Omega$ in the previous steps, i.e. consider the general case. From now own, this procedure is considered implicitly. $\square$

Lemma 3.2. *Let $u \in L^2\big((0,T); D\big((-\Delta)^{(1-s)/2}\big)\big) \cap L^\infty(\Omega_T)$, with $s \in (0,1)$. If for each function $\gamma \in L^2((0,T) \times \partial\Omega)$ and any C^1-deformation Ψ*

$$\operatorname*{ess\,lim}_{\tau \to 0^+} \int_0^T \int_{\partial\Omega_\tau} u(t,r) \nabla \mathcal{K} u(t,r) \cdot \nu_\tau(r) \gamma(t, \Psi_\tau^{-1}(r)) \, drdt, \tag{3.9}$$

exists, then

$$\operatorname*{ess\,lim}_{\tau \to 0^+} \int_0^T \int_{\partial\Omega} u(t, \Psi_\tau(r)) \nabla \mathcal{K} u(t, \Psi_\tau(r)) \cdot \nu_\tau(\Psi_\tau(r)) \gamma(t,r) \, drdt,$$

exists and both limits are equal.

Proof. 1. Fix any point $x \in \partial\Omega$ and since Ω has boundary of class C^2, there exists a neighbourhood W of x in $\mathbb{R}^n$, an open set $U \subset \mathbb{R}^{n-1}$ and a C^2 mapping $\zeta : U \to \partial\Omega \cap W$, which is C^1-diffeomorphism.

2. Now, set $\Gamma = \partial\Omega \cap W$, $\Gamma_\tau = \Psi_\tau(\Gamma)$, and consider $\gamma \in L^2((0,T) \times \Gamma)$. Then, from (3.9) and using the Area Formula, we have

$$\begin{aligned}
&\operatorname*{ess\,lim}_{\tau \to 0^+} \int_0^T \int_{\Gamma_\tau} u(t,r) \nabla \mathcal{K} u(t,r) \cdot \nu_\tau(r) \gamma(t, \Psi_\tau^{-1}(r)) \, dr \, dt \\
&= \operatorname*{ess\,lim}_{\tau \to 0^+} \int_0^T \int_\Gamma u(t, \Psi_\tau(r)) \nabla \mathcal{K} u(t, \Psi_\tau(r)) \cdot \nu_\tau(\Psi_\tau(r)) J[\Psi_\tau](r) \gamma(t,r) \, dr dt
\end{aligned}$$

where $J[\Psi_\tau]$ is defined in (3.8), thus we obtain that

$$u(t, \Psi_\tau(r)) \nabla \mathcal{K} u(t, \Psi_\tau(r)) \cdot \nu_\tau(\Psi_\tau(r)) J[\Psi_\tau](r), \tag{3.10}$$

converges weakly in $L^2((0,T) \times \Gamma)$. Therefore we have that (3.10) is uniformly bounded, with respect to $\tau > 0$ small enough.

On the other hand, we observe that

$$\begin{aligned}
&\int_0^T \int_\Gamma u(t, \Psi_\tau(r)) \nabla \mathcal{K} u(t, \Psi_\tau(r)) \cdot \nu_\tau(\Psi_\tau(r)) \gamma(t,r) \, dr dt \\
&= \int_0^T \int_{\Gamma_\tau} u(t,r) \nabla \mathcal{K} u(t,r) \cdot \nu_\tau(r) \gamma(t, \Psi_\tau^{-1}(r)) \, dr dt \\
&+ \int_0^T \int_\Gamma g(\tau, r) u(t, \Psi_\tau(r)) \nabla \mathcal{K} u(t, \Psi_\tau(r)) \cdot \nu_\tau(\Psi_\tau(r)) J[\Psi_\tau](r) \gamma(t,r) \, dr dt,
\end{aligned}$$

where $g(\tau, r) := (1 - J[\Psi_\tau])/J[\Psi_\tau]$. Therefore, passing to the limit as $\tau \to 0$ in the above equation, and taking into account (3.10), we have

$$\begin{aligned}
&\operatorname*{ess\,lim}_{\tau \to 0} \int_0^T \int_\Gamma u(t, \Psi_\tau(r)) \nabla \mathcal{K} u(t, \Psi_\tau(r)) \cdot \nu_\tau(\Psi_\tau(r)) \gamma(t,r) \, dr dt, \\
&= \operatorname*{ess\,lim}_{\tau \to 0} \int_0^T \int_{\Gamma_\tau} u(t,r) \nabla \mathcal{K} u(t,r) \cdot \nu_\tau(r) \gamma(t, \Psi_\tau^{-1}(r)) \, dr \, dt,
\end{aligned}$$

where we have used the Dominated Convergence Theorem.

3. Finally, since $\partial\Omega$ is a compact set, applying a standard partition of unity argument, we may exchange the set Γ by $\partial\Omega$ in the previous steps, i.e. consider the general case. From now own, this procedure is considered implicitly. □

Now we show that the essential limit in (3.2) and the boundary condition (3.3) make sense.

Proposition 3.1. *Let $u \in L^2\left((0,T); D\big((-\Delta)^{(1-s)/2}\big)\right) \cap L^\infty(\Omega_T)$ and assume that u satisfies* (3.1), *then:*

1. *There exists a function $\bar{u} \in L^\infty(\Omega)$, such that*

$$\operatorname*{ess\,lim}_{t\to 0^+} \int_\Omega u(t,x)\zeta(x)dx = \int_\Omega \bar{u}(x)\zeta(x)dx, \tag{3.11}$$

for each $\zeta \in L^1(\Omega)$.

2. *For each $\gamma \in L^2((0,T)\times\partial\Omega)$, and any C^1−deformation Ψ, the*

$$\operatorname*{ess\,lim}_{\tau\to 0^+} \int_0^T \int_{\partial\Omega} u(t,\Psi_\tau(r))\, \nabla\mathcal{K}u(t,\Psi_\tau(r))\cdot\nu_\tau(\Psi_\tau(r))\, \gamma(t,r)dr\, dt \tag{3.12}$$

exists.

Proof. 1. First, let $u \in L^2\left((0,T); D\big((-\Delta)^{(1-s)/2}\big)\right) \cap L^\infty(\Omega_T)$ be a function satisfying (3.1). Then, we consider the following sets:

i) Let $\mathcal{E}$ be a countable dense subset of $C_c^1(\Omega)$. For each $\zeta \in \mathcal{E}$, we define the set of full measure in $(0,T)$ by

$$E_\zeta := \Big\{ t \in (0,T)/\, t \text{ is a Lebesgue point of } I(t) = \int_\Omega u(t,x)\zeta(x)dx \Big\},$$

and consider

$$E := \bigcap_{\zeta\in\mathcal{E}} E_\zeta,$$

which is a full measure in $(0,T)$.

ii) Let $\mathcal{F}$ be a countable dense subset of $C_c^\infty((0,T)\times\mathbb{R}^n)$. For each $\gamma \in \mathcal{F}$, we define the set of full measure in $(0,1)$ by

$$F_\gamma = \Big\{ \tau \in (0,1)/\tau \text{ is a Lebesgue point of } J(\tau) \Big\},$$

where

$$J(\tau) = \int_0^T \int_{\partial\Omega_\tau} u(t,r)\nabla\mathcal{K}u(t,r)\cdot\nu_\tau(r)\gamma(t,\Psi_\tau^{-1}(r))\, dr\, dt,$$

which makes sense thank to Lemma 3.1, moreover consider

$$F := \bigcap_{\gamma\in\mathcal{F}} F_\gamma,$$

which also is a full measure in $(0,1)$.

2. To prove (3.11), let $\zeta \in \mathcal{E}$ and consider the set E defined above. Then, for each $t \in E$, $\|u(t,\cdot)\|_\infty \le C$. Thus we can find a sequence $\{t_m\}$, $t_m \in E$, $m \in \mathbb{N}$, $t_m \to 0$ as $m \to \infty$ and a function $\bar{u} \in L^\infty(\Omega)$, such that $u(t_m,\cdot) \to \bar{u}(\cdot)$ weakly-* in $L^\infty(\Omega)$ as $m \to \infty$.

If $c \in E$, then for large enough m, $t_m < c$. We fix such $t_m < c$ and set $\gamma_j(t) = H_j(t - t_m) - H_j(t - c)$, where the sequence $H_j(\cdot)$, $j \in \mathbb{N}$ is defined at the Appendix. Therefore, taking in (3.1) $\phi(t,x) = \gamma_j(t)\zeta(x)$, we have

$$\iint_{\Omega_T} u(t,x)\gamma_j'(t)\zeta(x)dxdt = \iint_{\Omega_T} u\nabla\mathcal{K}u \cdot \nabla\zeta(x)\,\gamma_j(t)\,dx\,dt. \tag{3.13}$$

Then passing to the limit in (3.13) as $j \to \infty$, and taking into account that, t_m, c are Lebesgue points of the function $I(t)$, also that $\gamma_j(t)$ converges pointwise to the characteristic function of the interval $[t_m, c)$, we obtain

$$I(t_m) - I(c) = \int_{t_m}^{c}\int_\Omega u\nabla\mathcal{K}u \cdot \nabla\zeta(x)dxdt,$$

which implies in the limit as $m \to \infty$ that

$$\int_\Omega \bar{u}(x)\zeta(x)dx - I(c) = \int_0^c\int_\Omega u\nabla\mathcal{K}u \cdot \nabla\zeta(x)dxdt,$$

for all $c \in E$. Therefore, in view of the density of $\mathcal{E}$ in $L^1(\Omega)$, we have

$$\lim_{E \ni t \to 0} I(t) = \int_\Omega \bar{u}(x)\zeta(x)dx,$$

for each $\zeta \in L^1(\Omega)$, which proves (3.11).

3. Now, we show (3.12). First, recall item 3 in the proof of Lemma 3.2. Let $\gamma \in \mathcal{F}$, consider F, and define $S := \Psi(F \times \partial\Omega)$. For τ_1, $\tau_2 \in F$, with $\tau_1 < \tau_2$, define $\zeta_j(\tau) = H_j(\tau - \tau_1) - H_j(\tau - \tau_2)$, $j \in \mathbb{N}$, and take in (3.1) $\phi(t,x)$ defined by

$$\phi(t,x) = \begin{cases} \gamma(t, \Psi^{-1}_{h(x)}(x))\zeta_j(h(x)), & \text{for } x \in S, \\ \qquad\qquad\qquad\qquad 0\,, & \text{for } x \in \Omega \setminus S, \end{cases}$$

where $h(x)$ is the level set associated with the deformation Ψ, which is defined at the Appendix. Then, we have

$$\begin{aligned} &\int_0^T\int_S u(t,x)\ \gamma_t(t, \Psi^{-1}_{h(x)}(x))\ \zeta_j(h(x))\ dxdt \\ &= \int_0^T\int_S u(t,x)\nabla\mathcal{K}u(t,x) \cdot \nabla\gamma(t, \Psi^{-1}_{h(x)}(x))\ \zeta_j(h(x))\ dxdt \\ &+ \int_0^T\int_S u(t,x)\nabla\mathcal{K}u(t,x) \cdot \nabla h(x)\ \zeta_j'(h(x))\ \gamma(t, \Psi^{-1}_{h(x)}(x))\ dxdt. \end{aligned}$$

Consequently, applying the Coarea Formula for the function h, we obtain

$$\int_0^1 \zeta_j(\tau) \int_0^T \int_{\partial\Omega_\tau} u(t,r) \frac{\gamma_t(t, \Psi_\tau^{-1}(r))}{|\nabla h(r)|} \, dr \, dt \, d\tau$$

$$= \int_0^1 \zeta_j(\tau) \int_0^T \int_{\partial\Omega_\tau} u(t,r) \nabla \mathcal{K} u(t,r) \cdot \frac{\nabla \gamma(t, \Psi_{h(\cdot)}^{-1}(\cdot))(r)}{|\nabla h(r)|} \, dr \, dt \, d\tau$$

$$+ \int_0^1 \zeta_j'(\tau) \int_0^T \int_{\partial\Omega_\tau} u(t,r) \nabla \mathcal{K} u(t,r) \cdot \nu_\tau(r) \gamma(t, \Psi_\tau^{-1}(r)) \, dr \, dt \, d\tau.$$

Therefore, applying the Dominated Convergence Theorem in the above equation, we get in the limit as $j \to \infty$

$$J(\tau_2) + \int_0^{\tau_2} \Phi(\tau) d\tau = J(\tau_1) + \int_0^{\tau_1} \Phi(\tau) d\tau, \tag{3.14}$$

for all τ_1, $\tau_2 \in F$ and $\gamma \in \mathcal{F}$, where $\Phi(\tau)$ is given by

$$\int_0^T \int_{\partial\Omega_\tau} u(t,r) \left(\frac{\gamma_t(t, \Psi_\tau^{-1}(r))}{|\nabla h(r)|} - \nabla \mathcal{K} u(t,r) \cdot \frac{\nabla \gamma(t, \Psi_{h(\cdot)}^{-1}(\cdot))(r)}{|\nabla h(r)|} \right) dr \, dt.$$

On the other hand, due to $\mathcal{F}$ is dense in $C_c^\infty((0,T) \times \mathbb{R}^n)$, we have that (3.14), still hold for $\gamma \in C_c^\infty((0,T) \times \mathbb{R}^n)$, consequently we obtain that

$$\lim_{F \ni \tau \to 0^+} \int_0^T \int_{\partial\Omega_\tau} u(t,r) \nabla \mathcal{K} u(t,r) \cdot \nu_\tau(r) \gamma(t, \Psi_\tau^{-1}(r)) \, dr \, dt \tag{3.15}$$

exists for all $\gamma \in C_c^\infty((0,T) \times \mathbb{R}^n)$.

Now, using the same technique of the Lemma 3.2 and without loss of generality, we obtain from (3.15) that

$$\operatorname{ess} \lim_{\tau \to 0^+} \int_0^T \int_{\partial\Omega_\tau} u(t,r) \nabla \mathcal{K} u(t,r) \cdot \nu_\tau(r) \gamma(t, \Psi_\tau^{-1}(r)) \, dr \, dt$$

$$= \operatorname{ess} \lim_{\tau \to 0^+} \int_0^T \int_{\partial\Omega} u(t, \Psi_\tau(r)) \nabla \mathcal{K} u(t, \Psi_\tau(r)) \cdot \nu_\tau(\Psi_\tau(r)) J[\Psi_\tau](r) \gamma(t,r) \, dr dt$$

where $J[\Psi_\tau]$ is defined in (3.8). Therefore, we have

$$u(t, \Psi_\tau(r)) \nabla \mathcal{K} u(t, \Psi_\tau(r)) \cdot \nu_\tau(\Psi_\tau(r)) J[\Psi_\tau](r), \tag{3.16}$$

is uniformly bounded in $L^2((0,T) \times \Gamma)$, with respect to $\tau > 0$ small enough. Thus, in view of the density of $C_c^\infty((0,T) \times \mathbb{R}^n)$ in $L^2((0,T) \times \partial\Omega)$ and together with (3.16), we obtain that (3.15) hold for all $\gamma \in L^2((0,T) \times \partial\Omega)$. Finally, due to Lemma 3.2, we obtain (3.12). □

3.2 Equivalence Theorem

In this section, we state and prove the Theorem, which expresses in convenient way the concept of (weak) solution of the IBVP (1.1) as given by Definition 3.1.

Theorem 3.1 (Equivalence Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ an open bounded set. A measurable and bounded function $u \in L^2\big((0,T); D\big((-\Delta)^{(1-s)/2}\big)\big)$, is a weak solution of the IBVP* (1.1) *if, and only if, satisfies*

$$\iint_{\Omega_T} u(t,x)\ (\partial_t \phi - \nabla \mathcal{K}u \cdot \nabla \phi) dxdt + \int_\Omega u_0(x)\, \phi(0,x)\ dx = 0, \qquad (3.17)$$

for each test function $\phi \in C_c^\infty((-\infty,T) \times \mathbb{R}^n)$.

Proof. First, let us recall the sets $\mathcal{E}, \mathcal{F}$ and respectively E, F in the proof of Proposition 3.1, also item 3 in the proof of Lemma 3.2.

1. Assume that u satisfies (3.17), then we show that u verifies (3.1)–(3.3). To show (3.1), it is enough to consider test functions $\phi \in C_c^\infty(\Omega_T)$. In order to show (3.2), let us consider $\phi(t,x) = \gamma_j(t)\zeta(x)$, $\gamma_j(t) = H_j(t+t_0) - H_j(t-t_0)$ for any $t_0 \in E$ (fixed), and $\zeta \in \mathcal{E}$. Then, from (3.17) we have

$$\iint_{\Omega_T} u(t,x)\gamma_j'(t)\zeta(x)\ dxdt + \int_\Omega u_0(x)\zeta(x)\ dx$$
$$= \iint_{\Omega_T} u(t,x)\nabla\mathcal{K}u(t,x) \cdot \nabla\zeta(x)\gamma_j(t)\ dxdt.$$

Passing to the limit in the above equation as $j \to \infty$, and taking into account that t_0 is Lebesque point of $I(t)$, we obtain

$$\int_\Omega u(t_0,x)\zeta(x)dx = \int_\Omega u_0(x)\zeta(x)dx - \int_0^{t_0}\int_\Omega u\nabla\mathcal{K}u \cdot \nabla\zeta(x)dxdt, \quad (3.18)$$

where we have used the Dominated convergence Theorem. Since $t_0 \in E$ is arbitrary, and in view of the density of $\mathcal{E}$ in $L^1(\Omega)$, it follows from (3.18) that

$$\operatorname*{ess\,lim}_{t\to 0} \int_\Omega u(t,x)\zeta(x)dxdt = \int_\Omega u_0(x)\zeta(x)\ dx$$

for all $\zeta \in L^1(\Omega)$, which shows (3.2). Finally, let us show (3.3). Similarly to item 3 in the proof of Proposition 3.1, we choose

$$\phi(t,x) = \begin{cases} \gamma(t, \Psi_{h(x)}^{-1}(x))\zeta_j(h(x)), & \text{for } x \in S, \\ 0\ , & \text{for } x \in \Omega \setminus S, \end{cases}$$

where $\gamma \in \mathcal{F}$, $\zeta_j(\tau) = H_j(\tau + \tau_0) - H_j(\tau - \tau_0)$, with $\tau_0 \in F$, and $S = \Psi(F \times \partial\Omega)$. Therefore, from (3.17) we obtain

$$\begin{aligned}
&\int_0^T \int_S u(t,x)\ \partial_t \gamma(t, \Psi^{-1}_{h(x)}(x))\ \zeta_j(h(x))\ dxdt \\
&\qquad = \int_0^T \int_S u(t,x)\ \nabla \mathcal{K} u(t,x) \cdot \nabla \gamma(t, \Psi^{-1}_{h(x)}(x)) \zeta_j(h(x))\ dxdt \\
&\qquad + \int_0^T \int_S u(t,x)\ \nabla \mathcal{K} u(t,x) \cdot \nabla h(x) \zeta_j'(h(x)) \gamma(t, \Psi^{-1}_{h(x)}(x))\ dxdt.
\end{aligned}$$

On other hand, applying the Coarea Formula for the function h in the above equation, we have

$$\begin{aligned}
&\int_0^1 \zeta_j(\tau) \int_0^T \int_{\partial\Omega_\tau} u(t,r) \frac{\partial_t \gamma(t, \Psi_\tau^{-1}(r))}{|\nabla h(r)|}\, dr\, dt\, d\tau \\
&= \int_0^1 \zeta_j(\tau) \int_0^T \int_{\partial\Omega_\tau} u(t,r) \nabla \mathcal{K} u(t,r) \cdot \frac{\nabla \gamma(t, \Psi^{-1}_{h(x)}(x))(r)}{|\nabla h(r)|}\, dr\, dt\, d\tau \\
&+ \int_0^1 \zeta_j'(\tau) \int_0^T \int_{\partial\Omega_\tau} u(t,r) \nabla \mathcal{K} u(t,r) \cdot \nu_\tau(r) \gamma(t, \Psi_\tau^{-1}(r))\, dr\, dt\, d\tau.
\end{aligned}$$

Then, passing the limit in the above equation as $j \to \infty$ and taking into account that τ_0 is Lebesque point of $J(\tau)$, and also that $\zeta_j(t)$ converge pointwise to the characteristic function of the interval $[-\tau_0, \tau_0)$, we obtain

$$\begin{aligned}
&\int_0^{\tau_0} \int_0^T \int_{\partial\Omega_\tau} u(t,r) \frac{\partial_t \gamma(t, \Psi_\tau^{-1}(r))}{|\nabla h(r)|}\, dr\, dt\, d\tau \\
&= \int_0^{\tau_0} \int_0^T \int_{\partial\Omega_\tau} u(t,r) \nabla \mathcal{K} u(t,r) \cdot \frac{\nabla \gamma(t, \Psi^{-1}_{h(x)}(x))(r)}{|\nabla h(r)|}\, dr\, dt\, d\tau + J(\tau_0)
\end{aligned}$$

for all $\tau_0 \in F$. Then, for each $\tau \in F$ we have

$$|J(\tau)| \leq C\, |\Psi((0,\tau) \times \partial\Omega)|\,,$$

where C is a positive constant, which does not depend on τ. Hence passing to the limit as $\tau \to 0$, we obtain

$$\lim_{F \ni \tau \to 0} \int_0^T \int_{\partial\Omega_\tau} u(t,r) \nabla \mathcal{K} u(t,r) \cdot \nu_\tau(r) \gamma(t, \Psi_\tau^{-1}(r))\, dr\, dt = 0,$$

for all $\gamma \in \mathcal{F}$. By approximation, we conclude that the above limit holds for any $\gamma \in L^2((0,T) \times \partial\Omega)$. Finally, applying Lemma 3.2 we obtain (3.3).

2. Now, let us consider: (3.1)–(3.3) $\Rightarrow$ (3.17). The idea is similar what we have done before, for completeness we give the main points. Firstly consider $j \in \mathbb{N}$ sufficiently large and take for any $t_0 \in E$

$$\phi(t,x) = \psi(t,x) H_j(t - t_0),$$

where $\psi \in C_c^\infty((-\infty,T) \times \Omega))$, $H_j(t)$ as considered before. Then, from (3.1) we obtain

$$\iint_{\Omega_T} u(t,x)\partial_t\psi(t,x)H_j(t-t_0)\ dxdt + \iint_{\Omega_T} u(t,x)\ \psi(t,x)\ H_j'(t-t_0)\ dxdt$$
$$- \iint_{\Omega_T} u(t,x)\nabla\mathcal{K}u(t,x)\cdot\nabla\psi(t,x)H_j(t-t_0)\ dxdt = 0.$$

Passing to the limit as $j \to \infty$, and taking into account that t_0 is a Lebesgue point of $I(t)$, also that $H_j(\cdot - t_0)$ converge pointwise to the Heaviside function $H(\cdot - t_0)$, after that, taking the limit as $E \ni t_0 \to 0$ and using (3.2), we have

$$\begin{aligned}\iint_{\Omega_T} u(t,x)\partial_t\psi(t,x)\ dxdt + \int_\Omega u_0(x)\ \psi(0,x)\ dx&\\ - \iint_{\Omega_T} u(t,x)\nabla\mathcal{K}u(t,x)\cdot\nabla\psi(t,x)\ dxdt = 0,&\end{aligned} \tag{3.19}$$

for all $\psi \in C_c^\infty((-\infty,T) \times \Omega)$. In particular, for

$$\psi(t,x) = \phi(t,x)(1 - \zeta_j(h(x)),$$

where $\phi \in C_c^\infty((-\infty,T) \times \mathbb{R}^n)$, $h(x)$ as above and we consider the function $\zeta_j(\tau) = H_j(\tau + \tau_0) - H_j(\tau - \tau_0)$, where $\tau_0 \in F$. Then, from (3.19) we obtain

$$\begin{aligned}\iint_{\Omega_T} u(t,x)\partial_t\phi(t,x)(1-\zeta_j(h(x)))\ dxdt + \int_\Omega u_0(x)\ \phi(0,x)(1-\zeta_j(h(x)))\ dx&\\ - \iint_{\Omega_T} u(t,x)\nabla\mathcal{K}u(t,x)\cdot\nabla\phi(t,x)\ (1-\zeta_j(h(x)))\ dxdt&\\ + \iint_{\Omega_T} u(t,x)\nabla\mathcal{K}u(t,x)\cdot\nabla h(x)\ \zeta_j'(h(x))\phi(t,x)\ dxdt = 0.&\end{aligned}$$

Finally, we use the Coarea Formula for the function h in the last integral of the above equation, and pass to limit as $j \to \infty$. Therefore, we obtain for all $\phi \in C_c^\infty((-\infty,T) \times \mathbb{R}^n)$

$$\iint_{\Omega_T} u(t,x)(\partial_t\phi(t,x) - \nabla\mathcal{K}u(t,x)\cdot\nabla\phi(t,x))\,dxdt + \int_\Omega u_0(x)\phi(0,x)dxdt = 0,$$

where we have used (3.3). $\square$

3.3 Basic estimates

Here we describe (formally) the main basic estimates, which are required to show existence of weak solutions of the IBVP (1.1).

1. *Conservation of mass*:

$$\int_\Omega u(t,x)\ dx = \int_\Omega u_0(x)\ dx,$$

for all $t \in (0,T)$.

2. *First energy estimate*:

$$\int_\Omega u(t,x)\log u(t,x)\ dx + \int_0^t \int_\Omega |\nabla \mathcal{H} u(t',x)|^2\ dxdt' = \int_\Omega u_0(x)\log u_0(x)\ dx,$$

for all $t \in (0,T)$.

3. *Second energy estimate*:

$$\frac{1}{2}\int_\Omega |\mathcal{H}u(t,x)|^2 dx + \int_{t_0}^t \int_\Omega u(t',x)|\nabla \mathcal{K} u(t',x)|^2 dxdt' = \frac{1}{2}\int_\Omega |\mathcal{H}u(t_0,x)|^2 dx,$$

where $0 \leq t_0 < t \leq T$, and recall that $\mathcal{H} = \mathcal{K}^{1/2}$.

4. *Conservation of positivity*: If the initial condition u_0 is positive, then the solution $u(t,\cdot)$ of (1.1) is positive for all times.

   Indeed, we assume $u_0 > 0$ (without loss of generality). For any $0 < t \leq T$ (fixed), let $x(t)$ be a point where $u(t,\cdot)$ is a minimum, which is to say

$$u(t,x) \geq u(t,x(t)) \ \ for\ all\ x \in \Omega.$$

   For each $\delta > 0$, we consider

$$\varphi_\delta(w) = \begin{cases} (w^2+\delta^2)^{1/2} - \delta, & \text{for } 0 \leq w, \\ 0 \quad , & \text{for } w \leq 0. \end{cases} \tag{3.20}$$

   Then, $\varphi_\delta(w)$ converges to $w^+ = \max\{w,0\}$ as $\delta \to 0^+$. Now, multiplying the first equation in (1.1) by $\varphi'_\delta(u)$, we obtain

$$\frac{d}{dt}\varphi_\delta(u) = \nabla u(t,x(t)) \cdot \nabla \mathcal{K} u(t,x(t))\ \varphi'_\delta(u) \\ + u(t,x(t))\ \varphi'_\delta(u)\ \Delta \mathcal{K} u(t,x(t)).$$

The first term in the right hand side of the above equation is zero, since $(x(t), t)$ is a point where $u(t, \cdot)$ is minimum. For the second term, we recall that $-\Delta\mathcal{K} = (-\Delta)^{1-s}$, hence due to Lemma 2.1, it follows that

$$-\Delta\mathcal{K}u(t, x(t)) = \frac{1}{\Gamma(s-1)} \int_0^\infty (e^{r\Delta}u(t, x(t)) - u(t, x(t)))\frac{dr}{r^{2-s}}.$$

Now, $\Gamma(s-1) < 0$ because $s - 1 < 0$, and using the strong maximum principle for the heat equation, we obtain that

$$e^{r\Delta}u(t, x(t)) - u(t, x(t)) \geq 0$$

and thus $\Delta\mathcal{K}u \geq 0$. Moreover, $u\ \varphi_\delta'(u) \geq 0$, hence we get $\frac{d}{dt}\varphi_\delta(u) \geq 0$. Therefore,

$$\varphi_\delta(u(t)) \geq \varphi_\delta(u_0). \tag{3.21}$$

Then passing to the limit in (4) as $\delta \to 0$, we obtain $u^+(t) \geq u_0$, which implies that $u(t) > 0$.

5. L^∞ *estimate*: Similar to the above description, it is not difficult to show that, the L^∞ norm of u does not increase in time.

Chapter 4

Existence of Weak Solution

4.1 Parabolic regularization

In order to show existence of weak solutions as given by Definition 3.1, we regularize the IBVP (1.1). First, we add a Laplacian term in the first equation of (1.1), and eliminate the degeneracy by raising the level set $\{u = 0\}$ in the diffusion coefficient. Then, we regularize all the coefficients. Specifically, we take small numbers $\delta,\ \mu \in (0, 1)$ and consider

$$\begin{aligned}
\partial_t u_{\mu,\delta} - \delta \Delta u_{\mu,\delta} &= \operatorname{div}(d(u_{\mu,\delta}) \nabla \mathcal{K} u_{\mu,\delta}) \qquad \text{in } \Omega_T, &(4.1)\\
u_{\mu,\delta}(0, \cdot) &= u_{0\delta} \qquad \text{in } \Omega, &(4.2)\\
u_{\mu,\delta} &= 0 \qquad \text{on } (0, T) \times \partial\Omega, &(4.3)
\end{aligned}$$

where $d(\lambda) = \lambda + \mu$, $u_{0\delta}$ is a non-negative smooth and bounded approximation of the initial data $u_0 \geq 0$, satisfying compatibility conditions.

Now, for $\delta,\ \mu > 0$ fixed, we study the parabolic perturbation (4.1)–(4.3). First, we make use of the well known results of existence, uniqueness and uniform L^∞ bounds for quasilinear parabolic problems. Therefore, for each $\delta,\ \mu > 0$, there exists a unique classical solution $u_{\mu,\delta} \in C^2(\Omega_T) \cap C\big(\bar{\Omega}_T\big)$ of the IBVP (4.1)–(4.3), (see [10], p. 449). Then, we consider the following

Theorem 4.1. *For each $\mu, \delta > 0$, let $u := u_{\mu,\delta}$ be the unique classical solution of* (4.1)–(4.3)*. Then, u satisfies:*

(1) *For all $\phi \in C_c^\infty((-\infty, T) \times \mathbb{R}^n)$,*

$$\iint_{\Omega_T} u(t,x)(\partial_t \phi(t,x) + \delta \Delta \phi(t,x))\ dxdt + \int_\Omega u_{0\delta}(x)\ \phi(0,x)\ dx = \iint_{\Omega_T} d(u(t,x)) \nabla \mathcal{K} u(t,x) \cdot \nabla \phi(t,x)\ dxdt. \tag{4.4}$$

(2) *For each* $t \in (0,T)$*, we have*

$$\|u(t)\|_\infty \le \|u_0\|_\infty, \tag{4.5}$$

and the conservation of the "total mass"

$$\int_\Omega u(t,x)\ dx = \int_\Omega u_{0\delta}(x)\ dx \le \int_\Omega u_0(x)\ dx. \tag{4.6}$$

Furthermore, $0 \le u(t,x)$*, for all* $(t,x) \in \Omega_T$

(3) *The first energy estimate: For* $\eta(\lambda) := (\lambda+\mu)\log(1+(\lambda/\mu)) - \lambda$ $(\lambda \ge 0)$*, and all* $t \in (0,T)$*,*

$$\begin{aligned}\int_\Omega \eta(u(t))\, dx + \delta \int_0^t \int_\Omega \frac{|\nabla u|^2}{d(u)}\, dx\, dt\ + \int_0^t \int_\Omega |\nabla \mathcal{H} u|^2\, dxdt \\ = \int_\Omega \eta(u_{0\delta}) dx.\end{aligned} \tag{4.7}$$

(4) *The second energy estimate: For all* $0 < t_1 < t_2 < T$*,*

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\mathcal{H}u(t_2,x)|^2\, dx + \delta \int_{t_1}^{t_2} \int_\Omega |\nabla \mathcal{H} u|^2\, dxdt \\ &+ \int_{t_1}^{t_2} \int_\Omega d(u)\, |\nabla \mathcal{K} u|^2\ dxdt = \frac{1}{2}\int_\Omega |\mathcal{H}u(t_1,x)|^2\, dx.\end{aligned} \tag{4.8}$$

(5) *For all* $v \in H_0^1(\Omega)$*,*

$$\int_0^T \langle \partial_t u(t), v\rangle dt = -\delta \iint_{\Omega_T} \nabla u \cdot \nabla v\, dxdt + \iint_{\Omega_T} d(u) \nabla \mathcal{K} u \cdot \nabla v\, dxdt. \tag{4.9}$$

Proof. 1. First, let us show (4.4). For each $k \in \mathbb{N}$, consider ξ_k be given by (5.1) at the Appendix. Then, for each test function $\phi \in C_c^\infty((-\infty,T)\times\mathbb{R}^n)$, we multiply (4.1) by $\xi_k \phi$ and integrate over Ω_T. After integration by parts, and using that $\xi_k(\cdot), u(t,\cdot) = 0$ on $\partial\Omega$, we obtain

$$\begin{aligned}&\iint_{\Omega_T} \Big\{ u(t,x)\partial_t \phi + \delta u(t,x)\Delta\phi - d(u(t,x))\nabla \mathcal{K} u(t,x)\cdot\nabla\phi \Big\}\ \xi_k(x)\ dxdt \\ &\quad + \int_\Omega u_{0\delta}(x)\phi(0)\xi_k(x) dx = \iint_{\Omega_T} d(u(t,x))\ \phi \nabla \mathcal{K} u(t,x)\cdot\nabla\xi_k(x) dxdt \\ &\quad - \delta \iint_{\Omega_T} u(t,x)\nabla\phi\cdot\nabla\xi_k(x) dxdt + \delta \iint_{\Omega_T} \phi \nabla u(t,x)\cdot\nabla\xi_k(x) dxdt \\ &\quad = I_1 - I_2 + I_3,\end{aligned} \tag{4.10}$$

with obvious notation. Let us observe the I_1 term, we have

$$|I_1| \le (\|u\|_\infty + 1)\|\phi\|_\infty \Big(\iint_{\Omega_T} |\nabla \mathcal{K} u(t,x)|^2 dxdt \Big)^{1/2} \Big(\iint_{\Omega_T} |\nabla \xi_k(x)|^2 dxdt \Big)^{1/2},$$

where we have used the uniform limitation of $u(t,x)$, and Holder's inequality. Therefore, due to Lemma 5.1 (Appendix), we have

$$\lim_{k\to\infty} I_1 = 0.$$

Similarly, we obtain that I_2, I_3 go to zero as $k \to \infty$. Then, passing to the limit as $k \to \infty$ in (4.10), again applying Lemma 5.1, we get (4.4).

2. Statement (4.5) follows directly. Indeed, we already have $\|u\|_\infty \le \|u_{0\varepsilon}\|_\infty$, hence $u(t,x) \le \|u_0\|_\infty$. Moreover, the conservation of positivity can be established with the same ideas used in Section 3.3, item 4.

3. In order to prove (4.6), we multiply (4.1) by $\xi_k(x)$, and integrate over Ω. Then, after integration by parts, we have

$$\frac{\partial}{\partial t} \int_\Omega u(t,x)\xi_k(x)\, dx = -\int_\Omega \big(\delta\ \nabla u(t,x) + d(u(t,x)) \nabla \mathcal{K} u(t,x)\big) \cdot \nabla \xi_k(x)\ dx.$$

Now, we integrate the above equation over $(0,t)$ to obtain

$$\begin{aligned}&\int_\Omega \big(u(t,x) - u_{0,\delta}(x)\big)\xi_k(x)\ dx \\ &\qquad = -\int_0^t \int_\Omega \big(\delta\ \nabla u(t',x) + d(u(t',x)) \nabla \mathcal{K} u(t',x)\big) \cdot \nabla \xi_k(x)\ dxdt'.\end{aligned}$$

Finally, we pass to the limit as $k \to \infty$ in the above equation, and argue similarly to item 1.

4. To get the first energy estimate (4.7), we multiple equation (4.1) by $\eta'(u)$ and integrate on Ω. Then, after integration by parts and taking into account that $\eta'(0) = 0$, we have

$$\frac{\partial}{\partial t} \int_\Omega \eta(u) dx = -\delta \int_\Omega \frac{|\nabla u|^2}{d(u)}\ dx - \int_\Omega |\nabla \mathcal{H} u|^2\, dx.$$

Then, we integrate over $(0,t)$, for all $0 < t < T$, to obtain (4.7).

5. To prove (4.8), we multiply (4.1) by $\xi_k \mathcal{K} u$, integrate over Ω and take into account that $\xi_k = 0$ on $\partial\Omega$, we obtain

$$\int_\Omega \xi_k \frac{\partial u}{\partial t} \mathcal{K} u\, dx + \delta \int_\Omega \nabla u \cdot \nabla(\xi_k \mathcal{K} u)\, dx + \int_\Omega d(u) |\nabla \mathcal{K} u|^2\, dx = 0.$$

Passing to the limit as $k \to \infty$ and using the Lemma 5.1, it follows that

$$\frac{1}{2}\frac{\partial}{\partial t} \int_\Omega |\mathcal{H} u(t)|^2 dx + \delta \int_\Omega |\nabla \mathcal{H} u|^2\, dx + \int_\Omega d(u) |\nabla \mathcal{K} u|^2\ dx = 0.$$

Finally, integrating over (t_1,t_2) we get the second energy estimate (4.8), for all $0<t_1<t_2<T$.

6. It remains to show (4.9), which follows applying the same techniques above, so the proof is omitted. The proof of the Theorem 4.1 is complete. $\square$

4.2 Limit transition

The main issue in this section is to pass to the limit in (4.4), which is to say, as the two parameters δ, μ go to zero. Then, we use the Equivalence Theorem 3.1 to show a solution of the IBVP (1.1).

As a first step, we define $u_\delta := u_{\mu,\delta}$ (fixing $\mu>0$). Then, we consider the following

Proposition 4.1. *Let $\{u_\delta\}_{\delta>0}$ be the classical solutions of (4.1)–(4.3). Then, there exists a subsequence of $\{u_\delta\}_{\delta>0}$, which weakly converges to some function $u \in L^2\big((0,T); D\big((-\Delta)^{(1-s)/2}\big)\big) \cap L^\infty(\Omega_T)$, satisfying*

(1) *For all $\varphi \in C_c^\infty((-\infty,T)\times\mathbb{R}^n)$*

$$\begin{aligned}\iint_{\Omega_T} u(t,x)\partial_t\varphi(t,x) + \int_\Omega u_0(x)\varphi(0,x)dx \\ = \iint_{\Omega_T} d\,(u(t,x))\,\nabla\mathcal{K}u(t,x)\cdot\nabla\varphi(t,x)dxdt.\end{aligned} \tag{4.11}$$

(2) *For almost all $t\in(0,T)$,*

$$\|u(t)\|_\infty \le \|u_0\|_\infty, \quad \textit{and} \tag{4.12}$$

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx. \tag{4.13}$$

Furthermore, $0\le u(t,x)$ a.e in Ω_T.

(3) *The first energy estimate: For $\eta(\lambda) := (\lambda+\mu)\log(1+(\lambda/\mu))-\lambda$ $(\lambda\ge 0)$, and almost all $t\in(0,T)$,*

$$\int_\Omega \eta(u(t))dx + \int_0^t\int_\Omega |\nabla\mathcal{H}u|^2\ dxdt' \le \int_\Omega \eta(u_0)\ dx. \tag{4.14}$$

(4) *Second energy estimate: For almost all $0<t_1<t_2<T$,*

$$\frac{1}{2}\int_\Omega |\mathcal{H}u(t_2)|^2dx + \int_{t_1}^{t_2}\int_\Omega d(u)|\nabla\mathcal{K}u|^2\,dx\,dt \le \frac{1}{2}\int_\Omega |\mathcal{H}u(t_1)|^2dx. \tag{4.15}$$

(5) *For each* $v \in H_0^1(\Omega)$,

$$\int_0^T \langle \partial_t u, v \rangle dt = \iint_{\Omega_T} d(u) \nabla \mathcal{K} u \cdot \nabla v \, dxdt. \tag{4.16}$$

Proof. 1. To prove (4.11), we pass to the limit in (4.4) as $\delta \to 0^+$. Therefore, we need to show compactness of the sequence $\{u_\delta\}_{\delta>0}$. From (4.5), it follows that $\{u_\delta\}_{\delta>0}$ is (uniformly) bounded in $L^\infty(\Omega_T)$. Then, it is possible to select a subsequence, still denoted by $\{u_\delta\}$, converging weakly-$\star$ to u in $L^\infty(\Omega_T)$, i.e.

$$\lim_{\delta \to 0^+} \int_{\Omega_T} u_\delta(t,x) \phi(t,x) \, dtdx = \int_{\Omega_T} u(t,x) \phi(t,x) \, dtdx,$$

for all $\phi \in L^1(\Omega_T)$, which is enough to pass to the limit in the first integral in the left hand side of (4.4).

The integral in right hand side of (4.4) contains the product $d(u_\delta) \nabla \mathcal{K} u_\delta$, so we need more than weak convergence. From (4.8), we have

$$\iint_{\Omega_T} |\nabla \mathcal{K} u_\delta|^2 \, dxdt \leq \frac{C}{\mu},$$

where C is a positive constant which does not depend on δ. Therefore, the right-hand side is (uniformly) bounded in $L^2(\Omega_T)$ w.r.t. δ. Thus we obtain (along suitable subsequence) that, $\nabla \mathcal{K} u_\delta$ converges weakly to $\nabla \mathcal{K} u$ in $L^2(\Omega_T)$, where we have used the uniqueness of the limit. Recall that, at this point $\mu > 0$ is fixed. Let us now, applying the Aubin-Lions' Theorem, prove strong convergence for $\{u_\delta\}_{\delta>0}$ in $L^2(\Omega_T)$. First, from (2.19) we have $(-\Delta)^{(1-s)/2} u_\delta = \nabla \mathcal{H} u_\delta$ in norm-$L^2(\Omega_T)$, moreover due to (4.7), we obtain

$$\iint_{\Omega_T} \left| (-\Delta)^{(1-s)/2} u_\delta(t,x) \right|^2 dxdt = \iint_{\Omega_T} |\nabla \mathcal{H} u_\delta|^2 dxdt \leq C.$$

Then, it is possible to select a subsequence, still denoted by $\{u_\delta\}$, converging weakly to u in $L^2\big((0,T); D\big((-\Delta)^{(1-s)/2}\big)\big)$. Furthermore, from (4.7)–(4.9) we have

$$\int_0^T \|\partial_t u_\delta\|^2_{H^{-1}(\Omega)} \, dt \leq C \; (\|u_0\|_\infty + \mu). \tag{4.17}$$

Thus, the right-hand side of (4.17) is (uniformly) bounded in $L^2((0,T); H^{-1}(\Omega))$ w.r.t. δ. Passing to a subsequence, it follows that $\partial_t u_\delta$ converges weakly to $\partial_t u$ in $L^2(0,T; H^{-1}(\Omega))$. Then, applying the Aubin-Lions compactness Theorem (see [12], Lemma 2.48) it follows that, u_δ converges to u (along suitable subsequence) strongly in $L^2(\Omega_T)$ as δ goes to zero, which is enough to show that, $d(u_\delta) \nabla \mathcal{K} u_\delta$ converges weakly to $d(u) \nabla \mathcal{K} u$ as $\delta \to 0^+$. Therefore, the equality (4.11) follows.

2. In order to prove $0 \leq u(t,x)$ a.e in Ω_T and (4.12), we use respectively the positivity of u_δ and (4.5). Moreover, recall that u_δ converges strong to u in $L^2(\Omega_T)$ and therefore (for a subsequence) u_δ converges almost everywhere to u in Ω_T. Then, passing to the limit in $0 \leq u_\delta$ and (4.5) as $\delta \to 0^+$, we obtain the assertions.

3. To show (4.13). First, recall that u_δ converges almost everywhere to u in Ω_T, moreover u_δ is bounded in $L^\infty(\Omega_T)$ w.r.t. δ, then applying the Dominated Convergence Theorem, it follows that

$$\lim_{\delta \to 0^+} \int_\Omega u_\delta(t,x) dx = \int_\Omega u(t,x) dx,$$

for almost all $t \in (0,T)$. A similar result is obtained for $u_{0\delta}$. Finally, passing to the limit in (4.6) as $\delta \to 0^+$, we get (4.13).

4. To prove the first energy estimate (4.14), we pass to the limit in (4.7) as $\delta \to 0^+$. Due to u_δ converges almost everywhere to u in Ω_T, and η is a continuous function, it follows that $\eta(u_\delta)$ converges almost everywhere to $\eta(u)$ in Ω_T. Moreover, u_δ is bounded in $L^\infty(\Omega_T)$ w.r.t. δ, then for almost all $t \in (0,T)$

$$\lim_{\delta \to 0^+} \int_\Omega \eta(u_\delta(t)) \; dx = \int_\Omega \eta(u(t)) \; dx,$$

where we have used the Dominated Convergence Theorem. We can proceed in a similar way as before for the sequence $u_{0,\delta}$.

On the other hand, since $\nabla \mathcal{H} u_\delta$ is bounded in $L^2(\Omega)$ (see (4.7)), it is possible to select a subsequence, still denoted by $\nabla \mathcal{H} u_\delta$ converges weakly to $\nabla \mathcal{H} u$ in $L^2(\Omega_T)$, which implies

$$\int_0^t \int_\Omega |\nabla \mathcal{H} u|^2 \, dxdt' \leq \liminf_{\delta \to 0^+} \int_0^t \int_\Omega |\nabla \mathcal{H} u_\delta|^2 \, dxdt'$$

for almost all $t \in (0,T)$. Also observe that the second integral in the left hand side of (4.7) is positive for all $\delta > 0$, hence we throw it out. Therefore passing to the limit in (4.7) as δ tends to zero, we obtain the assertion.

5. To show the second energy estimate (4.15), we pass to the limit in (4.8) as δ goes to zero. First, we have to study the convergence of each integral involved in (4.8). One notes that, due to the continuity in $L^2(\Omega_T)$ of the $\mathcal{H}$ operator, and the strong convergence of u_δ, we obtain $\mathcal{H} u_\delta$ converge strong to $\mathcal{H} u$ in $L^2(\Omega_T)$. Consequently, it is possible to select a subsequence, still denoted by $\mathcal{H} u_\delta(t)$ such that, for almost all $t \in (0,T)$

$$\lim_{\delta \to 0^+} \int_\Omega |\mathcal{H} u_\delta(t,x)|^2 \, dx = \int_\Omega |\mathcal{H} u(t,x)|^2 \, dx.$$

On the other hand, since $\nabla \mathcal{H} u_\delta$ is bounded in $L^2(\Omega_T)$ (see (4.7)), thus the second integral in (4.8) tend to zero as $\delta \to 0^+$. Finally, the convergence of the third integral follows applying the same technique used in item 1. Then, passing to the limit in (4.8) as $\delta \to 0^+$, we obtain (4.15).

6. It remains to prove (4.16). To show that, we argue similarly as in the proof of the item 1, so we pass to the limit in (4.9) as $\delta \to 0^+$, and the proof is concluded. $\square$

Remark 4.1. *We remarks that, the function u, obtained in the previous proposition, depend on the fixed parameter μ. From now on, for each $\mu > 0$ we write u_μ instead of u.*

4.3 Existence of weak solution

In this section, we prove the existence of (weak) solution of the IBVP (1.1) as given by the Definition 3.1.

Theorem 4.2. *Let $u_0 \in L^\infty(\Omega)$ be a non-negative function. Then, there exists a weak solution $u \in L^2\big((0,T); D\big((-\Delta)^{(1-s)/2}\big)\big) \cap L^\infty(\Omega_T)$ of the IBVP (1.1).*

Proof. First, we consider the sequence $\{u_\mu\}_{\mu>0}$, which satisfies for each $\mu > 0$, (4.11)–(4.16). Then, we proceed to pass to the limit in (4.11) as $\mu \to 0^+$, and conclude due to Equivalence Theorem 3.1 the solvability of the IBVP (1.1).

From (4.12), it follows that $\{u_\mu\}_{\mu>0}$ is (uniformly) bounded in $L^\infty(\Omega_T)$. Then, it is possible to select a subsequence, still denoted by $\{u_\mu\}$, converging weakly-$\star$ to u in $L^\infty(\Omega_T)$, which is enough to pass to the limit in the first integral in the left hand side of (4.11).

Now, let us consider the integral in the right hand side of (4.11), which contains the product $d(u_\mu)\nabla\mathcal{K}u_\mu$. First, we recall that

$$\eta(\lambda) = (\lambda+\mu)\big(\log(\lambda+\mu) - \log(\mu)\big) - \lambda \qquad (\forall \lambda \geq 0),$$

then from (4.13) and (4.14), we obtain for almost all $t \in (0,T)$

$$\begin{aligned} \int_0^t \int_\Omega |\nabla\mathcal{H}u_\mu|^2 \, dxdt + \int_\Omega u_\mu(t)\log(u_\mu(t)+\mu) \, dx& \\ \leq \int_\Omega (u_0+\mu)\log(u_0+\mu) \, dx&, \end{aligned} \tag{4.18}$$

where we have used that $\mu \int_\Omega [\log(\mu+u_\mu) - \log(\mu)] \, dx \geq 0$ for all $\mu > 0$.

Since $f = f^+ - f^-$, where $f^\pm = \max\{\pm f, 0\}$, it follows from (4.18) that

$$\begin{aligned}\int_0^t \int_\Omega |\nabla \mathcal{H} u_\mu|^2 \, dxdt + \int_\Omega u_\mu(t) \log^+(u_\mu(t) + \mu) \, dx \\ &\leq \int_\Omega (u_0 + \mu) \log(u_0 + \mu) \, dx + \int_\Omega u_\mu(t) \log^-(u_\mu(t) + \mu) dx \\ &\leq C,\end{aligned}$$

where $C > 0$ is a constant which does not depend on μ, since u_μ is bounded in $L^\infty(\Omega_T)$ w.r.t. μ, and $\int_\Omega u_\mu(t) \log^-(u_\mu(t) + \mu) dx$ is bounded w.r.t. μ (small enough). Consequently, we have that $\nabla \mathcal{H} u_\mu$ is (uniformly) bounded in $L^2(\Omega_T)$.

On the other hand, from (2.19) we have $\nabla \mathcal{K} u_\mu = \mathcal{H}((-\Delta)^{1/2} \mathcal{H} u_\mu)$ in norm-$L^2(\Omega_T)$, hence from the continuity of the $\mathcal{H}$ operator, it follows that

$$\iint_{\Omega_T} |\nabla \mathcal{K} u_\mu|^2 \, dxdt \leq \iint_{\Omega_T} |\nabla \mathcal{H} u_\mu|^2 \, dxdt.$$

Therefore, $\nabla \mathcal{K} u_\mu$ is (uniformly) bounded in $L^2(\Omega_T)$, and thus we obtain (along suitable subsequence) that $\nabla \mathcal{K} u_\mu$ converges weakly to $\nabla \mathcal{K} u$ in $L^2(\Omega_T)$.

We also need strong convergence for $\{u_\mu\}_{\mu > 0}$ in $L^2(\Omega_T)$. To show that, we apply again the Aubin-Lions compactness Theorem. First, from (2.19) we have $(-\Delta)^{(1-s)/2} u_\mu = \nabla \mathcal{H} u_\mu$ in morm-$L^2(\Omega_T)$, and thus from the boundedness of $\nabla \mathcal{H} u_\mu$ in $L^2(\Omega_T)$, we have

$$\iint_\Omega |(-\Delta)^{(1-s)/2} u_\mu(t, x)|^2 \, dxdt \leq C.$$

Then, it is possible to select a subsequence, still denoted by $\{u_\mu\}$, which converges weakly to u in $L^2\left(0, T; D\big((-\Delta)^{(1-s)/2}\big)\right)$. Moreover, from (4.16) with the boundedness of $\nabla \mathcal{K} u_\mu$ in $L^2(\Omega_T)$, and the uniform limitation of u_μ, we have

$$\int_0^T \|\partial_t u_\mu\|^2_{H^{-1}(\Omega)} \, dt \leq C. \tag{4.19}$$

Passing to a subsequence, we obtain that

$$\partial_t u_\mu \text{ converges weakly to } \partial_t u \text{ in } L^2(0, T; H^{-1}(\Omega)).$$

Applying the Aubin-Lions Compactness Theorem, we get that u_μ converges strongly to u (along suitable sequence) in $L^2(\Omega_T)$. Consequently, we obtain that $d(u_\mu) \nabla \mathcal{K} u_\mu$ converges weakly to $d(u) \nabla \mathcal{K} u$ as $\mu \to 0^+$, which is to say, we are ready to pass to the limit in (4.11) as $\mu \to 0^+$ to get

$$\iint_{\Omega_T} u(t, x) \partial_t \varphi(t, x) \, dxdt - \iint_{\Omega_T} u \nabla \mathcal{K}(u) \cdot \nabla \varphi dxdt + \int_\Omega u_0(x) \varphi(0, x) dx = 0,$$

for all $\phi \in C_c^\infty((-\infty, T) \times \mathbb{R}^n)$. According to the Equivalence Theorem 3.1, we have obtained the solvability of the IVBP (1.1). $\square$

Corollary 4.1. *The weak solution u of the IBVP* (1.1) *given by Theorem 4.2, satisfies:*

(1) *For almost all $t \in (0, T)$, we have*

$$\|u(t)\|_\infty \leq \|u_0\|_\infty, \quad \textit{and} \tag{4.20}$$

$$\int_\Omega u(x,t) dx = \int_\Omega u_0(x)\ dx. \tag{4.21}$$

Moreover, $0 \leq u(t, x)$ a.e. in $(0, T) \times \Omega$.

(2) *The first energy estimate: For almost all $t \in (0, T)$,*

$$\int_0^t \int_\Omega |\nabla \mathcal{H} u|^2\ dx dt' + \int_\Omega u(t) \log(u(t))\ dx \leq \int_\Omega u_0 \log(u_0)\ dx. \tag{4.22}$$

(3) *The second energy estimate: For almost all $0 < t_1 < t_2 < T$,*

$$\frac{1}{2} \int_\Omega |\mathcal{H} u(t_2)|^2\ dx + \int_{t_1}^{t_2} \int_\Omega u |\nabla \mathcal{K} u|^2\, dx\, dt \leq \frac{1}{2} \int_\Omega |\mathcal{H} u(t_1)|^2 dx. \tag{4.23}$$

Proof. The proof can be established following almost the same lines of items (2)–(5) in the proof of Proposition 4.1. Therefore, we omit it here. □

Remark 4.2. *The uniqueness question of week solutions to* (1.1) *remains open. It seems that, even if to the Cauchy problem studied in [2]. Certainly, this is one of the most important point to be considered in near future.*

Chapter 5

Appendix

5.1 Deformation

Let us fix here some notation and background used in the thesis. Following [16], we first consider the notion of C^1- (admissible) deformations, which is used to give the correct notion of traces.

Definition 5.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. A C^1-map $\Psi : [0,1] \times \partial\Omega \to \overline{\Omega}$ is said a C^1- admissible deformation, when satisfies the following conditions:*

(1) *For all $r \in \partial\Omega$, $\Psi(0,r) = r$.*

(2) *The derivative of the map $[0,1] \ni \tau \mapsto \Psi(\tau, r)$ at $\tau = 0$ is not orthogonal to $\nu(r)$, for each $r \in \partial\Omega$.*

Moreover, for each $\tau \in [0,1]$, we denote: Ψ_τ the mapping from $\partial\Omega$ to $\overline{\Omega}$, given by $\Psi_\tau(r) := \Psi(\tau, r)$; $\partial\Omega_\tau = \Psi_\tau(\partial\Omega)$; ν_τ the unit outward normal field in $\partial\Omega_\tau$. In particular, $\nu_0(x) = \nu(x)$ is the unit outward normal field in $\partial\Omega$.

Remark 5.1. *As mentioned in [16], it must be recognized that domains with C^1 boundaries always have C^1-admissible deformations. Since in our case $\partial\Omega$ is C^2, it is enough to take $\Psi(\tau, r) = r - \epsilon\tau\nu(r)$ for sufficiently small $\epsilon > 0$. Therefore, under the above conditions, we just call through the paper: C^1-deformations instead of C^1-admissible deformations.*

Now, we define a *level set function associated with the deformation Ψ*.

To begin, let $\Omega \subset \mathbb{R}^n$ be an open set with (at least) C^1 boundary, and assume that Ψ is a C^1- deformation. Then, for each $x \in \partial\Omega$ there exists a neighbourhood W of x in $\mathbb{R}^n$, an open set $U \subset \mathbb{R}^{n-1}$ and a C^1 diffeomorphism mapping $\zeta : U \to \partial\Omega \cap W$. On the other hand, we define $\psi : [0,1] \times U \longrightarrow \overline{\Omega}$ by

$$\psi(\tau, y) := \Psi(\tau, \zeta(y)),$$

which is a C^1 function. Moreover the Jacobian of ψ in $(0, y)$, satisfies

$$J\psi(0,y) = J[\zeta](y)\,|\partial_\tau \Psi(0,\zeta(y)) \cdot \nu(\zeta(y))| > 0$$

for all $y \in U$. Then, applying the Inverse Function Theorem and passing to a smaller neighbourhood if necessary (still denoted by U), there exists $\varrho > 0$ such that, the function $\psi : [0,\varrho) \times U \longrightarrow \overline{\Omega}$ is a C^1 diffeomorphism onto its image.

Since $\partial\Omega$ is compact, we can find finitely many points $x_i \in \partial\Omega$, corresponding sets $W_i \subset \mathbb{R}^n$; $U_i \subset \mathbb{R}^{n-1}$ and functions $\gamma_i \in C^1(U_i)$ $(i = 1, \cdots, m)$, such that $\partial\Omega \subset \cup_{i=1}^m W_i$ and

$$\gamma_i : U_i \longrightarrow \partial\Omega \cap W_i.$$

Moreover, there exit $\varrho_i > 0$, $(i = 1, \ldots, m)$, such that, $\psi_i : [0,\varrho_i) \times U_i \longrightarrow \overline{\Omega}$ is a C^1 diffeomorphism onto its image, where $\psi_i(\tau, y) := \Psi(\tau, \gamma_i(y))$.

Now, we consider $\varrho = \min\{\varrho_i;\, i = 1, \cdots, m\}$ and $V_i := \Psi([0,\varrho) \times \gamma_i(U_i))$. Define $h_i : V_i \to [0,\varrho)$, as follow $h_i(x) := \pi_1 \circ \psi_i^{-1}(x)$, for $x \in V_i$, where $\pi_1 : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}$, given by $\pi_1(a,b) = a$.

In particular, if $x \in \partial\Omega_\tau \cap V_i$, we obtain that $h_i(x) = \tau$. Due to the function ψ_i being a C^1 diffeomorphism, we can see that the function h_i is a C^1 function. Then, we define the *level set function associated with the deformation* Ψ, that is to say, the function

$$h : \bigcup_{i=1}^{m} V_i \to [0,\varrho)$$

by setting $h(x) = h_i(x)$, if $x \in V_i$, which is clearly a C^1 function. Without loss of generality, we may assume $\varrho = 1$. Moreover, since the function ψ_i is a C^1 diffeomorphism, we have that $\nabla h(x) \neq 0$ for all $x \in \bigcup_{i=1}^m V_i$, and also $\nabla h(r)$ is parallel to $\nu_\tau(r)$ on $\partial\Omega_\tau$.

To follow, we define some auxiliary functions and their respective properties, which are important to show existence of solutions of the IBPV (1.1).

1. Let $\rho > 0$ be sufficiently small, and define

$$s(x) := \begin{cases} h(x), & \text{if } x \in \Omega, \\ -h(x), & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

2. For each $k \in \mathbb{N}$, and all $x \in \mathbb{R}^n$, define ξ_k by

$$\xi_k(x) := 1 - \exp\left(-k\; s(x)\right). \tag{5.1}$$

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with C^2 boundary. Then, it follows that:*

1. *The function $s(x)$ is Lipschitz continuous in $\mathbb{R}^n$, and C^1 on the closure of $\{x \in \mathbb{R}^n : |s(x)| < \rho\}$.*

2. *The sequence $\{\xi_k\}$ satisfies*

$$\lim_{k\to+\infty} \int_\Omega |1-\xi_k|^2 dx = 0, \quad \textit{and} \quad \lim_{k\to+\infty} \int_\Omega |\nabla \xi_k|^2 dx = 0. \tag{5.2}$$

Proof. See Málek, Necas, Rokyta and Ruzicka [12] p.129. □

Finally, let us consider the following approching sequences.

Choose a non-negative function $\gamma \in C_c^1(\mathbb{R})$, with support contained in $[0,1]$, such that, $\int \gamma(t)dt = 1$. Then, we set for $j \in \mathbb{N}$

$$\delta_j(t) := j\ \gamma(jt), \quad H_j(t) = \int_0^t \delta_j(s)\ ds,$$

hence for each $j \geq 1$, $H_j'(t) = \delta_j(t)$. Clearly, the sequence $\{H_j'\}$ converges as $j \to \infty$ to the Dirac δ-measure in $\mathcal{D}'(\mathbb{R})$, while the sequence $\{H_j\}$ converges pointwise to the Heaviside function

$$H(t) = \begin{cases} 1, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

5.2 Interpolation spaces: J-method

We recall the basic definitions and results about interpolation between Banach spaces, following Adams [1] Chapter 7.

Let X_0, X_1 be Banach spaces each of which is (continuously) imbedded in a Hausdorff topological vector space $\mathcal{E}$, and whose intersection is nontrivial.

The intersection $X_0 \cap X_1$ and the algebraic sum $X_0 + X_1$, are themselves Banach spaces with respect to the norms

$$\begin{aligned} \|u\|_{X_0\cap X_1} &= \max\{\|u\|_{X_0}, \|u\|_{X_1}\}, \\ \|u\|_{X_0+X_1} &= \inf\{\|u_0\|_{X_0} + \|u_1\|_{X_1} \mid u = u_0 + u_1 \text{ with } u_0 \in X_0,\ u_1 \in X_1\}. \end{aligned}$$

Define moreover the space (Bochner integral)

$$L_\star^q := L^1\left((0,+\infty); \frac{dt}{t} : X_0 + X_1\right) \quad for\ any\ q \geq 1.$$

The J-method. Define the $J(t,u) - norm$ by

$$J(t,u) = \max\{\|u\|_{X_0}, t\|u\|_{X_1}\}\,. \tag{5.3}$$

The J-norm is clearly equivalent to $\|u\|_{X_0 \cap X_1}$. If $0 \leq \theta \leq 1$ and $1 \leq q \leq \infty$ we denote by

$$X_{\theta,q} := [X_0, X_1]_{\theta,q}$$

the space of all $u \in X_0 + X_1$ such that

$$u = \int_0^\infty f(t)\frac{dt}{t}$$

for some $f \in L^1_\star$ having values in $X_0 \cap X_1$, and such that the function

$$t \to \frac{J(t, f(t))}{t^\theta} \in L^q_\star$$

In the case $q = 2$, we will simplify the notation as follows:

$$X_{\theta,2} := [X_0, X_1]_{\theta,2} = [X_0, X_1]_\theta = X_\theta. \tag{5.4}$$

Now, we enunciate a Theorem, which shows that $X_{\theta,q}$ is non-trivial Banach space. The Theorem is called *J-Method*

Theorem 5.1 (The J-Method)**.** *If either* $1 < q \leq \infty$ *and* $0 < \theta < 1$ *or* $q = 1$ *and* $0 \leq \theta \leq 1$ *, then* $X_{\theta,q} = [X_0, X_1]_{\theta,q}$ *is a non-trivial Banach space with the norm*

$$\|u\|_{X_{\theta,q,J}} = \inf_{f \in S(u)} \|f\|_{L^q_\star} = \inf_{f \in S(u)} \left(\int_0^\infty \left(t^{-\theta} J(t, f(t)) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \quad if\ q < \infty$$

where

$$S(u) = \left\{ f \in L^1_\star \mid u \int_0^{+\infty} f(t)\frac{dt}{t} \right\}$$

Furthermore,

$$\|u\|_{X_0 \cap X_1} \leq \|u\|_{\theta,q,J} \leq \|u\|_{X_0 + X_1}$$

so that $X_0 \cap X_1 \hookrightarrow [X_0, X_1]_{\theta,q} \hookrightarrow X_0 + X_1$ *with continuous injections, so that* $[X_0, X_1]_{\theta,q}$ *is an intermediate space between* X_0 *and* X_1*.*

Proof. This is Theorem. 7.13 of Adams [1], proof at p. 211. □

We are interested in a discrete version of the above theorem, but in a slightly more general form than the one given in Theorem 7.14 of [1]

Theorem 5.2 (The Discrete version of the J-method). *Let μ_k be an increasing sequence $0 < \mu_k < \mu_{k+1} \longrightarrow \infty$, such that $0 < \mu_{k+1}/\mu_k \leq \Lambda_0 < \infty$. Let either $1 < q \leq \infty$ and $0 < \theta < 1$ or $q = 1$ and $0 \leq \theta \leq 1$. Then a function $u \in X_0 + X_1$ belongs to $X_{\theta,q} = [X_0, X_1]_{\theta,q}$ if and only if $u = \sum_{k\geq 1} u_k$, where the series converges in $X_0 + X_1$, and the sequence*

$$U_k = \mu_k^{-\theta} J(\mu_k, u_k) \in \ell^q(\mathbb{N}).$$

In this case, the norm $\|u\|_{X_{\theta,q,J}}$ is equivalent to

$$\|u\|_{\theta,q,JD} = \inf \left\{ \|U_k\|_{\ell^q(\mathbb{N})} \mid u = \sum_{k\geq 0} u_k \right\}.$$

Proof. The proof of this Theorem could be found it in M. Bonforte, Y. Sire, L. Vazquez [15] p. 37. □

5.3 Area and Coarea Formulas

In this section, we recall the basic definitions to enunciate the area and coarea formulas, following Evans and Gariepy [7] Chapter 3.

Here, we consider $f : \mathbb{R}^n \to \mathbb{R}^m$ a Lipschitz continuous mappings

Definition 5.2. *Let $\Omega \subset \mathbb{R}^n$.*

(1) *A function $f : \Omega \to \mathbb{R}^m$ is called Lipschitz continuous provided*

$$|f(x) - f(y)| \leq C|x - y|,$$

for some constant C and all $x, y \in A$.

(2) *A function $f : \Omega \to \mathbb{R}^m$ is called locally Lipschitz continuous if for each compact $K \subset \Omega$, there exists a constant C_K such that*

$$|f(x) - f(y)| \leq C_K|x - y|,$$

for all $x, y \in K$.

Now, we enunciate Rademacher's Theorem, which say that a Lipschitz continuous function is differentiable $\mathcal{L}^n$-a.e.

Theorem 5.3 (Rademacher's Theorem). *Assume that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a locally Lipschitz continuous function. Then f is differentiable $\mathcal{L}^n - a.e.$*

Proof. See Evans and Gariepy [7] p 103. □

Thank to Rademacher's Theorem, we can define the Jacobian of Lipschitz continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$. Indeed, by Theorem 5.3, we obtain that f is differentiable $\mathcal{L}^n$-a.e., and therefore $Df(x)$ exists, and can be regarded as a linear mapping from $\mathbb{R}^n$ into $\mathbb{R}^m$, for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$. Then the following definition make sense.

Definition 5.3. *Assume $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz continuous map*

(1) *If $n \leq m$, we define the Jacobian of f by*

$$Jf(x) = \sqrt{\det\big([Df(x)]^*[Df(x)]\big)}$$

(2) *If $n \geq m$, we define the Jacobian of f by*

$$Jf(x) = \sqrt{\det\big([Df(x)][Df(x)]^*\big)}$$

where $[Df(x)]^*$ is the adjoint of $Df(x)$.

Now, we are ready to enunciate the Area Formula and Coarea Formula for Lipschitz function.

First we start with the Area Formula and its extension, as follow

Theorem 5.4 (Area formula)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz continuous, with $n \leq m$. Then, for any $\mathcal{L}^n$-measurable set $\Omega \subset \mathbb{R}^n$, we have*

(1) *$f(\Omega)$ is $\mathcal{H}^n$-measurable.*

(2) *The mapping $y \to \mathcal{H}^0\big(\Omega \cap f^{-1}\{y\}\big)$ is $\mathcal{H}^n$-measurable on $\mathbb{R}^m$ and*

$$\int_\Omega Jf(x)\,dx = \int_{\mathbb{R}^m} \mathcal{H}^0\big(\Omega \cap f^{-1}\{y\}\big)d\mathcal{H}^n(y), \tag{5.5}$$

where the mapping $y \to \mathcal{H}^0\big(\Omega \cap f^{-1}\{y\}\big)$ is called multiplicity function.

Proof. See Evans and Gariepy [7] p 119. □

Observe that (5.5) can be rewrite as follows

$$\int_{\mathbb{R}^n} \mathcal{X}_\Omega(x) Jf(x)dx = \int_{\mathbb{R}^m} \left\{ \sum_{x \in f^{-1}(y)} \mathcal{X}_\Omega(x) \right\} d\mathcal{H}^n(y),$$

where $\mathcal{X}_\Omega$ is the characteristic function of Ω. The next corollary generalize the Area Formula if we change $\mathcal{X}_\Omega$ for any $g : \mathbb{R}^n \to \mathbb{R}$, $\mathcal{L}^n$-summable function.

Corollary 5.1 (Changing variables)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz continuous, $n \leq m$. Then for each $\mathcal{L}^n$-summable function $g : \mathbb{R}^n \to \mathbb{R}$,*

$$\int_{\mathbb{R}^n} g(x) Jf(x) dx = \int_{\mathbb{R}^m} \left\{ \sum_{x \in f^{-1}(y)} g(x) \right\} d\mathcal{H}^n(y). \tag{5.6}$$

Proof. See Evans and Gariepy [7] p 122. □

Finally we enunciate the Coarea Formula and its extension. we have

Theorem 5.5 (Coarea formula)**.** *. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz continuous, $n \geq m$. Then, for any $\mathcal{L}^n$ -measurable set $\Omega \subset \mathbb{R}^n$, we have*

(1) *$\Omega \cap f^{-1}\{y\}$ is $\mathcal{H}^{n-m}$-measurable for $\mathcal{L}^n$-a.e. y.*

(2) *The mapping $y \to \mathcal{H}^{n-m}\big(\Omega \cap f^{-1}\{y\}\big)$ is $\mathcal{H}^n$-measurable on $\mathbb{R}^m$ and*

$$\int_{\Omega} Jf(x) = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(\Omega \cap f^{-1}\{y\}) d\mathcal{H}^n(y). \tag{5.7}$$

Proof. See Evans and Gariepy [7] p 134. □

Observe that (5.7) can be rewrite as follows

$$\int_{\mathbb{R}^n} \mathcal{X}_\Omega(x) Jf(x) dx = \int_{\mathbb{R}^m} \left\{ \int_{f^{-1}\{y\}} \mathcal{X}_\Omega d\mathcal{H}^{n-m} \right\} dy,$$

where $\mathcal{X}_\Omega$ is the characteristic function of Ω. The next corollary generalize the Coarea Formula if we change $\mathcal{X}_\Omega$ for any $g : \mathbb{R}^n \to \mathbb{R}$, $\mathcal{L}^n$-summable function.

Corollary 5.2 (Integration over level sets)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz, $n \geq m$. Then for each $\mathcal{L}^n$-summable function $g : \mathbb{R}^n \to \mathbb{R}$,*

(1) *$g|_{f^{-1}\{y\}}$ is $\mathcal{H}^{n-m}$ summable for $\mathcal{L}^m$-a.e. y, and*

(2)

$$\int_{\mathbb{R}^n} g(x) Jf(x) dx = \int_{\mathbb{R}^m} \left\{ \int_{f^{-1}\{y\}} g \, d\mathcal{H}^{n-m} \right\} dy.$$

Proof. See Evans and Gariepy [7] p 139. □

5.4 Aubin-Lions's Theorem

In this section, we recall the basic definitions to enunciate Aubin-Lions Theorem, following Málek, Necas, Rokyta and Ruzicka [12] Chapter 1.

Let X, Y be two Banach spaces equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$.

Definition 5.4. *Let X and Y be Banach spaces, $X \subset Y$. We say that X is (continuously) imbedded into Y, written*

$$X \hookrightarrow Y,$$

if only if there exists $c > 0$ such that $\|x\|_Y \leq c\|x\|_X$, for all $x \in X$.

Definition 5.5. *Let X and Y be Banach spaces, $X \subset Y$. We say that X is compactly imbedded into Y, written*

$$X \hookrightarrow\hookrightarrow Y,$$

provided

(1) *$X \hookrightarrow Y$.*

(2) *The identity map $I : X \to Y$ is compact, i.e. $I(B)$ is compact in Y for every bounded subset B of X.*

Let X be a Banach space and $T > 0$. The space $L^p((0,T);X)$, $1 \leq p \leq \infty$ we denote the space of all measurable functions $u : I \to X$ for which the norm

$$\|u\|_{L^p((0,T),X)} = \left\{ \int_0^T \|u(t)\|_X^p dt \right\}^{1/p}, \quad p < \infty,$$

or

$$\|u\|_{L^\infty((0,T),X)} = \operatorname{ess} \sup_{t\in(0,T)} \|u(t)\|_X, \quad p = \infty,$$

respectively, is finite. That space are called Bochner spaces. Now we enunciate the Aubin-Lions's Theorem.

Theorem 5.6 (Aubin-Lions)**.** *Let $1 < \alpha, \beta < \infty$. Let X be a Banach space, and let X_0, X_1 be separable and reflexive Banach spaces. Provided that $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$ we have*

$$\left\{ v \in L^\alpha((0,T);X_0); \frac{dv}{dt} \in L^\beta((0,T);X_1) \right\} \hookrightarrow\hookrightarrow L^\alpha((0,T);X).$$

Proof. See Málek, Necas, Rokyta and Ruzicka [12] p.36. □

Bibliography

[1] R. A. Adams, J. F. Fournier, *Sobolev spaces. Second edition*, Pure and Applied Mathematics (Amsterdam), 140 Elsevier/Academic Press, Amsterdam, 2003.

[2] L. Caffarelli and J.L. Vazquez, *Nonlinear Porous Medium Flow with Fractional Potential Pressure*, Arch. Rational Mech. Anal. 202 (2011), 537–565.

[3] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations 32 (2007), no. 7-9, 1245-1260.

[4] A. Capella, J. Dávila, L. Dupaigne, Y. Sire, *Regularity of radial extremal solutions for some non local semilinear equations*, Preprint, arXiv:1004.1906.

[5] X. Cabré and J. Tan, *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Advances in Mathematics, 224 (2010), 2052–2093.

[6] E. B. Davies. *Spectral theory and differential operators*, Cambridge Studies in Advanced Mathematics, 42. Cambridge University Press, Cambridge, 1995.

[7] L.C. Evans and R.F. Gariepy, *Measure Theory and fine property of function*, CRC Press, Boca Raton, Florida, 1992.

[8] H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.

[9] S. Kim, Ki-Ahm Lee , *Smooth solution for the porous medium equation in a bounded domain*, J. Differential Equation 347(2009) 1064-1095.

[10] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'ceva, *Linear and quasilinear equations of parabolic type.* American Mathematical Society, Providence RJ, 1968.

[11] J.L. Lions, E. Magenes, *Problemes aux limites non-homogenes et application*, V.1, Dunod, Paris, 1968.

[12] J. Málek, J. Necas, M. Rokyta and M. Ruzicka, *Weak and Measure-valued solutions to evolutionary PDEs*, Chapman and Hall, London, 1996.

[13] F. Otto, *Ein Randwertproblem fur skalare Erhaltungassatze*, Thesis, Universitat Bonn, 1993.

[14] Luc Tartar, *An Introduction to Sobolev Spaces and Interpolation Spaces*, Lecture Notes of the Unione Matematica Italiana, Springer

[15] M. Bonforte, Y. Sire, J. L. Vázquez, *Existence, Uniqueness and Asymptotic behaviour for fractional porous medium equations on bounded domains*, Manuscript submitted to AIMS' Journals.

[16] W. Neves, E. Panov, J. Silva, *Strong Traces for Conservations Laws with general non-autonomous flux*, submitted.

[17] W. Noll, *A mathematical theory of the mechanical behavior of continuous media*, Arch. Rational Mech. Anal. 2 (1958), 197–226.

[18] J.L. Vázquez, *The Dirichlet problem for the porous medium equation in bounded domains. Asymptotic behavior*, Springer-Verlag 2004.

[19] J.L. Vázquez, *Recent progress in the theory of Nonlinear Diffusion with Fractional Laplacian operator*, Discrete Contin. Dyn. Ser S7(2014) no 4, 857-885.