On Magnetic Flows

Freddy Pablo Castro Vicente

Tese de Doutorado apresentada ao Programa de Pós-graduação do Instituto de Matemática, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Doutor em Matemática.

Orientador: Alexander Eduardo Arbieto Mendoza

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Neste trabalho, estudamos algumas propriedades genéricas de fluxos magnéticos. Basicamente o fluxo magnético é definido como o fluxo geodésico perturbado por uma 2-forma. Tal forma é obtida pelo levantamento de uma 2-forma na variedade, somada à 2-forma canônica no fibrado tangente da variedade.

J. A. G. Miranda estudou certas propriedades genéricas de fluxos magnéticos. Ele estudou em [26] duas propriedades genéricas dos fluxos magnéticos em superfícies, a saber, o teorema de Kupka-Smale e o teorema dos k-Jets do mapa de Poincaré. Em [27] ele estudou a entropia topológica dos fluxos magnéticos em superfícies mostrando a positividade desta na presença de uma órbita fechada não-hiperbólica ou na presença de infinitas órbitas fechadas.

O objetivo desta tese é estender os resultados acima para variedades suaves em dimensão qualquer. Para isto são usadas técnicas de *teoria do controle geométrico* introduzidas por Rifford e Ruggiero em [38] e Lazrag, Rifford e Ruggiero em [21]. Em [38] é obtido a versão do teorema de Kupka-Smale no contexto de fluxos Hamilton-Tonelli em variedades suaves de dimensão qualquer, que generaliza o trabalho de Oliveira [30] em superfícies. Em [21] é obtido uma versão do lema de Franks no contexto de fluxos geodésicos, aqui eles no precisam da condição genérica na curvatura como no trabalho de Contreras in [8], nesse sentido é melhor.

Nosso principal resultado é, referente a propriedade dos fluxos magnéticos. Basicamente diz:

Teorema.- *Existe um conjunto aberto e denso de fluxos magnéticos definidos numa variedade suave de dimensão qualquer, tal que, tem entropia topológica positiva ou tem um número finito de órbitas fechadas.*

On Magnetic Flows

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In this work, we study some generic properties of magnetic flows. Basically the magnetic flow is defined as the geodesic flow perturbed by a 2-form. Such form is obtained by the lifting of a 2-form in the manifold added to the canonical 2-form in the tangent bundle of the manifold.

J. A. G. Miranda has studied certain generic properties of magnetic flow. He studied in [26] two generic properties of magnetic flows on surfaces, namely the Kupka-Smale's theorem and the k-Jets theorem of the Poincaré map. In [27] he studied the topological entropy of magnetic flows on surfaces showing its positivity in the presence of a closed non-hyperbolic orbit or in the presence of infinite closed orbits.

The objective of this thesis is to extend the above results to smooth manifolds in any dimension. For this are used techniques of *geometric control theory* introduced by Rifford and Ruggiero in [38] and Lazrag, Rifford and Ruggiero in [21]. In [38] is obtained a version of Kupka-Smale's theorem in the context of Hamilton-Tonelli flows in smooth manifolds of any dimension, which generalizes the work of Oliveira in [30] in surfaces. In [21] a version of the Franks' lemma is obtained in the context of geodetic flows, here they do not need the generic condition in the curvature as in the work of Contreras in [8], in that sense it is better.

Our main result is, concerning the property of the magnetic flows. Basically says:

Theorem.- *There is an open and dense set of magnetic flows defined in a smooth manifold of any dimension, such that it has positive topological entropy or has a finite number of closed orbits.*

Contents

Chapter 1

Introduction

One of the main questions in the study of dynamical systems is, how chaotic is a system? But another natural question is, does it mean that a system is chaotic? There are indeed different definitions to say that a system is chaotic. One of the main examples of chaotic systems is the Smale's horseshoe which has an infinite number of periodic orbits, and this is a hyperbolic set. A definition of chaos that we will use in this work is that of topological entropy, in short, it measures the exponential growth rate of the number of periodic orbits of the system in question. One of the main properties is that this is a topological invariant and is somewhat stable.

On the other hand, in the study of hyperbolic dynamics, we have the famous work of Ricardo Mañé on C^1 structural stability [25], where the pillars of the demonstration since work are the Franks' Lemma and the Pugh's Closing Lemma, the latter using the Kupka-Smale's theorem. In fact, these two results: the Kupka-Smale's theorem [34] and the Franks' lemma [11] are of paramount importance in the study of systems in different contexts. We also have that Newhouse proof the C^1 structural stability in the context of symplectic diffeomorphisms, see [28].

There are many works regarding positive topological entropy of systems in different contexts. One of the main ones is the work of Contreras [8] in geodesic flows that states that almost all geodesic flow has positive topological entropy. But our main reference are the works of J. A. G. Miranda [26] and [27], where he studies the topological entropy of the magnetic flows in surfaces. Our work is

Theorem A: *There is an open and dense set of magnetic flows defined in a smooth manifold of any dimension, such that, it has positive topological entropy or has a finite number of closed orbits.*

This is a extension of Miranda's results. Our main tool will be the *geometric control theory* developed by Lazrag, Rifford and Ruggiero in [38] and [21]. But we must always contrast with the proof in geodetic flows of Contreras, where the main difference is the non-existence of Rademacher's theorem [37] in the context of magnetic flows, this is,

Problem: *Is it true that almost every magnetic flow has infinite closed orbits at every level of energy?*

On the other hand, we must highlight the great power of the method using geometric control theory to solve perturbations of a linear system of ordinary differential equations generated by the nature of magnetic curvature. The first time these methods were used in [38] was to obtain the Kupka-Smale's theorem in the context of Hamilton-Tonelli flows defined in smooth manifolds of any dimension. Then in [21], it was also used to obtain the Franks' Lemma of geodetic flows without the need of generic condition in the curvature as in [8].

In Chapter 2, we define the magnetic flows and make a contrast with the geodesic flows, we define the magnetic fields of Jacobi and the magnetic injective radius. We will also study the method of geometric control theory that we will use. Finally, we recall some dynamic and ergodic properties of C^1 diffeomorphisms. Briefly, given a smooth closed 2-form Ω in M, the magnetic flow is a Hamiltonian flow in the tangent bundle TM, denote by ϕ_t^Ω satisfying the following equation

$$\begin{cases} x' & = & v, \\ \dfrac{Dv}{dt} & = & Y^\Omega(v). \end{cases}$$

In Chapter 3, we will obtain *the magnetic tubular neighborhood*, that satisfies the classical properties, as in the case of geodesic flows. The most difficult part is obtain the matrix Ω-magnetic curvature as symmetric matrix, since Ω is skew-symmetric. The tubular neighborhood implies *the perturbative Lemma* that is the key to all the work. Thus, for each perturbation of Ω, define its linear part of Poincaré map. The perturbative lemma states that this application is an open map on the symplectic matrix.

In Chapter 4 we will study the version of KupKa-Smale's theorem in this context of magnetic flows. The original theorem was made for smooth diffeomorphisms by I. Kupka [17] and S. Smale [43] in 1963, separately. However best known version of the proof this theorem is of M. Peixoto [34]. This theorem also exists in other contexts, for example smooth vector fields, also in smooth flows such as the geodesic flows by D. Anosov [4] and G. Contreras [9], Hamiltonian flows by R. C. Robinson [39] and [40], also the magnetic flows on surfaces by J. A. Miranda [26]. In all the methods of Peixoto are used. In simple language, this theorem says that

Theorem B: *Almost every magnetic flow satisfy these two properties:*

1. *all closed orbits are hyperbolic or elliptic,*
2. *all heteroclinic points are transversal.*

In Chapter 5 we will study one of the important generic results in relation to the spaces of k-Jets of the map of Poincaré of a closed orbit of the magnetic flow. The motivation is always the analogy to geodesic flows, in W. Klingenberg and F. Takens [15] extend the bumpy metric theorem including conditions in the k-jets of Poincaré map of closed orbits of the geodesic flow. On the other hand F. Takens [45] also obtains the analogous result in the Hamiltonian context, that is, for Hamiltonian flows. In simple language, this result says that

Theorem C: *Let $\mathcal{Q}$ an open and dense invariant subset of space k-jets. Almost every magnetic flow, it has the k-jet of the Poincaré map belong to $\mathcal{Q}$.*

In Chapter 6, last, is divided into two parts. The first part we will study the version of *Franks's Lemma* in this context of magnetic flows in smooth manifolds of any dimension. The original version of this lemma was made for diffeomorphisms in smooth manifolds of any dimension and proved by Franks [11]. Contreras and Paternain [9] proved the version of this lemma in the context of flows geodesic on surfaces, later Contreras [8] proved it for any dimension. Then Miranda [27] proved this result in this context of magnetic flows but in surfaces. The idea here is to generalize this result. In simple words says

In the second part of this chapter, we will study a property of dichotomy for the magnetic flows. Basically this says that

there exists a open and dense set of magnetic flows such that either the topological entropy is positive or the closed set of closed orbits is a hyperbolic set.

This result is another way of writing Miranda's work on surfaces [27]. On the other hand, if we contrast with geodesic flows, Contreras [8] has the version of this result. In the case that the geodesic flows have an infinite number of periodic orbits, he use the Rademacher's theorem [37], in order to perturb and still obtain an infinite quantity of closed orbits and passing through the *Smale's spectral decomposition theorem for flows* he obtains a basic non-trivial hyperbolic set, then positive topological entropy. A major difference in our context is, that there is no version of Rademacher's theorem in magnetic flows.

Chapter 2

Preliminaries

In this chapter we will give the definitions and basic results that will be necessary in the rest of this work.

Throughout all this work, we always denote M be a compact, connected, boundaryless Riemaniann manifold of dimension $m := n + 1$ with the smooth Riemannian metric $g := \langle \cdot, \cdot \rangle$. This metric induces a geodesic distance on the manifold, that we will denote by d, in such a way that the pair (M, d) is a complete metric space.

We denote by ∇ the Levi-Civita connection associated the metric, by R the Riemannian curvature tensor and $\pi : TM \to M$ denote the canonical projection.

A starting point for our studies is the concept of a geodetic curve in manifolds. Roughly speaking, a geodesic is a curve that locally minimizes the distance between two points.

By analytically, in local coordinates, such curves are solutions of a nonlinear second-order ordinate differential equation. Moreover, they can be seen as the projection of the orbits of a flow, said geodesic, in the tangent bundle TM.

From this point of view, other important dynamic systems, derived from the physics, can be seen in this way. We will now outline some of these equations. After this, we will define several objects that will be used in the thesis.

2.1 The equations of the magnetic flow

A curve $\gamma : (-\varepsilon, \varepsilon) \to M$, for small $\varepsilon > 0$, is a geodesic if it satisfies

$$\frac{D}{dt}\gamma' = 0.$$

From the point of view of TM, we can see this second-order equation as the following system of first-order equations

$$\begin{cases} x' &= v, \\ \frac{D}{dt}v &= 0. \end{cases} \tag{2.1}$$

Let $Y : TM \to TM$, a linear skew-symmetric bundle map. This application will be said to be a *magnetic field* (or *Lorentz force*) and gives rise to what we call the magnetic flow, which models the motion of a charge particle and unit mass under the effect of the magnetic field. As above, the equation in TM is

$$\begin{cases} x' &= v, \\ \frac{D}{dt}v &= Y(v). \end{cases} \tag{2.2}$$

2.2 Riemannian Geometry

In this section we will collect several geometry results that will be used later.

Recall that a Jacobi field J is a field on a geodesic $\gamma : [0, T] \to M$ that satisfies the following differential equation

$$\frac{D^2}{dt^2}J + R(\gamma', J)\gamma' = 0.$$

We also recall that a Jacobi field can be seen as a variational field on a geodesic, obtained by a variation of the same by geodesics. Let $p = \gamma(0) \in M$, then we say that $\gamma(t_0)$, with $t_0 > 0$, is a *conjugate point* of p if there is a Jacobi field J, not identically null, such that $J(0) = J(t_0) = 0$. The conjugate points have a deep relation with the negativity of a bilinear form which is the form of index

$$I(V, W) = \int_0^T \left\{ \left\langle \frac{D}{dt}V, \frac{D}{dt}W \right\rangle - \langle R(\gamma', V)\gamma', W\rangle \right\} dt.$$

Proposition 2.2.1. *(Morse index) If i_T is the maximum dimension of a subspace in which the quadratic form associated with I is defined negative then i_T is the number of points conjugated to $\gamma(0)$, where we counted these points with multiplicity.*

The proof of proposition can be found in [10], ch. 11.

On the other hand, the index form has relations with the second derivative of the functional energy along a geodesic. Given a closed curve $\alpha : [0, T] :\to M$, the *energy* of c is given by

$$\mathcal{E}(\alpha) = \int_0^T \|\alpha'(s)\|^2 ds.$$

Proposition 2.2.2. *(The formula of the second variation) Suppose α is a geodesic. If α_s is a part-differentiable geodesic variation of α, for closed curves, and V is the variational field associated with this variation then*

$$\frac{1}{2}\frac{d^2\mathcal{E}}{ds^2}\bigg|_{s=0} = I(V, V).$$

The proof of this proposition can be found in [10], ch. 9.

2.3 Symplectic Geometry and Hamiltonian Flows

Later we will present the equations mentioned above from other points of view. For this we need the language of symplectic geometry.

Definition 2.3.1. We say that a 2-form ω on M is *symplectic* if it is closed, $d\omega = 0$, and not degenerate, this is, if $\omega_x(u, v) = 0$ for all $v \in T_xM$ then $u = 0$.

Thus, if ω is a symplectic form then we say that (M, ω) is a symplectic manifold.

For our purposes it is interesting to obtain a description of TTM. In particular, we would like to obtain a reasonable Riemannian metric in TM. One of the most useful is the *Sasaki metric*, which is obtained by the decomposition of TTM in the so-called vertical and horizontal bundles, which we present below.

The *vertical bundle* is simply the kernel of derivative canonical projection, that is, $V := \ker(d\pi)$. Geometrically, if $\theta = (x, v) \in TM$ then V_θ is the space tangent to fiber $\pi^{-1}(x)$ at point θ, this is, $V_\theta = T_\theta T_x M$.

The horizontal bundle is also defined as the kernel of a certain application, known as the connection map. As the manifold M is Riemannian we can define the application connection

$$K : TTM \to TM,$$

as follows. Given $\xi \in T_\theta TM$, let $Z : (-\varepsilon, \varepsilon) \to TM$ *adapted to* ξ, this is, $Z(t) = (\alpha(t), \beta(t))$ where $\alpha : (-\varepsilon, \varepsilon) \to M$, $\alpha = \pi \circ Z$ and β is a vector field along α. Such that $Z(0) = \theta$ and $Z'(0) = \xi$. We defined

$$K_\theta(\xi) = \nabla_{\alpha'}\beta\big|_{t=0}.$$

Then the *horizontal bundle* on TM, as the bundle on TM whose fiber in θ is given by

$$H_\theta := \ker(K_\theta).$$

Proposition 2.3.2.

$$TTM = H \oplus V.$$

The proof of this proposition can be found in [31], p. 13. Through this decomposition we can define the Sasaki metric as follows. If $\xi \in T_\theta TM$ then via the decomposition above, we can write $\xi = (\xi_1, \xi_2)$, where $\xi_1 = \xi_h = d_\theta\pi(\xi)$ and $\xi_2 = \xi_v = K_\theta(\xi)$.

Definition 2.3.3. The *Sasaki metric* in TM is given by

$$\langle\langle \xi, \eta \rangle\rangle_\theta := \langle \xi_1, \eta_1 \rangle_x + \langle \xi_2, \eta_2 \rangle_x \,,$$

for all $\xi, \eta \in T_\theta TM$. In addition, it makes H_θ and V_θ orthogonal.

A *Hamiltonian* is a smooth function $H : TM \to \mathbb{R}$. By contraction of the symplectic form, we can define a field X_H, which we will call the *Hamiltonian field.* That is, X_H is the only field that satisfies:

$$d_\theta H(\cdot) = \omega_\theta(X_H, \cdot).$$

By compactness, this field generates a smooth complete flow φ_t in TM that we will call the *Hamiltonian flow* associated with H. It is simple to note that the symplectic form is preserved by this flow, see [31], p. 10.

If c is a regular value of H then $T^cM := H^{-1}(c)$ is said a *energy level.* Which is an invariant submanifold by the Hamiltonian flows. Indeed:

$$(H(\varphi_t(\theta)))'(s) = d_{\varphi_s(\theta)}H(X_H(\varphi_s(\theta))) = \omega_{\varphi_s(\theta)}(X_H(\varphi_s(\theta)), X_H(\varphi_s(\theta))) = 0,$$

then $H(\varphi_t(\theta)) = H(\theta) = c$. So we can consider $\varphi_t : T^cM \to T^cM$.

Let V be a vector space and ω a symplectic form in it. Note that, necessarily, the dimension of V must be even. Thus, $\dim V = 2n$.

Definition 2.3.4. We will say that a subspace $L \subset V$ is *Lagrangian* if $\dim L = n$ and $\omega|_{L\times L} = 0$.

Let $\Lambda(V)$ be the set formed by all the Lagrangian subspaces of (V, ω). This set has the manifold structure and is called the *Grassmannian manifold of the Lagrangian subspaces.*

2.4 Magnetic flows

In this section we present the equations from another point of view, using the Hamiltonian language. Also presented some properties of the magnetic flows that is the object of study in all this work.

The *geodetic flow* is a Hamiltonian flow with respect to the Hamiltonian $H : TM \to \mathbb{R}$, as $H(x, v) = \dfrac{1}{2}\langle v, v\rangle_x$ (kinetic energy), and the symplectic canonical form ω_0 of TM (”pullback” of the symplectic canonical form of cotangent bundle T^*M by the metric), such that for every $\xi, \eta \in T_\theta TM = H_\theta \oplus V_\theta$ we have

$$\omega_0(\xi, \eta) = \langle \xi_1, \eta_2\rangle - \langle \xi_2, \eta_1\rangle .$$

For more information see [31].

Given a smooth closed 2-form Ω in M, we defined the symplectic form $\omega_\Omega := \omega_0 + \pi^*\Omega$ in TM that is called the *twist symplectic structure.* It is not difficult to show that ω_Ω is a symplectic form on TM. We call the Hamiltonian flow with respect to the Hamiltonian energy and ω_Ω of *magnetic flow* with respect to Ω and denote by $\phi_t := \phi_t^\Omega : TM \to TM$. The *magnetic field* associated with Ω is denote by $X := X^\Omega$.

A direct calculation shows that vertical bundle is Lagrangian with respect to the symplectic form ω_Ω. In addition, this bundle presents the twist property with respect to the magnetic flow, as shown in [29].

Proposition 2.4.1. *(Twist property of vertical bundle) Let E be a Lagrangian subspace of $T_\theta TM$. The subset given by*

$$\{t \in \mathbb{R} : d_\theta \phi_t(E) \cap V_{\phi_t(\theta)} \neq \{0\}\}$$

is discrete.

Let $Y := Y(\Omega) : TM \to TM$ be the bundle linear map definite as

$$\Omega_x(u, v) = \langle Y_x(u), v \rangle_x ,$$

for all $u, v \in T_xM$, it is called the *Lorentz force*. Note that $Y_x : T_xM \to T_xM$ is a linear skew-symmetric map, for all $x \in M$. Also that the map $(x, v) \mapsto Y_x(v)$ is a $(1, 1)$-tensor. We recall some important equations satisfied by the Lorentz force, see [12] and [23].

Lemma 2.4.2. *For all $u, v, w \in T_xM$, we have that*

1. $\nabla_u(Y(v)) = (\nabla_u Y)(v) + Y(\nabla_u v)$
2. $\langle (\nabla_w Y)(u), v\rangle + \langle u, (\nabla_w Y)(v)\rangle = 0,$
3. $\langle (\nabla_w Y)(u), v\rangle + \langle (\nabla_v Y)(w), u\rangle + \langle (\nabla_u Y)(v), w\rangle = 0.$

Now we will deduce the previous equation. Denote ω_Ω by simply ω

$$\begin{aligned} d_\theta H(\xi) &= \omega_\theta(X(\theta), \xi) \\ &= (\omega_0)_\theta(X(\theta), \xi) + (\pi^*\Omega)(X(\theta), \xi) \\ &= (\omega_0)_\theta(X(\theta), \xi) + \Omega_x(d_\theta\pi(X(\theta)), d_\theta\pi(\xi)) \\ &= (\omega_0)_\theta(X(\theta), \xi) + \langle Y_x(d_\theta\pi(X(\theta)), d_\theta\pi(\xi)\rangle_x \end{aligned}$$

holds for every $\xi \in T_\theta TM$ and if we write $\xi = (\xi_1, \xi_2)$ and $X = (X_1, X_2)$, so

$$\langle \xi_2, v \rangle_x = \langle X_1(\theta), \xi_2 \rangle_x - \langle X_2(\theta), \xi_1 \rangle_x + \langle Y_x(X_1(\theta)), \xi_1 \rangle_x \,,$$

therefore

$$X(\theta) = (v, Y_x(v)) \in H(\theta) \oplus V(\theta),$$

for every $\theta = (x, v) \in TM$.

Note that if $c > 0$ then the vector field X has no singularities in T^cM. To simplify the notation, we still denote by ϕ_t the restriction of the magnetic flow to the energy level T^cM.

It is easily seen from this equation that a curve is an integral curve of X if and only if it is of the form $t \mapsto (\gamma(t), \gamma'(t)) \in TM$ and satisfies the equation

$$\frac{D}{dt}\gamma' = Y_\gamma(\gamma'), \tag{2.3}$$

that is equivalence to 2.2. when we have no Lorentz force or $\Omega \equiv 0$, we obtain the geodesic equation 2.1. A curve that satisfies 2.3 is called the Ω-*magnetic geodesic*.

Now we are going to deduce the Jacobi equation magnetic. Let $\theta \in TM$, $\xi \in T_\theta TM$ and the curve $Z : (-\epsilon, \epsilon) \to TM$ adapted to ξ and consider the variation $f(s,t) = \pi(\phi_t(Z(s)))$. Set $J_\xi(t) := \dfrac{\partial f}{\partial s}(0,t)$, $\gamma_s(t) := f(s,t)$ and $\gamma_0 = \gamma$, where γ is a Ω-magnetic geodesic of θ and denote $\theta_t = \phi_t(\theta) = (\gamma(t), \dot\gamma(t))$. From the well know identity:

$$\frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial s} = \frac{D}{dt}\frac{D}{dt}\frac{\partial f}{\partial s} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t},$$

and

$$\nabla_{\gamma_s'}\gamma_s' = Y_{\gamma_s}(\gamma_s'),$$

we obtain

$$\frac{D^2}{dt^2}J_\xi + R(\gamma', J_\xi)\gamma' = \left.\frac{D}{ds}\right|_{s=0} (Y_{\gamma_s}(\gamma_s'))\,.$$

Note that the map $(x, v) \mapsto Y_x(v)$ is a (1,1)-tensor. Thus using the covariant derivative ∇ on (1,1)-tensor induced by the Riemannian connection we obtain

$$\frac{D}{ds}Y(\gamma_s') = (\nabla_{J_\xi}Y)(\gamma_s') + Y(J_\xi'),$$

and we deduce the Jacobi equation

$$\frac{D^2}{dt^2}J_\xi + R(\gamma', J_\xi)\gamma' - (\nabla_{J_\xi}Y)(\gamma') - Y\left(\frac{D}{dt}J_\xi\right) = 0.$$

Lemma 2.4.3. *If $\xi \in T_\theta TM$, then $d_\theta\phi_t(\xi) = \left(J_\xi(t), \frac{D}{dt}J_\xi(t)\right)$ in $H_{\theta_t} \oplus V_{\theta_t}$*

Proof. Consider as before $f(s,t) = \pi \circ \phi_t \circ Z(s)$, then

$$\frac{\partial f}{\partial s}(s,t) = d\pi(\phi_t \circ Z(s)) \cdot d\phi_t(Z(s)) \cdot Z'(s)$$

Now we take $s = 0$, thus $J_\xi(t) = d_{\theta_t}\pi \cdot d_\theta\phi_t(\xi)$. Remember that $\phi_t(Z(s)) = (\gamma_s(t), \gamma_s'(t))$, then $K_{\theta_t}(d_\theta\phi_t(\xi)) = \nabla_{\gamma_s'(t)}\gamma_s'(t)\Big|_{s=0}$, but

$$\begin{aligned}
\frac{\partial^2 f}{\partial t \partial s}(s,t) &= \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial t}(s,t)\right) = \frac{\partial}{\partial s}\left(\pi \circ X \circ \phi_t \circ Z(s)\right) \\
&= \frac{\partial}{\partial s}\left(\pi \circ X(\gamma_s(t), \gamma_s'(t))\right) = \frac{\partial}{\partial s}(\gamma_s'(t))
\end{aligned}$$

For $s = 0$, we have that $\frac{D}{dt}J_\xi(t) = \frac{\partial}{\partial s}(\gamma_s'(t))\Big|_{s=0} = K_{\theta_t}(d_\theta\phi_t(\xi))$ □

Note that if $\xi \in T_\theta T^cM$, then $d_\theta\phi_t(\xi) \in T_\theta T^cM$ for all $t \in \mathbb{R}$. Thus, we have that

$$\begin{aligned}
0 &= d_{\theta_t}H(d_\theta\phi_t(\xi)) = d_{\theta_t}H\left(J_\xi(t), \frac{D}{dt}J_\xi(t)\right) = \omega\left(X(\phi_t(\theta)), \left(J_\xi(t), \frac{D}{dt}J_\xi(t)\right)\right) \\
&= \left\langle \gamma'(t), \frac{D}{dt}J_\xi(t)\right\rangle - \langle Y(\gamma'(t)), J_\xi(t)\rangle + \Omega(\gamma'(t), J_\xi) = \left\langle \gamma'(t), \frac{D}{dt}J_\xi(t)\right\rangle
\end{aligned}$$

So

$$\left\langle \frac{D}{dt}J_\xi, \gamma'\right\rangle = 0$$

Definition 2.4.4. We say that J is a *Jacobi field* under Ω along γ if hold

$$\frac{D^2}{dt^2}J + R(\gamma', J)\gamma' - (\nabla_J Y)(\gamma') - Y\left(\frac{D}{dt}J\right) = 0. \tag{2.4}$$

and

$$\left\langle \frac{D}{dt}J, \gamma'\right\rangle = 0 \tag{2.5}$$

Note that from equation 2.4 we can see that

$$\begin{aligned}
\left\langle \frac{D}{dt}J, \gamma'\right\rangle' &= \left\langle \frac{D^2}{dt^2}J, \gamma'\right\rangle + \left\langle \frac{D}{dt}J, Y(\gamma')\right\rangle \\
&= \left\langle -R(\gamma', J)\gamma' + (\nabla_J Y)(\gamma') + Y\left(\frac{D}{dt}J\right), \gamma'\right\rangle - \left\langle Y\left(\frac{D}{dt}J\right), \gamma')\right\rangle \\
&= \langle (\nabla_J Y)(\gamma'), \gamma'\rangle \\
&= 0.
\end{aligned}$$

Therefore, it is enough to check condition 2.5 at a point.

Let $\overline{\Omega}^2(M)$ be the set of all smooth closed 2-form on M endowed with the C^r-topology. Let $H^2(M,\mathbb{R})$ the cohomology class and denote by $i(M,g)$ injectivity radius of (M,g). For Ω a smooth closed 2-form in M, since for $x \in M$, $\Omega_x : T_xM \times T_xM \to \mathbb{R}$ is a bilinear map

$$\|\Omega_x\| := \sup\{|\Omega_x(u,v)| : u,v \in T_xM \text{ with } \|u\| = \|v\| = 1\},$$

then $|\Omega_x(u,v)| \leq \|\Omega_x\|\|u\|\|v\|$ and $\|\Omega\|_{C^0} := \sup_{x\in M} \|\Omega_x\|$.

Lemma 2.4.5. *Given $c > 0$ and $\Omega \in \overline{\Omega}^2(M)$, let $K = K(c,\Omega) \in \mathbb{R}$ be defined as $K = \min\{1/(\|\Omega\|_{C^0} + 1)^2, i(M,g)/2c\}$. Then $\pi \circ \phi_t^\Omega(\theta) : [0,K) \to M$ is injective, for every $\theta \in T^cM$.*

The proof of this Lemma is equal to the Lemma 2.1 of [26]. The $K(c,\Omega)$ will be called the *magnetic injectivity radius.*

2.5 Examples

In this section, we want to illustrate some examples of magnetic flows.

Example 2.5.1. *In fact, every geodesic flow is a magnetic flow, considering $\Omega = 0$, since we have that the Lorentz force disappears, $Y = 0$. Thus any 0-magnetic geodetic curve is a geodesic curve in M.*

Example 2.5.2. *Now we consider, Ω be a smooth exact 2-form in M, this is, there exist a η smooth 1-form in M such that $\Omega = d\eta$. In this case the magnetic flows is called exact magnetic flows. We can define a Lagrangian as $L_\eta : TM \to \mathbb{R}$, where*

$$L_\eta(x,v) = H(x,v) - \eta_x(v)$$

The corresponding Euler-Lagrangian equation is

$$\frac{D}{dt}\langle\gamma'(t),\cdot\rangle = d\eta_x(\gamma'(t)) = \left\langle Y^\eta_{\gamma(t)}(\gamma'(t)),\cdot\right\rangle,$$

then

$$\frac{D}{dt}\gamma'(t) = Y^\eta_{\gamma(t)}\gamma'(t).$$

Example 2.5.3. *Let $n = 1$, this is, M be a surface. This case $\overline{\Omega}^2(M) = \Omega^2(M)$ is a $C^\infty(M)$-linear space with dimension 1. Thus, if Ω denote a area form of M, we have that $\Omega^2(M) = \{f\Omega : f \in C^\infty(M)\}$. For each $c > 0$ and $f \in C^\infty(M)$ we can define $\phi_t^f : T^cM \to T^cM$ the magnetic flow. For the area form, the Lorentz force is denote by $i : TM \to TM$ be a linear bundle as, for each $x \in M$ and $v \in T_xM$, we have that iv is the angle of rotation $+\pi/2$, hence $\{v, iv\}$ is a positive oriented orthogonal basis for T_xM. For $f \in \mathbb{C}^\infty(M)$ the Lorentz force is $Y = fi$, this is, for each $x \in M$ and $v \in T_xM$, we have that $Y_x(v) = f(x)iv$.*

Let $\gamma : \mathbb{R} \to M$ be a f-magnetic geodesic, this is, the curve γ that satisfies

$$\nabla_{\gamma'}\gamma' = f(\gamma)i\gamma'.$$

As $\{\gamma', i\gamma'\}$ is a positive oriented orthogonal basis for $T_\gamma M$. Denote $J = x\gamma' + yi\gamma'$ the Jacobi field under f along γ and consider $c = 1/2$, by 2.4 and 2.5 we have that

$$\begin{cases} x' - fy &= 0, \\ y'' + (K + f^2 - \nabla_{i\gamma'}f)\, y &= 0, \end{cases}$$

where K is the sectional curvature along γ. Take $f = -K = 1$, As equations above are

$$\begin{cases} x' &= y, \\ y'' &= 0, \end{cases}$$

the solutions have the form $x(t) = 1/2at^2 + bt + c$ and $y(t) = at + b$. In this conditions (γ, γ') has no conjugate points.

There are many more examples of magnetic flows that come from physics. The interested reader can see [44], for more examples.

2.6 Geometric control theory

In this section we state the result of [20] and [21]. This is referent at *Geometric Control Theory* that we will use for obtain our results. This method was created in [21] but here state a better version.

The first version of this type of result appeared in [38], called *first order controllability theorem* and was used to extend [30], that is, the Kupka-Smale's theorem in the context of Hamilton-Tonelli flows in manifolds of any dimension. Therefore, in [21] we find another version this result, more elaborate, called *second order controllability theorem*", it was used to obtain the Franks' lemma for geodetic flows, improving the Franks' lemma in [8]. Our case, these techniques of geometric control theory, in particular the second order controllability theorem, are very important because it will be used to obtain the Kupka-Smale's theorem and the Franks' lemma for magnetic flows in any dimension.

Let us a consider a *bilinear control system* on $M_{2n}(\mathbb{R})$ (with $n, k \geq 1$), of the form

$$W'(t) = A(t)W(t) + \sum_{i=1}^{k} u_i(t)B_iW(t) \text{ for a.e. } t, \tag{2.6}$$

where the *state* $W(t) \in M_{2n}(\mathbb{R})$, the *control* $u(t) \in \mathbb{R}^k$, $t \in [0,T] \mapsto A(t) \in M_{2n}(\mathbb{R})$ (with $T > 0$) is a smooth maps, and $B_1, \dots, B_k \in M_{2n}(\mathbb{R})$.

Given $\overline{W} \in M_{2n}(\mathbb{R})$ and $u \in L^2\left([0,T]; \mathbb{R}^k\right)$, the Cauchy problem

$$\begin{cases} W'(t) = A(t)W(t) + \sum_{i=1}^{k} u_i(t)B_iW(t) \text{ for a.e. } t \in [0,T], \\ W(0) = \overline{W}, \end{cases}$$

there exists a unique solution $W_{\overline{W},u}(\cdot)$. The *End-Point mapping* associated with $\overline{W}$ in time $T > 0$ is defined as

$$\begin{array}{rccc} E^{\overline{W},T} & : & L^2([0,T];\mathbb{R}^k) & \longrightarrow & M_{2n}(\mathbb{R}) \\ & & u & \longmapsto & W_{\overline{W},u}(T). \end{array}$$

It is a smooth mapping whose differential can be expressed in terms of the linearized control systems. Give $\overline{W} \in M_{2n}(\mathbb{R})$ the differential of $E^{\overline{W},T}$ at $u \equiv 0$ is given by the linear operator

$$\begin{array}{rccc} D_0E^{\overline{W},T} & : & L^2([0,T];\mathbb{R}^k) & \longrightarrow & M_{2n}(\mathbb{R}) \\ & & v & \longmapsto & X(T), \end{array}$$

where $X(\cdot)$ is the unique solution to the Cauchy problem

$$\begin{cases} X'(t) = A(t)X(t) + \sum_{i=1}^{k} v_i(t)B_i\overline{W}(t), \text{ for a.e. } t \in [0,T], \\ X(0) = 0, \end{cases}$$

where $\overline{W}(\cdot) := W_{\overline{W},0}(\cdot)$. Note that if we denote by $S(\cdot)$ the solution to the Cauchy problem, that is the solution of the homogeneous equations associated to 2.6

$$\begin{cases} S'(t) & = & A(t)S(t) \text{ for every } t \in [0,T], \\ S(0) & = & I_{2n}, \end{cases}$$

and that the following is the constant variation formula.

$$D_0 E^{\overline{W},T} \cdot v = \sum_{i=1}^{k} S(T) \int_0^T v_i(t) S(t)^{-1} B_i \overline{W}(t) dt,$$

for every $v \in L^2([0,T];\mathbb{R}^k)$. Let $Sp(n)$ be the symplectic group in $M_{2n}(\mathbb{R})$, that is the smooth submanifold of matrices $W \in M_{2n}(\mathbb{R})$ satisfying

$$W^* \mathbb{J} W = \mathbb{J} \text{ where } \mathbb{J} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

$Sp(n)$ has dimension $p := n(2n+1)$. Denote by $\mathcal{S}(2n)$ the set of $2n \times 2n$ symmetric matrices in $M_{2n}(\mathbb{R})$. The tangent spaces to $Sp(n)$ at the identity matrix is given by

$$\mathfrak{sp}(n) := T_{I_{2n}} Sp(n) = \{X \in M_{2n}(\mathbb{R}) : \mathbb{J}X \in \mathcal{S}(2n)\}.$$

Therefore, if there holds

$$A(t), B_1, \ldots, B_k \in \mathfrak{sp}(n) \text{ for all } t \in [0,T], \tag{2.7}$$

then $Sp(n)$ is invariant with respect to (2.6), that is for every $\overline{W} \in Sp(n)$ and $u \in L^2([0,T];\mathbb{R}^k)$,

$$W_{\overline{W},u}(t) \in Sp(n) \text{ for all } t \in [0,T].$$

In particular, this means that for every $\overline{W} \in Sp(n)$, the End-Point mapping $E^{\overline{W},T}$ is valued in $Sp(n)$. Given $\overline{W} \in Sp(n)$, we are interested in local controllability properties of (2.6) around 0. More precisely, we have the following result of [21].

Proposition 2.6.1. *Let $T > 0$, and for every θ in some set of parameters Θ let $t \in [0,T] \to A^\theta(t)$ be a smooth mapping and $B_1^\theta, \ldots, B_k^\theta \in M_{2n}(\mathbb{R})$ satisfying (2.7) such that*

$$B_i^\theta B_j^\theta = 0 \text{ for every } i,j = 1,\ldots,k. \tag{2.8}$$

Define for every $\theta \in \Theta$ the k sequences of smooth mapping $\{B_1^{\theta,j}\}, \dots, \{B_k^{\theta,j}\} : [0,T] \to \mathfrak{sp}(n)$ as

$$\begin{cases} B_i^{\theta,0} = B_i^{\theta} \\ B_i^{\theta,j}(t) = \dot{B}_i^{\theta,j-1}(t) + \left[B_i^{\theta,j-1}, A \right](t), \end{cases} \tag{2.9}$$

for every $t \in [0,T]$ and every $i = 1, \dots, k$ and assume that the following properties are satisfied for every $\theta \in \Theta$:

$$\left[B_i^{\theta,j}(0), B_i^{\theta} \right] \in Span \left\{ B_r^{\theta,s}(0) : r = 1, \dots, k, s \geq 0 \right\}, \tag{2.10}$$

for every $i = 1, \dots, n$, $j = 1, 2$ and

$$Span \left\{ B_i^{\theta,j}(0), \left[B_i^{\theta,1}(0), B_l^{\theta,1}(0) \right] : i, l = 1, \dots, k \text{ and } j = 0, 1, 2 \right\} = \mathfrak{sp}(n). \tag{2.11}$$

Assume moreover, that the sets

$$\{B_i^{\theta} : i = 1, \dots, k, \theta \in \Theta\} \subset M_{2n}(\mathbb{R})$$

and

$$\{t \in [0,T] \mapsto A^{\theta}(t) : \theta \in \Theta\} \subset C^2([0,T]; M_{2n}(\mathbb{R}))$$

are compact. Then, there are $\mu, K > 0$ such that for every $\theta \in \Theta$, every $\overline{W} \in Sp(n)$ and every $W \in B(\overline{W}^{\theta}(T), \mu) \cap Sp(n)$ ($\overline{W}^{\theta}(T)$ denotes the solution at time T of the control system (2.6) with parameter θ starting from $\overline{W}$), there is $u \in C^{\infty}([0,T]; \mathbb{R}^k)$ with support in $[0,T]$ satisfying

$$E_{\theta}^{\overline{W},T}(u) = W \text{ and } \|u\|_{C^2} \leq K|X - \overline{W}(T)|^{1/2}$$

($E_{\theta}^{\overline{W},T}$ denotes the End-Point mapping associated with the control system (2.6) with parameter θ).

Let us briefly explain this result from where it comes from and give some important observations. The problem is to find conditions such that the End-Point mapping is locally open at $\overline{u} = 0$. The first version from result appeared in [38], which is called *first-order controllability theorem* and concludes that the End-Point mapping is a submersion in $\overline{u} = 0$. Then in [21] is found the second version from result called the *second-order controllability theorem*, where neighborhood sizes are estimated such that the End-Point mapping is locally open at $\overline{u} = 0$. Finally, in this same paper is the parametric version of the second order controllability theorem, which is the theorem 2.6.1.

Observation 2.6.2. *We give some remarks on the conclusion of the theorem 2.6.1*

1. *This parametric version follow of second-order controllability theorem and the fact that smooth controls with support in* $(0, T)$ *are dense in* $L^2([0, T]; \mathbb{R}^k)$ *and compactness.*

2. *The constant* $\mu > 0$ *exist due the Inverse Function Theorem applied to the orthogonal projection* $M_{2n}(\mathbb{R}) \hookrightarrow T_X Sp(n)$ *restricted to* $Sp(n)$*, thus this constant is uniform.*

3. *The constant* $K > 0$ *exist due the theorem 4.2.6 in [20], where given conditions to* $F : \mathcal{U} \to \mathbb{R}^N$ *be a locally open map at* $\overline{u}$*, here* $\mathcal{U}$ *is an open set in a Banach space and* F *is of class* C^2*.*

4. *In fact,* $\mu > 0$ *and* $K > 0$ *are independents.*

2.7 Dynamics and Entropy

In this section, we introduce some important fact above of Dynamics Systems and Ergodic Theory, that we use in our results. For this sections we refer the reader to see [13].

Let $c > 0$ and Ω be a smooth closed 2-form on M. Consider the magnetic flow ϕ_t^Ω in T^cM and $\theta = (x, v) \in T^cM$ such that $\theta_t = \phi_t^\Omega(\theta) = (\gamma(t), \gamma'(t))$ is a closed orbit in T^cM with period $T_\theta > 0$, where γ is a closed Ω-magnetic geodesic in M. We can define the *Poicaré map* $\mathcal{P} := P(\Omega, \theta, \Sigma)$ as following: one can choose a local hypersurface Σ in T^cM through θ and transversal to θ_t such that there are open neighborhoods Σ_0 and Σ_{T_θ} of θ in Σ and a differentiable mapping $\varsigma : \Sigma_0 \to \mathbb{R}$ with $\varsigma(\theta) = T_\theta$ such that the map $\mathcal{P} : \Sigma_0 \to \Sigma_{T_\theta}$ given by $\vartheta \mapsto \phi^\Omega_{\varsigma(\vartheta)}(\vartheta)$, is a diffeomorphism.

Definition 2.7.1. We say that θ_t is

1. *degenerate* if its linearized Poincaré map $d_\theta\mathcal{P}$ has an eigenvalue which is a root unity.

2. *hyperbolic* if its linearized Poincaré map $d_\theta\mathcal{P}$ has not eigenvalue of modulus 1.

3. *elliptic* if it is non degenerate and non hyperbolic.

4. $\mathfrak{c}$-*elliptic*, for $\mathfrak{c} > 0$, if it is elliptic and the linearized Poincaré map $d_\theta \mathcal{P}$ has precisely $2\mathfrak{c}$ eigenvalues of modulus 1.

Let θ_t and ϑ_t be two hyperbolic closed orbits of ϕ_t^Ω in T^cM. We say that a orbit σ_t is a *heteroclinic orbit* from $\theta_{[0,T_\theta]}$ to $\vartheta_{[0,T_\vartheta]}$ if

$$\lim_{t\to-\infty} d\left(\theta_{[0,T_\theta]}, \sigma_t\right) = 0 \text{ and } \lim_{t\to+\infty} d\left(\vartheta_{[0,T_\vartheta]}, \sigma_t\right) = 0.$$

We say that the orbit σ_t is *homoclinic* if exists $s \in [0, T_\theta]$, such that $\theta_s = \vartheta$.

Definition 2.7.2. Let θ_t be a hyperbolic closed orbit of ϕ_t^Ω in T^cM. The *weak stable* and *weak unstable* manifolds of $\theta_{[0,T_\theta]}$ are

$$W^s\left(\theta_{[0,T_\theta]}\right) := \left\{\vartheta \in T^cM : \lim_{t\to+\infty} d\left(\theta_{[0,T_\theta]}, \vartheta_t\right) = 0\right\},$$

and

$$W^u\left(\theta_{[0,T_\theta]}\right) := \left\{\vartheta \in T^cM : \lim_{t\to-\infty} d\left(\theta_{[0,T_\theta]}, \vartheta_t\right) = 0\right\},$$

respectively. These are $(n+1)$-dimensional invariant immersed submanifolds of T^cM.

Another important concept in dynamic systems is the hyperbolic set that generalizes the concept of hyperbolic periodic orbit.

These sets are very important for the study of dynamical systems, in particular, it helps us to study the chaotic behavior of the systems, which is concentrated in the set of periodic orbits. There are many results for general systems, but in the case of magnetic flows we have for example that the sets $W^s\left(\theta_{[0,T_\theta]}\right)$ and $W^u\left(\theta_{[0,T_\theta]}\right)$ are $n+1$-dimensional invariant immersed submanifolds of T^cM. Then a heteroclinic orbit σ_t is an orbit in the intersection $W^s\left(\theta_{[0,T_\theta]}\right) \cap W^u\left(\vartheta_{[0,T_\vartheta]}\right)$. We say that the heteroclinic orbit σ_t is *transverse* if $W^s\left(\theta_{[0,T_\theta]}\right)$ and $W^u\left(\vartheta_{[0,T_\vartheta]}\right)$ are transversal at $\sigma_{[0,T_\sigma]}$.

Definition 2.7.3. A *hyperbolic set* is a compact ϕ_t^Ω-invariant (i.e. $\phi_t^\Omega(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$) subset $\Lambda \subset T^cM$ such that the restriction the tangent bundle of T^cM to Λ has a splitting

$$T_\Lambda T^cM = E^s \oplus \left\langle X^\Omega \right\rangle \oplus E^u,$$

where $\left\langle X^\Omega \right\rangle$ is the subspace generated by the vector field X^Ω of ϕ_t^Ω, E^s and E^u are $d\phi_t^\Omega$ invariant subbundles and there are constant $C, \lambda > 0$ such that

1. $\left|d\phi_t^\Omega(\xi)\right| \le Ce^{-\lambda t}|\xi|$ for all $t > 0$, $\xi \in E^s$,

2. $\left|d\phi_{-t}^\Omega(\xi)\right| \le Ce^{-\lambda t}|\xi|$ for all $t > 0$, $\xi \in E^u$

A classic example of a hyperbolic set is the so-called *Smale Horseshoe*

Other definitions that we will use are the following. Let $\Lambda \subset T^cM$ be a compact subset and ϕ_t^Ω-invariant. We say that Λ is a *locally maximal invariant set* if there exist a neighborhood U of Λ in T^cM such that

$$\Lambda = \bigcap_{t\in\mathbb{R}} \phi_t^\Omega(U).$$

We say that Λ is a *nontrivial hyperbolic basic set* if is a locally maximal compact invariant subset which is hyperbolic and it has a dense orbit and which is not a single closed orbit

The simplest invariant which measures the complexity of magnetic flow ϕ_t^Ω in T^cM is its topological entropy which we denote by $h_{top}(\Omega, c)$. The topological entropy measures the difficulty in predicting the position of an orbit given an approximation of its initial state. Given $\theta \in T^cM$ and $T, \delta > 0$, define the (δ, T)-dynamic ball about θ as

$$B\left(\theta, \delta, T\right) := \left\{\vartheta \in T^cM : d(\theta_t, \vartheta_t) < \delta, \text{ for all } t \in [0, T]\right\},$$

Let $N_\delta(T)$ be the minimal quantity of (δ, T)-dynamic balls needed to cover T^cM. The topological entropy is the limit on δ of the exponential growth rate of $N_\delta(T)$ as:

$$h_{top}(\Omega, c) := \lim_{t\to+\infty} \delta \to 0 \limsup_{T\to+\infty} \frac{1}{T} \log N_\delta(T). \tag{2.12}$$

Thus, if $h_{top}(\Omega, c) > 0$, some dynamic balls must contract exponentially at least in one direction.

A way of obtaining positive topological entropy is by showing that the flow has a nontrivial hyperbolic basic set. Using symbolic dynamics one shows that if a flow contains a nontrivial hyperbolic basic set then it has positive topological entropy. It also has infinitely many periodic orbits and their number grows exponentially with their period, namely

$$h_{top}(\Omega, c) \ge h_{top}(\phi_t^\Omega|_\Lambda) = \lim_{T\to+\infty} \frac{1}{T} \log n(T) > 0,$$

where $n(T)$ is the number of periodic orbits in Λ with period T.

In this last part we will assume the discrete dynamics of a C^r-diffeomorphism $f : M \to M$ and let p be a hyperbolic point in M. The following result affirms the existence of horseshoes Smale type.

Proposition 2.7.4. *(Theorem 6.5.5 in [13]) If q is a transverse homoclinic point of p, then in an arbitrarily small neighborhood of p there exists a horseshoe for some iterate of f. Furthermore the hyperbolic invariant set in this horseshoe contains an iterate of q.*

A consequence of perturbation theory is that the property of having a horseshoe is open. Let Λ be a hyperbolic subset in M of f. The following states the continuous variation of stable and unstable sets hyperbolic spaces.

Proposition 2.7.5. *(Proposition 6.4.4 in [13]) The dimensions of subspaces E_x^s and E_x^u are locally constant and those subspaces change continuously with x in Λ.*

Chapter 3

The Main Perturbative Lemma

In this chapter we are going to present one of the most important parts, for the rest of work, which is the next one. Given a segment of Ω-magnetic geodetic γ, we are going to construct a *magnetic tubular neighborhood* around it, that satisfies the classical properties, as in the case of geodesic flows. The most difficult part to do this is to construct a suitable base of the tangent bundle in the segment γ that depends on Ω, and once we can obtain one of the most important properties for our objectives, is that the matrix Ω-magnetic curvature in γ well a symmetric matrix. The difficulty is due to the Lorentz Force, since that this is an skew-symmetric linear map in the fibers.

The tubular neighborhood is going to help us get the famous *Perturbative Lemma* that is the key to all the work. To achieve this result we need to define the so-called set of perturbations $\tilde{\Omega}$ of Ω, such that these are supported in the tubular neighborhood of γ and preserve γ. Thus, we can define an application as follows, for each perturbation, consider its linear part of Poincaré map. The perturbative lemma states that this application is an open map on the symplectic matrix.

A direct consequence of this perturbative lemma is the version of *Franks' Lemma* for this context of magnetic flows. But we show this result in a future chapter. We will use the notations of first chapter and also the results presented.

3.1 Special coordinates and Magnetic curvature

In this section we define the special coordinates type Fermi coordinates, we obtain a coordinate system of a piece of magnetic geodesic where we present the magnetic curvature matrix. We will use some methods of Gouda [12].

Let $c > 0$ and Ω be a smooth closed 2-form on M. Consider the magnetic flow ϕ_t^Ω in T^cM and $\theta = (x, v) \in T^cM$ such that $\theta_t = \phi_t^\Omega(\theta) = (\gamma(t), \gamma'(t))$ is a orbit segment in T^cM with $t \in [0, \tau]$ and $0 < \tau < K(c, \Omega)$, where γ is a segment of a Ω-magnetic geodesic in M and remember that $K(c, \Omega)$ is the magnetic injectivity radius.

Let $\Sigma \subset T^cM$ be a local transversal section to X^Ω in the energy level T^cM at the point θ. The linearized Poincaré map is a linear symplectic mapping. Let $\delta\Omega \in \overline{\Omega}^2(M)$ such that $(\delta\Omega)_{\gamma(t)} = 0$ for every $t \in [0, \tau]$, then $\phi_t^{\Omega+\delta\Omega}$ preserves the orbit segment θ_t and its energy level. If $\delta\Omega$ is small enough in a neighborhood of $\gamma([0, \tau])$, the Poincaré map $\mathcal{P}(\Omega + \delta\Omega) := P(\Omega + \delta\Omega, \theta, \Sigma) : \Sigma_0 \to \Sigma_\tau$ associated to the magnetic flow of $\Omega + \delta\Omega$ in T^cM and its differential $d_\theta\mathcal{P}(\Omega + \delta\Omega) : T_\theta\Sigma_0 \to T_{\theta_\tau}\Sigma_\tau$ are well-defined. Our aim is to show that the set of $d_\theta P(\Omega + \delta\Omega)$ for $\delta\Omega$ as above small enough contains as open subset of the set of linear symplectic matrices from $T_\theta\Sigma$ onto $T_{\theta_\tau}\Sigma_\tau$.

Let $v_1 := v/\sqrt{2c}$ and let us choose $v_2, \ldots, v_m \in T_xM$ such that $v_1, v_2, \ldots, v_m$ is an orthonormal basis in T_xM. We define a vector field V_i along γ as a solution of the differential equation

$$\begin{cases} V_i'(t) = Y_{\gamma(t)}(V_i(t)), \\ V_i(0) = v_i. \end{cases}$$

In particular $V_1 = \gamma'/\sqrt{2c}$. Note that

$$\begin{aligned} \langle V_i, V_j\rangle' &= \langle V_i', V_j\rangle + \langle V_i, V_j'\rangle \\ &= \langle Y(V_i), V_j\rangle + \langle V_i, Y(V_j)\rangle \\ &= \langle Y(V_i), V_j\rangle - \langle Y(V_i), V_j\rangle \\ &= 0. \end{aligned}$$

Thus $V_1, \cdots, V_m$ are orthonormal vector fields along γ (type Fermi coordinates).

We know that $Y_x : T_xM \to T_xM$ is an $m \times m$ skew-symmetric linear mapping for each $x \in M$. Let $x \in M$ and fix $v \in T_xM$, $v \neq 0$, we can define $Pr_\theta : T_xM \to v^\perp$ the map natural projection, where $v^\perp = \{u \in T_xM : \langle u, v\rangle_x = 0\}$, is easy see that $\langle Pr_\theta(u), w\rangle = \langle Pr_\theta(u), Pr_\theta(w)\rangle$, for all $u, w \in T_xM$, since if $w \in T_xM$, we can write $w = Pr_\theta(w) + (w - Pr_\theta(w)) \in v^\perp \oplus \langle v\rangle$, then $\langle Pr_\theta(u), w - Pr_\theta(w)\rangle = 0$. Thus we obtain that $(Pr_\theta)^* = Pr_\theta$.

We define also $(Y_\perp)_x : T_xM \to T_xM$ as $(Y_\perp)_x = Pr_\theta Y_x Pr_\theta$, is clear that

1. $(Y_\perp)_x(v) = 0$, because $Pr_\theta(v) = 0$,
2. $(Y_\perp)_x(v^\perp) \subset v^\perp$, because if $u \in v^\perp$, $(Y_\perp)_x(u) = Pr_\theta Y(u) \in v^\perp$,
3. $(Y_\perp)_x^* = -(Y_\perp)_x$, because $Y^* = -Y$.

Remember that $0 < \tau < K(c, \Omega)$, now we define for each $t \in [0, \tau]$, the follow linear map $P_t : T_{\gamma(t)}M \to T_{\gamma(t)}M$ as

$$P_t = \exp\left(\frac{1}{2}\int_0^t (Y_\perp)_{\gamma(s)} ds\right),$$

it is clear that this map is a linear isomorphism and we have of 3. that $P_t^{-1} = P_t^*$ i.e. the map P_t is an orthogonal linear map, so it takes an orthogonal base on an orthogonal basis. Thus we have that

$$e_1(t) := P_t^{-1}V_1(t), \ldots, e_m(t) := P_t^{-1}V_m(t) \tag{3.1}$$

is an orthonormal basis of $T_{\gamma(t)}M$.

Consider the differentiable map $\Phi : [0, \tau] \times \mathbb{R}^n \to M$ given by

$$\Phi(x_1, x_2, \ldots, x_m) = \exp_{\gamma(x_1)}\left(\sum_{i=2}^{m} x_i e_i(x_1)\right),$$

where $\exp_x : T_xM \to M$ denotes the Riemannian exponential map. This map has maximal rank at $(x_1, 0, \ldots, 0)$, $x_1 \in [0, \tau]$. Since $\gamma(t)$ has no self-intersections on $t \in [0, \tau]$, there exists a neighborhood V of $[0, \tau] \times \{0\}$. Then $\psi := \Phi^{-1}|_V$ is a diffeomorphism, if $U := \Phi(V)$ then (U, ψ) is a local coordinate chart where $\gamma(t) = (t, 0)$, $g_{ij}(t, 0) = \delta_{ij}$ and the Christoffel symbols are $\Gamma_{ij}^k(t, 0) = 0$, since the first partial derivatives of g_{ij} vanish at $(t, 0)$.

Let

$$\overline{Y}_{ij}(t) := \left\langle V_i(t), Y_{\gamma(t)}(V_j(t))\right\rangle_{\gamma(t)} \text{ and } Y_{ij}(t) := \left\langle e_i(t), Y_{\gamma(t)}(e_j(t))\right\rangle_{\gamma((t))},$$

denote $\overline{Y}(t) = \left(\overline{Y}_{ij}(t)\right)$ and $Y(t) = (Y_{ij}(t))$ are the matrices representations of $Y_{\gamma(t)}$ at coordinates $V_i(t)$ and $e_i(t)$ respectively. Thus we have that

$$Y(t) = P_t^{-1}\overline{Y}(t)P_t. \tag{3.2}$$

In these coordinates note that $e_1(t) = V_1(t) = \gamma'(t)/\sqrt{2c}$, since $\left(\overline{Y}_\perp\right)_{\gamma(t)}$ has zeros in the first column and first row. Moreover note that

$$P_t' = \frac{1}{2}P_t(Y_\perp)_{\gamma(t)}.$$

We are now going to consider an arbitrary Jacobi field written in these coordinates in order to obtain the matrix of magnetic curvature, the idea is to do the same work done for the geodetic flow. Remember the definition of Jacobi field in 2.4 and 2.5

At these coordinates, the covariant derivative and the common derivative coincide. In order not to overload the notation, we will avoid typing time t.

Let J be a Jacobi field under Ω along γ arbitrary. Let J expressed as $J = \sum_{j=1}^{m} f_j e_j$ where each f_j is a smooth function along γ. Then

$$J' = \sum_{j=1}^{m}\left(f_j'e_j + f_je_j'\right),$$

thus we have that

$$J'' = \sum_{j=1}^{m}\left(f_j''e_j + 2f_j'e_j' + f_je_j''\right), \tag{3.3}$$

but $e_1' = V_1' = \overline{Y}(V_1) = Y(e_1)$ and for $j = 2,\dots m$ we have of 3.1 and 3.2 that

$$\begin{aligned}
e_j' &= (P^{-1})'V_j + P^{-1}V_j' \\
&= P^{-1}\overline{Y}(V_j) - \frac{1}{2}P^{-1}\overline{Y}_\perp V_j \\
&= P^{-1}\overline{Y}P(e_j) - \frac{1}{2}P^{-1}\overline{Y}_\perp P(e_j) \\
&= Y(e_j) - \frac{1}{2}Y_\perp(e_j),
\end{aligned}$$

observe that, since $Y_\perp(e_1) = 0$ then

$$e_j' = Y(e_j) - \frac{1}{2}Y_\perp(e_j) \tag{3.4}$$

for all $j = 1, 2, \ldots, m$. Also have that for all $j = 1, 2, \ldots, m$

$$\begin{aligned} e_j'' &= \nabla_{\dot\gamma}(Y(e_j)) - \frac{1}{2}(Y_\perp(e_j))' \\ &= (\nabla_{\dot\gamma}Y)(e_j) + Y(e_j') - \frac{1}{2}Y_\perp'(e_j) - \frac{1}{2}Y_\perp(e_j'), \end{aligned}$$

thus we have that

$$e_j'' = (\nabla_{\dot\gamma}Y)(e_j) + Y(e_j') - \frac{1}{2}Y_\perp'(e_j) - \frac{1}{2}Y_\perp Y(e_j) + \frac{1}{4}Y_\perp^2(e_j). \tag{3.5}$$

Since J is a Jacobi field, this satisfies a equation (2.4), moreover apply (3.5) and (3.4) in (3.3) we have that

$$\begin{aligned} J'' + \sum_{j=1}^{m}\left(f_j R(\gamma', e_j)\gamma' - f_j(\nabla_{e_j}Y)(\gamma') - f_j'Y(e_j) - f_jY(e_j')\right) &= 0, \\ \sum_{j=1}^{m}\Big\{f_j''e_j + f_j'(Y - Y_\perp)(e_j) + f_j\big[R(\gamma', e_j)\gamma' + & \\ (\nabla_{\gamma'}Y)(e_j) - (\nabla_{e_j}Y)(\gamma') - \frac{1}{2}Y_\perp'(e_j) - \frac{1}{2}Y_\perp Y(e_j) + \frac{1}{4}Y_\perp^2(e_j)\big]\Big\} &= 0, \end{aligned}$$

denote by

$$\begin{aligned} R_{ij} &:= \langle R(\gamma', e_i)\gamma', e_j\rangle = \langle R(\gamma', e_j)\gamma', e_i\rangle = \langle e_i, R(\gamma', e_j)\gamma'\rangle, \\ (Y')_{ij} &:= \langle e_i, Y'(e_j)\rangle = \langle e_i, (\nabla_{\gamma'}Y)(e_j)\rangle, \\ (\partial Y)_{ij} &:= \sqrt{2c}(\nabla_{e_j}Y)_{i1} = \sqrt{2c}\left\langle e_i, (\nabla_{e_j}Y)(e_1)\right\rangle = \left\langle e_i, (\nabla_{e_j}Y)(\gamma')\right\rangle, \\ (Y_\perp')_{ij} &:= \langle e_i, Y_\perp'(e_j)\rangle, \\ (Y_\perp Y)_{ij} &:= \langle e_i, Y_\perp Y(e_j)\rangle, \\ (Y_\perp^2)_{ij} &:= \left\langle e_i, Y_\perp^2(e_j)\right\rangle. \end{aligned}$$

Note que $(Y_\perp Y)_{ij} = (YY_\perp)_{ij}$ for all $i, j = 2, \ldots, m$, moreover $(Y_\perp Y)_{1j} = 0$ for all $j = 1, 2, \ldots, m$ and $(YY_\perp)^* = Y_\perp Y$. Thus we have that, if $f = (f_1, f_2, \ldots, f_m)$, then

$$f'' + (Y - Y_\perp)f' + (R + Y' - \partial Y - \frac{1}{2}Y_\perp' - \frac{1}{2}Y_\perp Y + \frac{1}{4}Y_\perp^2)f = 0. \tag{3.6}$$

The first line of the equation (3.6) is written as

$$\begin{aligned} f_1'' + \sum_{j=1}^{m} \left(Y_{1j} f_j' + Y_{1j}' f_j \right) &= 0, \\ \left(f_1' + \sum_{j=2}^{m} Y_{1j} f_j \right)' &= 0. \end{aligned}$$

Since $Y_{11} = 0$ and the (2.5) have that $\langle J', \dot{\gamma} \rangle = 0$ then

$$f_1' = -\sum_{j=2}^{m} Y_{1j} f_j. \tag{3.7}$$

For $i \neq 1$ and of (3.7) in (3.6) we have that

$$\begin{aligned} f_i'' + Y_{i1} f_1' + \sum_{j=2}^{m} \Bigg(R_{ij} + Y_{ij}' - (\partial Y)_{ij} - \frac{1}{2} (Y_\perp')_{ij} & \\ -\frac{1}{2} (Y_\perp Y)_{ij} + \frac{1}{4} (Y_\perp^2)_{ij} \Bigg) f_j &= 0, \\ f_i'' + \sum_{j=2}^{m} (R_{ij} + \frac{1}{2} Y_{ij}' - (\partial Y)_{ij} - \frac{1}{4} (Y^2)_{ij} - Y_{i1} Y_{1j}) f_j &= 0, \end{aligned}$$

since $Y_{ij} = (Y_\perp)_{ij}$ for all $i, j = 2, 3, \ldots, m$, and if we denote $\widetilde{Y}_{ij} := Y_{i1} Y_{1j}$, then the new equation is

$$f'' + \left(R + \frac{1}{2} Y' - \partial Y - \frac{1}{4} Y^2 - \widetilde{Y} \right) f = 0. \tag{3.8}$$

Note here, that the matrices are of order $n \times n$, also that R, Y^2 and $\widetilde{Y}$ are $n \times n$ symmetric matrices. We show that $\partial Y - \frac{1}{2} Y'$ is also a $n \times n$ symmetric matrix, for $i, j = 2, 3, \ldots, m$ we have that

$$\left(\frac{1}{2} Y' - \partial Y \right)_{ij} = \frac{1}{2} \langle e_i, Y'(e_j) \rangle - \left\langle e_i, (\nabla_{e_j} Y)(\gamma') \right\rangle,$$

then we have that

$$\left(\frac{1}{2} Y' - \partial Y \right)_{ij} - \left(\frac{1}{2} Y' - \partial Y \right)_{ji} =$$

$$= \langle (\nabla_{\gamma'} Y)(e_j), e_i \rangle + \left\langle \left(\nabla_{e_j} Y \right)(e_i), \gamma' \right\rangle + \langle (\nabla_{e_i} Y)(\gamma'), e_j \rangle = 0,$$

since Ω is closed ($d\Omega = 0$).

On the other hand, as $Y_{ij} = \langle e_i, Y(e_j)\rangle = -\Omega(e_i, e_j)$ and $Y = -\Omega$ seen as matrix. So we define the matrix *magnetic curvature of* Ω as

$$K^{\Omega}(t) := R_{\gamma(t)} + \partial\Omega_{\gamma(t)} - \frac{1}{2}\Omega'_{\gamma(t)} - \frac{1}{4}\Omega^2_{\gamma(t)} - \widetilde{\Omega}_{\gamma(t)}, \tag{3.9}$$

it is a $n \times n$ symmetric matrix, where $\partial\Omega$ and $\widetilde{\Omega}$ are similarly defined as the matrix Ω. Then

$$f'' + K^{\Omega} f = 0.$$

We shall study the real $(n \times n)$-matrix differential equation along γ,

$$X'' + K^{\Omega} X = 0. \tag{3.10}$$

It is equivalent to

$$\begin{pmatrix} X \\ X' \end{pmatrix}' = \begin{pmatrix} 0 & I_{n\times n} \\ -K^{\Omega} & 0 \end{pmatrix} \begin{pmatrix} X \\ X' \end{pmatrix}$$

Let $W = \begin{pmatrix} X \\ X' \end{pmatrix}$, then

$$W'(t) = \begin{pmatrix} 0 & I_{n\times n} \\ -K^{\Omega}(t) & 0 \end{pmatrix} W(t). \tag{3.11}$$

Thus, finally we have that

Lemma 3.1.1. *Let $\theta_{[0,\tau]}$ be a orbit segment of the magnetic flow of Ω without self-intersection. Exists a local coordinate chart (U, ψ), such that $\psi = (x_1 = t, x_2, \ldots, x_m)$, $\psi(x) = 0$ and $\gamma(t) = (t, 0, \ldots, 0)$, satisfying (3.11), where the matrix $W(t)$ represents a basis of Jacobi fields and its derivatives defined in the orbit, and the matrix $K^{\Omega}(t)$ represents the magnetic curvature.*

In the case of geodesic flow i.e. $\Omega \equiv 0$, we have the same matrix with K^{Ω} being the Riemannian curvature matrix which is always a $n \times n$ symmetric matrix. In our case does not run Fermi coordinates so we had to make a rotation of the Fermi's coordinates in function of Ω and simultaneously obtain a $n \times n$ symmetric matrix. This lemma use Gouda's method [12] and generalizes the [26].

Example 3.1.2. *For $m = 3$ and $2c = 1$, we have that*

$$\Omega := \begin{pmatrix} 0 & \alpha & -\beta \\ -\alpha & 0 & \sigma \\ \beta & -\sigma & 0 \end{pmatrix} \quad \textit{and} \quad R := \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & b & c \end{pmatrix}$$

then, since Ω is closed, we have that $\partial_1\sigma + \partial_2\beta + \partial_3\alpha = 0$. Thus

$$\partial\Omega = \begin{pmatrix} 0 & 0 & 0 \\ \partial_1\alpha & \partial_2\alpha & \partial_3\alpha \\ -\partial_1\beta & -\partial_2\beta & -\partial_3\beta \end{pmatrix}$$

and

$$\widetilde{\Omega} = \begin{pmatrix} \alpha^2 & -\alpha\beta \\ -\alpha\beta & \beta^2 \end{pmatrix}.$$

Thus the equation (3.8) can be written as

$$f'' + \left[\begin{pmatrix} a & b \\ b & c \end{pmatrix} + \begin{pmatrix} -\partial_2\alpha & \partial_2\beta + \frac{1}{2}\partial_1\sigma \\ \partial_2\beta + \frac{1}{2}\partial_1\sigma & \partial_3\beta \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} \frac{5}{4}\alpha^2 + \frac{1}{4}\sigma^2 & -\frac{5}{4}\alpha\beta \\ -\frac{5}{4}\alpha\beta & \frac{5}{4}\beta^2 + \frac{1}{4}\sigma^2 \end{pmatrix} \right] f = 0$$

where $f = \begin{pmatrix} f^2 \\ f^3 \end{pmatrix}$.

3.2 Local perturbations of the magnetic flow

In this section we obtain the Perturbation Theorem, which is the part more important in this work. We will use the method in [38] but we will have that find the form suitable of the perturbations of Ω. The most important tool in this section is *Geometric Control Theory*, which has already been used in many different aspects and areas, more specifically to see [19], [20], [21] and [38].

Suppose that θ_t be a closed orbit of ϕ_t^{Ω} in T^cM of period $T_\theta > 0$. Applying Lemma 3.1.1 to a piece of θ_t, thus there exists a local coordinate chart (U, ψ). We may assume that $\psi(x) = 0$ and $d_x\psi{\cdot}v = (1, 0..., 0)$, then $\theta_t = \phi_t^{\Omega}(x, v) = (\psi^{-1}(t, 0, \ldots 0), (d_x\psi)^{-1}(1, 0, \ldots, 0)) : [0, \tau] \to T^cM$, for some $0 < \tau < K = K(c, \Omega)$. We need to study generic perturbations of Ω in the neighborhood U of $\gamma = \psi^{-1}(t, 0, \ldots 0)$.

Let $0 < \epsilon << \dfrac{1}{2}$ fix, such that $U_\epsilon := \psi\left([0, \tau] \times (-\epsilon, \epsilon)^n\right) \subset U$. Let a family of smooth function $u_{ij} : [0, \tau] \to \mathbb{R}$ such that

$$Supp(u_{ij}) \subset (0, \tau) \text{ for every } i \leq j \text{ in } 2, \ldots, m.$$

We have that consider $f : [0, +\infty) \to [0, +\infty)$ a smooth function bump such that

$$f \equiv \begin{cases} 1, & \text{in } \left[0, \dfrac{1}{2}\right] \\ \\ 0, & \text{in } \left[\dfrac{4}{5}, +\infty\right] \end{cases}. \tag{3.12}$$

Note that $f'(r) = 0$ if $r \in (0, 1/2)$.Now we define a family of smooth perturbations $f_i : M \to \mathbb{R}$ with support in U_ϵ by

$$f_1(\psi(x)) = \frac{1}{\sqrt{2c}} \sum_{2 \leq i < j \leq m} u_{ij}(x_1) x_i x_j f(\|\hat{x}_1\|), \tag{3.13}$$

$$f_i(\psi(x)) = -\frac{1}{\sqrt{2c}} \int_0^{x_1} u_{ii}(s) ds x_i f(\|\hat{x}_1\|), \tag{3.14}$$

for $i = 2, \ldots, m$, where $x = (x_1, x_2, \ldots, x_m)$ and $\hat{x}_1 = (x_2, x_3, \ldots, x_m)$. Note here that as $x \in U_\epsilon$, we have that $\|\hat{x}_1\| < \epsilon << \dfrac{1}{2}$, so $f(\|\hat{x}_1\|) = 1$ and $f'(\|\hat{x}_1\|) = 0$, this will be used in the following calculations. Now consider the 1-form in M define by $\eta := \sum_{k=1}^{m} f_k dx_k$ with support in U_ϵ. Then taking $\delta\Omega := d\eta$ with support in U_ϵ, which view as matrix is

$$(\delta\Omega) = \begin{pmatrix} 0 & \dfrac{\partial f_1}{\partial x_2} - \dfrac{\partial f_2}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_m} - \dfrac{\partial f_m}{\partial x_1} \\ \dfrac{\partial f_2}{\partial x_1} - \dfrac{\partial f_1}{\partial x_2} & 0 & \cdots & \dfrac{\partial f_2}{\partial x_m} - \dfrac{\partial f_m}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} - \dfrac{\partial f_1}{\partial x_m} & \dfrac{\partial f_m}{\partial x_2} - \dfrac{\partial f_2}{\partial x_m} & \cdots & 0 \end{pmatrix}.$$

Now we writing $(\delta\Omega)_{ij}$ in coordinates:

$$\begin{aligned}
(\delta\Omega)_{21} &= \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \\
&= -\frac{1}{\sqrt{2c}} u_{22}(x_1) x_2 f(\|\hat{x}_1\|) - \frac{1}{\sqrt{2c}} \left(\sum_{l=3}^{m} u_{2l}(x_1) x_l f(\|\hat{x}_1\|) \right. \\
&\quad \left. + \sum_{3\le k<l\le m} u_{ij}(x_1) x_k x_l x_2 \frac{f'(\|\hat{x}_1\|)}{\|\hat{x}_1\|} \right) \\
&= -\frac{1}{\sqrt{2c}} \sum_{l=2}^{m} u_{2l}(x_1) x_l,
\end{aligned}$$

remember that $f(\|\hat{x}_1\|) = 1$ and $f'(\|\hat{x}_1\|) = 0$. Now for each $i = 3, \ldots, m$ have that

$$\begin{aligned}
(\delta\Omega)_{i1} &= \frac{\partial f_i}{\partial x_1} - \frac{\partial f_1}{\partial x_i} \\
&= -\frac{1}{\sqrt{2c}} u_{ii}(x_1) x_i f(\|\hat{x}_1\|) - \frac{\partial}{\partial x_i} \frac{1}{\sqrt{2c}} \left(\sum_{k=2}^{i-1} u_{ki}(x_1) x_k x_i \right. \\
&\quad \left. + \sum_{l=i+1}^{m} u_{il}(x_1) x_i x_l + \text{ terms wihtout } x_i \right) f(\|\hat{x}_1\|) \\
&= -\frac{1}{\sqrt{2c}} u_{ii}(x_1) x_i f(\|\hat{x}_1\|) - \frac{1}{\sqrt{2c}} \left(\sum_{k=2}^{i-1} u_{ki}(x_1) x_k + \sum_{l=i+1}^{m} u_{il}(x_1) x_l \right) f(\|\hat{x}_1\|) \\
&\quad + \frac{1}{\sqrt{2c}} \sum_{2\le k<l\le m} u_{kl}(x_1) x_k x_l x_i \frac{f'(\|\hat{x}_1\|)}{\|\hat{x}_1\|} \\
&= -\frac{1}{\sqrt{2c}} \left(\sum_{k=2}^{i-1} u_{ki}(x_1) x_k + \sum_{l=i}^{m} u_{il}(x_1) x_l \right),
\end{aligned}$$

and as for each $2 \leq i < j \leq m$ we have that

$$\frac{\partial f_i}{\partial x_j} = -\frac{1}{\sqrt{2c}} \int_0^{x_1} u_{ii}(s) ds x_j \frac{f'(\|\hat{x}_1\|)}{\|\hat{x}_1\|} = 0,$$

so we have that $(\delta\Omega)_{ij} = 0$, all this in U_ϵ, note that $(\delta\Omega)$ is a $n \times n$ skew-symmetric matrix of the follow form

$$\frac{1}{\sqrt{2c}} \begin{pmatrix} 0 & \sum_{j=2}^{m} u_{2j}(x_1)x_j & \ldots & \sum_{i=2}^{m} u_{im}(x_1)x_i \\ -\sum_{j=2}^{m} u_{2j}(x_1)x_j & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\sum_{i=2}^{m} u_{im}(x_1)x_i & 0 & \ldots & 0 \end{pmatrix}.$$

Notice that in $x = \gamma(t) = (t, 0)$, we have that $(\delta\Omega)$ is the matrix zero. Remember the equation 3.9, we need see $(\delta\Omega)$ without the first column and the first row, i.e. $(\delta\Omega_{ij})_2^m \in M_n(\mathbb{R})$ this is the matrix zero in U_ϵ, so $(\delta\Omega)^2 = 0$ in U_ϵ and particularly in γ. We also have that $\widetilde{\delta\Omega} = 0 = (\delta\Omega)'$ in γ. Finally, we only need to see the form of the matrix $\partial(\delta\Omega)$, for this remember the definition $(\partial(\delta\Omega))_{ij} := \sqrt{2c}\, \partial_j (\delta\Omega)_{i1}$, thus we have that, for example

$$(\partial(\delta\Omega))_{2j} = \sqrt{2c}\, \partial_j (\delta\Omega)_{21} = \sqrt{2c}\, \partial_j \left(-\frac{1}{\sqrt{2c}} \sum_{l=2}^{m} u_{2l}(x_1) x_l \right) = -u_{2j}(x_1), \text{ for } j = 2, \ldots, m.$$

Following this process we have to $(\partial(\delta\Omega))_{ij} = -u_{ij}(x_1)$, for every $i, j = 2, \ldots, m$. Thus the matrix $\partial(\delta\Omega)$ in $\gamma(t)$, have the form, taking $x_1 = t$

$$- \begin{pmatrix} u_{22}(t) & u_{23}(t) & \ldots & u_{2m}(t) \\ u_{23}(t) & u_{33}(t) & \ldots & u_{3m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ u_{2m}(t) & u_{3m}(t) & \ldots & u_{mm}(t) \end{pmatrix},$$

we denote $U(t) = -\partial(\delta\Omega)_{\gamma(t)}$ which is a $n \times n$ symmetric $n \times n$-matrix, remember that $m = n + 1$.

Example 3.2.1. *For $m = 3$ and $2c = 1$, we have that*

$$\begin{aligned} f_1(x_1, x_2, x_3) &= u_{23}(x_1)x_2x_3f(\|\hat{x}_1\|), \\ f_2(x_1, x_2, x_3) &= -\int_0^{x_1} u_{22}(s)ds x_2 f(\|\hat{x}_1\|) \text{ and} \\ f_3(x_1, x_2, x_3) &= -\int_0^{x_1} u_{33}(s)ds x_3 f(\|\hat{x}_1\|). \end{aligned}$$

Take $\eta = f_1dx_1 + f_2dx_2 + f_3dx_3$, then $\delta\Omega = d\eta =$

$$\left(\frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}\right) dx_1dx_2 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) dx_1dx_3 + \left(\frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2}\right) dx_2dx_3.$$

In matrix is

$$(\delta\Omega) = \begin{pmatrix} 0 & \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} & \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \\ \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} & 0 & \frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} & \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} & 0 \end{pmatrix}.$$

Note that

$$\begin{aligned} \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} &= u_{22}(x_1)x_2 + u_{23}(x_1)x_3, \\ \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} &= u_{23}(x_1)x_2 + u_{33}(x_1)x_3 \end{aligned}$$

and $\frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2} = 0$, thus we have that the matrix of $\delta\Omega$ in U_ϵ is

$$\begin{pmatrix} 0 & u_{22}(x_1)x_2 + u_{23}(x_1)x_3 & u_{23}(x_1)x_2 + u_{33}(x_1)x_3 \\ -u_{22}(x_1)x_2 - u_{23}(x_1)x_3 & 0 & 0 \\ -u_{23}(x_1)x_2 - u_{33}(x_1)x_3 & 0 & 0 \end{pmatrix}$$

and

$$U(t) = \begin{pmatrix} u_{22}(t) & u_{23}(t) \\ u_{23}(t) & u_{33}(t) \end{pmatrix}$$

Remember the equation 3.9, for $\Omega+\delta\Omega$, we have that

$$\begin{aligned} K^{\Omega+\delta\Omega}(t) &= R_{\gamma(t)}+\partial(\Omega+\delta\Omega)_{\gamma(t)}-\frac{1}{2}(\Omega+\delta\Omega)'_{\gamma(t)}-\frac{1}{4}(\Omega+\delta\Omega)^2_{\gamma(t)}-(\widetilde{\Omega+\delta\Omega})_{\gamma(t)} \\ &= R_{\gamma(t)}+\partial\Omega_{\gamma(t)}+\partial(\delta\Omega)_{\gamma(t)}-\frac{1}{2}\Omega'_{\gamma(t)}-\frac{1}{2}(\delta\Omega)'_{\gamma(t)}-\frac{1}{4}\Omega^2_{\gamma(t)}-\frac{1}{2}\Omega_{\gamma(t)}(\delta\Omega)_{\gamma(t)} \\ &\quad -\frac{1}{4}(\delta\Omega)^2_{\gamma(t)}-\widetilde{\Omega}_{\gamma(t)}-(\Omega_{i1}(\delta\Omega)_{1j})_{\gamma(t)}-((\delta\Omega)_{i1}(\Omega+\delta\Omega)_{1j})_{\gamma(t)} \end{aligned}$$

For the previously seen we have to $\partial(\delta\Omega)_{\gamma(t)}=-U(t)$, $(\delta\Omega)'_{\gamma(t)}=0$, $(\delta\Omega)_{\gamma(t)}=0$ and also that $(\delta\Omega)_{i1}=(\delta\Omega)_{1j}=0$ in $\gamma(t)$, for every $t\in[0,\tau]$. So we have that

$$K^{\Omega+\delta\Omega}=R_{\gamma(t)}+\partial\Omega_{\gamma(t)}-\frac{1}{2}\Omega'_{\gamma(t)}-\frac{1}{4}\Omega^2_{\gamma(t)}-\widetilde{\Omega}_{\gamma(t)}-U(t),$$

thus we have that

$$K^{\Omega+\delta\Omega}(t)=K^{\Omega}(t)-U(t), \tag{3.15}$$

and the cohomology class $[\delta\Omega]=0$ this is $[\Omega]=[\Omega+\delta\Omega]$ in $H^2(M,\mathbb{R})$. Since $(\delta\Omega)_{\gamma(t)}=0$, for every $t\in[0,\tau]$, then the trajectory θ_t is an orbit of the magnetic flow of $\Omega+\delta\Omega$ and the level energy is preserved. Using lemma 3.1.1, proposition 2.4.3 and by the Jacobi equation, we have that

$$d_\theta\mathcal{P}(\Omega+\delta\Omega)(\tau)(J(0),J'(0))=(J(\tau),J'(\tau)),$$

where $J:[0,\tau]\to\mathbb{R}^n$ is solution to the Jacobi equation and by (3.11) we have that

$$J''(t)+K^{\Omega+\delta\Omega}(t)J(t)=0,\ \text{for every } t\in[0,\tau].$$

In other terms, $d_\theta\mathcal{P}(\Omega+\delta\Omega)(\tau)$ is equal to the $n\times n$ symplectic matrix $W(\tau)$ given by the solution $W:[0,\tau]\to Sp(n)$ at time τ of the following Cauchy problem:

$$\begin{cases} W'(t)=A(t)W(t)+\displaystyle\sum_{2\le i\le j\le m}u_{ij}(t)B_{ij}W(t),\ \text{for all } t\in[0,\tau], \\ W(0)=I_{2n}, \end{cases}$$

where the $2n\times 2n$ matrices $A(t),B_{ij}$ are defined by

$$A(t):=\begin{pmatrix} 0 & I_n \\ -K^{\Omega}(t) & 0 \end{pmatrix} \tag{3.16}$$

for every $t \in [0,\tau]$ and

$$B_{ij} := \begin{pmatrix} 0 & 0 \\ \mathcal{B}_{ij} & 0 \end{pmatrix},$$

where all the $\mathcal{B}_{ij}$ are $n \times n$-symmetric matrices, such that for all $2 \leq i \leq j \leq m$ are defined by

$$\mathcal{B}_{ij} = \begin{cases} E_{(i-1)(i-1)}, & \text{if } i = j, \\ E_{(i-1)(j-1)} + E_{(j-1)(i-1)}, & \text{if } i < j, \end{cases}$$

where $\{E_{kl} : 1 \leq k, l \leq n = m-1\}$ is the canonic basic of the set of $n \times n$-matrix, $M_n(\mathbb{R})$. In fact $\{\mathcal{B}_{ij} : 2 \leq i \leq j \leq m\}$ is the canonic basic of the set of $n \times n$-symmetric matrix $\mathcal{S}(n)$. Thus we have that

$$U(t) = \sum_{2 \leq i \leq j \leq m} u_{ij}(t)\mathcal{B}_{ij} \text{ and } \begin{pmatrix} 0 & 0 \\ & \\ U(t) & 0 \end{pmatrix} = \sum_{2 \leq i \leq j \leq m} u_{ij}(t) B_{ij}.$$

Example 3.2.2. *Suppose that $n = 2$ and $2c = 1$. Remember the equation 3.11 in $M_2(\mathbb{R})$: $X''(t) + K^{\Omega+\delta\Omega}(t)X(t) = 0$, this imply that*

$$\begin{cases} X'(t) &= X'(t), \\ X''(t) &= -K^{\Omega+\delta\Omega}(t)X(t). \end{cases}$$

Taking $W(t) = \begin{pmatrix} X(t) \\ X'(t) \end{pmatrix}$, the before equation is equivalent to

$$W'(t) = \begin{pmatrix} 0 & I_{2\times 2} \\ -K^{\Omega+\delta\Omega}(t) & 0 \end{pmatrix} W,$$

in $M_4(\mathbb{R})$, but $K^{\Omega+\delta\Omega}(t) = K^{\Omega}(t) - U(t)$, then

$$\begin{aligned} W'(t) &= \begin{pmatrix} 0 & I_{2\times 2} \\ -K^{\Omega}(t) & 0 \end{pmatrix} W(t) + \begin{pmatrix} 0 & 0 \\ U(t) & 0 \end{pmatrix} W(t), \\ W'(t) &= A(t)W(t) + \begin{pmatrix} 0 & 0 \\ U(t) & 0 \end{pmatrix} W(t), \end{aligned}$$

From example 3.2.1, we have that

$$
\begin{aligned}
U(t) &= u_{22}(t)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + u_{23}(t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + u_{33}(t)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= u_{22}(t)\mathcal{B}_{22} + u_{23}(t)\mathcal{B}_{23} + u_{33}(t)\mathcal{B}_{33}.
\end{aligned}
$$

This we have that

$$
\begin{aligned}
W'(t) &= A(t)W(t) + \begin{pmatrix} 0 & 0 \\ u_{22}(t)\mathcal{B}_{22} + u_{23}(t)\mathcal{B}_{23} + u_{33}(t)\mathcal{B}_{33} & 0 \end{pmatrix} W(t) \\
W'(t) &= A(t)W(t) + u_{22}(t)\begin{pmatrix} 0 & 0 \\ \mathcal{B}_{22} & 0 \end{pmatrix} W(t) + u_{23}(t)\begin{pmatrix} 0 & 0 \\ \mathcal{B}_{23} & 0 \end{pmatrix} W(t) \\
&\quad + u_{33}(t)\begin{pmatrix} 0 & 0 \\ \mathcal{B}_{33} & 0 \end{pmatrix} W(t) \\
W'(t) &= A(t)W(t) + u_{22}(t)B_{22}W(t) + u_{23}(t)B_{23}W(t) + u_{33}(t)B_{33}W(t) \\
W'(t) &= A(t)W(t) + \sum_{2\le i\le j\le 3} u_{ij}B_{ij}W(t)
\end{aligned}
$$

This last equation is the called control system.

Since our control system has the form (2.6), the result in the Section 2.6 is apply. By compactness of $\Theta := M$ and regularity of the magnetic flow, the compactness assumption in Proposition 2.6.1 are satisfied. It remains to check that assumptions (2.8), (2.10) and (2.11) are hold. This is the same procedure as in [20] and [21].

First we check immediately that

$$B_{ij}B_{kl} = 0, \text{ for every } 2 \le i \le j \le m \text{ and } 2 \le k \le l \le m.$$

So, assumption (2.8) is satisfied. Since the B_{ij} do not depend on time, we check easily that the matrices $B^0_{ij}, B^1_{ij}, B^2_{ij}$ associated to our system are given by

$$\begin{cases} B^0_{ij} = B_{ij}, \\ \\ B^1_{ij} = \left[B_{ij}, A(t)\right], \\ \\ B^2_{ij}(t) = \left[\left[B_{ij}, A(t)\right], A(t)\right], \end{cases}$$

for every $t \in [0, \tau]$ and $2 \le i \le j \le m$. An easy computation yields for any $2 \le i \le j \le m$ and any $t \in [0, \tau]$,

$$B^1_{ij} = [B_{ij}, A(t)] = \begin{pmatrix} -\mathcal{B}_{ij} & 0 \\ \\ 0 & \mathcal{B}_{ij} \end{pmatrix}$$

and

$$B^2_{ij}(t) = [[B_{ij}, A(t)], A(t)] = \begin{pmatrix} 0 & -2\mathcal{B}_{ij} \\ \\ -\mathcal{B}_{ij}K^{\Omega}(t) - K^{\Omega}(t)\mathcal{B}_{ij} & 0 \end{pmatrix},$$

note that B^0_{ij}, B^1_{ij} are constant. Then we get for any $2 \le i \le j \le m$,

$$[B^1_{ij}(0), B_{ij}] = 2 \begin{pmatrix} 0 & 0 \\ \\ (\mathcal{B}_{ij})^2 & 0 \end{pmatrix} \in Span\left\{B^0_{kl}(0) = B_{kl} : 2 \le k \le l \le m\right\},$$

and

$$[B^2_{ij}(0), B_{ij}] = 2 \begin{pmatrix} -(\mathcal{B}_{ij})^2 & 0 \\ \\ 0 & (\mathcal{B}_{ij})^2 \end{pmatrix} \in Span\left\{B^1_{kl}(0) : 2 \le k \le l \le m\right\},$$

because

$$(\mathcal{B}_{ij})^2 = \begin{cases} \mathcal{B}_{ii}, & \text{if } i = j \\ \\ \mathcal{B}_{ii} + \mathcal{B}_{jj}, & \text{if } i < j \end{cases}.$$

So assumption (2.10) is satisfied. It remains to show that (2.11) holds. We first notice that for any $2 \leq i \leq j \leq m$ and $2 \leq k \leq l \leq m$, we have

$$\begin{aligned}
[B^1_{ij}(0), B^1_{kl}(0)] &= [[B_{ij}, A(0)], [B_{kl}, A(0)]] \\
&= \begin{pmatrix} [\mathcal{B}_{ij}, \mathcal{B}_{kl}] & 0 \\ & \\ 0 & [\mathcal{B}_{ij}, \mathcal{B}_{kl}] \end{pmatrix},
\end{aligned}$$

with

$$[\mathcal{B}_{ij}, \mathcal{B}_{kl}] = \delta_{il}\mathcal{C}_{jk} + \delta_{jk}\mathcal{C}_{il} + \delta_{ik}\mathcal{C}_{jl} + \delta_{jl}\mathcal{C}_{ik}, \tag{3.17}$$

where $\mathcal{C}_{ij}$ is the $n \times n$ skew-symmetric matrix defined by

$$\mathcal{C}_{ij} = \begin{cases} E_{(i-1)(j-1)} - E_{(j-1)(i-1)}, & \text{if } i < j, \\ 0, & \text{if } i = j. \end{cases}$$

We define

$$\mathbf{A} := Span\left\{B^0_{ij}(0), B^1_{ij}(0), B^2_{ij}(0), [B^1_{ij}(0), B^1_{kl}(0)] : 2 \leq i, j \leq k, l \leq m\right\}.$$

It is sufficient to show that the space $\mathbf{A} \subset M_{2n}(\mathbb{R})$ satisfies that $\mathbf{A} \subset \mathfrak{sp}(n)$ and has dimension $p = n(2n+1)$. First since the set matrices $\mathcal{B}_{ij}$ with $2 \leq i \leq j \leq m$ forms a basis of the vector space of $n \times n$ symmetric matrices $S(n)$ we check easily by the formulas that the vector space

$$\mathbf{A}_1 := Span\left\{B_{ij}, B^2_{ij}(0) = [[B_{ij}, A(0)], A(0)] : 2 \leq i \leq j \leq m\right\}$$

has dimension $n(n+1)$. We check easily that the vector spaces

$$\mathbf{A}_2 := Span\left\{B^1_{ij} : 2 \leq i \leq j \leq m\right\} = Span\left\{\begin{pmatrix} -\mathcal{B}_{ij} & 0 \\ 0 & \mathcal{B}_{ij} \end{pmatrix} : 2 \leq i \leq j \leq m\right\}$$

and

$$\begin{aligned}
\mathbf{A}_3 &:= Span\left\{[B^1_{ij}(0), B^1_{kl}(0)] : 2 \leq i, j \leq k, l \leq m\right\} \\
&= Span\left\{\begin{pmatrix} [\mathcal{B}_{ij}, \mathcal{B}_{kl}] & 0 \\ 0 & [\mathcal{B}_{ij}, \mathcal{B}_{kl}] \end{pmatrix} : 2 \leq i, j \leq k, l \leq m\right\}
\end{aligned}$$

are orthogonal to $\mathbf{A}_1$ with respect to the scalar product $P \cdot Q = tr(P^*Q)$. So, we need to show that $\mathbf{A}_2 + \mathbf{A}_3$ has dimension n^2. By the above formulas, we have that $\mathbf{A}_2$ and $\mathbf{A}_3$ are orthogonal. The space $\mathbf{A}_2$ has the same dimension as $S(n)$, that is $n(n+1)/2$. Moreover, by (3.17) for every $2 \leq i = k < j < l \leq m$, we have

$$[\mathcal{B}_{ij}, \mathcal{B}_{kl}] = \mathcal{C}_{jl}.$$

The space spanned by the matrices of the form

$$\begin{pmatrix} \mathcal{C}_{jl} & 0 \\ 0 & \mathcal{C}_{jl} \end{pmatrix},$$

with $2 \leq j < l \leq m$ has dimension $n(n-1)/2$. This shows that $\mathbf{A}_3$ has dimension at least $n(n-1)/2$ and so $\mathbf{A}_2 \oplus \mathbf{A}_3$ has dimension n^2. Thus we have proved the following result. Let $\mathcal{F}$ the set of $\delta\Omega = d\eta$ where the $\eta \in \Omega^1(M)$ defined as above i.e. $supp(\delta\Omega) \subset U$ and $\delta\Omega = 0$ in γ. We can consider the follow map

$$\begin{aligned} S_{\tau,\theta} \; : \; & \mathcal{F} \longrightarrow Sp(n), \\ & \delta\Omega \longmapsto d_\theta \mathcal{P}(\Omega + \delta\Omega)(\tau). \end{aligned}$$

Theorem 3.2.3. *(Perturbative Theorem) Let $c > 0$ and $\Omega \in \overline{\Omega}^2(M)$ and $0 < \tau < K(c, \Omega)$. There is $R, K > 0$ (depending on c, Ω and τ) such that the following property holds: For each $\theta \in T^c M$ and $r \in (0, R)$, $\mathcal{F}$ as defined above, we have that*

$$B_{Kr}(S_{\tau,\theta}(0)) \cap Sp(n) \subset S_{\tau,\theta}\left(B_r^{C^1}(0) \cap \mathcal{F}\right).$$

where $B_\delta^{C^1}(0) \subset \overline{\Omega}^2(M)$ is the open ball of radius δ centered at $0 \in \overline{\Omega}^2(M)$ in the C^r topology.

Basically this theorem say that $S_{\tau,\theta}$ is a open map. This is the technical result that we need to demonstrate our results.

Remember that, en this section θ be a closed orbit of period T_θ. Suppose also that $K/2 < \tau < K$. We have that the number of self-intersection the closed magnetic geodesic $\gamma : [0, T_\theta] \to M$ is finite. There exist $l \in \mathbb{N}$ such that $T_\theta = l\tau$, we define $\gamma_i(t) := \gamma(t + i\tau)$ for all $t \in [0, \tau]$ and $i = 0, \ldots, l-1$. We can choose $U_i \subset M$ open, as in the lemma 3.1.1, disjoint sets for $i = 0, \ldots, l-1$, such that

$$U_i \cap \gamma((0,\tau)) \subset \gamma((0,\tau)) \text{ and } U_i \cap \gamma_j = \emptyset, \text{ for all } i \neq j$$

Denote $\mathbf{U} := \bigcup_{i=0}^{l-1} U_i$, we consider the map

$$\begin{aligned} S_\theta \ : \ \mathcal{F} &\longrightarrow \prod_{i=0}^{l-1} Sp(n), \\ \delta\Omega &\longmapsto d_\theta \mathcal{P}(\Omega + \delta\Omega)(T_\theta) = \prod_{i=0}^{l-1} d_{\theta_{i\tau}} \mathcal{P}_i(\Omega + \delta\Omega)(\tau), \end{aligned}$$

where $\mathcal{P}_i(\Omega + \delta\Omega)$ is the Poincaré map from $\Sigma_{i\tau}$ to $\Sigma_{(i+1)\tau}$. Applying l times the theorem 3.2.3, we prove the follow corollary.

Corolary 3.2.4. *Let Ω be a smooth closed 2-form on M and $\mathcal{U}$ an open neighbourhood of Ω in the C^1 topology. Suppose that $\theta_t \subset T^cM$ is a closed orbit of ϕ_t^Ω, with minimal period T_θ. Then choosing τ, l and $\mathbf{U}$ as above, the image of set $\{\mathcal{U} \setminus \{\Omega\}\} \cap \mathcal{F}$ by the map S_θ is an open neighbourhood of $S_\theta(0)$ in $\prod_{i=0}^{l-1} Sp(n)$.*

Chapter 4

The Kupka-Smale Property

In this chapter we are going to study the version of KupKa-Smale's theorem in this context of magnetic flows. The original theorem was made for smooth diffeomorphisms by I. Kupka [17] and S. Smale [43] in 1963, separately, each has a different version of the demonstration. However best known version of the proof this theorem is of M. Peixoto [34], he unified in a single improved show of both that is used until now. This theorem is one of the pillars for demonstration of the C^1 structural stability of Mañé [25]. This theorem also exists in other contexts, for example smooth vector fields, also in smooth flows such as the geodesic flows by D. Anosov [4] and G. Contreras [9], Hamiltonian flows by R. C. Robinson [39] and [40], also the magnetic flows on surfaces by J. A. Miranda [26]. In all the methods of Peixoto are used.

In simple language, this theorem says that almost every magnetic flow satisfies two properties. The first one says that all closed orbit is non-degenerate and the second says that every heteroclinic point is transverse. Recall that an non-degenerate closed orbit is either hyperbolic or elliptical. The difference with original version of the theorem is that not possible to destroy the ellipticity of closed orbit due to Robinson [41].

In the first section of this chapter we announce this theorem and then proceed to demonstrate the first property. In the second section we prove the second property. We are based on the methods of M. Peixoto [34], D. Anosov [4] and J. A. Miranda [26]. Notice here that we generalize Miranda's theorem.

4.1 The Kupka-Smale theorem for magnetic flows

In this section we will write accurately the version of Kupka-Smale's theorem in the context of magnetic flows defined in manifolds of any dimension. Next we are going to prove the first property of the theorem. Basically says that for almost all magnetic flow, we have that, all closed orbit is non-degenerate, and for this we will use the perturbative theorem 3.2.3.

Recall that a subset $\mathcal{R} \subset \overline{\Omega}^2(M)$ is called a C^r-residual if it contains a countable intersection of open and dense subsets in the C^r-topology.

Now we come to define the concept of when a property is satisfied by almost all magnetic flow, see [27] and [26].

Definition 4.1.1. We say that a property **P** is C^r-generic for magnetic flows if, for each $c > 0$. there exist a set $\mathcal{R}(c) \subset \overline{\Omega}^2(M)$, such that following holds.

1. The subset $\mathcal{R}_h(c) := \{\Omega \in \mathcal{R}(c) : [\Omega] = h\}$ is C^r-residual in $\overline{\Omega}^2_h(M) := \{\Omega \in \overline{\Omega}^2(M) : [\Omega] = h\}$, for all $h \in H^2(M, \mathbb{R})$ and
2. The flow ϕ_t^Ω has the property **P** in T^cM, for all $\Omega \in \mathcal{R}(c)$.

This definition is due to J. A. G. Miranda. Note that a generic property is also generic for subclasses of magnetic flows given by 2-form with the same cohomology class in $H^2(M, \mathbb{R})$. In particular,the same happens for the family of exact magnetic flows. Now if we are in the conditions to announce the main result of this chapter.

Theorem 4.1.2. *(Kupka-Smale) The following property:*

1. *all closed orbits are hyperbolic or elliptic,*
2. *all heteroclinic points are transversal*

are C^r-generic for magnetic flows, with $1 \leq r \leq \infty$.

Let $c > 0$ and Ω be a smooth closed 2-form on M. Consider the magnetic flow ϕ_t^Ω in T^cM and $\theta = (x, v) \in T^cM$ such that $\theta_t = \phi_t^\Omega(\theta) = (\gamma(t), \gamma'(t))$ be a closed orbit in T^cM with period $T_\theta > 0$, where γ is a closed Ω-magnetic geodesic in M.

We consider the following subset $\mathcal{N}(t) = \mathcal{N}(\theta_t) \subset T_{\theta_t}T^cM$,for every $t \in [0, T_\theta]$, be the subspace

$$\mathcal{N}(t) := \left\{\xi \in T_{\theta_t}T^cM : \langle \xi_1, \gamma'(t)\rangle_{\gamma(t)} = 0\right\}.$$

If $\xi = X^\Omega(\theta_t)$, then $\xi_1 = \gamma'(t)$, therefore the subspace $\mathcal{N}(t)$ is transversal to X^Ω along of θ_t, note that $V(\theta_t) \subset \mathcal{N}(t)$. Hence the dimension of $\mathcal{N}(t)$ is $2n$ and

$$T_{\theta_t}T^cM = \mathcal{N}(t) \oplus \left\langle X^\Omega(\theta_t)\right\rangle.$$

Therefore, the restriction of the twisted form $\omega_{\theta_t} := \omega_\Omega|_{\theta_t}$ to $\mathcal{N}(t)$ is a non-degenerate 2-form. Note that $\mathcal{N}(\theta)$ does not depend on the 2-form Ω. For $i = 2, \ldots, m$, we have that $(e_i(t), 0), (0, e_i(t)) \in H(\theta_t) \oplus V(\theta_t)$, then $(e_i(t), 0), (0, e_i(t)) \in \mathcal{N}(t)$ and

$$\begin{aligned}
\omega_{\theta_t}((e_i(t), 0), (e_j(t)), 0) &= \Omega_{ij}, \\
\omega_{\theta_t}((0, e_i(t)), (0, e_j(t))) &= 0 \text{ and} \\
\omega_{\theta_t}((e_i(t), 0), (0, e_j(t))) &= \delta_{ij}.
\end{aligned}$$

Thus, we have that

$$(e_2(t), 0), \ldots (e_m(t), 0), (0, e_2(t)), \ldots (0, e_m(t)),$$

is an basis of $\mathcal{N}(t)$, for every $t \in [0, T_\theta]$. We say that a closed orbit is *non-degenerate of order* $k \in \mathbb{N}$, if the derivate of the kth iterated on the linearized Poincaré map $d_\theta\mathcal{P}^k(\Omega) : \Sigma \to \Sigma$, has no eigenvalues equal 1.

Given $a, c > 0$ and $k \in \mathbb{N}$, let $\mathcal{G}^k(c, a)$, be the subset of every $\Omega \in \overline{\Omega}^2(M)$ such that all closed orbits of ϕ_t^Ω in T^cM, with minimal period $< a$, are non-degenerate of order k. Thus the first part of the Theorem 4.1.2 can be reduces to following proposition. Note that

$$\mathcal{G}^k(c, a) = \bigcap_{i=1}^{k} \mathcal{G}^1(c, ia),$$

also that

$$\mathcal{R}_1(c) := \bigcap_{n\in\mathbb{N}} \mathcal{G}^1(c, n).$$

Proposition 4.1.3. *Given $c, a > 0$ and $r \in \mathbb{N}$, the subset $\mathcal{G}^1(c,a) \subset \overline{\Omega}^2(M)$ is a open and dense subset in the C^r topology. Moreover, for each $h \in H^2(M,R)$, the subset $\mathcal{G}^1_h(c,a)$ is C^r-dense subset of $\overline{\Omega}^2_h(M)$.*

The prove this proposition is following the ideas of Miranda [26], so for this we need some lemmas that we will statement.

Remember that $\theta_t = (\gamma(t), \gamma'(t)) = \phi^{\Omega}_t(\theta)$ is a closed orbit in T^cM with minimal period $T_\theta > 0$ in T^cM. For each $i = 2, \ldots, m$ consider a function $f_i \in C^\infty(M)$ with support in U, neighborhood of $\gamma([0, T_\theta])$ and defined

$$f_i(x_1, \ldots, x_m) = \frac{1}{\sqrt{2c}} \int_0^{x_1} u_i(s) ds x_i f\left(\|\hat{x}_1\|\right)$$

in local coordinates as in the lemma 3.1.1 and f as 3.12. Let $\eta_i := f_i dx_i$ be a smooth 1-form in M, hence $d\eta_i = u_i(x_1) x_i dx_1 dx_i$ in U. Let is consider

$$\begin{aligned} \gamma_i(s,t) &:= \pi \circ \phi^{\Omega + s(d\eta_i)}_t(\theta), \text{ for } s \in (-\varepsilon, \varepsilon), \\ V_i(t) &:= \frac{\partial}{\partial s}\Big|_{s=0} \gamma_i(s,t), \end{aligned}$$

hence $\gamma(t) = \gamma_i(0,t)$ and $V_i(t)$ is a vector field along the magnetic geodesic $\gamma(t)$. Then

$$\begin{aligned} Z_i(t) &:= \frac{\partial}{\partial s}\Big|_{s=0} \phi^{\Omega + s(d\eta_i)}_t(\theta) \\ &= (V_i(t), V_i'(t)) \in H(\theta_t) \oplus V(\theta_t). \end{aligned}$$

Since that $d\eta_i \equiv 0$ in γ, then

$$\frac{D}{ds}\Big|_{s=0} \left(\frac{D}{dt} \gamma_i'(s,t) \right) = \frac{D}{ds}\Big|_{s=0} \left(Y_{\gamma_i(s,t)}(\gamma_i'(s,t)) \right),$$

thus we have that $V_i(t)$ satisfied the Jacobi equation (2.4) for Ω, note that $e_i(0) = e_i(T_\theta)$ for every $i = 1, \ldots, m$, thus we have that

$$\begin{cases} \begin{pmatrix} V_{i,\perp}(t) \\ V'_{i,\perp}(t) \end{pmatrix}' = A(t) \begin{pmatrix} V_{i,\perp}(t) \\ V'_{i,\perp}(t) \end{pmatrix} + u_i(t) \begin{pmatrix} 0 \\ e_i \end{pmatrix}, \text{ for every } t \in [0, T_\theta] \\ \\ V_{i,\perp}(0) = V'_{i,\perp}(0) = 0, \end{cases}$$

where $V_{i,\perp}(t) = (V_{i,2}(t), \ldots, V_{i,m}(t))$ and $A(t)$ as before in 3.16. If $S(t)$ is the fundamental matrix of the correspondent homogeneous equation, then

$$\begin{pmatrix} V_{i,\perp} \\ V'_{i,\perp} \end{pmatrix} (T_\theta) = S(T_\theta) \int_0^{T_\theta} u_i(t) S(t)^{-1} \begin{pmatrix} 0 \\ e_i \end{pmatrix} dt.$$

Fix $t_0 \in (0, T_\theta)$ and $0 < \lambda < \varepsilon < T_\theta - t_0$ such that $\gamma([t_0 - \varepsilon, t_0 + \varepsilon])$ does not have self-intersection points. Let $\delta_\lambda : \mathbb{R} \to \mathbb{R}$ be a C^∞-approximation of the Dirac delta at the point t_0. Chose $u_i(t) = \delta'_\lambda(t)$ and $\widetilde{u}_i(t) = \delta_\lambda(t)$, we have that, for $(e_i, 0), (0, e_i) \in \mathcal{N}(T_\theta) = \mathcal{N}(\theta)$

$$d_\theta \mathcal{P}(\Omega)(T_\theta)(e_i, 0) = (V_{i,\perp}(T_\theta), V'_{i,\perp}(T_\theta))$$

and

$$d_\theta \mathcal{P}(\Omega)(T_\theta)(0, e_i) = (\widetilde{V}_{i,\perp}(T_\theta), \widetilde{V}'_{i,\perp}(T_\theta)),$$

since

$$\frac{d}{dt}\left(S(t)^{-1} \begin{pmatrix} 0 \\ e_i \end{pmatrix} \right) = -S(t)^{-1} \begin{pmatrix} 0 & I \\ K^{\Omega + d\eta_i}(t) & 0 \end{pmatrix} \begin{pmatrix} 0 \\ e_i \end{pmatrix} = -S(t)^{-1} \begin{pmatrix} e_i \\ 0 \end{pmatrix}.$$

Thus we have the following result.

Lemma 4.1.4. *Suppose that $\phi_t^\Omega(\theta)$ is a closed orbit of minimal period $T_\theta > 0$ on $T^c M$. Then there is $\eta_2, \ldots \eta_m$ 1-forms in M such that*

$$Z_i := \frac{d}{ds}\Big|_{s=0} \left(\phi_{T_\theta}^{\Omega + s d\eta_i}(\theta) \right) \text{ for every } i = 2, \ldots, m,$$

are a basis of $\mathcal{N}(\theta)$.

This is similar to Lemma 3.3 in [26]. Which implies the following result.

Lemma 4.1.5. *Let $\Omega_0 \in \overline{\Omega}^2(M)$ and $\theta_0 \in T^c M$ such that $\phi_t^{\Omega_0}(\theta_0)$ is a closed orbit of minimal period $t_0 > 0$ Then the map*

$$\begin{array}{rcccc} ev & : & T^c M \times \mathbb{R} \times \overline{\Omega}^2_{[\Omega_0]}(M) & \longrightarrow & T^c M \times T^c M \supset \Delta, \\ & & (\theta, t, \Omega) & \longmapsto & (\theta, \phi_t^\Omega(\theta)), \end{array}$$

is transversal to the diagonal $\Delta \subset T^c M \times T^c M$ in the point $(\theta_0, t_0, \Omega_0)$.

Proof. By hypothesis $ev(\theta_0, t_0, \Omega_0) = (\theta_0, \phi_{t_0}^{\Omega_0}(\theta_0)) = (\theta_0, \theta_0) \in \Delta$. Computing the derivate of the map ev, we obtain

$$\begin{aligned} d_{(\theta_0,t_0,\Omega_0)}ev \cdot (\delta\theta, \delta t, \delta\Omega) &= d_{(\theta_0,t_0)}(\theta_0, \phi_{t_0}^{\Omega_0}(\theta_0)) \cdot (\delta\theta, \delta t) + d_{\Omega_0}(\theta_0, \phi_{t_0}^{\Omega_0}(\theta_0)) \cdot \delta\Omega \\ &= \big(\delta\theta, d_{\theta_0}\phi_{t_0}^{\Omega_0} \cdot \delta\theta + \delta t X^{\Omega_0}(\theta_0)\big) + \left(0, \left.\frac{d}{ds}\right|_{s=0} \phi_{t_0}^{\Omega_0+s\delta\Omega}(\theta_0)\right) \end{aligned}$$

since

$$\begin{aligned} T_{(\theta_0,\theta_0)}(T^cM \times T^cM) &= T_{\theta_0}T^cM \times T_{\theta_0}T^cM = T_{(\theta_0,\theta_0)}\Delta \oplus (\{0\} \times T_{\theta_0}T^cM) \\ &= T_{(\theta_0,\theta_0)}\Delta \oplus \big(\{0\} \times \big(\langle X^{\Omega_0}(\theta_0)\rangle \oplus \mathcal{N}(\theta_0)\big)\big) \\ &= T_{(\theta_0,\theta_0)}\Delta \oplus \big(\{0\} \times \langle X^{\Omega_0}(\theta_0)\rangle\big) \oplus (\{0\} \times \mathcal{N}(\theta_0)) \end{aligned}$$

Using the previous lemma, we have that exists Z_i the form $\left.\frac{d}{ds}\right|_{s=0} \phi_{t_0}^{\Omega_0+s\delta\Omega}(\theta_0)$ such that $\{Z_2, \ldots, Z_m\}$ is a basis of $\mathcal{N}(\theta_0)$. □

This is similar to Lemma 3.4 in [26]. The perturbative Theorem (3.2.3) and its Corollary (3.2.4), together with the previous lemmas implies the following result.

Lemma 4.1.6. *Let $\Omega_0 \in \mathcal{G}^1(c, a)$ and $k \in \mathbb{N}$. Then there exists a $\Omega \in \mathcal{G}^k_{[\Omega_0]}(c, a)$, such that Ω is arbitrarily C^r-close to Ω_0.*

Proof. Since $\Omega_0 \in \mathcal{G}^1(c, a)$, have all closed orbits of flow $\phi_t^{\Omega_0}\big|_{T^cM}$, with minimal period $< a$, are non-degenerate of order 1, this is, the derivate of the mth iterated on the linearized Poincaré map has no eigenvalues equal to 1. Then for this closed orbits, we have that $ev(\Omega_0)$ is transversal to Δ in (θ_0, t_0), where $t_0 < a$ and $\phi_{t_0}^{\Omega_0}(\theta_0) = \theta_0$ and other orbit $ev(\Omega_0)(\theta, t) \notin \Delta$. Thus we have that $ev(\Omega_0) \pitchfork_{T^cM\times[0,a]} \Delta$. Then the magnetic flow $\phi_t^{\Omega_0}$ has a finite number of closed orbits of period less that a. Let $\theta_1, \ldots, \theta_l$ be such closed orbits and $t_1, \ldots, t_l$ be its minimal periods. For each $i = 1, \ldots, l$, as in the corollary 3.2.4, we have that $S_i = S_{\theta_i} : \mathcal{F}_i \to Sp(n)$ be defined by $S_i(\delta\Omega) = d_{\theta_i}P(\Omega_0 + \delta\Omega)$. Then, by this corollary, for all C^r-open neighbourhood $\mathcal{U}$ of 0, the subset $S_i(\mathcal{U} \cap \mathcal{F}_i) \subset Sp(n)$ are open neighbourhood of $d_{\theta_i}P(\Omega_0)$, for $i+1, \ldots, l$. Hence, given $k \in \mathbb{N}$, for each $i = 1, \ldots, l$ there exists a linear map $A_i \in S_i(\mathcal{U}) \cap \mathcal{F}_i$, such that the mth iterated on A_i does not admit an eigenvalue equal to 1. Therefore, if $\delta\Omega_i \in S_i^{-1}(A_i) \cap \mathcal{U}$, we have that $\Omega = \delta\Omega_1 + \cdots + \delta\Omega_l$ satisfies the lemma.

$\square$

This is similar to Lemma 3.5 in [26]. The part of opening in the proposition 4.1.3 is a consequence of dynamic properties and transversality due to [1], we only show the part of density.

Proof. **Proof of the Proposition 4.1.3. Density:**

Let $\Omega \in \overline{\Omega}^2(M)$. Take $k = k(a, \Omega) \in \mathbb{N}$ such that $(k-1)K < 2a \leq kK$ and $\mathcal{U}$ a C^r open neighborhood of Ω such that, if $\widehat{\Omega} \in \mathcal{U}$, then

$$\|\widehat{\Omega}\|_{C^0} < \|\Omega\|_{C^0} + 1,$$

thus $\mathcal{U} \subset \mathcal{G}^l(c, K)$, for every $l \in \mathbb{N}$, in particular

$$\Omega \in \mathcal{U} \subset \mathcal{G}^1(c, K)$$

Consider the map

$$\begin{array}{rcccl} ev & : & T^cM \times \mathbb{R} \times \mathcal{U}_{[\Omega]} & \longrightarrow & T^cM \times T^cM \supset \Delta, \\ & & (\theta, t, \widehat{\Omega}) & \longmapsto & (\theta, \phi_t^{\widehat{\Omega}}(\theta)). \end{array}$$

The Lemma 4.1.5, implies that, if $ev(\theta_0, t_0, \Omega_0) \in \Delta$, then $ev \pitchfork_{(\theta_0, t_0, \Omega_0)} \Delta$. Hence $ev(\Omega_0) \pitchfork_{T^cM \times [0, 3K/2]} \Delta$. So due to Abraham's Theorem of Transversality, see [2], we have that the set of every $\Omega_0 \in \mathcal{U}_{[\Omega]}$ such that $ev(\Omega_0) \pitchfork_{T^cM \times [0,3K/2]} \Delta$ is dense in $\mathcal{U}_{[\Omega]}$. Then, there is $\widehat{\Omega}_1 \in \mathcal{U}_{[\Omega]}$ such that

$$ev(\widehat{\Omega}_1) \pitchfork_{T^cM \times [0,3K/2]} \Delta \text{ and } \|\Omega - \widehat{\Omega}\|_{C^r} < \frac{\varepsilon}{2k}.$$

Lemma 4.1.6, implies that there is $\Omega_1 \in \mathcal{G}^k_{[\Omega]}(c, 3k/2)$ with $\|\Omega_1 - \widehat{\Omega}_1\|_{C^r} < \dfrac{\varepsilon}{2k}$. Hence $\|\Omega - \Omega_1\|_{C^r} < \dfrac{\varepsilon}{k}$. We can take $\Omega_1 \in \mathcal{U}_{[\Omega]}$ and consider $\mathcal{U}_1 = \mathcal{U} \cap \mathcal{G}^k_{[\Omega]}(c, 3K/2)$ and

$$\begin{array}{rcccl} ev & : & T^cM \times \mathbb{R} \times \mathcal{U}_1 & \longrightarrow & T^cM \times T^cM \supset \Delta, \\ & & (\theta, t, \widehat{\Omega}) & \longmapsto & (\theta, \phi_t^{\widehat{\Omega}}(\theta)). \end{array}$$

Suppose that $ev(\theta_0, t_0, \Omega_0) \in \Delta$. Let T_0 be the minimal period of the closed orbit $\phi_t^{\Omega_0}(\theta_0)$. If $2T_0 \leq 3K$ then $ev(\Omega_0) \pitchfork_{(\theta_0, lT_0)} \Delta$, for every $1 \leq l \leq k$. Since $\mathcal{U}_1 \subset \mathcal{U}_{[\Omega]}$, we have that

$K < T_0$ and $t_0 < kT_0$. Therefore, $ev(\Omega_0) \pitchfork_{(\theta_0,t_0)} \Delta$. If $2T_0 \in (3K, 4K]$ then $t_0 = T_0$ and, by Lemma 4.1.5 have that $ev(\Omega_0) \pitchfork_{(\theta_0,t_0,\Omega_0)} \Delta$, hence $ev(\Omega_0) \pitchfork_{T^cM\times[0,2K]} \Delta$. So due to Abraham's Theorem of Transversality, see [2], we have that there is $\widehat{\Omega}_2 \in \mathcal{U}_1$, such that

$$ev(\widehat{\Omega}_2) \pitchfork_{T^cM\times[0,2k]} \Delta \text{ and } \|\Omega_1 - \widehat{\Omega}_2\|_{C^r} < \frac{\varepsilon}{2k}.$$

Lemma 4.1.6, implies that there is $\Omega_2 \in \mathcal{G}^k_{[\Omega]}(c, 2K)$ with $\|\Omega_2 - \widehat{\Omega}_2\|_{C^r} < \dfrac{\varepsilon}{2k}$. Hence $\|\Omega_1 - \Omega_2\|_{C^r} < \dfrac{\varepsilon}{k}$. Repeating the same arguments for $2 < l \leq k-1$, we obtain $\Omega_l \in \mathcal{G}^k_{[\Omega]}(c, l(K/2) + K)$, with $\|\Omega_l - \Omega_{l-1}\|_{C^r} < \dfrac{\varepsilon}{k}$.

Finally, since $\mathcal{G}^k_{[\Omega]}(c, (k-1)K/2 + K) \subset \mathcal{G}^1_{[\Omega]}(c, a)$ and $\|\Omega - \Omega_k\|_{C^r} < \varepsilon$, we have that $\Omega \in \overline{\mathcal{G}^1_{[\Omega]}(c, a)}$, this prove the proposition 4.1.3 and thus the first part of Theorem 4.1.2. $\square$

So we have proved the following result.

Theorem 4.1.7. *The property: all closed orbits are non-degenerate, is C^r-generic for magnetic flows $1 \leq r \leq \infty$.*

An immediate consequence is as follows.

Corolary 4.1.8. *Given $c > 0$, there exist a residual set $\mathcal{O}(c)$ in $\overline{\Omega}^2(M)$, such that if $\Omega \in \mathcal{O}(c)$ then the number of periodic orbits of ϕ^Ω_t in T^cM with period $\leq T$ is finite, for all $T > 0$.*

4.2 Transversal heteroclinic points

In this section we will prove the second part of Kupka-Smale's theorem 4.1.2, which is equivalent to saying that the stable and unstable manifolds of all closed orbits intersect transversely.

For each $c, a > 0$, we define $\mathcal{K}(c, a)$ the set of all $\Omega \in \mathcal{G}^1(c, a)$ such that, for every hyperbolic closed orbits $\theta_t, \vartheta_t \subset T^cM$, of period $< a$, $W^u_a(\theta_t) \pitchfork_{T^cM} W^s_a(\vartheta_t)$. Since the stable and unstable manifolds of a hyperbolic closed orbit depend continuously on part compact, we have that $\mathcal{K}(c, a)$ is an open subset of $\overline{\Omega}^2(M)$, for all $a, c > 0$. Thus to complete the proof of Theorem 4.1.2 is sufficient to prove that, for every $\Omega \in \overline{\Omega}^2(M)$, the set $\mathcal{K}_{[\Omega]}(c,a)$ is dense in $\mathcal{G}^1_{[\Omega]}(c,a)$. It is enough to prove the existence of a local perturbation for Ω that preserve the orbits θ_t and ϑ_t and such that the perturbation local manifolds $W^u_a(\theta_t)$ and $W^s_a(\vartheta_t)$ are transversal in a fundamental domain of $W^u_a(\theta_t)$.

Lemma 4.2.1. *(Lemma 3.6 in [26]) Let $\sigma \in W^u_a(\theta_t) \subset T^cM$ be such that the restriction $\pi|_{W^u(\theta_t)}$ is a diffeomorphism in a neighbohood $U \subset W^u(\theta_t)$ of the point σ. Let $V \subset \overline{V} \subset U$ be sufficiently small neighbohood of σ in $W^u(\theta_t)$. Then there is an exact 2-form $d\eta$, with norm arbitrarily small in the C^r topology $(1 \leq r \leq \infty)$, such that*

1. *$Supp(d\eta) \subset \pi(U)$,*
2. *θ_t and ϑ_t are hyperbolic closed orbits of the magnetic flow associated with $\widehat{\Omega} = \Omega + d\eta$,*
3. *$\sigma \in \widehat{W^u_a(\theta_t)}$, where $\widehat{W^u_a(\theta_t)}$ denotes the local stable manifold of θ_t for the flow $\phi^{\widehat{\Omega}}_t$,*
4. *the connected component of $\widehat{W^u_a(\theta_t)} \cap V$ that contains the point σ and $\widehat{W^s(\theta_t)}$ are transversal.*

We will prove the Lemma 4.2.1 later. From the general theory of the Hamiltonian systems we know that, $W^s(\theta_t), W^s(\theta_t) \subset T^cM$ are Hamiltonians submanifolds of TM, with the symplectic twist form ω_Ω.

Proof. **Proof of the density of $\mathcal{K}(c,a)$:**

Let $\mathcal{D} \subset W^u_a(\theta)$ be a fundamental domain of $W^u_a(\theta)$ and $\sigma \in \mathcal{D}$. By the inverse function theorem we know that $\pi|_{W^u(\theta)}$ is a local diffeomorphism in σ if, and only if, $T_\sigma W^u(\theta) \cap V_\sigma = \{0\}$. As $W^u(\theta)$ is a Lagrangian submanifold we have, from Lemma 2.4.1, that $\{t \in \mathbb{R} : d_\sigma \phi^\Omega_t(T_\sigma W^u(\theta)) \cap V_{\phi^\Omega_t(\sigma)} \neq \{0\}\}$, is discrete. Then there exists $t(\sigma) > 0$ arbitrarily close to 0, such that $\pi|_{W^u(\theta)}$ is a diffeomorphism in a neighborhood $U_{t(\sigma)} \subset W^u(\theta)$ of the point $\phi^\Omega_{t(\sigma)}(\sigma)$. Since $\Omega \in \mathcal{G}^1(c,a)$, we can assume that $\pi(\phi^\Omega_{-t(\sigma)}(U_{t(\sigma)}))$ does not intersect any closed orbit of period $\leq a$. Let $W_\sigma \subset \mathcal{D}$ be a neighborhood of σ such that $\sigma \in W_\sigma \subset \overline{W}_\sigma \subset \phi^\Omega_{-t(\sigma)}(U_{t(\sigma)})$. Then, we can take a finite number of points $\sigma_1, \ldots, \sigma_l$ such that the neighborhood $W_1, \ldots, W_l$ cover the fundamental domain $\mathcal{D}$ and such that the points $\phi^{\Omega}_{t_i}(\sigma_i)$ and the neighborhoods $\mathcal{V}_i = \phi^{\Omega}_{t_i}(W_i) \subset U_i$ satisfy the hypothesis in Lemma 4.2.1, for each $i = 1, \ldots, l$.

Applying Lemma 4.2.1 to $\phi^{\Omega}_{t_1}(\sigma_1) \in \mathcal{V}_1 \subset \overline{\mathcal{V}}_1 \subset U_1$, we obtain an exact 2-form $d\eta_1 \in \overline{\Omega}^2(M)$, with C^r-norm arbitrarily small, such that $Supp(d\eta) \subset \pi(U_1)$ and the connected component of $\widehat{W^u(\theta_t)} \cap \overline{\mathcal{V}}_1$ that contain $\phi^{\Omega}_{t_1}(\sigma_1)$ is transversal to $\widehat{W^s(\vartheta_t)}$. Since $\mathcal{G}^1(c, a)$ is open in $\overline{\Omega}^2(M)$, we can assume that $\Omega + d\eta \in \mathcal{G}^1(c, a)$.

The transversality condition on compact subsets is an open condition. Hence, we can successively apply Lemma 4.2.1 in $\mathcal{V}_i$, to obtain an exact 2-form $d\eta_i \in \overline{\Omega}^2(M)$, with C^r-norm small, and such that the invariant manifolds are transversal in $\overline{\mathcal{V}}_1 \cup \ldots \cup \overline{\mathcal{V}}_i$, for $1 \leq i \leq l$.

Since the number of closed orbits of period $< a$ is finite, repeating the same arguments for each possible pair of hyperbolic orbits of period $< a$, in such a way that the perturbation supports are isolated, we obtain an exact 2-form $d\eta$ in M, with C^r-norm arbitrarily small, such that $\Omega + d\eta \in \mathcal{K}(c, a)$. This prove the density of $\mathcal{K}(c, a)$.

$\square$

The proof of Lemma 4.2.1 follow the same ideas that Miranda [26], in fact, the prove depend only the following result where the dimension of manifold is important. We finish this section with the prove this Lemma.

Recall that a submanifold $\mathcal{N}$ of a symplectic manifold $(\mathcal{M}^{2n}, \omega)$ is *Lagrangian* when $\dim(\mathcal{N}) = n$ and $i^*_{\mathcal{N}}\omega \equiv 0$, where $i_{\mathcal{N}} : \mathcal{N} \to \mathcal{M}$ denotes the inclusion map, see the definition 2.3.4. Let $H : \mathcal{M} \to \mathbb{R}$ be a Hamiltonian of class C^2 and $c \in \mathbb{R}$, the following are easy consequence of the definitions and Darboux coordinates, see, for example, the appendix of [9].

1. Suppose $\mathcal{N} \subset H^{-1}(c)$ be a submanifold de dimension n. Then $\mathcal{N}$ is Lagrangian if and only if the Hamiltonian vector field X_H is tangent to $\mathcal{N}$.

2. If $\mathcal{N} \subset H^{-1}(c)$ is Lagrangian and $\theta \in \mathcal{N}$, such that $X_H(\theta) \neq 0$. Then there exist a

neighborhood $U \subset \mathcal{M}$ of θ and a coordinate system $(x, y) : U \to \mathbb{R}^n \times \mathbb{R}^n$ such that $\omega = \sum_i dx_i \wedge dy_i$, $\mathcal{N} \cap U = [y \equiv 0]$ and $X_H|_{\mathcal{N}} = \partial/\partial x_1$.

Lemma 4.2.2. *Let $\mathcal{N}$ and $\mathcal{N}_0$ be two Lagrangian submanifolds contained in an energy level c of a Hamiltonian $H : \mathcal{M} \to \mathbb{R}$ on a symplectic manifold $(\mathcal{M}^{2n}, \omega)$, Let $\theta \in \mathcal{N}$ be a non-singular point for the Hamiltonian vector field X_H. Let $(t, x, y) : U \to [0, 1] \times [-\varepsilon, \varepsilon]^{n-1} \times [-\varepsilon, \varepsilon]^n$ be the Darboux coordinates for $\mathcal{N}$ in a neighborhood U of $\theta \in \mathcal{N}$. Then, given $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon$, there exist a sequence of submanifolds $\mathcal{N}_k \subset H^{-1}(c)$ with dimension n, such that*

1. *$\mathcal{N}_k \to \mathcal{N}$ in the C^∞-topology,*
2. *$\mathcal{N} \cap A = \mathcal{N}_k \cap A$, where $x = (x_2, \ldots x_n)$, $y = (y_1, \ldots, y_n)$ and*
$$A = \{(t, x, y) \in \mathbb{R}^{2n} : \|x\| \geq \varepsilon_1 \text{ or } 0 \leq t \leq 1/4\},$$
3. *$\mathcal{N}_k$ are invariant in $A \cup B$, where*
$$B = \{(t, x, y) \in \mathbb{R}^{2n} : \|x\| \leq \varepsilon \text{ and } 1/2 \leq t \leq 1\},$$
4. *$\mathcal{N}_k \cap C$ are invariant and transversal to $\mathcal{N}_0$, where*
$$C = \{(t, x, y) \in \mathbb{R}^{2n} : \|x\| \leq \varepsilon_2 \text{ and } 1/2 \leq t \leq 1\},$$
5. *$\displaystyle\int_{\mathcal{N}_k} i_k^* \omega = 0$, where $i_k : \mathcal{N}_k \hookrightarrow U$ is the inclusion.*

Proof. Let $\alpha : [-\varepsilon, \varepsilon] \to [0, 1]$ and $\beta : [0, 1] \to [0, 1]$ be smooth functions, such that

$$\alpha \equiv \begin{cases} 0, & \text{in } \mathbb{R} \setminus [-\varepsilon_1, \varepsilon_1] \\ \\ 1, & \text{in } [-\varepsilon_2, \varepsilon_2] \end{cases}, \int \alpha = 0 \text{ and } \beta \equiv \begin{cases} 0, & \text{in } \left[1, \dfrac{1}{4}\right] \\ \\ 1, & \text{in } \left[\dfrac{1}{2}, 1\right] \end{cases}.$$

For $s = (s_2, \ldots, s_n) \in \mathbb{R}^{n-1}$ with $\|s\|$ small, consider $f_s : [0, 1] \times [-\varepsilon, \varepsilon]^{n-1} \to \mathbb{R}^n$ defined as

$$f_s(t, x) = \big(f_s^1(t, x), s_2\alpha(x_2)\beta(t), \ldots, s_n\alpha(x_n)\beta(t)\big)$$

and f_s^1 is defined by:

$$H(t, x, f_s(t, x)) = c. \tag{4.1}$$

Since the curves $t \mapsto (t, x, 0, 0) \subset \mathcal{N}$ are solutions of the Hamiltonian system $(\mathcal{M}, \omega, H)$, we have that $H(t, x, 0, 0) = c$ and $\dfrac{\partial H}{\partial y_1}(t, x, 0, 0) \neq 0$. By the implicit function theorem, for any s with $\|s\|$ sufficiently small, we can solve equation (4.1) for $(t, x) \mapsto f_s^1(t, x)$ with f_s^1 of class C^∞.

We define $\mathcal{N}_s = \{(t, x, f_s(t, x)) \in \mathbb{R}^{2n} : (t, x) \in [0, 1] \times [-\varepsilon, \varepsilon]^{n-1}\}$. By construction the supports of the maps f_s are fixed and $\lim_{s \to 0} f_s = 0$. Therefore, $\mathcal{N}_s \to \mathcal{N}$ in the C^∞-topology when $s \to 0$. Since $f_s(t, x) = 0$ for every $(t, x) \in A$, then $\mathcal{N}_s \cap A = \mathcal{N} \cap A$. Moreover, we have that

$$\begin{aligned} i_s^* \omega &= i_s^*(dt \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n) \\ &= -s_2 \alpha(x_2) \beta'(t) dt \wedge dx_2 - \cdots - s_n \alpha(x_n) \beta'(t) dt \wedge dx_n. \end{aligned}$$

for every s with $\|s\|$ small, where $i_s : \mathcal{N}_s \hookrightarrow U$ denote the inclusion. Since $\beta'(t) = 0$ for every $t \in [1/2, 1]$, the submanifolds $\mathcal{N}_s \cap B$ are Lagrangian. Hence $H(\mathcal{N}_s) = c$. Note that 2-form $i_s^* \omega$ has compact support and

$$\int_{\mathcal{N}_s} i_s^* \omega = -\sum_{i=2}^{n} s_i \int_0^1 \beta'(t) \left(\int_{-\varepsilon}^{\varepsilon} \alpha(x_i) dx_i \right) dt = 0,$$

for every s with $\|s\|$ small. Observe that $\mathcal{N}_s \cap C = [\widehat{y} = s] \cap H^{-1}(c)$, where $\widehat{y} = (y_2, \ldots, y_n)$. It is a basic fact about transversality that $\mathcal{N}_s \cap C$ and $\mathcal{N}_0 \subset H^{-1}(c)$ are transversal in $H^{-1}(c)$ if and only if $\|s\|$ is small, is a regular value of the map $\rho|_{\mathcal{N}_0}$, where $\rho(y) = \widehat{y}$. Then, by Sard's theorem, we have that there is a sequence $s_n \to 0$ for which $\mathcal{N}_{s_n}$ satisfy the theorem.

$\square$

Chapter 5

k-Jets

In this chapter we are going to study one of the important generic results in relation to the spaces of k-Jets of the map of Poincaré of a closed orbit of the magnetic flow. The motivation is always the analogy to geodetic flows, remember that a Riemannian metric is *bumpy* if all closed orbits of the geodetic flow generated by this metric, is non-degenerate. The bumpy metric theorem states that generically every Riemannian metric is bumpy, this theorem was proved by R. Abraham [3], but we can find a complete proof by D. Anosov [4]. Then W. Klingenberg and F. Takens [15] extend the bumpy metric theorem including conditions in the k-jets of Poincaré map of closed orbits of the geodetic flow. On the other hand F. Takens [45] also obtains the analogous result in the Hamiltonian context, that is, for Hamiltonian flows.

In the setting of magnetic flows there is a version of this result on surfaces by J. A. Miranda [26]. Basically we are going to extend this result for manifolds of any dimension.

In the first section we will define the k-Jets later we announce the k-Jets' theorem and see some properties. In the second section we study the perturbation of k-jet with respect to Ω be exact and in the last section we are going to proved the theorem of k-Jets. Note that at first everything is made for exact Ω, but at the end of prove we use the fact that locally Ω is always exact. Note also here that the main reference here is J. A. Miranda [26].

5.1 The k-jet space for magnetic flows

In this section we are going to define the space of k-Jets for symplectic maps. Subsequently, we are going to announce the k-Jets' theorem, taking into account the definition of when a property is C^r-generic for magnetic flows, defined in the previous chapter, definition 4.1.1. We are going to define the concept of when a family of symplectic linear maps is k-general and we will study some of its properties in relation to the Poincaré map of closed orbit of magnetic flows.

We consider $(\mathbb{R}^{2n}, \omega_0)$ as the canonical symplectic linear space of dimension $2n$. Let $Diff_{\omega_0}(\mathbb{R}^{2n}, 0)$ be the space of smooth symplectic diffeomorphisms $f : (\mathbb{R}^{2n}, \omega_0) \to (\mathbb{R}^{2n}, \omega_0)$ such that $f(0) = 0$. Given $k \in \mathbb{N}$, we define the following equivalence relation in $Diff_{\omega_0}(\mathbb{R}^{2n}, 0)$, denoted by $\sim_k$:

$$f \sim_k g \Leftrightarrow \text{ the Taylor polynomials of degree } k \text{ at zero are equal.}$$

Let $f \in Diff_{\omega_0}(\mathbb{R}^{2n}, 0)$, we define the *$k$-jet* of f as the equivalence class of f with respect to the relation $\sim_k$, which we denote as $j^k(f) = j^k(f)(0)$, i.e.

$$j^k(f) := \left\{ g \in Diff_{\omega_0}(\mathbb{R}^{2n}, 0) : f \sim_k g \right\}.$$

The *space of symplectic k-jets at zero* is define as the set of all equivalence classes with respect to the relation $\sim_k$ of elements of $Diff_{\omega_0}(\mathbb{R}^{2n}, 0)$, which we denoted by $J^k_s(n)$, i.e.

$$J^k_s(n) := \left\{ j^k(f) : f \in Diff_{\omega_0}(\mathbb{R}^{2n}, 0) \right\}.$$

Note that $J^k_s(n)$ is a vector space that it is also a Lie group, with the product defined by

$$j^k(f) \cdot j^k(g) = j^k(f \circ h), \text{ for every } f, g \in Diff_{\omega_0}(\mathbb{R}^{2n}, 0).$$

Thus we have that the invertible elements form a Lie group. When $k = 1$, we can identify $J^1_s(n)$ with the classic Lie group $Sp(n)$. Let $Q \subset J^k_s(n)$ be a subset of the space of symplectic k-jets. We say that Q is *invariant* if $\sigma \cdot Q \cdot \sigma = Q$, for all $\sigma \in J^k_s(n)$.

In the same way we can define $\mathcal{J}^k_s(n)$ be the space of the k-jets at 0 of symplectic vector fields that are zero at 0. Let us see that this is the Lie algebra of $J^k_s(n)$ via the *map exponential.* We define the bracket $[\cdot, \cdot]^k : \mathcal{J}^k_s(n) \times \mathcal{J}^k_s(n) \to \mathcal{J}^k_s(n)$ by $[j^k(X), j^k(Y)]^k =$

$-j^k([X,Y])$. Since X, Y are zero in the origin, $[\cdot,\cdot]^k$ depends only on the k-jets of X and Y. Then $[\cdot,\cdot]^k$ defines a Lie algebra structure in $\mathcal{J}_s^k(n)$. Moreover, $\mathcal{J}_s^k(n)$ is the Lie algebra of $J_s^k(n)$ and the exponential map $\exp : \mathcal{J}_s^k(n) \to J_s^k(n)$ is given by $\exp(tj^k(X)) = j^k(\phi_t)$, where ϕ_t is the local flow associated with X at zero. For more information about the k-jets spaces see, for example,[7] and [16].

Let $c > 0$ and Ω be a smooth closed 2-form on M. Consider the magnetic flow ϕ_t^Ω in T^cM and $\theta = (x,v) \in T^cM$ such that $\theta_t = \phi_t^\Omega(\theta) = (\gamma(t),\gamma'(t))$ is a closed orbit in T^cM with period $T_\theta > 0$, where γ is a closed Ω-magnetic geodesic in M. We know that the map of Poincaré $\mathcal{P} := P(\Omega,\theta,\Sigma)$ is a symplectic map, where $\Sigma \subset T^cM$ is a local transverse section at the point θ, with the symplectic structure of ω_Ω. Therefore, using Darboux coordinates, we can assume that $j^k(\mathcal{P}) \in J_s^k(n)$. Remember that M have dimension $m = n+1$.

In the same way as in [15] the fact that the k-jet of the Poincaré map $\mathcal{P}$ of a closed orbit θ_t belongs to an invariant subset $Q \subset J_s^k(n)$ is independent of the chosen section Σ and of the chosen coordinates on Σ; hence, the k-jet of the Poincaré map of θ belong to Q is well defined.

We now state the local perturbation result for magnetic flows on manifolds of any dimension, similarly to the result of Klingenberg and Takens [15] for geodesic flows, Takens [45] for Hamiltonian flows, Miranda [26] for magnetic flows on surfaces and of Carballo and Miranda [6] Tonelli Hamiltonian.

Theorem 5.1.1. *Let $Q \subset J_s^k(n)$ be an open and invariant subset such that*

$$j^k(P(\Omega,\theta,\Sigma)) \in \overline{Q}.$$

Then there exists an exact 2-form $d\eta \in \Omega^2(M)$, arbitrary C^r-close to zero, with $r > k$, such that θ_t is a closed orbit of $\phi_t^{\Omega+d\eta}$ and $j^k(P(\Omega,\theta,\Sigma)) \in Q$.

This is similar to Theorem 1.3 [26]. In this chapter we will prove the Theorem 5.1.1 Let us first describe the general case. Let N be an arbitrary manifold and X be a smooth vector field on N with $\psi_t : N \to N$ the corresponding flow. Let $\alpha : [0,T] \to N$ be a segment of an orbit of ψ_t and let $\Sigma(t)$ be a family of local transversal sections with

$\alpha(t) \in \Sigma(t)$ such that $\Sigma(t) = f^{-1}(t)$ in neighborhood of $\alpha(t)$, where t is a regular value for some smooth function $f : N \to \mathbb{R}$.

Given $k > 0$, let Y be the vector field in N satisfying $j^{k-1}(Y)(\alpha(t)) = 0$, for all $t \in [0, T]$ and $\alpha(0), \alpha(T) \notin Supp(Y)$. Let $P_t : \Sigma(0) \to \Sigma(t)$ and $P'_t : \Sigma(0) \to \Sigma(t)$ be the Poincaré maps on open neighborhood of $\alpha(0) \in \Sigma(0)$ On $\Sigma(t)$ with respect to X and $X+Y$, respectively. For each $t \in [0, T]$, we consider a map $S_t(Y) : (\Sigma(0), \alpha(0)) \to (\Sigma(0), \alpha(0))$ defined as $S_t(Y) = P_t^{-1} \circ P'_t$.

Since $\Sigma(t)$ is transversal to X, we can decompose Y locality as $Y = Y_1 + Y_2$, such that way that $Y_1|_{\Sigma(t)}$ is tangent to $\Sigma(t)$, for all $t \in [0, T]$, and Y_2 is a multiple de X. Let Y_t be the non-autonomous vector field on $\Sigma(0)$ defined by $Y_t = (P_t)^*(Y_1|_{\Sigma(t)}) = (P_t)^{-1} \circ Y_1|_{\Sigma}(t) \circ P_t$. The following proposition shows as a relation between the k-jets of the map $S_t(Y)$ and the vector field Y_t. The proof can be seen in [15], Section 2.

Proposition 5.1.2. *The k-jet of the map $S_t(Y)$ at $\alpha(0)$ is equal to the k-jet of the flow in the time t correspondent to the vector field Y_t.*

For $k \in \mathbb{N}$ we denote by $\mathbb{R}_k[x, y]$ the set of all real homogeneous polynomials of degree k in the $2n$ variables $x = (x_1, \dots, x_n)$, $y = (y_1 \dots, y_n)$. This is a real vector space of dimension $d = d(2n, k) = \binom{2n-1+k}{k}$. Remember that n is fix since $\dim(M) = n + 1$.

We fix the polynomial $F(x_1 \dots, x_n, y_1 \dots, y_n) = x_1^k$ and we define

$$G_k = \left\{(\sigma_1, \dots, \sigma_d) \in Sp(n)^d : \{F \circ \sigma_1, \dots, F \circ \sigma_d\} \text{ is a basis of } \mathbb{R}_k[x, y]\right\}.$$

The following proposition is proof in [6], Section 2.

Proposition 5.1.3. *(Proposition 6 in [6]) For each $k \in \mathbb{N}$, the subset G_k is open and dense in $Sp(n)^d$.*

We define when a family of symplectic matrix is called k-general.

Definition 5.1.4. We say that a one-parameter family $\sigma : [a, b] \to Sp(n)$ of class C^r is *k-general* of class C^r, when $\sigma(a) = I$ and there exists times $t_1, \dots, t_d \in [a, b]$ such that $(\sigma_{t_1}, \dots, \sigma_{t_d}) \in G_k$.

Example 5.1.5. *The following example is a k-general family for any $k \in \mathbb{N}$. Let $\sigma : [0,1] \to Sp(n)$ be given by*

$$\sigma_t(x,y) = \left(x_1 + \cdots + t^{n-1}x_n + t^n y_1 + \cdots + t^{2n-1}y_n, x_2, \ldots, x_n, y\right).$$

Then $\sigma_0 = I$ and

$$F \circ \sigma_t(x,y) = \left(x_1 + \cdots + t^{n-1}x_n + t^n y_1 + \cdots + t^{2n-1}y_n\right)^k$$

$$= \sum_{0 \le i_1 \le \cdots \le i_{2n-1} \le k} t^{\sum_{j=1}^{2n-1} i_j} \binom{i_2}{i_1} \cdots \binom{k}{i_{2n-1}} x_1^{k-i_{2n-1}} \ldots x_n^{i_{n+1}-i_n} y_1^{i_n - i_{n-1}} \ldots y_n^{i_1}.$$

Since

$$\left\{\binom{i_2}{i_1} \cdots \binom{k}{i_{2n-1}} x_1^{k-i_{2n-1}} \ldots x_n^{i_{n+1}-i_n} y_1^{i_n-i_{n-1}} \ldots y_n^{i_1} : 0 \le i_1 \le \cdots \le i_{2n-1} \le k\right\}$$

is a basis for $\mathbb{R}_k[x,y]$, the coordinates of $F \circ \sigma_t(x,y)$ in this basis are

$$\left\{t^{i_1+\cdots+i_{2n-1}} : 0 \le i_1 \le \cdots \le i_{2n-1} \le k\right\} = (1, t, \ldots, t^{d-1}) \in \mathbb{R}^d.$$

It is easy to see that there exists values $t_1, \ldots, t_d \in [0,1]$ such that the vectors $(1, t_l, \ldots, t_l^{d-1})$, with $l = 1, \ldots, d$, are linearity independent vectors in $\mathbb{R}^d$. hence the σ_t is a k-general family.

By combining theorem 3.2.3 and the above proposition, we obtain the following result. For a prove see [26] in the section 4.

Proposition 5.1.6. *Let $\gamma : \mathbb{R} \to M$ be a Ω-magnetic geodesic. Given $k, r \in \mathbb{N}$ and $K/2 < \tau < K(= K(c,\Omega))$, there exists an exact 2-form $d\xi$, with norm arbitrarily small in the C^r-topology, such that the one parameter family $t \mapsto d\mathcal{P}_t(\Omega + d\xi)$ defined in $[0,\tau]$ is k-general, where $\mathcal{P}_t(\Omega + d\eta) : \Sigma_0 \to \Sigma_t$ is the map Poincaré of orbit segment $\gamma|_{[0,t]}$ of $\Omega + d\eta$ in T^cM, for every $t \in [0,\tau]$.*

This is similar to Proposition 4.3 in [26].

5.2 Perturbation of the k-jet

In this section we are going to study how the k-jet of the map Poincaré is perturbed in relation to the perturbation of Ω. One important hypothesis is the concept of a one-parameter family k-general. All this is for the case that Ω is exact. See [26].

Given an exact 2-form $d\eta \in \Omega^2(M)$, we define the *Magnetic Lagrangian* $L_\eta : TM \to \mathbb{R}$ induced by $d\eta$ as

$$L_\eta(x, v) = \frac{1}{2} \langle v, v \rangle_x - \eta_x(v).$$

Note that L_η is convex and superlinear. Therefore, the Euler-Lagrangian flow of L_η is conjugated to a Hamiltonian flows in (T^*M, ω_0) via the Legendre transformation $\mathcal{L} : TM \to T^*M$, defined by

$$\mathcal{L}(x, v) = \left(x, \frac{\partial L_\eta}{\partial v}(x, v) \right).$$

The correspondent Hamiltonian $H_\eta : T^*M \to \mathbb{R}$, which we call the *Magnetic Hamiltonian*, is given by

$$H_\eta(x, p) = \frac{1}{2} |p + \eta|_x^2,$$

where $|\cdot|_x$ denote the norm induced by the metric g in T_x^*M. Let X_η be the Hamiltonian vector field of H_η and ψ_t^η be the flow generated by the field X_η.

Let $\gamma : [0, \tau] \to M$ be a segment of a $d\eta$-magnetic geodesic without self-intersection points, ($\tau < K = K(c, \Omega)$). Then

$$\Gamma(t) := \mathcal{L}(\gamma(t), \dot{\gamma}(t)) = \left(\gamma(t), \frac{\partial L_\eta}{\partial v}(\gamma(t), \dot{\gamma}(t)) \right) = (\gamma(t), p(t))$$

is a segment of an orbit of the flow ψ_t^η.

We will now choose a local coordinate system in a neighborhood of $\Gamma([0, \tau])$ in T^*M in order to describe the Hamiltonian H_η and its vector field. Since $\gamma([0, \tau]) \subset M$ has no self-intersection points, we can choose coordinates (x_0, x) with $x = (x_1, \ldots, x_n)$, in a neighborhood $U \subset M$ of $\gamma([0, \tau])$ such that $\gamma(t) = (t, 0)$ and $\{\partial/\partial x_0, \partial/\partial x_1, \ldots, \partial/\partial x_n\}|_{(t,0)}$ is a orthogonal basis for $T_{(t,0)}M$, for all $t \in [0, \tau]$. For each $(x_0, x) \in U$, let $\{dx_0, dx_1, \ldots, dx_n\} \subset T^*_{(x_0,x)}M$ be the dual basis for $\{\partial/\partial x_0, \partial/\partial x_1, \ldots, \partial/\partial x_n\} \subset T_{(x_0,x)}M$. Then, if $p \in T^*_{(x_0,x)}M$, we define y_i by $p = \sum_i y_i dx_i$ and we have a natural chart

$$(x_0, x, y_0, y) = (x_0, x_1 \ldots, x_n, y_0, y_1 \ldots, y_n)$$

of $\pi^{-1}(U) \subset T^*M$. In these coordinates we have $\omega_0 = dx_0 \wedge dy_0 + dx \wedge dy$,

$$H_\eta(x_0, x, y_0, y) = \frac{1}{2} \sum_{i,j=0}^{n} g^{ij}(x_0, x)(y_i + \eta_i)(y_j + \eta_j),$$

$$X_\eta = \sum_{i=0}^{n} \left(\frac{\partial H_\eta}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial H_\eta}{\partial x_i} \frac{\partial}{\partial y_i} \right)$$

where $[g^{ij}]_{0 \leq i,j \leq n}$ denotes the inverse matrix of the coefficients of the metric g with respect to the coordinate system (x_0, x) in U and

$$\Gamma(t) = (t, 0, \ldots, 0, 1 - \eta_0(t, 0), -\eta_1(t, 0), \ldots, -\eta_n(t, 0))$$

with the 1-form $\eta|_U = \eta_0 dx_0 + \eta_1 dx_1 + \ldots, \eta_n dx_n$.

Let $\delta : \mathbb{R} \to \mathbb{R}$ and $\beta : \mathbb{R}^n \to \mathbb{R}$ are smooth functions satisfying

1. $supp(\delta) \subset (0, \tau)$,
2. $supp(\beta) \subset B_\varepsilon(0)$, with $\varepsilon > 0$ sufficiently small, and
3. $j^{k+1}(\beta)(0) = x_1^{k+1}$, is a homogeneous of degree $k + 1$.

Given $k > 1$, we define the subset $\mathcal{F}^k \subset \overline{\Omega}^2(M)$ as the subset of the exact 2-form $d\xi$, with $supp(\xi) \subset U \subset M$ such that, in the local coordinates (x_0, x), the 1-form ξ is given by

$$\xi(x_0, x) = \xi_0(x_0, x) dx_0 = \delta(x_0)\beta(x) dx_0.$$

We will consider perturbations of the type $(\eta + \xi)$ for the magnetic Hamiltonian H_η.

Since $j_s^k(\beta)(0) = 0$, we have the $d\xi(\gamma(t)) = d\xi(t, 0) = 0$. Hence, $\gamma(t)$ is a magnetic geodesic for $d(\eta + \xi)$. In H_η, substituting η by $(\eta + \xi)$ in coordinates obtain

$$H_{(\eta+\xi)} = H_\eta + \xi_0 \frac{\partial H_\eta}{\partial y_0} + \frac{\xi_0^2}{2} g^{00}.$$

We denote

$$F_\xi = F(\xi, k) = \xi_0 \left(\frac{\partial H_\eta}{\partial y_0} \right) + \left(\frac{\xi_0^2}{2} \right) g^{00} = \xi_0 \left(\frac{\partial H_\eta}{\partial y_0} \right) + \mathcal{O}(x_1^{2(k+1)}).$$

Locally $H_{\eta+\xi} = H_\eta + F_\xi$, for all $d\eta \in \mathcal{F}^k$. Note that $j^k(F_\xi)(\Gamma(t)) = 0$ for all $t \in [0, \tau]$ and $\Gamma(0), \Gamma(\tau) \notin Supp(F_\xi)$.

We set $[0, \tau] \to \Lambda(t) = \{x_0 = t\}$ the family of local hypersurfaces in T^*M along $\Gamma(t)$. Since $\dot{\Gamma}(t) = X_\eta(\Gamma(t))$, then $\partial H_\eta/\partial y_0 \neq 0$. Therefore, $\Lambda(t)$ is a local transversal section at the point $\Gamma(t)$, for each $t \in [0, \tau]$. Let Y_ξ be the Hamiltonian vector field of the Hamiltonian $F_\xi : T^*M \to \mathbb{R}$, satisfies the conditions of general case (Proposition 5.1.2). We consider the map $S_t : (\Lambda(0), \Gamma(0)) \to (\Lambda(0), \Gamma(0))$ defined by $S_t = S_t(\xi) = S_t(Y_\xi) = P_t^{-1} \circ P'_t$ where P_t and P'_t denote the Poincaré maps for the Hamiltonian field X_η and $X_{(\eta+\xi)}$, with $d\xi \in \mathcal{F}^k$, respectively.

Since $\xi_0(x_0, x) = \delta(x_0)\beta(x)$ and $j^k(\beta)(0) = 0$ we have that, if

$$Y_\xi^1 := -\frac{\partial F_\xi}{\partial x_1}\frac{\partial}{\partial y_1},$$

and $Y_\xi^2 := Y_\xi - Y_\xi^1$, then $j^k(Y_\xi^2)(\Gamma(t)) = 0$ and $j^{k+1}(Y_\xi^1)(\Gamma(t)) \neq 0$ for all $t \in [0, \tau]$. Moreover, since $\Lambda(t) = \{x_0 = t\}$, the vector field $Y_\xi^1|_{\Lambda(t)}$ is tangent to $\Lambda(t)$, for all $t \in [0, \tau]$. Therefore we have the following of the proposition 5.1.2.

Corolary 5.2.1. *The $j^k(S_T(\xi))(\Gamma(0))$ is equal to the k-jet of the flow at time t associated with the non-autonomous vector field $P_t^*(Y_\xi^1|_{\Lambda(t)})$ at the point $\Gamma(0)$.*

This is similar to Remark 2 in [26]. Consider $\Sigma(t) := \Lambda(t) \cap H_\eta^{-1}(H_\eta(\Gamma(0))) \subset T^*M$ submanifold for all $t \in [0, \tau]$. Then the canonical 2-form ω_0 induces a symplectic structure on $\Sigma(t)$ and the restriction $P_t|_{\Sigma(0)} : \Sigma(0) \to \Sigma(t)$ is a symplectic map for all $t \in [0, \tau]$. Since $\Lambda(s) \cap H_{\eta+\xi}^{-1}(H_{\eta+\xi}(\Gamma(0))) = \Sigma(s)$ for $s = 0, \tau$, the restriction $P'_T|_{\Sigma(0)} : \Sigma(0) \to \Sigma(\tau)$ is also a symplectic map.

Note that $\partial H_\eta/\partial y_0(\Gamma(t)) = 1$ for all $t \in [0, \tau]$. For each $t \in [0, \tau]$ there is open set $U_t \subset \mathbb{R}^{2n}$ and a function $\alpha_t : U_t \to \mathbb{R}$ such that

$$\Sigma(t) = \{(t, x, \alpha_t(x, y), y) \in \Lambda(t) : (x, y) \in U_t\}.$$

Note that $\omega_0|_{\Sigma(t)} = dx \wedge dy$. For each $d\xi \in \mathcal{F}^k$ and $t \in [0, \tau]$, we define

$$F_{\xi,t}(x, y) = F_\xi|_{\Sigma(t)} = \xi_0(t, x)\frac{\partial H_\eta}{\partial y_0}(t, x, \alpha_t(x, y), y) + \mathcal{O}(x_1^{2(k+1)}),$$

and we denote by $Z_{\xi,t}$ the Hamiltonian field correspondent to the Hamiltonian $F_{\xi,t}$: $\Sigma(t) \to \mathbb{R}$. We have that

$$j^{k+1}(F_{\xi,t})(\Gamma(t)) = j^{k+1}(\xi_0)(t,0) = \delta(t)x_1^{k+1},$$

take $F(x,y) = x_1^{k+1}$.

Proposition 5.2.2. *The $j^k(S_t|_{\Sigma(0)})(\Gamma(0))$ is equal to the k-jet in $\Gamma(0)$ of the Hamiltonian flow at time t correspondent to the non-autonomous Hamiltonian $\delta(t)F \circ (P_t|_{\Sigma(0)})$ in $\Sigma(0)$.*

This is similar to Proposition 4.4 in [26]. We define the map

$$\begin{aligned} S_\tau^k \ : \ & \mathcal{F}^k \longrightarrow J_s^k(n), \\ & d\xi \longmapsto j^k(S_\tau(\xi)|_{\Sigma(0)})(\Gamma(0)). \end{aligned}$$

We have that $S_\tau^k(\mathcal{F}^k) \subset \ker(\pi_k)$, where the map $\pi_k : J_s^{k+1}(n) \to J_s^k(n)$ is the canonical projection.

Proposition 5.2.3. *Suppose that the one parameter family $t \in [0,\tau] \mapsto d_{\Gamma(0)}P_t|_{\Sigma(0)} \in Sp(n)$ is $(k+1)$-general for some $k > 1$. Then $S_T^k(\mathcal{F}^k)$ is an open subset of $\ker(\pi_k)$.*

The proof of these propositions are similar as in [26].

5.3 Proof of theorem 5.1.1

In this section we are going to give the proof of the theorem 5.1.1, which we do in two parts. The first part is for the exact case and in the second part we use the fact that locally a closed form is exact. See [26].

Let us first consider the exact case:

Let $\gamma : \mathbb{R} \to M$ be a $d\eta$-magnetic geodesic of period T_θ (where $\theta = (\gamma(0), \gamma'(0))$) and let $H_\eta : T^*M \to \mathbb{R}$ be the correspondent magnetic Hamiltonian. Since the number of self-intersection points is finite, we can choose $\tau \in (0, T_\theta]$, such that the segment $\gamma([0,\tau])$ does not contain self-intersection of the curve γ. Let U be a tubular neighbourhood of

$\gamma([0,\tau])$ in M, sufficiently small, such that $U \cap \gamma = \gamma([0,T])$. It follows that we can choose a local coordinates system (x_0, x) in U and a family of local transversal sections $\Sigma(t) \subset H_\eta^{-1}(H_\eta(\Gamma(0))) \subset T^*M$ as in the last section. By proposition 5.1.6, there exists an exact close 2-form $d\overline{\eta}$ arbitrarily C^r-close to $d\eta$, for $r > k$, such that the correspondent one parameter family $[0,\tau] \to d_{\Gamma(0)}P_t|_{\Sigma(0)}$ is l-general, for any $l = 2, \ldots, k+1$.

We set $\mathcal{F}^l = \mathcal{F}^l(U, \gamma, \overline{\eta}, \tau) \subset \overline{\Omega}^2(M)$, for $l = 2, \ldots, k+1$, as the last section, and

$$\mathcal{F} = \mathcal{F}(U, \gamma, \tau) = \left\{ d\xi \in \overline{\Omega}^2(M) : d\eta|_{\gamma([0,\tau])} \equiv 0 \text{ and } suppd\eta \subset U \right\}.$$

It is easy to see that $\mathcal{F}^l \subset \mathcal{F}$ for all $l = 1, \ldots, d$. We define the map

$$\begin{aligned} S \ : \ \mathcal{F} &\longrightarrow J_s^k(n), \\ d\xi &\longmapsto j^k(S_\tau(\xi))(\Gamma(0)). \end{aligned}$$

By theorem 3.2.3 and proposition 5.2.3, we have that S is an open map in a neighbourhood of $0 \in \mathcal{F}$. Since $j^k(P(d\eta, \theta, \Sigma)) \in \overline{Q}$, the openness of S in a neighbourhood of zero implies that there exists an exact 2-form $d\xi$ arbitrarily C^r-close to zero, with $r > k$, such that the $S(d\xi)$ is an element of the set $j^k(P(\theta, \Sigma, d\overline{\eta}))^{-1} \cdot Q$, then $j^k(P(\theta, \Sigma, d(\eta + \overline{\xi}))) \in Q$, this prove the theorem for the exact case.

We now consider the non-exact case:

Let Ω be a non-exact 2-form on M and let γ, τ, U and $\mathcal{F} = \mathcal{F}(U, \gamma, \tau)$ be as in the exact case. We can suppose without loss of generality that there is $\varepsilon > 0$, such that the segment $\gamma : [-\varepsilon, \tau + \varepsilon] \to M$ does not has self-intersection points. Let $N \subset M$ be an open tubular neighbourhood of $\gamma(-\varepsilon, \tau + \varepsilon)$ and by reducing U, if necessary, we can assume that $U \subset N$. Since N is a tubular neighbourhood of $\gamma(-\varepsilon, \tau + \varepsilon)$, we have $\Omega|_N = d\eta$, for some 1-form η in $N \subset M$. Hence, the restriction of the magnetic flow to $TN = T_N M \subset TM$ is an exact magnetic flow. Then it is conjugated to a Hamiltonian flow in (T^*N, ω_0) for the Hamiltonian $H_\eta : T^*N \to \mathbb{R}$, defined as before. Observe that, by definition of the perturbation space $\mathcal{F}$, if an exact 2-form $d\xi \in \mathcal{F}$ then ξ has compact support in $U \subset N$. Therefore, ξ can be extended to a global exact 1-form on M, still denoted by ξ, as zero outside N. Since the k-jet at one point of a map is totally determined by its restriction in any neighbourhood of that point, it is enough to prove the theorem for a segment $\gamma([0,\tau])$

in the open neighbourhood N. Therefore, the proof of the theorem is reduced to the already proven one, which is the exact case.

We conclude this chapter by saying that the following corollaries is obtained using the corollary 4.1.8 and the definition 4.1.1

Corolary 5.3.1. *Given an open and dense invariant subset $Q \subset J_s^k(n)$, the property:*

$$P_Q : \text{ the } k\text{-jet of the Poincaré map belong to } Q$$

is a C^r-generic property for magnetic flows, with $k < r \leq \infty$.

This is the desired generic property, since it is the property that other flows satisfy such as geodetic flows by [15], Hamiltonian flows by [45], Tonelli Hamiltonian flows by [6] and magnetic flows on surfaces by [26].

Chapter 6

Franks' Lemma and Positive Entropy

This last chapter is divided into two parts. In the first part we are going to study the version of *Franks's Lemma* in this context of magnetic flows in manifolds of any dimension. The original version of this lemma was made for diffeomorphisms in manifolds of any dimension and proved by Franks. This lemma has many applications, but one of the most important is made by Mañé [25] in the C^1-structurally stable. A long time later, Contreras and Paternain [9] proved the version of this lemma in the context of flows geodesic on surfaces, a few years later Contreras [8] proved it for any dimension. Then Miranda [27] proved this result in this context of magnetic flows but in surfaces. The idea here is to generalize this result.

Oliveira proved in [30] the Kupka-Smale's theorem in the context of Tonelli flows on surfaces. Then using the methods of geometric control theory Rifford and Ruggiero in [38] obtain a perturbative result which implies and generalizes the of Oliveira for manifolds of any dimension. This is the first time we see that the methods of geometric control theory make a big difference in the study of conservative systems. On the other hand, using these same methods, Lazrag [19] and [20] improves the proved of Franks' lemma for geodetic flows made by Contreras. However these methods were improved by Lazrag, Rifford and Ruggiero in [21] to obtain the version of Franks' lemma for geodetic flows. The difference with the work of Contreras [8] is that his Franks' lemma works on an open and dense set of Riemannian metrics. On the contrary, Lazrag, Rifford and Ruggiero [21] obtain the result on all the Riemannian metrics, besides for Contreras it is not difficulty, since its main result is that generically every Riemannian metric has geodesic flow with positive entropy.

The second part of this chapter, we are going to study a property of dichotomy for the magnetic flows. Basically this says that there exists a set of open and dense magnetic flows such that either the topological entropy is positive or the closed set of closed orbits is a hyperbolic set. This result is another way of writing Miranda's work on surfaces [27]. On the other hand, if we contrast with geodetic flows, Contreras [8] has the version of this result and his method is to split the prove into two parts. The first works with the geodesic flows that do not have periodic hyperbolic orbit, where he gets to perturb and find a basic non-trivial hyperbolic set, thus a type Smale's horseshoe which implies positive topological entropy. In the second part he studies geodetic flows with an infinite number of periodic orbits, this is due to the Kupka-Smale theorem for geodetic flows. Using Rademacher's theorem [37], in order to perturb and still obtain an infinite quantity of closed orbits and passing through the *Smale's spectral decomposition theorem for flows* he obtains a basic non-trivial hyperbolic set, then also positive topological entropy. A major difference in our context is, that there is no version of Rademacher's theorem in magnetic flows, this is the main reason.

6.1 Franks' lemma for magnetic flows

In this section we are going to prove the Franks' lemma in this context of magnetic flows on manifolds of any dimension. As we explained earlier, the importance of geometric control theory, in particular the works of Lazrag, Rifford and Ruggiero [21]. The idea is to first obtain the perturbative result, which we already get, which is the theorem 3.2.3, where we use section 2.6.

Let $c > 0$, Ω be a smooth closed 2-form so M. We set $\theta = (x, v) \in T^cM$, with $\gamma : [0, \tau] \to M$ magnetic geodesic such that $\gamma(0) = x$ and $\dot{\gamma}(0) = v$, where $0 < \tau < K(c, \Omega)$.

Considering the definitions of $\mathcal{F}$ and $S_{\tau,\theta}$ of the section 3.2, under these conditions we can use the theorem 3.2.3. In this case, there is $K, R > 0$ such that if $r \in (0, R)$

$$B_{Kr}(S_{\tau,\theta}(0)) \cap Sp(n) \subset S_{\tau,\theta}\left(B_r^{C^1}(0) \cap \mathcal{F}\right).$$

This will prove the Franks' lemma for magnetic flows in any dimension.

Suppose that $\theta_t = (\gamma(t), \gamma'(t)) \subset T^cM$ is a closed orbit and let $T_\theta > 0$ be its minimal period. By Lemma 2.4.5, $K := K(c, \Omega) < T_\theta$ and the number of self-intersection points of γ is finite. We fix $\tau \in (K/2, K]$, such that $T_\theta = l\tau$, with $l \in \mathbb{N}$, denote $\gamma_i(t) = \gamma(t + i\tau)$. There exists $U_i \subset M$ open and disjoint sets for $0 \leq i \leq l-1$, such that

$$U_i \cap \gamma((0,\tau)) \subset \gamma((0,\tau)), \text{ and } U_i \cap U_j = \emptyset, \text{ for every } i \neq j.$$

For $\mathbf{U} = \bigcup_{i=0}^{l-1} U_i$, we consider the map

$$\begin{aligned} S_\theta \ : \ \mathcal{F} &\longrightarrow \prod_{i=0}^{l-1} Sp(n), \\ d\eta &\longmapsto d_\theta \mathcal{P}(\Omega + d\eta)(T_\theta) = \prod_{i=0}^{l-1} d_{\theta_{i\tau}} \mathcal{P}_i(\Omega + d\eta), \end{aligned}$$

where $\mathcal{P}_i$ is the Poincaré map from $\Sigma_{i\tau}$ to $\Sigma_{(i+1)\tau}$. Applying l times Theorem 3.2.3, we prove the following corollary.

Corolary 6.1.1. *(Franks' Lemma) Let $c > 0$, Ω be a smooth closed 2-form on M and θ in T^cM such that $\phi_t^\Omega(\theta)$ be a closed orbit in T^cM with minimal period T_θ. Let l and S_θ be defined as above. If $\mathcal{U}$ an open neighborhood of Ω in the C^1 topology, then there exist $r > 0$ that depend of $c > 0$, Ω and $\mathcal{U}$ such that the the image of the set $\mathcal{U} \cap \mathcal{F}$ under the map S_θ contains a product of balls of radius r center at $(S_{0,\theta}(\Omega), \ldots, S_{l-1,\theta}(\Omega)) \in \prod_{i=0}^{l-1} Sp(n)$.*

On the other hand, if we consider $\mathbf{F} := \{\alpha_1, \ldots, \alpha_N\}$ be a finite set of Ω-magnetic geodesic that are transverse to γ. We have the following result.

Proposition 6.1.2. *For any tubular neighborhood U of γ and any set $\mathbf{F}$ of transverse Ω-magnetic geodesic, the support of the C^1 perturbation can be contained in $U \setminus V$ for some neighborhood V of the transverse Ω-magnetic geodesic $\mathbf{F}$.*

The Franks' lemma will be useful for the study of the magnetic flows with infinity many closed orbits in sn energy level.

6.2 Star Magnetic flows

In this section, first we are going to state the main result of this last chapter which is the property of the magnetic flows defined in manifolds of any dimension. The proof of this result will be made in two parts, the first in this section and the second in last.

Remember the section 2.7 for definitions. Thus our main result is the following.

Theorem 6.2.1. *There exists an open and dense set in the C^1-topology of smooth closed 2-form in M whose magnetic flow have positive topological entropy or have a finite number of periodic orbits.*

If the version of Rademacher's theorem exist in our context, we could improve our main result and thus obtain a version of Contreras's theorem in our context. Although we know that at very high energy levels the exact magnetic flow can be seen as geodetic flow and thus we can improve our result, only for high energy levels in exact magnetic flows.

Let $c > 0$ and Ω be a smooth closed 2-form in M, we define $\mathcal{P}(\Omega, c)$ be the set of all closed orbit of ϕ_t^Ω in T^cM and $Per(\Omega, c)$ be the union of $\alpha(\mathbb{R})$ for all $\alpha \in \mathcal{P}(\Omega, c)$. We now denote by $\mathcal{H}^1(M, c)$ the set of smooth closed 2-form Ω in M such that α is hyperbolic closed orbit, for all $\alpha \in \mathcal{P}(\Omega, c)$, this set is endowed with the C^1-topology. Let $h \in H^2(M, \mathbb{R})$, consider

$$\mathcal{F}_h^1(M, c) := int_{C^1}\mathcal{H}^1(M, c) \cap \overline{\Omega}_h^2(M),$$

and $\mathcal{F}^1(M, c) := \bigcup\{\mathcal{F}_h^1(M, c) : h \in H^2(M, \mathbb{R})\}$.

Definition 6.2.2. We say that the magnetic flow ϕ_t^Ω (or Ω) is *star* if $\Omega \in \mathcal{F}^1(M, c)$.

Note that $\overline{Per(\Omega, c)} \subset T^cM$ is a compact and invariant subset.

Theorem 6.2.3. *If Ω is star, then $\overline{Per(\Omega, c)} \subset T^cM$ is a hyperbolic set.*

In this section we are going to prove the theorem 6.2.3. In fact we are going to prove the local version of this result, where we will use the Franks' lemma and the stably hyperbolic of the symplectic linear maps.

6.2.1 Symplectic hyperbolic stability

In this section we will study the hyperbolic stability of a family of periodic sequences of symplectic linear maps. Our main reference here is Contreras [9], section 8.

We say that a linear map $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is *hyperbolic* if there exists a splitting $\mathbb{R}^{2n} = E^s \oplus E^u$ and an iterate $L \in \mathbb{N}$ such that $T(E^s) = E^s$, $T(E^u) = E^u$ and

$$\left\|T^L|_{E^s}\right\| < \frac{1}{2} \text{ and } \left\|(T|_{E^u})^{-L}\right\| < \frac{1}{2}.$$

The subspaces E^s and E^u are called the *stable subspace* and *unstable subspace* respect of T.

We say that a sequence $\tau : \mathbb{Z} \to Sp(n)$ is *periodic* if there exists $l \geq 1$ such that $\tau_{i+l} = \tau_i$ for all $i \in \mathbb{Z}$. We say that a periodic sequence τ is *hyperbolic* if the linear map $\prod_{i=1}^{l} \tau_i$ is hyperbolic. In this case the stable and unstable subspaces of $\prod_{i=0}^{l-1} \tau_{i+j}$ are denoted by $E_j^s(\tau)$ and $E_j^u(\tau)$ respectively.

We say that a family $\tau = \{\tau^\alpha\}_{\alpha\in\mathcal{A}}$ of sequences in $Sp(n)$ is *bounded* if there exists $R > 0$ such that $\|\tau_i^\alpha\| < R$ for all $\alpha \in \mathcal{A}$ and $i \in \mathbb{Z}$. Given two families of periodic sequences in $Sp(n)$, $\tau = \{\tau^\alpha\}_{\alpha\in\mathcal{A}}$ and $\eta = \{\eta^\alpha\}_{\alpha\in\mathcal{A}}$, we say that they are *periodically equivalent* if they have the same indexing set $\mathcal{A}$ and for all $\alpha \in \mathcal{A}$ the periods of τ^α and η^α coincide. Given two periodically equivalent families of periodic sequences in $Sp(n)$, $\tau = \{\tau^\alpha\}_{\alpha\in\mathcal{A}}$ and $\eta = \{\eta^\alpha\}_{\alpha\in\mathcal{A}}$, define

$$d(\tau,\eta) := \sup\left\{\|\tau_i^\alpha - \eta_i^\alpha\| : \alpha \in \mathcal{A}, i \in \mathbb{Z}\right\}.$$

We say that a family $\tau = \{\tau^\alpha\}_{\alpha\in\mathcal{A}}$ is *hyperbolic* if for all $\alpha \in \mathcal{A}$, the periodic sequence τ^α is hyperbolic. We say that a hyperbolic periodic family τ is *stably hyperbolic* if there exists $\varepsilon > 0$ such that any periodically equivalent family η satisfying $d(\tau,\eta) < \varepsilon$ is also hyperbolic.

Finally, we say that a family of periodic sequences τ is *uniformly hyperbolic* if there exist $K > 0$, $0 < \lambda < 1$ and invariant subspaces $E_i^s(\tau^\alpha), E_i^u(\tau^\alpha), \alpha \in \mathcal{A}, i \in \mathbb{Z}$, such that

$$\left\|\prod_{i=0}^{l-1} \tau_{i+j}^\alpha|_{E_j^s(\tau^\alpha)}\right\| < K\lambda^l \text{ and } \left\|\left(\prod_{i=0}^{l-1} \tau_{i+j}^\alpha|_{E_j^u(\tau^\alpha)}\right)^{-1}\right\| < K\lambda^l,$$

for all $\alpha \in \mathcal{A}$, $j \in \mathbb{Z}$ and $l \in \mathbb{N}$. Observe that in this case the sequence τ is hyperbolic and the subspaces $E_i^s(\tau^\alpha)$, $E_i^u(\tau^\alpha)$ necessarily coincide with the stable and unstable subspaces of the map $\prod_{j=0}^{l-1} \tau_{i+j}^\alpha$.

We are now ready to state the following result.

Theorem 6.2.4. *If τ^α is a stably hyperbolic family of periodic sequences of bounded symplectic linear maps then it is uniformly hyperbolic.*

For a prove see theorem 8.1 in [8]

6.2.2 Proof of theorem 6.2.3

We will state and proved the local version of the theorem 6.2.3.

Given $c > 0$, $U \subset T^cM$ an open set and Ω be a smooth closed 2-form in M. Let $\mathcal{P}(\Omega, c, U)$ be the set of closed orbits of ϕ_t^Ω completely contained in U and $Per(\Omega, c, U)$ be the union of $\alpha(\mathbb{R})$ for all $\alpha \in \mathcal{P}(\Omega, c, U)$. We denote by $\mathcal{H}^1(U, c)$ the set of smooth closed 2-forms Ω on M such that α is a hyperbolic closed orbits, for all $\alpha \in \mathcal{P}(\Omega, c, U)$, this set is endowed with the C^1-topology. Let $h \in H^2(M, \mathbb{R})$, consider

$$\mathcal{F}_h^1(U, c) := int_{C^1}\mathcal{H}^1(U, c) \cap \overline{\Omega}_h^2(M),$$

and $\mathcal{F}^1(U, c) := \bigcup\{\mathcal{F}_h^1(U, c) : h \in H^2(M, \mathbb{R})\}$.

Definition 6.2.5. We say that Ω is *star* in U if $\Omega \in \mathcal{F}^1(U, c)$.

The following proposition is a local version that implies theorem 6.2.3

Proposition 6.2.6. *If Ω is star in U, then $\overline{Per(\Omega, c, U)} \subset T^cM$ is a hyperbolic set.*

Proof. There exist $h \in H^2(M, \mathbb{R})$ such that $\Omega \in \mathcal{F}_h^1(U, c)$. Let $K = K(c, \Omega)$ the magnetic injectivity radius. For each $\alpha \in \mathcal{A} := \mathcal{P}(\Omega, c, U)$, exist $\theta = (x, v) \in T^cM$ such that $\alpha(t) = \phi_t^\Omega(\theta) = (\gamma(t), \gamma'(t)) \in T^cM$, for every $t \in \mathbb{R}$. Let T_α be the minimal periodic of α and $l = l(\alpha, \Omega)$ in $\mathbb{N}$ such that $T_\alpha = l\tau$ for some $\tau \in (K/2, K]$. Let for each $i = 0, \ldots, l-1$

$$\mathcal{N}(i, \alpha) := \left\{\xi \in T_{\alpha(i\tau)}T^cM : \langle d\pi(\xi), \gamma'(i\tau)\rangle = 0\right\}.$$

Note that $T_{\alpha(i\tau)}T^cM = \mathcal{N}(i,\alpha) \oplus \langle X^{\Omega}(\alpha(i\tau)) \rangle$ and the restriction of the twist symplectic form ω_Ω in $\mathcal{N}(i,\alpha)$ is non-degenerate. Let $\Sigma_i \subset TU$ be the local transversal section at $\alpha(i\tau)$ such that $T_{\alpha(i\tau)}\Sigma_i = \mathcal{N}(i,\alpha)$. Let $\tau^\alpha : \mathbb{Z} \to Sp(n)$ be the periodic sequence of period l such that $\tau_i^\alpha = dP(\Omega,\theta,\Sigma_i,\Sigma_{i+1}) : \mathcal{N}(i,\alpha) \to \mathcal{N}(i+1,\alpha)$ the linearized Poincaré maps. Note that $\tau_i^\alpha = S^{\Omega}_{\tau,\theta_{i\tau}}(0)$ as in theorem 3.2.3.

Lemma 6.2.7. *The family $\tau = \{\tau^\alpha\}_{\alpha\in\mathcal{A}}$ is stably hyperbolic.*

Proof. Since $\Omega \in \mathcal{F}_h^1(U,c)$, there exist $\mathcal{U} \subset \overline{\Omega}_h^2(M)$ a C^1-neighbourhood of Ω such that, if $\varpi \in \mathcal{U}$, then β is a closed orbit hyperbolic, for every $\beta \in \mathcal{P}(\varpi,c)$. By the hyperbolic analytic continuation, $\mathcal{A}$ and $\mathcal{P}(\varpi,c)$ are bijective, for every $\varpi \in \mathcal{U}$. Let $\alpha \in \mathcal{A}$, then $\beta(\alpha) \in \mathcal{P}(\varpi,c)$ intersects the sections Σ_i, $i = 1,\ldots,l$. Therefore, we can cut $\beta(\alpha)$ into the same number if segments as α. So for each $\varpi \in \mathcal{U}$ and $\beta(\alpha)$ we can apply all before, then we can say that $\tau = \tau(\varpi)$ is hyperbolic.

We suppose that $\{\tau(\varpi)\}_{\varpi\in\mathcal{U}}$ is not stably hyperbolic, then there is a periodically equivalent family η with $d(\tau,\eta)$ arbitrarily small which is not hyperbolic. Then there exists α in $\mathcal{A}$ and a sequence of linear symplectic maps $\eta_i^\alpha : \mathcal{N}(i,c) \to \mathcal{N}(i+1,c)$ such that τ_i^α and η_t^α are closed arbitrarily and $\prod_{i=1}^{l} \eta_i^\alpha$ is not hyperbolic. We now will use the corollary 6.1.1. Note that the perturbation space in theorem 3.2.3 preserves α. By corollary 6.1.1 there exist a 2-form ϖ in $\mathcal{U}$ such that $\alpha \in \mathcal{P}(\varpi,c)$ and $\eta_i^\alpha = S_\theta^\varpi(0)$. Since $d_\theta P(\varpi,\theta_\alpha,\Sigma_0,\Sigma_0) = \prod_{i=1}^{l} S_{i,\theta}(\varpi) = \prod_{i=1}^{l} \eta_i^\alpha$, then α is not hyperbolic for the magnetic flow of ϖ. This contradicts the choice of $\mathcal{U}$. □

Then, from Theorem 6.2.4 we obtain a hyperbolic splitting on $\mathcal{P}(\Omega,c,U)$. The Hyperbolicity condition implies the continuity of the splitting in $Per(\Omega,c,U)$, see for example the proposition 6.4.4 in [13] for diffeomorphisms. Then the splitting extends continuously to the closure $\overline{Per(\Omega,c,U)}$ and the extension is also a hyperbolic set. □

Thus we can conclude that, $\overline{Per(\Omega,c)}$ is a hyperbolic set, for every Ω star in M. Now we will study the case when Ω is not star.

6.3 Elliptic closed magnetic geodesic

In this section we will prove the result main theorem 6.2.1. We really need to study non-star magnetic flows. For that, first we can obtain the following, using theorem 4.1.2 and the corollary 5.3.1.

Theorem 6.3.1. *Let $Q \in J^k_s(n)$ be open, dense and invariant. Then the following property:*

1. *all closed orbits are hyperbolic or elliptic,*
2. *all k-jet of the Poincaré map belong to Q*
3. *all heteroclinic points are transversal*

are C^r-generic for magnetic flows, with $k \leq r \leq \infty$.

Since countable intersections of residual subsets are residual, in Theorem 6.3.1 we can replace Q by a residual invariant subset in $J^k_s(n)$.

The idea for the demonstration is to find a suitable set Q such that given an Ω by the theorem 6.3.1 there exists a perturbation of Ω such that it has an elliptic closed orbit. Then by Le Calvez [22] and Contreras [8] we can find, near this elliptical orbit, another hyperbolic orbit with a transversal homoclinic point.

6.3.1 Symplectic twist maps

In this section we are going to study the twist property of the Poincaré map.

Let $c > 0$ and Ω be a smooth closed 2-form in M, if $\theta \in T^cM$ such that $\theta_t = \phi^\Omega_t(\theta)$ is a closed orbit in T^cM. Let $\mathcal{P} := P(\Omega, \theta, \Sigma)$ the Poincaré map. Remember the following defines. We say that θ_t is

1. *degenerate* if $d\mathcal{P}$ has an eigenvalue which is a root of unity.
2. *hyperbolic* if $d\mathcal{P}$ has not eigenvalue of modulus 1.

3. *elliptic* if is non degenerate and non hyperbolic.

4. $\mathfrak{c}$-*elliptic* if is elliptic and $d\mathcal{P}$ has precisely $2\mathfrak{c}$ eigenvalues of modulus 1.

Suppose that θ is a $\mathfrak{c}$-elliptic periodic point, $\mathfrak{c} \leq n$. Let $T_\theta\Sigma = E^s \oplus E^c \oplus E^u$ be the decomposition into the stable, center and unstable subspaces for $d\mathcal{P}$. This is, E^s, E^c and E^u are invariant under $d\mathcal{P}$ and $d\mathcal{P}|_{E^s}$ has only eigenvalues ρ of modulus $|\rho| < 1$, $d\mathcal{P}|_{E^c}$ has only eigenvalues ρ of modulus $|\rho| = 1$ and $d\mathcal{P}|_{E^u}$ has only eigenvalues ρ of modulus $|\rho| > 1$. Then there are local embeddings $W^s : (\mathbb{R}^{n-\mathfrak{c}}, 0) \to (\Sigma, \theta)$, $W^c : (\mathbb{R}^{2\mathfrak{c}}, 0) \to (\Sigma, \theta)$ and $W^u : (\mathbb{R}^{n-\mathfrak{c}}, 0) \to (\Sigma, \theta)$, such that $T_\theta W^\sigma = E^\sigma$, $\sigma \in \{s, c, u\}$, which are locally invariant under $\mathcal{P}$. They are called stable, center and unstable manifolds for (Σ, θ). The stable and unstable manifolds are unique, but the center manifold may not be unique. If $\mathcal{P}$ is of class C^k (resp. C^1) then W^s, W^u, are C^k (resp. C^1). If $\mathcal{P}$ is of class C^k (resp. C^1) then W^c can be chosen C^k (resp. C^r , with r arbitrarily large) on a sufficiently small neighborhood of θ. The submanifolds W^s, W^u are isotropic with respect to the twisted symplectic structure ω_Ω (i.e. $\omega_\Omega|_{E^s} \equiv 0$ and $\omega_\Omega|_{E^u} \equiv 0$) because $\mathcal{P}$ preserves ω_Ω and $d\mathcal{P}$ (resp. $d\mathcal{P}^{-1}$) asymptotically contracts tangent vectors in W^s (resp. W^u). The restriction $\omega_\Omega|_{E^c}$ is non degenerate (see [41]) and hence $\mathcal{P}|_{W^c}$ is a symplectic map on a sufficiently small neighborhood of θ.

Let $\rho_1, \ldots, \rho_\mathfrak{c}, \overline{\rho}_1, \ldots, \overline{\rho}_\mathfrak{c}$ be the eigenvalues of $\mathcal{P}$ with modulus 1.

Definition 6.3.2. We say that θ is 4-elementary if

$$\prod_{i=1}^{\mathfrak{c}} \rho_i^{\nu_i} \neq 1 \text{ whenerver } 1 \leq \sum_{i=1}^{\mathfrak{c}} |\nu_i| \leq 4.$$

In this case there are symplectic coordinates $(x_1, \ldots, x_\mathfrak{c}, y_1, \ldots, y_\mathfrak{c})$ in W^c such that $\omega_\Omega|_{W^c} = \sum_{i=1}^{\mathfrak{c}} dy_i \wedge dx_i$ and $\mathcal{P}|_{W^c}$ is written in the *Birkhoff normal form* $\mathcal{P}(x, y) = (X, Y)$, where

$$Z_k = e^{2\pi i \vartheta_k} z_k + f_k(z), \text{ where } \vartheta_k = a_k + \sum_{l=1}^{\mathfrak{c}} \beta_{kl} |z_l|^2,$$

$z = x + iy$, $Z = X + iY$, $\rho_k = e^{2\pi i a_k}$ and $f(z) = f(x, y)$ has vanishing derivatives up to order 3 at the origin, note that $a_k \in [0, 1)$ is not rational.

Definition 6.3.3. We say that θ is weakly monotonous if the matrix $\beta = (\beta_{kl})$ is non-singular.

The property $\det(\beta) \neq 0$ is independent of the particular choice of normal form. In these coordinates, the matrix β can be detected from the 3-jet of $\mathcal{P}$ at $\theta = (0,0)$ and it can be seen that the property of be 4-elementary and weakly monotonous is open and dense in the jet space $J^3_s(\mathfrak{c})$.

Consider the following maps

$$\begin{tikzcd}
(x,y) & & (\vartheta,\varrho) & & (\vartheta,\varrho^2/\varepsilon)=(\vartheta,r) \\[-20pt]
\mathbb{D}^* \arrow[rr, "\mathcal{A}"] \arrow[d, "f_c"] & & \mathbb{T}^{\mathfrak{c}} \times \mathbb{R}^{\mathfrak{c}}_+ \arrow[rr, "\mathcal{B}"] & & \mathbb{T}^{\mathfrak{c}} \times \mathbb{R}^{\mathfrak{c}}_+ \arrow[d, "F_\varepsilon"] \\
\mathbb{D}^* \arrow[rr, "\mathcal{A}"'] & & \mathbb{T}^{\mathfrak{c}} \times \mathbb{R}^{\mathfrak{c}}_+ \arrow[rr, "\mathcal{B}"'] & & \mathbb{T}^{\mathfrak{c}} \times \mathbb{R}^{\mathfrak{c}}_+
\end{tikzcd}$$

where $\mathbb{D} = \{(x,y) \in \mathbb{R}^{\mathfrak{c}} \times \mathbb{R}^{\mathfrak{c}} : |x|^2 + |y|^2 < 1\}$, $\mathbb{D}^* = \mathbb{D} \setminus \{(0,0)\}$, $f_c = \mathcal{P}|_{W^c}$ in the above coordinates, $\mathbb{T}^{\mathfrak{c}} = \mathbb{R}^{\mathfrak{c}}/\mathbb{Z}^{\mathfrak{c}}$ and $\mathcal{A}^{-1}$ is given by $x_k = \varrho_k \cos(2\pi\vartheta_k)$, $y_k = \varrho_k \sin(2\pi\vartheta_k)$. Since the coordinates in Birkhoff normal form are symplectic, the map f_c preserves the form $\omega_\Omega = dx \wedge dy$. Let $\mathcal{C} = \mathcal{B} \circ \mathcal{A} : \mathbb{D}^* \to \mathbb{T}^{\mathfrak{c}} \times \mathbb{R}^{\mathfrak{c}}_+$ be given by $\mathcal{C}(x,y) = (\vartheta, r)$, $r_k = \varrho_k^2/\varepsilon$. Then $\mathcal{C}^*(rd\vartheta) = \dfrac{1}{2\pi\varepsilon}(xdy - ydx) =: \lambda_\varepsilon$. Since $\mathbb{D}$ is simply connected, $f_c^*(\lambda_\varepsilon) - \lambda_\varepsilon$ is exact. Therefore $F_\varepsilon^*(rd\vartheta) - rd\vartheta$ is exact.

Let $\mathcal{F}_\varepsilon(\vartheta, r) := (\vartheta + a + \varepsilon\beta r, r)$ be the symplectic diffeomorphism given by the first term in Birkhoff normal form in the coordinates (ϑ, r). Its N-th iterate is given by $\mathcal{F}_\varepsilon^N(\vartheta, r) = (\vartheta + Na + \varepsilon N\beta r, r)$. This is a totally integrable (see [5]) weakly monotonous (i.e. $\det(\varepsilon N\beta) \neq 0$) twist map of $\mathbb{T}^{\mathfrak{c}} \times \mathbb{R}^{\mathfrak{c}}_+$. Let $\mathbb{B}_\delta$ open ball in $\mathbb{R}^{\mathfrak{c}}_+$ with center point $\dfrac{1}{2\mathfrak{c}}(1, \ldots, 1)$ and radius δ. In the Moser's Appendix in [14], J. Moser proves that given $\varsigma > 0$ there exist $\delta > 0$, $N \in \mathbb{N}$ and $\varepsilon > 0$ such that

1. $\|F_\varepsilon^N - \mathcal{F}_\varepsilon^N\|_{C^1} < \varsigma$ in $\mathbb{T}^{\mathfrak{c}} \times \mathbb{B}_\delta$ and
2. there exists a torus $\mathcal{T}^{\mathfrak{c}}$ radially transformed by F_ε^N in $\mathbb{T}^{\mathfrak{c}} \times \mathbb{B}_\delta$, i.e. $\mathcal{T}^{\mathfrak{c}} = \{(\vartheta, r(\vartheta)) : \vartheta \in \mathbb{T}^{\mathfrak{c}}\} \subset \mathbb{T}^{\mathfrak{c}} \times \mathbb{B}_\delta$ such that $F_\varepsilon^N(\vartheta, r(\vartheta)) = (\vartheta, R(\vartheta))$ for some $R : \mathbb{T}^{\mathfrak{c}} \to \mathbb{R}^{\mathfrak{c}}_+$

Let $\mathfrak{f}_N$ be a generating function for F_ε^N, i.e. a function $\mathfrak{f}_N : \mathbb{R}^{\mathfrak{c}} \times \mathbb{B}_\delta \to \mathbb{R}$ such that

$d\mathfrak{f}_N = \left(F_\varepsilon^N\right)^* (rd\vartheta) - rd\vartheta$. On the radially transformed torus $\mathcal{T}^\mathfrak{c}$ we have that

$$d\mathfrak{f}_N(\vartheta, r(\vartheta)) = (R(\vartheta) - r(\vartheta))d\vartheta$$

Then critical points of $d\mathfrak{f}_N|_{\mathcal{T}^\mathfrak{c}}$ correspond to fixed points of F_ε^N in $\mathbb{T}^\mathfrak{c}$.

Let $\mathcal{Q} \subset J_s^3(n)$ be the set of 3-jets of C^3 symplectic automorphisms T of $\mathbb{R}^n \times \mathbb{R}^n$ which fix the origin and are such that

1. The eigenvalues of $d_0 T$ are all different.
2. the eigenvalues of modulus 1 satisfy 4-elementary.
3. The coefficient of the Birkhoff normal form satisfy the weakly monotonous condition.

It is well known that $\mathcal{Q}$ is residual and invariant in all $J_s^k(n)$, $k \geq 3$, thus we can use $\mathcal{Q}$ in theorem 6.3.1.

6.3.2 Proof of theorem 6.2.1

In this last section we are going to obtain positive topological entropy from non-star magnetic flows.

Let $c > 0$ and Ω be a smooth closed 2-form in M non-star, by theorem 6.3.1, we have that there exist $\tilde{\Omega}$ arbitrarily C^r-near to Ω ($r \geq 4$) such that the magnetic flow $\tilde{\phi}_t$ of $\tilde{\Omega}$ in $T^c M$ have a elliptic closed orbit θ_t and satisfying (2) and (3) of theorem 6.3.1.

We can suppose that θ is a $\mathfrak{c}$-elliptic periodic point with $\mathfrak{c} \leq n$, as stated in the before section, Moser proves that there exist a subset $\mathbb{T}^\mathfrak{c} \times \mathbb{B}_\delta$ neighborhood of θ and iterate $N \in \mathbb{N}$ such that the N-th iterate F_ε^N of the Poincaré map $F_\varepsilon = \mathcal{P}|_{W^c}$ is a weakly monotonous twist map with fixed points which is C^1 near to a totally integrable twist map $\mathcal{F}_\varepsilon^N$ in these conditions, we can to use the theorem 4.1 in [8], thus we obtain that F has a 1-elliptic periodic point $\tilde{\theta}$ near θ. Since the central manifold is normally hyperbolic, by lemme 8.6 in [5], the periodic point $\tilde{\theta}$ will also be 1-elliptic for the whole Poincaré map $\mathcal{P}$.

As the central manifold W^c have dimension 2, we can to use the following result

Proposition 6.3.4. *(Le Calvez [22]) Let f be a diffeomorphism of the annulus $\mathbb{R} \times S^1$ such that it is a twist map, it is area preserving, the form $f^*(rd\vartheta) - rd\vartheta$ is exact and:*

1. *If x is a periodic point for f and q is its least period, the eigenvalues of $d_x f^q$ are not roots of unity.*
2. *The stable and unstable manifolds of hyperbolic periodic orbits of f intersect transversally.*

Then f has periodic orbits with homoclinic points.

Thus we have that $\mathcal{P}|_{W^c}$ have hyperbolic orbits with homoclinic points. This hyperbolic periodic orbit will be hyperbolic in the Poincaré section (see [5]). A homoclinic point in the central manifold is also a homoclinic point in the Poincaré section, and it must be transversal by the Kupka-Smale condition. Since the fact of have homoclinic point is stable, so there exists $\tilde{\mathcal{U}}$ neighborhood of $\tilde{\Omega}$ such that if $\kappa \in \tilde{\mathcal{U}}$ then ϕ_t^κ has a closed orbit in T^cM where those Poincaré map $\mathcal{P}_\kappa$ has a homoclinic point, then ϕ_t^κ has positive topology entropy $h_{top}(\kappa, c) > 0$.

Using the following result

Proposition 6.3.5. *Let $A \subset X$ such that for every $x \in X$ and $\varepsilon > 0$ there exists $a \in A$ and V neighbourhood of a with $d(a, x) < \varepsilon$, then A contain a subset open and dense.*

We have that for $c > 0$ there exist a subset open and dense in $\overline{\Omega}^2(M) \setminus \mathcal{F}^1(M, c)$ such that if Ω belong such subset, ϕ_t^Ω in T^cM have positive entropy topological. Thus, for all $c > 0$, there exist a subset C^1-open and dense in $\overline{\Omega}^2(M)$ such that if Ω belong such subset, ϕ_t^Ω in T^cM either have positive topological entropy or the closed set of periodic orbits is a hyperbolic set. Finally we can take the union respect to $c > 0$ and obtain our result.

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