Control and Stabilization of the Gear–Grimshaw system and the generalized KdV-Burgers equation

by

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To Sindy, my wife,
whose love, confidence and patience
are a constant source of inspiration and encouragement.

and

To the memory of my old friend and second father, Gabriel Gil Carvajal (1948-2016),
who always supported me and believed I could do great things.
I submitted my thesis two months after he passed away.
My dedication to him is a small way of saying
thanks and I miss you very much.
A wonderful human being, smart and generous person.

Abstract

In this thesis, we prove some well-posedness, controllability and stabilization results for a class for dispersive partial differential equations. First, we consider a coupled system of two Korteweg-de Vries equations (KdV), the so-called Gear-Grimshaw system, posed on a bounded domain. We obtain the exact controllability provided by a suitable configuration of the controls position in the boundary. For two kind of boundary conditions, namely, Neumann and Dirichlet-Neumann, we prove the existence of the so-called *critical length phenomenon* (see for instance, [78]). Next, we prove the well-posedness, exponential stabilization and controllability of the Korteweg-de Vries Burgers equation posed on the whole real line. Initially, we prove that, under the effect of a damping term, the solution of the generalized Korteweg-de Vries Burgers equation decays exponentially when the exponent p in the nonlinear term ranges over the interval $[1, 5)$. We also give and answer to the exact controllability problem in the energy space for solutions in $L^2_{loc}(\mathbb{R}^2)$.

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I wish to express my gratitude to many people and institutions that made it possible for me to pursue a life of studies.

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CHAPTER

ONE

INTRODUCTION

We begin this section with a review of some general results of control theory concerning the relations among of controllability, observability and stabilization of dynamical systems. One can refer to [31] and [72] for an introduction and fruitful results on the subject.

1.1 Classical Control Theory

A control system is a dynamical system on which one can act by using suitable controls. There are a lot of problems that appear when studying a control system. But the most common ones are the controllability problem and the stabilization problem.

1.1.1 Controllability and Observability

The controllability problem, roughly speaking, may be formulated as follows: Consider an evolution system (either described in terms of Partial or Ordinary Differential Equations (PDE/ODE)). We are allowed to act on the trajectories of the system by means of a suitable control (the right hand side of the system, the boundary conditions, etc.). Then, given a time interval $t \in (0,T)$, and initial and final states we have to find a control, such that the solution matches both the initial state at time $t = 0$ and the final one at time $t = T$.

Controllability of finite-dimensional linear control systems

Let us start with the case of Ordinary Differential Equations, recalling some well known results in the finite dimensional context. Let $n, m \in N^*$ and $T > 0$. We consider the linear system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad t \in (0,T), \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where A is a real $(n \times n)$ matrix, B is a real $(n \times m)$ matrix and x^0 a vector in $\mathbb{R}^n$. The function $x : [0,T] \to \mathbb{R}^n$ represents the *state* and $u : [0,T] \to \mathbb{R}^m$ the *control*.

Given an initial datum $x^0 \in \mathbb{R}^n$ and a vector function $u \in L^2(0,T,\mathbb{R}^m)$, the system (1.1) has a unique solution $x \in H^1(0,T,\mathbb{R}^n)$ characterized by the variation of constants formula:

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds, \quad \forall t \in [0,T].$$

Definition 1.1.1. *System* (1.1) *is* ***controllable*** *in time* $T > 0$*, if given any initial and final data* $x^0, x^1 \in \mathbb{R}^n$*, respectively, there exists* $u \in L^2(0, T, \mathbb{R}^m)$*, such that the solution of* (1.1) *satisfies* $x(T) = x^1$*.*

Note that m is the number of controls entering in the system, while n stands for the number of components of the state to be controlled. In applications, it is desirable to make the number of controls m to be as small as possible. But this, of course, may affect the control properties of the system.

If we define the set of reachable states

$$R(T, x^0) = \left\{ x(T) \in \mathbb{R}^n : \; x \text{ solution of } (1.1) \text{ with } u \in L^2(0, T, \mathbb{R}^m) \right\},$$

the controllability property is equivalent to the fact that $R(T, x^0) = \mathbb{R}^n$ for any $x^0 \in \mathbb{R}^n$. It is well known that, if such a linear system is controllable from x^0 in time $T > 0$, then it is controllable in time T', for every $T' > 0$, and from every initial state $x'^0 \in \mathbb{R}^n$. Let us now define the controllability Gramian of the control system (1.1).

Definition 1.1.2. *The controllability Gramian of the control system* (1.1) *is the symmetric* $(n \times n)$ *matrix*

$$\int_0^T e^{(T-t)A} B B^* e^{(T-t)A^*} \, dt,$$

It is also well known that the system (1.1) is controllable in time T if and only if the *Gramian* of the system, is non-singular. The next results, due to Rudolph Kalman, Yu-Chi Ho and Kumpati Narendra.

Theorem 1.1.1 ([53]). *The linear time varying control system* (1.1) *is controllable if and only if its controllability Gramian is invertible.*

Since we are in finite dimension, it is easy to see that the Gramianis invertible if and only if there exists f $c > 0$ such that

$$\int_0^T \left\| B^* e^{(T-t)A} x^0 \right\| dt \geq c \|x^0\|^2$$

for every $x^0 \in \mathbb{R}^n$ (observablity inequality). The necessary and sufficient condition for controllability given in Theorem 1.1.1 requires computing a matrix, which might be quite difficult (and even impossible) in many cases, even for simple linear control systems. However, there exists a controllability criterion which is much simpler to check. It is the Kalman condition, due to R. E. Kalman and gives a complete answer to the problem of exact controllability of finite dimensional linear systems. It shows, in particular, that the time of control is irrelevant.

Theorem 1.1.2 ([60]). *System* (1.1) *is exactly controllable in some time* T *if and only if*

$$rank(B, AB, ..., A^{n-1}B) = n,$$

Consequently, if system (1.1) *is controllable in some time* $T > 0$ *it is controllable in any time.*

Controllability of infinite dimensional linear control system

We introduce some known results on controllability of infinite dimensional linear control systems in Banach spaces. Let X be a reflexive Banach space and $S(t)$ denote a strongly continuous semigroup on X, of generator $(A, D(A))$. Let X_{-1} denote the completion of X

for norm $\|x\|_{-1} = \|(\lambda I - A)^{-1}x\|$, where $\lambda \in \rho(A)$ is fixed. The space X_{-1} is isomorphic to $(D(A^*))'$, the dual space of $D(A^*)$. The semigroup $S(t)$ extends to a semigroup on X_{-1}, still denoted by $S(t)$, whose generator is an extension of the operator A, also denoted by A. With this notation, A is a linear operator from X to X_{-1}.

Let U be a reflexive Banach space. A linear continuous operator $B : U \to X_{-1}$ is called the control operator. Note that B is said to be bounded if $B \in \mathcal{L}(U, X_{-1})$, and is called unbounded otherwise. The control operator B is admissible for the semigroup $S(t)$ if every solution of

$$\begin{cases} y' = Ay(t) + Bu(t), \\ y(0) = y_0, \end{cases} \tag{1.2}$$

where $u \in L^2(0, +\infty, U)$, satisfies $y(t) \in X$, for every $t \geq 0$. The solution of equation (1.2) is understood in the mild sense, i.e,

$$y(t) = S(t)y_0 + \int_0^T S(t-s)Bu(s)ds,$$

for every $t \geq 0$. For $T > 0$, define $L_T : L^2(0, T; U) \to X_{-1}$ by

$$L_T u = \int_0^T S(T-s)Bu(s)ds.$$

A control operator $B \in \mathcal{L}(U, X_{-1})$ is admissible, if and only if $Im L_T \subset X$, for some $T > 0$.

In contrast to the case of linear finite dimensional control systems, there exist many types of controllability properties. We provide three different notions.

Let $B \in \mathcal{L}(U, X_{-1})$ denote an admissible control operator.

Definition 1.1.3. *For $y_0 \in X$ and $T > 0$, the system (1.2) is said to be* ***exactly controllable*** *from y_0 in time T if, for every $y_1 \in X$, there exists a control $u \in L^2(0, T; U)$ so that the solution of (1.2), with $y(0) = y_0$, satisfies $y(T) = y_1$.*

Definition 1.1.4. *The system (1.2) is said to be* ***approximately controllable*** *from y_0 in time T if, for every $y_1 \in X$, there exist $\varepsilon > 0$ and a control $u \in L^2(0, T; U)$ so that the solution of (1.2), with $y(0) = y_0$, satisfies*

$$\|y(T) - y_1\|_X \leq \varepsilon.$$

Definition 1.1.5. *For $T > 0$, the system (1.2) is said to be* ***null controllable*** *if, for every $y_0 \in X$, there exists a control $u \in L^2(0, T; U)$ so that the solution of (1.2), with $y(0) = y_0$, satisfies $y(T) = 0$.*

Clearly, exact controllability implies null controlability and approximate controllability. However, the converse is false in general. In finite dimension, i.e., when $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, the three concepts are equivalent.

The proofs of the next results cited here are classical, and they can be found, for example in [32, 63, 72, 98, 99]. These tests are based on the HUM method due to J.-L. Lions [63]. A first result ensure that the controllability can be given as follows:

Theorem 1.1.3. *The system (1.2) is exactly controllable in time $T > 0$ if and only if there exists a constant $c > 0$, such that*

$$\int_0^T \|B^* S^*(t) y_0\|_U^2 \, dt \geq c\|y_0\|_X^2, \quad \forall y_0 \in X. \tag{1.3}$$

(1.3) is called an *observability inequality*. Such inequality means that the map

$$\Gamma : y_0 \mapsto B^*S^*(\cdot)y_0,$$

is boundedly invertible, i.e, it is possible to recover a complete information about the initial state y^0 from a measure on $[0,T]$ of the output $B[S^*(\cdot)]t$ (*observability property*).

Theorem 1.1.4. *The system* (1.2) *is null controllable in time* $T > 0$ *if and only if there exists a constant* $c > 0$ *such that*

$$\int_0^T \|B^*S^*(t)y_0\|_U^2 \, dt \geq c\|S^*(T)y_0\|_X^2, \quad \forall y_0 \in X. \tag{1.4}$$

(1.4) is a weak observability inequality, i.e, only $S^*(T)y_0$ may be recovered, not y_0.

For linear equations, controllability is achieved, in general, by proving an observability inequality. Several methods can be used to derive such an the observability inequality, including Carleman estimates, the method of multipliers, microlocal analysis, among others. We refer the reader to the excellent textbook [29], [40], [92] and references therein.

The Hilbert Uniqueness Method (HUM):

The well known Hilbert Uniqueness Method (in short HUM) was introduced in [63] consisting of minimizing a cost function, namely, the L^2-norm of the control. First at all, we associate to the boundary-initial value problem

$$\begin{cases} \dot{z} = Az + Bu, \\ z(0) = 0, \end{cases} \tag{1.5}$$

its adjoint problem, obtained by taking the distributional adjoint of the operator $\partial_t - A$, namely $-\partial_t - A^*$:

$$\begin{cases} \dot{y} = -A^*y, \\ y(T) = y_T. \end{cases} \tag{1.6}$$

Note that (1.6) is without control and backwards in time. Assume that B is admissible and the control system (1.2) is null controllable in time T. We define the space H as the completion of $D(\Omega)$ (with Ω being a domain where system (1.2) acts on) with respect to the norm

$$\|\psi\|_H = \left(\int_0^T \|B^*S^*(t)\psi\|_U^2 dt \right)^{1/2}.$$

Let $y_0 \in X$. For every $\psi \in H$, we define

$$J(\psi) = \frac{1}{2} \int_0^T \|B^*S^*(t)\psi\|_U^2 dt + \langle S^*(T)\psi, y_0 \rangle_X .$$

Clearly, the functional J is strictly convex, and from the observability inequality (1.4), it is coercive in H. Then, it follows that J has a unique minimizer $\varphi \in H$. Define the control u by

$$u(t) = B^*S^*(T - t)\varphi,$$

for every $t \in [0,T]$, and let $y(\cdot)$ be the solution of (1.2), associated with the control u. Then, one has $y(T) = 0$ and, moreover, u is the control of minimal L^2-norm, among all controls whose associated trajectory satisfies $y(T) = 0$.

This proves that observability implies controllability, and gives a way to construct the control of minimal of L^2-norm. This is more or less the contents of the Hilbert Uniqueness Method.

1.1.2 Stability

For the context of the stability of partial differential equations, let us consider the abstract system

$$\begin{cases} \dot{x} = Ax + Bu, \\ x(0) = x^0, \end{cases} \tag{1.7}$$

where A is a linear operator defined in a state space and B is a control operator that allows us to act on the system through a control u.

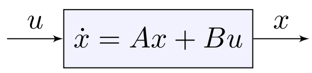
The stabilization problem consists of finding a feedback operator F, such that, with the control

$$u = Fx, \tag{1.8}$$

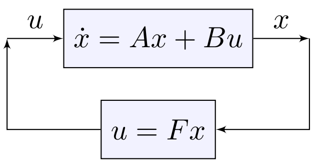
the solution of the *closed-loop problem* (1.7) tends to zero, as t tends to $+\infty$. If we are interested in exponential stability, i.e, we should prove the existence of two positive constants λ and c, such that for any x^0, we have that

$$\|x(t)\| \leq Ce^{-\lambda t}\|x^0\|, \quad \forall t \geq 0,$$

for a suitable norm $\|\cdot\|$ on the state space. The terminology *closed-loop* comes from the following diagrams (see [84, p. 113]): If a control $u(t)$ is defined externally, we can compute the solution $x(t)$ through (1.7), i.e, the loop is open as it may be represented by the diagram



In this case (1.7) is called a *open-loop system*. Furthermore, if the control $u(t)$ is contructed from $x(t)$ through (1.8), the loop is closed and we have to modify the above diagram to



Definition 1.1.6. *The system* (1.7) *is said* ***stabilizable*** *if there exist an operator F and positive constants λ and C, such that the solutions of the closed-loop problem* (1.7) *satisfy* (1.8). *Moreover, if for any positive λ, there exist an operator $F = F(\lambda)$ and a positive constant C, such that the solutions of* (1.7) *satisfy* (1.8), *we said that the closed-loop problem* (1.7) *is* ***completely stabilizable****.*

The following results show the strong relation between stabilization and controllability.

Theorem 1.1.5. *Assume that A generates a group of operators. Then,*

1. *If the system* (1.7) *is null controllable, then it is exponentially stable.*
2. *The following properties are equivalent:*
   - *(i) The system* (1.7) *is exactly controllable in some time $T > 0$;*
   - *(ii) The system* (1.7) *is null controllable in some time $T > 0$;*
   - *(iii) The system* (1.7) *is completely stabilizable.*

The first result was given by Datko [36]. The implication $(i) \Rightarrow (ii)$ is obvious, $(i) \Rightarrow (iii)$ is due to Slemrod [89], whereas the implication $(iii) \Rightarrow (i)$ is due to Megan [66] and Zabczyk [98, Theorem 3.4 p. 229].

1.2 Contribution of the Thesis

This thesis is composed of five chapters and comprises two main parts, numbered I and II. The first part deals with the boundary controllability of the Gear-Grimshaw model and the second part studies the well-posedness, controllability and stability of the Korteweg-de Vries Burgers equation posed in a unbounded domain. Most of the results presented in the thesis are published or submitted for publication [17, 18, 42, 43].

PART I

In this part, we investigate the boundary controllability properties of the nonlinear dispersive system

$$\begin{cases} u_t + uu_x + u_{xxx} + av_{xxx} + a_1vv_x + a_2(uv)_x = 0, & \text{in } (0,L)\times(0,T), \\ cv_t + rv_x + vv_x + abu_{xxx} + v_{xxx} + a_2buu_x + a_1b(uv)_x = 0, & \text{in } (0,L)\times(0,T), \\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L), \end{cases} \tag{1.9}$$

In (1.9), a_1, a_2, a, b, c and r are real constants, $u = u(x,t)$ and $v = v(x,t)$ are real-valued functions of the two variables x and t and subscripts indicate partial differentiation. In order to provide the tools to handle with the controllability problem, we assume that the coefficients a, b, c and r satisfy

$$b, c \text{ and } r \text{ are positive and } 1 - a^2b > 0. \tag{1.10}$$

System (1.16) was derived by Gear and Grimshaw in [45] as a model to describe strong interactions of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion (we also refer to [10, 88] for an extensive discussion on the physical relevance of the system). This somewhat complicated model has the structure of a pair of Korteweg-de Vries (KdV) equations coupled through both dispersive and nonlinear effects and has been object of intensive research in recent year. It is a special case of a broad class of nonlinear evolution equations for which the well-posedness theory associated to the pure initial-value problem posed on the whole real line $\mathbb{R}$, or on a finite interval with periodic boundary conditions, has been intensively investigated. By contrast, the mathematical theory pertaining to the study of the boundary value problem is considerably less advanced, specially in what concerns the study of the controllability properties.

The first and second chapters are dedicated to study the controllability properties of system (1.9) with two kinds of boundary conditions, namely, Neumann and Dirichlet-Neumann boundary conditions, respectively.

- Neumann boundary controllability of the Gear-Grimshaw system with critical size restrictions on the spatial domain

In **Chapter Two**, we study the boundary controllability of the Gear-Grimshaw system, posed on a finite domain $(0, L)$, with the Neumann boundary condition:

$$\begin{cases} u_{xx}(0,t) = h_0(t),\ u_x(L,t) = h_1(t),\ u_{xx}(L,t) = h_2(t), \\ v_{xx}(0,t) = g_0(t),\ v_x(L,t) = g_1(t),\ v_{xx}(L,t) = g_2(t). \end{cases} \tag{1.11}$$

The boundary functions h_i and g_i, for $i = 0, 1, 2$, are considered as control inputs acting on the boundary conditions. Our purpose is to see weather we can force the solutions of the system to have certain properties by choosing appropriate control inputs. More precisely, we are mainly concerned with the following exact control problem:

Given $T > 0$ and $u^0, v^0, u^1, v^1 \in L^2(0, L)$, can one find appropriate control inputs h_i, g_i, for $i = 0, 1, 2$, such that the corresponding solution (u, v) of (1.9)-(1.11) satisfies

$$(u(x, T), v(x, T)) = (u^1(x), v^1(x))? \tag{1.12}$$

We first prove that the corresponding linearized system around the origin is exactly controllable in $(L^2(0, L))^2$ when $h_2(t) = g_2(t) = 0$. In this case, the exact controllability property is derived for any $L > 0$ with control functions $h_0, g_0 \in H^{-\frac{1}{3}}(0, T)$ and $h_1, g_1 \in L^2(0, T)$. If we change the position of the controls and consider $h_0(t) = h_2(t) = 0$ (resp. $g_0(t) = g_2(t) = 0$) we obtain the result with control functions $g_0, g_2 \in H^{-\frac{1}{3}}(0, T)$ and $h_1, g_1 \in L^2(0, T)$ if and only if the length L of the spatial domain $(0, L)$ does not belong to a countable set. In all cases the regularity of the controls are sharp in time. If only one control act in the boundary condition, $h_0(t) = g_0(t) = h_2(t) = g_2(t) = 0$ and $g_1(t) = 0$ (resp. $h_1(t) = 0$), the linearized system is proved to be exactly controllable for small values of the length L and large time of control T. Finally, the nonlinear system is shown to be locally exactly controllable *via* the contraction mapping principle, if the associated linearized systems are exactly controllable.

- Boundary controllability of a nonlinear coupled system of two Korteweg-de Vries equations with critical size restrictions on the spatial domain

Chapter Three is concerned with the study of the the Gear-Grimshaw system (1.9) satisfying the following boundary conditions

$$\begin{cases} u(0,t) = h_0(t), \ u(L,t) = h_1(t), \ u_x(L,t) = h_2(t), \\ v(0,t) = g_0(t), \ v(L,t) = g_1(t), \ v_x(L,t) = g_2(t). \end{cases} \tag{1.13}$$

The functions h_0, h_1, h_2, g_0, g_1 and g_2 are the control inputs and u_0, v_0 the initial data. As Chapter 1, our purpose is to see whether one can force the solutions of those systems to have certain desired properties by choosing appropriate control inputs. More precisely, we are concerned with the control problem (1.12).

The results obtained improve the controllability results obtained by Cerpa *et al.* in [26] and by Micu *et al.* in [69] for a nonlinear coupled system of two Korteweg-de Vries (KdV) equations posed on a bounded interval. Initially, in [69], the authors proved that the nonlinear system is exactly controllable by using four boundary controls without any restriction on the length L of the interval. Later on, in [26], two boundary controls were considered to prove that the same system is exactly controllable for small values of the length L and large time of control T. Here, we use the ideas contained in [17] to prove that, with another configuration of four controls, it is possible to obtain the existence of the so-called *critical length phenomenon* for the linear system, i. e., whether the system is controllable depends on the length of the spatial domain. In addition, when we consider only one control input, the boundary controllability still holds for suitable values of the length L and time of control T. In both cases, the control spaces are sharp due a technical lemma which reveals a hidden regularity for the solution of the adjoint system.

PART II

The second part of this thesis, is devoted to investigate the asymptotic behavior and the controllability properties for the Korteweg-de Vries Burgers equation posed in the whole line.

- On the well-posedness and asymptotic behavior of the generalized Korteweg-de Vres Burgers equation

In **Chapter Four**, we investigate the well-posedness and the exponential stability of the generalized Korteweg-de Vries Burgers (GKdV-B) equation on the whole real line under the effect of a damping term by a function $b = b(x)$. More precisely,

$$\begin{cases} u_t + u_{xxx} - u_{xx} + a(u)u_x + b(x)u = 0 & \text{in } \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}, \end{cases} \tag{1.14}$$

where $a = a(x)$ is a positive real-valued function that satisfies the growth conditions

$$\begin{cases} |a^{(j)}(\mu)| \leq C(1 + |\mu|^{p-j}), \quad \forall \mu \in \mathbb{R}, \text{ for some } \ C > 0, \\ j = 0, 1 \ \text{ if } \ 1 \leq p < 2 \text{ and } j = 0, 1, 2 \ \text{ if } \ p \geq 2. \end{cases} \tag{1.15}$$

We obtain the global well-posedness in $H^s(\mathbb{R})$ for $1 \leq p < 2$ and $s \in [0, 3]$ and in $L^2(\mathbb{R})$ for $2 \leq p < 5$ and $H^3(\mathbb{R})$ for $p \geq 2$. Here, p denotes the power in the nonlinear term. The exponential stabilization is obtained for a definite damping term ($1 \leq p < 2$) by using multiplier techniques combined with interpolation. Under the effect of a localized damping term ($2 \leq p < 5$) we obtain a similar result by multiplier techniques combined with compactness arguments, reducing the problem to prove a unique continuation property for weak solutions.

Our analysis was inspired by the results obtained by Cavalcanti et al. for KdV-Burgers equation [23] and by Rosier and Zhang for the generalized KdV equation posed on a bounded domain [81] (see also [61]). In this context, we refer to the survey [80] for a quite complete review on the state of art.

- Controllability Aspects of the Korteweg-de Vries Burgers Equation on the Unbounded Domain

The aim of the **Chapter Five** is to consider the controllability problem of the linear system associated to Korteweg-de Vries Burgers equation posed in the whole space. The KdV-Burgers equation has been used in a study of wave propagation through liquid field elastic tube and for a description of shallow water waves on viscous fluid. Following the ideas contained in [79], we obtain a sort of exact controllability for solutions in $L^2_{loc}(\mathbb{R}^2)$. The proof of our main result has several ingredients as Carlemann's estimate, observability inequality and an approximation theorem.

Part I

Controllability of the Gear–Grimshaw System

SETTING OF THE PROBLEM

The goal of the first part is to investigate the boundary controllability properties of the nonlinear dispersive system

$$\begin{cases} u_t + uu_x + u_{xxx} + av_{xxx} + a_1vv_x + a_2(uv)_x = 0, & \text{in } (0,L)\times(0,T), \\ cv_t + rv_x + vv_x + abu_{xxx} + v_{xxx} + a_2buu_x + a_1b(uv)_x = 0, & \text{in } (0,L)\times(0,T), \\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L), \end{cases} \tag{1.16}$$

with the following boundary conditions

$$\begin{cases} u_{xx}(0,t) = h_0(t),\ u_x(L,t) = h_1(t),\ u_{xx}(L,t) = h_2(t), \\ v_{xx}(0,t) = g_0(t),\ v_x(L,t) = g_1(t),\ v_{xx}(L,t) = g_2(t). \end{cases} \tag{1.17}$$

and

$$\begin{cases} u(0,t) = h_0(t),\ u(L,t) = h_1(t),\ u_x(L,t) = h_2(t), \\ v(0,t) = g_0(t),\ v(L,t) = g_1(t),\ v_x(L,t) = g_2(t). \end{cases} \tag{1.18}$$

In (1.16), a_1, a_2, a, b, c and r are real constants, $u = u(x,t)$ and $v = v(x,t)$ are real-valued functions of the two variables x and t and subscripts indicate partial differentiation. The boundary functions h_i and g_i, for $i = 0, 1, 2$, are considered as control inputs acting on the boundary conditions. Our purpose is to see weather we can force the solutions of the system to have certain properties by choosing appropriate control inputs. More precisely, we are mainly concerned with the following exact control problem:

Given $T > 0$ and $u^0, v^0, u^1, v^1 \in L^2(0,L)$, can one find appropriate control inputs h_i, g_i, for $i = 0, 1, 2$, such that the corresponding solution (u, v) of (1.16)-(1.17) *(resp.* (1.16)-(1.18)*) satisfies*

$$(u(x,T), v(x,T)) = (u^1(x), v^1(x))? \tag{1.19}$$

If one can always find control inputs to guide the system from any given initial state (u_0, v_0) to any given terminal state (u_1, v_1), then the system is said to be exactly controllable. However, being different from other systems, the length L of the spatial domain may play a crucial role in determining the controllability of the system, specially when some configurations of four controls input are allowed to be used. This phenomenon, the so-called *critical length phenomenon*, was observed for the first time by Rosier [78] while studying the boundary controllability for the KdV equation. Throughout the paper we will provide a detailed explanation of such phenomenon but, roughly speaking, Rosier proved the existence of a finite dimensional subspace M of $L^2(0,L)$, which is not reachable by the KdV system, when starting from the origin, if L belongs to a countable set of critical lengths.

In order to provide the tools to handle with this problem, we assume that the coefficients a, b, c and r satisfy

$$b, c \text{ and } r \text{ are positive and } 1 - a^2b > 0. \tag{1.20}$$

System (1.16) was derived by Gear and Grimshaw in [45] as a model to describe strong interactions of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion (we also refer to [10, 88] for an extensive discussion on the physical relevance of the system). This somewhat complicated model has the structure of a pair of Korteweg-de Vries (KdV) equations coupled through both dispersive and nonlinear effects and has been object of intensive research in recent year. It is a special case of a broad class of nonlinear evolution equations for which the well-posedness theory associated to the pure initial-value problem posed on the whole real line $\mathbb{R}$, or on a finite interval with periodic boundary conditions, has been intensively investigated. By contrast, the mathematical theory pertaining to the study of the boundary value problem is considerably less advanced, specially in what concerns the study of the controllability properties. As far as we know, the controllability results for system (1.16) was first obtained in [68], when the model is posed on a periodic domain and $r = 0$. In this case, a diagonalization of the main terms allows to the decouple the corresponding linear system into two scalar KdV equations and use the previous results available in the literature. Later on, assuming that (1.20) holds, Micu *et al.* [69] proved the local exact boundary controllability property for the nonlinear system, posed on a bounded interval, considering the following boundary conditions:

$$\begin{cases} u(0,t) = 0,\ u(L,t) = f_1(t),\ u_x(L,t) = f_2(t), \\ v(0,t) = 0,\ v(L,t) = k_1(t),\ v_x(L,t) = k_2(t). \end{cases} \tag{1.21}$$

The analysis developed in [69] was inspired by the results obtained by Rosier in [78] for the scalar KdV equation. It combines the analysis of the linearized system and the Banach's fixed point theorem. Their main result reads as follows:

Theorem A (Micu *et al.* [69]) *Let $L > 0$ and $T > 0$. Then there exists a constant $\delta > 0$, such that, for any initial and final data $u^0, v^0, u^1, v^1 \in L^2(0,L)$ verifying*

$$||(u^0, v^0)||_{(L^2(0,L))^2} \leq \delta \quad \text{and} \quad ||(u^1, v^1)||_{(L^2(0,L))^2} \leq \delta,$$

there exist four control functions $f_1, k_1 \in H_0^1(0,T)$ and $f_2, k_2 \in L^2(0,T)$, such that the solution

$$(u,v) \in C([0,T];(L^2(0,L))^2) \cap L^2(0,T;(H^1(0,L))^2) \cap H^1(0,T;(H^{-2}(0,L))^2)$$

of (1.16)-(1.21) *verifies* (1.19).

The proof of Theorem A combines the analysis of the linearized system and the Banach fixed point theorem. It is important to point out that, in order to analyze the linearized system, the authors follow the classical duality approach [39, 63] and, therefore, the exact controllability property is equivalent to an observability inequality for the solutions of the adjoint system. The problem is then reduced to prove a nonstandard unique continuation property of the eigenfunctions of the corresponding differential operator.

An improvement of Theorem A was made by Cerpa *et al.*, in [26]. The authors considered the system (1.16)-(1.18) with only two control inputs acting on the Neumann boundary conditions, that is,

$$\begin{cases} u(0,t) = 0,\ u(L,t) = 0,\ u_x(L,t) = h_2(t), \\ v(0,t) = 0,\ v(L,t) = 0,\ v_x(L,t) = g_2(t). \end{cases} \tag{1.22}$$

In this case, the analysis of the linearized system is much more complicated, therefore the authors used a direct approach based on the multiplier technique that gives the observability inequality for small values of the length L and large time of control T. The fixed point argument, as well as, the existence and regularity results needed in order to consider the nonlinear system run exactly in the same way as in [69].

Theorem B (Cerpa *et al.* [26]) *Let us suppose that $T, L > 0$ satisfy*

$$1 > \frac{\max\{b,c\}}{\min\left\{b(1-\varepsilon^2), \left(1-\frac{a^2 b}{\varepsilon^2}\right)\right\}}\left\{\frac{rL^2}{3c\pi^2} + \frac{L^3}{3T\pi^2}\right\}$$

where

$$\varepsilon = \sqrt{\frac{-(1-b)+\sqrt{(1-b)^2+4a^2b^2}}{2b}}.$$

Then, there exists a constant $\delta > 0$, such that, for any initial and final data (u^0, v^0), $(u^1, v^1) \in (L^2(0,L))^2$ verifying

$$||(u^0, v^0)||_{(L^2(0,L))^2} \leq \delta \quad \text{and} \quad ||(u^1, v^1)||_{(L^2(0,L))^2} \leq \delta,$$

there exist two control functions $h_2, g_2 \in L^2(0,T)$, with $h_0 = g_0 = h_1 = g_1 = 0$, such that the solution

$$(u,v) \in C([0,T];(L^2(0,L))^2) \cap L^2(0,T;(H^1(0,L))^2) \cap H^1(0,T;(H^{-2}(0,L))^2)$$

of (1.16)-(1.22) *verifies* (1.19).

The program of this work was carried out for a particular choice of boundary conditions and aims to establish as a fact that such a model predicts the interesting controllability properties initially observed for the KdV equation. Therefore, to introduce the reader to the theory developed for KdV with the boundary conditions of types (1.17) and (1.18), we present below a summary of the results achieved in [78] and [15], respectively.

Rosier, in [78], studied the following boundary control problem for the KdV equation posed on the finite domain $(0, L)$

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0 & \text{in } (0,L)\times(0,T), \\ u(0,t) = 0,\ u(L,t) = 0,\ u_x(L,t) = g(t) & \text{in } (0,T), \\ u(x,0) = u^0(x) & \text{in } (0,L), \end{cases} \tag{1.23}$$

where the boundary value function $g(t)$ is considered as a control input. First, the author studies the associated linear system

$$\begin{cases} u_t + u_x + u_{xxx} = 0 & \text{in } (0,L)\times(0,T), \\ u(0,t) = 0,\ u(L,t) = 0,\ u_x(L,t) = g(t) & \text{in } (0,T), \\ u(x,0) = u^0(x) & \text{in } (0,L) \end{cases} \tag{1.24}$$

and discovered the so-called *critical length* phenomena, i.e., whether the system (1.24) is exactly controllable depends on the length L of the spatial domain $(0, L)$. More precise, the following result was proved:

Theorem B (Rosier [78]) *The linear system* (1.24) *is exactly controllable in the space $L^2(0,L)$ if and only if the length L of the spatial domain $(0,L)$ does not belong to the set*

$$\mathcal{N} := \left\{\frac{2\pi}{\sqrt{3}}\sqrt{k^2+kl+l^2} \,:\, k,\, l \in \mathbb{N}^*\right\}. \tag{1.25}$$

Then, by using a fixed point argument, the controllability result was extended to the nonlinear system when $L \notin \mathcal{N}$.

Theorem C (Rosier [78]) *Let $T > 0$ be given. If $L \notin \mathcal{N}$, there exists $\delta > 0$, such that, for any $u^0, u^T \in L^2(0,L)$ with*

$$||u^0||_{L^2(0,L)} + ||u^T||_{L^2(0,L)} \leq \delta,$$

one can find a control input $g \in L^2(0,T)$*, such that the nonlinear system* (1.23) *admits a unique solution*

$$u \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$$

satisfying

$$u(x,T) = u^T(x).$$

More recently, in [15], Caicedo *et al.* investigated the boundary control problem of the KdV equation with new boundary conditions, namely, the Neumann boundary conditions:

$$\begin{cases} u_t + (1+\beta)u_x + u_{xxx} = 0 & \text{in } (0,L)\times(0,T), \\ u_{xx}(0,t) = 0, \;\; u_x(L,t) = h(t), \;\; u_{xx}(L,t) = 0 & \text{in } (0,T), \\ u(x,0) = u^0(x) & \text{in } (0,L). \end{cases} \tag{1.26}$$

In (1.26), β is a given real constant and g a control input. For any $\beta \neq -1$, the authors obtained the following set of *critical lengths*

$$\mathcal{R}_\beta := \left\{ \frac{2\pi}{\sqrt{3(1+\beta)}}\sqrt{k^2+kl+l^2} \; : k, \, l \, \in \mathbb{N}^* \right\} \cup \left\{ \frac{k\pi}{\sqrt{\beta+1}} : k \in \mathbb{N}^* \right\}, \tag{1.27}$$

and proved that the following result holds:

Theorem D (Caicedo *et al.* [15])

(i) If $\beta \neq -1$*, the linear system (1.26) is exactly controllable in the space* $L^2(0,L)$ *if and only if the length* L *of the spatial domain* $(0,L)$ *does not belong to the set* $\mathcal{R}_\beta$*.*

(ii) If $\beta = -1$*, then the system (1.26) is not exact controllable in the space* $L^2(0,L)$ *for any* $L > 0$*.*

In addition, for the nonlinear system

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0 & \text{in } (0,L)\times(0,T), \\ u_{xx}(0,t) = 0, \;\; u_x(L,t) = h(t), \;\; u_{xx}(L,t) = 0 & \text{in } (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,L), \end{cases} \tag{1.28}$$

the result below was proved by using a fixed point argument:

Theorem E (Caicedo *et al.* [15]) *Let* $T > 0$*,* $\beta \neq -1$ *and* $L \notin \mathcal{R}_\beta$ *be given. There exists* $\delta > 0$*, such that, for any* $u^0, u^T \in L^2(0,L)$ *with*

$$||u^0 - \beta||_{L^2(0,L)} + ||u^T - \beta||_{L^2(0,L)} \leq \delta,$$

one can find a control input $h \in L^2(0,T)$*, such that the system* (1.28) *admits unique solution*

$$u \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$$

satisfying

$$u(x,T) = u^T(x).$$

Both theorems, Theorems B and D, were proved following the classical duality approach [39, 63] which reduces the problem to obtain an observability inequality for the solutions of the corresponding adjoint system. Then, the controllability is obtained with the aid of a compactness argument that leads the issue to a nonstandard unique continuation principle for the eigenfunctions of the differential operator associated to the model.

The critical lengths in (1.25) and (1.27) are such that there are eigenfunctions of the linear scalar problem for which the observability inequality associated to the adjoint system fails[1]. However, in [15], the authors encountered some difficulties that require special attention. For instance, the adjoint system of the linear system (1.26) is given by

$$\begin{cases} \psi_t + (1+\beta)\psi_x + \psi_{xxx} = 0 & \text{in } (0,L)\times(0,T), \\ (1+\beta)\psi(0,t) + \psi_{xx}(0,t) = 0 & \text{in } (0,T), \\ (1+\beta)\psi(L,t) + \psi_{xx}(L,t) = 0 & \text{in } (0,T), \\ \psi_x(0,t) = 0 & \text{in } (0,T), \\ \psi(x,T) = \psi^T(x) & \text{in } (0,L). \end{cases} \tag{1.29}$$

The exact controllability of system (1.26) is equivalent to the following observability inequality for the adjoint system (1.29):

$$||\psi^T||_{L^2(0,L)} \leq C||\psi_x(L,\cdot)||_{L^2(0,T)},$$

for some $C > 0$. Nonetheless, the usual multiplier method and compactness arguments used to deal with the system (1.29) only lead to

$$||\psi^T||^2_{L^2(0,L)} \leq C_1||\psi_x(L,\cdot)||^2_{L^2(0,T)} + C_2||\psi(L,\cdot)||^2_{L^2(0,T)}, \tag{1.30}$$

where C_1 and C_2 are positive constants. In order to absorb the extra term present in (1.30), Caicedo *et al.* derived a technical result, which reveals some hidden regularity (sharp trace regularities) for solutions of the adjoint system (1.29):

Theorem F (Caicedo *et al.* [15]) *For any $\psi^T \in L^2(0,L)$, the solution*

$$\psi \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$$

of the problem (1.29) possesses the following sharp trace properties

$$\sup_{x\in(0,L)} ||\partial_x^r \psi(x,\cdot)||_{H^{\frac{1-r}{3}}(0,T)} \leq C_r||\psi^T||_{L^2(0,L)}, \tag{1.31}$$

for $r = 0,1,2$, where C_r are positive constants.

Estimate (1.31) is then combined with a compactness argument to remove the extra term in (1.30). We remark that the sharp Kato smoothing properties obtained by Kenig, Ponce and Vega [55] for the solutions of the KdV equation posed on the line, played an important role in the proof of the previous result. The same strategy has been successfully applied by Cerpa *et al.* [27] for the study of a similar boundary controllability problem.

In Chapter Two and Three, we address the boundary controllability problem for the Gear-Grimshaw system with boundary conditions (1.17) and (1.18), respectively.

[1] In the case of $L \in \mathcal{N}$ (resp. $L \in \mathcal{R}_\beta$), Rosier (resp. Caicedo *et al.* in [15]) proved in [78] that the associated linear system (1.24) is not controllable; there exists a finite-dimensional subspace of $L^2(0,L)$, denoted by $\mathcal{M} = \mathcal{M}(L)$, which is unreachable from 0 for the linear system. More precisely, for every nonzero state $\psi \in \mathcal{M}$, $g \in L^2(0,T)$ and $u \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$ satisfying (1.24) and $u(\cdot,0) = 0$, one has $u(\cdot,T) \neq \psi$. A spatial domain $(0,L)$ is called *critical* for the system (1.24) (resp. (1.26)) if its domain length $L \in \mathcal{N}$ (resp. $L \in \mathcal{R}_\beta$).

CHAPTER

TWO

NEUMANN BOUNDARY CONTROLLABILITY OF THE GEAR-GRIMSHAW SYSTEM WITH CRITICAL SIZE RESTRICTIONS ON THE SPATIAL DOMAIN

In this chapter we study the boundary controllability of the Gear-Grimshaw system, posed on a finite domain $(0, L)$, with Neumann boundary condition. We first prove that the corresponding linearized system around the origin is exactly controllable in $(L^2(0, L))^2$ when $h_2(t) = g_2(t) = 0$. In this case, the exact controllability property is derived for any $L > 0$ with control functions $h_0, g_0 \in H^{-\frac{1}{3}}(0, T)$ and $h_1, g_1 \in L^2(0, T)$. If we change the position of the controls and consider $h_0(t) = h_2(t) = 0$ (resp. $g_0(t) = g_2(t) = 0$) we obtain the result with control functions $g_0, g_2 \in H^{-\frac{1}{3}}(0, T)$ and $h_1, g_1 \in L^2(0, T)$ if and only if the length L of the spatial domain $(0, L)$ does not belong to a countable set. In all cases the regularity of the controls are sharp in time. If only one control act in the boundary condition, $h_0(t) = g_0(t) = h_2(t) = g_2(t) = 0$ and $g_1(t) = 0$ (resp. $h_1(t) = 0$), the linearized system is proved to be exactly controllable for small values of the length L and large time of control T. Finally, the nonlinear system is shown to be locally exactly controllable via the contraction mapping principle, if the associated linearized systems are exactly controllable.

2.1 Main result and notations

We are now in position to return considerations to the control properties of the system (1.16). First, we prove that the corresponding linear system with the following boundary conditions

$$\begin{cases} u_{xx}(0,t) = h_0(t), \ u_x(L,t) = h_1(t), \ u_{xx}(L,t) = 0, \\ v_{xx}(0,t) = g_0(t), \ v_x(L,t) = g_1(t), \ v_{xx}(L,t) = 0, \end{cases}$$

is exactly controllable in $(L^2(0, L))^2$ with controls h_0, $g_0 \in H^{-\frac{1}{3}}(0, T)$ and h_1, $g_1 \in L^2(0, T)$. In this case, no restriction on the length L of the spatial domain is required. However, if we change the position of the controls a critical size restriction can appear. This is the case when we consider the following boundary conditions

$$\begin{cases} u_{xx}(0,t) = 0, \ u_x(L,t) = h_1(t), \ u_{xx}(L,t) = 0, \\ v_{xx}(0,t) = g_0(t), \ v_x(L,t) = g_1(t), \ v_{xx}(L,t) = g_2(t). \end{cases}$$

In this case, the exact controllability result in $(L^2(0, L))^2$ is derived with controls g_0, $g_2 \in H^{-\frac{1}{3}}(0, T)$ and h_1, $g_1 \in L^2(0, T)$ if and only if the length L does not belong of the following set

$$\mathcal{F}_r := \left\{ 2\pi k\sqrt{\frac{1-a^2b}{r}} : k \in \mathbb{N}^* \right\} \cup \left\{ \pi\sqrt{\frac{(1-a^2b)\alpha(k,l,m,n,s)}{3r}} : k,l,m,n,s \in \mathbb{N}^* \right\}, \tag{2.1}$$

where

$$\begin{aligned}\alpha := \alpha(k,l,m,n,s) =& 5k^2 + 8l^2 + 9m^2 + 8n^2 + 5s^2 + 8kl + 6km \\ & + 4kn + 2ks + 12ml + 8ln + 3ls + 12mn + 6ms + 8ns.\end{aligned}$$

As in [15], the hidden regularity for the corresponding adjoint system (1.16) was required. Here, the result is given in Proposition 2.2.4, which is the key point to prove the controllability result.

Finally, for small values of the length L and large time of control T we derive a exact controllability result in $(L^2(0,L))^2$ by assuming that the controls $g_1(t) = 0$ (resp. $h_1(t) = 0$) and $g_0(t) = g_2(t) = 0$. In this case, the analysis of the linearized system is much more complicated, therefore we use a direct approach based on the multipliers technique, as in [26]. In all cases, the result obtained for the linear system allows to prove the local controllability property of the nonlinear system (1.16) by means of a fixed point argument.

The analysis describe above are summarized in the main result, Theorem 2.1.1. However, in order to make the reading easier, throughout this chapter, we use the following notation for the boundary functions:

$$\begin{aligned}
&\vec{h}_1 = (0,h_1,0),\ \vec{g}_1 = (g_0,g_1,g_2) \ \text{ and } \ \vec{h}_2 = (h_0,h_1,h_2),\ \vec{g}_2 = (0,g_1,0),\\
&\vec{h}_3 = (h_0,h_1,0),\ \vec{g}_3 = (g_0,g_1,0) \ \text{ and } \ \vec{h}_4 = (0,h_1,h_2),\ \vec{g}_4 = (0,g_1,g_2),\\
&\vec{h}_5 = (0,h_1,0),\ \vec{g}_5 = (0,0,0) \qquad \text{and } \ \vec{h}_6 = (0,0,0),\ \vec{g}_6 = (0,g_1,0).
\end{aligned}$$

We also introduce the space $\mathcal{X} := (L^2(0,L))^2$ endowed with the inner product

$$\langle (u,v),(\varphi,\psi)\rangle := \int_0^L u(x)\varphi(x)dx + \frac{b}{c}\int_0^L v(x)\psi(x)dx, \qquad \forall (u,v),(\varphi,\psi) \in \mathcal{X},$$

and the spaces

$$\mathcal{H}_T := H^{-\frac{1}{3}}(0,T) \times L^2(0,T) \times H^{-\frac{1}{3}}(0,T)$$

and

$$\mathcal{Z}_T := C([0,T];(L^2(0,L))^2) \cap L^2(0,T,(H^1(0,L))^2)$$

endowed with their natural inner products.

Thus, our main result reads as follows:

Theorem 2.1.1. *Let $T > 0$. Then, there exists $\delta > 0$, such that, for any $(u^0,v^0),(u^1,v^1) \in \mathcal{X} := (L^2(0,L))^2$ verifying*

$$\|(u^0,v^0)\|_{\mathcal{X}} + \|(u^1,v^1)\|_{\mathcal{X}} \leq \delta,$$

the following holds:

(i) If $L \in (0,\infty) \setminus \mathcal{F}_r$, one can find $\vec{h}_i, \vec{g}_i \in \mathcal{H}_T$, for $i = 1,2$, such that the system (1.16)-(1.17) admits a unique solution $(u,v) \in \mathcal{Z}_T$ satisfying (1.19).

(ii) For any $L > 0$, one can find $\vec{h}_i, \vec{g}_j \in \mathcal{H}_T$, for $j = 3,4$, such that the system (1.16)-(1.17) admits a unique solution $(u,v) \in \mathcal{Z}_T$, satisfying (1.19).

(iii) Let $T > 0$ and $L > 0$ satisfying

$$1 > \frac{\beta C_T}{T}\left[L + \frac{r}{c}\right],$$

where C_T is the constant in (2.21) and β is the constant given by the embedding $H^{\frac{1}{3}}(0,T) \subset L^2(0,T)$. Then, one can find $\vec{h}_k, \vec{g}_k \in \mathcal{H}_T$, for $k = 5, 6$, such that the system (1.16)-(1.17) admits a unique solution $(u, v) \in \mathcal{Z}_T$, satisfying (1.19).

Before close this section, we observe that the exact controllability result given in Theorem A holds without any restriction of the Length L. However, we believe that, with another configuration of the controls, it is possible to prove the existence of a critical set for the system (1.16).

The chapter is organized as follows:

— In Section 2.2, we show that the system (1.16)-(1.17) is locally well-posed in $\mathcal{Z}_T$, whenever $(u^0, v^0) \in (L^2(0,L))^2$, h_0, $g_0 \in H^{-\frac{1}{3}}(\mathbb{R}^+)$, h_1, $g_1 \in L^2(\mathbb{R}^+)$ and h_2, $g_2 \in H^{-\frac{1}{3}}(\mathbb{R}^+)$. Various linear estimates, including hidden regularities, are presented for solutions of the corresponding linear system. As we pointed out before, such estimates will play important roles in studying the controllability properties.

— In Section 2.3, the boundary control system (1.16) is investigated for its controllability. We investigate first the linearized system and its corresponding adjoint system for their controllability and observability. In particular, the hidden regularities for the solutions of the adjoint system presented in the Section 2.2 are used to prove observability inequalities associated to the control problem.

— The proof of our main result, Theorem 2.1.1, is presented in Section 2.4. Finally, the chapter ends with an appendix, where the proof of a technical lemma used in the paper is furnished.

2.2 Well-posedness

2.2.1 Linear System

In this section, we establish the well-posedness of the linear system associated to (1.16)-(1.17):

$$\begin{cases} u_t + u_{xxx} + a v_{xxx} = 0, & \text{in } (0,L)\times(0,T),\\ v_t + \frac{r}{c}v_x + \frac{ab}{c}u_{xxx} + \frac{1}{c}v_{xxx} = 0, & \text{in } (0,L)\times(0,T),\\ u_{xx}(0,t) = h_0(t),\ u_x(L,t) = h_1(t),\ u_{xx}(L,t) = h_2(t), & \text{in } (0,T),\\ v_{xx}(0,t) = g_0(t),\ v_x(L,t) = g_1(t),\ v_{xx}(L,t) = g_2(t), & \text{in } (0,T),\\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L). \end{cases} \tag{2.2}$$

We begin by considering the following linear non-homogeneous boundary value problem

$$\begin{cases} u_t + u_{xxx} + a v_{xxx} = f, & \text{in } (0,L)\times(0,T),\\ v_t + \frac{ab}{c}u_{xxx} + \frac{1}{c}v_{xxx} = s, & \text{in } (0,L)\times(0,T),\\ u_{xx}(0,t) = h_0(t),\ u_x(L,t) = h_1(t),\ u_{xx}(L,t) = h_2(t), & \text{in } (0,T),\\ v_{xx}(0,t) = g_0(t),\ v_x(L,t) = g_1(t),\ v_{xx}(L,t) = g_2(t), & \text{in } (0,T),\\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L), \end{cases} \tag{2.3}$$

with the notation introduced in Section 1. Then, the next proposition shows that the problem (2.3) is well-posed in the space $\mathcal{X}$.

Proposition 2.2.1. *Let $T>0$ be given. Then, for any (u^0,v^0) in $\mathcal{X}$, f,s belong to $L^1(0,T;L^2(0,L))$ and $\overrightarrow{h},\overrightarrow{g}\in\mathcal{H}_T$, problem (2.3) admits a unique solution $(u,v)\in\mathcal{Z}_T$, with*

$$\partial_x^k u,\partial_x^k v\in L^\infty_x(0,L;H^{\frac{1-k}{3}}(0,T)),\quad k=0,1,2. \tag{2.4}$$

Moreover, there exists $C>0$, such that

$$\|(u,v)\|_{\mathcal{Z}_T}+\sum_{k=0}^{2}\|(\partial_x^k u,\partial_x^k v)\|_{L^\infty_x(0,L;(H^{\frac{1-k}{3}}(0,T))^2)}\le C\left\{\|(u^0,v^0)\|_{\mathcal{X}}\right.$$
$$\left.+\|(\overrightarrow{h},\overrightarrow{g})\|_{\mathcal{H}_T}+\|(f,s)\|_{L^1(0,T;(L^2(0,L))^2)}\right\}.$$

Proof. We diagonalize the main term in (2.2) and consider the change of variable

$$\begin{cases} u=2a\widetilde{u}+2a\widetilde{v},\\ v=\left(\left(\frac{1}{c}-1\right)+\lambda\right)\widetilde{u}+\left(\left(\frac{1}{c}-1\right)-\lambda\right)\widetilde{v},\end{cases}$$

where $\lambda=\sqrt{\left(\frac{1}{c}-1\right)^2+\frac{4a^2b}{c}}$. Thus, we can transform the linear system (2.3) into

$$\begin{cases} \widetilde{u}_t+\alpha_-\widetilde{u}_{xxx}=\widetilde{f},\\ \widetilde{v}_t+\alpha_+\widetilde{v}_{xxx}=\widetilde{s},\\ \widetilde{u}_{xx}(0,t)=\widetilde{h}_0(t),\ \widetilde{u}_x(L,t)=\widetilde{h}_1(t),\ \widetilde{u}_{xx}(L,t)=\bar{h}_2(t),\\ \widetilde{v}_{xx}(0,t)=\widetilde{g}_0(t),\ \widetilde{v}_x(L,t)=\widetilde{g}_1(t),\ \widetilde{v}_{xx}(L,t)=\widetilde{g}_2(t),\\ \widetilde{u}(x,0)=\widetilde{u}^0(x),\quad \widetilde{v}(x,0)=\widetilde{v}^0(x),\end{cases} \tag{2.5}$$

where $\alpha_\pm=-\frac{1}{2}\left(\left(\frac{1}{c}-1\right)\pm\lambda\right)$ and

$$\begin{cases} \widetilde{f}=-\frac{1}{2}\left(\frac{\alpha_+}{a\lambda}f+\frac{1}{\lambda}s\right),\quad \widetilde{u}_0=-\frac{1}{2}\left(\frac{\alpha_-}{a\lambda}u^0-\frac{1}{\lambda}v^0\right),\quad \widetilde{h}_i=-\frac{1}{2}\left(\frac{\alpha_-}{a\lambda}h_i-\frac{1}{\lambda}g_i\right),\quad i=0,1,2,\\ \widetilde{s}=-\frac{1}{2}\left(\frac{\alpha_-}{a\lambda}f-\frac{1}{\lambda}s\right),\quad \widetilde{v}_0=\frac{1}{2}\left(\frac{\alpha_+}{a\lambda}u^0-\frac{1}{\lambda}v^0\right),\quad \widetilde{g}_i=\frac{1}{2}\left(\frac{\alpha_+}{a\lambda}h_i-\frac{1}{\lambda}g_i\right),\quad i=0,1,2.\end{cases}$$

Note that condition (1.20) guarantees that $\alpha_\pm$ are nonzero. Therefore, system (2.5) can be decoupled into two single KdV equations as follows:

$$\begin{cases} \widetilde{u}_t+\alpha_-\widetilde{u}_{xxx}=\widetilde{f},\\ \widetilde{u}_{xx}(0,t)=\widetilde{h}_0(t),\ \widetilde{u}_x(L,t)=\widetilde{h}_1(t),\ \widetilde{u}_{xx}(L,t)=\widetilde{h}_2(t),\\ \widetilde{u}(0,x)=\widetilde{u}^0(x)\end{cases} \tag{2.6}$$

and

$$\begin{cases} \widetilde{v}_t+\alpha_+\widetilde{v}_{xxx}=\widetilde{s},\\ \widetilde{v}_{xx}(0,t)=\widetilde{g}_0(t),\ \widetilde{v}_x(L,t)=\widetilde{g}_1(t),\ \widetilde{v}_{xx}(L,t)=\widetilde{g}_2(t),\\ \widetilde{v}(x,0)=\widetilde{v}^0(x).\end{cases} \tag{2.7}$$

Here, we consider the solutions written in the form $\{W^\pm_{bdr}(t)\}_{t\ge0}$ that will be called *the boundary integral operator*. For this purpose we use a lemma, which can be found in [22, Lemma 2.4] (see also [15, Lemma 2.1]), for solutions of (2.6) (or (2.7)). For the sake of completeness we will present the proof in the Appendix A.1.

Lemma 2.2.2. *The solution u of the IBVP (2.6) (or (2.7)), when $\widetilde{f}=0$, $\widetilde{s}=0$ and null initial data, can be written in the form*

$$u(x,t)=[W^+_{bdr}\vec{\widetilde{h}}](x,t):=[W^+_{bdr}\vec{h}](x,t):=\sum_{j,m=1}^{3}[W^+_{j,m}h_m](x,t),$$

where

$$[W^+_{j,m}h](x,t) \equiv [U_{j,m}h](x,t) + \overline{[U_{j,m}h](x,t)} \tag{2.8}$$

with

$$[U_{j,m}h](x,t) \equiv \frac{1}{2\pi}\int_0^{+\infty} e^{i\rho^3 t}e^{\lambda_j^+(\rho)x}3\rho^2[Q^+_{j,m}h](\rho)d\rho \tag{2.9}$$

for $j = 1,3,\ m = 1,2,3$ *and*

$$[U_{2,m}h](x,t) \equiv \frac{1}{2\pi}\int_0^{+\infty} e^{i\rho^3 t}e^{-\lambda_2^+(\rho)(1-x)}3\rho^2[Q^+_{2,m}h](\rho)d\rho \tag{2.10}$$

for $m = 1,2,3$. *Here*

$$[Q^+_{j,m}h](\rho) := \frac{\Delta^+_{j,m}(\rho)}{\Delta^+(\rho)}\hat{h}^+(\rho), \qquad [Q^+_{2,m}h](\rho) = \frac{\Delta^+_{2,m}(\rho)}{\Delta^+(\rho)}e^{\lambda_2^+(\rho)}\hat{h}^+(\rho) \tag{2.11}$$

for $j = 1,3$ *and* $m = 1,2,3$. *Here* $\hat{h}^+(\rho) = \hat{h}(i\rho^3)$, $\Delta^+(\rho)$ *and* $\Delta^+_{j,m}(\rho)$ *are obtained from* $\Delta(s)$ *and* $\Delta_{j,m}(s)$ *by replacing* s *with* $i\rho^3$ *and* $\lambda_j^+(\rho) = \lambda_j(i\rho^3)$ *where*

$$\Delta = \lambda_1\lambda_2\lambda_3\left(\lambda_1(\lambda_3-\lambda_2)e^{-\lambda_1} + \lambda_2(\lambda_1-\lambda_3)e^{-\lambda_2} + \lambda_3(\lambda_2-\lambda_1)e^{-\lambda_3}\right);$$

$$\Delta_{1,1} = e^{-\lambda_1}\lambda_2\lambda_3(\lambda_3-\lambda_2),\ \Delta_{2,1} = e^{-\lambda_2}\lambda_1\lambda_3(\lambda_1-\lambda_3),\ \Delta_{3,1} = e^{-\lambda_3}\lambda_1\lambda_2(\lambda_2-\lambda_1);$$
$$\Delta_{1,2} = \lambda_2^2\lambda_3^2(e^{\lambda_2}-e^{\lambda_3}),\ \Delta_{2,2} = \lambda_1^2\lambda_3^2(e^{\lambda_3}-e^{\lambda_1}),\ \Delta_{3,2} = \lambda_1^2\lambda_2^2(e^{\lambda_1}-e^{\lambda_2});$$
$$\Delta_{1,3} = \lambda_2\lambda_3(\lambda_2e^{\lambda_3}-\lambda_3e^{\lambda_2}),\ \Delta_{2,3} = \lambda_1\lambda_3(\lambda_3e^{\lambda_1}-\lambda_1e^{\lambda_3}),\ \Delta_{3,3} = \lambda_1\lambda_2(\lambda_1e^{\lambda_2}-\lambda_2e^{\lambda_1}).$$

Since

$$(\widetilde{u}^0,\widetilde{v}^0)\in\mathcal{X}, \quad (\widetilde{f},\widetilde{s})\in L^1(0,T;(L^2(0,L))^2) \text{ and } \overrightarrow{h}, \overrightarrow{g}\in\mathcal{H}_T,$$

by [15, Propositions 2.2 and 2.5], we obtain the existence of $(\widetilde{u},\widetilde{v})\in\mathcal{Z}_T$, solution of the system (2.5), such that

$$\partial_x^k\widetilde{u}, \partial_x^k\widetilde{v} \in L^\infty_x(0,L;H^{\frac{1-k}{3}}(0,T)), \quad k=0,1,2,$$

and

$$\|(\widetilde{u},\widetilde{v})\|_{\mathcal{Z}_T} + \sum_{k=0}^{2}\|(\partial_x^k\widetilde{u},\partial_x^k\widetilde{v})\|_{L^\infty_x(0,L;(H^{\frac{1-k}{3}}(0,T))^2)} \leq C\left\{\|(\widetilde{u}^0,\widetilde{v}^0)\|_{\mathcal{X}} + \|(\overrightarrow{h},\overrightarrow{g})\|_{\mathcal{H}_T} + \|(\widetilde{f},\widetilde{s})\|_{L^1(0,T;(L^2(0,L))^2)}\right\},$$

for some constant $C>0$. Furthermore, we can write $\widetilde{u}$ and $\widetilde{v}$ in its integral form as follows

$$\widetilde{u}(t) = W_0^-(t)\widetilde{u}^0 + W_{bdr}^-(t)\overrightarrow{h} + \int_0^t W_0^-(t-\tau)\widetilde{f}(\tau)d\tau,$$

$$\widetilde{v}(t) = W_0^+(t)\widetilde{v}^0 + W_{bdr}^+(t)\overrightarrow{g} + \int_0^t W_0^+(t-\tau)\widetilde{s}(\tau)d\tau,$$

where $\{W_0^\pm(t)\}_{t\geq 0}$ are the C_0-semigroup in the space $L^2(0,L)$ generated by the linear operators

$$A^\pm = -\alpha_\pm g''',$$

with domain

$$D(A^\pm) = \{g\in H^3(0,L) : g''(0) = g'(L) = g''(L) = 0\},$$

and $\{W^{\pm}_{bdr}(t)\}_{t\geq 0}$ are the operator given in Lemma 2.2.2 (see also [15, Lemma 2.1] for more details). Then, by change of variable we can easily verify that

$$\begin{cases} u(t) = W_0^-(t)u^0 + W_{bdr}^-(t)\overrightarrow{h} + \displaystyle\int_0^t W_0^-(t-\tau)f(\tau)d\tau, \\ v(t) = W_0^+(t)v^0 + W_{bdr}^+(t)\overrightarrow{g} + \displaystyle\int_0^t W_0^+(t-\tau)s(\tau)d\tau \end{cases}$$

and the result follows. □

The global well-posedness of the system (2.2) is obtained using a fixed point argument.

Proposition 2.2.3. *Let $T>0$ be given. Then, for any $(u^0,v^0)\in\mathcal{X}$ and $\overrightarrow{h},\overrightarrow{g}\in\mathcal{H}_T$, problem (2.2) admits a unique solution $(u,v)\in\mathcal{Z}_T$ with*

$$\partial_x^k u, \partial_x^k v \in L^\infty_x(0,L;H^{\frac{1-k}{3}}(0,T)), \quad k=0,1,2.$$

Moreover, there exist $C>0$, such that

$$\|(u,v)\|_{\mathcal{Z}_T} + \sum_{k=0}^{2}\|(\partial_x^k u,\partial_x^k v)\|_{L^\infty_x(0,L;(H^{\frac{1-k}{3}}(0,T))^2)} \leq C\left\{\|(u^0,v^0)\|_{\mathcal{X}} + \|(\overrightarrow{h},\overrightarrow{g})\|_{\mathcal{H}_T} \right.$$
$$\left. + \|(f,s)\|_{L^1(0,T;(L^2(0,L))^2)}\right\}.$$

Proof. Let

$$\mathcal{F}_T := \left\{(u,v)\in\mathcal{Z}_T : (u,v)\in L^\infty_x(0,L;(H^{\frac{1-k}{3}}(0,T))^2), k=0,1,2\right\}$$

equipped with the norm

$$\|(u,v)\|_{\mathcal{F}_T} = \|(u,v)\|_{\mathcal{Z}_T} + \sum_{k=0}^{2}\|(\partial_x^k u,\partial_x^k v)\|_{L^\infty_x(0,L;(H^{\frac{1-k}{3}}(0,T))^2)}.$$

Let $0<\beta\leq T$ to be determined later. For each $u,v\in\mathcal{F}_\beta$, consider the problem

$$\begin{cases} \omega_t + \omega_{xxx} + a\eta_{xxx} = 0, & \text{in } (0,L)\times(0,\beta), \\ \eta_t + \frac{ab}{c}\omega_{xxx} + \frac{1}{c}\eta_{xxx} = -\frac{r}{c}v_x, & \text{in } (0,L)\times(0,\beta), \\ \omega_{xx}(0,t) = h_0(t),\ \omega_x(L,t) = h_1(t),\ \omega_{xx}(L,t) = h_2(t), & \text{in } (0,\beta), \\ \eta_{xx}(0,t) = g_0(t),\ \eta_x(L,t) = g_1(t),\ \eta_{xx}(L,t) = g_2(t), & \text{in } (0,\beta), \\ \omega(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L). \end{cases} \tag{2.12}$$

According to Proposition 2.2.1, we can define the operator

$$\Gamma : \mathcal{F}_\beta \to \mathcal{F}_\beta, \quad \text{given by} \quad \Gamma(u,v) = (\omega,\eta),$$

where (ω,η) is the solution of (2.12). Moreover,

$$\|\Gamma(u,v)\|_{\mathcal{F}_\beta} \leq C\left\{\|(u^0,v^0)\|_{\mathcal{X}} + \|(\overrightarrow{h},\overrightarrow{g})\|_{\mathcal{H}_\beta} + \|(0,v_x)\|_{L^1(0,\beta;(L^2(0,L))^2)}\right\}, \tag{2.13}$$

where the positive constant C depends only on T. Since

$$\|(0,v_x)\|_{L^1(0,\beta;L^2(0,L))} \leq \beta^{\frac{1}{2}}\|(u,v)\|_{\mathcal{F}_\beta},$$

we obtain a positive constant $C>0$, such that

$$\|\Gamma(u,v)\|_{\mathcal{F}_\beta} \leq C\left\{\|(u^0,v^0)\|_{\mathcal{X}} + \|(\overrightarrow{h},\overrightarrow{g})\|_{\mathcal{H}_\beta}\right\} + C\beta^{\frac{1}{2}}\|(u,v)\|_{\mathcal{F}_\beta}. \tag{2.14}$$

Let (u,v) belongs to

$$\in B_r(0) := \left\{(u,v) \in \mathcal{F}_\beta : \|(u,v)\|_{\mathcal{F}_\beta} \leq r\right\},$$

with $r = 2C\left\{\|(u^0,v^0)\|_{\mathcal{X}} + \|(\overrightarrow{h},\overrightarrow{g})\|_{\mathcal{H}_\beta}\right\}$. Choosing $\beta > 0$, satisfying

$$C\beta^{\frac{1}{2}} \leq \frac{1}{2}, \tag{2.15}$$

from (2.14) we obtain

$$\|\Gamma(u,v)\|_{\mathcal{F}_\beta} \leq r.$$

The above estimate allows us to conclude that

$$\Gamma : B_r(0) \subset \mathcal{F}_\beta \to B_r(0).$$

On the other hand, note that $\Gamma(u_1,v_1) - \Gamma(u_2,v_2)$ solves the following system

$$\begin{cases} \omega_t + \omega_{xxx} + a\eta_{xxx} = 0, & \text{in } (0,L)\times(0,\beta), \\ \eta_t + \frac{ab}{c}\omega_{xxx} + \frac{1}{c}\eta_{xxx} = -\frac{r}{c}(v_{1x} - v_{2x}), & \text{in } (0,L)\times(0,\beta), \\ \omega_{xx}(0,t) = \omega_x(L,t) = \omega_{xx}(L,t) = 0, & \text{in } (0,\beta), \\ \eta_{xx}(0,t) = \eta_x(L,t) = \eta_{xx}(L,t) = 0, & \text{in } (0,\beta), \\ \omega(x,0) = 0, \quad v(x,0) = 0, & \text{in } (0,L). \end{cases}$$

Again, from Proposition 2.2.1 and (2.15), we have

$$\begin{aligned} \|\Gamma(u_1,v_1) - \Gamma(u_2,v_2)\|_{\mathcal{F}_\beta} &\leq C\|(0, v_{1x} - v_{2x})\|_{L^1(0,\beta;(L^2(0,L))^2)} \\ &\leq C\beta^{\frac{1}{2}}\|(u_1,v_1) - (u_2,v_2)\|_{\mathcal{F}_\beta} \\ &\leq \frac{1}{2}\|(u_1,v_1) - (u_2,v_2)\|_{\mathcal{F}_\beta}. \end{aligned}$$

Hence, $\Gamma : B_r(0) \to B_r(0)$ is a contraction and, by Banach fixed point theorem, we obtain a unique $(u,v) \in B_r(0)$, such that

$$\Gamma(u,v) = (u,v) \in \mathcal{F}_\beta,$$

and (2.13) holds, for all $t \in (0,\beta)$. Since the choice of β is independent of (u^0,v^0), the standard continuation extension argument yields that the solution (u,v) belongs to $\mathcal{F}_T$. The proof is complete. □

Adjoint System

Consider the following homogeneous initial-value problem associated to (1.16)-(1.17):

$$\begin{cases} u_t + u_{xxx} + av_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ v_t + \frac{r}{c}v_x + \frac{ab}{c}u_{xxx} + \frac{1}{c}v_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ u_{xx}(0,t) = u_x(L,t) = u_{xx}(L,t) = 0, & \text{in } (0,T), \\ v_{xx}(0,t) = v_x(L,t) = v_{xx}(L,t) = 0, & \text{in } (0,T), \\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L). \end{cases} \tag{2.16}$$

In order to introduce the backward system associated to (2.16), we multiply the first equation of (2.16) by φ, the second one by ψ and integrate over $(0,L)\times(0,T)$. Assuming that the functions u, v, φ and ψ are regular enough to justify all the computations, we obtain, after integration by parts, the following identity:

$$
\begin{aligned}
&\int_0^L (u(x,T)\varphi(x,T)+v(x,T)\psi(x,T))\,dx-\int_0^L \left(u^0(x)\varphi(x,0)+v^0(x)\psi(x,0)\right)dx=\\
&\quad\int_0^T\int_0^L u(x,t)\left(\varphi(x,t)+\varphi_{xxx}(x,t)+\frac{ab}{c}\psi_{xxx}(x,t)\right)dxdt\\
&\quad+\int_0^T\int_0^L v(x,t)\left(\psi(x,t)+\frac{r}{c}\psi(x,t)+a\varphi_{xxx}(x,t)+\frac{1}{c}\psi_{xxx}(x,t)\right)dxdt\\
&\quad-\int_0^T u_{xx}(L,t)\left(\varphi(L,t)+\frac{ab}{c}\psi(L,t)\right)dt+\int_0^T u_{xx}(0,t)\left(\varphi(0,t)+\frac{ab}{c}\psi(0,t)\right)dt\\
&\quad+\int_0^T u_x(L,t)\left(\varphi_x(L,t)+\frac{ab}{c}\psi_x(L,t)\right)dt-\int_0^T u_x(0,t)\left(\varphi_x(0,t)+\frac{ab}{c}\psi_x(0,t)\right)dt\\
&\quad-\int_0^T u(L,t)\left(\varphi_{xx}(L,t)+\frac{ab}{c}\psi_{xx}(L,t)\right)dt+\int_0^T u(0,t)\left(\varphi_{xx}(0,t)+\frac{ab}{c}\psi_{xx}(0,t)\right)dt\\
&\quad-\int_0^T v_{xx}(L,t)\left(a\varphi(L,t)+\frac{1}{c}\psi(L,t)\right)dt+\int_0^T v_{xx}(0,t)\left(a\varphi(0,t)+\frac{1}{c}\psi(0,t)\right)dt\\
&\quad+\int_0^T v_x(L,t)\left(a\varphi_x(L,t)+\frac{1}{c}\psi_x(L,t)\right)dt-\int_0^T v_x(0,t)\left(a\varphi_x(0,t)+\frac{1}{c}\psi_x(0,t)\right)dt\\
&\quad-\int_0^T v(L,t)\left(a\varphi_{xx}(L,t)+\frac{1}{c}\psi_{xx}(L,t)+\frac{r}{c}\psi(L,t)\right)dt\\
&\quad+\int_0^T v(0,t)\left(a\varphi_{xx}(0,t)+\frac{1}{c}\psi_{xx}(0,t)+\frac{1}{c}\psi(0,t)\right)dt.
\end{aligned}
$$

Having the previous equality in hands, we consider backward system as follows

$$
\begin{cases}
\varphi_t+\varphi_{xxx}+\frac{ab}{c}\psi_{xxx}=0, & \text{in } (0,L)\times(0,T),\\
\psi_t+\frac{r}{c}\psi_x+a\varphi_{xxx}+\frac{1}{c}\psi_{xxx}=0, & \text{in } (0,L)\times(0,T)
\end{cases}
\tag{2.17}
$$

satisfying the boundary conditions,

$$
\begin{cases}
a\varphi_x(0,t)+\frac{1}{c}\psi_x(0,t)=0, & \text{in } (0,T),\\
\varphi_x(0,t)+\frac{ab}{c}\psi_x(0,t)=0, & \text{in } (0,T),\\
\varphi_{xx}(L,t)+\frac{ab}{c}\psi_{xx}(L,t)=0, & \text{in } (0,T),\\
\varphi_{xx}(0,t)+\frac{ab}{c}\psi_{xx}(0,t)=0, & \text{in } (0,T),\\
a\varphi_{xx}(L,t)+\frac{1}{c}\psi_{xx}(L,t)+\frac{r}{c}\psi(L,t)=0, & \text{in } (0,T),\\
a\varphi_{xx}(0,t)+\frac{1}{c}\psi_{xx}(0,t)+\frac{r}{c}\psi(0,t)=0, & \text{in } (0,T)
\end{cases}
\tag{2.18}
$$

and the final conditions

$$
\varphi(x,T)=\varphi^1(x),\qquad \psi(x,T)=\psi^1(x),\qquad \text{in } (0,L).
\tag{2.19}
$$

Since the coefficients satisfy $1-a^2b>0$, we can deduce from the first and second equations of (2.18) that the above boundary conditions can be written as

$$
\begin{cases}
\varphi_x(0,t)=\psi_x(0,t)=0, & \text{in } (0,T),\\
\varphi_{xx}(L,t)+\frac{ab}{c}\psi_{xx}(L,t)=0, & \text{in } (0,T),\\
\varphi_{xx}(0,t)+\frac{ab}{c}\psi_{xx}(0,t)=0, & \text{in } (0,T),\\
a\varphi_{xx}(L,t)+\frac{1}{c}\psi_{xx}(L,t)+\frac{r}{c}\psi(L,t)=0, & \text{in } (0,T),\\
a\varphi_{xx}(0,t)+\frac{1}{c}\psi_{xx}(0,t)+\frac{r}{c}\psi(0,t)=0, & \text{in } (0,T).
\end{cases}
\tag{2.20}
$$

The following proposition is the key to prove the controllability of the linear system (2.2). The result ensures the hidden regularity for the solution of the adjoint system (2.17)-(2.20).

Proposition 2.2.4. *For any $(\varphi^1, \psi^1) \in \mathcal{X}$, the system (2.17)-(2.20) admits a unique solution $(\varphi, \psi) \in \mathcal{Z}_T$, such that it has the following sharp trace properties*

$$\begin{cases} \sup\limits_{0<x<L} \|\partial_x^k \varphi(x, \cdot)\|_{H^{\frac{1-k}{3}}(0,T)} \leq C_T \|\varphi^1\|_{L^2(0,L)}, \\ \sup\limits_{0<x<L} \|\partial_x^k \psi(x, \cdot)\|_{H^{\frac{1-k}{3}}(0,T)} \leq C_T \|\psi^1\|_{L^2(0,L)}, \end{cases} \tag{2.21}$$

for $k = 0, 1, 2$, where C_r is a positive constant.

Proof. Proceeding as in the proof of Proposition 2.2.3, we obtain the result. Indeed, first we consider the change of variable $t \to T - t$ and $x \to L - x$, then for any (φ, ψ) in $\mathcal{Z}_T$, we consider the system

$$\begin{cases} u_t + u_{xxx} + \frac{ab}{c} v_{xxx} = 0, & \text{in } (0, L) \times (0, T), \\ v_t + a u_{xxx} + \frac{1}{c} v_{xxx} = -\frac{r}{c} v_x, & \text{in } (0, L) \times (0, T), \\ \varphi(x, 0) = \varphi^0(x),\ \psi(x, 0) = \psi^0(x), & \text{in } (0, L), \end{cases}$$

with boundary conditions

$$\begin{cases} u_x(L, t) = v_x(L, t) = 0, & \text{in } (0, T), \\ u_{xx}(L, t) = -\frac{ab}{c} \psi_{xx}(L, t), & \text{in } (0, T), \\ u_{xx}(0, t) = -\frac{ab}{c} \psi_{xx}(0, t), & \text{in } (0, T), \\ v_{xx}(L, t) = -ac\varphi_{xx}(L, t) - r\psi(L, t), & \text{in } (0, T), \\ v_{xx}(0, t) = -ac\varphi_{xx}(0, t) - r\psi(0, t), & \text{in } (0, T). \end{cases}$$

By using a fixed point argument the result is achieved. □

The adjoint system possesses a relevant estimate as described below.

Proposition 2.2.5. *Any solution (φ, ψ) of the adjoint system (2.17)-(2.20) satisfies*

$$\begin{aligned} \|(\varphi^1, \psi^1)\|_{\mathcal{X}}^2 \leq{}& \frac{1}{T} \|(\varphi, \psi)\|_{L^2(0,T;\mathcal{X})} + \frac{1}{2} \|\varphi_x(L, \cdot)\|_{L^2(0,T)}^2 + \frac{b}{2c} \|\psi_x(L, \cdot)\|_{L^2(0,T)}^2 \\ &+ \frac{br}{c^2} \|\psi(L, \cdot)\|_{L^2(0,T)}^2 + \frac{1}{2} \left\| \varphi_x(L, \cdot) + \frac{ab}{c} \psi_x(L, \cdot) \right\|_{L^2(0,T)}^2 \\ &+ \frac{b}{2c} \left\| a\varphi_x(L, \cdot) + \frac{1}{c} \psi_x(L, \cdot) \right\|_{L^2(0,T)}^2, \end{aligned} \tag{2.22}$$

with initial data $(\varphi^1, \psi^1) \in \mathcal{X}$.

Proof. Multiplying the first equation of (2.17) by $-t\varphi$, the second one by $-\frac{b}{c} t\psi$ and integrating by parts over $(0, T) \times (0, L)$, we obtain

$$\begin{aligned} \frac{T}{2} \int_0^L \varphi^2(x, T) dx ={}& \frac{1}{2} \int_0^T \int_0^L \varphi^2(x, t) dx dt + \frac{ab}{c} \int_0^T \int_0^L t \varphi_{xxx}(x, t) \psi(x, t) dx dt \\ &- \int_0^T t \left[\varphi_{xx}(x, t)\varphi(x, t) - \frac{1}{2} \varphi_x^2(x, t) + \frac{ab}{c} \psi_{xx}(x, t)\varphi(x, t) - \frac{ab}{c} \psi_x(x, t)\varphi_x(x, t) \right. \\ &\left. + \frac{ab}{c} \psi(x, t)\varphi_{xx}(x, t) \right]_0^L dt \end{aligned}$$

and

$$\frac{Tb}{2c}\int_0^L \psi^2(x,T)dx = \frac{b}{2c}\int_0^T\int_0^L \psi^2(x,t)dxdt - \frac{ab}{c}\int_0^T\int_0^L t\varphi_{xxx}(x,t)\psi(x,t)dxdt$$
$$-\int_0^T t\left[\frac{b}{c^2}\psi_{xx}(x,t)\psi(x,t) - \frac{b}{2c^2}\psi_x^2(x,t) + \frac{br}{2c^2}\psi^2(x,t)\right]_0^L dt.$$

Adding the above identities, it follows that

$$\frac{T}{2}\|(\varphi^1,\psi^1)\|_{\mathcal{X}}^2 = \frac{1}{2}\|(\varphi,\psi)\|_{L^2(0,T;\mathcal{X})}^2$$
$$-\int_0^T t\left[\frac{b}{c}\psi(x,t)\left(a\varphi_{xx}(x,t) + \frac{1}{c}\psi_{xx}(x,t) + \frac{r}{c}\psi(x,t)\right)\right]_0^L dt$$
$$-\int_0^T t\left[\frac{b}{2c}\psi_x(x,t)\left(a\varphi_x(x,t) + \frac{1}{c}\psi_x(x,t)\right) - \frac{1}{2}\varphi_x(x,t)\left(\varphi_x(x,t) + \frac{ab}{c}\psi_x(x,t)\right)\right]_0^L dt$$
$$+\int_0^T t\left[\varphi(x,t)\left(\varphi_{xx}(x,t) + \frac{ab}{c}\psi_{xx}(x,t)\right) - \frac{br}{2c^2}\psi^2(x,t)\right]_0^L dt.$$

Then, from (2.20), we obtain

$$\frac{T}{2}\|(\varphi^1,\psi^1)\|_{\mathcal{X}}^2 \leq \frac{1}{2}\|(\varphi,\psi)\|_{L^2(0,T;\mathcal{X})}^2 + \frac{bT}{2c}\int_0^T \psi_x(L,t)\left(a\varphi_x(L,t) + \frac{1}{c}\psi_x(L,t)\right)dt$$
$$+\frac{T}{2}\int_0^T \varphi_x(L,t)\left(\varphi_x(L,t) + \frac{ab}{c}\psi_{x,t}(L,t)\right)dt$$
$$+\frac{brT}{2c^2}\int_0^T \psi^2(L,t)dt - \frac{brT}{2c^2}\int_0^T \psi^2(0,t)dt.$$

Finally, (2.22) is obtained by applying Young's inequality on the right hand side of the above inequality. □

2.2.2 Nonlinear System

In this subsection, attention will be given to the full nonlinear system (1.16)-(1.17). The proof of the lemma below is available in [11, Lemma 3.1] and, therefore, we will omit it.

Lemma 2.2.6. *There exists a constant $C > 0$, such that, for any $T > 0$ and $(u,v) \in \mathcal{Z}_T$,*

$$\|uv_x\|_{L^1(0,T;L^2(0,L))} \leq C(T^{\frac{1}{2}} + T^{\frac{1}{3}})\|u\|_{\mathcal{Z}_T}\|v\|_{\mathcal{Z}_T}.$$

We first show that system (1.16)-(1.17) is locally well-posed in the space $\mathcal{Z}_T$.

Theorem 2.2.7. *For any $(u^0,v^0) \in \mathcal{X}$ and $\overrightarrow{h} = (h_0,h_1,h_2)$, $\overrightarrow{g} = (g_0,g_1,g_2) \in \mathcal{H}_T$, there exists $T^* > 0$, depending on $\|(u^0,v^0)\|_{\mathcal{X}}$, such that the problem (1.16)-(1.17) admits a unique solution $(u,v) \in \mathcal{Z}_{T^*}$ with*

$$\partial_x^k u, \partial_x^k v \in L_x^\infty(0,L;H^{\frac{1-k}{3}}(0,T^*)), \quad k = 0,1,2.$$

Moreover, the corresponding solution map is Lipschitz continuous.

Proof. Let

$$\mathcal{F}_T = \left\{(u,v) \in \mathcal{Z}_T : (u,v) \in L_x^\infty(0,L;(H^{\frac{1-k}{3}}(0,T))^2), k = 0,1,2\right\}$$

equipped with the norm

$$\|(u,v)\|_{\mathcal{F}_T} = \|(u,v)\|_{\mathcal{Z}_T} + \sum_{k=0}^{2} \|(\partial_x^k u, \partial_x^k v)\|_{L_x^\infty(0,L;(H^{\frac{1-k}{3}}(0,T))^2)}.$$

Let $0 < T^* \leq T$ to be determined later. For each $u, v \in \mathcal{F}_{T^*}$, consider the problem

$$\begin{cases} \omega_t + \omega_{xxx} + a\eta_{xxx} = f(u,v), & \text{in } (0,L)\times(0,T^*), \\ \eta_t + \frac{ab}{c}\omega_{xxx} + \frac{1}{c}\eta_{xxx} = s(u,v), & \text{in } (0,L)\times(0,T^*), \\ \omega_{xx}(0,t) = h_0(t),\ \omega_x(L,t) = h_1(t),\ \omega_{xx}(L,t) = h_2(t), & \text{in } (0,T^*), \\ \eta_{xx}(0,t) = g_0(t),\ \eta_x(L,t) = g_1(t),\ \eta_{xx}(L,t) = g_2(t), & \text{in } (0,T^*), \\ \omega(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L), \end{cases} \tag{2.23}$$

where

$$f(u,v) = -a_1(vv_x) - a_2(uv)_x$$

and

$$s(u,v) = -\frac{r}{c}v_x - \frac{a_2 b}{c}(uu_x) - \frac{a_1 b}{c}(uv)_x.$$

Since $\|v_x\|_{L^1(0,T^*;L^2(0,L))} \leq \beta^{\frac{1}{2}}\|v\|_{\mathcal{Z}_{T^*}}$, from Lemma 2.2.6 we deduce that $f(u,v)$ and $s(u,v)$ belong to $L^1(0,T^*;L^2(0,L))$ and satisfy

$$\|(f,s)\|_{L^1(0,T^*;(L^2(0,L))^2)} \leq C_1((T^*)^{\frac{1}{2}} + (T^*)^{\frac{1}{3}})\left(\|u\|^2_{\mathcal{Z}_{T^*}} + (\|u\|_{\mathcal{Z}_{T^*}} + 1)\|v\|_{\mathcal{Z}_{T^*}} + \|v\|^2_{\mathcal{Z}_{T^*}}\right), \tag{2.24}$$

for some positive constant C_1. Then, according to Proposition 2.2.1, we can define the operator

$$\Gamma : \mathcal{F}_{T^*} \to \mathcal{F}_{T^*}, \quad \text{given by} \quad \Gamma(u,v) = (\omega,\eta),$$

where (ω,η) is the solution of (2.23). Moreover,

$$\|\Gamma(u,v)\|_{\mathcal{F}_{T^*}} \leq C\left\{\|(u^0,v^0)\|_{\mathcal{X}} + \|(\overrightarrow{h},\overrightarrow{g})\|_{\mathcal{H}_{T^*}} + \|(f,s)\|_{L^1(0,T^*;(L^2(0,L))^2)}\right\}, \tag{2.25}$$

where the positive constant C depends only on T^*. Combining (2.24) and (2.25), we obtain

$$\begin{aligned}\|\Gamma(u,v)\|_{\mathcal{F}_{T^*}} \leq & C\left\{\|(u^0,v^0)\|_{\mathcal{X}} + \|(\overrightarrow{h},\overrightarrow{g})\|_{\mathcal{H}_{T^*}}\right\} \\ & + CC_1((T^*)^{\frac{1}{2}} + (T^*)^{\frac{1}{3}})\left(\|u\|^2_{\mathcal{Z}_{T^*}} + (\|u\|_{\mathcal{Z}_{T^*}} + 1)\|v\|_{\mathcal{Z}_{T^*}} + \|v\|^2_{\mathcal{Z}_{T^*}}\right).\end{aligned}$$

Let (u,v) belongs to

$$B_r(0) := \left\{(u,v) \in \mathcal{F}_{T^*} : \|(u,v)\|_{\mathcal{F}_{T^*}} \leq r\right\},$$

where $r = 2C\left\{\|(u^0,v^0)\|_{\mathcal{X}} + \|(\overrightarrow{h},\overrightarrow{g})\|_{\mathcal{H}_T}\right\}$. From the estimate above, it follows that

$$\|\Gamma(u,v)\|_{\mathcal{F}_{T^*}} \leq \frac{r}{2} + CC_1((T^*)^{\frac{1}{2}} + (T^*)^{\frac{1}{3}})\,(3r+1)\,r. \tag{2.26}$$

Then, by choosing $T^* > 0$, such that

$$CC_1((T^*)^{\frac{1}{2}} + (T^*)^{\frac{1}{3}})\,(3r+1) \leq \frac{1}{2}, \tag{2.27}$$

from (2.26), we have

$$\|\Gamma(u,v)\|_{\mathcal{F}_{T^*}} \leq r.$$

Thus, we conclude that

$$\Gamma : B_r(0) \subset \mathcal{F}_{T^*} \to B_r(0).$$

On the other hand, $\Gamma(u_1,v_1)-\Gamma(u_2,v_2)$ solves the system

$$\begin{cases} \omega_t+\omega_{xxx}+a\eta_{xxx}=f(u_1,v_1)-f(u_2,v_2), & \text{in } (0,L)\times(0,T^*),\\ \eta_t+\frac{ab}{c}\omega_{xxx}+\frac{1}{c}\eta_{xxx}=s(u_1,v_1)-s(u_2,v_2), & \text{in } (0,L)\times(0,T^*),\\ \omega_{xx}(0,t)=\omega_x(L,t)=\omega_{xx}(L,t)=0, & \text{in } (0,T^*),\\ \eta_{xx}(0,t)=\eta_x(L,t)=\eta_{xx}(L,t)=0, & \text{in } (0,T^*),\\ \omega(x,0)=0, \quad v(x,0)=0, & \text{in } (0,L), \end{cases}$$

where, $f(u,v)$ and $s(u,v)$ were defined in (2.23). Note that

$$\begin{aligned} |f(u_1,v_1)-f(u_2,v_2)| \le C_2|\,((v_2-v_1)v_{2,x}+v_1(v_2-v_1)_x+(u_2(v_2-v_1))_x \\ +((u_2-u_1)v_1)_x)\,| \end{aligned}$$

and

$$\begin{aligned} |s(u_1,v_1)-s(u_2,v_2)| \le C_2|\,((v_2-v_1)_x+(u_2-u_1)u_{2,x}+u_1(u_2-u_1)_x \\ +(u_2(v_2-v_1))_x+((u_2-u_1)v_1)_x)\,|, \end{aligned}$$

for some positive constant C_2. Then, Proposition 2.2.1 and Lemma 2.2.6, give us the following estimate

$$\begin{aligned} \|\Gamma(u_1,v_1)-\Gamma(u_2,v_2)\|_{\mathcal{F}_{T^*}} &\le C\|(f(u_1,v_1)-f(u_2,v_2),s(u_1,v_1)-s(u_2,v_2))\|_{L^1(0,T^*,[L^2]^2)} \\ &\le C_3((T^*)^{\frac12}+(T^*)^{\frac13})(8r+1)\|(u_1-u_2,v_1-v_2)\|_{\mathcal{F}_{T^*}}, \end{aligned}$$

for some positive constant C_3. Choosing T^*, satisfying (2.27) and such that

$$C_3((T^*)^{\frac12}+(T^*)^{\frac13})(8r+1)\le\frac12,$$

we obtain

$$\|\Gamma(u_1,v_1)-\Gamma(u_2,v_2)\|_{\mathcal{F}_{T^*}}\le\frac12\|(u_1-u_2,v_1-v_2)\|_{\mathcal{F}_{T^*}}.$$

Hence, $\Gamma: B_r(0)\to B_r(0)$ is a contraction and, by Banach fixed point theorem, we obtain a unique $(u,v)\in B_r(0)$, such that $\Gamma(u,v)=(u,v)\in\mathcal{F}_{T^*}$ and, therefore, the proof is complete. □

Remark 2.2.8. *From the proof of Proposition 2.2.1, we deduce that the solution of the system (1.16)-(1.17) can be written as*

$$\begin{aligned} \begin{pmatrix} u(t)\\ v(t)\end{pmatrix} = W_0(t)\begin{pmatrix} u^0(x)\\ v^0(x)\end{pmatrix} + W_{bdr}(t)\begin{pmatrix} \vec{h}\\ \vec{g}\end{pmatrix} \\ -\int_0^t W_0(t-\tau)\begin{pmatrix} a_1(vv_x)(\tau)+a_2(uv)_x(\tau)\\ \frac{r}{c}v_x(\tau)+\frac{a_2b}{c}(uu_x)(\tau)+\frac{a_1b}{c}(uv)_x(\tau)\end{pmatrix}d\tau, \end{aligned}$$

with

$$W_0(t)=\begin{pmatrix} W_0^-(t) & 0\\ 0 & W_0^+(t)\end{pmatrix} \quad \text{and} \quad W_{bdr}(t)=\begin{pmatrix} W_{bdr}^-(t) & 0\\ 0 & W_{bdr}^+(t)\end{pmatrix},$$

where $\{W_0^\pm(t)\}_{t\ge0}$ are the C_0-semigroup in the space $L^2(0,L)$ generated by the linear operators

$$A^\pm=-\alpha_\pm g''',$$

where

$$\alpha_\pm=-\frac12\left(\left(\frac1c-1\right)\pm\sqrt{\left(\frac1c-1\right)^2+\frac{4a^2b}{c}}\right),$$

with domain

$$D(A^\pm)=\{g\in H^3(0,L): g''(0)=g'(L)=g''(L)=0\},$$

and $\{W_{bdr}^\pm(x)\}_{t\ge0}$ is the operator defined in Lemma 2.2.2.

2.3 Exact Boundary Controllability for the Linear System

In this section, we study the existence of controls $\overrightarrow{h} := (h_0, h_1, h_2)$ and $\overrightarrow{g} := (g_0, g_1, g_2) \in \mathcal{H}_T$, such that the solution (u, v) of the system

$$\begin{cases} u_t + u_{xxx} + av_{xxx} = 0 & \text{in } (0,L)\times(0,T), \\ v_t + \frac{r}{c}v_x + \frac{ab}{c}u_{xxx} + \frac{1}{c}v_{xxx} = 0 & \text{in } (0,L)\times(0,T), \\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L), \end{cases} \tag{2.28}$$

with the boundary conditions

$$\begin{cases} u_{xx}(0,t) = h_0(t),\ u_x(L,t) = h_1(t),\ u_{xx}(L,t) = h_2(t) & \text{in } (0,T), \\ v_{xx}(0,t) = g_0(t),\ v_x(L,t) = g_1(t),\ v_{xx}(L,t) = g_2(t) & \text{in } (0,T), \end{cases} \tag{2.29}$$

satisfies

$$u(\cdot,T) = u^1(\cdot), \qquad \text{and} \qquad v(\cdot,T) = v^1(\cdot). \tag{2.30}$$

More precisely, we have the following definition:

Definition 2.3.1. *Let $T > 0$. System (2.28)-(2.29) is exactly controllable in time T if for any initial and final data (u^0, v^0) and (u^1, v^1) in $\mathcal{X}$, there exist control functions $\overrightarrow{h} = (h_0, h_1, h_2)$ and $\overrightarrow{g} = (g_0, g_1, g_2)$ in $\mathcal{H}_T$, such that the solution of (2.28)-(2.29) satisfies (2.30).*

Remark 2.3.1. *Without loss of generality, we shall consider only the case $u^0 = v^0 = 0$. Indeed, let (u^0, v^0), (u^1, v^1) in $\mathcal{X}$ and $\overrightarrow{h}$, $\overrightarrow{g}$ in $\mathcal{H}_T$ be controls which lead the solution $(\widetilde{u}, \widetilde{v})$ of (2.28) from the zero initial data to the final state $(u^1, v^1) - (u(T), v(T))$, where (u, v) is the mild solution corresponding to (2.28)-(2.29) with initial data (u^0, v^0). It follows immediately that these controls also lead to the solution $(\widetilde{u}, \widetilde{v}) + (u, v)$ of (2.28)-(2.29) from (u^0, v^0) to the final state (u^1, v^1).*

In the following pages, we will analyze the exact controllability of the system (2.28)-(2.29) for different combinations of four controls and one control.

2.3.1 Four Controls

Case 1

Consider the following boundary conditions:

$$\begin{cases} u_{xx}(0,t) = h_0(t),\ u_x(L,t) = h_1(t),\ u_{xx}(L,t) = 0 & \text{in } (0,T), \\ v_{xx}(0,t) = g_0(t),\ v_x(L,t) = g_1(t),\ v_{xx}(L,t) = 0 & \text{in } (0,T). \end{cases} \tag{2.31}$$

We first give an equivalent condition for the exact controllability property.

Lemma 2.3.2. *For any (u^1, v^1) in $\mathcal{X}$, there exist four controls $\overrightarrow{h} = (h_0, h_1, 0)$ and $\overrightarrow{g} = (g_0, g_1, 0)$ in $\mathcal{H}_T$, such that the solution (u, v) of (2.28)-(2.31) satisfies (2.30) if and only if*

$$\begin{aligned} \int_0^L \left(u^1(x)\varphi^1(x) + v^1(x)\psi^1(x)\right) dx = & \int_0^T h_0(t)\left(\varphi(0,t) + \frac{ab}{c}\psi(0,t)\right) dt \\ & + \int_0^T h_1(t)\left(\varphi_x(L,t) + \frac{ab}{c}\psi_x(L,t)\right) dt \\ & + \int_0^T g_0(t)\left(a\varphi(0,t) + \frac{1}{c}\psi(0,t)\right) dt \\ & + \int_0^T g_1(t)\left(a\varphi_x(L,t) + \frac{1}{c}\psi_x(L,t)\right) dt, \end{aligned} \tag{2.32}$$

for any (φ^1, ψ^1) *in* $\mathcal{X}$*, where* (φ, ψ) *is the solution of the backward system* (2.17)-(2.20) *with initial data* (φ^1, ψ^1)*.*

Proof. The relation (2.32) is obtained by multiplying the equations in (2.28) by the solution (φ, ψ) of (2.17)-(2.20), integrating by parts and using the boundary conditions (2.31). □

The following observability inequality plays a fundamental role for the study of the controllability properties.

Proposition 2.3.3. *For* $T > 0$ *and* $L > 0$*, there exists a constant* $C := C(T, L) > 0$*, such that*

$$\|(\varphi^1, \psi^1)\|_{\mathcal{X}}^2 \leq C \left\{ \left\| (-\Delta_t)^{\frac{1}{6}} \left(\varphi(0, \cdot) + \frac{ab}{c} \psi(0, \cdot) \right) \right\|_{L^2(0,T)}^2 \right.$$
$$+ \left\| \varphi_x(L, \cdot) + \frac{ab}{c} \psi_x(L, \cdot) \right\|_{L^2(0,T)}^2 + \left\| (-\Delta_t)^{\frac{1}{6}} \left(a\varphi(0, \cdot) + \frac{1}{c} \psi(0, \cdot) \right) \right\|_{L^2(0,T)}^2$$
$$\left. + \left\| a\varphi_x(L, \cdot) + \frac{1}{c} \psi_x(L, \cdot) \right\|_{L^2(0,T)}^2 \right\}, \quad (2.33)$$

for any $(\varphi^1, \psi^1) \in \mathcal{X}$*, where* (φ, ψ) *is a solution of* (2.17)-(2.20) *with initial data* (φ^1, ψ^1)*, where* $\Delta_t := \partial_t^2$*.*

Proof. We argue by contradiction, as in [78, Proposition 3.3], and suppose that (2.33) does not hold. In this case, we obtain a sequence $\{(\varphi_n^1, \psi_n^1)\}_{n \in \mathbb{N}}$, satisfying

$$1 = \|(\varphi_n^1, \psi_n^1)\|_{\mathcal{X}}^2 \geq n \left\{ \left\| (-\Delta_t)^{\frac{1}{6}} \left(\varphi_n(0, \cdot) + \frac{ab}{c} \psi_n(0, \cdot) \right) \right\|_{L^2(0,T)}^2 \right.$$
$$+ \left\| \varphi_{n,x}(L, \cdot) + \frac{ab}{c} \psi_{n,x}(L, \cdot) \right\|_{L^2(0,T)}^2 + \left\| (-\Delta_t)^{\frac{1}{6}} \left(a\varphi_n(0, \cdot) + \frac{1}{c} \psi_n(0, \cdot) \right) \right\|_{L^2(0,L)}^2$$
$$\left. + \left\| a\varphi_{n,x}(L, \cdot) + \frac{1}{c} \psi_{n,x}(L, \cdot) \right\|_{L^2(0,T)}^2 \right\}. \quad (2.34)$$

Consequently, (2.34) imply that

$$\begin{cases} \varphi_n(0, \cdot) + \frac{ab}{c} \psi_n(0, \cdot) \to 0 & \text{in} \quad H^{\frac{1}{3}}(0, T), \\ a\varphi_n(0, \cdot) + \frac{1}{c} \psi_n(0, \cdot) \to 0 & \text{in} \quad H^{\frac{1}{3}}(0, T), \\ \varphi_{n,x}(L, \cdot) + \frac{ab}{c} \psi_{n,x}(L, \cdot) \to 0 & \text{in} \quad L^2(0, T), \\ a\varphi_{n,x}(L, \cdot) + \frac{1}{c} \psi_{n,x}(L, \cdot) \to 0 & \text{in} \quad L^2(0, T), \end{cases} \quad (2.35)$$

as $n \to \infty$. Since $1 - a^2 b > 0$, (2.35) guarantees that the following convergences hold

$$\begin{cases} \varphi_n(0, \cdot) \to 0, \quad \psi_n(0, \cdot) \to 0 & \text{in} \quad H^{\frac{1}{3}}(0, T), \\ \varphi_{n,x}(L, \cdot) \to 0, \quad \psi_{n,x}(L, \cdot) \to 0 & \text{in} \quad L^2(0, T), \end{cases} \quad (2.36)$$

as $n \to \infty$. The next steps are devoted to pass the strong limit in the left hand side of (2.34). First, observe that from Proposition 2.2.4 we deduce that $\{(\varphi_n, \psi_n)\}_{n \in \mathbb{N}}$ is bounded in $L^2(0, T; (H^1(0, L))^2)$. Then, (2.17) implies that $\{(\varphi_{t,n}, \psi_{t,n})\}_{n \in \mathbb{N}}$ is bounded in $L^2(0, T; (H^{-2}(0, L))^2)$ and the compact embedding

$$H^1(0, L) \hookrightarrow L^2(0, L) \hookrightarrow H^{-2}(0, L)$$

allows us to conclude that $\{(\varphi_n,\psi_n)\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(0,T;\mathcal{X})$. Consequently, we obtain a subsequence, still denoted by the same index n, satisfying

$$(\varphi_n,\psi_n)\to(\varphi,\psi) \text{ in } L^2(0,T;\mathcal{X}), \text{ as } n\to\infty. \tag{2.37}$$

On the other hand, (2.21) and (2.34) imply that the sequences

$$\{\varphi_n(0,\cdot)\}_{n\in\mathbb{N}} \text{ and } \{\psi_n(0,\cdot)\}_{n\in\mathbb{N}} \text{ are bounded in } H^{\frac{1}{3}}(0,T).$$

Then, the following compact embedding

$$H^{\frac{1}{3}}(0,T)\hookrightarrow L^2(0,T) \tag{2.38}$$

guarantees that the above sequences are relatively compact in $L^2(0,T)$, that is, we obtain a subsequence, still denoted by the same index n, satisfying

$$\begin{cases}\varphi_n(0,\cdot)\to\varphi(0,\cdot) & \text{in} \quad L^2(0,T),\\ \psi_n(0,\cdot)\to\psi(0,\cdot) & \text{in} \quad L^2(0,T),\end{cases} \tag{2.39}$$

as $n\to\infty$. Then, from (2.36) and (2.39) we deduce that

$$\varphi(0,\cdot)=\psi(0,\cdot)=0.$$

Moreover, (2.21), (2.34) and (2.38) imply that $\{\varphi_n(L,t)\}_{n\in\mathbb{N}}$ and $\{\psi_n(L,t)\}_{n\in\mathbb{N}}$ are relatively compact in $L^2(0,T)$. Hence, we obtain a subsequence, still denoted by the same index, satisfying

$$\begin{cases}\varphi_n(L,\cdot)\to\varphi(L,\cdot) \text{ in } L^2(0,T),\\ \psi_n(L,\cdot)\to\psi(L,\cdot) \text{ in } L^2(0,T),\end{cases} \tag{2.40}$$

as $n\to\infty$. In addition, according to Proposition 2.2.5, we have

$$\begin{aligned}\|(\varphi_n^1,\psi_n^1)\|_{\mathcal{X}}^2 \leq{}& \frac{1}{T}\|(\varphi_n,\psi_n)\|_{L^2(0,T;\mathcal{X})}+\frac{1}{2}\|\varphi_{n,x}(L,\cdot)\|^2_{L^2(0,T)}\\ &+\frac{b}{2c}\|\psi_{n,x}(L,\cdot)\|^2_{L^2(0,T)}+\frac{br}{c^2}\|\psi_n(L,\cdot)\|^2_{L^2(0,T)}\\ &+\frac{1}{2}\left\|\varphi_{n,x}(L,\cdot)+\frac{ab}{c}\psi_{n,x}(L,\cdot)\right\|^2_{L^2(0,T)}+\frac{b}{2c}\left\|a\varphi_{n,x}(L,\cdot)+\frac{1}{c}\psi_{n,x}(L,\cdot)\right\|^2_{L^2(0,T)}.\end{aligned}$$

Then, from (2.35), (2.36), (2.37) and (2.40) we conclude that $\{(\varphi_n^1,\psi_n^1)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}$ and, therefore, we get

$$(\varphi_n^1,\psi_n^1)\to(\varphi^1,\psi^1) \text{ in } \mathcal{X}, \text{ as } n\to\infty. \tag{2.41}$$

Thus, Proposition 2.2.4 together with (2.41) imply that

$$\begin{cases}\varphi_{n,x}(L,\cdot)\to\varphi_x(L,\cdot) \text{ in } L^2(0,T),\\ \psi_{n,x}(L,\cdot)\to\psi_x(L,\cdot) \text{ in } L^2(0,T)\end{cases} \tag{2.42}$$

and

$$\begin{cases}\varphi_{n,xx}(L,\cdot)+\frac{ab}{c}\psi_{n,xx}(L,\cdot)\longrightarrow\varphi_{xx}(L,\cdot)+\frac{ab}{c}\psi_{xx}(L,\cdot),\\ \varphi_{n,xx}(0,\cdot)+\frac{ab}{c}\psi_{n,xx}(0,\cdot)\longrightarrow\varphi_{xx}(0,\cdot)+\frac{ab}{c}\psi_{xx}(0,\cdot),\\ a\varphi_{n,xx}(L,\cdot)+\frac{1}{c}\psi_{n,xx}(L,\cdot)+\frac{r}{c}\psi_n(L,\cdot)\longrightarrow a\varphi_{xx}(L,\cdot)+\frac{1}{c}\psi_{xx}(L,\cdot)+\frac{r}{c}\psi(L,\cdot),\\ a\varphi_{n,xx}(0,\cdot)+\frac{1}{c}\psi_{n,xx}(0,\cdot)+\frac{r}{c}\psi_n(0,\cdot)\longrightarrow a\varphi_{xx}(0,\cdot)+\frac{1}{c}\psi_{xx}(0,\cdot)+\frac{r}{c}\psi(0,\cdot),\end{cases}$$

in $L^2(0,T)$ as $n \to \infty$. Since (φ_n, ψ_n) is a solution of the adjoint system, we obtain that

$$\begin{cases} \varphi_{xx}(L,\cdot) + \frac{ab}{c}\psi_{xx}(L,\cdot) = 0, \\ \varphi_{xx}(0,\cdot) + \frac{ab}{c}\psi_{xx}(0,\cdot) = 0, \\ a\varphi_{xx}(L,\cdot) + \frac{1}{c}\psi_{xx}(L,\cdot) + \frac{r}{c}\psi(L,\cdot) = 0, \\ a\varphi_{xx}(0,\cdot) + \frac{1}{c}\psi_{xx}(0,\cdot) + \frac{r}{c}\psi(L,\cdot) = 0. \end{cases}$$

On the other hand, from (2.36) and (2.42), we have

$$\varphi_x(L,\cdot) = \psi_x(L,\cdot) = 0.$$

Finally, we obtain that (φ,ψ) is a solution of

$$\begin{cases} \varphi_t + \varphi_{xxx} + a\frac{ab}{c}\psi_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ \psi_t + \frac{r}{c}\psi_x + a\varphi_{xxx} + \frac{1}{c}\psi_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ a\varphi_{xx}(L,t) + \frac{1}{c}\psi_{xx}(L,t) + \frac{r}{c}\psi(L,t) = 0, & \text{in } (0,T), \\ a\varphi_{xx}(0,t) + \frac{1}{c}\psi_{xx}(0,t) + \frac{r}{c}\psi(0,t) = 0, & \text{in } (0,T), \\ \varphi_{xx}(L,t) + \frac{ab}{c}\psi_{xx}(L,t) = 0, & \text{in } (0,T). \\ \varphi_{xx}(0,t) + \frac{ab}{c}\psi_{xx}(0,t) = 0, & \text{in } (0,T). \\ \varphi_x(0,t) = \psi_x(0,t) = 0, & \text{in } (0,T), \\ \varphi(x,T) = \varphi^1(x), \qquad \psi(x,T) = \psi^1(x), & \text{in } (0,L), \end{cases} \tag{2.43}$$

satisfying the additional boundary conditions

$$\varphi(0,t) = \psi(0,t) = \varphi_x(L,t) = \psi_x(L,t) = 0 \ \text{ in } \ (0,T) \tag{2.44}$$

and

$$\|(\varphi^1,\psi^1)\|_{\mathcal{X}} = 1. \tag{2.45}$$

Observe that (2.45) implies that the solutions of (2.43)-(2.44) can not be identically zero. However, by Lemma 2.3.4 below, one can conclude that $(\varphi,\psi) = (0,0)$, which drive us to a contradiction. □

Lemma 2.3.4. *For any $T > 0$, let N_T denote the space of the initial states $(\varphi^1,\psi^1) \in \mathcal{X}$, such that the solution of (2.43) satisfies (2.44). Then, $N_T = \{0\}$.*

Proof. The proof uses the same arguments as those given in [78].

If $N_T \neq \{0\}$, the map $(\varphi^1,\psi^1) \in N_T \to A(N_T) \subset \mathbb{C}N_T$ (where $\mathbb{C}N_T$ denote the complexification of N_T) has (at least) one eigenvalue. Hence, there exist $\lambda \in \mathbb{C}$ and $\varphi_0, \psi_0 \in H^3(0,L)\setminus\{0\}$, such that

$$\begin{cases} \lambda\varphi_0 + \varphi_0''' + \frac{ab}{c}\psi_0''' = 0, & \text{in } (0,L), \\ \lambda\psi_0 + \frac{r}{c}\psi_0' + a\varphi_0''' + \frac{1}{c}\psi_0''' = 0, & \text{in } (0,L), \\ \varphi_0'(x) = \psi_0'(x) = 0, & \text{in } \{0,L\}, \\ a\varphi_0''(x) + \frac{1}{c}\psi_0''(x) + \frac{r}{c}\psi_0(x) = 0, & \text{in } \{0,L\}, \\ \varphi_0''(x) + \frac{ab}{c}\psi_0''(x) = 0, & \text{in } \{0,L\}, \\ \varphi_0(0) = \psi_0(0) = 0. \end{cases}$$

The notation $\{0,L\}$, used above, mean that the expression is applied in 0 and L.

Since $1 - a^2b > 0$, the above system becomes

$$\begin{cases} \lambda\varphi_0 + \varphi_0''' + \frac{ab}{c}\psi_0''' = 0, & \text{in } (0,L), \\ \lambda\psi_0 + \frac{r}{c}\psi_0' + a\varphi_0''' + \frac{1}{c}\psi_0''' = 0, & \text{in } (0,L), \\ \varphi_0(0) = \varphi_0'(0) = \varphi_0''(0) = 0, \\ \psi_0(0) = \psi_0'(0) = \psi_0''(0) = 0. \end{cases} \tag{2.46}$$

By straightforward computations we see that $(\varphi_0,\psi_0)=(0,0)$ is the unique solution of (2.46) for all $L>0$, which concludes the proof of Lemma 2.3.4 and Proposition 2.3.3. □

The following theorem gives a positive answer for the control problem:

Theorem 2.3.5. *Let $T>0$ and $L>0$. Then, the system (2.28)-(2.31) is exactly controllable in time T.*

Proof. Let us denote by Γ the linear and bounded map defined by

$$\begin{array}{rccc}\Gamma: & L^2(0,L)\times L^2(0,L) & \longrightarrow & L^2(0,L)\times L^2(0,L)\\ & (\varphi^1(\cdot),\psi^1(\cdot)) & \longmapsto & \Gamma(\varphi^1(\cdot),\psi^1(\cdot))=(u(\cdot,T),v(\cdot,T)),\end{array}$$

where (u,v) is the solution of (2.28)-(2.31), with

$$\begin{cases} h_0(t)=(-\Delta_t)^{\frac{1}{3}}\left(\varphi(0,t)+\frac{ab}{c}\psi(0,t)\right), & h_1(t)=\varphi_x(L,t)+\frac{ab}{c}\psi_x(L,t),\\ g_0(t)=(-\Delta_t)^{\frac{1}{3}}\left(a\varphi(0,t)+\frac{1}{c}\psi(0,t)\right), & g_1(t)=a\varphi_x(L,t)+\frac{1}{c}\psi_x(L,t),\end{cases} \tag{2.47}$$

and (φ,ψ) the solution of the system (2.17)-(2.20) with $\Delta_t=\partial_t^2$ and initial data (φ^1,ψ^1). According to Lemma 2.3.2 and Proposition 2.3.3, we obtain

$$\begin{aligned}(\Gamma(\varphi^1,\psi^1),(\varphi^1,\psi^1))_{(L^2(0,L))^2}=&\left\|\varphi_x(L,\cdot)+\frac{ab}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}\\ &+\left\|a\varphi_x(L,\cdot)+\frac{1}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}\\ &+\left((-\Delta_t)^{\frac{1}{3}}\left(\varphi(0,\cdot)+\frac{ab}{c}\psi(0,\cdot)\right),\varphi(0,\cdot)+\frac{ab}{c}\psi(0,\cdot)\right)_{L^2(0,T)}\\ &+\left((-\Delta_t)^{\frac{1}{3}}\left(a\varphi(0,\cdot)+\frac{1}{c}\psi(0,\cdot)\right),a\varphi(0,\cdot)+\frac{1}{c}\psi(0,\cdot)\right)_{L^2(0,T)}\\ =&\left\|\varphi_x(L,\cdot)+\frac{ab}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}+\left\|a\varphi_x(L,\cdot)+\frac{1}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}\\ &+\left\|(-\Delta_t)^{\frac{1}{6}}\left(a\varphi(0,\cdot)+\frac{1}{c}\psi(0,\cdot)\right)\right\|^2_{L^2(0,T)}\\ &+\left\|(-\Delta_t)^{\frac{1}{6}}\left(\varphi(0,\cdot)+\frac{ab}{c}\psi(0,\cdot)\right)\right\|^2_{L^2(0,T)}\\ \geq& C^{-1}\|(\varphi^1,\psi^1)\|^2_{\mathcal{X}}.\end{aligned}$$

Thus, by the Lax-Milgram theorem, Γ is invertible. Consequently, for given $(u^1,v^1)\in(L^2(0,L))^2$, we can define $(\varphi^1,\psi^1):=\Gamma^{-1}(u^1,v^1)$ to solve the system (2.17)-(2.20) and get $(\varphi,\psi)\in\mathcal{Z}_T$. Then, if $h_0(t)$, $h_1(t)$, $g_0(t)$ and $g_1(t)$ are given by (2.47), the corresponding solution (u,v) of the system (2.28)-(2.31), satisfies

$$(u(\cdot,0),v(\cdot,0))=(0,0)\quad\text{and}\quad(u(\cdot,T),v(\cdot,T))=(u^1(\cdot),v^1(\cdot)).$$

□

Remark 2.3.6. *An important question is whether the exact controllability holds, in time $T>0$, when we consider the boundary condition with another configuration, for example,*

$$\begin{cases} u_{xx}(0,t)=0 & u_x(L,t)=h_1(t) & u_{xx}(L,t)=h_2(t), & \text{in } (0,T),\\ v_{xx}(0,t)=0, & v_x(L,t)=g_1(t), & v_{xx}(L,t)=g_2(t), & \text{in } (0,T).\end{cases} \tag{2.48}$$

Observe that, in this case it would be necessary to prove that the following observability inequality

$$\|(\varphi^1,\psi^1)\|^2_{\mathcal{X}} \leq C\left\{\left\|(-\Delta_t)^{\frac{1}{6}}\left(\varphi(L,\cdot)+\frac{ab}{c}\psi(L,\cdot)\right)\right\|^2_{L^2(0,T)} + \left\|\varphi_x(L,\cdot)+\frac{ab}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)} + \left\|(-\Delta_t)^{\frac{1}{6}}\left(a\varphi(L,\cdot)+\frac{1}{c}\psi(L,\cdot)\right)\right\|^2_{L^2(0,T)} + \left\|a\varphi_x(L,\cdot)+\frac{1}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}\right\},$$

holds for any (φ^1,ψ^1) in $\mathcal{X}$, where (φ,ψ) is solution of (2.17)-(2.20) with initial data (φ^1,ψ^1). It can be done using Proposition 2.2.4 together with the contradiction argument used in the proof of Proposition 2.3.3. Thus, the next result about the exact controllability of the system (2.28)-(2.48) also holds:

Theorem 2.3.7. *Let $T>0$ and $L>0$. Then, the system (2.28)-(2.48) is exactly controllable in time T.*

Case 2

We consider the following boundary conditions:

$$\begin{cases} u_{xx}(0,t)=0 & u_x(L,t)=h_1(t) \quad u_{xx}(L,t)=0, & \text{in } (0,T), \\ v_{xx}(0,t)=g_0(t), & v_x(L,t)=g_1(t), \quad v_{xx}(L,t)=g_2(t), & \text{in } (0,T). \end{cases} \tag{2.49}$$

First, as in subsection above, we give an equivalent condition for the exact controllability property. It can be done using the same idea of the proof of Lemma 2.3.2.

Lemma 2.3.8. *For any (u^1,v^1) in $\mathcal{X}$, there exist four controls $\overrightarrow{h}=(0,h_1,0)$ and $\overrightarrow{g}=(g_0,g_1,g_2)$ in $\mathcal{H}_T$, such that the solution (u,v) of (2.28)-(2.49) satisfies (2.30) if and only if*

$$\begin{aligned}\int_0^L (u^1(x)\varphi^1(x)+v^1(x)\psi^1(x))dx = &\int_0^T g_0(t)\left(a\varphi(0,t)+\frac{1}{c}\psi(0,t)\right)dt \\ &+\int_0^t g_1(t)\left(a\varphi_x(L,t)+\frac{1}{c}\psi_x(L,t)\right)dt \\ &-\int_0^T g_2(t)\left(a\varphi(L,t)+\frac{1}{c}\psi(L,t)\right)dt \\ &+\int_0^T h_1(t)\left(\varphi_x(L,t)+\frac{ab}{c}\psi_x(L,t)\right)dt,\end{aligned} \tag{2.50}$$

for any (φ^1,ψ^1) in $\mathcal{X}$, where (φ,ψ) is the solution of the backward system (2.17)-(2.20).

To prove the exact controllability property, it suffices to prove the following observability inequality:

Proposition 2.3.9. *Let $T>0$ and $L\in(0,\infty)\setminus\mathcal{F}_r$, where $\mathcal{F}_r$ is given by (2.1). Then, there exists a constant $C(T,L)>0$, such that*

$$\|(\varphi^1,\psi^1)\|^2_{\mathcal{X}} \leq C\left\{\left\|(-\Delta_t)^{\frac{1}{6}}\left(a\varphi(0,\cdot)+\frac{1}{c}\psi(0,\cdot)\right)\right\|^2_{L^2(0,T)} + \left\|\varphi_x(L,\cdot)+\frac{ab}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)} + \left\|(-\Delta_t)^{\frac{1}{6}}\left(a\varphi(L,\cdot)+\frac{1}{c}\psi(L,\cdot)\right)\right\|^2_{L^2(0,T)} + \left\|a\varphi_x(L,\cdot)+\frac{1}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}\right\},$$

for any (φ^1, ψ^1) *in* $\mathcal{X}$*, where* (φ, ψ) *is solution of* (2.17)-(2.20) *with initial data* (φ^1, ψ^1)*, where* $\Delta_t := \partial_t^2$*.*

Proof. We proceed as in the proof of Proposition 2.3.3 using the contradiction argument. Therefore, we will summarize it. Firstly, we show that the sequences $\{(\varphi_n^1, \psi_n^1)\}_{n\in\mathbb{N}}$,

$$\{a\varphi_n(0,\cdot) + \frac{1}{c}\psi_n(0,\cdot)\}_{n\in\mathbb{N}},$$

$$\{a\varphi_n(L,\cdot) + \frac{1}{c}\psi_n(L,\cdot)\}_{n\in\mathbb{N}},$$

$$\{a\varphi_{n,x}(L,\cdot) + \frac{1}{c}\psi_{n,x}(L,\cdot)\}_{n\in\mathbb{N}}$$

and

$$\{\varphi_{n,x}(L,\cdot) + \frac{ab}{c}\psi_{n,x}(L,\cdot)\}_{n\in\mathbb{N}},$$

are relatively compact in $\mathcal{X}$ and $L^2(0,T;\mathcal{X})$, respectively. Next, we proceed as in the proof of Proposition 2.3.3 to get that

$$a\varphi_n(0,\cdot) + \frac{1}{c}\psi_n(0,\cdot) \to 0,$$

$$a\varphi_n(L,\cdot) + \frac{1}{c}\psi_n(L,\cdot) \to 0,$$

$$\varphi_{n,x}(L,\cdot) \to 0, \ \ \psi_x(L,\cdot) \to 0,$$

as $n \to \infty$, and

$$||(\varphi,\psi)||_{(L^2(0,L))^2} = 1.$$

Finally, combining the hidden regularity of the solutions of the adjoint system (2.21) and the compact embedding $H^{\frac{1}{3}}(0,T) \hookrightarrow L^2(0,T)$, we conclude that (φ, ψ) satisfies

$$\begin{cases} \varphi_t + \varphi_{xxx} + \frac{ab}{c}\psi_{xxx} = 0 & \text{in } (0,L)\times(0,T), \\ \psi_t + \frac{r}{c}\psi_x + a\varphi_{xxx} + \frac{1}{c}\psi_{xxx} = 0 & \text{in } (0,L)\times(0,T), \\ \varphi_{xx}(L,t) + \frac{ab}{c}\psi_{xx}(L,t) = 0 & \text{in } (0,T), \\ \varphi_{xx}(0,t) + \frac{ab}{c}\psi_{xx}(0,t) = 0 & \text{in } (0,T), \\ a\varphi_{xx}(L,t) + \frac{1}{c}\psi_{xx}(L,t) + \frac{r}{c}\psi(L,t) = 0 & \text{in } (0,T), \\ a\varphi_{xx}(0,t) + \frac{1}{c}\psi_{xx}(0,t) + \frac{r}{c}\psi(0,t) = 0 & \text{in } (0,T), \\ \varphi_x(0,t) = \psi_x(0,t) = 0 & \text{in } (0,T), \\ \varphi(x,T) = \varphi^1(x), \ \ \psi(x,T) = \psi^1(x) & \text{in } (0,L) \end{cases} \tag{2.51}$$

and

$$\begin{cases} a\varphi(L,t) + \frac{1}{c}\psi(L,t) = 0 & \text{in } (0,T), \\ a\varphi(0,t) + \frac{1}{c}\psi(0,t) = 0 & \text{in } (0,T), \\ \varphi_x(L,t) = \psi_x(L,t) = 0 & \text{in } (0,T), \\ \|(\varphi,\psi)\|_{\mathcal{X}} = 1. \end{cases} \tag{2.52}$$

Notice that the solutions of (2.51)-(2.52) can not be identically zero. Therefore, from Lemma 2.3.10, one can conclude that $(\varphi, \psi) = (0,0)$, which drive us to a contradiction. □

Lemma 2.3.10. *For any* $T > 0$*, let* N_T *denote the space of the initial states* $(\varphi^1, \psi^1) \in \mathcal{X}$*, such that the solution of* (2.51) *satisfies* (2.52)*. Then, for* $L \in (0,\infty) \setminus \mathcal{F}_r$*,* $N_T = \{0\}$*.*

Proof. By the same arguments given in [78], if $N_T \neq \{0\}$, the map $(\varphi^1, \psi^1) \in N_T \to A(N_T) \subset \mathbb{C}N_T$ has (at least) one eigenvalue. Hence, there exist $\lambda \in \mathbb{C}$ and $\varphi_0, \psi_0 \in H^3(0,L) \setminus \{0\}$, such that

$$\begin{cases} \lambda\varphi_0 + \varphi_0''' + \frac{ab}{c}\psi_0''' = 0, & \text{in } (0,L), \\ \lambda\psi_0 + \frac{r}{c}\psi_0' + a\varphi_0''' + \frac{1}{c}\psi_0''' = 0, & \text{in } (0,L), \\ a\varphi_0(x) + \frac{1}{c}\psi_0(x) = 0, & \text{in } \{0,L\}, \\ \varphi_0'(x) = \psi_0'(x) = 0, & \text{in } \{0,L\}, \\ \varphi_0''(x) + \frac{ab}{c}\psi_0''(x) = 0, & \text{in } \{0,L\}, \\ a\varphi_0''(x) + \frac{1}{c}\psi_0''(x) + \frac{r}{c}\psi_0(x) = 0, & \text{in } \{0,L\}. \end{cases} \tag{2.53}$$

To conclude the proof of the Lemma 2.3.10, we prove that this does not hold if $L \in (0,\infty) \setminus \mathcal{F}_r$. To simplify the notation, henceforth we denote $(\varphi_0, \psi_0) := (\varphi, \psi)$.

Lemma 2.3.11. *Let $L > 0$. Consider the assertion*

$(\mathcal{N}): \quad \exists\lambda \in \mathbb{C}, \exists\varphi, \psi \in H^3(0,L) \setminus (0,0), \quad$ *such that*

$$\begin{cases} \lambda\varphi + \varphi''' + \frac{ab}{c}\psi''' = 0, & \textit{in } (0,L), \\ \lambda\psi + \frac{r}{c}\psi' + a\varphi''' + \frac{1}{c}\psi''' = 0, & \textit{in } (0,L), \\ a\varphi(x) + \frac{1}{c}\psi(x) = 0, & \textit{in } \{0,L\}, \\ \varphi'(x) = \psi'(x) = 0, & \textit{in } \{0,L\}, \\ \varphi''(x) + \frac{ab}{c}\psi''(x) = 0, & \textit{in } \{0,L\}, \\ a\varphi''(x) + \frac{1}{c}\psi''(x) + \frac{r}{c}\psi(x) = 0, & \textit{in } \{0,L\}. \end{cases}$$

Then, $(\mathcal{N})$ holds if and only if $L \in \mathcal{F}_r$.

Proof. We use an argument similar to the one used in [78, Lemma 3,5]. Let us introduce the notation $\hat{\varphi}(\xi) = \int_0^L e^{-ix\xi}\varphi(x)dx$ and $\hat{\psi}(\xi) = \int_0^L e^{-ix\xi}\psi(x)dx$. Then, multiplying the first and the second equations in $(\mathcal{N})$ by $e^{-ix\xi}$ and integrating by part in $(0,L)$, it follows that

$$\begin{aligned} &\left((i\xi)^3 + \lambda\right)\hat{\varphi}(\xi) + \frac{ab}{c}(i\xi)^3\hat{\psi}(\xi) \\ &\qquad + \Bigg[\Bigg(\left(\varphi''(x) + \frac{ab}{c}\psi''(x)\right) + (i\xi)\left(\varphi'(x) + \frac{ab}{c}\psi'(x)\right) \\ &\qquad\qquad + (i\xi)^2\left(\varphi(x) + \frac{ab}{c}\psi(x)\right)\Bigg)e^{-ix\xi}\Bigg]_0^L = 0 \end{aligned}$$

and

$$\begin{aligned} &\left(\frac{1}{c}(i\xi)^3 + \frac{r}{c}(i\xi) + \lambda\right)\hat{\psi}(\xi) + a(i\xi)^3\hat{\varphi}(\xi) + \Bigg[\Bigg(\left(a\varphi''(x) + \frac{1}{c}\psi''(x) + \frac{r}{c}\psi(\xi)\right) \\ &\qquad + (i\xi)\left(a\varphi'(x) + \frac{1}{c}\psi'(x)\right) + (i\xi)^2\left(a\varphi(x) + \frac{1}{c}\psi(x)\right)\Bigg)e^{-ix\xi}\Bigg]_0^L = 0. \end{aligned}$$

The boundary conditions allow us to conclude that

$$\begin{cases} [(i\xi)^3 + \lambda]\hat{\varphi}(\xi) + \dfrac{ab}{c}(i\xi)^3\hat{\psi}(\xi) = (i\xi)^2\left(\varphi(0) + \dfrac{ab}{c}\psi(0) - \left(\varphi(L) + \dfrac{ab}{c}\psi(L)\right)e^{-iL\xi}\right), \\ \\ \dfrac{1}{c}[(i\xi)^3 + r(i\xi) + c\lambda]\hat{\psi}(\xi) + a(i\xi)^3\hat{\varphi}(\xi) = 0. \end{cases} \tag{2.54}$$

Then, from the first equation in (2.54), we obtain

$$\hat{\varphi}(\xi) = \frac{(i\xi)^2\left(\alpha + \beta e^{-iL\xi}\right)}{(i\xi)^3 + \lambda} - \frac{ab(i\xi)^3\hat{\psi}(\xi)}{c\left((i\xi)^3 + \lambda\right)}, \tag{2.55}$$

where $\alpha = \varphi(0) + \frac{ab}{c}\psi(0)$ and $\beta = -\varphi(L) - \frac{ab}{c}\psi(L)$. Replacing the above expression in the second equation in (2.54) it follows that

$$\frac{1}{c}\left[(i\xi)^3 + r(i\xi) + c\lambda - \frac{a^2b(i\xi)^6}{(i\xi)^3 + \lambda}\right]\hat{\psi}(\xi) = -\frac{a(i\xi)^5\left(\alpha + \beta e^{-iL\xi}\right)}{(i\xi)^3 + \lambda}.$$

Thus,

$$\hat{\psi}(\xi) = -\frac{ac(i\xi)^5\left(\alpha + \beta e^{-iL\xi}\right)}{(1 - a^2b)(i\xi)^6 + r(i\xi)^4 + (c+1)\lambda(i\xi)^3 + r\lambda(i\xi) + c\lambda^2}. \tag{2.56}$$

Replacing (2.56) in (2.55), we obtain

$$\hat{\varphi}(\xi) = \frac{(i\xi)^2\left((i\xi)^3 + r(i\xi) + c\lambda\right)\left(\alpha + \beta e^{-iL\xi}\right)}{(1 - a^2b)(i\xi)^6 + r(i\xi)^4 + (c+1)\lambda(i\xi)^3 + r\lambda(i\xi) + c\lambda^2}.$$

Setting $\lambda = ip$, $p \in \mathbb{C}$, from the previous identities we can write $\hat{\psi}(\xi) = -i\left[acf(\xi)\right]$ and $\hat{\varphi}(\xi) = -ig(\xi)$, where

$$\begin{cases} f(\xi) = \dfrac{\xi^5\left(\alpha + \beta e^{-iL\xi}\right)}{P(\xi)}, \\[2ex] g(\xi) = \dfrac{\xi^2\left(\xi^3 - r\xi - cp\right)\left(\alpha + \beta e^{-iL\xi}\right)}{P(\xi)}, \end{cases}$$

with

$$P(\xi) := (1 - a^2b)\xi^6 - r\xi^4 - (c+1)p\xi^3 + rp\xi + cp^2.$$

Using Paley-Wiener theorem (see [97, Section 4, page 161]) and the usual characterization of $H^2(\mathbb{R})$ functions by means of their Fourier transforms, we see that $(\mathcal{N})$ is equivalent to the existence of $p \in \mathbb{C}$ and $(\alpha, \beta) \in \mathbb{C}^2 \setminus (0, 0)$, such that

(i) f and g are entire functions in $\mathbb{C}$,

(ii) $\displaystyle\int_{\mathbb{R}} |f(\xi)|^2(1 + |\xi|^2)^2 d\xi < \infty$ and $\int_{\mathbb{R}} |g(\xi)|^2(1 + |\xi|^2)^2 d\xi < \infty$,

(iii) $\forall \xi \in \mathbb{C}$, we have that $|f(\xi)| \le c_1(1 + |\xi|)^k e^{L|Im\xi|}$ and $|g(\xi)| \le c_1(1 + |\xi|)^k e^{L|Im\xi|}$, for some positive constants c_1 and k.

Notice that if (i) holds true, then (ii) and (iii) are satisfied. Recall that f and g are entire functions if and only if the roots $\xi_0, \xi_1, \xi_2, \xi_3, \xi_4$ and ξ_5 of $P(\xi)$ are roots of $\xi^5\left(\alpha + \beta e^{-iL\xi}\right)$ and $\xi^2(\xi^3 - r\xi - cp)\left(\alpha + \beta e^{-iL\xi}\right)$.

Let us first assume that $\xi = 0$ is not a root of $P(\xi)$. Thus, it is sufficient to consider the case when $\alpha + \beta e^{-iL\xi}$ and $P(\xi)$ share the same roots. Observe that the roots of $\alpha + \beta e^{-iL\xi}$ are simple, unless $\alpha = \beta = 0$ (Indeed, in this case $\varphi(0) + \frac{ab}{c}\psi(0) = 0$ and $\varphi(L) + \frac{ab}{c}\psi(L) = 0$ and using the system (2.53) we conclude that $(\varphi, \psi) = (0, 0)$, which is a contradiction). Then, (i) holds provided that the roots of $P(\xi)$ are simple. Therefore, it follows that $(\mathcal{N})$ is equivalent to the existence of complex numbers p and ξ_0 and positive integers k, l, m, n and s, such that, if we set

$$\xi_1 = \xi_0 + \frac{2\pi}{L}k, \quad \xi_2 = \xi_1 + \frac{2\pi}{L}l, \quad \xi_3 = \xi_2 + \frac{2\pi}{L}m, \quad \xi_4 = \xi_3 + \frac{2\pi}{L}n \quad \text{and} \quad \xi_5 = \xi_4 + \frac{2\pi}{L}s, \tag{2.57}$$

$P(\xi)$ can be written as follows

$$P(\xi) = (\xi - \xi_0)(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4)(\xi - \xi_5).$$

In particular, we obtain the following relations:

$$\xi_0 + \xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 = 0, \tag{2.58}$$

$$\begin{aligned} \xi_0(\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5) + \xi_1(\xi_2 + \xi_3 + \xi_4 + \xi_5) + \xi_2(\xi_3 + \xi_4 + \xi_5) \\ + \xi_3(\xi_4 + \xi_5) + \xi_4\xi_5 = -\frac{r}{1 - a^2b} \end{aligned} \tag{2.59}$$

and

$$\xi_0\xi_1\xi_2\xi_3\xi_4\xi_5 = \left(\frac{c}{1 - a^2b}\right) p^2.$$

(2.57) and (2.58) imply that

$$\begin{aligned} \xi_0 + \left(\xi_0 + \frac{2\pi}{L}k\right) + \left(\xi_0 + \frac{2\pi}{L}(k + l)\right) + \left(\xi_0 + \frac{2\pi}{L}(k + l + m)\right) \\ + \left(\xi_0 + \frac{2\pi}{L}(k + l + m + n)\right) + \left(\xi_0 + \frac{2\pi}{L}(k + l + m + n + s)\right) = 0. \end{aligned}$$

Straightforward computations lead to

$$\xi_0 = -\frac{\pi}{3L}(5k + 4l + 3m + 2n + s). \tag{2.60}$$

On the other hand, from (2.59), we obtain

$$\begin{aligned} \xi_0\left(5\xi_0 + \frac{2\pi}{L}(5k + 4l + 3m + 2n + s)\right) \\ + \left(\xi_0 + \frac{2\pi}{L}k\right)\left(4\xi_0 + \frac{2\pi}{L}(4k + 4l + 3m + 2n + s)\right) \\ + \left(\xi_0 + \frac{2\pi}{L}(k + l)\right)\left(3\xi_0 + \frac{2\pi}{L}(3k + 3l + 3m + 2n + s)\right) \\ + \left(\xi_0 + \frac{2\pi}{L}(k + l + m)\right)\left(2\xi_0 + \frac{2\pi}{L}(2k + 2l + 2m + 2n + s)\right) \\ + \left(\xi_0 + \frac{2\pi}{L}(k + l + m + n)\right)\left(\xi_0 + \frac{2\pi}{L}(k + l + m + n + s)\right) = -\frac{r}{1 - a^2b}. \end{aligned}$$

Thus, we have

$$15\xi_0^2 + \frac{2\pi}{L}(25k + 20 + 15m + 10n + 5s)\xi_0 + \frac{4\pi^2}{L^2}\eta = -\frac{r}{1 - a^2b}, \tag{2.61}$$

where

$$\begin{aligned} \eta = k(10k + 10l + 9m + 7n + 4s) + l(6k + 6l + 6m + 5n + 3s) \\ + m(3k + 3l + 3m + 3n + 2s) + n(k + l + m + n + s). \end{aligned}$$

Replacing (2.60) in (2.61), we obtain

$$\frac{3rL^2}{1 - a^2b} = \pi^2 \left(5(5k + 4l + 3m + 2n + s)^2 - 12\eta\right).$$

From the discussion above, we can conclude that

$$\begin{cases} L = \pi\sqrt{\dfrac{(1-a^2b)\alpha(k,l,m,n,s)}{3r}}, \\ \\ \xi_0 = -\dfrac{\pi}{3}(5k+4l+3m+2n+s), \\ \\ p = \sqrt{\dfrac{(1-a^2b)\xi_0\xi_1\xi_2\xi_3\xi_4\xi_5}{c}}, \end{cases} \tag{2.62}$$

where

$$\begin{aligned} \alpha(k,l,m,n,s) :=& 5k^2+8l^2+9m^2+8n^2+5s^2+8kl+6km+4kn+2ks+12ml \\ &+8ln+3ls+12mn+6ms+8ns. \end{aligned}$$

Now, we assume that $\xi_0 = 0$ is a root of $P(\xi)$. Then, it follows that $p = 0$ and

$$\begin{cases} f(\xi) = \dfrac{\xi^5\left(\alpha+\beta e^{-iL\xi}\right)}{(1-a^2b)\xi^6 - r\xi^4} = \dfrac{\xi\left(\alpha+\beta e^{-iL\xi}\right)}{(1-a^2b)\xi^2 - r}, \\ \\ g(\xi) = \dfrac{\xi^2\left(\xi^3 - r\xi\right)\left(\alpha+\beta e^{-iL\xi}\right)}{(1-a^2b)\xi^6 - r\xi^4} = \dfrac{\left(\xi^2 - r\right)\left(\alpha+\beta e^{-iL\xi}\right)}{\xi\left((1-a^2b)\xi^2 - r\right)}. \end{cases}$$

In this case, $(\mathcal{N})$ holds if and only if f and g satisfy (i), (ii) and (iii). Thus, (i) holds provided that

$$\xi_0 = 0, \quad \xi_1 = \sqrt{\frac{r}{1-a^2b}} \quad \text{and} \quad \xi_2 = -\sqrt{\frac{r}{1-a^2b}}$$

are roots of $\alpha+\beta e^{-iL\xi}$. Therefore, we can write $\xi_1 = \xi_0 + \frac{2\pi}{L}k$, for $k \in \mathbb{Z}$. Consequently, it follows that

$$L = 2\pi k\sqrt{\frac{1-a^2b}{r}}. \tag{2.63}$$

Finally, from (2.62) and (2.63), we deduce that $(\mathcal{N})$ holds if and only if $L \in \mathcal{F}_r$, where $\mathcal{F}_r$ is given by (2.1). This completes the proof of Lemma 2.3.11, Lemma 2.3.10 and, consequently, the proof of Proposition 2.3.9. □

The next result gives a positive answer for the control problem, and can be proved using the same ideas presented in Theorem 2.3.5 and, thus, we will omit the proof.

Theorem 2.3.12. *Let $T > 0$ and $L \in (0,\infty)\setminus\mathcal{F}_r$, where $\mathcal{F}_r$ is given by (2.1). Then, the system (2.28)-(2.49) is exactly controllable in time T.*

Remark 2.3.13. *As in the previous subsection, the question here is whether system (2.28)-(2.64) is exactly controllable with another configuration of the boundary condition, for example,*

$$\begin{cases} u_{xx}(0,t) = h_0(t), & u_x(L,t) = h_1(t), & u_{xx}(L,t) = h_2(t) & \text{in } (0,T), \\ v_{xx}(0,t) = 0, & v_x(L,t) = g_1(t), & v_{xx}(L,t) = 0 & \text{in } (0,T). \end{cases} \tag{2.64}$$

The answer for this question is positive if we prove that the following observability inequality

$$\begin{aligned} \|(\varphi^1,\psi^1)\|^2_{\mathcal{X}} \le C\Bigg\{ &\left\|(-\Delta)^{\frac{1}{6}}\left(\varphi(0,\cdot)+\frac{ab}{c}\psi(0,\cdot)\right)\right\|^2_{L^2(0,T)} + \left\|\varphi_x(L,\cdot)+\frac{ab}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)} \\ &+ \left\|(-\Delta)^{\frac{1}{6}}\left(\varphi(L,\cdot)+\frac{ab}{c}\psi(L,\cdot)\right)\right\|_{L^2(0,T)} + \left\|a\varphi_x(L,\cdot)+\frac{1}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}\Bigg\}, \end{aligned}$$

holds, for any (φ^1, ψ^1) *in* $\mathcal{X}$*, where* (φ, ψ) *is solution of* (2.17)-(2.20) *with initial data* (φ^1, ψ^1)*. Note that it can be proved using Proposition 2.2.4 together with the contradiction argument as in the proof of Proposition 2.3.9. Thus, the exact controllability result is also true in this case.*

Theorem 2.3.14. *Let* $T > 0$ *and* $L \in (0, \infty) \setminus \mathcal{F}_r$*. Then, the system* (2.28)-(2.64) *is exactly controllable in time* T*.*

2.3.2 One Control

In this subsection, we intend to prove the exact controllability of the system by using only one boundary control h_1 or g_1 and fixing $h_0 = h_2 = g_0 = g_2 = 0$, namely,

$$\begin{cases} u_{xx}(0,t) = 0 & u_x(L,t) = h_1(t), & u_{xx}(L,t) = 0, & \text{in } (0,T), \\ v_{xx}(0,t) = 0, & v_x(L,t) = 0, & v_{xx}(L,t) = 0, & \text{in } (0,T). \end{cases} \tag{2.65}$$

or

$$\begin{cases} u_{xx}(0,t) = 0 & u_x(L,t) = 0 & u_{xx}(L,t) = 0, & \text{in } (0,T), \\ v_{xx}(0,t) = 0, & v_x(L,t) = g_1(t), & v_{xx}(L,t) = 0, & \text{in } (0,T). \end{cases} \tag{2.66}$$

The result below give us an equivalent condition for the exact controllability and the proof is analogous to the proof of the Lemma 2.3.2.

Lemma 2.3.15. *For any* (u^1, v^1) *in* $\mathcal{X}$*, there exist one control* $\overrightarrow{h} = (0, h_1, 0)$ *and* $\overrightarrow{g} = (0,0,0)$ *(resp.* $\overrightarrow{h} = (0,0,0)$ *and* $\overrightarrow{g} = (0, g_1, 0)$*) in* $\mathcal{H}_T$*, such that the solution* (u, v) *of* (2.28)-(2.65) *(resp.* (2.28)-(2.66)*) satisfies* (2.30) *if and only if*

$$\int_0^L (u^1(x)\varphi^1(x) + v^1(x)\psi^1(x))dx = \int_0^T h_1(t)\left[\varphi_x(L,t) + \frac{ab}{c}\psi_x(L,t)\right] dt$$

$$\left(\textit{resp.} \quad \int_0^L (u^1(x)\varphi^1(x) + v^1(x)\psi^1(x))dx = \int_0^T g_1(t)\left[a\varphi_x(L,t) + \frac{1}{c}\psi_x(L,t)\right] dt\right)$$

for any (φ^1, ψ^1) *in* $\mathcal{X}$*, where* (φ, ψ) *is the solution of the backward system* (2.17)-(2.20)*.*

Note that using the change of variable $x' = L - x$ and $t' = T - t$, the system (2.17)-(2.20) is equivalent to the following forward system

$$\begin{cases} \varphi_t + \varphi_{xxx} + \frac{ab}{c}\psi_{xxx} = 0, & \text{in } (0,L) \times (0,T), \\ \psi_t + \frac{r}{c}\psi_x + a\varphi_{xxx} + \frac{1}{c}\psi_{xxx} = 0, & \text{in } (0,L) \times (0,T), \\ \varphi(x,0) = \varphi^0(x), \ \psi(x,0) = \psi^0(x), & \text{in } (0,L), \end{cases} \tag{2.67}$$

with boundary conditions

$$\begin{cases} \varphi_{xx}(x,t) + \frac{ab}{c}\psi_{xx}(x,t) = 0, & \text{in } \{0,L\} \times (0,T), \\ a\varphi_{xx}(x,t) + \frac{1}{c}\psi_{xx}(x,t) + \frac{r}{c}\psi(x,t) = 0, & \text{in } \{0,L\} \times (0,T), \\ \varphi_x(L,t) = \psi_x(L,t) = 0, & \text{in } (0,T). \end{cases} \tag{2.68}$$

It is well know (according to the previous sections) that the observability inequality

$$\|(\varphi^0, \psi^0)\|^2_{\mathcal{X}} \le C \left\| \varphi_x(0,\cdot) + \frac{ab}{c}\psi_x(0,\cdot) \right\|^2_{L^2(0,T)} \tag{2.69}$$

or

$$\|(\varphi^0, \psi^0)\|^2_{\mathcal{X}} \le C \left\| a\varphi_x(0,\cdot) + \frac{1}{c}\psi_x(0,\cdot) \right\|^2_{L^2(0,T)} \tag{2.70}$$

plays a fundamental role for the study of the controllability. To prove (2.69) (resp. (2.70)), we use a direct approach based on the multiplier technique that gives us the observability inequality for small values of the length L and large time of control T.

Proposition 2.3.16. *Let us suppose that $T>0$ and $L>0$ satisfy*

$$1 > \frac{\beta C_T}{T}\left[L+\frac{r}{c}\right], \tag{2.71}$$

where C_T is the constant in (2.21) and β is the constant given by the embedding $H^{\frac{1}{3}}(0,T)\subset L^2(0,T)$. Then, there exists a constant $C(T,L)>0$, such that for any (φ^0,ψ^0) in $\mathcal{X}$ the observability inequality (2.69) (resp. (2.70)) holds, where (φ,ψ) is solution of (2.67)-(2.68) with initial data (φ^0,ψ^0).

Proof. We multiply the first equation in (2.67) by $(T-t)\varphi$, the second one by $\frac{b}{c}(T-t)\psi$ and integrate over $(0,T)\times(0,L)$, to give us:

$$\begin{aligned}
\frac{T}{2}\int_0^L(\varphi_0^2(x)+\frac{b}{c}\psi_0^2(x))dx =& \frac{1}{2}\int_0^T\int_0^L\left(\varphi^2(x,t)+\frac{b}{c}\psi^2(x,t)\right)dxdt\\
&+\int_0^T(T-t)\left[\varphi(L,t)\left(\varphi_{xx}(L,t)+\frac{ab}{c}\psi_{xx}(L,t)\right)\right]dt\\
&-\int_0^T(T-t)\left[\varphi(0,t)\left(\varphi_{xx}(0,t)+\frac{ab}{c}\psi_{xx}(0,t)\right)\right]dt\\
&+\int_0^T(T-t)\left[\frac{b}{c}\psi(L,t)\left(a\varphi_{xx}(L,t)+\frac{\psi_{xx}(L,t)}{c}+\frac{r}{2c}\psi(L,t)\right)\right]dt\\
&+\int_0^T(T-t)\left[-\frac{b}{c}\psi(0,t)\left(a\varphi_{xx}(0,t)+\frac{\psi_{xx}(0,t)}{c}+\frac{r}{2c}\psi(0,t)\right)\right]dt\\
&+\frac{1}{2}\int_0^T(T-t)\left[\varphi_x^2(0,t)+\frac{2ab}{c}\psi_x(0,t)\varphi_x(0,t)+\frac{b}{c^2}\psi_x^2(0,t)\right]dt.
\end{aligned}$$

From the boundary conditions (2.68), we have that

$$\begin{aligned}
\|(\varphi^0,\psi^0)\|^2_{\mathcal{X}} \leq& \frac{1}{T}\|(\varphi,\psi)\|^2_{L^2(0,T;\mathcal{X})}+\frac{br}{c^2T}\|\psi(0,\cdot)\|^2_{L^2(0,T)}-\frac{br}{c^2}\int_0^T\frac{T-t}{T}\psi(L,t)^2dt\\
&+\int_0^T\left[\varphi_x^2(0,t)+\frac{2ab}{c}\psi_x(0,t)\varphi_x(0,t)+\frac{b}{c^2}\psi_x^2(0,t)\right]dt,\\
\leq& \frac{1}{T}\|(\varphi,\psi)\|^2_{L^2(0,T;\mathcal{X})}+\frac{\beta br}{c^2T}\|\psi(0,\cdot)\|^2_{H^{\frac{1}{3}}(0,T)}\\
&+\frac{1}{a^2b}\left\|\varphi_x(0,\cdot)+\frac{ab}{c}\psi_x(0,\cdot)\right\|^2_{L^2(0,T)},
\end{aligned}$$

$$\left(\text{resp. } \|(\varphi^0,\psi^0)\|^2_{\mathcal{X}}\leq\frac{1}{T}\|(\varphi,\psi)\|^2_{L^2(0,T;\mathcal{X})}+\frac{\beta br}{c^2T}\|\psi(0,\cdot)\|^2_{H^{\frac{1}{3}}(0,T)}+\frac{1}{a^2}\left\|a\varphi_x(0,\cdot)+\frac{1}{c}\psi_x(0,\cdot)\right\|^2_{L^2(0,T)}\right)$$

where β is the constant given by the compact embedding $H^{\frac{1}{3}}(0,T)\subset L^2(0,T)$. On the other hand, note that $L^\infty(0,L)\subset L^2(0,L)$, thus

$$\|\varphi(\cdot,t)\|^2_{L^2(0,L)}\leq L\|\varphi(\cdot,t)\|^2_{L^\infty(0,L)}, \quad \text{and} \quad \|\psi(\cdot,t)\|^2_{L^2(0,L)}\leq L\|\psi(\cdot,t)\|^2_{L^\infty(0,L)}, \tag{2.72}$$

Hence,

$$
\begin{aligned}
\|(\varphi,\psi)\|^2_{L^2(0,T;\mathcal{X})} &= \int_0^T \left\{ \|\varphi(\cdot,t)\|^2_{L^2(0,L)} + \frac{b}{c}\|\psi(\cdot,t)\|^2_{L^2(0,L)} \right\} dt \\
&\le L \int_0^T \left\{ \|\varphi(\cdot,t)\|^2_{L^\infty(0,L)} + \frac{b}{c}\|\psi(\cdot,t)\|^2_{L^\infty(0,L)} \right\} dt \\
&\le L\beta \|\varphi\|^2_{H^{\frac{1}{3}}(0,T;L^\infty(0,L))} + \frac{bL\beta}{c}\|\psi\|^2_{H^{\frac{1}{3}}(0,T;L^\infty(0,L))}.
\end{aligned}
$$

Thanks to the Proposition 2.2.4, we obtain

$$
\begin{aligned}
\|(\varphi^0,\psi^0)\|^2_{\mathcal{X}} \le & \frac{L\beta C_T}{T}\|\varphi^0\|^2_{L^2(0,L)} + \frac{bL\beta C_T}{cT}\|\psi^0\|^2_{L^2(0,L)} + \frac{\beta C_T br}{c^2T}\|\psi^0\|^2_{L^2(0,L)} \\
& \qquad + \frac{1}{a^2b}\left\| \varphi_x(0,\cdot) + \frac{ab}{c}\psi_x(0,\cdot)\right\|^2_{L^2(0,T)} \\
\le & \frac{L\beta C_T}{T}\|(\varphi^0,\psi^0)\|^2_{\mathcal{X}} + \frac{\beta C_T r}{cT}\|(\varphi^0,\psi^0)\|^2_{\mathcal{X}} \\
& \qquad + \frac{1}{a^2b}\left\| \varphi_x(0,\cdot) + \frac{ab}{c}\psi_x(0,\cdot)\right\|^2_{L^2(0,T)}.
\end{aligned}
$$

Finally, it follows that

$$
\|(\varphi^0,\psi^0)\|^2_{\mathcal{X}} \le K \left\| \varphi_x(0,\cdot) + \frac{ab}{c}\psi_x(0,\cdot)\right\|^2_{L^2(0,T)}
$$

under the condition

$$
K = \frac{1}{a^2b}\left(1 - \frac{\beta C_T}{T}\left[L + \frac{r}{c}\right]\right)^{-1} > 0. \tag{2.73}
$$

□

From the observability inequality (2.69), the following result holds.

Theorem 2.3.17. *Let $T>0$ and $L>0$ satisfying (2.71). Then, the system (2.28)-(2.65) (resp. (2.28)-(2.66)) is exactly controllable in time T.*

Proof. Consider the map

$$
\begin{array}{rccc}
\Gamma: & L^2(0,L)\times L^2(0,L) & \longrightarrow & L^2(0,L)\times L^2(0,L) \\
 & (\varphi^1(\cdot),\psi^1(\cdot)) & \longmapsto & \Gamma(\varphi^1(\cdot),\psi^1(\cdot)) = (u(\cdot,T),v(\cdot,T))
\end{array}
$$

where (u,v) is the solution of (2.28)-(2.49), with

$$
\begin{cases}
h_1(t) = \varphi_x(L,t) + \frac{ab}{c}\psi_x(L,t), \\
g_1(t) = a\varphi_x(L,t) + \frac{1}{c}\psi_x(L,t),
\end{cases}
$$

and (φ,ψ) is the solution of the system (2.17)-(2.20) with initial data (φ^1,ψ^1). By (2.69) (resp. (2.70)) and the Lax-Milgram theorem, the proof is achieved. □

2.4 The Nonlinear Control System

We are now in position to prove our main result considering several configurations of the control in the boundary conditions. Let $T > 0$, from Theorems 2.3.5, 2.3.7, 2.3.12, 2.3.14 and 2.3.17, we can define the bounded linear operators

$$\Lambda_i : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{H}_T \times \mathcal{H}_T \qquad (i = 1, 2, 3, 4, 5, 6),$$

such that, for any $(u^0, v^0) \in \mathcal{X}$ and $(u^1, v^1) \in \mathcal{X}$,

$$\Lambda_i \left(\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} \right) := \begin{pmatrix} \vec{h}_i \\ \vec{g}_i \end{pmatrix},$$

where $\vec{h}_i$ and $\vec{g}_i$ were defined in the Introduction.

Proof of Theorem 2.1.1. We treat the nonlinear problem (1.16)-(1.17) using a classical fixed point argument.

According to Remark 2.2.8, the solution can be written as

$$\begin{aligned} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} =& W_0(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + W_{bdr}(t) \begin{pmatrix} \vec{h}_i \\ \vec{g}_i \end{pmatrix} \\ &- \int_0^t W_0(t-\tau) \begin{pmatrix} a_1(vv_x)(\tau) + a_2(uv)_x(\tau) \\ \frac{r}{c} v_x(\tau) + \frac{a_2 b}{c}(uu_x)(\tau) + \frac{a_1 b}{c}(uv)_x(\tau) \end{pmatrix} d\tau, \end{aligned}$$

for $i = 1, 2, 3, 4, 5, 6$, where $\{W_0(t)\}_{t\geq 0}$ and $\{W_{bdr}(t)\}_{t\geq 0}$ are the operators defined in Proposition 2.2.1. We only analyze the case $i = 1$, since the other cases are analogous we will omit them.

For $u, v \in \mathcal{Z}_T$, let us define

$$\begin{pmatrix} \upsilon \\ \nu(T, u, v) \end{pmatrix} := \int_0^T W_0(T-\tau) \begin{pmatrix} a_1(vv_x)(\tau) + a_2(uv)_x(\tau) \\ \frac{a_2 b}{c}(uu_x)(\tau) + \frac{a_2 b}{c}(uv)_x(\tau) \end{pmatrix} d\tau$$

and consider the map

$$\begin{aligned} \Gamma \begin{pmatrix} u \\ v \end{pmatrix} =& W_0(t) \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} + W_{bdr}(x)\Lambda_1 \left(\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} + \begin{pmatrix} \upsilon \\ \nu(T, u, v) \end{pmatrix} \right) \\ &- \int_0^t W_0(t-\tau) \begin{pmatrix} a_1(vv_x)(\tau) + a_2(uv)_x(\tau) \\ \frac{r}{c} v_x(\tau) + \frac{a_2 b}{c}(uu_x)(\tau) + \frac{a_1 b}{c}(uv)_x(\tau) \end{pmatrix} d\tau. \end{aligned}$$

If we choose

$$\begin{pmatrix} \vec{h}_1 \\ \vec{g}_1 \end{pmatrix} = \Lambda_1 \left(\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} + \begin{pmatrix} \upsilon \\ \nu(T, u, v) \end{pmatrix} \right), \tag{2.74}$$

from Theorem 2.3.12, we get

$$\Gamma \begin{pmatrix} u \\ v \end{pmatrix} \Big|_{t=0} = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix}$$

and

$$\Gamma \begin{pmatrix} u \\ v \end{pmatrix} \Big|_{t=T} = \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} + \begin{pmatrix} \upsilon \\ \nu(T, u, v) \end{pmatrix} - \begin{pmatrix} \upsilon \\ \nu(T, u, v) \end{pmatrix} = \begin{pmatrix} u^1 \\ v^1 \end{pmatrix}.$$

Now we prove that the map Γ is a contraction in an appropriate metric space, then its fixed point (u, v) is the solution of (1.16)-(1.17) with $\vec{h}_1$ and $\vec{g}_1$ defined by (2.74), satisfying (1.19). In order to prove the existence of the fixed point we apply the Banach fixed point theorem to the restriction of Γ on the closed ball

$$B_r = \left\{ (u, v) \in \mathcal{Z}_T : \|(u, v)\|_{\mathcal{Z}_T} \leq r \right\},$$

for some $r > 0$.

(i) Γ *maps* B_r *into itself.*

Using Proposition 2.2.3 there exists a constant $C_1 > 0$, such that

$$\left\|\Gamma\begin{pmatrix} u \\ v \end{pmatrix}\right\|_{\mathcal{Z}_T} \leq C_1 \left\{ \left\|\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}\right\|_{\mathcal{X}} + \left\|\Lambda_1\left(\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} + \begin{pmatrix} v \\ \nu(T,u,v) \end{pmatrix}\right)\right\|_{\mathcal{H}_T} \right\}$$
$$+ C_1 \left\{ \int_0^t \left\| \begin{pmatrix} a_1(vv_x)(\tau) + a_2(uv)_x(\tau) \\ \frac{r}{c}v_x(\tau) + \frac{a_2 b}{c}(uu_x)(\tau) + \frac{a_1 b}{c}(uv)_x(\tau) \end{pmatrix} \right\|_{\mathcal{X}} d\tau \right\}.$$

Moreover, since

$$\left\|\Lambda_1\left(\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} + \begin{pmatrix} v \\ \nu(T,u,v) \end{pmatrix}\right)\right\|_{\mathcal{H}_T} \leq C_2 \left\{ \left\|\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}\right\|_{\mathcal{X}} + \left\|\begin{pmatrix} u^1 \\ v^1 \end{pmatrix}\right\|_{\mathcal{X}} + \left\|\begin{pmatrix} v \\ \nu(T,u,v) \end{pmatrix}\right\|_{\mathcal{X}} \right\},$$

applying Lemma 2.2.6, we can deduce that

$$\left\|\Gamma\begin{pmatrix} u \\ v \end{pmatrix}\right\|_{\mathcal{Z}_T} \leq C_3\delta + C_4(r+1)r,$$

where C_4 is a constant depending only on T. Thus, choosing r and δ such that

$$r = 2C_3\delta$$

and

$$2C_3C_4\delta + C_4 \leq \frac{1}{2},$$

the operator Γ maps B_r into itself for any $(u,v) \in \mathcal{Z}_T$.

(ii) Γ *is contractive.*

Proceeding as in the proof of Theorem 2.2.7, we obtain

$$\left\|\Gamma\begin{pmatrix} u \\ v \end{pmatrix} - \Gamma\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}\right\|_{\mathcal{Z}_T} \leq C_5(r+1)r \left\|\begin{pmatrix} u-\widetilde{u} \\ v-\widetilde{v} \end{pmatrix}\right\|_{\mathcal{Z}_T},$$

for any $(u,v), (\widetilde{u},\widetilde{v}) \in B_r$ and a constant C_5 depending only on T. Thus, taking $\delta > 0$, such that

$$\gamma = 2C_3C_5\delta + C_5 < 1,$$

we obtain

$$\left\|\Gamma\begin{pmatrix} u \\ v \end{pmatrix} - \Gamma\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}\right\|_{\mathcal{Z}_T} \leq \gamma \left\|\begin{pmatrix} u-\widetilde{u} \\ v-\widetilde{v} \end{pmatrix}\right\|_{\mathcal{Z}_T}.$$

Therefore, the map Γ is a contraction. Thus, from (i), (ii) and the Banach fixed point theorem, Γ has a fixed point in B_r and its fixed point is the desired solution. The proof of Theorem 2.1.1 is, thus, complete. □

CHAPTER

THREE

BOUNDARY CONTROLLABILITY OF A NONLINEAR COUPLED SYSTEM OF TWO KORTEWEG–DE VRIES EQUATIONS WITH CRITICAL SIZE RESTRICTIONS ON THE SPATIAL DOMAIN: DIRICHLET–NEUMANN BOUNDARY CONDITIONS

This chapter is devoted to improve the controllability results obtained by Cerpa et al. in [26] and by Micu et al. in [69] for a nonlinear coupled system of two Korteweg-de Vries (KdV) equations posed on a bounded interval. Initially, in [69], the authors proved that the nonlinear system is exactly controllable by using four boundary controls without any restriction on the length L of the interval. Later on, in [26], two boundary controls were considered to prove that the same system is exactly controllable for small values of the length L and large time of control T. Here, we use the ideas contained in [17] to prove that, with another configuration of four controls, it is possible to prove the existence of the so-called critical length phenomenon for the linear system, i. e., whether the system is controllable depends on the length of the spatial domain. In addition, when we consider only one control input, the boundary controllability still holds for suitable values of the length L and time of control T. In both cases, the control spaces are sharp due to a technical lemma which reveals a hidden regularity for the solution of the adjoint system.

3.1 Main result and notations

The results obtained in this Chapter were motivated by the results obtained in [26] and [69]. Although the analysis developed by the authors can be compared to the analysis developed by Rosier [78] for the KdV equation, the problem related to the existence of critical lengths addressed by Rosier was not studied, more precisely, the existence of the so-called *critical length phenomenon.* Indeed, Rosier proved that the linear KdV equation is exactly controllable by means of a single boundary control except when L lies in a countable set of critical lengths. This was done using the classical duality approach and the critical lengths found by Rosier are such that there are eigenvalues of the linear problem for which the observability inequality leading to the controllability fails.

Having all these results in hands, a natural question to be asked here is the following one.

Critical Length Phenomenon: Does the system (1.16)-(1.18) present the *critical length* *phenomenon* ?

Recall that we consider the system (1.16) with the following four controls

$$\begin{cases} u(0,t)=0, & u(L,t)=0, & u_x(L,t)=h_2(t) & \text{in } (0,T), \\ v(0,t)=g_0(t), & v(L,t)=g_1(t), & v_x(L,t)=g_2(t) & \text{in } (0,T). \end{cases} \tag{3.1}$$

As conjectured by Capistrano–Filho *et al.* in [17], indeed we can prove that system (1.16)-(3.1) is controllable if and only if the length L of the spatial domain $(0,L)$ does not belong to a new countable set, i. e.,

$$L \notin \mathcal{F}'_r := \left\{ \pi\sqrt{\frac{(1-a^2b)\alpha(k,l,m,n,s)}{3r}} : k,l,m,n,s \in \mathbb{N} \right\}, \tag{3.2}$$

where

$$\begin{aligned} \alpha := \alpha(k,l,m,n,s) =& 5k^2+8l^2+9m^2+8n^2+5s^2+8kl+6km \\ &+4kn+2ks+12ml+8ln+3ls+12mn+6ms+8ns. \end{aligned}$$

Furthermore, it is possible to get the controllability of the system by using only one control

$$\begin{cases} u(0,t)=0, & u(L,t)=0, & u_x(L,t)=h_2(t) & \text{in } (0,T), \\ v(0,t)=0, & v(L,t)=0, & v_x(L,t)=0 & \text{in } (0,T), \end{cases}$$

under the condition

$$L < \frac{\min\{b,c\}}{\max\{b,c\}\beta C_T} T, \tag{3.3}$$

where C_T is the positive constant in (3.37) and β is the constant given by the embedding $H^{\frac{1}{3}}(0,T) \subset L^2(0,T)$.

The analysis described above is summarized in the main result of this chapter, Theorem 3.1.1. In order to make the reading of the proof easier, throughout the chapter we use the following notation for the boundary functions:

$$\vec{h}_1 = (0,0,h_2), \quad \vec{g}_1 = (g_0,g_1,g_2) \quad \text{and} \quad \vec{h}_2 = (0,0,h_2), \quad \vec{g}_2 = (0,0,0).$$

We also introduce the spaces of the boundary functions as follows

$$\mathcal{H}_T := H^{\frac{1}{3}}(0,T) \times H^{\frac{1}{3}}(0,T) \times L^2(0,T) \tag{3.4}$$

and

$$\mathcal{Z}_T := C([0,T];(L^2(0,L))^2) \cap L^2(0,T,(H^1(0,L))^2), \tag{3.5}$$

endowed with their natural inner products. Finally, we consider the space $\mathcal{X} := (L^2(0,L))^2$ endowed with the inner product

$$\langle (u,v),(\varphi,\psi)\rangle := \frac{b}{c}\int_0^L u(x)\varphi(x)dx + \int_0^L v(x)\psi(x)dx, \qquad \forall (u,v),(\varphi,\psi) \in \mathcal{X}.$$

With the notation above, we can answer the question mentioned in previous subsection as follows:

Theorem 3.1.1. *Let $T>0$ and $L>0$. Then, there exists $\delta>0$ depending on L, such that for (u^0,v^0), (u^1,v^1) in $\mathcal{X}$ verifying*

$$\|(u^0,v^0)\|_{\mathcal{X}} + \|(u^1,v^1)\|_{\mathcal{X}} \leq \delta,$$

the following holds:

(i) *If* $L \in (0,+\infty)\backslash\mathcal{F}'_r$*, then, one can find* $\vec{h}_1, \vec{g}_1 \in \mathcal{H}_T$*, such that the solution* $(u,v) \in \mathcal{Z}_T$ *of the system* (1.16)-(1.18) *satisfies* (1.19).

(ii) *If* $L > 0$ *fulfills* (3.3)*, then, one can find* $\vec{h}_2, \vec{g}_2 \in \mathcal{H}_T$*, such that the solution* $(u,v) \in \mathcal{Z}_T$ *of the system* (1.16)-(1.18) *satisfies* (1.19).

Theorem 3.1.1 will be proved using the same approach that Capistrano–Filho *et al.* used to establish Theorem C. In order to deal with the linearized system, we also use the classical duality approach [39,63] which reduces the problem to prove an observability inequality for the solutions of the corresponding adjoint system associated to (1.16)-(1.18):

$$\begin{cases} \varphi_t + \varphi_{xxx} + \frac{ab}{c}\psi_{xxx} = 0, & \text{in } (0,L)\times(0,T),\\ \psi_t + \frac{r}{c}\psi_x + a\varphi_{xxx} + \frac{1}{c}\psi_{xxx} = 0, & \text{in } (0,L)\times(0,T),\\ \varphi(0,t) = \varphi(L,t) = \varphi_x(0,t) = 0, & \text{in } (0,T),\\ \psi(0,t) = \psi(L,t) = \psi_x(0,t) = 0, & \text{in } (0,T),\\ \varphi(x,T) = \varphi^1(x), \quad \psi(x,T) = \psi^1(x), & \text{in } (0,L). \end{cases} \tag{3.6}$$

Similarly, as in [17], one will encounter some difficulties that demand special attention. To prove assertion (i) we need to prove a hidden regularity for the solutions of the system of the linear system (3.6). In our case, the result is given by the following lemma.

Lemma 3.1.2 (Kato sharp trace regularities)**.** *For any* $(\varphi^0, \psi^0) \in \mathcal{X}$*, the system* (3.6) *admits a unique solution* $(\varphi,\psi) \in \mathcal{Z}_T$*, such that it possess the following sharp trace properties*

$$\sup_{0\le x\le L} \|(\partial_x^k \varphi(x,\cdot), \partial_x^k \psi(x,\cdot))\|_{(H^{\frac{1-k}{3}}(0,T))^2} \le C_T \|(\varphi^0,\psi^0)\|_{(L^2(0,L))^2}, \quad \text{for } k = 0,1,2. \tag{3.7}$$

The sharp Kato smoothing properties of solutions of the Cauchy problem of the KdV equation posed on the whole line $\mathbb{R}$ due to Kenig, Ponce and Vega [55] will play an important role in the proof of Lemma 3.1.2. In what concerns the assertion (ii), the observability inequality for the solutions of (3.6) is proved using multipliers together with the Lemma 3.1.2. It is precisely the hidden regularity (sharp trace regularity) given by Lemma 3.1.2 that enable us to prove Theorem B with less controls.

The program of this work was carried out for the particular choice of boundary control inputs and aims to establish as a fact that such a model has the interesting qualitative properties initially observed for the KdV equation. Consideration of this issue for nonlinear dispersive equations has received considerable attention, specially the problems related to the study of the controllability properties.

The plan of this chapter is as follows.

—- In Section 3.2, we show that the linear system associated to (1.16)-(1.18) is globally well-posed in $\mathcal{Z}_T$. Additionally, we present various estimates, among them Lemma 3.1.2 for the solution of the adjoint system.

—- Section 3.3 is intended to show the controllability of the linear system associated with (1.16) when four controls are considered in the boundary conditions. Moreover, when only one function is a control input the boundary controllability result is also proved. Here, the hidden regularities for the solutions of the adjoint system presented in the Section 2 are used to prove observability inequalities associated to the control problem.

—- In Section 3.4, we prove the local well-posedness of the system (1.16)-(1.18) in $\mathcal{Z}_T$. After that, the exact boundary controllability of the nonlinear system is proved *via* contraction mapping principle.

—- Finally, Section 3.5 contains some remarks and related problems.

3.2 Well-posedness

3.2.1 Linear homogeneous system

Firstly, we establish the well-posedness of the initial-value problem of the linear system associated to (1.16)-(1.18):

$$\begin{cases} u_t + u_{xxx} + a v_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ v_t + \frac{r}{c} v_x + \frac{ab}{c} u_{xxx} + \frac{1}{c} v_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ u(0,t) = u(L,t) = u_x(L,t) = 0, & \text{in } (0,T), \\ v(0,t) = v(L,t) = v_x(L,t) = 0, & \text{in } (0,T), \\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L). \end{cases} \tag{3.8}$$

Let us define the operator A by

$$A\begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} \partial_{xxx} & a\partial_{xxx} \\ \dfrac{ab}{c}\partial_{xxx} & \frac{r}{c}\partial_x + \frac{1}{c}\partial_{xxx} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{3.9}$$

with domain

$$D(A) = \left\{(u,v) \in (H^3(0,L))^2 \, : u(0) = v(0) = u(L) = v(L) = u_x(L) = v_x(L) = 0\right\} \subset \mathcal{X}.$$

The linear system (3.8) can be written in abstract form as

$$\begin{cases} U_t = AU, \\ U(0) = U_0, \end{cases} \tag{3.10}$$

where $U := (u,v)$ and $U_0 := (u^0, v^0)$. We denote by A^* the adjoint operator of A, defined by

$$A^*\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \partial_{xxx} & \frac{ab}{c}\partial_{xxx} \\ a\partial_{xxx} & \dfrac{r}{c}\partial_x + \dfrac{1}{c}\partial_{xxx} \end{pmatrix}\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \tag{3.11}$$

with domain

$$D(A^*) = \left\{(\varphi,\psi) \in (H^3(0,L))^2 : \varphi(0) = \psi(0) = \varphi(L) = \psi(L) = \varphi_x(0) = \psi_x(0) = 0\right\} \subset \mathcal{X}.$$

The following results can be found in [69].

Proposition 3.2.1. *The operator A and its adjoint A^* are dissipative in $\mathcal{X}$.*

As a consequence, we have that (see Corol. 4.4, page 15, in [77]):

Theorem 3.2.2. *Let $U_0 \in \mathcal{X}$. There exists a unique (weak) solution $U = S(\cdot)U_0$ of (3.8) such that*

$$U \in C([0,T];\mathcal{X}) \cap H^1\left(0,T;(H^{-2}(0,L))^2\right). \tag{3.12}$$

Moreover, if $U_0 \in D(A)$ then (3.8) has a unique (classical) solution U such that

$$U \in C([0,T];D(A)) \cap C^1((0,T);\mathcal{X}).$$

The next result reveals a gain of regularity for the weak solutions given by Theorem 3.2.2.

Theorem 3.2.3. *Let (u^0,v^0) in $\mathcal{X}$ and (u,v) the weak solution of (3.8). Then,*

$$(u,v) \in L^2(0,T;(H^1(0,L))^2)$$

and there exists a positive constant c_0 such that

$$\|(u,v)\|_{L^2(0,T;(H^1(0,L))^2)} \leq c_0 \|(u^0,v^0)\|_{\mathcal{X}}.$$

Moreover, there exist two positive constants c_1 and c_2 such that

$$\|(u_x(0,\cdot), v_x(0,\cdot))\|^2_{\mathcal{X}} \leq c_1 \|(u^0,v^0)\|^2_{\mathcal{X}}.$$

and

$$\|(u^0,v^0)\|^2_{\mathcal{X}} \leq \frac{1}{T}\|(u,v)\|^2_{L^2(0,T;\mathcal{X})} + c_2\|(u_x(0,\cdot), v_x(0,\cdot))\|^2_{\mathcal{X}}.$$

3.2.2 Linear nonhomogeneous system

In this subsection, we study the nonhomogeneous system corresponding to (1.16)-(1.18):

$$\begin{cases} u_t + u_{xxx} + av_{xxx} = 0, & \text{in } (0,L)\times(0,T),\\ v_t + \frac{r}{c}v_x + \frac{ab}{c}u_{xxx} + \frac{1}{c}v_{xxx} = 0, & \text{in } (0,L)\times(0,T),\\ u(0,t) = h_0(t),\ u(L,t) = h_1(t),\ u_x(L,t) = h_2(t), & \text{in } (0,T),\\ v(0,t) = g_0(t),\ v(L,t) = g_1(t),\ v_x(L,t) = g_2(t), & \text{in } (0,T),\\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L). \end{cases} \tag{3.13}$$

The next well-posedness result can be found in [69, Theorems 2.3, 2.4].

Theorem 3.2.4. *There exists a unique linear and continuous map*

$$\Psi : \mathcal{X} \times (H_0^1(0,T))^2 \times (H_0^1(0,T))^2 \times (L^2(0,T))^2 \to C([0,T];\mathcal{X}) \cap L^2(0,T;(H^1(0,L))^2)$$

such that, for any (u^0,v^0) in $D(A)$ and h_i, g_i in $C_0^2[0,T]$, with $i = 0,1,2$,

$$\Psi((u^0,v^0),(h_0,g_0,h_1,g_1,h_2,g_2)) = (u,v)$$

where (u,v) is the unique classical solution of (3.13). Moreover, there exists a positive constant $C > 0$ such that

$$\|(u,v)\|^2_{C([0,T];\mathcal{X})} + \|(u,v)\|_{L^2(0,T;(H^1(0,L))^2)} \le C\left[\|(u^0,v^0)\|^2_{\mathcal{X}} + \sum_{i=0}^{2}(\|h_i\|_{H^1(0,T)} + \|g_i\|_{H^1(0,T)})\right].$$

Our main goal in this subsection is to improve Theorem 3.2.4. We will obtain some important trace estimates, using a new tool, which reveals the sharp Kato smoothing (or hidden regularity) for the solution of system (3.13). In order to do that, we consider the system

$$\begin{cases} u_t + u_{xxx} + av_{xxx} = f, & \text{in } (0,L)\times(0,T),\\ v_t + \frac{ab}{c}u_{xxx} + \frac{1}{c}v_{xxx} = s, & \text{in } (0,L)\times(0,T),\\ u(0,t) = h_0(t),\ u(L,t) = h_1(t),\ u_x(L,t) = h_2(t), & \text{in } (0,T),\\ v(0,t) = g_0(t),\ v(L,t) = g_1(t),\ v_x(L,t) = g_2(t), & \text{in } (0,T),\\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L), \end{cases} \tag{3.14}$$

where $f = f(x,t)$ and $s = s(x,t)$. Then, we have the following result:

Proposition 3.2.5. *Let $T > 0$ be given, for any (u^0,v^0) in $\mathcal{X}$, f,s in $L^1(0,T;L^2(0,L))$ and $\overrightarrow{h} := (h_0,h_1,h_2)$, $\overrightarrow{g} := (g_0,g_1,g_2)$ in $\mathcal{H}_T$, the IBVP (3.14) admits a unique solution $(u,v) \in \mathcal{Z}_T$, with*

$$\partial_x^k u, \partial_x^k v \in L_x^\infty(0,L;H^{\frac{1-k}{3}}(0,T)), \quad k = 0,1,2. \tag{3.15}$$

Moreover, there exists $C > 0$, such that

$$\|(u,v)\|_{\mathcal{Z}_T} + \sum_{k=0}^{2}\|(\partial_x^k u, \partial_x^k v)\|_{L_x^\infty(0,L;H^{\frac{1-k}{3}}(0,T))} \le C\left\{\|(u^0,v^0)\|_{(L^2(0,L))^2} + \|(\overrightarrow{h},\overrightarrow{g})\|_{\mathcal{H}_T} + \|(f,s)\|_{L^1(0,T;L^2(0,L))}\right\}. \tag{3.16}$$

To prove the Proposition 3.2.5, we need an auxiliary result.

Proposition 3.2.6. *Consider the following nonhomogeneous Korteweq-de Vries equation*

$$\begin{cases} u_t + \alpha u_{xxx} = f, & \text{in } (0,L)\times(0,T), \\ u(0,t) = h_0(t),\ u(L,t) = h_1(t),\ u_x(L,t) = h_2(t), & \text{in } (0,T), \\ u(x,0) = u^0(x), & \text{in } (0,L). \end{cases} \tag{3.17}$$

For any $u^0 \in L^2(0,L)$, $f \in L^1(0,T;L^2(0,L))$, $\overrightarrow{h} := (h_0,h_1,h_2) \in \mathcal{H}_T$ and $\alpha > 0$, the IBVP (3.17) admits a unique solution

$$u \in \mathcal{X}_T := C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$$

with

$$\partial_x^k u \in L_x^\infty(0,L;H^{\frac{1-k}{3}}(0,T)), \quad k = 0,1,2. \tag{3.18}$$

Moreover, there exists $C>0$, such that

$$\|u\|_{\mathcal{X}_T} + \sum_{k=0}^{2} \|\partial_x^k u\|_{L_x^\infty(0,L;H^{\frac{1-k}{3}}(0,T))} \leq C\left\{\|u^0\|_{L^2(0,L)} + \|(\overrightarrow{h},\overrightarrow{g})\|_{\mathcal{H}_T} + \|(f,s)\|_{L^1(0,T;L^2(0,L))}\right\}. \tag{3.19}$$

Proof. When $k = 0,1$ the result was proved by Bona, Sun and Zhang in [11]. Therefore, for the sake of completeness, we prove the result for the case when $k = 2$.

Proceeding as in [11], it is sufficient to prove that the solution v of the following linear non-homogeneous boundary value problem,

$$\begin{cases} v_t + \alpha v_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ v(0,t) = h_0(t),\ v(L,t) = h_1(t),\ v_x(L,t) = h_2(t), & \text{in } (0,T), \\ v(x,0) = 0, & \text{in } (0,L). \end{cases} \tag{3.20}$$

satisfies

$$\sup_{0\leq x\leq L} \|\partial_x^2 v(x,\cdot)\|_{H^{-\frac{1}{3}}(0,T)} \leq C_T \|\overrightarrow{h}\|_{\mathcal{H}_T}. \tag{3.21}$$

Indeed, applying the Laplace transform with respect to t, (3.20) is converted to

$$\begin{cases} s\hat{v}(x,s) + \alpha \hat{v}_{xxx}(x,s) = 0, & \text{in } (0,L)\times(0,T), \\ \hat{v}(0,s) = \hat{h}_0(s),\ \hat{v}(L,s) = \hat{h}_1(s),\ \hat{v}_x(L,s) = \hat{h}_2(s), & \text{in } (0,T), \\ \hat{v}(x,0) = 0, & \text{in } (0,L). \end{cases} \tag{3.22}$$

where

$$\hat{v}(x,s) = \int_0^\infty e^{-st} v(x,t)dt \quad \text{and} \quad \hat{h}_j(s) = \int_0^\infty e^{-st} h_j(t)dt, \qquad j = 0,1,2.$$

The solution $\hat{v}(x,s)$ can be written in the form $\hat{v}(x,s) = \sum_{j=0}^{2} c_j(s)e^{-\lambda_j(s)x}$, where $\lambda_j(s)$ are the solutions of the characteristic equation $s + \alpha\lambda^3 = 0$ and $c_j(s)$, solve the linear system

$$\begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_0} & e^{\lambda_1} & e^{\lambda_2} \\ \lambda_0 e^{\lambda_0} & \lambda_1 e^{\lambda_1} & \lambda_2 e^{\lambda_2} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \hat{h}_0 \\ \hat{h}_1 \\ \hat{h}_2 \end{pmatrix}.$$

Using the Cramer rule, we obtain $c_j(s) = \frac{\Delta_j(s)}{\Delta(s)}$, $j = 0,1,2$, where $\Delta(s)$ is the determinant of the coefficient matrix and $\Delta_j(s)$ the determinants of the matrices that are obtained by replacing the ith-column by the column vector $\overrightarrow{h} := (\hat{h}_0(s), \hat{h}_1(s), \hat{h}_2(s))$. Taking the inverse Laplace transform of $\hat{v}$, yields

$$v(x,t) = \frac{1}{2\pi i}\sum_{j=0}^{2}\int_{r-i\infty}^{r+i\infty} e^{st}\frac{\Delta_j(s)}{\Delta(s)}e^{\lambda_j(s)x}ds$$

for any $r > 0$. Note that, v may also be written in the form

$$v(x,t) = \sum_{m=0}^{2} v_m(x,t), \tag{3.23}$$

where $v_m(x,t)$ solves (3.20) with $h_j = 0$ when $j \neq m$, $m, j = 0, 1, 2$. Thus, v_m take the form

$$v_m(x,t) = \frac{1}{2\pi i}\sum_{j=0}^{2}\int_{r-i\infty}^{r+i\infty} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_m(s)ds := [W_m(t)h_m(t)](x) \tag{3.24}$$

where $\Delta_{j,m}(s)$ is obtained from $\Delta_j(s)$ by letting $\hat{h}_m \equiv 1$ and $\hat{h}_j \equiv 0$, for $j \neq m$, $j, m = 0, 1, 2$. Moreover, note that, the right-hand sides are continuous with respect to r for $r \geq 0$. As the left-hand sides do not depend on r, it follows that we may take $r = 0$. Thus, we can write v_m as

$$v_m(x,t) = v_m^+(x,t) + v_m^-(x,t), \tag{3.25}$$

where

$$v_m^+(x,t) = \frac{1}{2\pi i}\sum_{j=0}^{2}\int_{0}^{i\infty} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_m(s)ds,$$

$$v_m^-(x,t) = \frac{1}{2\pi i}\sum_{j=0}^{2}\int_{-i\infty}^{0} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_m(s)ds.$$

Making the substitution $s = i\alpha\rho^3L^3$ with $\rho \geq 0$ in the characteristic equation, the three roots are given in terms of ρ by

$$\lambda_0(\rho) = iL\rho, \quad \lambda_1(\rho) = -iL\rho\left(\frac{1+i\sqrt{3}}{2}\right), \quad \lambda_2(\rho) = -iL\rho\left(\frac{1-i\sqrt{3}}{2}\right).$$

Thus, v_m^+ and v_m^- have the following representation,

$$v_m^+(x,t) = \frac{3\alpha L^3}{2\pi}\sum_{j=0}^{2}\int_{0}^{\infty} e^{i\alpha\rho^3L^3t}\frac{\Delta_{j,m}^+(\rho)}{\Delta^+(\rho)}e^{\lambda_j^+(\rho)x}\hat{h}_m^+(\rho)\rho^2d\rho \quad \text{and} \quad v_m^-(x,t) = \overline{v_m^+(x,t)}, \tag{3.26}$$

where $\Delta_{j,m}^+(\rho) = \Delta_{j,m}(i\alpha\rho^3L^3)$, $\Delta^+(\rho) = \Delta(i\alpha\rho^3L^3)$, $\lambda_j^+(\rho) = \lambda_j(i\alpha\rho^3L^3)$ and $\hat{h}_m^+(\rho) = \hat{h}_m(i\alpha\rho^3L^3)$. Thus, we have

$$\begin{aligned}
\partial_x^2 v_m^+(x,t) &= \frac{3\alpha L^3}{2\pi}\sum_{j=0}^{2}\int_{0}^{\infty} e^{i\alpha\rho^3L^3t}(\lambda_j^+(\rho))^2\frac{\Delta_{j,m}^+(\rho)}{\Delta^+(\rho)}e^{\lambda_j^+(\rho)x}\hat{h}_m^+(\rho)\rho^2d\rho \\
&= \frac{1}{2\pi}\sum_{j=0}^{2}\int_{0}^{\infty} e^{i\mu t}(\lambda_j^+(\theta(\mu)))^2\frac{\Delta_{j,m}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}e^{\lambda_j^+(\theta(\mu))x}\hat{h}_m^+(\theta(\mu))d\mu,
\end{aligned}$$

where $\theta(\mu)$ is the real solution of $\mu = \alpha\rho^3L^3$, for $\rho \geq 0$. Here

$$\lambda_0(\rho) = iL\rho, \quad \lambda_1(\rho) = -iL\rho\left(\frac{1+i\sqrt{3}}{2}\right), \quad \lambda_2(\rho) = -iL\rho\left(\frac{1-i\sqrt{3}}{2}\right).$$

Applying Plancherel Theorem (with respect to t), yields for any $x \in (0, L)$,

$$\begin{aligned}
\|\partial_x^2 v_m^+(x,\cdot)\|^2_{H^{-\frac{1}{3}}(0,T)} &\leq \frac{1}{2\pi}\sum_{j=0}^{2}\int_0^\infty |\mu|^{-\frac{2}{3}}\left|(\lambda_j^+(\theta(\mu)))^2\frac{\Delta_{j,m}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}e^{\lambda_j^+(\theta(\mu))x}\hat{h}_m^+(\theta(\mu))\right|^2 d\mu \\
&= \frac{1}{2\pi}\sum_{j=0}^{2}\int_0^\infty \alpha^{-\frac{2}{3}}\rho^{-2}L^{-2}\left|(\lambda_j^+(\rho))^2\frac{\Delta_{j,m}^+(\rho)}{\Delta^+(\rho)}e^{\lambda_j^+(\rho)x}\right|^2\left|\hat{h}_m^+(\rho)\right|^2(3\alpha L^3\rho^2)d\rho, \\
&= \frac{3\alpha^{-\frac{1}{3}}L}{2\pi}\sum_{j=0}^{2}\int_0^\infty \left|(\lambda_j^+(\rho))^2\frac{\Delta_{j,m}^+(\rho)}{\Delta^+(\rho)}e^{\lambda_j^+(\rho)x}\right|^2\left|\hat{h}_m^+(\rho)\right|^2 d\rho.
\end{aligned}$$

On the other hand, note that

$$\sup_{0\leq x\leq L}\left|e^{\lambda_0^+(\rho)x}\right| \leq C, \quad \sup_{0\leq x\leq L}\left|e^{\lambda_1^+(\rho)x}\right| \leq Ce^{\frac{\sqrt{3}}{2}\rho L}, \quad \sup_{0\leq x\leq L}\left|e^{\lambda_2^+(\rho)x}\right| \leq Ce^{-\frac{\sqrt{3}}{2}\rho L}.$$

Then, it follows that

$$\begin{aligned}
\|\partial_x^2 v_m^+(x,\cdot)\|^2_{H^{-\frac{1}{3}}(0,T)} &\leq C\left\{\int_0^\infty \rho^4\left|\frac{\Delta_{0,m}^+(\rho)}{\Delta^+(\rho)}\right|^2\left|\hat{h}_m^+(\rho)\right|^2 d\rho\right. \\
&++\left.\int_0^\infty \rho^4 e^{\sqrt{3}\rho L}\left|\frac{\Delta_{1,m}^+(\rho)}{\Delta^+(\rho)}\right|^2\left|\hat{h}_m^+(\rho)\right|^2 d\rho\int_0^\infty \rho^4 e^{-\sqrt{3}\rho L}\left|\frac{\Delta_{2,m}^+(\rho)}{\Delta^+(\rho)}\right|^2\left|\hat{h}_m^+(\rho)\right|^2 d\rho\right\}.
\end{aligned}$$

Using the estimates of $\left|\frac{\Delta_{j,m}^+(\rho)}{\Delta^+(\rho)}\right|$ proved in [11], that is,

$\frac{\Delta_{0,0}^+(\rho)}{\Delta^+(\rho)} \sim e^{-\frac{\sqrt{3}}{2}\rho L}$	$\frac{\Delta_{1,0}^+(\rho)}{\Delta^+(\rho)} \sim e^{-\sqrt{3}\rho L}$	$\frac{\Delta_{2,0}^+(\rho)}{\Delta^+(\rho)} \sim 1$
$\frac{\Delta_{0,1}^+(\rho)}{\Delta^+(\rho)} \sim 1$	$\frac{\Delta_{1,1}^+(\rho)}{\Delta^+(\rho)} \sim e^{-\frac{\sqrt{3}}{2}\rho L}$	$\frac{\Delta_{2,1}^+(\rho)}{\Delta^+(\rho)} \sim 1$
$\frac{\Delta_{0,2}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}$	$\frac{\Delta_{1,2}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}e^{-\frac{\sqrt{3}}{2}\rho L}$	$\frac{\Delta_{2,2}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}$

(3.27)

we obtain

$$\begin{aligned}
\|\partial_x^2 v_0^+(x,\cdot)\|^2_{H^{-\frac{1}{3}}(0,T)} &\leq C\int_0^\infty \rho^4\left|\hat{h}_0^+(\rho)\right|^2 d\rho = C\int_0^\infty \rho^4\left|\hat{h}_0(i\alpha\rho^3L^3)\right|^2 d\rho \\
&= C\int_0^\infty \rho^4\left|\int_0^\infty e^{-i\alpha\rho^3L^3t}h_0(t)dt\right|^2 d\rho.
\end{aligned}$$

Setting $\mu = \alpha\rho^3L^3$, it follows that

$$\begin{aligned}
\|\partial_x^2 v_0^+(x,\cdot)\|^2_{H^{-\frac{1}{3}}(0,T)} &= C\int_0^\infty \rho^4\left|\int_0^\infty e^{-ia\rho^3L^3t}h_0(t)dt\right|^2 d\rho \\
&\leq C\int_0^\infty \mu^{\frac{2}{3}}\left|\int_0^\infty e^{-i\mu t}h_0(t)dt\right|^2 d\mu \\
&\leq C\|h_0\|^2_{H^{\frac{1}{3}}(\mathbb{R}^+)}.
\end{aligned}$$

Similarly, we obtain estimates for $\partial_x^2 v_1$ and $\partial_x^2 v_2$ in $H^{-\frac{1}{3}}(0,T)$. Indeed,

$$\|\partial_x^2 v_1^+(x,\cdot)\|^2_{H^{-\frac{1}{3}}(0,T)} \leq C\|h_1\|^2_{H^{\frac{1}{3}}(\mathbb{R}^+)}$$

and

$$\|\partial_x^2 v_2^+(x,\cdot)\|^2_{H^{-\frac{1}{3}}(0,T)} \leq C\int_0^\infty \rho^2 \left|\hat{h}_1^+(\rho)\right|^2 d\rho \leq C\|h_2\|^2_{L^2(\mathbb{R}^+)}.$$

Thus, (3.21) follows from (3.23), (3.25) and (3.26). We also observe that, as in [11, Theorem 2.10], the solutions can be written in the form of the boundary integral operator W_{bdr} as follows

$$v(x,t) = [W_{bdr}\overrightarrow{h}](x,t) = \sum_{i=0}^{2}[W_j(t)h_j](x), \tag{3.28}$$

where W_j is defined in (3.24). □

Proof of Proposition 3.2.5. Consider the change of variable

$$\begin{cases} u = 2a\widetilde{u} + 2a\widetilde{v}, \\ v = \left(\left(\frac{1}{c}-1\right)+\lambda\right)\widetilde{u} + \left(\left(\frac{1}{c}-1\right)-\lambda\right)\widetilde{v} \end{cases} \tag{3.29}$$

with $\lambda = \sqrt{\left(\frac{1}{c}-1\right)^2 + \frac{4a^2b}{c}}$. Thus, we can transform the linear system (3.14) into

$$\begin{cases} \widetilde{u}_t + \alpha_-\widetilde{u}_{xxx} = \widetilde{f}, \\ \widetilde{v}_t + \alpha_+\widetilde{v}_{xxx} = \widetilde{s}, \\ \widetilde{u}(0,t) = \widetilde{h}_0(t),\ \widetilde{u}(L,t) = \widetilde{h}_1(t),\ \widetilde{u}_x(L,t) = \bar{h}_2(t), \\ \widetilde{v}(0,t) = \widetilde{g}_0(t),\ \widetilde{v}(L,t) = \widetilde{g}_1(t),\ \widetilde{v}_x(L,t) = \widetilde{g}_2(t), \\ \widetilde{u}(x,0) = \widetilde{u}^0(x), \quad \widetilde{v}(x,0) = \widetilde{v}^0(x), \end{cases} \tag{3.30}$$

where $\alpha_\pm = -\frac{1}{2}\left(\left(\frac{1}{c}-1\right)\pm\lambda\right)$ and

$$\begin{cases} \widetilde{f} = -\frac{1}{2}\left(\frac{\alpha_+}{a\lambda}f + \frac{1}{\lambda}s\right), \quad \widetilde{u}_0 = -\frac{1}{2}\left(\frac{\alpha_-}{a\lambda}u^0 - \frac{1}{\lambda}v^0\right), \quad \widetilde{h}_i = -\frac{1}{2}\left(\frac{\alpha_-}{a\lambda}h_i - \frac{1}{\lambda}g_i\right), \quad i=0,1,2, \\ \widetilde{s} = -\frac{1}{2}\left(\frac{\alpha_-}{a\lambda}f - \frac{1}{\lambda}s\right), \quad \widetilde{v}_0 = \frac{1}{2}\left(\frac{\alpha_+}{a\lambda}u^0 - \frac{1}{\lambda}v^0\right), \quad \widetilde{g}_i = \frac{1}{2}\left(\frac{\alpha_+}{a\lambda}h_i - \frac{1}{\lambda}g_i\right), \quad i=0,1,2. \end{cases}$$

The system (3.30) can be decoupled into two KdV equations as follows:

$$\begin{cases} \widetilde{u}_t + \alpha_-\widetilde{u}_{xxx} = \widetilde{f}, \\ \widetilde{u}(0,t) = \widetilde{h}_0(t),\ \widetilde{u}(L,t) = \widetilde{h}_1(t), \\ \widetilde{u}_x(L,t) = \widetilde{h}_2(t), \\ \widetilde{u}(0,x) = \widetilde{u}^0(x) \end{cases} \quad \text{and} \quad \begin{cases} \widetilde{v}_t + \alpha_+\widetilde{v}_{xxx} = \widetilde{s}, \\ \widetilde{v}(0,t) = \widetilde{g}_0(t),\ \widetilde{v}(L,t) = \widetilde{g}_1(t), \\ \widetilde{v}_x(L,t) = \widetilde{g}_2(t), \\ \widetilde{v}(x,0) = \widetilde{v}^0(x). \end{cases} \tag{3.31}$$

Note that for $\alpha_\pm$ to be nonzero, it is sufficient to assume that $a^2b \neq 1$. Then, it is easy to see that

$$(\widetilde{u}^0, \widetilde{v}^0) \in \mathcal{X}, \quad (\widetilde{f}, \widetilde{s}) \in L^1(0,T;(L^2(0,L))^2), \quad \overrightarrow{\widetilde{h}}, \overrightarrow{\widetilde{g}} \in \mathcal{H}_T.$$

By Proposition 3.2.6, we obtain the existence of $(\widetilde{u}, \widetilde{v})$, solution of the system (3.31) belongs to $\mathcal{Z}_T$, such that

$$\partial_x^k\widetilde{u}, \partial_x^k\widetilde{v} \in L_x^\infty(0,L;H^{\frac{1-k}{3}}(0,T)), \quad k=0,1,2$$

and

$$\|(\widetilde{u},\widetilde{v})\|_{\mathcal{Z}_T} + \sum_{k=0}^{2}\|(\partial_x^k\widetilde{u}, \partial_x^k\widetilde{v})\|_{L_x^\infty(0,L;H^{\frac{1-k}{3}}(0,T))} \leq C\left\{\|(\widetilde{u}^0,\widetilde{v}^0)\|_{(L^2(0,L))^2} + \|(\overrightarrow{\widetilde{h}}, \overrightarrow{\widetilde{g}})\|_{\mathcal{H}_T} + \|(\widetilde{f},\widetilde{s})\|_{L^1(0,T;L^2(0,L))}\right\}.$$

Furthermore, as in [11], we can write $\widetilde{u}$ and $\widetilde{v}$ in its integral form:

$$\widetilde{u}(t) = W_0^-(t)\widetilde{u}^0 + W_{bdr}^-(t)\overrightarrow{h} + \int_0^t W_0^-(t-\tau)\widetilde{f}(\tau)d\tau,$$

$$\widetilde{v}(t) = W_0^+(t)\widetilde{v}^0 + W_{bdr}^+(t)\overrightarrow{g} + \int_0^t W_0^+(t-\tau)\widetilde{s}(\tau)d\tau,$$

where $\{W_0^\pm(t)\}_{t\geq 0}$ is the C_0-semigroup in the space $L^2(0,L)$ generated by the linear operator

$$A^\pm = -\alpha_\pm g''',$$

with domain

$$D(A^\pm) = \{g \in H^3(0,L) : g(0) = g(L) = g'(L) = 0\},$$

and $\{W_{bdr}^\pm(t)\}_{t\geq 0}$ is the operator given in (3.28). By using the change of variable, it is easy to see that

$$\begin{cases} u(t) = W_0^-(t)u^0 + W_{bdr}^-(t)\overrightarrow{h} + \int_0^t W_0^-(t-\tau)f(\tau)d\tau, \\ v(t) = W_0^+(t)v^0 + W_{bdr}^+(t)\overrightarrow{g} + \int_0^t W_0^+(t-\tau)s(\tau)d\tau. \end{cases}$$

Therefore, the prove is complete. □

By using standard fixed point argument together with Propositions 3.2.5 and 3.2.6 we show the global well-posedness of the system (3.13).

Theorem 3.2.7. *Let $T > 0$ be given. For any (u^0, v^0) in $\mathcal{X}$ and $\overrightarrow{h} := (h_0, h_1, h_2)$, $\overrightarrow{g} := (g_0, g_1, g_2)$ in $\mathcal{H}_T$, the IBVP (3.13) admits a unique solution $(u, v) \in \mathcal{Z}_T$, with*

$$\partial_x^k u, \partial_x^k v \in L_x^\infty(0, L; H^{\frac{1-k}{3}}(0,T)), \quad k = 0, 1, 2.$$

Moreover, there exist $C > 0$, such that

$$\|(u,v)\|_{\mathcal{Z}_T} + \sum_{k=0}^{2} \|(\partial_x^k u, \partial_x^k v)\|_{L_x^\infty(0,L;H^{\frac{1-k}{3}}(0,T))} \leq C\left\{\|(u^0,v^0)\|_{(L^2(0,L))^2} + \|(\overrightarrow{h}, \overrightarrow{g})\|_{\mathcal{H}_T} + \|(f,s)\|_{L^1(0,T;L^2(0,L))}\right\}. \quad (3.32)$$

3.2.3 Adjoint system

We can now study the properties of the adjoint system of (3.8):

$$\begin{cases} \varphi_t + \varphi_{xxx} + \frac{ab}{c}\psi_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ \psi_t + \frac{r}{c}\psi_x + a\varphi_{xxx} + \frac{1}{c}\psi_{xxx} = 0, & \text{in } (0,L)\times(0,T), \end{cases} \quad (3.33)$$

with the boundary conditions,

$$\begin{cases} \varphi(0,t) = \varphi(L,t) = \varphi_x(0,t) = 0, & \text{in } (0,T), \\ \psi(0,t) = \psi(L,t) = \psi_x(0,t) = 0, & \text{in } (0,T) \end{cases} \quad (3.34)$$

and the final conditions

$$\varphi(x,T) = \varphi^1(x), \qquad \psi(x,T) = \psi^1(x), \qquad \text{in } (0,L). \quad (3.35)$$

Observe that, applying the change of variable $t = T - t$, we obtain

$$\begin{cases} \varphi_t - \varphi_{xxx} - \frac{ab}{c}\psi_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ \psi_t - \frac{r}{c}\psi_x - a\varphi_{xxx} - \frac{1}{c}\psi_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ \varphi(0,t) = \varphi(L,t) = \varphi_x(0,t) = 0, & \text{in } (0,T), \\ \psi(0,t) = \psi(L,t) = \psi_x(0,t) = 0, & \text{in } (0,T), \\ \varphi(x,0) = \varphi^0(x), \quad \psi(x,0) = \psi^0(x), & \text{in } (0,L). \end{cases} \tag{3.36}$$

Moreover, remark that the change of variable $x = L - x$ reduces system (3.36) to (3.13). Therefore, the properties of the solutions of (3.36) are similar to the ones deduced in Theorem 3.2.7.

Proposition 3.2.8. *For any $(\varphi^0, \psi^0) \in \mathcal{X}$, the system (3.36) admits a unique solution $(\varphi, \psi) \in \mathcal{Z}_T$, such that it possess the following sharp trace properties*

$$\begin{cases} \sup\limits_{0\le x\le L} \|\partial_x^k \varphi(x,\cdot)\|_{H^{\frac{1-k}{3}}(0,T)} \le C_T \|\varphi^0\|_{L^2(0,L)}, \\ \sup\limits_{0\le x\le L} \|\partial_x^k \psi(x,\cdot)\|_{H^{\frac{1-k}{3}}(0,T)} \le C_T \|\psi^0\|_{L^2(0,L)}, \end{cases} \tag{3.37}$$

for $k = 0, 1, 2$, where C_T increases exponentially in T.

In what concerned system (3.33)–(3.35), it possesses the sharp hidden regularity (3.37) a relevant result as described above. Moreover, we have the following estimate:

Proposition 3.2.9. *Any solution (φ, ψ) of the adjoint system (3.33)–(3.35) satisfies*

$$\begin{aligned} \|(\varphi^1,\psi^1)\|_{\mathcal{X}}^2 \le & \frac{C}{T}\|(\varphi,\psi)\|^2_{L^2(0,T;\mathcal{X})} + \frac{1}{2}\|\varphi_x(L,\cdot)\|^2_{L^2(0,T)} + \frac{b}{2c}\|\psi_x(L,\cdot)\|^2_{L^2(0,T)} \\ & + \frac{1}{2}\left\|\varphi_x(L,\cdot) + \frac{ab}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)} + \frac{b}{2c}\left\|a\varphi_x(L,\cdot) + \frac{1}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}, \end{aligned} \tag{3.38}$$

with $(\varphi^1, \psi^1) \in \mathcal{X}$ and $C = \frac{\max\{b,c\}}{\min\{b,c\}}$.

Proof. Multiplying the first equation of (3.33) by $-t\varphi$, the second one by $-\frac{b}{c}t\psi$ and integrating by parts in $(0,T)\times(0,L)$, we obtain

$$\begin{aligned} \frac{C_1 T}{2}\|(\varphi^1,\psi^1)\|^2_{\mathcal{X}} \le & \frac{C_2}{2}\|(\varphi,\psi)\|^2_{L^2(0,T;\mathcal{X})} \\ & - \int_0^T t\left[\frac{b}{c}\psi(x,t)\left(a\varphi_{xx}(x,t) + \frac{1}{c}\psi_{xx}(x,t) + \frac{r}{c}\psi(x,t)\right)\right. \\ & \qquad - \frac{b}{2c}\psi_x(x,t)\left(a\varphi_x(x,t) + \frac{1}{c}\psi_x(x,t)\right) \\ & \qquad - \frac{1}{2}\varphi_x(x,t)\left(\varphi_x(x,t) + \frac{ab}{c}\psi_x(x,t)\right) \\ & \qquad \left. + \varphi(x,t)\left(\varphi_{xx}(x,t) + \frac{ab}{c}\psi_{xx}(x,t)\right) - \frac{br}{2c^2}\psi^2(x,t)\right]_0^L dt, \end{aligned}$$

where $C_1 = \min\{b,c\}$ and $C_2 = \max\{b,c\}$. From (3.34) and applying Young inequality, (3.38) is obtained. ◻

3.3 Exact Boundary Controllability: Linear System

3.3.1 Four controls

Considerations are first given to the boundary controllability of the linear system

$$\begin{cases} u_t + u_{xxx} + av_{xxx} = 0 & \text{in } (0,L)\times(0,T), \\ v_t + \frac{r}{c}v_x + \frac{ab}{c}u_{xxx} + \frac{1}{c}v_{xxx} = 0 & \text{in } (0,L)\times(0,T), \\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L) \end{cases} \tag{3.39}$$

satisfying the boundary conditions

$$\begin{cases} u(0,t) = 0, & u(L,t) = 0, & u_x(L,t) = h_2(t) & \text{in } (0,T), \\ v(0,t) = g_0(t), & v(L,t) = g_1(t), & v_x(L,t) = g_2(t) & \text{in } (0,T), \end{cases} \tag{3.40}$$

which employ $\overrightarrow{h}_1 := (0,0,h_2)$ and $\overrightarrow{g}_1 := (g_0, g_1, g_2) \in \mathcal{H}_T$.

Theorem 3.3.1. *Let $L \in (0,\infty) \setminus \mathcal{F}'_r$, where $\mathcal{F}'_r$ is defined by (3.2) and $T > 0$ be given. There exists a bounded linear operator*

$$\Psi : \quad [L^2(0,L)]^2 \times [L^2(0,L)]^2 \longrightarrow \quad \mathcal{H}_T \times \mathcal{H}_T$$

such that for any $(u^0, v^0) \in [L^2(0,L)]^2$ and $(u^1, v^1) \in [L^2(0,L)]^2$, if one chooses

$$(\overrightarrow{h}_1, \overrightarrow{g}_1) = \Psi((u^0,v^0),(u^1,v^1)),$$

then the system (3.39)-(3.40) admits a solution $(u,v) \in \mathcal{Z}_T$ satisfying

$$u(\cdot,T) = u^1(\cdot), \qquad \text{and} \qquad v(\cdot,T) = v^1(\cdot). \tag{3.41}$$

To prove the previous result we first establish the following observability for the corresponding adjoint system (3.33)-(3.35).

Proposition 3.3.2. *For $T > 0$ and $L \in (0,\infty) \setminus \mathcal{F}'_r$. There exists a constant $C(T,L) > 0$, such that*

$$\begin{aligned} \|(\varphi^1,\psi^1)\|^2_{\mathcal{X}} \leq C \Bigg\{ & \left\| (-\Delta_t)^{-\frac{1}{6}} \left(a\varphi_{xx}(L,\cdot) + \frac{1}{c}\psi_{xx}(L,\cdot) \right) \right\|^2_{L^2(0,T)} \\ & + \left\| \varphi_x(L,\cdot) + \frac{ab}{c}\psi_x(L,\cdot) \right\|^2_{L^2(0,T)} + \left\| (-\Delta_t)^{-\frac{1}{6}} \left(a\varphi_{xx}(0,\cdot) + \frac{1}{c}\psi_{xx}(0,\cdot) \right) \right\|^2_{L^2(0,T)} \\ & + \left\| a\varphi_x(L,\cdot) + \frac{1}{c}\psi_x(L,\cdot) \right\|^2_{L^2(0,T)} \Bigg\}, \end{aligned} \tag{3.42}$$

for any $(\varphi^1,\psi^1) \in \mathcal{X}$, where (φ,ψ) is solution of (3.33)-(3.35).

Proof. We proceed as in [78, Proposition 3.3]. Let us suppose that (3.42) does not hold. In this case, it follows that there exists a sequence $\{(\varphi^1_n,\psi^1_n)\}_{n\in\mathbb{N}}$, such that

$$\begin{aligned} 1 = \|(\varphi^1_n,\psi^1_n)\|^2_{\mathcal{X}} \geq n \Bigg\{ & \left\| (-\Delta_t)^{-\frac{1}{6}} \left(a\varphi_{n,xx}(L,\cdot) + \frac{1}{c}\psi_{n,xx}(L,\cdot) \right) \right\|^2_{L^2(0,T)} \\ & + \left\| \varphi_{n,x}(L,\cdot) + \frac{ab}{c}\psi_{n,x}(L,\cdot) \right\|^2_{L^2(0,T)} + \left\| (-\Delta_t)^{-\frac{1}{6}} \left(a\varphi_{n,xx}(0,\cdot) + \frac{1}{c}\psi_{n,xx}(0,\cdot) \right) \right\|^2_{L^2(0,L)} \\ & + \left\| a\varphi_{n.x}(L,\cdot) + \frac{1}{c}\psi_{n,x}(L,\cdot) \right\|^2_{L^2(0,T)} \Bigg\}. \end{aligned} \tag{3.43}$$

where, for each $n\in\mathbb{N}$, $\{(\varphi_n,\psi_n)\}_{n\in\mathbb{N}}$ is the solution of (3.33)-(3.35). Inequality (3.43) imply that

$$\begin{cases} (-\Delta_t)^{-\frac{1}{6}}\left(a\varphi_{n,xx}(0,\cdot)+\frac{1}{c}\psi_{n,xx}(0,\cdot)\right)\to 0 & \text{in} \quad L^2(0,T),\\ (-\Delta_t)^{-\frac{1}{6}}\left(a\varphi_{n,xx}(L,\cdot)+\frac{1}{c}\psi_{n,xx}(L,\cdot)\right)\to 0 & \text{in} \quad L^2(0,T),\\ \varphi_{n,x}(L,\cdot)+\frac{ab}{c}\psi_{n,x}(L,\cdot)\to 0 & \text{in} \quad L^2(0,T),\\ a\varphi_{n,x}(L,\cdot)+\frac{1}{c}\psi_{n,x}(L,\cdot)\to 0 & \text{in} \quad L^2(0,T). \end{cases} \tag{3.44}$$

Since $1-a^2b>0$, from the convergence of the sequences in the third and fourth lines of (3.44), we obtain

$$\begin{cases} a\varphi_{n,xx}(0,\cdot)+\frac{1}{c}\psi_{n,xx}(0,\cdot)\to 0 & \text{in} \quad H^{-\frac{1}{3}}(0,T),\\ a\varphi_{n,xx}(L,\cdot)+\frac{1}{c}\psi_{n,xx}(L,\cdot)\to 0 & \text{in} \quad H^{-\frac{1}{3}}(0,T),\\ \varphi_{n,x}(L,\cdot)\to 0 & \text{in} \quad L^2(0,T),\\ \psi_{n,x}(L,\cdot)\to 0 & \text{in} \quad L^2(0,T). \end{cases} \tag{3.45}$$

From (3.37) and (3.43), we obtain that $\{(\varphi_n,\psi_n)\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;(H^1(0,L))^2)$. On the other hand, system (3.33) implies that the sequence $\{(\varphi_{t,n},\psi_{t,n})\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;(H^{-2}(0,L))^2)$, and the compact embedding

$$H^1(0,L)\hookrightarrow_{cc} L^2(0,L)\hookrightarrow H^{-2}(0,L), \tag{3.46}$$

allows us to conclude that $\{(\varphi_n,\psi_n)\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(0,T;\mathcal{X})$ and consequently, we obtain a subsequence, still denoted by the same index n, satisfying

$$(\varphi_n,\psi_n)\to(\varphi,\psi) \text{ in } L^2(0,T;\mathcal{X}), \text{ as } n\to\infty. \tag{3.47}$$

Furthermore, (3.37) implies that sequences $\{\varphi_n(0,\cdot)\}_{n\in\mathbb{N}}$, $\{\varphi_n(L,\cdot)\}_{n\in\mathbb{N}}$, $\{\psi_n(0,\cdot)\}_{n\in\mathbb{N}}$ and $\{\psi_n(L,\cdot)\}_{n\in\mathbb{N}}$ are bounded in $H^{\frac{1}{3}}(0,T)$. Then, the embedding

$$H^{\frac{1}{3}}(0,T)\hookrightarrow_{cc} L^2(0,T) \tag{3.48}$$

guarantees that the above sequences are relatively compact in $L^2(0,T)$. Thus, we obtain a subsequence, still denoted by the same index n, satisfying

$$\begin{cases} \varphi_n(0,\cdot)\to\varphi(0,\cdot), \quad \varphi_n(L,\cdot)\to\varphi(L,\cdot) & \text{in} \quad L^2(0,T),\\ \psi_n(0,\cdot)\to\psi(0,\cdot), \quad \psi_n(L,\cdot)\to\psi(L,\cdot) & \text{in} \quad L^2(0,T). \end{cases} \tag{3.49}$$

From (3.34), we deduce that

$$\begin{cases} \varphi(0,\cdot)=\varphi(L,\cdot)=0,\\ \psi(0,\cdot)=\psi(L,\cdot)=0. \end{cases}$$

In addition, according to Proposition 3.2.9, we have

$$\begin{aligned} \|(\varphi_n^1,\psi_n^1)\|_{\mathcal{X}}^2 \leq & \frac{C}{T}\|(\varphi_n,\psi_n)\|_{L^2(0,T;\mathcal{X})}+\frac{1}{2}\|\varphi_{n,x}(L,\cdot)\|_{L^2(0,T)}^2+\frac{b}{2c}\|\psi_{n,x}(L,\cdot)\|_{L^2(0,T)}^2\\ & +\frac{1}{2}\left\|\varphi_{n,x}(L,\cdot)+\frac{ab}{c}\psi_{n,x}(L,\cdot)\right\|_{L^2(0,T)}^2+\frac{b}{2c}\left\|a\varphi_{n,x}(L,\cdot)+\frac{1}{c}\psi_{n,x}(L,\cdot)\right\|_{L^2(0,T)}^2. \end{aligned}$$

Then, from (3.45) and (3.47) if follows that $\{(\varphi_n^1,\psi_n^1)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}$. Thus,

$$(\varphi_n^1,\psi_n^1)\to(\varphi^1,\psi^1) \text{ in } \mathcal{X}, \text{ as } n\to\infty. \tag{3.50}$$

Proposition 3.2.8 together with (3.50), imply that

$$\begin{cases} \varphi_{n,x}(0,\cdot) \to \varphi_x(0,\cdot) & \text{in } L^2(0,T), \text{ as } n \to \infty, \\ \varphi_{n,x}(L,\cdot) \to \varphi_x(L,\cdot) & \text{in } L^2(0,T), \text{ as } n \to \infty, \\ \psi_{n,x}(0,\cdot) \to \psi_x(0,\cdot) & \text{in } L^2(0,T), \text{ as } n \to \infty, \\ \psi_{n,x}(L,\cdot) \to \psi_x(L,\cdot) & \text{in } L^2(0,T), \text{ as } n \to \infty \end{cases}$$

and

$$\begin{cases} a\varphi_{n,xx}(0,\cdot) + \frac{1}{c}\psi_{n,xx}(0,\cdot) \to a\varphi_{xx}(0,\cdot) + \frac{1}{c}\psi_{xx}(0,\cdot) & \text{in } H^{-\frac{1}{3}}(0,T), \text{ as } n \to \infty, \\ a\varphi_{n,xx}(L,\cdot) + \frac{1}{c}\psi_{n,xx}(L,\cdot) \to a\varphi_{xx}(L,\cdot) + \frac{1}{c}\psi_{xx}(L,\cdot) & \text{in } H^{-\frac{1}{3}}(0,T), \text{ as } n \to \infty. \end{cases}$$

Finally, taking $n \to \infty$, from (3.33)-(3.35) and (3.45), we obtain that (φ, ψ) is solution of

$$\begin{cases} \varphi_t + \varphi_{xxx} + \frac{ab}{c}\psi_{xxx} = 0, & \text{in } (0,L) \times (0,T), \\ \psi_t + \frac{r}{c}\psi_x + a\varphi_{xxx} + \frac{1}{c}\psi_{xxx} = 0, & \text{in } (0,L) \times (0,T), \\ \varphi(0,t) = \varphi(L,t) = \varphi_x(0,t) = 0, & \text{in } (0,T). \\ \psi(0,t) = \psi(L,t) = \psi_x(0,t) = 0, & \text{in } (0,T). \\ \varphi(x,T) = \varphi^1(x), \qquad \psi(x,T) = \psi^1(x), & \text{in } (0,L), \end{cases} \tag{3.51}$$

satisfying the additional boundary conditions

$$\begin{cases} \varphi_x(L,t) = \psi_x(L,t) = 0, & \text{in } (0,T), \\ a\varphi_{xx}(0,\cdot) + \frac{1}{c}\psi_{xx}(0,\cdot) = 0, & \text{in } (0,T), \\ a\varphi_{xx}(L,\cdot) + \frac{1}{c}\psi_{xx}(L,\cdot) = 0, & \text{in } (0,T), \end{cases} \tag{3.52}$$

and, from (3.43), we get

$$\|(\varphi^1, \psi^1)\|_{\mathcal{X}} = 1. \tag{3.53}$$

Notice that (3.53) implies that the solutions of (3.51)-(3.52) can not be identically zero. However, from the following Lemma, one can conclude that $(\varphi, \psi) = (0,0)$, which drive us to contradicts (3.53). □

Lemma 3.3.3. *For any $T > 0$, let N_T denote the space of the initial states $(\varphi^1, \psi^1) \in \mathcal{X}$, such that the solution of (3.51) satisfies (3.52). Then, $N_T = \{0\}$.*

Proof. The proof uses the same arguments as those given in [78]. Therefore, if $N_T \neq \{0\}$, the map $(\varphi^1, \psi^1) \in N_T \to A(N_T) \subset \mathbb{C}N_T$ (where $\mathbb{C}N_T$ denote the complexification of N_T) has (at least) one eigenvalue, hence, there exists $\lambda \in \mathbb{C}$ and $\varphi_0, \psi_0 \in H^3(0,L) \setminus \{0\}$, such that

$$\begin{cases} \lambda\varphi_0 + \varphi_0''' + \frac{ab}{c}\psi_0''' = 0, & \text{in } (0,L), \\ \lambda\psi_0 + \frac{r}{c}\psi_0' + a\varphi_0''' + \frac{1}{c}\psi_0''' = 0, & \text{in } (0,L), \\ \varphi_0(0) = \varphi_0(L) = \varphi_0'(0) = \varphi_0'(L) = 0, \\ \psi_0(0) = \psi_0(L) = \psi_0'(0) = \psi_0'(L) = 0, \\ a\varphi_0''(0) + \frac{1}{c}\psi_0''(0) = 0, \\ a\varphi_0''(L) + \frac{1}{c}\psi_0''(L) = 0. \end{cases} \tag{3.54}$$

To conclude the proof of the Lemma, we prove that this does not hold if $L \in (0,\infty)\setminus\mathcal{F}_r'$. □

To simplify the notation, henceforth we denote $(\varphi_0, \psi_0) := (\varphi, \psi)$. Moreover, the notation $\{0, L\}$ means that the function is applied to 0 and L, respectively.

Lemma 3.3.4. *Let $L > 0$ and consider the assertion*

$(\mathcal{N})$: $\exists \lambda \in \mathbb{C}, \exists (\varphi, \psi) \in (H^3(0,L))^2 \setminus (0,0)$ *such that*

$$\begin{cases} \lambda\varphi + \varphi''' + \frac{ab}{c}\psi''' = 0, & in\ (0,L), \\ \lambda\psi + \frac{r}{c}\psi' + a\varphi''' + \frac{1}{c}\psi''' = 0, & in\ (0,L), \\ \varphi(x) = \psi(x) = 0, & in\ \{0,L\}, \\ \varphi'(x) = \psi'(x) = 0, & in\ \{0,L\}, \\ a\varphi''(x) + \frac{1}{c}\psi''(x) = 0, & in\ \{0,L\}. \end{cases}$$

Then, $(\mathcal{N})$ holds if and only if $L \in \mathcal{F}'_r$.

Proof. We use an argument which is similar to the one used in [78, Lemma 3,5]. Let us introduce the notation $\hat{\varphi}(\xi) = \int_0^L e^{-ix\xi}\varphi(x)dx$ and $\hat{\psi}(\xi) = \int_0^L e^{-ix\xi}\psi(x)dx$. Then, multiplying the equations by $e^{-ix\xi}$, integrating by parts over $(0,L)$ and using the boundary conditions, we have

$$\begin{cases} [(i\xi)^3 + \lambda]\hat{\varphi}(\xi) + \dfrac{ab}{c}(i\xi)^3\hat{\psi}(\xi) = \varphi''(0) + \dfrac{ab}{c}\psi''(0) - \left(\varphi''(L) + \dfrac{ab}{c}\psi''(L)\right)e^{-iL\xi}, \\ \dfrac{1}{c}[(i\xi)^3 + r(i\xi) + c\lambda]\hat{\psi}(\xi) + a(i\xi)^3\hat{\varphi}(\xi) = 0. \end{cases} \tag{3.55}$$

From the first equation in (3.55), we have

$$\hat{\varphi}(\xi) = \frac{\left(\alpha + \beta e^{-iL\xi}\right)}{(i\xi)^3 + \lambda} - \frac{ab(i\xi)^3\hat{\psi}(\xi)}{c\left((i\xi)^3 + \lambda\right)}, \tag{3.56}$$

where $\alpha = \varphi''(0) + \frac{ab}{c}\psi''(0)$ and $\beta = -\varphi''(L) - \frac{ab}{c}\psi''(L)$. Replacing (3.56) in the second equation of (3.55), it follows that

$$\frac{1}{c}\left[(i\xi)^3 + r(i\xi) + c\lambda - \frac{a^2b(i\xi)^6}{(i\xi)^3 + \lambda}\right]\hat{\psi}(\xi) = -\frac{a(i\xi)^3\left(\alpha + \beta e^{-iL\xi}\right)}{(i\xi)^3 + \lambda}.$$

Therefore,

$$\hat{\psi}(\xi) = -\frac{ac(i\xi)^3\left(\alpha + \beta e^{-iL\xi}\right)}{(1 - a^2b)(i\xi)^6 + r(i\xi)^4 + (c+1)\lambda(i\xi)^3 + r\lambda(i\xi) + c\lambda^2}. \tag{3.57}$$

Having (3.57) in hands, from (3.56) we obtain

$$\hat{\varphi}(\xi) = \left(1 + \frac{a^2b(i\xi)^6}{(1 - a^2b)(i\xi)^6 + r(i\xi)^4 + (c+1)\lambda(i\xi)^3 + r\lambda(i\xi) + c\lambda^2}\right)\frac{\left(\alpha + \beta e^{-iL\xi}\right)}{(i\xi)^3 + \lambda},$$

hence,

$$\hat{\varphi}(\xi) = \frac{\left((i\xi)^3 + r(i\xi) + c\lambda\right)\left(\alpha + \beta e^{-iL\xi}\right)}{(1 - a^2b)(i\xi)^6 + r(i\xi)^4 + (c+1)\lambda(i\xi)^3 + r\lambda(i\xi) + c\lambda^2}.$$

Setting $\lambda = ip$, $p \in \mathbb{C}$, we have that $\hat{\psi}(\xi) = -acif(\xi)$ and $\hat{\varphi}(\xi) = ig(\xi)$, where

$$\begin{cases} f(\xi) = \dfrac{\xi^3\left(\alpha + \beta e^{-iL\xi}\right)}{P(\xi)}, \\ g(\xi) = \dfrac{\left(\xi^3 - r\xi - cp\right)\left(\alpha + \beta e^{-iL\xi}\right)}{P(\xi)}. \end{cases}$$

with

$$P(\xi) := (1 - a^2b)\xi^6 - r\xi^4 - (c+1)p\xi^3 + rp\xi + cp^2.$$

Using Paley-Wiener theorem ([97, Section 4, page 161]) and the usual characterization of $H^2(\mathbb{R})$ functions by means of their Fourier transforms, we see that $(\mathcal{N})$ is equivalent to the existence of $p \in \mathbb{C}$ and $(\alpha, \beta) \in \mathbb{C}^2 \setminus (0,0)$, such that

(i) f and g are entire functions in $\mathbb{C}$,

(ii) $\displaystyle\int_{\mathbb{R}} |f(\xi)|^2(1+|\xi|^2)^2 d\xi < \infty$ and $\int_{\mathbb{R}} |g(\xi)|^2(1+|\xi|^2)^2 d\xi < \infty$,

(iii) $\forall \xi \in \mathbb{C}$, we have that $|f(\xi)| \leq c_1(1+|\xi|)^k e^{L|Im\xi|}$ and $|g(\xi)| \leq c_1(1+|\xi|)^k e^{L|Im\xi|}$, for some positive constants c_1 and k.

Notice that if (i) holds true, then (ii) and (iii) are satisfied. Recall that f and g are entire functions if only if, the roots $\xi_0, \xi_1, \xi_2, \xi_3, \xi_4$ and ξ_5 of $P(\xi)$ are roots of $\xi^3\left(\alpha+\beta e^{-iL\xi}\right)$ and $(\xi^3 - r\xi - cp)\left(\alpha+\beta e^{-iL\xi}\right)$.

Let us first assume that $\xi = 0$ is not root of $P(\xi)$. Thus, it is sufficiently to consider the case when $\alpha+\beta e^{-iL\xi}$ and $P(\xi)$ share the same roots. Since the roots of $\alpha+\beta e^{-iL\xi}$ are simple, unless $\alpha=\beta=0$ (Indeed, it implies that $\varphi''(0)+\frac{ab}{c}\psi''(0)=0$ and $\varphi''(L)+\frac{ab}{c}\psi''(L)=0$, thus, using the system (3.54), we conclude that $(\varphi,\psi)=(0,0)$, which is a contradiction). Then, (i) holds provided that the roots of $P(\xi)$ are simple. Thus, we conclude tha $(\mathcal{N})$ is equivalent to the existence of complex numbers p, ξ_0 and positive integers k, l, m, n and s, such that, if we set

$$\xi_1 = \xi_0 + \frac{2\pi}{L}k, \quad \xi_2 = \xi_1 + \frac{2\pi}{L}l, \quad \xi_3 = \xi_2 + \frac{2\pi}{L}m, \quad \xi_4 = \xi_3 + \frac{2\pi}{L}n \quad \text{and} \quad \xi_5 = \xi_4 + \frac{2\pi}{L}s, \tag{3.58}$$

we have

$$P(\xi) = (\xi-\xi_0)(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)(\xi-\xi_5). \tag{3.59}$$

In particular, we obtain the following relations:

$$\xi_0+\xi_1+\xi_2+\xi_3+\xi_4+\xi_5 = 0, \tag{3.60}$$

$$\begin{aligned}\xi_0(\xi_1+\xi_2+\xi_3+\xi_4+\xi_5)+\xi_1(\xi_2+\xi_3+\xi_4+\xi_5)+\xi_2(\xi_3+\xi_4+\xi_5)&\\ +\xi_3(\xi_4+\xi_5)+\xi_4\xi_5 &= -\frac{r}{1-a^2b},\end{aligned} \tag{3.61}$$

$$\xi_0\xi_1\xi_2\xi_3\xi_4\xi_5 = \left(\frac{c}{1-a^2b}\right)p^2. \tag{3.62}$$

Some calculations lead to

$$\begin{cases} L = \pi\sqrt{\dfrac{(1-a^2b)\alpha(k,l,m,n,s)}{3r}}, \\ \\ \xi_0 = -\dfrac{\pi}{3}(5k+4l+3m+2n+s), \\ \\ p = \sqrt{\dfrac{(1-a^2b)\xi_0\xi_1\xi_2\xi_3\xi_4\xi_5}{c}}, \end{cases} \tag{3.63}$$

where

$$\begin{aligned}\alpha(k,l,m,n,s) := 5k^2+8l^2+9m^2+8n^2+5s^2+8kl+6km+4kn+2ks+12ml&\\ +8ln+3ls+12mn+6ms+8ns.&\end{aligned}$$

Finally, we assume that $\xi_0 = 0$ is a root of $P(\xi)$. In this case, it follows that $p = 0$ and, therefore,

$$\begin{cases} f(\xi) = \dfrac{\xi^3\left(\alpha+\beta e^{-iL\xi}\right)}{(1-a^2b)\xi^6 - r\xi^4} = \dfrac{\left(\alpha+\beta e^{-iL\xi}\right)}{\xi\left((1-a^2b)\xi^2 - r\right)}, \\ \\ g(\xi) = \dfrac{\left(\xi^3 - r\xi\right)\left(\alpha+\beta e^{-iL\xi}\right)}{(1-a^2b)\xi^6 - r\xi^4} = \dfrac{\left(\xi^2 - r\right)\left(\alpha+\beta e^{-iL\xi}\right)}{\xi^3\left((1-a^2b)\xi^2 - r\right)}. \end{cases}$$

Then, $(\mathcal{N})$ holds if and only if f and g satisfy (i), (ii) and (iii). Thus (i) holds provided that

$$\xi_0 = 0, \quad \xi_1 = \sqrt{\frac{r}{1-a^2b}} \quad \text{and} \quad \xi_2 = -\sqrt{\frac{r}{1-a^2b}}$$

are roots of $\alpha + \beta e^{-iL\xi}$. Note that, zero must be root of multiplicity three, which leads to a contradiction. Thus, $\xi = 0$ is not root of $P(\xi)$. Finally, from (3.63), we deduce that $(\mathcal{N})$ holds if and only if $L \in \mathcal{F}'_r$. This completes the proof of Lemma 3.3.4, Lemma 3.3.3 and, consequently, the proof of Proposition 3.3.2. □

Proof of Theorem 3.3.1. Without loss of generality, we assume that $(u^0, v^0) = (0, 0)$. Moreover, it is easy to see that the the solution (u, v) of (3.39)-(3.40) satisfies (3.41) if and only if

$$\begin{aligned}\int_0^L \left(u^1(x)\varphi^1(x) + v^1(x)\psi^1(x)\right) dx = &\int_0^T g_0(t)\left(a\varphi_{xx}(0,t) + \frac{1}{c}\psi_{xx}(0,t)\right) dt \\ &- \int_0^T g_1(t)\left(a\varphi_{xx}(L,t) + \frac{1}{c}\psi_{xx}(L,t)\right) dt \\ &+ \int_0^T g_2(t)\left(a\varphi_{x}(L,t) + \frac{1}{c}\psi_{x}(L,t)\right) dt \\ &+ \int_0^T h_2(t)\left(\varphi_{x}(L,t) + \frac{ab}{c}\psi_{x}(L,t)\right) dt\end{aligned} \tag{3.64}$$

for any $(\varphi^1, \psi^1) \in \mathcal{X}$, where (φ, ψ) is the solution of the system (3.33)-(3.34), with initial data (φ^1, ψ^1). Relation (3.64) is obtained by multiplying the equations in (3.39) by the solution (φ, ψ) of (3.33)-(3.34), integrating by parts and using the boundary conditions (3.40).

Thus, in order to obtain the desired result, we introduce the linear bounded map Γ as follows

$$\begin{array}{rccc}\Gamma: & L^2(0,L) \times L^2(0,L) & \longrightarrow & L^2(0,L) \times L^2(0,L) \\ & (\varphi^1(\cdot), \psi^1(\cdot)) & \longmapsto & \Gamma(\varphi^1(\cdot), \psi^1(\cdot)) = (u(\cdot,T), v(\cdot,T)),\end{array}$$

where (u, v) is the solution of (3.39)-(3.40), with

$$\begin{cases} g_0(t) = (-\Delta_t)^{-\frac{1}{3}}\left(a\varphi_{xx}(0,t) + \frac{1}{c}\psi_{xx}(0,t)\right), & g_2(t) = a\varphi_x(L,t) + \frac{1}{c}\psi_x(L,t), \\ g_1(t) = -(-\Delta_t)^{-\frac{1}{3}}\left(a\varphi_{xx}(L,t) + \frac{1}{c}\psi_{xx}(L,t)\right), & h_2(t) = \varphi_x(L,t) + \frac{ab}{c}\psi_x(L,t), \end{cases} \tag{3.65}$$

being (φ, ψ) the solution of the system (3.33)-(3.34) with initial data (φ^1, ψ^1) and $\Delta_t = \partial_t^2$.

According to Proposition 3.3.2

$$
\begin{aligned}
(\Gamma(\varphi^1,\psi^1),(\varphi^1,\psi^1))_{(L^2(0,L))^2} =& \left\|\varphi_x(L,\cdot)+\frac{ab}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}\\
&+\left\|a\varphi_x(L,\cdot)+\frac{1}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}\\
&+\left((-\Delta_t)^{\frac13}\left(\varphi_{xx}(0,\cdot)+\frac{ab}{c}\psi_{xx}(0,\cdot)\right),\varphi_{xx}(0,\cdot)+\frac{ab}{c}\psi_{xx}(0,\cdot)\right)_{L^2(0,T)}\\
&+\left((-\Delta_t)^{\frac13}\left(a\varphi_{xx}(0,\cdot)+\frac{1}{c}\psi_{xx}(0,\cdot)\right),a\varphi_{xx}(0,\cdot)+\frac{1}{c}\psi_{xx}(0,\cdot)\right)_{L^2(0,T)}\\
=&\left\|\varphi_x(L,\cdot)+\frac{ab}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}+\left\|a\varphi_x(L,\cdot)+\frac{1}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}\\
&+\left\|(-\Delta_t)^{\frac16}\left(a\varphi_{xx}(0,\cdot)+\frac{1}{c}\psi_{xx}(0,\cdot)\right)\right\|^2_{L^2(0,T)}\\
&+\left\|(-\Delta_t)^{\frac16}\left(\varphi_{xx}(0,\cdot)+\frac{ab}{c}\psi_{xx}(0,\cdot)\right)\right\|^2_{L^2(0,T)}\\
\geq& C^{-1}\|(\varphi^1,\psi^1)\|^2_{\mathcal{X}},
\end{aligned}
$$

i.e, Γ is coercive. Then, by Lax-Milgram theorem, Γ is invertible. Consequently, for given $(u^1,v^1)\in\mathcal{X}$, we can define $(\varphi^1,\psi^1):=\Gamma^{-1}(u^1,v^1)$ to solve the system (3.33)-(3.34) and get $(\varphi,\psi)\in\mathcal{Z}_T$. Thus, if $h_0(t)$, $h_1(t)$, $g_0(t)$ and $g_1(t)$ are given by (3.65), the corresponding solution (u,v) of the system (3.39)-(3.40), satisfies

$$
(u(\cdot,0),v(\cdot,0))=(0,0)\quad\text{and}\quad(u(\cdot,T),v(\cdot,T))=(u^1(\cdot),=v^1(\cdot)).
$$

□

3.3.2 One control

Consider the boundary controllability of the linear system employing only one control input h_2 and fixing $h_0=h_1=g_0=g_1=0$, namely,

$$
\begin{cases}
u(0,t)=0 & u(L,t)=0, \quad u_x(L,t)=h_2(t), \quad \text{in } (0,T),\\
v(0,t)=0, & v(L,t)=0, \quad v_x(L,t)=0, \quad \text{in } (0,T).
\end{cases}
\tag{3.66}
$$

Note that by using the change of variable $x'=L-x$ and $t'=T-t$, the system (3.33)-(3.35) is equivalent to the following forward system

$$
\begin{cases}
\varphi_t+\varphi_{xxx}+\frac{ab}{c}\psi_{xxx}=0, & \text{in } (0,L)\times(0,T),\\
\psi_t+\frac{r}{c}\psi_x+a\varphi_{xxx}+\frac{1}{c}\psi_{xxx}=0, & \text{in } (0,L)\times(0,T),\\
\varphi(x,0)=\varphi^0(x),\ \psi(x,0)=\psi^0(x), & \text{in } (0,L),
\end{cases}
\tag{3.67}
$$

with boundary conditions

$$
\begin{cases}
\varphi(0,t)=\varphi(L,t)=\varphi_x(L,t)=0, & \text{in } (0,T),\\
\psi(0,t)=\psi(L,t)=\psi_x(L,t)=0, & \text{in } (0,T).
\end{cases}
\tag{3.68}
$$

In this case, the observability inequality

$$
\|(\varphi^0,\psi^0)\|^2_{\mathcal{X}}\leq C\left\|\varphi_x(0,\cdot)+\frac{ab}{c}\psi_x(0,\cdot)\right\|^2_{L^2(0,T)}
\tag{3.69}
$$

plays a fundamental role for the study of the controllability. To prove (3.69) we use a direct approach based on the multiplier technique and the estimates given by the hidden regularity. Such estimates give us the observability inequality for some values of the length L and time of control T.

Proposition 3.3.5. *Let us suppose that $T > 0$ and $L > 0$ satisfy*

$$L < \frac{\min\{b, c\}}{\max\{b, c\}\beta C_T} T, \tag{3.70}$$

where C_T is the constant in (3.37) and β is the constant given by the embedding $H^{\frac{1}{3}}(0, T) \subset L^2(0, T)$. Then, there exists a constant $C(T, L) > 0$, such that for any (φ^0, ψ^0) in $\mathcal{X}$ the observability inequality (3.69) holds, for any (φ, ψ) solution of (3.67)-(3.68) with initial data (φ^0, ψ^0).

Proof. We multiply the first equation in (3.67) by $(T - t)\varphi$, the second one by $\frac{b}{c}(T - t)\psi$ and integrate over $(0, T) \times (0, L)$. Thus, we obtain

$$\begin{aligned} \frac{T}{2} \int_0^L \left(\varphi_0^2(x) + \frac{b}{c}\psi_0^2(x)\right) dx &= \frac{1}{2} \int_0^T \int_0^L \left(\varphi^2(x, t) + \frac{b}{c}\psi^2(x, t)\right) dxdt \\ &+ \frac{1}{2} \int_0^T (T - t) \left[\varphi_x^2(0, t) + \frac{2ab}{c}\psi_x(0, t)\varphi_x(0, t) + \frac{b}{c^2}\psi_x^2(0, t)\right] dt. \end{aligned}$$

Consequently,

$$\|(\varphi^0, \psi^0)\|_{\mathcal{X}}^2 \leq \frac{C}{T}\|(\varphi, \psi)\|_{L^2(0,T;\mathcal{X})}^2 + C_1 \left\|\varphi_x(0, \cdot) + \frac{ab}{c}\psi_x(0, \cdot)\right\|_{L^2(0,T)}^2, \tag{3.71}$$

where $C = \frac{\max\{b,c\}}{\min\{b,c\}}$ and $C_1 = C_1(a, b, c) > 0$. On the other hand, note that

$$\|\varphi(\cdot, t)\|_{L^2(0,L)}^2 \leq L\|\varphi(\cdot, t)\|_{L^\infty(0,L)}^2, \quad \text{and} \quad \|\psi(\cdot, t)\|_{L^2(0,L)}^2 \leq L\|\psi(\cdot, t)\|_{L^\infty(0,L)}^2.$$

Hence,

$$\begin{aligned} \|(\varphi, \psi)\|_{L^2(0,T;\mathcal{X})}^2 &\leq L \int_0^T \left\{\frac{b}{c}\|\varphi(\cdot, t)\|_{L^\infty(0,L)}^2 + \|\psi(\cdot, t)\|_{L^\infty(0,L)}^2\right\} dt & (3.72) \\ &\leq \frac{bL\beta}{c}\|\varphi\|_{H^{\frac{1}{3}}(0,T;L^\infty(0,L))}^2 + L\beta\|\psi\|_{H^{\frac{1}{3}}(0,T;L^\infty(0,L))}^2 & (3.73) \end{aligned}$$

where β is the constant given by the compact embedding $H^{\frac{1}{3}}(0, T) \subset L^2(0, T)$. Combining (3.71), (3.72) and Proposition 3.2.8, we obtain

$$\|(\varphi^0, \psi^0)\|_{\mathcal{X}}^2 \leq \frac{L\beta C_T C}{T}\|(\varphi^0, \psi^0)\|_{\mathcal{X}}^2 + C_1 \left\|\varphi_x(0, \cdot) + \frac{ab}{c}\psi_x(0, \cdot)\right\|_{L^2(0,T)}^2.$$

Finally, we obtain

$$\|(\varphi^0, \psi^0)\|_{\mathcal{X}}^2 \leq K \left\|\varphi_x(0, \cdot) + \frac{ab}{c}\psi_x(0, \cdot)\right\|_{L^2(0,T)}^2$$

under the condition

$$K = C_1 \left(1 - \frac{CC_T\beta L}{T}\right)^{-1} > 0. \tag{3.74}$$

□

From the observability inequality (3.69), the following result holds.

Theorem 3.3.6. *Let $T > 0$ and $L > 0$ satisfying (3.70). Then, the system (3.39)-(3.66) is exactly controllable in time T.*

Proof. We can proceed following the same ideas presented in the proof of Theorem 2.3.12. In this case, we consider the map

$$\begin{array}{rccc} \Gamma: & L^2(0,L)\times L^2(0,L) & \longrightarrow & L^2(0,L)\times L^2(0,L) \\ & (\varphi^1(\cdot),\psi^1(\cdot)) & \longmapsto & \Gamma(\varphi^1(\cdot),\psi^1(\cdot)) = (u(\cdot,T),v(\cdot,T)) \end{array}$$

where (u,v) is the solution of (3.39)-(3.66), with $h_2(t) = \varphi_x(L,t)+\frac{ab}{c}\psi_x(L,t)$ and (φ,ψ) is the solution of the system (3.33)-(3.35) with initial data (φ^1,ψ^1). Then, the observability inequality (3.69) guarantees that Γ is coercive and, consequently, by using Lax-Milgram theorem, the proof is achieved. □

3.4 Exact Controllability: The Nonlinear Control System

3.4.1 Well-posedness of the nonlinear system

In this subsection, attention will be given to the full nonlinear initial boundary value problem (IBVP)

$$\begin{cases} u_t + uu_x + u_{xxx} + av_{xxx} + a_1vv_x + a_2(uv)_x = 0, & \text{in } (0,L)\times(0,T), \\ cv_t + rv_x + vv_x + abu_{xxx} + v_{xxx} + a_2buu_x + a_1b(uv)_x = 0, & \text{in } (0,L)\times(0,T), \\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L), \end{cases} \tag{3.75}$$

with the boundary conditions

$$\begin{cases} u(0,t) = h_0(t),\ u(L,t) = h_1(t),\ u_x(L,t) = h_2(t), \\ v(0,t) = g_0(t),\ v(L,t) = g_1(t),\ v_x(L,t) = g_2(t). \end{cases} \tag{3.76}$$

We show that the IBVP (3.75)-(3.76) is locally well-posed in the space $\mathcal{Z}_T$.

Theorem 3.4.1. *Let $T > 0$ be given. For any $(u^0,v^0) \in \mathcal{X}$ and $\overrightarrow{h} := (h_0,h_1,h_2)$, $\overrightarrow{g} := (g_0,g_1,g_2) \in \mathcal{H}_T$, there exists $T^* \in (0,T]$ depending on $\|(u^0,v^0)\|_{\mathcal{X}}$, such that the IBVP (3.75)-(3.76) admits a unique solution $(u,v) \in \mathcal{Z}_{T^*}$ with*

$$\partial_x^k u, \partial_x^k v \in L_x^\infty(0,L;H^{\frac{1-k}{3}}(0,T^*)), \quad k = 0,1,2.$$

Moreover, the corresponding solution map is Lipschitz continuous.

Proof. Let

$$\mathcal{F}_T = \left\{(u,v)\in\mathcal{Z}_T : (u,v)\in L_x^\infty(0,L;(H^{\frac{1-k}{3}}(0,T))^2), k=0,1,2\right\}$$

be a Banach space equipped with the norm

$$\|(u,v)\|_{\mathcal{F}_T} = \|(u,v)\|_{\mathcal{Z}_T} + \sum_{k=0}^{2}\|(\partial_x^k u,\partial_x^k v)\|_{L_x^\infty(0,L;(H^{\frac{1-k}{3}}(0,T))^2)}.$$

Let $0 < T^* \le T$ to be determined later. For each $u,v \in \mathcal{F}_{T^*}$, consider the problem

$$\begin{cases} \omega_t + \omega_{xxx} + a\eta_{xxx} = f(u,v), & \text{in } (0,L)\times(0,T^*), \\ \eta_t + \frac{ab}{c}\omega_{xxx} + \frac{1}{c}\eta_{xxx} = s(u,v), & \text{in } (0,L)\times(0,T^*), \\ \omega(0,t) = h_0(t),\ \omega(L,t) = h_1(t),\ \omega_x(L,t) = h_2(t), & \text{in } (0,T^*), \\ \eta(0,t) = g_0(t),\ \eta(L,t) = g_1(t),\ \eta_x(L,t) = g_2(t), & \text{in } (0,T^*), \\ \omega(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L), \end{cases} \tag{3.77}$$

where

$$f(u,v) = -a_1(vv_x) - a_2(uv)_x$$

and

$$s(u,v) = -\frac{r}{c}v_x - \frac{a_2 b}{c}(uu_x) - \frac{a_1 b}{c}(uv)_x.$$

Since $\|v_x\|_{L^1(0,\beta;L^2(0,L))} \leq \beta^{\frac{1}{2}}\|v\|_{\mathcal{Z}_\beta}$, from [11, Lemma 3.1] we deduce that $f(u,v)$ and $s(u,v)$ belong to $L^1(0,T^*;L^2(0,L))$ and

$$\|(f,s)\|_{L^1(0,T^*;(L^2(0,L))^2)} \leq C_1((T^*)^{\frac{1}{2}} + (T^*)^{\frac{1}{3}})\left(\|u\|^2_{\mathcal{Z}_{T^*}} + (\|u\|_{\mathcal{Z}_{T^*}}+1)\|v\|_{\mathcal{Z}_{T^*}} + \|v\|^2_{\mathcal{Z}_{T^*}}\right),$$

for some positive constant C_1. According to Proposition 3.2.5, we can define the operator

$$\Gamma : \mathcal{F}_{T^*} \to \mathcal{F}_{T^*} \quad \text{given by} \quad \Gamma(u,v) = (\omega,\eta),$$

where (ω,η) is the solution of (3.77). Moreover,

$$\|\Gamma(u,v)\|_{\mathcal{F}_{T^*}} \leq C\left\{\|(u^0,v^0)\|_{\mathcal{X}} + \|(\overrightarrow{h},\overrightarrow{g})\|_{\mathcal{H}^*_T} + \|(f,s)\|_{L^1(0,T^*;(L^2(0,L))^2)}\right\},$$

where the positive constant C depends only on T^*. Thus, we obtain

$$\begin{aligned}\|\Gamma(u,v)\|_{\mathcal{F}_{T^*}} \leq& C\left\{\|(u^0,v^0)\|_{\mathcal{X}} + \|(\overrightarrow{h},\overrightarrow{g})\|_{\mathcal{H}^*_T}\right\}\\ &+CC_1((T^*)^{\frac{1}{2}} + (T^*)^{\frac{1}{3}})\left(\|u\|^2_{\mathcal{Z}_{T^*}} + (\|u\|_{\mathcal{Z}_{T^*}}+1)\|v\|_{\mathcal{Z}_{T^*}} + \|v\|^2_{\mathcal{Z}_\beta}\right).\end{aligned}$$

Let $(u,v) \in B_r(0)$, where

$$B_r(0) := \left\{(u,v) \in \mathcal{F}_{T^*} : \|(u,v)\|_{\mathcal{F}_{T^*}} \leq r\right\},$$

with $r = 2C\left\{\|(u^0,v^0)\|_{\mathcal{X}} + \|(\overrightarrow{h},\overrightarrow{g})\|_{\mathcal{H}_T}\right\}$. It follows that

$$\|\Gamma(u,v)\|_{\mathcal{F}_{T^*}} \leq \frac{r}{2} + CC_1((T^*)^{\frac{1}{2}} + (T^*)^{\frac{1}{3}})\,(3r+1)\,r. \tag{3.78}$$

Choosing $T^* > 0$, such that

$$CC_1((T^*)^{\frac{1}{2}} + (T^*)^{\frac{1}{3}})\,(3r+1) \leq \frac{1}{2},$$

from (3.78), we have

$$\|\Gamma(u,v)\|_{\mathcal{F}_{T^*}} \leq r.$$

Therefore,

$$\Gamma : B_r(0) \subset \mathcal{F}_{T^*} \to B_r(0).$$

On the other hand, $\Gamma(u_1,v_1) - \Gamma(u_2,v_2)$ is the solution of system

$$\begin{cases} \omega_t + \omega_{xxx} + a\eta_{xxx} = f(u_1,v_1) - f(u_2,v_2), & \text{in } (0,L)\times(0,T^*),\\ \eta_t + \frac{ab}{c}\omega_{xxx} + \frac{1}{c}\eta_{xxx} = s(u_1,v_1) - s(u_2,v_2), & \text{in } (0,L)\times(0,T^*),\\ \omega(0,t) = \omega(L,t) = \omega_x(L,t) = 0, & \text{in } (0,T^*),\\ \eta(0,t) = \eta(L,t) = \eta_x(L,t) = 0, & \text{in } (0,T^*),\\ \omega(x,0) = 0, \quad v(x,0) = 0, & \text{in } (0,L). \end{cases}$$

Note that

$$\begin{aligned}|f(u_1,v_1) - f(u_2,v_2)| \leq C_2|\,&((v_2-v_1)v_{2,x} + v_1(v_2-v_1)_x\\ &+(u_2(v_2-v_1))_x + ((u_2-u_1)v_1)_x)\,|\end{aligned}$$

and

$$|s(u_1,v_1)-s(u_2,v_2)|\leq C_2|\,((v_2-v_1)_x+(u_2-u_1)u_{2,x}+u_1(u_2-u_1)_x \\ +(u_2(v_2-v_1))_x+((u_2-u_1)v_1)_x)\,|,$$

for some positive constant C_2. Proposition 3.2.5 and [11, Lemma 3.1] give us the following estimate

$$\|\Gamma(u_1,v_1)-\Gamma(u_2,v_2)\|_{\mathcal{F}_{T^*}}\leq C_3((T^*)^{\frac{1}{2}}+(T^*)^{\frac{1}{3}})(8r+1)\|(u_1-u_2,v_1-v_2)\|_{\mathcal{F}_{T^*}},$$

for some positive constant C_3. Choosing T^*, such that

$$C_3((T^*)^{\frac{1}{2}}+(T^*)^{\frac{1}{3}})(8r+1)\leq\frac{1}{2},$$

we obtain

$$\|\Gamma(u_1,v_1)-\Gamma(u_2,v_2)\|_{\mathcal{F}_{T^*}}\leq\frac{1}{2}\|(u_1-u_2,v_1-v_2)\|_{\mathcal{F}_{T^*}}.$$

Hence $\Gamma:B_r(0)\to B_r(0)$ is a contraction and, by Banach fixed point theorem, we obtain a unique $(u,v)\in B_r(0)$, such that $\Gamma(u,v)=(u,v)\in\mathcal{F}_{T^*}$ and, therefore, the proof is complete. □

We are now in position to prove our main result. First, define the bounded linear operators

$$\Lambda_i:\mathcal{X}\times\mathcal{X}\longrightarrow\mathcal{H}_T\times\mathcal{H}_T\qquad(i=1,2),\tag{3.79}$$

such that, for any $(u^0,v^0)\in\mathcal{X}$ and $(u^1,v^1)\in\mathcal{X}$,

$$\Lambda_i\left(\begin{pmatrix}u^0\\v^0\end{pmatrix},\begin{pmatrix}u^1\\v^1\end{pmatrix}\right):=\begin{pmatrix}\vec{h}_i\\\vec{g}_i\end{pmatrix},$$

where

(i) $\vec{h}_1=(0,0,h_2)$ and $\vec{g}_1=(g_0,g_1,g_2)$,

(ii) $\vec{h}_2=(0,0,h_2)$ and $\vec{g}_2=(0,0,0)$.

Proof of Theorem 3.1.1. According to Proposition 3.2.5 and [11, Theorem 2.10] the solution of (3.75)-(3.76) can be written as:

$$\begin{aligned}\begin{pmatrix}u(t)\\v(t)\end{pmatrix}=&W_0(t)\begin{pmatrix}u_0\\v_0\end{pmatrix}+W_{bdr}(t)\begin{pmatrix}\vec{h}_i\\\vec{g}_i\end{pmatrix}\\&-\int_0^t W_0(t-\tau)\begin{pmatrix}a_1(vv_x)(\tau)+a_2(uv)_x(\tau)\\\frac{r}{c}v_x(\tau)+\frac{a_2b}{c}(uu_x)(\tau)+\frac{a_1b}{c}(uv)_x(\tau)\end{pmatrix}d\tau,\end{aligned}$$

with $i=1,2$, where $\{W_0(t)\}_{t\geq0}$ and $\{W_{bdr}(t)\}_{t\geq0}$ are the operators defined in the proof of Proposition 3.2.5.

For $u,v\in\mathcal{Z}_T$, let us define

$$\begin{pmatrix}v\\\nu(T,u,v)\end{pmatrix}:=\int_0^T W_0(T-\tau)\begin{pmatrix}a_1(vv_x)(\tau)+a_2(uv)_x(\tau)\\\frac{a_2b}{c}(uu_x)(\tau)+\frac{a_2b}{c}(uv)_x(\tau)\end{pmatrix}d\tau.$$

Here, we consider the case $i=1$. The other case $i=2$ is analogous and, therefore, we will omit it. Consider the map

$$\begin{aligned}\Gamma\begin{pmatrix}u\\v\end{pmatrix}=&W_0(t)\begin{pmatrix}u^0\\v^0\end{pmatrix}+W_{bdr}(x)\Lambda_1\left(\begin{pmatrix}u^0\\v^0\end{pmatrix},\begin{pmatrix}u^1\\v^1\end{pmatrix}+\begin{pmatrix}v\\\nu(T,u,v)\end{pmatrix}\right)\\&-\int_0^t W_0(t-\tau)\begin{pmatrix}a_1(vv_x)(\tau)+a_2(uv)_x(\tau)\\\frac{r}{c}v_x(\tau)+\frac{a_2b}{c}(uu_x)(\tau)+\frac{a_1b}{c}(uv)_x(\tau)\end{pmatrix}d\tau.\end{aligned}$$

By choosing

$$\begin{pmatrix} \vec{h}_1 \\ \vec{g}_1 \end{pmatrix} = \Lambda_1 \left(\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} + \begin{pmatrix} v \\ \nu(T,u,v) \end{pmatrix} \right), \tag{3.80}$$

we get, from Theorem 3.3.1,

$$\Gamma \begin{pmatrix} u \\ v \end{pmatrix} \Big|_{t=0} = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix}$$

and

$$\Gamma \begin{pmatrix} u \\ v \end{pmatrix} \Big|_{t=T} = \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} + \begin{pmatrix} v \\ \nu(T,u,v) \end{pmatrix} - \begin{pmatrix} v \\ \nu(T,u,v) \end{pmatrix} = \begin{pmatrix} u^1 \\ v^1 \end{pmatrix}.$$

If we show that the map Γ is a contraction in an appropriate metric space, then its fixed point (u,v) is the solution of (3.75)-(3.76) with $\vec{h}_1$ and $\vec{g}_1$ defined by (3.80), satisfying $u(\cdot,T) = u^1(\cdot)$ and $v(\cdot,T) = v^1(\cdot)$. In order to prove the existence of the fixed point we apply the Banach fixed point theorem to the restriction of Γ on closed ball

$$B_r = \left\{ (u,v) \in \mathcal{Z}_T : \|(u,v)\|_{\mathcal{Z}_T} \leq r \right\},$$

for some $r > 0$.

(a) Γ *maps* B_r *in itself.*

Indeed, as in the proof of Theorem 3.4.1, we obtain that there exists a constant $C_1 > 0$ such that

$$\left\| \Gamma \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\mathcal{Z}_T} \leq C_1 \delta + C_2 (r+1) r,$$

where C_2 is a constant depending only on T. Thus, if we select r and δ satisfying

$$r = 2C_1 \delta$$

and

$$2C_1 C_2 \delta + C_2 \leq \frac{1}{2},$$

the operator Γ maps B_r into itself for any $(u,v) \in \mathcal{Z}_T$.

(b) Γ *is contractive.*

In fact, proceeding as in the proof of Theorem 3.4.1, we obtain

$$\left\| \Gamma \begin{pmatrix} u \\ v \end{pmatrix} - \Gamma \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} \right\|_{\mathcal{Z}_T} \leq C_3 (r+1) r \left\| \begin{pmatrix} u - \widetilde{u} \\ v - \widetilde{v} \end{pmatrix} \right\|_{\mathcal{Z}_T},$$

for any $(u,v), (\widetilde{u}, \widetilde{v}) \in B_r$ and C_3 constant depending only on T. Thus, choosing $\delta > 0$, such that

$$\gamma = 2C_2 C_3 \delta + C_3 < 1,$$

we obtain

$$\left\| \Gamma \begin{pmatrix} u \\ v \end{pmatrix} - \Gamma \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} \right\|_{\mathcal{Z}_T} \leq \gamma \left\| \begin{pmatrix} u - \widetilde{u} \\ v - \widetilde{v} \end{pmatrix} \right\|_{\mathcal{Z}_T}.$$

Therefore, the map Γ is a contraction.

Thus, from (a) and (b), Γ has a fixed point in B_r by the Banach fixed point Theorem and its fixed point is the desired solution. The proof of Theorem 3.1.1 is achieved. □

3.5 Further Comments

The following remarks are now in order:

• In [69], it was proved that the system (1.16) with the boundary conditions

$$\begin{cases} u(0,t)=0, & u(L,t)=h_1(t), & u_x(L,t)=h_2(t) & \text{in } (0,T), \\ v(0,t)=0, & v(L,t)=g_1(t), & v_x(L,t)=g_2(t) & \text{in } (0,T), \end{cases} \tag{3.81}$$

is exactly controllable in $L^2(0,L)$ when $h_1, g_1 \in H_0^1(0,T)$ and $h_2, g_2 \in L^2(0,T)$ (see Theorem A). By using the tools developed in this paper, more precisely, Lemma 3.1.2, an improvement of the regularity of the control can be obtained. In this case, the control (h_1, g_1, h_2, g_2) can be found in the space $H^{\frac{1}{3}}(0,T) \times H^{\frac{1}{3}}(0,T) \times L^2(0,T) \times L^2(0,T)$.

• Another case that can be treated is the following one

$$\begin{cases} u(0,t)=h_0(t), & u(L,t)=h_1(t), & u_x(L,t)=h_2(t) & \text{in } (0,T), \\ v(0,t)=0, & v(L,t)=0, & v_x(L,t)=g_2(t) & \text{in } (0,T). \end{cases} \tag{3.82}$$

By using the same ideas in the proof of Theorem 3.1.1, we can show that system (3.39)-(3.82) is exactly controllable for any time $T>0$ if $L \in (0,\infty) \setminus \mathcal{F}_r'$.

• Concerning the exact boundary controllability of the system (1.16) with one control, our approach can be applied to the following configuration:

$$\begin{cases} u(0,t)=0 & u(L,t)=0 & u_x(L,t)=0, & \text{in } (0,T), \\ v(0,t)=0, & v(L,t)=0, & v_x(L,t)=g_2(t), & \text{in } (0,T). \end{cases} \tag{3.83}$$

The proof of this case is analogous to (ii) of Theorem 3.1.1.

Part II

Controllability and Stability of the KdV–Burgers Equation Posed in the Whole Space

CHAPTER

FOUR

ON THE WELL–POSEDNESS AND ASYMPTOTIC BEHAVIOR OF THE GENERALIZED KDV-BURGERS EQUATION

In this chapter, we are concerned with the well-posedness and the exponential stabilization of the generalized Korteweg-de Vries Burgers equation, posed on the whole real line, under the effect of a damping term. Both problems are investigated when the exponent p in the nonlinear term ranges over the interval $[1, 5)$. *We first prove the global well-posedness in* $H^s(\mathbb{R})$, *for* $0 \le s \le 3$ *and* $1 \le p < 2$, *and in* $H^3(\mathbb{R})$, *when* $p \ge 2$. *For* $2 \le p < 5$, *we prove the existence of global solutions in the* L^2*-setting. Then, by using multiplier techniques and interpolation theory, the exponential stabilization is obtained with a indefinite damping term and* $1 \le p < 2$. *Under the effect of a localized damping term the result is obtained when* $2 \le p < 5$. *Combining multiplier techniques and compactness arguments it is shown that the problem of exponential decay is reduced to prove the unique continuation property of weak solutions. Here, the unique continuation is obtained via the usual Carleman estimate.*

4.1 Introduction

It is common knowledge that many physical problems, such as nonlinear shallow-water waves and wave motion in plasmas can be described by the family of the Korteweg-de Vries (KdV) equation. The KdV-type equations have also been used to describe a wide range of important physical phenomena related to acoustic waves in a harmonic crystal, quantum field theory, plasma physics and solid-state physics. In what concerns the study of wave propagation in a tube filled with viscous fluid or flow of the fluid containing gas bubbles, for example, the control equation can be reduced to the so-called KdV-Burgers equation [93]. It is commonly obtained from the KdV equation by adding a viscous term and combines nonlinearity, linear dissipation and dispersion terms:

$$u_t + \delta u_{xxx} - \nu u_{xx} + u u_x = 0, \quad t > 0, \, x \in \mathbb{R}.$$

Since δ and ν are positive numbers, the model can be viewed as a generalization of the KdV and Burgers equation. Particularly, the Burgers equation is a simple model equation for a variety of diffusion/dissipative processes in convection dominated systems, which include formation of weak shocks, traffic flow, turbulence, etc. If besides convective nonlinearity and dissipation/diffusion mechanism, the dispersion also plays its role over the spatial and temporal scales of interest, then the simplest nonlinear PDE governing the wave dynamics is the combination of both KdV and Burgers equation.

In this work we are concerned with the generalized KdV-Burgers equation (GKdV-B) under the effect of a damping term represented by a function $b = b(x)$, more precisely,

$$\begin{cases} u_t + u_{xxx} - u_{xx} + a(u)u_x + b(x)u = 0 & \text{in } \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}. \end{cases} \tag{4.1}$$

Our main purpose is to address two mathematical issues connected to the initial value problem (4.1): global well-posedness and large-time behavior of solutions. More precisely, we establish the well-posedness and the exponential decay of solutions in the classical Sobolev spaces H^s. Therefore, as usual, let us first consider the energy associated to the model, given by

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} u^2(x,t)dx.$$

Thus, at least formally, the solutions of (4.1) should satisfy

$$\frac{d}{dt}E(t) = -\int_{\mathbb{R}} u_x^2 dx - \int_{\mathbb{R}} b(x)u^2 dx, \tag{4.2}$$

for any positive t. Then, if we assume that $b(x) \geq b_0$, for some $b_0 > 0$, it is forward to infer that $E(t)$ converges to zero exponentially. By contrast, when the damping function b is allowed to change of sign or is effective on a subset of the domain, the problem is much more subtle. Moreover, whether (4.2) generates a flow that can be continued indefinitely in the temporal variable, defining a solution valid for all $t \geq 0$, is a nontrivial question.

In order to provide the tools to handle with both problems, we assume that $a = a(x)$ is a positive real-valued function that satisfies the growth conditions

$$\begin{cases} |a^{(j)}(\mu)| \leq C(1 + |\mu|^{p-j}), \quad \forall \mu \in \mathbb{R}, \text{ for some } C > 0, \\ j = 0, 1 \text{ if } 1 \leq p < 2 \text{ and } j = 0, 1, 2 \text{ if } p \geq 2, \end{cases} \tag{4.3}$$

except when u_0 belongs to $L^2(\mathbb{R})$ and $2 \leq p < 5$, (See below, Theorem 4.2.14 and Remark 4.2.15).

Moreover, in order to obtain the exponential stability in the case $1 \leq p < 2$, we take an indefinite damping satisfying

$$\begin{cases} b \in H^1(\mathbb{R}) \text{ and } b(x) \geq \lambda_0 + \lambda_1(x), \text{ almost everywhere, for some } \lambda_0 > 0 \text{ and} \\ \lambda_1 \in L^p(\mathbb{R}), \text{ such that } \|\lambda_1\|_{L^p(\mathbb{R})} < \left(\frac{\lambda_0}{c_p}\right)^{1-\frac{1}{2p}}, \text{ where } c_p = \left(1 - \frac{1}{2p}\right)\left(\frac{2}{p}\right)^{\frac{1}{2p-1}}. \end{cases} \tag{4.4}$$

Concerning the case $p \geq 2$, we consider a localized damping which acts everywhere but on a bounded subset of the line, more precisely,

$$\begin{gathered} b \in H^1(\mathbb{R}) \text{ is nonnegative and } b(x) \geq \lambda_0 > 0 \text{ almost everywhere in } (-\infty, \alpha) \cup (\beta, \infty), \\ \text{for some } \alpha, \beta \in \mathbb{R}, \text{ with } \alpha < \beta. \end{gathered} \tag{4.5}$$

Our analysis was inspired by the results obtained by Cavalcanti et al. for KdV-Burgers equation [23] and by Rosier and Zhang for the generalized KdV equation posed on a bounded domain [81] (see also [61]). In this context, we refer to the survey [80] for a quite complete review on the state of art.

When $1 \leq p < 2$ and $0 \leq s \leq 3$, we obtain the global well-posedness in the class $B_{s,T} = C([0,T]; H^s(\mathbb{R})) \cap L^2(0,T; H^{s+1}(\mathbb{R}))$ and prove that the solutions decay exponentially to zero in $H^s(\mathbb{R})$, where H^s denotes the classical Sobolev spaces. As it is known in the theory of dispersive wave equations, the results depend on the local theory, on the a priori estimates satisfied by the solutions and also on linear theory. Indeed, we combine the

Duhamel formula and a contraction-mapping principle to prove directly the local well-posedness. In order to get the global result we derive energy-type inequalities and make use of interpolation arguments. Those a priori estimates are sufficient to yield the global stabilization result and a strong smoothing property for solutions, $u \in C([\varepsilon, T]; H^s(\mathbb{R})) \cap L^2(\varepsilon, T; H^{s+1}(\mathbb{R}))$, for any $\varepsilon > 0$. Our analysis extends the results obtained in [23] from which we borrow some ideas involved in our proofs. When $p \geq 2$ we can use the same approach to prove that the global well-posedness also holds in $B_{3,T}$. In order to get the result in a stronger/weaker norm, we need a priori global estimates. However, the only available a priori estimate for (4.1) is the estimate provided by (4.2), which does not guarantee existence of global in time solutions. In fact, we do not know if the problem is locally well-posed in the energy space. Therefore, we restrict ourselves to the case $2 \leq p < 5$ to prove that the estimate provided by the energy dissipation law holds and establish the existence of global solutions in the space $C_\omega([0,T]; L^2(\mathbb{R})) \cap L^2(0,T; H^1(\mathbb{R}))$. The uniqueness remains an open problem. The main difficulty in this context comes from the structure of nonlinearities and the lack of regularity of the solutions we are dealing with. In what concerns the asymptotic behavior, we prove the exponential decay in the L^2-setting by following the approach used in [81]. It combines multiplier techniques and compactness arguments to reduce the problem to some unique continuation property for weak solutions. To overcome this problem we develop a Carleman inequality by modifying (slightly) a Carleman estimate obtained by Rosier in [79] to study the controllability properties of the KdV equation. It allows us to prove unique continuation property directly.

The program of this work was carried out for the particular choice of damping effect appearing in (4.1) and aims to establish as a fact that such a model predicts the interesting qualitative properties initially observed for the KdV-Burgers type equations. Consideration of this issue for nonlinear dispersive equations has received considerable attention, specially the problems on the time decay rate. At this respect, it is important to point out that the approach used here was successfully applied in the context of the KdV equation, posed on $\mathbb{R}^+$ and $\mathbb{R}$, under the effect of a localized damping term [24, 62, 76]. We also remark that, in the absence of the damping term b, the stabilization problem was addressed by Bona and Luo [6, 7], complementing the earlier studies developed in [1, 4, 35] and deriving sharp polynomial decay rates for the solutions. Later on, in [5, 85], the authors improved upon the foregoing theory. The asymptotic behavior was also discussed in the language of the global attractors [37, 38]. More precisely, the authors study the large time behaviour of the corresponding semigroup in constructing a global attractor.

The analysis described above was organized in two sections. In Section 4.2 we establish the global well-posedness results. The Section 4.3 is devoted to the stabilization problem. Finally, in the Appendix A.2, we prove a Carleman inequality. In all sections we split the results into several steps in order to make the reading easier.

4.2 Well-posedness.

First we consider the corresponding linear inhomogeneous initial value problem,

$$\begin{cases} u_t - u_{xx} + u_{xxx} + b(x)u = f & (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) = u_0(x) & x \in \mathbb{R}. \end{cases} \tag{4.6}$$

Setting

$$A_b := \partial_x^2 - \partial_x^3 - bI \text{ and } D(A_b) = H^3(\mathbb{R}), b \in L^\infty(\mathbb{R})$$

(4.6) can be written in the form

$$\begin{cases} u_t = A_b u + f \\ u(0) = u_0. \end{cases}$$

According to [23], A_b generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ of contractions in $L^2(\mathbb{R})$. Hence, if we consider the Banach space

$$\begin{cases} B_{s,T} := C([0,T]; H^s(\mathbb{R})) \cap L^2(0,T; H^{s+1}(\mathbb{R})) \\ \|u\|_{s,T} = \sup_{t\in[0,T]} \|u(t)\|_{H^s(\mathbb{R})} + \|\partial_x^{s+1} u\|_{L^2(0,T;L^2(\mathbb{R}))}, \end{cases} \tag{4.7}$$

the following result holds:

Proposition 4.2.1. *Let $T > 0$. If $u_0 \in L^2(\mathbb{R})$ and $f \in L^1(0,T; L^2(\mathbb{R}))$, (4.6) has a unique mild solution $u \in B_{0,T}$, and*

$$\|u\|_{0,T} \leq C_T \left\{ \|u_0\|_2 + \|f\|_{L^1(0,T;L^2(\mathbb{R}))} \right\}, \quad \textit{with} \quad C_T = 2e^{T\|b\|_\infty}.$$

Furthermore, the following energy identity holds for all $t \in [0,T]$:

$$\|u(t)\|_2^2 + 2\int_0^t \|u_x(s)\|_2^2 ds + 2\int_0^t \int_{\mathbb{R}} b(x)|u(x,s)|^2 dx ds = \|u_0\|_2^2 + 2\int_0^t \int_{\mathbb{R}} f(x,s)u(x,s) dx ds. \tag{4.8}$$

Proof. See [23, Proposition 4.1]. □

4.2.1 Case $1 \leq p < 2$.

In order to establish the well-posedness of (2.12) we need the following technical Lemmas, that will play an important role in the proofs:

Lemma 4.2.2 (Generalized Hölder inequality)**.** *Suppose that for $i = 1, 2, ..., n$, $f_i \in L^{p_i}$ and $\sum_{i=1}^n \frac{1}{p_i} = 1$. Then,*

$$\|f_1.f_2...f_n\|_{L^1} \leq \prod_{i=1}^n \|f_i\|_{L^{p_i}}. \tag{4.9}$$

Lemma 4.2.3. *Let $a \in C^0(\mathbb{R})$ be a function satisfying*

$$|a(\mu)| \leq C(1 + |\mu|^p), \quad \forall \mu \in \mathbb{R}, \tag{4.10}$$

with $0 \leq p < 2$. Then, there exists a positive constant C, such that, for any $T > 0$ and $u, v \in B_{0,T}$, we have

$$\|a(u)v_x\|_{L^1(0,T;L^2(\mathbb{R}))} \leq 2^{\frac{p}{2}} C T^{\frac{2-p}{4}} \|u\|_{0,T}^p \|v\|_{0,T} + CT^{\frac{1}{2}} \|v\|_{0,T}.$$

Proof. Recall that $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and

$$\|u\|_\infty^2 \leq 2\|u\|_2 \|u_x\|_2, \tag{4.11}$$

for all $u \in H^1(\mathbb{R})$. On the other hand, by (4.10),

$$\begin{aligned} \|a(u)v_x\|_{L^1(0,T;L^2(\mathbb{R}))} &\leq C\int_0^T \|\left(1 + |u(t)|^p\right) v_x(t)\|_2 dt \\ &\leq C\int_0^T \|v_x(t)\|_2 dt + C\int_0^T \|u(t)\|_\infty^p \|v_x(t)\|_2 dt. \end{aligned}$$

Using Hölder inequality (4.9) and (4.11), we have

$$\|a(u)v_x\|_{L^1(0,T;L^2(\mathbb{R}))} \leq CT^{\frac{1}{2}}\|v_x\|_{L^2(0,T;L^2)} + 2^{\frac{p}{2}}C\int_0^T \|u(t)\|_2^{\frac{p}{2}}\|u_x(t)\|_2^{\frac{p}{2}}\|v_x(t)\|_2 dt$$
$$\leq CT^{\frac{1}{2}}\|v_x\|_{L^2(0,T;L^2)} + 2^{\frac{p}{2}}C\|u\|_{C([0,T];L^2)}^{\frac{p}{2}}\int_0^T \|u_x(t)\|_2^{\frac{p}{2}}\|v_x(t)\|_2 dt.$$

Applying Lemma 4.2.2 with $\frac{p}{4}$, $\frac{2-p}{4}$ and $\frac{1}{2}$, it follows that

$$\|a(u)v_x\|_{L^1(0,T;L^2(\mathbb{R}))} \leq CT^{\frac{1}{2}}\|v\|_{0,T} + 2^{\frac{p}{2}}CT^{\frac{2-p}{4}}\|u\|_{0,T}^{\frac{p}{2}}\|u_x\|_{L^2(0,T;L^2(\mathbb{R}))}^{\frac{p}{2}}\|v_x\|_{L^2(0,T;L^2(\mathbb{R}))}$$
$$\leq 2^{\frac{p}{2}}CT^{\frac{2-p}{4}}\|u\|_{0,T}^{p}\|v\|_{0,T} + CT^{\frac{1}{2}}\|v\|_{0,T}.$$

□

Lemma 4.2.4. *For any $T > 0$, $b \in L^\infty(\mathbb{R})$ and $u, v, w \in B_{0,T}$, we have*

(i) $\|bu\|_{L^1(0,T;L^2(\mathbb{R}))} \leq T^{\frac{1}{2}}\|b\|_\infty\|u\|_{0,T}$,

(ii) $\|uw_x\|_{L^1(0,T;L^2(\mathbb{R}))} \leq 2^{\frac{1}{2}}T^{\frac{1}{4}}\|u\|_{0,T}\|w\|_{0,T}$.

If $1 \leq p < 2$,

(iii) $\|u|v|^{p-1}w_x\|_{L^1(0,T;L^2(\mathbb{R}))} \leq 2^{\frac{p}{2}}T^{\frac{2-p}{4}}\|u\|_{0,T}\|w\|_{0,T}\|v\|_{0,T}^{p-1}$,

(iv) Consider the map $M : B_{0,T} \to L^1(0,T;L^2(\mathbb{R}))$ defined by $Mu := a(u)u_x$. Then, M is locally Lipschitz continuous and

$$\|Mu - Mv\|\|_{L^1(0,T;L^2(\mathbb{R}))} \leq C\left\{2^{\frac{1}{2}}T^{\frac{1}{4}}\|u\|_{0,T} + 2^{\frac{p}{2}}T^{\frac{2-p}{4}}\left(\|u\|_{0,T}^{p} + \|u\|_{0,T}\|v\|_{0,T}^{p-1}\right.\right.$$
$$\left.\left. +\|v\|_{0,T}^{p}\right) + T^{\frac{1}{2}}\right\}\|u - v\|_{0,T},$$

where C is a positive constant.

Proof. (i) Using Hölder inequality, we have

$$\|bu\|_{L^1(0,T;L^2(\mathbb{R}))} \leq T^{\frac{1}{2}}\|b\|_\infty\|u\|_{L^2(0,T;L^2)} \leq T^{\frac{1}{2}}\|b\|_\infty\|u\|_{0,T}.$$

(ii) Combining (2.22) and Lemma 4.2.2 with $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$, it follows that

$$\|uw_x\|_{L^1(0,T;L^2(\mathbb{R}))} \leq \int_0^T \|u(t)\|_\infty\|w_x(t)\|_2 dt \leq 2^{\frac{1}{2}}\int_0^T \|u(t)\|_2^{\frac{1}{2}}\|u_x(t)\|_2^{\frac{1}{2}}\|w_x(t)\|_2 dt$$
$$\leq 2^{\frac{1}{2}}\|u(t)\|_{C([0,T];L^2)}^{\frac{1}{2}}\left(\int_0^T \|u_x(t)\|_2^2 dt\right)^{\frac{1}{4}}\left(\int_0^T \|w_x(t)\|_2^2 dt\right)^{\frac{1}{2}}T^{\frac{1}{4}}$$
$$\leq 2^{\frac{1}{2}}T^{\frac{1}{4}}\|u\|_{0,T}\|w_x\|_{0,T}.$$

(iii) We proceed as in (i) combining (2.22) and Lemma 4.2.2 with $\frac{1}{4}$, $\frac{p-1}{4}$, $\frac{2-p}{4}$ and $\frac{1}{2}$ to obtain

$$\|u|v|^{p-1}w_x\|_{L^1(0,T;L^2(\mathbb{R}))} \leq \int_0^T \|u(t)\|_\infty\|v(t)\|_\infty^{p-1}\|w_x(t)\|_2 dt$$
$$\leq 2^{\frac{p}{2}}\int_0^T \|u(t)\|_2^{\frac{1}{2}}\|u_x(t)\|_2^{\frac{1}{2}}\|v(t)\|_2^{\frac{p-1}{2}}\|v_x(t)\|_2^{\frac{p-1}{2}}\|w_x(t)\|_2 dt$$
$$\leq 2^{\frac{p}{2}}\|u\|_{0,T}^{\frac{1}{2}}\|v\|_{0,T}^{\frac{p-1}{2}}\int_0^T \|u_x(t)\|_2^{\frac{1}{2}}\|v_x(t)\|_2^{\frac{p-1}{2}}\|w_x(t)\|_2 dt$$
$$\leq 2^{\frac{p}{2}}\|u\|_{0,T}^{\frac{1}{2}}\|v\|_{0,T}^{\frac{p-1}{2}}\left(\int_0^T \|u_x\|_2^2 dt\right)^{\frac{1}{4}}\left(\int_0^T \|v_x\|_2^2 dt\right)^{\frac{p-1}{4}}\left(\int_0^T \|w_x\|_2^2 dt\right)^{\frac{1}{2}}T^{\frac{2-p}{4}}$$
$$\leq 2^{\frac{p}{2}}T^{\frac{2-p}{4}}\|u\|_{0,T}^{\frac{1}{2}}\|v\|_{0,T}^{\frac{p-1}{2}}\|u\|_{0,T}^{\frac{1}{2}}\|v\|_{0,T}^{\frac{p-1}{2}}\|w\|_{0,T},$$

which allows us to conclude the result.

(iv) Note that

$$\|Mu-Mv\|_{L^1(0,T;L^2(\mathbb{R}))} \leq \|(a(u)-a(v))u_x\|_{L^1(0,T;L^2(\mathbb{R}))} + \|a(v)(u-v)_x\|_{L^1(0,T;L^2(\mathbb{R}))}.$$

Using the Mean Valued Theorem, (ii), (iii) and Lemma 4.2.3, we have

$$\begin{aligned}\|Mu-Mv\|_{L^1(0,T;L^2(\mathbb{R}))} &\leq C\|(1+|u|^{p-1}+|v|^{p-1})|u-v|u_x\|_{L^1(0,T;L^2)} \\ &\quad + \|a(v)(u-v)_x\|_{L^1(0,T;L^2)} \\ &\leq C\Big\{2^{\frac{1}{2}}T^{\frac{1}{4}}\|u-v\|_{0,T}\|u\|_{0,T} + 2^{\frac{p}{2}}T^{\frac{2-p}{4}}\|u-v\|_{0,T}\|u\|_{0,T}^p \\ &+2^{\frac{p}{2}}T^{\frac{2-p}{4}}\|u-v\|_{0,T}\|u\|_{0,T}\|v\|_{0,T}^{p-1} + 2^{\frac{p}{2}}T^{\frac{2-p}{4}}\|u-v\|_{0,T}\|v\|_{0,T}^p + T^{\frac{1}{2}}\|u-v\|_{0,T}\Big\}.\end{aligned}$$

□

The above estimates lead to the following local existence result and a priori estimate:

Proposition 4.2.5. *Let a be a function $C^1(\mathbb{R})$ satisfying*

$$|a(\mu)| \leq C(1+|\mu|^p) \text{ and } |a'(\mu)| \leq C(1+|\mu|^{p-1}), \ \forall \mu \in \mathbb{R},$$

with $1 \leq p < 2$. Let $b \in L^\infty(\mathbb{R})$ and $u_0 \in L^2(\mathbb{R})$. Then, there exist $T > 0$ and a unique mild solution $u \in B_{0,T}$ of (2.12). Moreover,

$$\|u(t)\|_2^2 + 2\int_0^t \|u_x(s)\|_2^2 ds + 2\int_0^t \int_{\mathbb{R}} b(x)|u(x,s)|^2 dxds = \|u_0\|_2^2, \qquad \forall t \in [0,T]. \tag{4.12}$$

Proof. Let $T > 0$ to be determined later. For each $u \in B_{0,T}$ consider the problem

$$\begin{cases} v_t = A_b v - Mu \\ v(0) = u_0, \end{cases} \tag{4.13}$$

where $A_b v = \partial_x^2 v - \partial_x^3 v - bv$ and $Mu = a(u)u_x$. Since A_b generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ of contractions in $L^2(\mathbb{R})$, Lemma 4.2.3 and Proposition 4.2.1 allows us to conclude that (4.13) has a unique mild solution $v \in B_{0,T}$, such that

$$\|v\|_{0,T} \leq C_T\{\|u_0\|_2 + \|Mu\|_{L^1(0,T;L^2(\mathbb{R}))}\}, \tag{4.14}$$

where $C_T = 2e^{T\|b\|_\infty}$. Thus, we can define the operator

$$\Gamma : B_{0,T} \longrightarrow B_{0,T} \text{ given by } \Gamma(u) = v.$$

By using Lemma 4.2.3 and (2.39), we have

$$\|\Gamma u\|_{0,T} \leq C_T\{\|u_0\|_2 + 2^{p/2}CT^{\frac{2-p}{4}}\|u\|_{0,T}^{p+1} + CT^{\frac{1}{2}}\|u\|_{0,T}\}.$$

Thus, for $u \in B_R(0) := \{u \in B_{0,T} : \|u\|_{B_{0,T}} \leq R\}$, it follows that

$$\|\Gamma u\|_{0,T} \leq C_T\{\|u_0\|_2 + 2^{p/2}CT^{\frac{2-p}{4}}R^{p+1} + CT^{\frac{1}{2}}R.\}$$

Choosing $R = 2C_T\|u_0\|_2$, we obtain the following estimate

$$\|\Gamma u\|_{0,T} \leq \left(K_1 + \frac{1}{2}\right)R,$$

where $K_1 = K_1(T) = 2^{p/2}C_T C T^{\frac{2-p}{4}} R^p + C_T C T^{\frac{1}{2}}$. On the other hand, note that $\Gamma u - \Gamma w$ is solutions of

$$\begin{cases} v_t = A_b v - (Mu - Mw) \\ v(0) = 0. \end{cases}$$

Again, by applying Proposition 4.2.1, we have

$$\|\Gamma u - \Gamma w\|_{0,T} \leq C_T \|Mu - Mw\|_{L^1(0,T;L^2)}$$

and estimate (iv) in Lemma 4.2.4 allows us to conclude that

$$\|\Gamma u - \Gamma w\|_{0,T} \leq C_T C \left\{ 2^{\frac{1}{2}} T^{\frac{1}{4}} \|u\|_{0,T} + 2^{\frac{p}{2}} T^{\frac{2-p}{4}} \left(\|u\|_{0,T}^p + \|u\|_{0,T} \|w\|_{0,T}^{p-1} \right.\right.$$
$$\left.\left. + \|w\|_{0,T}^p \right) + T^{\frac{1}{2}} \right\} \|u - w\|_{0,T}.$$

Suppose that $u, w \in B_R(0)$ defined above. Then,

$$\|\Gamma u - \Gamma w\|_{B_{0,T}} \leq K_2 \|u - w\|_{B_{0,T}},$$

where $K_2 = K_2(T) = C_T C\{2^{\frac{1}{2}} T^{\frac{1}{4}} R + 3(2^{\frac{p}{2}}) T^{\frac{2-p}{4}} R^p + T^{\frac{1}{2}}\}$. Since $K_1 \leq K_2$, we can choose $T > 0$ to obtain $K_2 < \frac{1}{2}$ and

$$\begin{cases} \|\Gamma u\|_{B_{0,T}} \leq R \\ \|\Gamma u - \Gamma w\|_{B_{0,T}} < \frac{1}{2} \|u - w\|_{B_{0,T}} \end{cases}, \qquad \forall u, w \in B_R(0) \subset B_{0,T}.$$

Hence $\Gamma : B_R(0) \longrightarrow B_R(0)$ is a contraction and, by Banach fixed point theorem, we obtain a unique $u \in B_R(0)$, such that $\Gamma(u) = u$. Consequently, u is a unique local mild solution of (4.1) and

$$\|u\|_{B_{0,T}} \leq 2C_T \|u_0\|_2. \tag{4.15}$$

In order to prove (4.12) consider $v_n = \Gamma v_{n-1}$, $n \geq 1$. Since Γ is a contraction, we have

$$\lim_{n \to \infty} v_n = u \text{ in } B_{0,T}.$$

On the other hand, by (4.8) in Proposition 4.2.1, v_n verifies the identity

$$\|v_n(t)\|_2^2 + 2\int_0^t \|v_{nx}(s)\|_2^2 ds + 2\int_0^t \int_{\mathbb{R}} b(x)|v(x,s)|^2 dx ds$$
$$= \|u_0\|_2^2 + 2\int_0^t \int_{\mathbb{R}} M v_{n-1}(x,s) v_n(x,s) dx ds.$$

Then, taking the limit as $n \to \infty$, we get

$$\|u(t)\|_2^2 + 2\int_0^t \|u_x(s)\|_2^2 ds + 2\int_0^t \int_{\mathbb{R}} b(x)|u(x,s)|^2 dx ds = \|u_0\|_2^2$$

since the limit of the last term is $\int_0^t \int_{\mathbb{R}} Mu(x,s)u(x,s) dx ds = 0$. In fact,

$$\int_{\mathbb{R}} a(u(x)) u_x(x) dx = \int_{\mathbb{R}} [A(u(x))]_x dx$$

where

$$A(v) = \int_0^v a(s) ds.$$

□

From Proposition 4.2.5 we obtain our first global in time existence result:

Theorem 4.2.6. *Let a be a function $C^1(\mathbb{R})$ satisfying*

$$|a(\mu)| \leq C(1+|\mu|^p) \text{ and } |a'(\mu)| \leq C(1+|\mu|^{p-1}), \quad \forall \mu \in \mathbb{R},$$

with $1 \leq p < 2$. Let $b \in L^\infty(\mathbb{R})$ and $u_0 \in L^2(\mathbb{R})$. Then, there exist a unique global mild solution u of (4.1), such that, for each $T > 0$, there exist a nondecreasing continuous function $\beta_0 : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies

$$\|u\|_{0,T} \leq \beta_0(\|u_0\|_2)\|u_0\|_2. \tag{4.16}$$

Moreover, the following energy identity holds for all $t \geq 0$:

$$\|u(t)\|_2^2 + 2\int_0^t \|u_x(s)\|_2^2 ds + 2\int_0^t \int_{\mathbb{R}} b(x)|u(x,s)|^2 dx ds = \|u_0\|_2^2. \tag{4.17}$$

Proof. By Proposition 4.2.5, there exists a unique mild solution $u \in B_{0,T}$, for all $T < T_{max} \leq \infty$. Moreover,

$$\|u\|_{0,T} \leq 4e^{\|b\|_\infty t}\|u_0\|_2, \qquad \forall t \in [0, T_{max}),$$

which implies that u is a global mild solution of (4.1). On the other hand, (4.15) implies (4.16) with $\beta_0(s) = 2C_T$. The identity (4.17) is a direct consequence of (4.12) in Proposition 4.2.5. □

It follows from Theorem 4.2.6 that, for each fixed $T > 0$, the solution map

$$\mathcal{A} : L^2(\mathbb{R}) \to B_{0,T}, \quad \mathcal{A}u_0 = u \tag{4.18}$$

is well defined. Moreover, we have the following result:

Proposition 4.2.7. *The solution map (4.18) is locally Lipschitz continuous, i.e, there exists a continuous function $C_0 : \mathbb{R}^+ \times (0,\infty) \to \mathbb{R}^+$, nondecreasing in its first variable, such that, for all $u_0, v_0 \in L^2(\mathbb{R})$, we have*

$$\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,T} \leq C_0\left(\|u_0\|_2 + \|v_0\|_2, T\right)\|u_0 - v_0\|_2.$$

Proof. Let $0 < \theta \leq T$ and $n = \left[\frac{T}{\theta}\right]$. By Theorem 4.2.6,

$$\|\mathcal{A}u_0\|_{0,\theta} \leq 2C_\theta\|u_0\|_2, \tag{4.19}$$

and

$$\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,\theta} \leq C_\theta\left\{\|u_0 - v_0\|_2 + \|M(\mathcal{A}u_0) - M(\mathcal{A}v_0)\|_{L^1(0,\theta;L^2(\mathbb{R}))}\right\},$$

where $C_\theta = 2e^{\theta\|b\|_\infty}$. By Lemma 4.2.4,

$$\begin{aligned}\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,\theta} \leq{} & C_\theta\|u_0 - v_0\|_2 + C_\theta C\Big\{2^{\frac{1}{2}}\theta^{\frac{1}{4}}\|\mathcal{A}u_0\|_{0,\theta} \\ & + 2^{\frac{p}{2}}\theta^{\frac{2-p}{4}}\left(\|\mathcal{A}u_0\|_{0,\theta}^p + \|\mathcal{A}u_0\|_{0,\theta}\|\mathcal{A}v_0\|_{0,\theta}^{p-1} + \|\mathcal{A}v_0\|_{0,\theta}^p\right) + \theta^{\frac{1}{2}}\Big\}\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,\theta},\end{aligned}$$

and applying (4.19), it follows that

$$\begin{aligned}\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,\theta} &\leq C_\theta\|u_0 - v_0\|_2 + C_\theta C\Big\{2^{\frac{3}{2}}\theta^{\frac{1}{4}}C_\theta\|u_0\|_2 + 2^{\frac{3p}{2}}\theta^{\frac{2-p}{4}}C_\theta^p\left(\|u_0\|_2^p\right. \\ &\qquad \left.+\|u_0\|_2\|v_0\|_2^{p-1} + \|v_0\|_2^p\right) + \theta^{\frac{1}{2}}\Big\}\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,\theta} \\ &\leq C_T\|u_0 - v_0\|_2 + C_T C\theta^{\frac{2-p}{4}}\Big\{2^{\frac{3}{2}}\theta^{\frac{p-1}{4}}C_T(\|u_0\|_2 + \|v_0\|_2) \\ &\qquad +2^{\frac{3p}{2}}C_T^p\left(\|u_0\|_2 + \|v_0\|_2\right)^p + \theta^{\frac{p}{4}}\Big\}\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,\theta} \\ &\leq C_T\|u_0 - v_0\|_2 + C_T C\theta^{\frac{2-p}{4}}\Big\{2^{\frac{5}{2}}T^{\frac{p-1}{4}}C_T^2(\|u_0\|_2 + \|v_0\|_2) \\ &\qquad +2^{\frac{5p}{2}}C_T^{2p}\left(\|u_0\|_2 + \|v_0\|_2\right)^p + T^{\frac{p}{4}}\Big\}\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,\theta}.\end{aligned}$$

Choosing θ small enough, such that

$$\theta < \left[\frac{1}{2C_TC\left\{2^{\frac{5}{2}}T^{\frac{p-1}{4}}C_T\|u_0\|_2+\|v_0\|_2+2^{\frac{5p}{2}}C_T^{2p}(\|u_0\|_2+\|v_0\|_2)^p+T^{\frac{p}{4}}\right\}}\right]^{\frac{4}{2-p}}, \tag{4.20}$$

we have

$$\|\mathcal{A}u_0-\mathcal{A}v_0\|_{0,\theta}\leq 2C_T\|u_0-v_0\|_2. \tag{4.21}$$

Analogously, we can deduce that

$$\|\mathcal{A}u_0\|_{0,[k\theta,(k+1)\theta]}\leq 2C_\theta\|u(k\theta)\|_2, \qquad k=0,1,...,n-1,$$

where $\|\cdot\|_{0,[k\theta,(k+1)\theta]}$ denotes the norm of

$$B_{0,[k\theta,(k+1)\theta]}:=C\left([k\theta,(k+1)\theta];L^2(\mathbb{R})\right)\cap L^2(k\theta,(k+1)\theta;H^1(\mathbb{R})).$$

Moreover, by using the same arguments, we have

$$\begin{aligned}\|\mathcal{A}u_0-\mathcal{A}v_0\|_{0,[k\theta,(k+1)\theta]}&\leq C_T\|u(k\theta)-v(k\theta)\|_2+C_TC\theta^{\frac{2-p}{4}}\left\{2^{\frac{3}{2}}T^{\frac{p-1}{4}}C_T(\|u(k\theta)\|_2\right.\\&\left.+\|v(k\theta)\|_2)+2^{\frac{3p}{2}}C_T^p\left(\|u(k\theta)\|_2+\|v(k\theta)\|_2\right)^p+T^{\frac{p}{4}}\right\}\|\mathcal{A}u_0-\mathcal{A}v_0\|_{0,[k\theta,(k+1)\theta]}.\end{aligned}$$

Combining (4.19) and the above estimate, it follows that

$$\begin{aligned}\|\mathcal{A}u_0-\mathcal{A}v_0\|_{0,[k\theta,(k+1)\theta]}&\leq C_T\|u(k\theta)-v(k\theta)\|_2+C_TC\theta^{\frac{2-p}{4}}\left\{2^{\frac{5}{2}}T^{\frac{p-1}{4}}C_T^2(\|u_0\|_2+\|v_0\|_2)\right.\\&\left.+2^{\frac{5p}{2}}C_T^{2p}\left(\|u_0\|_2+\|v_0\|_2\right)^p+T^{\frac{p}{4}}\right\}\|\mathcal{A}u_0-\mathcal{A}v_0\|_{0,[k\theta,(k+1)\theta]}.\end{aligned}$$

Finally, from (4.20), we get

$$\|\mathcal{A}u_0-\mathcal{A}v_0\|_{0,[k\theta,(k+1)\theta]}\leq 2C_T\|u(k\theta)-v(k\theta)\|_2, \quad k=0,1,...,n-1. \tag{4.22}$$

On the other hand, note that (4.21) and (4.22) imply that

$$\|\mathcal{A}u_0-\mathcal{A}v_0\|_{0,[k\theta,(k+1)\theta]}\leq 2^kC_T^k\|u_0-v_0\|_2, \quad k=0,1,...,n-1,$$

and, therefore,

$$\|\mathcal{A}u_0-\mathcal{A}v_0\|_{0,[k\theta,(k+1)\theta]}\leq 2^nC_T^n\|u_0-v_0\|_2.$$

Finally,

$$\begin{aligned}\|\mathcal{A}u_0-\mathcal{A}v_0\|_{0,T}&\leq\sum_{k=0}^{n-1}\|\mathcal{A}u_0-\mathcal{A}v_0\|_{0,[k\theta,(k+1)\theta]}\leq\sum_{k=0}^{n-1}2^nC_T^n\|u_0-v_0\|_2\\&\leq 2^nC_T^nn\|u_0-v_0\|_2\leq C_0(\|u_0\|_2+\|v_0\|_2)\|u_0-v_0\|_2,\end{aligned}$$

where $C_0(s)=\frac{T}{\theta(s)}\left[2C_T\right]^{\frac{T}{\theta(s)}}$. □

Next, we will show well-posedness in $B_{3,T}$, with $1\leq p<2$. Therefore, let us first consider the following linearized problem given by

$$\begin{cases} v_t+v_{xxx}-v_{xx}+[a(u)v]_x+bv=0 & \text{in } \mathbb{R}\times(0,\infty)\\ v(0)=v_0 & \text{in } \mathbb{R}\times(0,\infty).\end{cases} \tag{4.23}$$

Then, we can establish the following proposition:

Proposition 4.2.8. *Let a be a function $C^1(\mathbb{R})$ satisfying*

$$|a(\mu)| \leq C(1+|\mu|^p) \text{ and } |a'(\mu)| \leq C(1+|\mu|^{p-1}), \quad \forall \mu \in \mathbb{R},$$

with $1 \leq p < 2$. Let $T > 0$, $b \in L^\infty(\mathbb{R})$, $u \in B_{0,T}$ and $v_0 \in L^2(\mathbb{R})$. Then, the problem (4.23) admits a unique solution $v \in B_{0,T}$, such that

$$\|v\|_{0,T} \leq \sigma(\|u\|_{0,T})\|v_0\|_2,$$

where $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function.

Proof. Let $0 < \theta \leq T$ and $u \in B_{0,T}$. The proof of the existence follows the steps of Proposition 4.2.5 and Theorem 4.2.6. Therefore, we will omit the details. First, note that Lemma 4.2.3 and Lemma 4.2.4 imply that $Nw := [a(u)w]_x \in L^1(0,\theta; L^2(\mathbb{R}))$, for all $w \in B_{0,\theta}$. Hence,

$$\|Nw\|_{L^1(0,\theta;L^2(\mathbb{R}))} \leq C\left\{2^{\frac{1}{2}}\theta^{\frac{1}{4}}\|u\|_{0,\theta}\|w\|_{0,\theta} + 2^{\frac{p+2}{2}}\theta^{\frac{2-p}{4}}\|u\|_{0,\theta}^p\|w\|_{0,\theta} + \theta^{\frac{1}{2}}\|w\|_{0,\theta}\right\}.$$

With the notation above, problem (4.23) takes the form

$$\begin{cases} v_t = A_b v - Nw \\ v(0) = u_0, \end{cases}$$

where $A_b v = \partial_x^2 v - \partial_x^3 v - bv$. Since A_b generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ of contractions in $L^2(\mathbb{R})$, by Proposition 4.2.1, (4.2.1) has a unique mild solution $v \in B_{0,\theta}$, such that

$$\|v\|_{0,\theta} \leq C_\theta\{\|v_0\|_2 + \|Nw\|_{L^1(0,\theta;L^2(\mathbb{R}))}\},$$

where $C_\theta = 2e^{\theta\|b\|_\infty}$. Thus, we can define the operator

$$\Gamma : B_{0,T} \longrightarrow B_{0,T} \text{ given by } \Gamma(w) = v.$$

Let $R > 0$ be a constant to be determined later and $w \in B_R(0) := \{w \in B_{0,\theta} : \|w\|_{B_{0,\theta}} \leq R\}$. Thus,

$$\|\Gamma w\|_{0,\theta} \leq C_T\{\|v_0\|_2 + \left(2^{\frac{1}{2}}C\theta^{\frac{1}{4}}\|u\|_{0,T} + 2^{\frac{p+2}{2}}C\theta^{\frac{2-p}{4}}\|u\|_{0,T}^p + \theta^{\frac{1}{2}}C\right)R\}.$$

By choosing $R = 2C_T\|v_0\|_2$, we have

$$\|\Gamma u\|_{0,\theta} \leq \left(K_1 + \frac{1}{2}\right)R,$$

where $K_1 = C_T C\left(2^{\frac{1}{2}}C\theta^{\frac{1}{4}}\|u\|_{0,T} + 2^{\frac{p+2}{2}}C\theta^{\frac{2-p}{4}}\|u\|_{0,T}^p + \theta^{\frac{1}{2}}\right)$. On the other hand, note that $\Gamma s - \Gamma w$ solves the problem

$$\begin{cases} v_t = A_b v - (Ns - Nw) \\ v(0) = 0. \end{cases}$$

Thus,

$$\|\Gamma s - \Gamma w\|_{0,\theta} \leq K_1\|s - w\|_{0,\theta}.$$

Choosing $\theta > 0$, such that $K_1 = K_1(\theta) < \frac{1}{2}$, we have

$$\begin{cases} \|\Gamma w\|_{B_{0,\theta}} \leq R \\ \|\Gamma s - \Gamma w\|_{B_{0,\theta}} < \frac{1}{2}\|s - w\|_{0,\theta} \end{cases}, \qquad \forall s, w \in B_R(0) \subset B_{0,\theta}.$$

Hence, $\Gamma : B_R(0) \longrightarrow B_R(0)$ is a contraction and, by Banach fixed point theorem, we obtain a unique $v \in B_R(0)$, such that $\Gamma(v) = v$. Consequently, v is a unique local mild solution of problem (4.23) and

$$\|v\|_{B_{0,\theta}} \leq 2C_T\|v_0\|_2.$$

Then, using standard arguments we may extend θ to T. Finally, the proof is completed defining $\sigma(s) = 2C_T$. ☐

The aforementioned result is proved below. We make use of Proposition 4.2.8 and classical energy-type estimates.

Theorem 4.2.9. *Let a be a function $C^1(\mathbb{R})$ satisfying*

$$|a(\mu)| \leq C(1+|\mu|^p) \text{ and } |a'(\mu)| \leq C(1+|\mu|^{p-1}), \quad \forall \mu \in \mathbb{R}, \tag{4.24}$$

with $1 \leq p < 2$. Let $T > 0$, $b \in H^1(\mathbb{R})$ and $u_0 \in H^3(\mathbb{R})$. Then, there exists a unique mild solution $u \in B_{3,T}$ of (2.12), such that

$$\|u\|_{3,T} \leq \beta_3(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})},$$

where $\beta_3 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing continuous function.

Proof. In order to make the reading easier, the proof will be done in several steps:

Step 1: $u \in L^2(0,T;H^3(\mathbb{R}))$

Since $u_0 \in H^3(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$, by Theorem 4.2.6, there exist a unique solution $u \in B_{0,T}$, such that

$$\|u\|_{0,T} \leq \beta_0(\|u_0\|_2)\|u_0\|_2. \tag{4.25}$$

We will show that $u \in B_{3,T}$. Let $v = u_t$. Then, v solves the problem

$$\begin{cases} v_t + v_{xxx} - v_{xx} + [a(u)v]_x + bv = 0 \\ v(0,x) = v_0, \end{cases}$$

where $v_0 = -\partial_x^3 u_0 + \partial_x^2 u_0 - a(u_0)\partial_x u_0 - bu_0$. Note that $v \in L^2(\mathbb{R})$ and there exists $C = C(\|u_0\|_2)$, satisfying

$$\|v_0\|_2 \leq C(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})}.$$

In fact, from (2.22) we can bound v_0 as follows:

$$\begin{aligned} \|v_0\|_2 &\leq \|\partial_x^3 u_0\|_2 + \|\partial_x^2 u_0\|_2 + \|a(u_0)\partial_x u_0\|_2 + \|bu_0\|_2 \\ &\leq C_1 \left\{ (1+\|b\|_{L^\infty(\mathbb{R})})\|u_0\|_{H^3(\mathbb{R})} + \|u_0\|_2^{\frac{p}{2}} \|\partial_x u_0\|_2^{\frac{p+2}{2}} \right\}. \end{aligned}$$

Recall the Gagliardo-Nirenberg inequality:

$$\|\partial_x^j u_0\|_2 \leq C\|\partial_x^m u_0\|_2^{\frac{j}{m}} \|u_0\|_2^{1-\frac{j}{m}}, \quad j \leq m, \quad \text{where } j,m = 0,1,2,3. \tag{4.26}$$

Applying (4.26) with $j = 1$ and $m = 2$, we have

$$\|v_0\|_2 \leq C_2 \left\{ (1+\|b\|_{L^\infty(\mathbb{R})})\|u_0\|_{H^3(\mathbb{R})} + \|u_0\|_2^{\frac{3p+2}{4}} \|\partial_x^2 u_0\|_2^{\frac{p+2}{4}} \right\}.$$

Then, Young inequality guarantees that

$$\|v_0\|_2 \leq C_3 \left\{ (1+\|b\|_{L^\infty(\mathbb{R})})\|u_0\|_{H^3(\mathbb{R})} + \|u_0\|_2^{\frac{4p}{2-p}} \|u_0\|_2 + \|\partial_x^2 u_0\|_2 \right\}.$$

Consequently, it gives

$$\|v_0\|_2 \leq C(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})}, \tag{4.27}$$

where $C(s) = C_3 \left\{ 2 + \|b\|_{L^\infty(\mathbb{R})} + s^{\frac{4p}{2-p}} \right\}$. Using Proposition 2.3.3, we see that $v \in B_{0,T}$ and

$$\|v\|_{0,T} \leq \sigma(\|u\|_{0,T})\|v_0\|_2,$$

where $\sigma(s) = 2C_T$. Combining (4.25) and (4.27), we get

$$\|v\|_{0,T} \leq \sigma(\beta_0(\|u_0\|_2)\|u_0\|_2)C(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})}. \tag{4.28}$$

Then,

$$u, u_t \in L^2(0,T; H^1(\mathbb{R})) \tag{4.29}$$

and, therefore,

$$u \in C([0,T]; H^1(\mathbb{R})) \hookrightarrow C([0,T]; C(\mathbb{R})). \tag{4.30}$$

On the other hand, note that $a(u)u_x, bu \in L^2(0,T; L^2(\mathbb{R}))$. In fact, from (4.30) it follows that

$$\begin{aligned}\|a(u)u_x\|^2_{L^2(0,T;L^2(\mathbb{R}))} &\leq C\left\{\int_0^T \|u_x\|_2^2 dx + \int_0^T \||u|^p u_x\|_2^2 dx\right\}\\ &\leq C\left\{1 + \|u\|^{2p}_{C(0,T;C(\mathbb{R}))}\right\}\|u\|^2_{0,T}\end{aligned}$$

and

$$\|bu\|_{L^2(0,T;L^2(\mathbb{R}))} \leq \|b\|_{L^\infty(\mathbb{R})}\|u\|^2_{L^2(0,T;L^2(\mathbb{R}))}.$$

Moreover, $u_{xxx} - u_{xx} = -u_t - a(u)u_x - bu$ in $D'(0,T,\mathbb{R})$. Hence,

$$u_{xxx} - u_{xx} = f \quad \in \quad L^2(0,T;L^2(\mathbb{R}), \quad \text{where } f := -u_t - a(u)u_x - bu.$$

Taking the Fourier transform, we have

$$\widehat{u} = \frac{\widehat{f} + \widehat{u}}{[1+\xi^2 - i\xi^3]} \tag{4.31}$$

and,

$$\|u(t)\|^2_{H^3(\mathbb{R})} \leq C_3\left\{\|f(t)\|_2^2 + \|u(t)\|_2^2\right\} \tag{4.32}$$

where $C_3 = 2\sup_{\xi\in\mathbb{R}} \dfrac{(1+\xi^2)^3}{(1+\xi^2)^2+\xi^6}$. Integrating (4.32) over $[0,T]$, we deduce that

$$u \in L^2(0,T; H^3(\mathbb{R})). \tag{4.33}$$

Step 2: $u \in B_{3,T}$

First, observe that, according to (4.29) and (4.33), we can apply [64, Thm 2.3] to obtain

$$u \in C([0,T]; H^2(\mathbb{R})).$$

This implies further

$$u_{xx}, bu \in C([0,T]; L^2(\mathbb{R})) \cap L^2(0,T; H^1(\mathbb{R})). \tag{4.34}$$

On the other hand, note that

$$\begin{aligned}\|a(u(t))u_x(t) - a(u(t_0))u_x(t_0)\|_2 &\leq \|[a(u(t)) - a(u(t_0))]u_x(t)\|_2\\ &\quad + \|a(u(t_0))[u_x(t) - u_x(t_0)]\|_2\\ &\leq C\left\{\|(1+|u(t)|^{p-1} + |u(t_0)|^{p-1})|u(t) - u(t_0)|u_x(t)\|_2\right.\\ &\quad \left.+\|(1+|u(t_0)|^p)|u_x(t) - u_x(t_0)|\|_2\right\}\\ &\leq C\left\{(1+\|u(t)\|_\infty^{p-1} + \|u(t_0)\|_\infty^{p-1})\|u(t) - u(t_0)\|_\infty\|u_x(t)\|_2\right.\\ &\quad \left.+(1+\|u(t_0)\|_\infty^p)\|u_x(t) - u_x(t_0)\|_2\right\}.\end{aligned}$$

Then, by (4.30) we have

$$\lim_{t\to t_0} \|a(u(t))u_x(t) - a(u(t_0))u_x(t_0)\|_2 = 0$$

and, therefore $a(u)u_x \in C([0,T]; L^2(\mathbb{R}))$. The results above also guarantee that

$$a(u)u_x \in C([0,T]; L^2(\mathbb{R})) \cap L^2(0,T; H^1(\mathbb{R})). \tag{4.35}$$

Indeed, it is sufficient to combine (4.30), (4.33) and the estimates

$$\|a'(u)u_x^2\|_{L^2(0,T;L^2(\mathbb{R}))} \leq C\left\{(1 + \|u\|^{p-1}_{C([0,T];C(\mathbb{R}))})\|u_x\|_{C([0,T];C(\mathbb{R}))}\|u_x\|_{L^2([0,T];L^2(\mathbb{R}))}\right\}$$

and

$$\|a(u)u_{xx}\|_{L^2(0,T;L^2(\mathbb{R}))} \leq C\left\{(1 + \|u\|^{p}_{C([0,T];C(\mathbb{R}))})\|u_{xx}\|_{L^2([0,T];L^2(\mathbb{R}))}\right\}.$$

Since

$$u_{xxx} = -u_t + u_{xx} - a(u)u_x - bu,$$

from the fact that $u_t \in C([0,T], L^2(\mathbb{R}))$, (4.29), (4.34) and (4.35), we obtain

$$u_{xxx} \in C([0,T], L^2(\mathbb{R})) \cap L^2(0,T; H^1(\mathbb{R})). \tag{4.36}$$

Moreover, since $u \in B_{0,T}$, from (4.36), it follows that $u \in B_{3,T}$.

Step 3: $\|u\|_{C([0,T];H^3(\mathbb{R}))} \leq \sigma_1(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})}$

First, note that, according to (4.32), the following estimate holds:

$$\|u(t)\|_{H^3(\mathbb{R})} \leq C_4\left\{\|u_t(t)\|_2 + \|a(u(t))u_x(t)\|_2 + \|bu(t)\|_2 + \|u(t)\|_2\right\}. \tag{4.37}$$

Next we combine (4.24), (4.11) and (4.26) with $j = 1$ and $m = 2$, to obtain

$$\begin{aligned}\|a(u(t))u_x(t)\|_2 &\leq C\left\{\|u_x(t)\|_2 + \|u(t)\|_2^{\frac{p}{2}}\|u_x(t)\|_2^{\frac{p+2}{2}}\right\}\\ &\leq C\left\{\|u_{xx}(t)\|_2^{\frac{1}{2}}\|u(t)\|_2^{\frac{1}{2}} + \|u(t)\|_2^{\frac{3p+2}{4}}\|u_{xx}(t)\|_2^{\frac{p+2}{4}}\right\}.\end{aligned}$$

Moreover, Young inequality gives

$$\|a(u(t))u_x(t)\|_2 \leq C_5\left(\|u(t)\|_2 + \|u(t)\|_2^{\frac{3p+2}{2-p}}\right) + \frac{1}{2C_4}\|u(t)\|_{H^3(\mathbb{R})}$$

Replacing the estimate above in (4.37) and taking the supremum in $[0,T]$, we get

$$\|u\|_{C([0,T];H^3(\mathbb{R}))} \leq 2C_4\left\{\|u_t\|_{0,T} + (C_6 + \|b\|_\infty)\|u\|_{0,T} + C_5\|u\|_{0,T}^{\frac{3p+2}{2-p}}\right\}.$$

Then, using (4.25) and (4.28) it follows that

$$\begin{aligned}\|u\|_{C([0,T];H^3(\mathbb{R}))} \leq 2C_4&\{\sigma(\beta_0(\|u_0\|_2)\|u_0\|_2)C(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})}\\ &+(C_6 + \|b\|_\infty)\beta_0(\|u_0\|_2)\|u_0\|_2 + C_5\beta_0^{\frac{3p+2}{2-p}}(\|u_0\|_2)\|u_0\|_2^{\frac{4p}{2-p}}\|u_0\|_2\Big\}\\ &= \sigma_1(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})},\end{aligned} \tag{4.38}$$

where $\sigma_1(s) = 2C_4\left\{\sigma(\beta_0(s)s)C(s) + (C_6 + \|b\|_\infty)\beta_0(s) + C_5\beta_0^{\frac{3p+2}{2-p}}(s)s^{\frac{4p}{2-p}}\right\}$.

Step 4: $\|u_{xxxx}\|_{L^2(0,T;L^2(\mathbb{R}))} \leq \sigma_5(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})}$

We know from (4.36) that $u \in L^2(0,T;H^4(\mathbb{R}))$. To prove the desired result, we differentiate the equation with respect to x to obtain

$$\begin{aligned}\|u_{xxxx}\|_{L^2(0;T,L^2(\mathbb{R}))} \leq \|v\|_{0,T} + T^{\frac{1}{2}}\|u\|_{C(0;T,H^3(\mathbb{R}))} + \|a'(u)u_x^2\|_{L^2(0;T,L^2(\mathbb{R}))} \\ + \|a(u)u_{xx}\|_{L^2(0;T,L^2(\mathbb{R}))} + \|[bu]_x\|_{L^2(0;T,L^2(\mathbb{R}))}. \quad (4.39)\end{aligned}$$

The next steps are denoted to estimate the terms on the right side of (4.39). First, observe that

$$\begin{aligned}\|[bu]_x\|_{L^2(0;T,L^2(\mathbb{R}))} &\leq \|b\|_{H^1(\mathbb{R})}\|u\|_{L^2(0;T,H^1(\mathbb{R}))} + \|b\|_{H^1(\mathbb{R})}\|u_x\|_{L^2(0;T,L^2(\mathbb{R}))} \\ &\leq 2\|b\|_{H^1(\mathbb{R})}\|u\|_{0,T}.\end{aligned}$$

Then, from (4.25), (4.28) and (4.38), we obtain

$$\|u_{xxxx}\|_{L^2(0;T,L^2(\mathbb{R}))} \leq \sigma_2(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})} + \|a'(u)u_x^2\|_{L^2(0;T,L^2(\mathbb{R}))} + \|a(u)u_{xx}\|_{L^2(0;T,L^2(\mathbb{R}))} \quad (4.40)$$

where $\sigma_2(s) = \sigma(\beta_0(s)s)C(s) + T^{\frac{1}{2}}\sigma_1(s) + 2\|b\|_{H^1(\mathbb{R})}\beta_0(s)$. Moreover, using (2.22) it follows that

$$\begin{aligned}\|a'(u(t))u_x^2(t)\|_2 &\leq C\left\{\|u_x^2(t)\|_2 + \||u(t)|^{p-1}u_x^2(t)\|_2\right\} \\ &\leq C_7\left\{\|u_x(t)\|_2^{\frac{3}{2}}\|u_{xx}(t)\|_2^{\frac{1}{2}} + \|u(t)\|_2^{\frac{p-1}{2}}\|u_x(t)\|_2^{\frac{p+2}{2}}\|u_{xx}(t)\|_2^{\frac{1}{2}}\right\} \\ &\leq C_7\left\{\|u_x(t)\|_2\|u(t)\|_{H^3(\mathbb{R})} + \|u(t)\|_2^{\frac{p-1}{2}}\|u_x(t)\|_2^{\frac{p+2}{2}}\|u_{xx}(t)\|_2^{\frac{1}{2}}\right\}.\end{aligned}$$

Then, Gagliardo-Nirenberg inequality (4.26) with $j=1$ and $m=3$ leads to

$$\|a'(u(t))u_x^2(t)\|_2 \leq C_8\left\{\|u_x(t)\|_2\|u(t)\|_{H^3(\mathbb{R})} + \|u(t)\|_2^{\frac{5p+2}{6}}\|u_{xxx}(t)\|_2^{\frac{p+4}{6}}\right\}.$$

Moreover, Young inequality gives

$$\|a'(u(t))u_x^2(t)\|_2^2 \leq C_9\left\{\|u_x(t)\|_2\|u(t)\|_{H^3(\mathbb{R})} + \|u(t)\|_2^{\frac{5p+2}{2-p}} + \|u_{xxx}(t)\|_2\right\},$$

which allows us to conclude that

$$\begin{aligned}\|a'(u)u_x^2\|_{L^2(0,T;L^2(\mathbb{R}))} \leq C_{10}\Big\{\|u\|_{C([0,T];H^3(\mathbb{R}))}\|u\|_{0,T} + T^{\frac{1}{2}}\|u\|_{0,T}^{\frac{(5p+2)}{2-p}} \\ + T^{\frac{1}{2}}\|u\|_{C([0,T];H^3(\mathbb{R}))}\Big\}.\end{aligned}$$

Hence

$$\|a'(u)u_x^2\|_{L^2(0,T;L^2(\mathbb{R}))} \leq \sigma_3(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})} \quad (4.41)$$

whit $\sigma_3(s) = C_{10}\left\{\sigma_1(s)\beta_0(s)s + T^{\frac{1}{2}}\beta_0^{\frac{(5p+2)}{2-p}}(s)s^{\frac{(5p+2)}{2-p}-1} + T^{\frac{1}{2}}\sigma_1(s)\right\}$. On the other hand, (4.11) yields

$$\begin{aligned}\|a(u(t))u_{xx}(t)\|_2 &\leq C_{11}\left\{\|u(t)\|_{H^3(\mathbb{R})} + \|u(t)\|_2^{\frac{p}{2}}\|u_x(t)\|_2^{\frac{p}{2}}\|u_{xx}(t)\|_2\right\} \\ &\leq C_{11}\left\{\|u\|_{C([0,T];H^3(\mathbb{R}))} + \|u\|_{0,T}^{\frac{p}{2}}\|u\|_{C([0,T];H^3(\mathbb{R}))}\|u_x(t)\|_2^{\frac{p}{2}}\right\}.\end{aligned}$$

It transpires that

$$\|a(u)u_{xx}\|_{L^2(0,T;L^2(\mathbb{R}))} \leq C_{12}\left\{T^{\frac{1}{2}}\|u\|_{C([0,T];H^3(\mathbb{R}))}\right.$$
$$\left. +\|u\|_{0,T}^{\frac{p}{2}}\|u\|_{C([0,T];H^3(\mathbb{R}))}\left(\int_0^T \|u_x(t)\|_2^p\right)^{\frac{1}{2}}\right\}$$
$$\leq C_{12}\left\{T^{\frac{1}{2}}\|u\|_{C([0,T];H^3(\mathbb{R}))} + T^{\frac{2-p}{4}}\|u\|_{0,T}^p\|u\|_{C([0,T];H^3(\mathbb{R}))}\right\}.$$

from which one obtains the inequality

$$\|a(u)u_{xx}\|_{L^2(0,T;L^2(\mathbb{R}))} \leq \sigma_4(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})}, \tag{4.42}$$

whit $\sigma_4(s) = C_{12}\left\{T^{\frac{1}{2}}\sigma_1(s) + T^{\frac{2-p}{4}}\beta_0^p(s)\sigma_1(s)s^p\right\}$. Consequently, (4.40), (4.41) and (4.42) lead to

$$\|u_{xxxx}\|_{L^2(0;T,L^2(\mathbb{R}))} \leq \sigma_5(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})}, \tag{4.43}$$

where $\sigma_5(s) = \sigma_2(s) + \sigma_3(s) + \sigma_4(s)$. Finally, using (4.38) and (4.43), we conclude that $u \in L^(0,T;H^4(\mathbb{R}))$ and

$$\|u\|_{3,T} \leq \beta_3(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})},$$

where $\beta_3(s) = \sigma_1(s) + \sigma_5(s)$. □

Next, we will show the well-posedness of the IVP (4.1) in the space $H^s(\mathbb{R})$ for $0 \leq s \leq 3$ and $1 \leq p < 2$. In order to do that, we will use a method introduced by Tartar [94] and adapted by Bona and Scott [8, Theorem 4.3] to prove the global well-posedness of the pure initial value problem for the KdV equation on the whole line in fractional order Sobolev spaces $H^s(\mathbb{R})$.

Let B_0 and B_1 be two Banach spaces, such that $B_1 \subset B_0$ with the inclusion map being continuous. For $f \in B_0$ and $t \geq 0$, let

$$K(f,t) = \inf_{g\in B_1}\{\|f-g\|_{B_0} + t\|g\|_{B_1}\}.$$

For $0 < \theta < 1$ and $1 \leq p \leq +\infty$, define

$$\mathbb{B}_{\theta,p} := [B_0, B_1]_{\theta,p} = \left\{f \in B_0 : \|f\|_{\theta,p} := \left(\int_0^\infty K(f,t)t^{-\theta p-1}dt\right)^{\frac{1}{p}} < \infty\right\}$$

with the usual modification for the case $p = \infty$. Then, $B_{\theta,p}$ is a Banach space with norm $\|\cdot\|_{\theta,p}$. Given two pairs (θ_1,p_1) and (θ_2,p_2) as above, we write $(\theta_1,p_1) \prec (\theta_2,p_2)$ when

$$\begin{cases} \theta_1 < \theta_2 & \text{or} \\ \theta_1 = \theta_2 & \text{and } p_1 > p_2. \end{cases}$$

If $(\theta_1,p_1) \prec (\theta_2,p_2)$, then $\mathbb{B}_{\theta_2,p_2} \subset \mathbb{B}_{\theta_1,p_1}$ with the inclusion map continuous.

Then, the following result holds:

Theorem 4.2.10. *Let B_0^j and B_1^j be Banach spaces such that $B_1^j \subset B_0^j$ with continuous inclusion mappings, for $j = 1,2$. Let α and q lie in the ranges $0 < \alpha < 1$ and $1 \leq q \leq \infty$. Suppose that $\mathcal{A}$ is a mapping satisfying*

(i) $\mathcal{A} : \mathbb{B}_{\alpha,q}^1 \to B_0^2$ *and, for* $f,g \in \mathbb{B}_{\alpha,q}^1$,

$$\|\mathcal{A}f - \mathcal{A}g\|_{B_0^2} \leq C_0\left(\|f\|_{\mathbb{B}_{\alpha,q}^1} + \|g\|_{\mathbb{B}_{\alpha,q}^1}\right)\|f-g\|_{B_0^1},$$

(ii) $\mathcal{A} : B_1^1 \to B_1^2$ *and, for* $h \in \mathbb{B}_1^1$,

$$\|\mathcal{A}h\|_{B_1^2} \leq C_1\left(\|h\|_{\mathbb{B}^1_{\alpha,q}}\right)\|h\|_{B_1^1},$$

where $C_j : \mathbb{R}^+ \to \mathbb{R}^+$ *are continuous nondecreasing functions, for* $j = 0, 1$. *Then, if* $(\theta, p) \geq (\alpha, q)$, $\mathcal{A}$ *maps* $\mathbb{B}^1_{\theta,p}$ *into* $\mathbb{B}^2_{\theta,p}$ *and, for* $f \in \mathbb{B}^1_{\theta,p}$, *we have*

$$\|\mathcal{A}f\|_{\mathbb{B}^2_{\theta,p}} \leq C\left(\|f\|_{\mathbb{B}^1_{\alpha,q}}\right)\|f\|_{\mathbb{B}^1_{\theta,p}},$$

where $C(r) = 4C_0(4r)^{1-\theta}C_1(3r)^{\theta}$, $r > 0$.

Proof. See [8, Theorem 4.3] □

This theorem leads to the main result of this section.

Theorem 4.2.11. *Let* a *be a* $C^1(\mathbb{R})$ *function satisfying*

$$|a(\mu)| \leq C(1 + |\mu|^p), \quad |a'(\mu)| \leq C(1 + |\mu|^{p-1}), \qquad \forall \mu \in \mathbb{R},$$

with $1 \leq p < 2$, *and let* $T > 0$ *and* $0 \leq s \leq 3$ *be given. In addition, assume that* $b \in L^\infty(\mathbb{R})$ *when* $s = 0$ *and* $b \in H^1(\mathbb{R})$ *when* $s > 0$. *Then, for any* $u_0 \in H^s(\mathbb{R})$, *the IVP (4.1) admits a unique solution* $u \in B_{s,T}$. *Moreover, there exists a nondecreasing continuous function* $\beta_s : \mathbb{R}^+ \to \mathbb{R}^+$, *such that*

$$\|u\|_{B_{s,T}} \leq \beta_s(\|u_0\|_2)\|u_0\|_{H^s(\mathbb{R})}.$$

Proof. We define

$$B_0^1 = L^2(\mathbb{R}), \quad B_0^2 = B_{0,T}, \quad B_1^1 = H^3(\mathbb{R}) \quad \text{and} \quad B_1^2 = B_{3,T}.$$

Thus,

$$\mathbb{B}^1_{\frac{s}{3},2} = [L^2(\mathbb{R}), H^3(\mathbb{R})]_{\frac{s}{3},2} = H^s(\mathbb{R}) \quad \text{and} \quad \mathbb{B}^2_{\frac{s}{3},2} = [B_{0,T}, B_{3,T}]_{\frac{s}{3},2} = B_{s,T}.$$

Combining Proposition 4.2.7 and Theorem 4.2.9 we obtain (i) and (ii) in Theorem 4.2.10. Then, Theorem 4.2.10 yields the result. □

Theorem 2.3.7 gives a strong smoothing property for the solutions of the problem.

Corollary 4.2.12. *Under the assumptions of Theorem 4.2.11, for any* $u_0 \in L^2(\mathbb{R})$, *the corresponding solution* u *of (4.1) belongs to*

$$B_{3,[\varepsilon,T]} = C([\varepsilon, T]; H^3(\mathbb{R})) \cap L^2(\varepsilon, T; H^4(\mathbb{R})),$$

for every $T > 0$ *and* $0 < \varepsilon < T$.

Proof. The same result was obtained for the generalized KdV and the KdV-Burgers equations in [81] and [23], respectively. Since the proof is analogous and follows from classical arguments we omit it. □

4.2.2 Case $p \geq 2$.

We first restrict ourselves to case $2 \leq p < 5$ to obtain the existence of solutions in the L^2-setting, i.e. finite energy solutions. Next we prove the global well-posedness in the space $B_{3,T}$.

First, we recall the following result which follows from the Egoroff theorem.

Lemma 4.2.13. *Let Ω be an open set in $\mathbb{R}^N$, and let $\{f_n\}$ be a sequence of functions in $L^p(\Omega)$ (with $1 < p < \infty$) such $f_n \rightharpoonup f$ in $L^p(\Omega)$ and $f_n(x) \to g(x)$ a.e. Then $f(x) = g(x)$ a.e.*

Unlike the case $1 \leq p < 2$, the next result is not obtained combining semigroup theory and fixed point arguments. Here, due to some technical problems, the solution is obtained as limit of the regular ones. We follows the ideas contained in [81].

Theorem 4.2.14. *Let a be a $C^1(\mathbb{R})$ function satisfying*

$$|a(\mu)| \leq C|\mu|^p \qquad |a'(\mu)| \leq C|\mu|^{p-1}, \qquad \forall \mu \in \mathbb{R}, \tag{4.44}$$

with $2 \leq p < 5$. Then, for any $u_0 \in L^2(\mathbb{R})$ the problem (2.12) admits at least one solution u, such that

$$u \in C_w([0,T]; L^2(\mathbb{R})) \cap L^2(0,T; H^1(\mathbb{R})), \quad \text{for all } T > 0.$$

Proof. Consider a sequence $\{a_n\} \in C_0^\infty(\mathbb{R})$, such that

$$|a_n^{(j)}(\mu)| \leq C(1 + |\mu|^{p-j}), \quad \forall \mu \in \mathbb{R}, \quad j = 0, 1, \tag{4.45}$$

$$a_n \longrightarrow a \quad \text{uniformly in each compact set in } \mathbb{R}. \tag{4.46}$$

Note that $|a_n(\mu)| \leq C_n(1 + |\mu|)$ and $|a_n'(\mu)| \leq C_n$. Then, for each n, Theorem 4.2.6 guarantees the existence of a function $u_n \in B_{0,T}$ solution of

$$\begin{cases} \partial_t u_n + \partial_x^3 u_n - \partial_x^2 u_n + a_n(u_n)\partial_x u_n + b(x)u_n = 0 \\ u_n(0,x) = u_0(x), \end{cases} \tag{4.47}$$

with $\|u_n\|_{0,T} \leq 2C_T\|u_0\|_{L^2(\mathbb{R})}$. Hence,

$$\{u_n\} \quad \text{is bounded in } C([0,T]; L^2(\mathbb{R})) \cap L^2(0,T; H^1(\mathbb{R})). \tag{4.48}$$

From (4.48) we obtain a function u and a subsequence, still denoted by the same index n, such that

$$u_n \rightharpoonup u \quad \text{in } L^\infty(0,T; L^2(\mathbb{R})) \text{ weak } * \tag{4.49}$$

$$u_n \rightharpoonup u \quad \text{in } L^2(0,T; H^1(\mathbb{R})) \text{ weak.} \tag{4.50}$$

In order to analyze the nonlinear term $a_n(u_n)\partial_x u_n$ we consider the functions

$$A(u) := \int_0^u a(v)dv \quad \text{and} \quad A_n(u) := \int_0^u a_n(v)dv. \tag{4.51}$$

Note that $a_n(u_n)\partial_x u_n = \partial_x[A_n(u_n)]$. Then, taking $\alpha \in \left(1, \frac{6}{p+1}\right)$ and proceeding as in [81, proof of Theorem 2,14]), we deduce that for each interval $I \subset \mathbb{R}$, the sequence $\{A_n(u_n)\}$ is bounded in $L^\alpha([0,T] \times I)$. Indeed,

$$|A_n(u)|^\alpha \leq C\left(2|u| + \frac{|u|^{p+1}}{p+1}\right)^\alpha \leq C'\left(|u|^\alpha + |u|^{\alpha(p+1)}\right), \tag{4.52}$$

where C and C' denote some positive constants which depend only on p and α. Therefore,

$$\begin{aligned}
\|A_n(u_n)\|^{\alpha}_{L^{\alpha}((0,T)\times I)} &\le C''\left(\|u_n\|^{\alpha}_{L^2(0,T;L^2(I))} + \int_0^T \|u_n(t)\|_{\infty}^{\alpha(p+1)-2}\|u_n(t)\|_2^2 dt\right)\\
&\le C''\left(\|u_n\|^{\alpha}_{0,T} + 2^{\frac{\alpha(p+1)-2}{2}}\int_0^T \|u_n(t)\|_2^{\frac{\alpha(p+1)+2}{2}}\|u_{nx}(t)\|_2^{\frac{\alpha(p+1)-2}{2}} dt\right)\\
&\le C''\left(\|u_n\|^{\alpha}_{0,T} + 2^{\frac{\alpha(p+1)-2}{2}} T^{\frac{6-\alpha(p+1)}{4}}\|u_n\|_{0,T}^{\frac{\alpha(p+1)+2}{2}}\|u_n\|_{0,T}^{\frac{\alpha(p+1)-2}{2}}\right)\\
&\le C''\left(\|u_n\|^{\alpha}_{0,T} + \|u_n\|^{\alpha(p+1)}_{0,T}\right)\\
&\le \widetilde{C}\left(\|u_0\|_2^{\alpha} + \|u_0\|_2^{\alpha(p+1)}\right),
\end{aligned} \tag{4.53}$$

where $\widetilde{C}$ is a positive constant. Consequently,

$$\{A_n(u_n)\} \quad \text{is bounded in } L^{\alpha}(0,T;H^{-1}(I)) \quad (\text{since } L^{\alpha}(I)\hookrightarrow H^{-1}(I))$$

and

$$\{a_n(u_n)\partial_x u_n\} = \{\partial_x[A_n(u_n)]\} \quad \text{is bounded in } L^{\alpha}(0,T;H^{-2}(I)). \tag{4.54}$$

Moreover, (4.48) and the fact that $1 < \alpha \le 2$ allow us to conclude that

$$\{\partial_x^3 u_n\}, \{\partial_x^2 u_n\} \text{ and } \{bu_n\} \text{ are bounded in } L^2(0,T;H^{-2}(\mathbb{R})) \subset L^{\alpha}(0,T;H^{-2}(\mathbb{R}))$$

and, therefore,

$$\partial_t u_n = -\partial_x^3 u_n + \partial_x^2 u_n - a_n(u_n)\partial_x u_n - bu_n \text{ is bounded in } L^{\alpha}(0,T;H^{-2}(I)). \tag{4.55}$$

Since $\{u_n\}$ is bounded in $L^{\alpha}(0,T;H^1(\mathbb{R}))$ and the first embedding in $H^1(I)\hookrightarrow L^2(I)\hookrightarrow H^{-2}(I)$ is compact, we can apply [91, Corollary 4, pag 85] to conclude that $\{u_n\}$ is relatively compact in $L^2(0,T;L^2(I))$. Using a diagonal process, we obtain a subsequence, still denoted by $\{u_n\}$, such that

$$u_n \longrightarrow u \text{ in } L^2(0,T;L^2_{loc}(\mathbb{R})) \text{ strongly and a.e.} \tag{4.56}$$

Moreover, by (4.50),

$$u_n \rightharpoonup u \quad \text{weak in } L^2(0,T;L^2(\mathbb{R})) \equiv L^2(\mathbb{R}\times(0,T))$$

and by applying Lemma 4.2.13, we obtain

$$u_n \longrightarrow u \text{ a.e in } \mathbb{R}\times(0,T). \tag{4.57}$$

Then, using (4.46), (4.51) and (4.57), it is easy to see that

$$A_n(u_n(x,t)) \longrightarrow A(u(x,t)) \text{ a.e in } \mathbb{R}\times(0,T).$$

Next, proceeding as in the previous steps and by applying Lemma 4.2.13, the following convergence holds

$$A_n(u_n) \rightharpoonup A(u) \text{ weak in } L^{\alpha}(0,T;L^{\alpha}_{loc}(\mathbb{R})).$$

Therefore, $A_n(u_n) \longrightarrow A(u)$ in $D'(\mathbb{R}\times(0,T))$ and, by taking the partial derivative, we obtain

$$a_n(u_n)\partial_x u_n \longrightarrow a(u)\partial_x u \text{ in } D'(\mathbb{R}\times(0,T)). \tag{4.58}$$

From (4.56) and (4.58), we can take the limit in (4.47) to conclude that u solves the equation (4.1) in the sense of distribution, i.e,

$$u_t + u_{xxx} - u_{xx} + a(u)u_x + bu = 0 \quad \text{in } D'(\mathbb{R}\times(0,T)). \tag{4.59}$$

On the other hand, by (4.48) and (4.55), we infer from [91, Corollary 4, pag 85] that $\{u_n\}$ is relatively compact in $C([0,T];H^{-1}_{loc}(\mathbb{R}))$. Therefore, there exists a subsequence (denoted by $\{u_n\}$), such that

$$u_n \longrightarrow u \text{ in } C([0,T];H^{-1}_{loc}(\mathbb{R})). \tag{4.60}$$

In particular, $u(x,0)=\lim_{n\to\infty}u_n(x,0)=u_0(x)$. Now, note that (4.50) yields

$$\begin{aligned}
&u_{xxx}\in L^2(0,T;H^{-2}(\mathbb{R}))\hookrightarrow L^\alpha(0,T;H^{-2}(\mathbb{R})),\\
&u_{xx}\in L^2(0,T;H^{-1}(\mathbb{R}))\hookrightarrow L^\alpha(0,T;H^{-2}(\mathbb{R})),\\
&bu\in L^2(0,T;H^{1}(\mathbb{R}))\hookrightarrow L^\alpha(0,T;H^{-2}(\mathbb{R})).
\end{aligned}$$

Finally, we claim that

$$a(u)u_x=[A(u)]_x\in L^\alpha(0,T;H^{-2}(\mathbb{R})), \tag{4.61}$$

for any $\alpha\in\left(1,\frac{6}{p+1}\right)$. In fact, first note that $\beta=\frac{\alpha(p+1)-2}{2}<2$, then (4.49) and (4.50) imply that $u\in L^\infty(0,T,L^2(\mathbb{R}))\cap L^2(0,T,H^1(\mathbb{R}))\subset L^\infty(0,T,L^2(\mathbb{R}))\cap L^\beta(0,T,H^1(\mathbb{R}))$. Moreover, by using (4.44) and (4.51), there exists $C=C(\alpha,p)>0$ such that $|A(u)|^\alpha\le C|u|^{\alpha(p+1)}$. Thus, we obtain

$$\begin{aligned}
\|A(u)\|^\alpha_{L^\alpha((0,T)\times\mathbb{R})}&=\int_0^T\int_{\mathbb{R}}|A(u)(x,t)|^\alpha dxdt\le C\int_0^T\int_{\mathbb{R}}|u(x,t)|^{\alpha(p+1)}dxdt\\
&\le C\int_0^T\|u(t)\|_\infty^{\alpha(p+1)-2}\|u(t)\|_2^2dt\\
&\le 2^{\frac{\alpha(p+1)-2}{2}}C\int_0^T\|u(t)\|_2^{\frac{\alpha(p+1)+2}{2}}\|u_x(t)\|_2^{\frac{\alpha(p+1)-2}{2}}dt\\
&\le 2^{\frac{\alpha(p+1)-2}{2}}C\|u\|^{\frac{\alpha(p+1)+2}{2}}_{L^\infty(0,T,L^2(\mathbb{R}))}\|u\|^\beta_{L^\beta(0,T,H^1(\mathbb{R}))}.
\end{aligned}$$

Then, it yields that

$$A(u)\in L^\alpha(0,T,L^\alpha(\mathbb{R})). \tag{4.62}$$

Furthermore, since $\alpha\in(1,2)$, it is easy to see that $L^\alpha(\mathbb{R})\subset H^{-1}(\mathbb{R})$. Indeed, take $v\in L^\alpha(\mathbb{R})$ with $1<\alpha<2$. Thus, for any $q>1$, it follows that

$$\|v\|^2_{H^{-1}(\mathbb{R})}=\int_{\mathbb{R}}(1+|\xi|^2)^{-1}|\hat{v}(\xi)|^2d\xi\le K\|\hat{v}\|^2_{L^{2q}(\mathbb{R})},$$

where $K=\left(\int_{\mathbb{R}}(1+|\xi|^2)^{-\frac{q}{q-1}}d\xi\right)^{\frac{q-1}{q}}$. In order to get K finite, we take $q=\frac{\alpha}{2(\alpha-1)}>1$ and applying Hausdorff-Young inequality

$$\|\hat{v}\|_{L^{\frac{\alpha}{\alpha-1}}(\mathbb{R})}\le C_\alpha\|v\|_{L^\alpha(\mathbb{R})},\quad \text{for some } C_\alpha>0,$$

we obtain

$$\|v\|^2_{H^{-1}(\mathbb{R})}\le M\|v\|^2_{L^\alpha(\mathbb{R})},$$

where M is a positive constant, which proof that $L^\alpha(\mathbb{R})\subset H^{-1}(\mathbb{R})$. Thus, (4.62) implies that $A(u)\in L^\alpha(0,T,H^{-1}(\mathbb{R}))$, proving the claim (4.61). Now, from (4.59), we deduce that $u_t\in L^\alpha(0,T;H^{-2}(\mathbb{R}))$, then $u\in W^{1,\alpha}(0,T;H^{-2}(\mathbb{R}))$. Since $\alpha>1$, we conclude that $u\in C([0,T];H^{-2}(\mathbb{R}))$. In particular, we obtain

$$u\in L^\infty(0,T;L^2(\mathbb{R}))\cap C_w([0,T];H^{-2}(\mathbb{R}))$$

and from Lemma 1.4 in [95, Ch III], it follows that $u\in C_w([0,T];L^2(\mathbb{R}))$. □

Remark 4.2.15. *When $2 \le p < 4$ we can prove the Theorem 4.2.14 with more general assumptions on the function $a(\cdot)$. More precisely,*

$$|a(\mu)| \le C(1 + |\mu|^p) \qquad |a'(\mu)| \le C(1 + |\mu|^{p-1}), \qquad \forall \mu \in \mathbb{R}.$$

The proof follows the same steps, except in (4.62)*. Indeed, we first claim that there exists $\alpha \in \left(1, \frac{6}{p+1}\right)$, such that*

$$A(u) \in L^\alpha(0, T, L^2(\mathbb{R})). \tag{4.63}$$

To proof it, note that

$$\begin{aligned}
\|A(u)\|^\alpha_{L^\alpha(0,T,L^2(\mathbb{R}))} &= \int_0^T \left(\int_{\mathbb{R}} |A(u)(x,t)|^2 dx\right)^{\alpha/2} dt &(4.64)\\
&\le C \int_0^T \left(\int_{\mathbb{R}} \left(|u(x,t)|^2 + |u(x,t)|^{2(p+1)}\right) dx\right)^{\alpha/2} dt\\
&\le C \left(\int_0^T \|u(t)\|^\alpha_{L^2(\mathbb{R})} dt + \int_0^T \left(\int_{\mathbb{R}} |u(x,t)|^{2(p+1)} dx\right)^{\alpha/2} dt\right)\\
&\le C \left(\|u\|^\alpha_{L^\alpha(0,T;L^2(\mathbb{R}))} + \int_0^T \|u(t)\|^{\alpha p}_{L^\infty(\mathbb{R})} \|u(t)\|^\alpha_{L^2(\mathbb{R})} dt\right)\\
&\le C \left(\|u\|^\alpha_{L^\alpha(0,T;L^2(\mathbb{R}))} + 2^{\frac{\alpha p}{2}} \int_0^T \|u(t)\|_2^{\frac{\alpha(p+2)}{2}} \|u_x(t)\|_2^{\frac{\alpha p}{2}} dt\right). &(4.65)
\end{aligned}$$

Since $1 \le \frac{4}{p} \le \frac{6}{p+1}$. we can pick any $\alpha \in \left(1, \frac{4}{p}\right)$ and by using (4.50)*, we obtain $u \in L^\infty(0, T, L^2(\mathbb{R})) \cap L^2(0, T, H^1(\mathbb{R})) \subset L^\alpha(0, T, L^2(\mathbb{R})) \cap L^{\frac{\alpha p}{2}}(0, T, H^1(\mathbb{R}))$. Hence,* (4.64) *implies that*

$$\|A(u)\|^\alpha_{L^\alpha(0,T,L^2(\mathbb{R}))} \le C \left(\|u\|^\alpha_{L^\alpha(0,T;L^2(\mathbb{R}))} + 2^{\frac{\alpha p}{2}} \|u\|^{\frac{\alpha(p+2)}{2}}_{L^\infty(0,T,L^2(\mathbb{R}))} \|u\|^{\frac{\alpha p}{2}}_{L^{\frac{\alpha p}{2}}(0,T,H^1(\mathbb{R}))}\right),$$

which proof (4.63)*. Consequently, we obtain* (4.61) *provided that $2 \le p < 4$.*

Definition 4.2.1. *Let $T > 0$, A function $u \in C_w([0, T]; L^2(\mathbb{R})) \cap L^2(0, T; H^1(\mathbb{R}))$ is said to be a weak solution of problem (4.1) if there exists a sequence $\{a_n\}$ of function in $C_0^\infty(\mathbb{R})$ satisfying (4.45) and (4.46) and a sequence of strong solution u_n to (4.47), such that (4.49),(4.50),(4.57) and (4.60) hold true.*

The proof of next result also requires an adaptation of Lemma 4.2.4 as follows.

Lemma 4.2.16. *For any $T > 0$, $p \ge 1$ and $u, v, w \in B_{3,T}$, such that $u_t, v_t, w_t \in B_{0,T}$, we have*

(i) $\|(a(u)v_x)_x\|_{L^2(0,T;L^2(\mathbb{R}))} \le CT^{\frac{1}{2}} \left\{\|u\|_{3,T}\|v\|_{3,T} + 2\|u\|^p_{3,T}\|v\|_{3,T} + \|v\|_{3,T}\right\},$

(ii)

$$\begin{aligned}
\|a(u)v_x\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} \le CT^{\frac{1}{2}} \Big\{&(\|v\|_{3,T} + \|v_t\|_{0,T}) + \|u\|^p_{3,T}(\|v\|_{3,T} + \|v_t\|_{0,T})\\
&+\|u_t\|_{0,T}\|v\|_{3,T} + \|u_t\|_{0,T}\|v\|_{3,T}\|u\|^{p-1}_{3,T}\Big\},
\end{aligned}$$

(iii) $\|uw_x\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} \le 2^{\frac{1}{2}} T^{\frac{1}{4}} \{\|u\|_{3,T}\|w\|_{3,T} + \|u_t\|_{0,T}\|w\|_{3,T} + \|u\|_{3,T}\|w_t\|_{0,T}\}.$

(iv) If $p \ge 2$, then

$$\begin{aligned}
\|u|w|^{p-1}v_x\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} \le T^{\frac{1}{2}} \Big\{&\|u\|_{3,T}\|w\|^{p-1}_{3,T}(\|v\|_{3,T} + \|v_t\|_{0,T})\\
&+\|v\|_{3,T}\|w\|^{p-1}_{3,T}\|u_t\|_{0,T} + (p-1)\|u\|_{3,T}\|w\|^{p-2}_{3,T}\|v\|_{3,T}\|w_t\|_{0,T}\Big\}.
\end{aligned}$$

Proof. First, note that if $u \in B_{3,T}$ we have

$$\left\{ \begin{array}{l} \partial_x^j u \in C([0,T]; H^{3-j}(\mathbb{R})) \hookrightarrow C([0,T]; C(\mathbb{R})) \\ \|\partial_x^j u\|_{C([0,T];C(\mathbb{R}))} \leq C\|u\|_{3,T} \end{array} \right. \quad , \quad j = 0,1,2,$$
$$\left\{ \begin{array}{l} \partial_x^j u \in L^2([0,T]; H^{4-j}(\mathbb{R})) \hookrightarrow L^2([0,T]; L^2(\mathbb{R})) \\ \|\partial_x^j u\|_{L^2([0,T];L^2(\mathbb{R}))} \leq C\|u\|_{3,T} \end{array} \right. \quad , \quad j = 0,1,2,3. \tag{4.66}$$

(i) (4.3) and (4.66) imply that

$$\begin{aligned} \|(a(u)v_x)_x\|_{L^2(0,T;L^2(\mathbb{R}))} \leq\ & C\{T^{\frac{1}{2}}\|u_x\|_{C([0,T];C(\mathbb{R}))}\|v_x\|_{C(0,T;L^2(\mathbb{R}))} \\ & + T^{\frac{1}{2}}\|u\|^{p-1}_{C([0,T];C(\mathbb{R}))}\|u_x\|_{C([0,T];C(\mathbb{R}))}\|v_x\|_{C(0,T;L^2(\mathbb{R}))} \\ & + T^{\frac{1}{2}}\|v_{xx}\|_{C([0,T];L^2(\mathbb{R}))} + T^{\frac{1}{2}}\|u\|^{p}_{C([0,T];C(\mathbb{R}))}\|v_{xx}\|_{C(0,T;L^2(\mathbb{R}))}\} \\ \leq\ & CT^{\frac{1}{2}}\left\{\|u\|_{3,T}\|v\|_{3,T} + 2\|u\|^p_{3,T}\|v\|_{3,T} + \|v\|_{3,T}\right\}. \end{aligned}$$

(ii) By (4.3), Hölder inequality and (4.66) we get

$$\begin{aligned} \|a(u)v_x\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} \leq\ & C\left\{\|v_x\|_{L^2(0,T;L^2(\mathbb{R}))}T^{\frac{1}{2}}\right. \\ & + \|u\|^p_{C([0,T];C(\mathbb{R}))}\|v_x\|_{L^2(0,T;L^2(\mathbb{R}))}T^{\frac{1}{2}} \\ & + \|v_x\|_{C([0,T];C(\mathbb{R}))}\|u_t\|_{L^2(0,T;L^2(\mathbb{R}))}T^{\frac{1}{2}} \\ & + \|u\|^{p-1}_{C([0,T];C(\mathbb{R}))}\|v_x\|_{C([0,T];C(\mathbb{R}))}\|u_t\|_{L^2(0,T;L^2(\mathbb{R}))}T^{\frac{1}{2}} \\ & \left. + T^{\frac{1}{2}}\|v_{tx}\|_{L^2(0,T;L^2(\mathbb{R}))} + \|u\|^p_{C([0,T];C(\mathbb{R}))}\|v_{tx}\|_{L^2(0,T;L^2(\mathbb{R}))}T^{\frac{1}{2}}\right\} \\ \leq\ & CT^{\frac{1}{2}}\left\{\|v\|_{3,T} + \|u\|^p_{3,T}\|v\|_{3,T} + \|v\|_{3,T}\|u_t\|_{0,T}\right. \\ & \left. + \|u\|^{p-1}_{3,T}\|v\|_{3,T}\|u_t\|_{0,T} + \|v_t\|_{0,T} + \|u\|^p_{3,T}\|v_t\|_{0,T}\right\}. \end{aligned}$$

(iii) It is a consequence of (ii) in Lemma 4.2.4.

(iv) Hölder inequality and (4.66) lead to the desired result:

$$\begin{aligned} \|u|w|^{p-1}v_x\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} \leq\ & \|u\|_{C([0,T];C(\mathbb{R}))}\|w\|^{p-1}_{C([0,T];C(\mathbb{R}))}\|v_x\|_{L^2(0,T;L^2(\mathbb{R}))}T^{\frac{1}{2}} \\ & + \|v_x\|_{C([0,T];C(\mathbb{R}))}\|w\|^{p-1}_{C([0,T];C(\mathbb{R}))}\|u_t\|_{L^2(0,T;L^2(\mathbb{R}))}T^{\frac{1}{2}} \\ & + (p-1)\|u\|_{C([0,T];C(\mathbb{R}))}\|w\|^{p-2}_{C([0,T];C(\mathbb{R}))}\|v_x\|_{C([0,T];C(\mathbb{R}))}\|w_t\|_{L^2(0,T;L^2(\mathbb{R}))}T^{\frac{1}{2}} \\ & + \|u\|_{C([0,T];C(\mathbb{R}))}\|w\|^{p-1}_{C([0,T];C(\mathbb{R}))}\|v_{tx}\|_{L^2(0,T;L^2(\mathbb{R}))}T^{\frac{1}{2}} \\ \leq\ & T^{\frac{1}{2}}\left\{\|u\|_{3,T}\|w\|^{p-1}_{3,T}\|v\|_{3,T} + \|v\|_{3,T}\|w\|^{p-1}_{3,T}\|u_t\|_{0,T}\right. \\ & \left. + (p-1)\|u\|_{3,T}\|w\|^{p-2}_{3,T}\|v\|_{3,T}\|w_t\|_{0,T} + \|u\|_{3,T}\|w\|^{p-1}_{3,T}\|v_t\|_{0,T}\right\}. \end{aligned}$$

□

Proposition 4.2.1 asserts that the inhomogeneous linear problem (4.6) is well-posedness and we have the existence of a mild solution. However, we can have a regular solution as shows the following result.

Proposition 4.2.17. *Let $T>0$, $b \in H^1(\mathbb{R})$ and $u_0 \in H^3(\mathbb{R})$. If $f \in W^{1,1}(0,T;L^2(\mathbb{R}))$ and $f_x \in L^2(0,T;L^2(\mathbb{R}))$, the inhomogeneous linear problem (4.6) has a unique regular solution $u \in B_{3,T}$, such that*

$$\|u\|_{3,T} \leq C_{3,T}\left\{\|u_0\|_{H^3(\mathbb{R})} + \|f\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}\right\}, \tag{4.67}$$

$u_t \in B_{0,T}$ and

$$\|u_t\|_{0,T} \leq C_{0,T}\left\{\|u_0\|_{H^3(\mathbb{R})} + \|f(0)\|_{L^2(\mathbb{R})} + \|f_t\|_{L^1(0,T;L^2(\mathbb{R}))}\right\}, \tag{4.68}$$

where $C_{3,T} = 2Ce^{\|b\|_\infty T}$ and $C_{0,T} = 2e^{\|b\|_\infty T}$.

Proof. By using the semigroup theory and the previous results, we obtain a unique regular solution $u \in C([0,T];H^3(\mathbb{R}))$. Therefore, we will prove that $u \in L^2(0,T,H^4(\mathbb{R}))$. Indeed, first note that $u_0 \in H^3(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$, hence applying the Proposition 4.2.1 it follows that $u \in B_{0,T}$ and

$$\|u\|_{0,T} \leq C_T\left\{\|u_0\|_{H^3(\mathbb{R})} + \|f\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}\right\} \tag{4.69}$$

where $C_T = 2e^{\|b\|_\infty T}$. On the other hand, note that u_t solves the problem

$$\begin{cases} v_t - v_{xx} + v_{xxx} + bv = f_t & \text{in } \mathbb{R}\times(0,\infty) \\ v(0) = v_0 & \text{in } \mathbb{R}\times(0,\infty), \end{cases}$$

where $v_0 = \partial_x^2 u_0 - \partial_x^3 u_0 - bu_0 + f(\cdot,0) \in L^2(\mathbb{R})$. Then, by applying the Proposition 2.3.9, we have $u_t \in B_{0,T}$ and

$$\|u_t\|_{0,T} \leq C_T\left\{\|u_0\|_{H^3(\mathbb{R})} + \|f(0)\|_2 + \|f_t\|_{L^1(0,T;L^2(\mathbb{R}))}\right\}, \tag{4.70}$$

obtaining (4.68). Moreover,

$$\begin{aligned}\|(bu)_x\|_{L^2(0,T;L^2(\mathbb{R}))} &\leq \|b_x\|_2\|u\|_{L^2(0,T;L^\infty(\mathbb{R}))} + \|b\|_\infty\|u_x\|_{L^2(0,T;L^2(\mathbb{R}))} \\ &\leq C\|b\|_{H^1(\mathbb{R})}\|u\|_{0,T},\end{aligned} \tag{4.71}$$

where C is the embedding constant of $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Since

$$\partial_x^4 u = \partial_x^3 u - \partial_x u_t - \partial_x(bu) + \partial_x f \quad \text{in } D'(\mathbb{R}), \quad \text{for all } t>0,$$

we have that $\partial_x^4 u \in L^2(0,T;L^2(\mathbb{R}))$, i.e. $u \in L^2(0,T;H^4(\mathbb{R}))$ and $u \in B_{3,T}$. In order to prove (4.67), we need some estimates. Note that, from (4.69), we get

$$\sup_{t\in[0,T]}\|u(t)\|_2 \leq C_T\left\{\|u_0\|_{H^3(\mathbb{R})} + \|f\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}\right\}. \tag{4.72}$$

Multiplying the equation in (4.6) by u_{xx} and integrating in $\mathbb{R}$ one obtains the inequality

$$\frac{1}{2}\frac{d}{dt}\|u_x(t)\|_2^2 + \|u_{xx}(t)\|_2^2 \leq \{\|f(t)\|_2 + \|bu(t)\|_2\}\|u_{xx}(t)\|_2.$$

Then, Young inequality leads to

$$\frac{1}{2}\frac{d}{dt}\|u_x(t)\|_2^2 + \frac{1}{2}\|u_{xx}(t)\|_2^2 \leq C\left\{\|f(t)\|_2^2 + \|b\|_\infty^2\|u(t)\|_2^2\right\}$$

Integrating on $[0,T]$, using (4.69) and the embedding

$$W^{1,1}(0,T)(0,T;L^2(\mathbb{R})) \hookrightarrow L^\infty(0,T;L^2(\mathbb{R})),$$

the solution can be estimated as follows

$$\sup_{t\in[0,T]}\|u_x(t)\|_2 \leq CC_T\left\{\|u_0\|_{H^3(\mathbb{R})} + \|f\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}\right\}. \tag{4.73}$$

A similar estimate is obtained by multiplying the equation by $\partial_x^4 u$, integrating in $\mathbb{R}$ and using Young inequality:

$$\frac{1}{2}\frac{d}{dt}\|u_{xx}(t)\|_2^2 + \frac{1}{2}\|u_{xxx}(t)\|_2^2 \le C\left\{\|f_x(t)\|_2^2 + \|(bu)_x(t)\|_2^2\right\}.$$

Integrating on $[0,T]$ and using (4.71) and (4.69), we have

$$\sup_{t\in[0,T]}\|u_{xx}(t)\|_2 \le CC_T\left\{\|u_0\|_{H^3(\mathbb{R})} + \|f\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}\right\}. \tag{4.74}$$

Since

$$\|u_{xxx}(t)\|_2 \le \|u_t(t)\|_2 + \|u_{xx}(t)\|_2 + \|bu(t)\|_2 + \|f(t)\|_2,$$

using (4.69), (4.70), (4.74) and the embedding above, we conclude that

$$\sup_{t\in[0,T]}\|u_{xxx}(t)\|_2 \le CC_T\left\{\|u_0\|_{H^3(\mathbb{R})} + \|f\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}\right\}. \tag{4.75}$$

Putting together (4.72), (4.73), (4.74) and (4.75), we have

$$\|u\|_{C([0,T];H^3(\mathbb{R}))} \le CC_T\left\{\|u_0\|_{H^3(\mathbb{R})} + \|f\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}\right\}. \tag{4.76}$$

On the other hand,

$$\begin{aligned}\|\partial_x^4 u\|_{L^2(0,T;L^2(\mathbb{R}))} &\le \|u_{xxx}\|_{L^2(0,T;L^2(\mathbb{R}))} + \|\partial_x u_t\|_{L^2(0,T;L^2(\mathbb{R}))} + \|(bu)_x\|_{L^2(0,T;L^2(\mathbb{R}))}\\ &\quad + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}\\ &\le T^{\frac{1}{2}}\|u\|_{C([0,T];H^3(\mathbb{R}))} + \|u_t\|_{0,T} + \|(bu)_x\|_{L^2(0,T;L^2(\mathbb{R}))}\\ &\quad + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}.\end{aligned}$$

The above inequality, (4.69) - (4.71) and (4.75) allow us to conclude that

$$\|\partial_x^4 u\|_{L^2(0,T;L^2(\mathbb{R}))} \le CC_T\left\{\|u_0\|_{H^3(\mathbb{R})} + \|f\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}\right\}. \tag{4.77}$$

(4.76) and (4.77) implies (4.67). □

Theorem 4.2.18. *Let $b \in H^1(\mathbb{R})$ and $a \in C^2(\mathbb{R})$ satisfying*

$$|a(\mu)| \le C(1+|\mu|^p),\ |a'(\mu)| \le C(1+|\mu|^{p-1}) \text{ and } |a''(\mu)| \le C(1+|\mu|^{p-2}), \quad \forall \mu \in \mathbb{R}, \tag{4.78}$$

with $p \ge 2$. Let $T > 0$ and $u_0 \in H^3(\mathbb{R})$. Then, there exists a unique solution $u \in B_{3,T}$ of (2.12), such that

$$\|u\|_{3,T} \le \eta_3(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})},$$

where $\eta_3 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing continuous function.

Proof. Let $0 < \theta \le T$ and $R > 0$ to be a constant to be determined later. Consider

$$S_{\theta,R} := \left\{(u,v) \in B_{3,\theta} \times B_{0,\theta} : v = u_t,\ \|(u,v)\|_{B_{3,\theta}\times B_{0,\theta}} := \|u\|_{3,\theta} + \|v\|_{0,\theta} \le R\right\}.$$

Then, for each $(u,u_t) \in S_{\theta,R} \subset B_{3,\theta} \times B_{0,\theta}$, consider the problems

$$\begin{cases} v_t = A_b v - a(u)u_x \\ v(0) = u_0 \end{cases} \tag{4.79}$$

and

$$\begin{cases} z_t = A_b z - [a(u)u_x]_t \\ z(0) = z_0, \end{cases} \tag{4.80}$$

with $z_0 = -u_{0xxx} + u_{0xx} - bu_0 - a(u_0)u_{0x} \in L^2(\mathbb{R})$ and $A_b v = \partial_x^2 v - \partial_x^3 v - bv$. Recall that A_b generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ of contractions in $L^2(\mathbb{R})$. Moreover, by Lemma 4.2.16 (i) and (ii), $a(u)u_x \in W^{1,1}(0,\theta; L^2(\mathbb{R}))$ and $[a(u)u_x]_x \in L^2(0,\theta; L^2(\mathbb{R}))$. Then, by Proposition 4.2.17 the problems (4.79) and (4.80) have a unique mild solution v, such that $(v, v_t) \in B_{3,\theta} \times B_{0,\theta}$ and

$$\|(v,v_t)\|_{B_{3,\theta}\times B_{0,\theta}} \leq C_\theta\{\|u_0\|_{H^3(\mathbb{R})} + \|a(u)u_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))} + \|[a(u)u_x]_x\|_{L^2(0,\theta;L^2(\mathbb{R}))}\}, \tag{4.81}$$

where $C_\theta = 2e^{\theta\|b\|_\infty}$. Thus, we can define the operator

$$\Gamma : S_{\theta,R} \subset B_{3,\theta} \times B_{0,\theta} \longrightarrow B_{3,\theta} \times B_{0,\theta} \quad \text{by} \quad \Gamma(u,u_t) = (v,v_t).$$

Since $C_\theta \leq C_T$, from (4.81) and Lemma 4.2.16, we have

$$\begin{aligned} \|\Gamma(u,u_t)\|_{B_{3,\theta}\times B_{0,\theta}} \leq\ & C_T\|u_0\|_{H^3(\mathbb{R})} + C_T C\theta^{\frac{1}{2}}\{(\|u\|_{3,\theta} + \|u_t\|_{0,\theta}) \\ & + \|u\|_{3,\theta}^p(\|u\|_{3,\theta} + \|u_t\|_{0,\theta}) + \|u_t\|_{0,\theta}\|u\|_{3,\theta} \\ & + \|u_t\|_{0,\theta}\|u\|_{3,\theta}^p\Big\} + C_T C\theta^{\frac{1}{2}}\Big\{\|u\|_{3,\theta}^2 + 2\|u\|_{3,\theta}^{p+1} + \|u\|_{3,T}\Big\} \\ \leq\ & C_T\|u_0\|_{H^3(\mathbb{R})} + C_T C\theta^{\frac{1}{2}}\left\{4R^{p+1} + 2R^2 + 2R\right\}. \end{aligned}$$

Choosing $R = 2C_T\|u_0\|_{H^3(\mathbb{R})}$, it follows that

$$\|\Gamma(u,u_t)\|_{B_{3,\theta}\times B_{0,\theta}} \leq \left(K_1 + \frac{1}{2}\right)R,$$

where $K_1(\theta) = C_T C\theta^{\frac{1}{2}}\{4R^p + 2R + 2\}$. On the other hand, let $(u,u_t), (w,w_t) \in S_{\theta,R}$ and note that $\Gamma(u,u_t) - \Gamma(w,w_t)$ is solutions of

$$\begin{cases} v_t = A_b v + [a(w)w_x - a(u)u_x] \\ v(0) = 0 \end{cases}$$

and

$$\begin{cases} z_t = A_b z + [a(w)w_x - a(u)u_x]_t \\ z(0) = 0. \end{cases}$$

Hence, from Lemma 4.2.16, the following estimate holds

$$\begin{aligned} \|\Gamma(u,u_t) - \Gamma(w,w_t)\|_{B_{3,\theta}\times B_{0,\theta}} \leq C_T\{&\|a(w)w_x - a(u)u_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))} \\ &+ \|[a(w)w_x - a(u)u_x]_x\|_{L^2(0,\theta;L^2(\mathbb{R}))}\}. \end{aligned} \tag{4.82}$$

The next steps are devoted to estimate the terms on the right hand side of (4.82):

$$\begin{aligned} \|a(w)w_x - a(u)u_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))} \leq\ & \|(a(w) - a(u))w_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))} \\ & + \|a(u)(w-u)_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))}. \end{aligned}$$

By using the Mean Valued Theorem and Lemma 4.2.16, we have

$$\begin{aligned}
\|a(w)w_x - a(u)u_x\|_{W^{1,1}(0,\theta;L^2)} \le & C\|(1+|u|^{p-1}+|w|^{p-1})|w-u|w_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))} \\
& + \|a(u)(w-u)_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))} \\
& + \| |w|^{p-1}|w-u|w_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))}\} + \|a(u)(w-u)_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))} \\
\le & C\left\{2^{\frac{1}{2}}\theta^{\frac{1}{4}}\{\|w-u\|_{3,\theta}\|w\|_{3,\theta} + \|(w-u)_t\|_{0,\theta}\|w\|_{3,\theta} + \|w-u\|_{3,\theta}\|w_t\|_{0,\theta}\}\right. \\
& + \theta^{\frac{1}{2}}\{\|w-u\|_{3,\theta}\|u\|_{3,\theta}^{p-1}(\|w\|_{3,\theta} + \|w_t\|_{0,\theta}) + \|(w-u)_t\|_{0,\theta}\|w\|_{3,\theta}\|u\|_{3,\theta}^{p-1} \\
& + (p-1)\|u_t\|_{0,\theta}\|w-u\|_{3,\theta}\|u\|_{3,\theta}^{p-2}\|w\|_{3,\theta}\} \\
& + \theta^{\frac{1}{2}}\{\|w-u\|_{3,\theta}\|w\|_{3,\theta}^{p-1}(\|w\|_{3,\theta} + \|w_t\|_{0,\theta}) \\
& \left. + \|w\|_{3,\theta}^{p}\|[w-u]_t\|_{0,\theta} + (p-1)\|w-u\|_{3,\theta}\|w\|_{3,\theta}^{p-1}\|w_t\|_{0,\theta}\}\right\} \\
& + C\theta^{\frac{1}{2}}\left\{(\|w-u\|_{3,\theta} + \|(w-u)_t\|_{0,\theta}) + \|u\|_{3,\theta}^{p}(\|w-u\|_{3,\theta} + \|(w-u)_t\|_{0,\theta})\right. \\
& \left. + \|u_t\|_{0,\theta}\|w-u\|_{3,\theta} + \|u_t\|_{0,\theta}\|w-u\|_{3,\theta}\|u\|_{3,\theta}^{p-1}\right\} \\
\le & K_2\|(w-u,(w-u)_t)\|_{B_{3,\theta}\times B_{0,\theta}}
\end{aligned} \tag{4.83}$$

where $K_2(\theta) = C\left\{(2^{\frac{3}{2}}\theta^{\frac{1}{4}} + \theta^{\frac{1}{2}})R + 2(p+1)\theta^{\frac{1}{2}}R^p + \theta^{\frac{1}{2}}\right\}$. To estimate the second term, note that

$$\begin{aligned}
[a(w)w_x - a(u)u_x]_x = & [a'(w) - a'(u)]w_x^2 + a'(u)[w-u]_x[w+u]_x \\
& + [a(w) - a(u)]w_{xx} + a(u)[w_{xx} - u_{xx}]
\end{aligned} \tag{4.84}$$

Then, from the Mean Valued Theorem, (4.66) and (4.78), we have the following estimates:

$$\begin{aligned}
\|[a'(w) - a'(u)]w_x^2\|_{L^2(0,\theta;L^2(\mathbb{R}))} \le & C\|[1+|w|^{p-2}+|u|^{p-2}]|w-u|w_x^2\|_{L^2(0,\theta;L^2(\mathbb{R}))} \\
\le & C\{\| |w-u|w_x^2\|_{L^2(0,\theta;L^2(\mathbb{R}))} + \| |w|^{p-2}|w-u|w_x^2\|_{L^2(0,\theta;L^2(\mathbb{R}))} \\
& + \| |u|^{p-2}|w-u|w_x^2\|_{L^2(0,\theta;L^2(\mathbb{R}))}\} \\
\le & C\theta^{\frac{1}{2}}\left\{\|w\|_{3,\theta}^2\|w-u\|_{3,\theta} + \|w\|_{3,\theta}^{p}\|w-u\|_{3,\theta} + \|u\|_{3,\theta}^{p-2}\|w\|_{3,\theta}^2\|w-u\|_{3,\theta}\right\},
\end{aligned}$$

$$\begin{aligned}
\|a'(u)[w-u]_x[w+u]_x\|_{L^2(0,\theta;L^2(\mathbb{R}))} \le & \|a'(u)[w-u]_x w_x\|_{L^2(0,\theta;L^2(\mathbb{R}))} \\
& + \|a'(u)[w-u]_x u_x\|_{L^2(0,\theta;L^2(\mathbb{R}))} \\
\le & C\{\|[w-u]_x w_x\|_{L^2(0,\theta;L^2(\mathbb{R}))} + \|[w-u]_x u_x\|_{L^2(0,\theta;L^2(\mathbb{R}))} \\
& + \|u|^{p-1}|[w-u]_x w_x\|_{L^2(0,\theta;L^2(\mathbb{R}))} + \| |u|^{p-1}[w-u]_x u_x\|_{L^2(0,\theta;L^2(\mathbb{R}))}\} \\
\le & C\theta^{\frac{1}{2}}\{\|w\|_{3,\theta}\|w-u\|_{3,\theta} + \|u\|_{3,\theta}\|w-u\|_{3,\theta} \\
& \left. + \|u\|_{3,\theta}^{p-1}\|w\|_{3,\theta}\|w-u\|_{3,\theta} + \|u\|_{3,\theta}^{p}\|w-u\|_{3,\theta}\right\}
\end{aligned}$$

and

$$\begin{aligned}
\|[a(w) - a(u)]w_{xx}\|_{L^2(0,\theta;L^2(\mathbb{R}))} \le C\theta^{\frac{1}{2}}\left\{\|w\|_{3,\theta}\|w-u\|_{3,\theta} + \|u\|_{3,\theta}^{p-1}\|w\|_{3,\theta}\|w-u\|_{3,\theta}\right. \\
\left. + \|w\|_{3,\theta}^{p}\|w-u\|_{3,\theta}\right\}
\end{aligned}$$

$$\|a(u)[w_{xx} - u_{xx}]\|_{L^2(0,\theta;L^2(\mathbb{R}))} \le C\theta^{\frac{1}{2}}\left\{\|w-u\|_{3,\theta} + \|u\|_{3,\theta}^{p}\|w-u\|_{3,\theta}\right\}.$$

The above estimates and (4.84), show that

$$\begin{aligned}
\|[a(w)w_x - a(u)u_x]_x\|_{L^2(0,\theta;L^2(\mathbb{R}))} \le & C\theta^{\frac{1}{2}}\left\{2\|w\|_{3,\theta}^{p} + \|w\|_{3,\theta}^2 + 2\|w\|_{3,\theta} + 2\|u\|_{3,\theta}^{p} + \|u\|_{3,\theta}\right. \\
& \left. + 2\|u\|_{3,\theta}^{p-1}\|w\|_{3,\theta} + \|u\|_{3,\theta}^{p-2}\|w\|_{3,\theta}^2 + 1\right\}\|w-u\|_{3,\theta} \\
\le & K_3\|(w-u,[w-u]_t)\|_{B_{3,\theta}\times B_{0,\theta}},
\end{aligned} \tag{4.85}$$

where $K_3(\theta) = C\theta^{\frac{1}{2}}\left\{7R^p + R^2 + 3R + 1\right\}$. From (4.82), (4.83) and (4.85), we get

$$\|\Gamma(u,u_t) - \Gamma(w,w_t)\|_{B_{3,\theta}\times B_{0,\theta}} \le K_4\|(w-u,[w-u]_t)\|_{B_{3,\theta}\times B_{0,\theta}},$$

where $K_4 = C_T(K_2+K_3) = C_T C\theta^{\frac{1}{2}}\left\{2(p+9)R^p + R^{p-1} + R^2 + 4R + 2\right\} + 2^{\frac{3}{2}}C_T C\theta^{\frac{1}{4}}R$. Note that $K_1 \le K_4$, therefore choosing $\theta > 0$, such that $K_4 < \frac{1}{2}$, it follows that

$$\begin{cases} \|\Gamma(u,u_t)\|_{B_{3,\theta}\times B_{0,\theta}} \le R \\ \|\Gamma(u,u_t) - \Gamma(w,w_t)\|_{B_{3,\theta}\times B_{0,\theta}} \le \frac{1}{2}\|(w-u,[w-u]_t)\|_{B_{3,\theta}\times B_{0,\theta}} \end{cases},$$

for all $(u,u_t),(w,w_t) \in S_{\theta,R} \subset B_{3,\theta}\times B_{0,\theta}$. Hence $\Gamma : S_{\theta,R} \longrightarrow S_{\theta,R}$ is a contraction and, by Banach fixed point theorem, we obtain a unique $(u,u_t) \in S_{\theta,R}$, such that $\Gamma(u,u_t) = (u,u_t)$. Thus, u is a unique local mild solution to problem (4.1) and satisfies

$$\|u\|_{3,\theta} \le 2C_T\|u_0\|_{H^3(\mathbb{R})}. \tag{4.86}$$

Moreover, (4.86) implies the solution does not blow-up in finite time and, by using standard arguments, we can extent θ to $[0,T]$. Finally, the proof is complete defining $\eta_3(s) = 2C_T$. □

4.3 Exponential stability

This section is devoted to prove the exponential decay of the solutions under the assumptions (4.4) and (4.5). We consider two cases: $1 \le p < 2$ and $2 \le p < 5$.

4.3.1 Case $1 \le p < 2$.

In order to make our work self-contained, we prove the following proposition which is simliar to Theorem 5.1 in [23].

Proposition 4.3.1. *Let b satisfying (4.4). Then, for any $u_0 \in L^2(\mathbb{R})$ and $1 \le p < 2$, the corresponding solution u of (4.1) is exponentially stable and it satisfies the decay estimate*

$$\|u(t)\|_2 \le e^{-2\lambda_0 t}\|u_0\|_2, \quad \forall t \ge 0. \tag{4.87}$$

Proof. We first consider $u_0 \in H^3(\mathbb{R})$ and u the corresponding smooth solution. Multiplying the equation in (4.1) by u and integrating in $\mathbb{R}$, we have

$$\frac{d}{dt}\|u(t)\|_2^2 + 2\|u_x(t)\|_2^2 = -2\int_{\mathbb{R}} b(x)|u(x,t)|^2 dx.$$

Hence, proceeding as in [23, Theorem 5.1], we obtain

$$\|u(t)\|_2 \le e^{-2\lambda_0 t}\|u_0\|_2.$$

Now, let $u_0 \in L^2(\mathbb{R})$ and u the corresponding mild solution given by Theorem 4.2.6. Consider $\{u_{n,0}\} \in H^3(\mathbb{R})$, such that

$$u_{n,0} \to u_0 \quad \text{in } L^2(\mathbb{R}).$$

Then, the corresponding strong solutions u_n satisfy the estimate

$$\|u_n(t)\|_2 \le e^{-2\lambda_0 t}\|u_{n,0}\|_2. \tag{4.88}$$

On the other hand, note that the identity (4.17) in Theorem 4.2.6 implies that for all $t \ge 0$

$$u_n \to u \quad \text{in } L^2(\mathbb{R}).$$

Taking the limit in (4.88), we obtain (4.87). □

Corollary 4.3.2. *Let $T > 0$, $u_0 \in L^2(\mathbb{R})$ and b satisfying (4.4). Then there exists a nondecreasing continuous function $\alpha_0 : \mathbb{R}^+ \to \mathbb{R}^+$, such that the corresponding solution u of problem (4.1) with $1 \le p < 2$ satisfies*

$$\|u\|_{0,[t,t+T]} \le \alpha_0(\|u_0\|_2)e^{-2\lambda_0 t}, \quad \forall t \ge 0.$$

Proof. Note that, after a change of variable, the restriction of u to $[t, t+T]$ is a solution of problem (4.1) with respect to the initial data $u(t)$. Then, by Theorem 4.2.11 and Proposition 4.3.1 we have

$$\|u\|_{0,[t,t+T]} \le \beta_0(\|u(t)\|_2)\|u(t)\|_2 \le \beta_0(e^{-2\lambda_0 t}\|u_0\|_2)\|u_0\|_2 e^{-2\lambda_0 t} \le \alpha_0(\|u_0\|_2)e^{-2\lambda_0 t},$$

where $\alpha_0(s) = \beta_0(s)s$. □

The next result was inspired by the ideas introduced in the proof of Theorem 6.1 in [23] and Proposition 3.9 in [81].

Proposition 4.3.3. *Let $T > 0$, $1 \le p < 2$, $a(0) = 0$ and b satisfying (4.4). Then, there exist $\gamma > 0$, $T_0 > 0$ and a nonnegative continuous function $\alpha_3 : \mathbb{R}^+ \to \mathbb{R}^+$, such that, for every $u_0 \in H^3(\mathbb{R})$, the corresponding solution u satisfies*

$$\|u(t)\|_{H^3(\mathbb{R})} \le \alpha_3(\|u_0\|_2, T_0)\|u_0\|_{H^3(\mathbb{R})}e^{-\gamma t}, \qquad \forall t \ge T_0. \tag{4.89}$$

Proof. Let $v = u_t$. Then, by Proposition 4.2.8 v solves linearized equation (4.23) with $v_0 = -\partial_x^3 u_0 + \partial_x^2 u_0 - a(u_0)\partial_x u_0 - bu_0$ and satisfies

$$\|v\|_{0,T} \le \sigma(\|u\|_{0,T})\|v_0\|_2. \tag{4.90}$$

After a change of variable, the restriction of v to [t, t + T] is a solution of problem (4.23) with respect to the initial data $v(t)$ and

$$\|v\|_{0,[t,t+T]} \le \sigma(\|u\|_{0,[t,t+T]})\|v(t)\|_2.$$

Applying Corollary 4.3.2, it follows that

$$\|v\|_{0,[t,t+T]} \le \sigma(\alpha_0(\|u_0\|_2)e^{-2\lambda_0 t})\|v(t)\|_2 \le \sigma(\alpha_0(\|u_0\|_2))\|v(t)\|_2. \tag{4.91}$$

On the other hand, the solution v may be written as

$$v(t) = S(t)v_0 - \int_0^t S(t-s)[a(u(s))v(s)]_x ds$$

where $S(t)$ is a C_0-semigroup of contraction in $L^2(\mathbb{R})$ generated by the operator A_b. Note that $v_1(t) = S(t)v_0$ is solution of the problem (4.23) with $a(u) = 0$. Then, proceeding as in the proof of Proposition 4.3.1, we have

$$\|v_1(t)\|_2 \le \|v_0\|e^{-2\lambda_0 t}, \qquad \forall t \ge 0. \tag{4.92}$$

Let us now denote $v_2(t) = \int_0^t S(t-s)[a(u(s))v(s)]_x ds$. Note that

$$\|v_2(T)\|_2 \le \|a'(u)u_x v\|_{L^1(0,T;L^2(\mathbb{R}))} + \|a(u)v_x\|_{L^1(0,T;L^2(\mathbb{R}))}.$$

Moreover, $a(0) = 0$ implies that $|a(u)| \le C(1 + |u|^{p-1})|u|$, for some $C > 0$. Thus, by using Lemma 4.2.4, the following holds

$$\begin{aligned}\|v_2(T)\|_2 &\le C\left\{\|(1+|u|^{p-1})u_x v\|_{L^1(0,T;L^2(\mathbb{R}))} + \|(1+|u|^{p-1})|u|v_x\|_{L^1(0,T;L^2(\mathbb{R}))}\right\} \\ &\le 2C\left\{2^{\frac{1}{2}}T^{\frac{1}{4}}\|u\|_{0,T}\|v\|_{0,T} + 2^{\frac{p}{2}}T^{\frac{2-p}{4}}\|u\|_{0,T}^p\|v\|_{0,T}\right\}.\end{aligned} \tag{4.93}$$

Using (4.90), (4.92) and (4.93), we obtain a positive constant K_T, such that

$$\|v(T)\|_2 \leq \left(e^{-2\lambda_0 T} + K_T(1 + \|u\|_{0,T}^{p-1})\|u\|_{0,T}\sigma(\|u\|_{0,T})\right)\|v_0\|_2.$$

With the notation introduced above, we consider the sequence $y_n(\cdot) = v(\cdot, nT)$ and introduce $w_n(\cdot, t) = v(\cdot, t + nT)$. For $t \in [0, T]$, w_n solves the problem

$$\begin{cases} \partial_t w_n + \partial_x^3 w_n - \partial_x^2 w_n + [a(u(\cdot + nT))w_n]_x + bw_n = 0 & \text{in } \mathbb{R} \times \mathbb{R}^+ \\ w_n(0) = y_n & \text{in } \mathbb{R}. \end{cases}$$

First, observe that we can obtain for y_n an estimate similar to the one obtained for $v(T)$:

$$\begin{aligned} \|y_{n+1}\|_2 = \|w_n(T)\|_2 &\leq e^{-2\lambda_0 T}\|w_0\|_2 + K_T(1 + \|u(\cdot + nT)\|_{0,T}^{p-1})\|u(\cdot + nT)\|_{0,T}\|w_n\|_{0,T} \\ &\leq \left\{e^{-2\lambda_0 T} + K_T(1 + \|u\|_{0,[nT,(n+1)T]}^{p-1})\|u\|_{0,[nT,(n+1)T]}\sigma(\|u\|_{0,[nT,(n+1)T]})\right\}\|y_n\|_2. \end{aligned} \tag{4.94}$$

On the other hand, we can take $\beta > 0$, small enough, such that

$$e^{-2\lambda_0 T} + K_T(1 + \beta^{p-1})\beta\sigma(\beta) < 1.$$

With this choice of β, Corollary 4.3.2 allows us to choose $N > 0$, large enough, satisfying

$$\|u\|_{0,[nT,(n+1)T]} \leq \alpha_0(\|u_0\|_2)e^{-2\lambda_0 nT} \leq \alpha_0(\|u_0\|_2)e^{-2\lambda_0 NT} \leq \beta, \qquad \forall n > N.$$

Thus, from (4.94) we obtain the following estimate

$$\|y_{n+1}\|_2 \leq r\|y_n\|_2, \qquad \forall n \geq N, \quad \text{where } 0 < r < 1,$$

which implies

$$\|v((n+k)T)\|_2 \leq r^k\|v(nT)\|_2, \qquad \forall n \geq N. \tag{4.95}$$

Let $T_0 = NT$ and $t \geq T_0$. Then, there exists $k \in \mathbb{N}$ and $\theta \in [0, T]$, satisfying

$$t = (N + k)T + \theta.$$

Then, from (4.91) and (4.95), it is found that

$$\begin{aligned} \|v(t)\|_2 \leq \|v\|_{0,[(N+k)T,(N+k+1)T]} &\leq \sigma(\alpha_0(\|u_0\|_2))\|v((N+k)T)\|_2 \\ &\leq \sigma(\alpha_0(\|u_0\|_2))r^{\frac{t-NT-\theta}{T}}\|v(T_0)\|_2 \\ &\leq \sigma(\alpha_0(\|u_0\|_2))r^{\frac{t-NT-\theta}{T}}\sigma(\alpha_0(T_0, \|u_0\|_2))\|v(0)\|_2 \\ &\leq \eta_1(\|u_0\|)e^{-\delta_1 t}\|v_0\|_2, \end{aligned}$$

where $\delta_1 = \frac{1}{T}\ln\left(\frac{1}{r}\right)$ and $\eta_1(s) = \sigma(\alpha_0(s))\sigma(\alpha_0(T_0, s))r^{-(N+1)}$. Invoking the estimate (4.27) in Theorem 4.2.9, and having in mind that $v = u_t$, we get

$$\|u_t(t)\|_2 \leq \eta_2(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})}e^{-\delta_1 t}, \qquad \forall t \geq T_0, \tag{4.96}$$

where $\eta_2(s) = \eta_1(s)C(s)$. On the other hand, note that

$$\|u_{xxx}(t)\|_2 \leq \|u_t(t)\|_2 + \|u_{xx}(t)\|_2 + \|a(u(t))u(t)\|_2 + \|b\|_\infty\|u(t)\|_2. \tag{4.97}$$

Estimating the nonlinear term as in the proof of Lemma 4.2.3,

$$\|a(u(t))u(t)\|_2 = \|u(t)^{p+1}u_x(t)\|_2 \leq \|u(t)^{p+1}\|_\infty\|u_x(t)\|_2 \leq 2^{\frac{p+1}{2}}\|u(t)\|_2^{\frac{p+1}{2}}\|u_x(t)\|_2^{\frac{p+3}{2}},$$

from (4.97) we obtain

$$\|u_{xxx}(t)\|_2 \leq \|u_t(t)\|_2 + \|u_{xx}(t)\|_2 + 2^{\frac{p+1}{2}}\|u(t)\|_2^{\frac{p+1}{2}}\|u_x(t)\|_2^{\frac{p+3}{2}} + \|b\|_\infty\|u(t)\|_2.$$

Using Gagliardo-Nirenberg and Young inequalities, it follows that

$$
\begin{aligned}
\|u_{xxx}(t)\|_2 \leq & \|u_t(t)\|_2 + C\|u_{xxx}(t)\|_2^{\frac{2}{3}}\|u(t)\|_2^{\frac{1}{3}} + 2^{\frac{p+1}{2}}C\|u(t)\|_2^{\frac{5p+9}{6}}\|u_{xxx}(t)\|_2^{\frac{p+3}{6}} \\
& + \|b\|_\infty\|u(t)\|_2 \\
\leq & \|u_t(t)\|_2 + \left(\frac{C}{3\varepsilon} + \|b\|_\infty\right)\|u(t)\|_2 + \frac{2^{\frac{p+1}{2}}(3-p)C}{6\varepsilon}\|u(t)\|_2^{\frac{5p+9}{3-p}} \\
& + \left(\frac{p+7}{6}\right)C\varepsilon\|u_{xxx}(t)\|_2.
\end{aligned}
$$

Choosing $\varepsilon = \frac{3}{C(p+7)}$, we have

$$
\begin{aligned}
\|u_{xxx}(t)\|_2 \leq 2\|u_t(t)\|_2 + 2\left(\frac{C^2(p+7)}{9} + \|b\|_\infty\right)\|u(t)\|_2 \\
+ \frac{2^{\frac{p+1}{2}}(3-p)(p+7)C^2}{18}\|u(t)\|_2^{\frac{5p+9}{3-p}}.
\end{aligned}
$$

Applying Proposition 4.3.1 and estimate (4.96), the following decay estimate holds

$$
\|u_{xxx}(t)\|_2 \leq \eta_3(\|u_0\|)\|u_0\|_{H^3(\mathbb{R})}e^{-\gamma t}, \qquad \forall t \geq T_0, \tag{4.98}
$$

where $\eta_3(s) = 2\eta_2(s) + \frac{2}{9}C^2(p+7) + 2\|b\|_\infty + \frac{1}{3}2^{\frac{p+1}{2}}C^2(3-p)(p+7)s^{\frac{6p-1}{3-p}}$ and $\gamma = \min\{\delta_1, 2\lambda_0\}$. Now, using Gagliardo-Nirenberg and Young inequalities it is easy to obtain

$$
\|u(t)\|_{H^3(\mathbb{R})} \leq C_1\left(\|u(t)\|_2 + \|u_{xxx}(t)\|_2\right).
$$

Finally, by Proposition 4.3.1 and (4.98) we obtain (4.89) with $\alpha_3(s) = C_1\left(1 + \eta_3(s)\right)$. □

Proposition 4.3.1 and 4.3.3, together with Corollary 4.2.12 and interpolation arguments give the main result of this section.

Theorem 4.3.4. *Let $T > 0$, $1 \leq p < 2$, $a(0) = 0$ and b satisfying (4.4). Then, there exist positive constants γ, ε_0 and a continuous nonnegative function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$, such that, for every $u_0 \in H^s(\mathbb{R})$, with $0 \leq s \leq 3$, the corresponding solution u satisfies*

$$
\|u(t)\|_{H^s(\mathbb{R})} \leq \alpha(T_0, \|u_0\|_2)\|u_0\|_{H^s(\mathbb{R})}e^{-\lambda t}, \qquad \forall t \geq T_0. \tag{4.99}
$$

Proof. By Corollary 4.2.12 the corresponding solution u belongs to $B_{0,[\varepsilon,T]}$, for all $\varepsilon \in (0, T]$. In particular, we choose $\varepsilon \leq T_0$, where T_0 is given by Proposition 4.3.3. Then, by using the interpolation inequality (2.43) in [64, pag. 19], we have

$$
\|u(t)\|_{H^s(\mathbb{R})} = \|u(t)\|_{[L^2(\mathbb{R}),H^3(\mathbb{R})]_{2,\frac{s}{3}}} \leq C\|u(t)\|_2^{1-\frac{s}{3}}\|u(t)\|_{H^3(\mathbb{R})}^{\frac{s}{3}}, \quad \forall t \geq \varepsilon.
$$

Finally, Propositions 4.3.1 and 4.3.3 give us that

$$
\|u(t)\|_{H^s(\mathbb{R})} \leq Ce^{-2\left(1-\frac{s}{3}\right)\lambda_0 t}\|u_0\|_2^{\left(1-\frac{s}{3}\right)}\alpha_3^{\frac{s}{3}}(\|u_0\|_2, T_0)e^{-\frac{s}{3}\gamma t}, \qquad \forall t \geq T_0.
$$

Observe that, by construction, $\gamma \leq 2\lambda_0$, therefore we obtain (4.99) with $\alpha(s, T_0) = C\alpha_3^{\frac{s}{3}}(s, T_0)$. □

4.3.2 Case $2 \leq p < 5$.

Along this section we assume that the damping function $b = b(x)$ does not change sign and satisfies (4.5). Under this condition, we prove the exponential decay of the solutions in the L^2-norm by using the so-called compactness-uniqueness argument. The key is to establish the unique continuation property for the solution of the GKdV-B equation. The proof of this unique continuation property is mainly based on a Carleman estimate.

The next Carleman estimate is based on the global Carleman inequality obtained for the KdV equation in [79].

Lemma 4.3.5 (Carleman's estimative). *Let T and L be positive numbers. Then, there exist a smooth positive function ψ on $[-L, L]$ (which depends on L) and positive constants $s_0 = s_0(L, T)$ and $C = C(L, T)$, such that, for all $s \geq s_0$ and any*

$$q \in L^2(0, T; H^3(-L, L)) \cap H^1(0, T; L^2(-L, L)) \tag{4.100}$$

satisfying

$$q(t, \pm L) = q_x(t, \pm L) = q_{xx}(t, \pm L) = 0, \text{ for } 0 \leq t \leq T, \tag{4.101}$$

we have

$$\int_0^T \int_{-L}^L \left\{ \frac{s^5}{t^5(T-t)^5}|q|^2 + \frac{s^3}{t^3(T-t)^3}|q_x|^2 + \frac{s}{t(T-t)}|q_{xx}|^2 \right\} e^{-\frac{2s\psi(x)}{t(T-t)}} dxdt$$
$$\leq C \int_0^T \int_{-L}^L |q_t - q_{xx} + q_{xxx}|^2 e^{-\frac{2s\psi(x)}{t(T-t)}} dxdt.$$

It is well known that the second order term $-q_{xx}$ and the first order term q_x can be absorbed by choosing s large enough and increasing the constant C in the Carleman estimate in [79]. However, we present a proof of the Carleman estimate in the Lemma 4.3.5 in the Appendix.

Lemma 4.3.6 (Unique continuation property). *Let T be a positive number. If $u \in L^\infty(0, T; H^1(\mathbb{R}))$ solves*

$$\begin{cases} u_t - u_{xx} + u_{xxx} + a(u)u_x = 0 & \text{in } \mathbb{R} \times (0, T) \\ u \equiv 0 & \text{in } (-\infty, -L) \cup (L, \infty) \times (0, T), \end{cases}$$

for some $L > 0$, with $a \in C(\mathbb{R})$ satisfying (4.3), then $u \equiv 0$ in $\mathbb{R} \times (0, T)$.

Proof. For $h > 0$, consider

$$u^h(x, t) = \frac{1}{h} \int_t^{t+h} u(x, s) ds.$$

Then, $u^h \in W^{1,\infty}(0, T', H^1(\mathbb{R}))$ and

$$u^h \to u \quad \text{in } L^\infty(0, T'; H^1(\mathbb{R})), \tag{4.102}$$

for any $T' < T$. Moreover, u^h solves

$$\begin{cases} u_t^h - u_{xx}^h + u_{xxx}^h + (a(u)u_x)^h = 0 & \text{in } \mathbb{R} \times (0, T') \\ u^h \equiv 0 & \text{in } (-\infty, -L) \cup (L, \infty) \times (0, T). \end{cases} \tag{4.103}$$

On the other hand, note that $u \in L^\infty(0, T, H^1(\mathbb{R}))$ implies $a(u)u_x \in L^\infty(0, T, L^2(\mathbb{R}))$. Indeed, since

$$\|a(u)u_x\|_{L^\infty(0,T,L^2(\mathbb{R}))} \leq C \left\{ \|u\|_{L^\infty(0,T,H^1(\mathbb{R}))} + \|u\|^p_{L^\infty(0,T,L^\infty(\mathbb{R}))} \|u\|_{L^\infty(0,T,H^1(\mathbb{R}))} \right\},$$

$(a(u)u_x)^h \in L^\infty(0,T,L^2(\mathbb{R}))$. Then, proceeding as in the proof of Theorem 4.2.9, we have

$$u^h \in L^\infty(0,T',H_0^3(-L,L)) \cap H^1(0,T',L^2(-L,L)).$$

Invoking the Lemma 4.3.5 we obtain $C, s_0 > 0$ and a positive function ψ, such that

$$\int_Q \left\{ \frac{s^5|u^h|^2}{t^5(T-t)^5} + \frac{s^3|u_x^h|^2}{t^3(T-t)^3} + \frac{s|u_{xx}^h|^2}{t(T-t)} \right\} e^{-\frac{2s\psi(x)}{t(T-t)}} dxdt$$
$$\leq C \int_Q |u_t^h - u_{xx}^h + u_{xxx}^h|^2 e^{-\frac{2s\psi(x)}{t(T-t)}} dxdt,$$

for all $s > s_0$ and $Q = (0,T') \times (-L,L)$. By (4.103),

$$\int_Q |u_t^h - u_{xx}^h + u_{xxx}^h|^2 e^{-\frac{2s\psi(x)}{t(T-t)}} dxdt = \int_Q |(a(u)u_x)^h|^2 e^{-\frac{2s\psi(x)}{t(T-t)}} dxdt$$
$$\leq \int_Q |a(u)u_x^h|^2 e^{-\frac{2s\psi(x)}{t(T-t)}} dxdt + \int_Q |(a(u)u_x)^h - a(u)u_x^h|^2 dxdt$$
$$\leq \|a(u)\|^2_{L^\infty(Q)} \int_Q |u_x^h|^2 e^{-\frac{2s\psi(x)}{t(T-t)}} dxdt + \|(a(u)u_x)^h - a(u)u_x^h\|^2_{L^2(Q)}.$$

Hence,

$$0 < \int_Q \left\{ \frac{s^5}{t^5(T-t)^5}|u^h|^2 + \left(\frac{s^3}{t^3(T-t)^3} - C\|a(u)\|^2_{L^\infty(Q)} \right) |u_x^h|^2 \right.$$
$$\left. + \frac{s}{t(T-t)}|u_{xx}^h|^2 \right\} e^{-\frac{2s\psi(x)}{t(T-t)}} dxdt \leq C\|(a(u)u_x)^h - a(u)u_x^h\|^2_{L^2(Q)}, \quad (4.104)$$

since, for s large enough, we obtain $\frac{s^3}{t^3(T-t)^3} - C\|a(u)\|^2_{L^\infty(Q)} > 0$.

Note that (4.102) guarantees that $a(u)u_x^h \to a(u)u_x$ in $L^2(0,T;L^2(-L.L))$, since $a(u) \in L^\infty(0,T',L^\infty(-L,L))$. Moreover, as $a(u)u_x \in L^2(0,T',L^2(-L,L))$ we have that $(a(u)u_x)^h \in W^{1,\infty}(0,T',L^2(-L,L))$ and $(a(u)u_x)^h \to a(u)u_x \in L^2(0,T',L^2(-L,L))$. Thus, passing to the limit in (4.104), we obtain that $u \equiv 0$ in $(-L,L) \times (0,T')$. Using (4.103) and since T' may be taken arbitrarily close to T, we have $u \equiv 0$ in $\mathbb{R} \times (0,T)$. □

Now we show that any weak solution of (4.1) decays exponentially to zero in the space $L^2(\mathbb{R})$.

Theorem 4.3.7. *Let a be a $C^2(\mathbb{R})$ function satisfying (4.44) with $1 \leq p < 5$ and b satisfying (4.5). Then, the system (4.1) is semiglobally uniformly exponentially stable in $L^2(\mathbb{R})$, i.e, for any $r > 0$ there exist two constants $C > 0$ and $\eta = \eta(r) > 0$, such that, for any $u_0 \in L^2(\mathbb{R})$, with $\|u_0\|_{L^2(\mathbb{R})} < r$, and any weak solution u of (4.1),*

$$\|u(t)\|_{L^2(\mathbb{R})} \leq C\|u_0\|_{L^2(\mathbb{R})}e^{-\eta t}, \quad t \geq 0.$$

Proof. First, note that the corresponding solution u of (4.1) satisfies the following estimate

$$\|u(t)\|^2_{L^2(\mathbb{R})} + 2\|u_x\|^2_{L^2(0,t;L^2(\mathbb{R}))} + 2\int_0^t \int_{\mathbb{R}} b(x)|u(x,\tau)|^2 dxd\tau = \|u_0\|_2^2, \quad (4.105)$$

On the other hand, multiplying the equation in (4.1) by $(T-t)u$ and integrating on $\mathbb{R} \times [0,T]$, we obtain

$$\frac{T}{2}\|u_0\|_2^2 = \frac{1}{2}\|u\|^2_{L^2(0,T;L^2(\mathbb{R}))} + \int_0^T \int_{\mathbb{R}} (T-t)|u_x(x,t)|^2 dxdt + \int_0^T \int_{\mathbb{R}} (T-t)b(x)|u(x,t)|^2 dxdt, \quad (4.106)$$

which implies that

$$\|u_0\|_2^2 \leq \frac{1}{T}\|u\|^2_{L^2(0,T;L^2(\mathbb{R}))} + 2\|u_x\|^2_{L^2(0,T;L^2(\mathbb{R}))} + 2\int_0^T \int_{\mathbb{R}} b(x)|u(x,t)|^2 dxdt. \quad (4.107)$$

Claim 4.3.8. *For any $T>0$ and $r>0$ there exist $C=C(r,T)$, such that, for any weak solution u of (4.1) with $\|u_0\|_2 \le r$, the following estimate holds:*

$$\int_0^T \int_\alpha^\beta |u(x,t)|^2 dxdt \le C\left(\|u_x\|^2_{L^2(0,T;L^2(\mathbb{R}))} + \int_0^T \int_{\mathbb{R}} b(x)|u(x,t)|^2 dxdt\right). \tag{4.108}$$

Proof. We argue by contradiction and suppose that (4.108) does not hold. Hence, there exist a sequence $\{u_n\}$ of weak solution in $C_w([0,T];L^2(\mathbb{R}))\cap L^2(0,T;H^1(\mathbb{R}))$ satisfying

$$\|u_n(0)\|_2 \le r$$

and such that

$$\lim_{n\to\infty} \frac{\|u_n\|^2_{L^2(0,T;L^2(\alpha,\beta))}}{\|\partial_x u_n\|^2_{L^2(0,T;L^2(\mathbb{R}))} + \int_0^T \int_{\mathbb{R}} b(x)|u_n|^2 dxd} = +\infty. \tag{4.109}$$

Define

$$\lambda_n := \|u_n\|_{L^2(0,T;L^2(\alpha,\beta))} \quad \text{and} \quad v_n(x,t) := \frac{u_n(x,t)}{\lambda_n}.$$

Then, v_n satisfies

$$\|v_n\|_{L^2(0,T;L^2(\alpha,\beta))} = 1, \quad \forall n\in\mathbb{N} \tag{4.110}$$

and it is a weak solution of

$$\begin{cases} \partial_t v_n + \partial_x^3 v_n - \partial_x^2 v_n + a(\lambda_n v_n)\partial_x v_n + b v_n = 0 \\ v_n(x,0) = \frac{u_n(x,0)}{\lambda_n}. \end{cases}$$

Moreover, from (4.106), we get

$$\lambda_n := \|u_n\|_{L^2(0,T;L^2(\alpha,\beta))} \le T^{\frac{1}{2}}\|u_n(0)\|_2 \le T^{\frac{1}{2}} r, \tag{4.111}$$

and (4.109) implies that

$$\lim_{n\to\infty} \|\partial_x v_n\|^2_{L^2(0,T;L^2(\mathbb{R}))} = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_0^T \int_{\mathbb{R}} b(x)|v_n|^2 dxdt = 0. \tag{4.112}$$

Furthermore, by (4.111) we obtain a subsequence, denoted by the same index n, and $\lambda \ge 0$, such that

$$\lambda_n \to \lambda.$$

On the other hand, note that

$$|a(\lambda_n \mu)| \le C'(1+|\mu|^p)$$

and $v_n(x,0)$ is bounded in $L^2(\mathbb{R})$. In fact, by (4.107) and (4.112) we obtain

$$\|v_n(0)\|_2^2 \le \frac{1}{T} + \frac{2}{\lambda_n^2}\left\{\|\partial_x u_n\|^2_{L^2(0,T;L^2(\mathbb{R}))} + \int_0^T \int_{\mathbb{R}} b(x)|u_n(x,t)|^2 dxdt\right\}. \tag{4.113}$$

Combining (4.112), (4.113) and (4.105) we conclude that $\{v_n\}$ is bounded in the space $L^\infty(0,T;L^2(\mathbb{R}))\cap L^2(0,T;H^1(\mathbb{R}))$. Hence, extracting a subsequence if needed, we have

$$v_n \rightharpoonup v \quad \text{in } L^\infty([0,T];L^2(\mathbb{R})) \text{ weak } *,$$

$$v_n \rightharpoonup v \quad \text{in } L^2([0,T];H^1(\mathbb{R})) \text{ weak},$$

as $n \to \infty$. In order to analyze the nonlinear term, we consider the function

$$A(v) := \int_0^v a(\lambda u)du, \qquad A_n(v) := \int_0^v a(\lambda_n u)du.$$

Proceeding as in the proof of Theorem 4.2.14, it is easy to see that $a(\lambda_n v_n)\partial_x v_n = \partial_x[A_n(v_n)]$ is bounded in $L^\alpha([0,T]; H^{-2}_{loc}(\mathbb{R}))$, for $\alpha \in \left(1, \frac{6}{p+1}\right)$, and $\partial_t v_n = -\partial_x^3 v_n + \partial_x^2 v_n - a(\lambda_n v_n)\partial_x v_n - bv_n$ is bounded in $L^\alpha([0,T]; H^{-2}_{loc}(\mathbb{R})) \hookrightarrow L^1(0,T; H^{-2}_{loc}(\mathbb{R}))$. Since $\{v_n\}$ is bounded in $L^2([0,T]; H^1(\mathbb{R}))$ and using Aubin-Lions Theorem (see [64]), we obtain a subsequence, such that

$$v_n \to v \text{ strong in } L^2((\alpha, \beta) \times (0,T)). \tag{4.114}$$

By (4.112), it follows that $v_n \to 0$ strong in $L^2((\mathbb{R} \setminus (\alpha,\beta)) \times (0,T))$. Therefore,

$$v_n \to v \text{ strong in } L^2(\mathbb{R} \times (0,T)), \tag{4.115}$$

with

$$v \equiv 0 \text{ on } \omega \times [0,T], \text{ where } \omega = \mathbb{R} \setminus (\alpha, \beta). \tag{4.116}$$

and

$$a(\lambda_n v_n)\partial_x v_n \longrightarrow a(\lambda v)\partial_x v \text{ in } D'(\mathbb{R} \times [0,T]).$$

Thus, v solves

$$v_t + v_{xxx} - v_{xx} + a(\lambda v)v_x + bv = 0, \quad \text{in } D'([0,T] \times \mathbb{R})$$

and from (4.110) abd (4.114) - (4.116), it follows that

$$\|v\|_{L^2(0,T;L^2(\mathbb{R}))} = 1. \tag{4.117}$$

Claim 4.3.9. *Let $0 < t_1 < t_2 < T$. Then, there exists $(t_1', t_2') \subset (t_1, t_2)$, such that $v \in L^\infty(t_1', t_2'; H^1(\mathbb{R}))$*

Proof. Let w_n be solution of

$$\begin{cases} \partial_t w_n - \partial_x^2 w_n + \partial_x^3 w_n + a_n(\lambda_n w_n)\partial_x w_n = 0 & \text{in } \mathbb{R} \times (0,T) \\ w_n(x,0) = v_n(x,0) & \text{in } \mathbb{R} \end{cases}$$

where $a_n \in C_0^\infty(\mathbb{R})$ satisfies (4.45) and (4.46). Proceeding as in the proof of the Theorem 4.2.14, we have that

$$w_n - v_n \to 0 \text{ in } C([0,T]; H^{-1}_{loc}(\mathbb{R})) \quad \text{and} \quad \|w_n\|_{L^2(0,T;H^1(\mathbb{R}))} \leq C. \tag{4.118}$$

Consider $\tau_n \in \left(t_1, \frac{t_1+t_2}{2}\right)$, such that

$$\tau_n \to \tau \text{ and } \|w_n(\tau_n)\|_{L^2(0,T;H^1(\mathbb{R}))} \leq C.$$

Hence, by Theorem 4.2.18,

$$\|w_n(\tau_n + \cdot)\|_{L^2(0,\varepsilon;H^1(\mathbb{R}))} \leq C, \tag{4.119}$$

for any $\varepsilon \leq T$. On the other hand, note that (4.118) implies that

$$w_n(\tau_n + \cdot) \to v(\tau + \cdot) \quad \text{in } C([0,\varepsilon]; H^{-1}_{loc}(\mathbb{R})) \tag{4.120}$$

for $\varepsilon < \frac{t_2 - t_1}{2}$. Thus by (4.119) and (4.120), $v \in L^\infty(\tau, \tau+\varepsilon; H^1(\mathbb{R}))$. □

Applying the claim above and Lemma 4.3.6, we deduce that $v = 0$ in $\mathbb{R} \times (t_1', t_2')$, where $(t_1', t_2') \subset (t_1, t_2)$. As t_2 can be arbitrary close to t_1, we obtain by continuity of v in $H^{-1}_{loc}(\mathbb{R})$ that $v \equiv 0$, which contradicts (4.117). □

Returning to the proof of the Theorem, note that (4.105) implies that

$$\|u(T)\|^2_{L^2(\mathbb{R})} + 2\lambda_0 \int_0^T \int_{\mathbb{R}\setminus(\alpha,\beta)} |u(x,t)|^2 dxdt \le \|u_0\|^2_{L^2(\mathbb{R})}.$$

It follows that

$$\frac{1}{2\lambda_0}\|u(T)\|^2_{L^2(\mathbb{R})} + \int_0^T \int_{\mathbb{R}} |u(x,t)|^2 dxdt \le \frac{1}{2\lambda_0}\|u_0\|^2_{L^2(\mathbb{R})} + \int_0^T \int_\alpha^\beta |u(x,t)|^2 dxdt.$$

By claim 4.3.8 and the monotonicity of $\|u(\cdot,t)\|^2_{L^2(\mathbb{R})}$, the following estimate holds

$$\left(\frac{1}{2\lambda_0} + T\right)\|u(T)\|^2_{L^2(\mathbb{R})} \le \frac{1}{2\lambda_0}\|u_0\|^2_{L^2(\mathbb{R})} + C(r,T)\left(\|u_x\|^2_{L^2(0,T;L^2(\mathbb{R}))} + \int_0^T \int_{\mathbb{R}} b(x)|u(x,t)|^2 dxdt\right),$$

by (4.105), it yields that

$$\left(\frac{1}{2\lambda_0} + T + \frac{C(r,T)}{2}\right)\|u(T)\|^2_{L^2(\mathbb{R})} \le \left(\frac{1}{2\lambda_0} + \frac{C(r,T)}{2}\right)\|u_0\|^2_{L^2(\mathbb{R})}$$

that is,

$$\|u(T)\|^2_{L^2(\mathbb{R})} \le \gamma \|u_0\|^2_{L^2(\mathbb{R})}, \quad \text{with } 0 < \gamma < 1.$$

Consequently,

$$\|u(kT)\|^2_{L^2(\mathbb{R})} \le \gamma^k \|u_0\|^2_{L^2(\mathbb{R})}, \quad \forall k \ge 0.$$

Moreover, for any $t \ge 0$, there exist $k > 0$, such that $kT \le t < (k+1)T$. Thus,

$$\begin{aligned}\|u(t)\|^2_{L^2(\mathbb{R})} &\le \|u(kT)\|^2_{L^2(\mathbb{R})} \le \gamma^k \|u_0\|^2_{L^2(\mathbb{R})} \\ &\le \gamma^{\frac{t}{T}}\gamma^{-1}\|u_0\|^2_{L^2(\mathbb{R})} \\ &\le \gamma^{-1}\|u_0\|^2_{L^2(\mathbb{R})} e^{-\eta t},\end{aligned}$$

where $\eta = -\frac{ln\gamma}{T} > 0$. □

The next result asserts that the system (4.1) is globally uniformly exponentially stable in $L^2(\mathbb{R})$. It means that the constant η in Proposition 4.3.7 is independent of r, when $\|u_0\|_{L^2(\mathbb{R})} \le r$.

Theorem 4.3.10. *Let a be a $C^2(\mathbb{R})$ function satisfying (4.3), with $1 \le p < 5$, and b satisfying (4.5). Then, the system (4.1) is globally uniformly exponentially stable in $L^2(\mathbb{R})$, i.e, there exist a positive constant η and a nonegative continuous function $\alpha : \mathbb{R} \to \mathbb{R}$, such that, for any $u_0 \in L^2(\mathbb{R})$ with $\|u_0\|_{L^2(\mathbb{R})} < r$ and any weak solution u of (4.1),*

$$\|u(t)\|_{L^2(\mathbb{R})} \le \alpha(\|u_0\|_{L^2(\mathbb{R})}) e^{-\eta t}, \quad t \ge \varepsilon.$$

for all $\varepsilon > 0$, where $\eta = \eta(\varepsilon)$.

Proof. By Theorem 4.3.7, there exist $\eta' = \eta'(\varepsilon) > 0$ and $C = C(\varepsilon) > 0$, such that

$$\|u(t)\|_{L^2(\mathbb{R})} \leq C\|u(\varepsilon)\|_{L^2(\mathbb{R})} e^{-\eta' t}, \quad t \geq \varepsilon.$$

If $\|u_0\|_{L^2(\mathbb{R})} \leq r$, again, by Proposition 4.3.7 there exist $C_r > 0$ and $\eta_r > 0$, satisfying

$$\|u(t)\|_{L^2(\mathbb{R})} \leq C_r\|u_0\|_{L^2(\mathbb{R})} e^{-\eta_r t}, \quad t \geq 0.$$

Thus, for all $t \geq \varepsilon$, we have

$$\begin{aligned}
\|u(t)\|_{L^2(\mathbb{R})} &\leq C\|u(\varepsilon)\|_{L^2(\mathbb{R})} e^{-\eta' t} \\
&\leq CC_r\|u_0\|_{L^2(\mathbb{R})} e^{-\eta_r \varepsilon} e^{-\eta' t} \\
&\leq \alpha(\|u_0\|_{L^2(\mathbb{R})}) e^{-\eta' t},
\end{aligned}$$

where $\alpha(s) = CC_r e^{-\eta_r \varepsilon} s$. □

CHAPTER

FIVE

CONTROLLABILITY ASPECTS OF THE KORTEWEG-DE VRIES BURGERS EQUATION ON UNBOUNDED DOMAINS

The aim of this work is to consider the controllability problem of the linear Korteweg-de Vries Burgers equation posed in the whole space. Following the ideas contained in [79], we obtain a sort of exact controllability for solutions in $L^2_{loc}(\mathbb{R}^2)$ by deriving an internal observability inequality. To establish the result, we combine a Carlemann estimate and an approximation theorem.

5.1 Introduction

In this chapter, we are concerned with the study of controllability properties of the linear Korteweg-de Vries Burgers (KdVB) equation:

$$\begin{cases} u_t - u_{xx} + u_{xxx} = 0 & \text{in } \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}. \end{cases} \tag{5.1}$$

The KdV-Burgers equation was derived by Su and Gardner [93] for a wide class of nonlinear system in the weak nonlinearity and long wavelength approximation. This equation has been obtained when including electron inertia effects in the description of weak nonlinear plasma waves [49]. The KdV-Burgers equation has also been used in a study of wave propagation through liquid field elastic tube [52] and for a description of shallow water waves on viscous fluid. The equation (5.1) can be thought of as a composition of the KdV and Burgers equation, involving dispersion and dissipation effects.

Our main purpose is to address the controllability of (5.1) in $L^2(\mathbb{R})$. The analysis developed here was inspired by the results obtained by Rosier in [79] for the linear Korteweg-de Vries equation in the half-space, which claims that the linear KdV is indeed exactly boundary controllable in $\Omega = (0,\infty)$ provided that the solutions are not required to be in $L^\infty(0,T;L^2(\Omega))$:

Theorem (Rosier [79, Theorem 1.3]) *Let T, ε, b be positive numbers, with $\varepsilon < T$. Let $L^2(\Omega, e^{-2bx}dx)$ denote the space of (class of) measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_0^\infty u^2(x)e^{-2bx}dx < \infty$. Let $u_0 \in L^2(\Omega)$ and $u_T \in L^2(\Omega, e^{-2bx}dx)$. Then, there exists a function*

$$u \in L^2_{loc}([0,T] \times [0,\infty]) \cap C([0,\varepsilon], L^2(\Omega)) \cap C([T-\varepsilon,T], L^2(\Omega, e^{-2bx}dx))$$

fulfilling

$$\begin{cases} u_t + u_x + u_{xxx} & = 0 \quad \text{in } \mathcal{D}'(\Omega \times (0,T)) \\ u|_{t=0} & = u_0, \\ u|_{t=T} & = u_T. \end{cases}$$

In the above Theorem, note that u is locally square integrable. Actually, for a certain function u_0 in $L^2(0,\infty)$ and $u_T = 0$ a trajectory u as above cannot be found in $L^\infty(0,T,L^2(0,\infty))$ (see [79, Theorem 1.2]). It means that the bad behavior of the trajectories as $x \to \infty$ is the price to be paid for getting the exact controllability in the half space Ω. The same kind of results occurs for the heat and Schrodinger equation.

Our results have affinities with the work of L. Rosier [79] and we adapt his ideas to prove the main result:

Theorem 5.1.1. *Let $\{S(t)\}_{t\geq 0}$ denote the continuous semigroup on $L^2(\mathbb{R})$ generated by the differential operator $A = \partial_x^2 - \partial_x^3$ with domain $H^3(\mathbb{R})$. Let T, ε positive numbers, such that $\varepsilon < \frac{T}{2}$. Let $u_0, u_T \in L^2(\mathbb{R})$. Then, there exists a function*

$$u \in L^2_{loc}(\mathbb{R}^2) \cap C([0,\varepsilon] \cup [T-\varepsilon, T], L^2(\mathbb{R}))$$

which solves

$$\begin{cases} u_t - u_{xx} + u_{xxx} = 0 & \text{in } \mathcal{D}'(\mathbb{R} \times (0,T)) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}, \\ u(x,T) = S(T)u_T(x) & \text{in } \mathbb{R}. \end{cases} \tag{5.2}$$

The proof of the main result combines Fursikov-Imanuvilov's approach [41] for the boundary controllability of the Burgers equation on bounded domains, which is based on a global Carleman estimate. In order to obtain the extension to some unbounded domain, we follow Rosays clever proof of Malgrange-Ehrenpreiss theorem [83], which uses an approximation theorem. The proof of the approximation theorem is based on two technical results, namely, Proposition 5.2.3 and Lemma A.3.3. The Proposition 5.2.3 refers to an observability inequality, which differs from the case of the KdV, was proved by using an internal Carleman estimate. The final step is a standard Mittag-Lefflers procedure. As in the case of the KdV, the nonlinear problem for KdV-Burgers is open.

Since the semigroup $S(\cdot)$ associated to KdV-Burgers is not a group in $L^2(\mathbb{R})$, the proof of the Theorem 5.1.1 does not give us the exact controllability directly in the whole space, i.e, we have that the solution u of the Cauchy problem (5.1), satisfies $u(T) = S(T)u_T$, for any u_0 and u_T in $L^2(\mathbb{R})$. However, with some minor modifications in the proof of Theorem 5.1.1 as in [79], we obtain a exact controllability result in the half-space, provided that $u_0 \in L^2(0,+\infty)$ and $u_T \in L^2((0,+\infty), e^{-2bx}dx)$, namely:

Theorem 5.1.2. *Let T, ε, b be positive numbers, with $\varepsilon < T$. Let $u_0 \in L^2((0,+\infty))$ and $u_T \in L^2((0,+\infty), e^{-2bx}dx)$. Then there exists a function*

$$u \in L^2_{loc}([0,T] \times (0,+\infty)) \cap C([0,\varepsilon], L^2((0,+\infty))) \cap C([T-\varepsilon, T], L^2((0,+\infty), e^{-2bx}dx))$$

fulfilling

$$\begin{cases} u_t - u_{xx} + u_{xxx} & = 0 \quad \text{in } \mathcal{D}'((0,+\infty) \times (0,T)) \\ u|_{t=0} & = u_0, \\ u|_{t=T} & = u_T. \end{cases}$$

The major difference with Rosier work is the internal observability. The techniques used to prove the Proposition 5.2.3 are different from those used in the proof of the observability inequality for the KdV equation. More precisely, we developed a Carleman inequality which allows us to prove directly the observability as in [20] and [46]. It seems difficult to use the *compactness-uniqueness argument* and *the Ingham's inequality approach* used by Rosier in [78,79], due to the lack of $L^2(\mathbb{R})$-estimates and the differential operator nature associated to KdV-Burger equation, respectively.

An important remark is related to the approximate controllability for Pde's in $L^2(\Omega)$, when Ω is an unbounded domain for dispersive models. In this case, the approximate controllability problem has a positive answer. The (simple) proof of the next Proposition can be found in the appendix of [79].

Proposition 5.1.3. *Consider a (real) constant coefficients differential operator* $Au = \sum_{i=0}^{n} a_i \frac{d^i u}{dx^i}$, *with domain* $D(A) = \{u \in L^2(\mathbb{R}) : Au \in L^2(\mathbb{R})\}$. *Assume that* $n \geq 2$ *(with* $a_n \neq 0$*) and that* A *generates a continuous semigroup* $\{S(t)\}_{t\geq 0}$ *on* $L^2(\mathbb{R})$. *Let* $T > 0$ *and* $L_1 < L_2$ *be some numbers. Set*

$$\mathcal{R} = \left\{ \int_0^T S(T-t) f(\cdot, t) dt; f \in L^2(\mathbb{R}^2), \operatorname{supp} f \subset [L_1, L_2] \times [0, T] \right\},$$

where $\operatorname{supp} f$ *denotes the support of* f. *Then* $\mathcal{R}$ *is a strict dense subspace of* $L^2(\mathbb{R})$.

As far as we known there exist a few works about controllability in unbounded domain for dispersive equations [79]. In particular, to heat equation, we can cite some recent results, (see, for instance, [16], [47], [71] and [73]).

Finally, let us mention that the exact boundary controllability of the linear KdV in $L^2(0, +\infty)$ fails to be true if we restrict ourselves to solutions with bounded energy, that is, which belong to $L^\infty(0, T, L^2(0, +\infty))$. An implicit formulation (that is, without specification of the boundary conditions) of this fact is given in [79, Theorem 1.2], which shows that even the (boundary) null-controllability fails to be true for solutions with bounded energy. This phenomenon is unknown for the linear KdV-Burgers equation (5.1). Furthermore, like for the KdV equation, the nonlinear case remains a open problem.

The chapter is organized in the following way:

- In section 5.2, we present an internal observability inequality for an appropriate initial value problem of the KdV-B via Carleman estimates.

- Section 5.3 is devoted to the proof of the Theorem 5.1.1.

- Finally, in the Appendix A.3, we prove some results we have used in the proof of Theorem 5.1.1.

5.2 Internal Observability

In this section, we follow the same approach as in [20] to prove a observability inequality for the linear Kdv-Burgers equation posed in a bounded domain. Consider the differential operator

$$A =: \partial_{xx} - \partial_{xxx}, \quad D(A) := \{u \in H^3(-L, L) : u(-L) = u(L) = u_x(L) = 0\}. \tag{5.3}$$

Proposition 5.2.1. *The operator* A *and its adjoint* A^* *are dissipative in* $L^2(-L, L)$.

Proof. It is easy to see that A^* is given by

$$A^* = \partial_{xx} + \partial_{xxx}, \quad D(A^*) := \left\{\varphi \in H^3(-L,L) : \varphi(-L) = \varphi(L) = \varphi_x(-L) = 0\right\}. \quad (5.4)$$

Let $u \in D(A)$, hence,

$$\begin{aligned}(Au,u)_{L^2} &= \int_{-L}^{L} u_{xx}u dx - \int_{-L}^{L} u_{xxx}u dx = -\int_{-L}^{L} u_x^2 dx + \int_{-L}^{L} u_{xx}u_x dx \\ &= -\int_{-L}^{L} u_x^2 dx - \frac{1}{2}u_x^2(-L) \leq 0.\end{aligned}$$

Then, A is a dissipative operator in $L^2(-L,L)$. Analogously, A^* is also a dissipative operator in $L^2(-L,L)$. □

The above proposition together with the density property of the domains $D(A)$ and $D(A^*)$ in $L^2(-L,L)$ and the closeness of the operator A ($A = A^{**}$), allow us to conclude that A generates a C_0 semigroup of contractions $\{S_L(t)\}_{t\geq 0}$ on $L^2(-L,L)$ (See [77]) which be denoted by $S_L(\cdot)$. Classical existence results then give us the global well-posedness in the space $L^2(-L,L)$.

Theorem 5.2.2. *Let $u_0 \in L^2(-L,L)$ and consider the initial boundary value problem*

$$\begin{cases} u_t = Au & \text{in } (0,T)\times(-L,L), \\ u(0,x) = u_0(x) & \text{in } (-L,L). \end{cases} \quad (5.5)$$

Then, there exists a unique (weak) solution $u = S_L(\cdot)u_0$ of (5.5) such that

$$u \in C\left([0,T]; L^2(-L,L)\right) \cap H^1\left(0,T; (H^{-2}(0,L))^2\right).$$

Moreover, if $u_0 \in D(A)$ then (5.5) has a unique (classical) solution u such that

$$u \in C([0,T]; D(A)) \cap C^1((0,T); L^2(-L,L)).$$

In general, the following observability inequality plays a fundamental role for the study of the controllability properties. In this case, it will be used to prove an approximation theorem stated in the next section.

Proposition 5.2.3. *Let l, L, T be positive numbers such that $l < L$. Then there exists a constant $C > 0$ such that, for every $u_0 \in L^2(-L,L)$, the solution of (5.5) satisfies*

$$\|u\|_{L^2(0,T,L^2(-L,L))} \leq C\|u\|_{L^2(0,T,L^2(\omega))}, \quad \omega = (-l,l). \quad (5.6)$$

for some $C > 0$.

The proof of the Proposition 5.2.3 was motivated by the works [20] and [46]. Following the methods developed in above papers, we prove the internal observability (5.6) by using a Carleman estimate. Before to present the proof of Proposition 5.2.3, we establish some preliminary results.

Carleman Estimate for the KdV-Burgers equation

In order to prove the internal observability for the KdV-Burgers equation, we follow closely the ideas present in [20]. In such work, the authors establishes an internal Carlmenan estimate for the non-homogeneous system:

$$\begin{cases} q_t + q_{xxx} = f & \text{in } (0,L)\times(0,T), \\ q(0,t) = q(L,t) = q_x(L,t) = 0 & \text{in } (0,T), \\ q(x,0) = q_0(x) & \text{in } (0,L). \end{cases} \quad (5.7)$$

where $f \in L^2(0,T;L^2(0,L))$. Note that a priori, the solution q of (5.7) does not have regularity enough to apply the Carleman estimate present in [20, Proposition 3.1] with $f = q_{xx}$. Hence, to get the desired Carleman estimate, we assume that $\omega = (l_1, l_2)$ with $-L < l_1 < l_2 < L$ and pick any function $\psi \in C^3([-L,L])$ with

$$\psi > 0 \text{ in } [-L,L], \tag{5.8}$$

$$|\psi'| > 0, \psi'' < 0, \text{ and } \psi'\psi''' < 0 \text{ in } [-L,L] \setminus \omega, \tag{5.9}$$

$$\psi'(-L) < 0 \text{ and } \psi'(L) > 0, \tag{5.10}$$

$$\min_{x\in[l_1,l_2]} \psi(x) = \psi(l_3) < \max_{x\in[l_1,l_2]} \psi(x) = \psi(l_2) = \psi(l_3), \qquad \max_{x\in[-L,L]} \psi(x) = \psi(-L) = \psi(L), \tag{5.11}$$

$$\psi(-L) \le \frac{4}{3}\psi(l_3). \tag{5.12}$$

The existence of such a function is guaranteed in [20].

Lemma 5.2.4 (**Carleman's inequality**). *Let $T > 0$. Then, there exist positive constants $s_0 = s_0(T,\omega)$ and $C = C(T,\omega)$, such that, for all $s \ge s_0$ and any $u_0 \in L^2(-L,L)$, the solution u of (5.5) fulfills*

$$\begin{aligned}
\int_0^T \int_{-L}^L &\left\{ \frac{s^5\psi^5}{t^5(T-t)^5}|u|^2 + \frac{s^3\psi^3}{t^3(T-t)^3}|u_x|^2 + \frac{s\psi}{t(T-t)}|u_{xx}|^2 \right\} e^{-\frac{2s\psi(x)}{t(T-t)}}\,dxdt \\
&+ \int_0^T \left\{ \frac{s^3\psi(L)^3}{t^3(T-t)^3}|u_x(-L)|^2 + \frac{s\psi}{t(T-t)}|u_{xx}(-L)|^2 \right\} e^{-\frac{2s\psi(L)}{t(T-t)}}\,dt \\
\le C \int_0^T \int_\omega &\left\{ \frac{s^5\psi^5}{t^5(T-t)^5}|u|^2 + \frac{s^3\psi^3}{t^3(T-t)^3}|u_x|^2 + \frac{s\psi}{t(T-t)}|u_{xx}|^2 \right\} e^{-\frac{2s\psi(x)}{t(T-t)}}\,dxdt
\end{aligned} \tag{5.13}$$

Proof. First, we suppose that $u_0 \in D(A)$, so that u belongs to space $C([0,T];D(A)) \cap C^1([0,T];L^2(-L,L))$. The general follows by a density argument. Let $u = u(x,t)$ and $\varphi(t,x) = \frac{\psi(x)}{t(T-t)}$, where ψ is a positive function satisfying (5.8)-(5.12). Consider

$$v := e^{-s\varphi}u \quad \text{and} \quad w := e^{-s\varphi}P(e^{s\varphi}v),$$

where P is the differential operator given by

$$P = \partial_t - \partial_x^2 + \partial_x^3.$$

Note that

$$\begin{aligned}
\partial_t(e^{s\varphi}v) &= e^{s\varphi}\left\{s\varphi_t v + v_t\right\}, \\
\partial_x(e^{s\varphi}v) &= e^{s\varphi}\left\{s\varphi_x v + v_x\right\}, \\
\partial_x^2(e^{s\varphi}v) &= e^{s\varphi}\left\{s\varphi_{xx}v + s^2\varphi_x^2 v + 2s\varphi_x v_x + u_{xx}\right\}, \\
\partial_x^3(e^{s\varphi}v) &= e^{s\varphi}\left\{s\varphi_{xxx}v + 3s^2\varphi_x\varphi_{xx}v + 3s\varphi_{xx}v_x + s^3\varphi_x^3 v + 3s^2\varphi_x^2 v_x + 3s\varphi_x v_{xx} + v_{xxx}\right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
P(e^{s\varphi}v) = e^{s\varphi}\Big\{&\left(s\varphi_t + s\varphi_{xxx} + 3s^2\varphi_x\varphi_{xx} + s^3\varphi_x^3 - s\varphi_{xx} - s^2\varphi_x^2\right)v \\
&+ \left(3s\varphi_{xx} + 3s^2\varphi_x^2 - 2s\varphi_x\right)v_x + \left(3s\varphi_x - 1\right)v_{xx} + v_{xxx} + v_t\Big\}
\end{aligned}$$

and

$$w = Av + Bv_x + Cv_{xx} + v_{xxx} + v_t,$$

Note that by definition of w and using the boundary conditions of (5.5), we have that

$$Av + Bv_x + Cv_{xx} + v_{xxx} + v_t = 0. \tag{5.14}$$

where

$$A = s(\varphi_t - \varphi_{xx} + \varphi_{xxx}) + 3s^2\varphi_x\varphi_{xx} + s^3\varphi_x^3 - s^2\varphi_x^2, \tag{5.15}$$
$$B = 3s\varphi_{xx} + 3s^2\varphi_x^2 - 2s\varphi_x, \tag{5.16}$$
$$C = 3s\varphi_x - 1. \tag{5.17}$$

Set $L_1 v := v_t + v_{xxx} + Bv_x$ and $L_2 v := Av + Cv_{xx}$. Thus, we have

$$2\int_0^T\int_{-L}^L L_1(v)L_2(v)dxdt \le \int_0^T\int_{-L}^L (L_1(v)+L_2(v))^2\,dxdt \ = 0. \tag{5.18}$$

In the following, our efforts will be devoted to compute the double product in the previous equation. Let us denote by $(L_i v)_j$ the j-th term of $L_i v$ and $Q = [0,T]\times[-L,L]$. Then, to compute the integrals on the right hand side of (5.18), we perform integration by part in x or t:

$$((L_1v)_1,(L_2v)_1)_{L^2(Q)} = -\frac{1}{2}\int_Q A_t v^2 dxdt,$$

$$((L_1v)_2,(L_2v)_1)_{L^2(Q)} = -\frac{1}{2}\int_Q A_{xxx}v^2dxdt + \frac{3}{2}\int_Q A_x v_x^2 dxdt + \frac{1}{2}\int_0^T A(-L)v_x^2(-L)dt$$

$$((L_1v)_3,(L_2v)_1)_{L^2(Q)} = -\frac{1}{2}\int_Q (AB)_x v^2 dxdt,$$

$$((L_1v)_2,(L_2v)_2)_{L^2(Q)} = -\frac{1}{2}\int_Q C_x v_{xx}^2 dxdt + \frac{1}{2}\int_0^T C(L)v_{xx}^2(L)dt - \frac{1}{2}\int_0^T C(-L)v_{xx}^2(-L)dt$$

$$((L_1v)_3,(L_2v)_2)_{L^2(Q)} = -\frac{1}{2}\int_Q (BC)_x v_x^2 dxdt - \frac{1}{2}\int_0^T B(-L)C(-L)v_x^2(-L)$$

By using (5.14), we have that

$$\begin{aligned}
((L_1v)_1,(L_2v)_2)_{L^2(Q)} &= -\frac{1}{2}\int_Q C\partial_t(v_x^2)dxdt - \int_Q C_x v_x v_t dxdt\\
&=\frac{1}{2}\int_Q C_t v_x^2 dxdt + \int_Q C_x v_x\left(Av + Bv_x + Cv_{xx} + v_{xxx}\right)dxdt\\
&=\frac{1}{2}\int_Q C_t v_x^2 dxdt + \frac{1}{2}\int_Q AC_x(v^2)_x dxdt + \int_Q BC_x v_x^2 dxdt + \frac{1}{2}\int_Q CC_x (v_x^2)_x dxdt\\
&\quad + \int_Q C_x v_x v_{xxx} dxdt\\
&=\frac{1}{2}\int_Q C_t v_x^2 dxdt - \frac{1}{2}\int_Q (AC_x)_x v^2 dxdt + \int_Q BC_x v_x^2 dxdt - \frac{1}{2}\int_Q (CC_x)_x v_x^2 dxdt\\
&\quad - \frac{1}{2}\int_0^T C(-L)C_x(-L)v_x^2(-L)dt + \frac{1}{2}\int_Q C_{xxx}v_x^2 dxdt + \frac{1}{2}\int_0^T C_{xx}(-L)v_x^2(-L)dt\\
&\quad - \int_Q C_x v_{xx}^2 dxdt - \int_0^T C_x(-L)v_x(-L)v_{xx}(-L)dt
\end{aligned}$$

applying Young inequality, it follows that

$$
\begin{aligned}
((L_1v)_1,(L_2v)_2)_{L^2(Q)} \geq & \frac{1}{2}\int_Q \{C_t+2BC_x-(CC_x)_x+C_{xxx}\}\, v_x^2 dxdt - \frac{1}{2}\int_Q (AC_x)_x v^2 dxdt \\
& - \int_Q C_x v_{xx}^2 dxdt - \frac{1}{2}\int_0^T C_x^2(-L)v_x^2(-L) - \frac{1}{2}\int_0^T v_{xx}^2(-L)dt \\
& + \frac{1}{2}\int_0^T \{C_{xx}(-L)-C(-L)C_x(-L)\}\, v_x^2(-L)dt
\end{aligned}
$$

Putting together the inequalities above, we have

$$
\begin{aligned}
2\int_0^T\int_{-L}^L L_1(v)L_2(v)dxdt \geq & -\int_Q \{A_t+A_{xxx}+(AB)_x+(AC_x)_x\}\, v^2 dxdt \\
& + \int_Q \{3A_x-(BC)_x+C_t+2BC_x-(CC_x)_x+C_{xxx}\}\, v_x^2 dxdt \\
& - 3\int_Q C_x v_{xx}^2 dxdt - \int_0^T (C(L)+1)v_{xx}^2(L)dt \\
& + \int_0^T \left\{A(-L)-B(-L)C(-L)-C(-L)C_x(-L)+C_{xx}(-L)-C_{xx}^2(-L)\right\} v_x^2(-L)dt.
\end{aligned}
$$

From (5.18), it follows that

$$
\int_Q \left\{Dv^2+Eu_x^2+Fv_{xx}^2\right\} dxdt + \int_0^T Gv_x^2(-L)dt + \int_0^T Hv_{xx}^2(-L)dt \leq 0 \tag{5.19}
$$

with

$$
D = -\left(A_t+A_{xxx}+(AB)_x+(C_xA)_x\right), \tag{5.20}
$$

$$
E = 3A_x+BC_x-B_xC-(CC_x)_x+C_{xxx}+C_t, \tag{5.21}
$$

$$
F = -3C_x, \tag{5.22}
$$

$$
G = A(-L)-B(-L)C(-L)-C(-L)C_x(-L)+C_{xx}(-L)-C_{xx}^2(-L), \tag{5.23}
$$

$$
H = -C(-L)-1. \tag{5.24}
$$

In order to make the reading easier, the proof will be done in several steps to estimate every terms in the integral (5.19):

Step 1: Estimation of $\int_Q Du^2 dxdt$.

First at all, note that

$$
\begin{aligned}
A_t =& s(\varphi_{tt}-\varphi_{xxt}+\varphi_{xxxt})+3s^2\varphi_{xt}\varphi_{xx}+3s^2\varphi_x\varphi_{xxt}+3s^3\varphi_x^2\varphi_{xt}-2s^2\varphi_x\varphi_{xt} \\
A_{xxx} =& s(\varphi_{txxx}+\varphi_{6x}-\varphi_{5x})+9s^2\varphi_{xxx}^2+12s^2\varphi_{xx}\varphi_{4x}+3s^2\varphi_x\varphi_{5x}+6s^3\varphi_{xx}^3 \\
& +18s^3\varphi_x\varphi_{xx}\varphi_{xxx}+3s^3\varphi_x^2\varphi_{4x}-6s^2\varphi_{xx}\varphi_{xxx}-2s^2\varphi_x\varphi_{4x} \\
AB =& 3s^2\varphi_{xx}\varphi_t+3s^2\varphi_{xx}\varphi_{xxx}-3s^2\varphi_{xx}^2+9s^3\varphi_x\varphi_{xx}^2+12s^4\varphi_x^3\varphi_{xx}-12s^3\varphi_x^2\varphi_{xx} \\
& +3s^3\varphi_x^2\varphi_t+12s^3\varphi_x^2\varphi_{xxx}+3s^5\varphi_x^5-5s^4\varphi_x^4-2s^2\varphi_x\varphi_t-2s^2\varphi_x\varphi_{xxx} \\
& -2s^2\varphi_x\varphi_{xx}+2s^3\varphi_x^3 \\
(AB)_x =& 3s^2\varphi_{xxx}\varphi_t+3s^2\varphi_{xx}\varphi_{xt}+3s^2\varphi_{xxx}^2+3s^2\varphi_{xx}\varphi_{4x}-8s^2\varphi_{xx}\varphi_{xxx}+9s^3\varphi_{xx}^3 \\
& +24s^3\varphi_x\varphi_{xx}\varphi_{xxx}+36s^4\varphi_x^2\varphi_{xx}^2+12s^4\varphi_x^3\varphi_{xxx}-24s^3\varphi_x\varphi_{xx}^2-12s^3\varphi_x^2\varphi_{xxx} \\
& +6s^3\varphi_x\varphi_{xx}\varphi_t+3s^3\varphi_x^2\varphi_{xt}+3s^3\varphi_x^2\varphi_{4x}+15s^5\varphi_x^4\varphi_{xx}-20s^4\varphi_x^3\varphi_{xx}-2s^2\varphi_{xx}\varphi_t \\
& -2s^2\varphi_x\varphi_{xt}-2s^2\varphi_x^2\varphi_{4x}-2s^2\varphi_{xx}^2-2s^2\varphi_x\varphi_{xxx}+6s^3\varphi_x^2\varphi_{xx} \\
AC_x =& 3s^2\varphi_{xx}\varphi_t+3s^2\varphi_{xx}\varphi_{xxx}-3s^2\varphi_{xx}^2+9s^3\varphi_x\varphi_{xx}^2+3s^4\varphi_x^3\varphi_{xx}-3s^3\varphi_x^2\varphi_{xx} \\
(AC_x)_x =& 3s^2\varphi_{xxx}\varphi_t+3s^2\varphi_{xx}\varphi_{xt}+3s^2\varphi_{xxx}^2+3s^2\varphi_{xx}\varphi_{4x}-6s^2\varphi_{xx}\varphi_{xxx}+9s^3\varphi_{xx}^3 \\
& +18s^3\varphi_x\varphi_{xx}\varphi_{xxx}+9s^4\varphi_x^2\varphi_{xx}^2+3s^4\varphi_x^3\varphi_{xxx}-6s^3\varphi_x\varphi_{xx}^2-3s^3\varphi_x^2\varphi_{xxx}.
\end{aligned}
$$

All these estimations give us that

$$D = -15s^5\varphi_x^4\varphi_{xx} + D_1 \tag{5.25}$$

where

$$\begin{aligned}
-D_1 =& s\varphi_{tt} + s\varphi_{xxxt} - s\varphi_{xxt} + 9s^2\varphi_{xt}\varphi_{xx} + 3s^2\varphi_x\varphi_{xxt} + 3s^3\varphi_x^2\varphi_{xt} - 4s^2\varphi_x\varphi_{xt}\\
&+ s\varphi_{txx} + s\varphi_{6x} - s\varphi_{5x} + 18s^2\varphi_{xxx}^2 + 15s^2\varphi_{xx}\varphi_{4x} + 3s^2\varphi_x\varphi_{5x} + 24s^3\varphi_{xx}^3\\
&+ 60s^3\varphi_x\varphi_{xx}\varphi_{xxx} + 6s^3\varphi_x^2\varphi_{4x} - 20s^2\varphi_{xx}\varphi_{xxx} - 4s^2\varphi_x\varphi_{4x} + 6s^2\varphi_{xxx}\varphi_t\\
&+ 45s^4\varphi_x^2\varphi_{xx}^2 + 15s^4\varphi_x^3\varphi_{xxx} - 30s^3\varphi_x\varphi_{xx}^2 - 15s^3\varphi_x^2\varphi_{xxx} + 6s^3\varphi_x\varphi_{xx}\varphi_t\\
&+ 3s^3\varphi_x^2\varphi_{xt} - 20s^4\varphi_x^3\varphi_{xx} - 2s^2\varphi_{xx}\varphi_t - 2s^2\varphi_{xx}^2 - 2s^2\varphi_x\varphi_{xxx} + 6s^3\varphi_x^2\varphi_{xx}.
\end{aligned}$$

Note that (5.8)-(5.12) imply that

$$|\varphi_t| \le K_1\varphi^2, \quad |\varphi_{tt}| \le K_2\varphi^3, \quad \text{and} \quad |\partial_x^k\varphi| \le C_k\varphi, \tag{5.26}$$

where K_1, K_2 and C_k are positive constants depending of L, ω and k. Therefore, there exist a constant $k_1 > 0$, such that

$$|D_1| \le k_1 s^4\varphi^4, \quad (x,t) \in (-L,L)\times(0,T)$$

and

$$|15s^5\varphi_x^4\varphi_{xx}| \le k_1 s^5\varphi^5, \quad (x,t) \in \omega\times(0,T).$$

We infer from (5.9) that for some $k_2 > 0$,

$$-15s^5\varphi_x^4\varphi_{xx} = -15s^5\frac{(\psi')^4\psi''}{t^5(T-t)^5} \ge k_2 s^5\varphi^5, \quad (x,t) \in ([-L,L]\setminus\omega)\times(0,T).$$

Taking (5.25) into a count and using the above estimates in the first integral in (5.19), we obtain

$$\begin{aligned}
\int_Q Dv^2dxdt &= -15\int_Q s^5\varphi_x^4\varphi_{xx}v^2dxdt + \int_Q D_1v^2dxdt\\
&\ge k_2\int_{(0,T)\times([-L,L]\setminus\omega)}(s\varphi)^5v^2dxdt - k_1\int_{(0,T)\times\omega}(s\varphi)^5v^2dxdt - k_1\int_Q(s\varphi)^4v^2dxdt\\
&\qquad = \int_Q\left\{k_2(s\varphi)^5 - k_1(s\varphi)^4\right\}v^2dxdt - (k_1+k_2)\int_{(0,T)\times\omega}(s\varphi)^5v^2dxdt.
\end{aligned}$$

Thus, there exist a positive constants C_1 and C_2, such that for any $s \ge s_1$ with s_1 large enough, we obtain

$$\int_Q Dv^2dxdt \ge C_1\int_Q(s\varphi)^5v^2dxdt - C_2\int_{(0,T)\times\omega}(s\varphi)^5v^2dxdt. \tag{5.27}$$

Step 2: Estimation for $\int_Q Ev_x^2dxdt$ and $\int_Q Fv_{xx}^2dxdt$.

Note that

$$\begin{aligned}
BC_x =& 9s^2\varphi_{xx}^2 + 9s^3\varphi_x^2\varphi_{xx} - 6s^2\varphi_x\varphi_{xx},\\
-B_xC =& -27s^2\varphi_x\varphi_{xxx} - 18s^3\varphi_x^2\varphi_{xx} + 12s^2\varphi_x\varphi_{xx} + 9s\varphi_{xxx} - 2s\varphi_{xx},\\
3A_x =& 3s\varphi_{xt} + 3s\varphi_{4x} - 3s\varphi_{xxx} + 9s^2\varphi_{xx}^2 + 9s^2\varphi_x\varphi_{xxx} + 9s^3\varphi_x^2\varphi_{xx} - 6s^2\varphi_x\varphi_{xx},\\
CC_x =& 9s^2\varphi_x\varphi_{xx} - 3s\varphi_{xx},\\
-(CC_x)_x =& -9s^2\varphi_{xx}^2 - 9s^2\varphi_x\varphi_{xxx} + 3s\varphi_{xxx},\\
C_{xxx} + C_t =& 3s\varphi_{4x} + 3s\varphi_{xt}.
\end{aligned}$$

Putting together these expressions, we have

$$E = 6s\varphi_{xt} + 6s\varphi_{4x} + 9s^2\varphi_{xx}^2 - 27s^2\varphi_x\varphi_{xxx} + 9s\varphi_{xxx} - 2s\varphi_{xx}.$$

We infer from (5.8)-(5.12) that for some $k_3 > 0$ and $k_4 > 0$,

$$9s^2\varphi_{xx}^2 - 27s^2\varphi_x\varphi_{xxx} = \frac{9s^2\left((\psi'')^2 - 3\psi'\psi'''\right)}{t^2(T-t)^2} \geq k_3(s\varphi)^2, \quad (x,t) \in ([-L,L]\setminus\omega)\times(0,T),$$
$$\left|9s^2\varphi_{xx}^2 - 27s^2\varphi_x\varphi_{xxx}\right| \leq k_4(s\varphi)^2, \quad (x,t)\in\omega\times(0,T),$$
$$|6s\varphi_{xt} + 6s\varphi_{4x} + 9s\varphi_{xxx} - 2s\varphi_{xx}| \leq k_4 s\varphi^2, \quad (x,t)\in[-L,L]\times(0,T).$$

By using the above estimates, we obtain

$$\begin{aligned}
\int_Q Ev_x^2dxdt =& \int_Q \left(9s^2\varphi_{xx}^2 - 27s^2\varphi_x\varphi_{xxx}\right)v_x^2dxdt \\
&+ \int_Q \left(6s\varphi_{xt} + 6s\varphi_{4x} + 9s\varphi_{xxx} - 2s\varphi_{xx}\right)v_x^2dxdt \\
\geq& k_3\int_{(0,T)\times([-L,L]\setminus\omega)}(s\varphi)^2v_x^2dxdt - k_4\int_{(0,T)\times\omega}(s\varphi)^2v_x^2dxdt - k_4\int_Q s\varphi^2v_x^2dxdt \\
=& \int_Q\left\{k_3(s\varphi)^2 - k_4s\varphi^2\right\}v_x^2dxdt - (k_3+k_4)\int_{(0,T)\times\omega}(s\varphi)^2v_x^2dxdt.
\end{aligned}$$

Thus, there exist positive constants C_2 and C_3, such that, for any $s \geq s_2$ with s_2 large enough, we obtain

$$\int_Q Ev_x^2dxdt \geq C_2\int_Q(s\varphi)^2v_x^2dxdt - C_3\int_{(0,T)\times\omega}(s\varphi)^3v_x^2dxdt. \tag{5.28}$$

Moreover, note that (5.9) implies that there exist $C_4 > 0$ and $C_5 > 0$ such that

$$F = -9s\varphi_{xx} = -\frac{9s\psi''}{t(T-t)} \geq C_4 s\varphi, \quad (x,t)\in([-L,L]\setminus\omega)\times(0,T)$$

and

$$|9s\varphi_{xx}| \leq C_5 s\varphi, \quad (x,t)\in(\omega)\times(0,T).$$

Furthermore,

$$\int_Q Fv_{xx}^2dxdt \geq C_4\int_Q s\varphi v_{xx}^2dxdt - (C_4+C_5)\int_{(0,T)\times\omega}s\varphi v_{xx}^2dxdt. \tag{5.29}$$

Step 3: Estimation for $\int_0^T Gu_x^2(L)dt$ and $\int_0^T Hu_{xx}^2(L)dt$:

First, observe that

$$\begin{aligned}
A =& s\varphi_t - s\varphi_{xx} + s\varphi_{xxx} + 3s^2\varphi_x\varphi_{xx} + s^3\varphi_x^3 - s^2\varphi_x^2, \\
-BC =& -9s^2\varphi_x\varphi_{xx} + 3s\varphi_{xx} + 9s^2\varphi_x^2 - 9s^3\varphi_x^3 - 2s\varphi_x, \\
-CC_x =& -9s^2\varphi_x\varphi_{xx} + 3s\varphi_{xx}, \\
C_{xx} =& 3s\varphi_{xxx}, \\
-C_x^2 =& -9s^2\varphi_{xx}^2.
\end{aligned}$$

The above identities imply that

$$G = -8s^3\varphi_x^3(-L) + G_1$$

where

$$G_1 = s\varphi_t(-L) + 5s\varphi_{xx}(-L) + 4s\varphi_{xxx}(-L) - 15s^2\varphi_x(-L)\varphi_{xx}(-L) + 8s^2\varphi_x^2(-L) \\ - 2s\varphi_x(-L) - 9s^2\varphi_{xx}^2(-L).$$

We infer from (5.8)-(5.12) that for some $k_5 > 0$ and $k_6 > 0$,

$$-8s^3\varphi_x^3(-L) = -\frac{8s^3(\psi')^3(-L)}{t^3(T-t)^3} \geq k_5(s\varphi(L))^3, \quad t \in (0,T).$$

and

$$|G_1| \leq k_6(s\varphi(L))^2, \quad t \in (0,T).$$

Then, it follows that

$$\int_0^T Gv_x^2(-L)dt \geq \int_0^T \left(k_5(s\varphi(L))^3 - k_5(s\varphi(L))^2\right) v_x^2(-L)dxdt.$$

Thus, there exists a positive constant C_6, such that, for any $s \geq s_3$ with s_3 large enough, we obtain

$$\int_0^T Gv_x^2(-L)dt \geq C_6 \int_0^T (s\varphi(L))^3 v_x^2(-L)dxdt. \tag{5.30}$$

Moreover, note that (5.10) implies that there exist $k_7 > 0$ such that

$$H = -3s\varphi_x(-L) - 2 \geq k_7 s\varphi(L) - 2, \quad t \in (0,T).$$

Furthermore,

$$\int_0^T Hv_{xx}^2(-L)dt \geq \int_0^T \left(k_7 s\varphi(L) - 2\right) v_{xx}^2(-L)dxdt.$$

Hence, there exists a positive constant C_7, such that for any $s \geq s_4$ with s_4 large enough, we obtain

$$\int_0^T Hv_{xx}^2(-L)dt \geq C_7 \int_0^T s\varphi(L)v_{xx}^2(-L)dxdt. \tag{5.31}$$

Combining (5.19) together with (5.27)-(5.31), we obtain

$$\int_Q \left\{ \frac{s^5\psi^5}{t^5(T-t)^5}|v|^2 + \frac{s^2\psi^2}{t^2(T-t)^2}|v_x|^2 + \frac{s\psi}{t(T-t)}|v_{xx}|^2 \right\} dxdt \\ + \int_0^T \left\{ \frac{s^3\psi(-L)^3}{t^3(T-t)^3}|v_x(-L)|^2 + \frac{s\psi(-L)}{t(T-t)}|v_{xx}(-L)|^2 \right\} dt \\ \leq C_8 \int_{(0,T)\times\omega} \left\{ \frac{s^5\psi^5}{t^5(T-t)^5}|v|^2 + \frac{s^3\psi^3}{t^3(T-t)^3}|v_x|^2 + \frac{s\psi}{t(T-t)}|v_{xx}|^2 \right\} dxdt \tag{5.32}$$

for some $C_8 > 0$. On the other hand, note that

$$\int_Q \frac{s^3\psi^3}{t^3(T-t)^3}v_x^2 dxdt = -\int_Q \frac{s^3\psi^3}{t^3(T-t)^3}vv_{xx}dxdt - \int_Q \frac{2s^3\psi^2\psi'}{t^3(T-t)^3}vv_x dxdt \\ \leq \int_Q \frac{s^5\psi^5}{2t^5(T-t)^5}|v|^2 dxdt + \int_Q \frac{s\psi}{2t(T-t)}|v_{xx}|^2 dxdt \\ + \max_{x\in[-L,L]}\{(\psi'(x))^2\} \int_Q \frac{s^4\psi^2}{t^4(T-t)^4}|v|^2 dxdt + \int_Q \frac{s^2\psi^2}{t^2(T-t)^2}|v_x|^2 dxdt$$

From (5.32) and using the fact that s is large enough, there exist $C>0$ such that

$$\begin{aligned}\int_Q \left\{\frac{s^5\psi^5}{t^5(T-t)^5}|v|^2+\frac{s^3\psi^3}{t^3(T-t)^3}|v_x|^2+\frac{s\psi}{t(T-t)}|v_{xx}|^2\right\}dxdt&\\ +\int_0^T\left\{\frac{s^3\psi(L)^3}{t^3(T-t)^3}|v_x(-L)|^2+\frac{s\psi}{t(T-t)}|v_{xx}(-L)|^2\right\}dt&\\ \leq C_8\int_{(0,T)\times\omega}\left\{\frac{s^5\psi^5}{t^5(T-t)^5}|v|^2+\frac{s^3\psi^3}{t^3(T-t)^3}|v_x|^2+\frac{s\psi}{t(T-t)}|v_{xx}|^2\right\}dxdt.&\end{aligned} \tag{5.33}$$

Returning to the original variable $v=e^{-s\varphi}u$, we conclude the proof of the Lemma. □

In order to prove the Proposition 5.2.3, consider following spaces

$$\begin{aligned} X_0 &:= L^2(0,T;H^{-2}(0,L)), \quad X_1 := L^2(0,T;H_0^2(0,L)),\\ \widetilde{X}_0 &:= L^1(0,T;H^{-1}(0,L)), \quad \widetilde{X}_1 := L^1(0,T;(H^3\cap H_0^2)(0,L)),\end{aligned}$$

and

$$\begin{aligned} Y_0 &:= L^2((0,T)\times(0,L))\cap C^0([0,T];H^{-1}(0,L)),\\ Y_1 &:= L^2(0,T;H^4(0,L))\cap C^0([0,T];H^3(0,L))\end{aligned}$$

equipped with their natural norm. For any $\theta\in[0,1]$, we define the complex interpolation space

$$X_\theta=(X_0,X_1)_{[\theta]}, \quad \widetilde{X}_\theta=\left(\widetilde{X}_0,\widetilde{X}_1\right)_{[\theta]}, \quad \text{and} \quad Y_\theta=(Y_0,Y_1)_{[\theta]}.$$

For instance, we obtain that

$$\begin{aligned} X_{1/2} &= L^2((0,T)\times(0,L)), \quad \widetilde{X}_{1/2}=L^1(0,T;H_0^1(0,L)),\\ X_{1/4} &= L^2(0,T;H^{-1}(0,L)), \quad \widetilde{X}_{1/4}=L^1(0,T;L^2(0,L)),\end{aligned}$$

and

$$\begin{aligned} Y_{1/2} &= L^2(0,T;H^2(0,L))\cap C^0([0,T];H^1(0,L)),\\ Y_{1/4} &= L^2(0,T;H^1(0,L))\cap C^0([0,T];L^2(0,L)).\end{aligned}$$

We introduce the following non-homogeneous system with null initial data:

$$\begin{cases} u_t-u_{xx}+u_{xxx}=f & \text{in } (-L,L)\times(0,T),\\ u(-L,t)=u(L,t)=u_x(L,t)=0 & \text{in } (0,T),\\ u(x,0)=0 & \text{in } (-L,L).\end{cases} \tag{5.34}$$

Lemma 5.2.5. *Let $\theta\in[1/4,1]$. If $f\in X_\theta\cup\widetilde{X}_\theta$, then the solution u of (5.34) belongs to Y_θ and there exists some constant $C>0$ such that*

$$\begin{aligned} \|u\|_{Y_\theta} &\leq C\|f\|_{X_\theta}, \quad \textit{for } f\in X_\theta\\ \|u\|_{Y_\theta} &\leq C\|f\|_{\widetilde{X}_\theta}, \quad \textit{for } f\in\widetilde{X}_\theta\end{aligned}$$

Proof. In order to prove the Lemma, we follow the same approach developed in [46]. Note that if $f\in L^2(0,T;H^{-1}(0,L))\cup L^1(0,T;L^2(0,L))$, then the solution u of (5.34) belongs to $C([0,T];L^2(0,L))\cap L^2(0,T;H^1(0,L))$. Indeed, we will suppose that f belongs to $C_0^\infty((0,T)\times(0,L))$ and the general case follows by density. Multiplying (5.34) by u and integrating in $(0,t)\times(-L,L)$ with $t\in(0,T)$, we obtain that

$$\frac{1}{2}\int_{-L}^{L}u^2(t)dx+\int_0^t\int_{-L}^{L}u_x^2dxds \leq \int_0^t \langle f(s),u(s)\rangle_{H^{-1}\times H_0^1}ds, \quad \text{for } f\in L^2(0,T;H^{-1}(0,L))$$

or

$$\frac{1}{2}\int_{-L}^{L}u^2(t)dx+\int_0^t\int_{-L}^{L}u_x^2dxds\leq\int_0^t\int_{-L}^{L}fudxds, \quad \text{for } f\in L^2(0,T;L^2(0,L)).$$

Taking the supreme in $[0,T]$ and using the Young inequality, there exist a constant $C_1>0$, such that

$$\begin{cases}\|u\|_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^1(0,L))}\leq C_1\|f\|_{L^2(0,T;H^{-1}(0,L))}, & \text{if } f\in L^2(0,T;H^{-1}(0,L))\\ \|u\|_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^1(0,L))}\leq C_1\|f\|_{L^1(0,T;L^2(0,L))}, & \text{if } f\in L^1(0,T;L^2(0,L))\end{cases} \tag{5.35}$$

Now, suppose that $f\in L^2(0,T;H_0^2(0,L))\cup L^1(0,T;(H^3\cap H_0^2)(0,L))$, we will prove that the solution u of (5.34) belongs to $C([0,T];H^3(0,L))\cap L^2(0,T;H^4(0,L))$. Again, we first consider f belongs to $C_0^\infty((0,T)\times(0,L))$ and the general case follows by density. Consider the differential operator

$$P=-\partial_x^2+\partial_x^3.$$

Let us apply the operator P to the equation (5.34). Thus, by using the boundary condition of the system and the fact that $Pu=f-u_t$, it follows that

$$\begin{cases}(Pu)_t-(Pu)_{xx}+(Pu)_{xxx}=Pf & \text{in } (-L,L)\times(0,T),\\ (Pu)(-L,t)=(Pu)(L,t)=(Pu)_x(L,t)=0 & \text{in } (0,T),\\ (Pu)(x,0)=0 & \text{in } (-L,L).\end{cases}$$

Since $Pf\in L^2(0,T;H^{-1}(0,L))\cup L^1(0,T;L^2(0,L))$, from (5.35) we infer that

$$\begin{cases}\|Pu\|_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^1(0,L))}\leq C\|Pf\|_{L^2(0,T;H^{-1}(0,L))}, & \text{if } Pf\in L^2(0,T;H^{-1})\\ \|Pu\|_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^1(0,L))}\leq C\|Pf\|_{L^1(0,T;L^2(0,L))}, & \text{if } Pf\in L^1(0,T;L^2),\end{cases} \tag{5.36}$$

for some $C>0$. Moreover, note that there exists $C_2>0$, such that

$$\begin{aligned}&\|u\|_{L^\infty(0,T;H^3(0,L))\cap L^2(0,T;H^4(0,L))}\\ &\quad\leq C_2\left(\|u\|_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^1(0,L))}+\|u_{xxx}\|_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^1(0,L))}\right)\\ &\quad\leq C_2\left(\|u\|_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^1(0,L))}+\|Pu\|_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^1(0,L))}\right.\\ &\qquad\left.+\|u_{xx}\|_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^1(0,L))}\right).\end{aligned} \tag{5.37}$$

On the other hand,

$$L^\infty(0,T;H^3(0,L))\underset{compact}{\hookrightarrow}L^\infty(0,T;H^2(0,L))\hookrightarrow L^\infty(0,T;L^2(0,L))$$

and

$$L^2(0,T;H^4(0,L))\underset{compact}{\hookrightarrow}L^2(0,T;H^3(0,L))\hookrightarrow L^2(0,T;H^1(0,L))$$

By using [91, Lemma 8], we have that

$$\begin{aligned}&\|u_{xx}\|_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^1(0,L))}\\ &\qquad\leq\varepsilon\|u\|_{L^\infty(0,T;H^3(0,L))\cap L^2(0,T;H^4(0,L))}+C(\varepsilon)\|u\|_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^1(0,L))}.\end{aligned}$$

for any $\varepsilon > 0$. Choosing an appropriate $\varepsilon > 0$ from (5.35), (5.36) and (5.37), we get

$$\begin{cases} \|u\|_{L^\infty(0,T;H^3(0,L))\cap L^2(0,T;H^4(0,L))} \leq C_3\|f\|_{L^2(0,T;H_0^2(0,L))}, & \text{if } f \in L^2(0,T;H_0^2) \\ \|u\|_{L^\infty(0,T;H^3(0,L))\cap L^2(0,T;H^4(0,L))} \leq C_3\|f\|_{L^1(0,T;H^3(0,L))}, & \text{if } f \in L^1(0,T;(H^3 \cap H_0^2)). \end{cases} \tag{5.38}$$

In order to complete the proof, let us define the linear map $A : f \mapsto u$. By (5.35) and (5.38), A continuously maps $X_{1/4}$ and $\widetilde{X}_{1/4}$ into $Y_{1/4}$, and X_1 and $\widetilde{X}_1$ into Y_1. Moreover, the norm of the operator A can be estimate as follows

$$\begin{array}{ll} \|A\|_{\mathcal{L}(X_{1/4},Y_{1/4})} \leq C_1, & \|A\|_{\mathcal{L}(\widetilde{X}_{1/4},Y_{1/4})} \leq C_1, \\ \|A\|_{\mathcal{L}(X_1,Y_1)} \leq C_3, & \|A\|_{\mathcal{L}(\widetilde{X}_1,Y_1)} \leq C_3. \end{array}$$

From classical interpolation arguments (see [3]), we have that A continuously maps X_θ and $\widetilde{X}_\theta$ to Y_θ, for any $\theta \in [1/4, 1]$. Moreover, there exists a positive constant C, such tat the corresponding operator norms satisfy

$$\|A\|_{\mathcal{L}(X_\theta,Y_\theta)} \leq C \quad \text{and} \quad \|A\|_{\mathcal{L}(\widetilde{X}_\theta,Y_\theta)} \leq C.$$

This completes the proof. □

Lemma 5.2.6. *Let $0 < l < L$ and $T > 0$, and s_0 be as in Proposition 5.2.4. Then, there exists a positive constant C, such that for any $s \geq s_0$ and any $u_0 \in L^2(-L,L)$, the solution u of (5.5) satisfies*

$$\int_Q s^5\check{\varphi}^5|u|^2e^{-2s\hat{\varphi}}dxdt \leq Cs^{10}\int_0^T e^{s(6\hat{\varphi}-8\check{\varphi})}\check{\varphi}^{31}\|u(\cdot,t)\|^2_{L^2(\omega)}dt, \tag{5.39}$$

where $Q = (0,T) \times (-L,L)$, $\omega = (-l,l)$,

$$\hat{\varphi}(t) = \max_{x\in[-L,L]} \frac{\psi(x)}{t(T-t)} = \frac{\psi(0)}{t(T-t)} \quad \textit{and} \quad \check{\varphi}(t) = \min_{x\in[-L,L]} \frac{\psi(x)}{t(T-t)} = \frac{\psi(l_3)}{t(T-t)}$$

Proof. With Lemma 5.2.5 in hands, we can follow the same approach as in [20] and [46] with minor changes. In fact, by using the estimates (3.30)-(3.40) in the proof of [20, Lemma 3.7], we have that

$$\int_Q s^5\check{\varphi}^5|u|^2e^{-2s\hat{\varphi}}dxdt \leq Cs^{10}\int_0^T e^{s(6\hat{\varphi}-8\check{\varphi})}\check{\varphi}^{31}\|u(\cdot,t)\|^2_{L^2(\omega)}dt$$
$$+ 2\varepsilon s^{-2}\int_0^T e^{-2s\hat{\varphi}}\check{\varphi}^{-9}\|u(\cdot,t)\|^2_{H^{8/3}(\omega)}dt, \tag{5.40}$$

for any $\varepsilon > 0$. From here, we denote by C, the different positive constants which may vary from place to place. Next, we will estimate adequately the integral term

$$\int_0^T e^{-2s\hat{\varphi}}\check{\varphi}^{-9}\|u(\cdot,t)\|^2_{H^{8/3}(\omega)}dt.$$

This is done by a bootstrap argument based on the smoothing effect of the KdV-Burgers equation given by Lemma 5.2.5. Indeed, consider $u_1(x,t) := \theta_1(t)u(x,t)$, where

$$\theta_1(t) = \check{\varphi}^{-1/2}e^{-s\hat{\varphi}}.$$

Thus, u_1 is the solution of

$$\begin{cases} u_{1,t} - u_{1,xx} + u_{1,xxx} = f_1 & \text{in } (-L,L)\times(0,T), \\ u(-L,t) = u(L,t) = u_x(L,t) = 0 & \text{in } (0,T), \\ u(x,0) = 0 & \text{in } (-L,L). \end{cases} \tag{5.41}$$

with $f_1(x,t) = \theta_{1,t}(t)u(x,t)$. Since $|\theta_{1,t}(t)| \leq Cs\check{\varphi}^{3/2}e^{-s\hat{\varphi}}$, we have that $f \in L^2((0,T) \times (-L,L))$ with

$$\|f_1\|^2_{L^2((0,T)\times(-L,L)} \leq Cs^2 \int_Q \check{\varphi}^3 e^{-2s\hat{\varphi}}|u|^2 dxdt, \tag{5.42}$$

for some constant $C > 0$ and all $s \geq s_0$. Then by Lemma 5.2.5, u_1 belongs to $Y_{1/2} = L^2(0,T;H^2(-L,L)) \cap L^\infty(0,T;H^1(-L,L))$. Thus, interpolating over theses spaces, we obtain that u_1 belongs to space $L^4(0,T;H^{3/2}(-L,L))$ and

$$\|u_1\|_{L^4(0,T;H^{3/2}(-L,L))} \leq \|f\|_{L^2((0,T)\times(0,L))}. \tag{5.43}$$

Now, consider $u_2(x,t) := \theta_2(t)u(x,t)$, where

$$\theta_2(t) = \check{\varphi}^{-5/2}e^{-s\hat{\varphi}}.$$

Then u_2 is solution of (5.41) with $f_2 = \theta_{2,t}(t)u(x,t)$ instead f_1. Note that

$$f_2 = \theta_{2,t}(t)\theta_1^{-1}(t)u_1(x,t),$$

therefore since s is large, from (5.26) the following estimate holds

$$|\theta_{2,t}(t)\theta_1^{-1}(t)| \leq Cs.$$

Thus, we have that

$$\|f_2\|_{L^2(0,T;H^{1/3}(-L,L))} \leq Cs\|u_1\|_{L^2(0,T;H^{1/3}(-L,L))}.$$

By using the embedding $H^{3/2}(-L,L) \hookrightarrow H^{1/3}(-L,L)$ and Holder inequality, it follows that f_2 belongs to $X_{7/12} = L^2(0,T;H^{1/3}(-L,L))$ and, consequently,

$$\|f_2\|^2_{L^2(0,T;H^{1/3}(-L,L))} \leq CT^{1/2}s^2\|u_1\|^2_{L^4(0,T;H^{3/2}(-L,L))}. \tag{5.44}$$

Thus by Lemma 5.2.5, $u_2 \in Y_{7/12} = L^2(0,T;H^{7/3}(-L,L)) \cap L^\infty(0,T;H^{4/3}(-L,L))$ and

$$\|u_2\|_{L^2(0,T;H^{7/3}(-L,L))\cap L^\infty(0,T;H^{4/3}(-L,L))} \leq C\|f_2\|_{L^2(0,T;H^{1/3}(-L,L))} \tag{5.45}$$

Finally, let $u_3(x,t) := \theta_3(t)u(x,t)$, where

$$\theta_3(t) = \check{\varphi}^{-9/2}e^{-s\hat{\varphi}}.$$

Thus u_3 is solution of (5.41) with $f_3 = \theta_{3,t}(t)u(x,t)$ instead f_1. Note that

$$f_2 = \theta_{3,t}(t)\theta_2^{-1}(t)u_2(x,t)$$

and

$$|\theta_{3,t}(t)\theta_2^{-1}(t)| \leq Cs.$$

Thus, we have that

$$\|f_3\|_{L^2(0,T;H^{2/3}(-L,L))} \leq Cs\|u_2\|_{L^2(0,T;H^{2/3}(-L,L))}. \tag{5.46}$$

As above, by the embedding $H^{7/3}(-L,L) \hookrightarrow H^{2/3}(-L,L)$, we have that $f_3 \in X_{8/12}$. Thus, by Lemma 5.2.5, u_3 belongs to $Y_{8/12} = L^2(0,T;H^{8/3}(-L,L)) \cap L^\infty(0,T;H^{5/3}(-L,L))$ with

$$\|u_3\|_{L^2(0,T;H^{8/3}(-L,L))\cap L^\infty(0,T;H^{5/3}(-L,L))} \leq C\|f_3\|_{L^2(0,T;H^{2/3}(-L,L))}. \tag{5.47}$$

From (5.43)-(5.47), it yields that

$$\|u_3\|^2_{L^2(0,T;H^{8/3}(-L,L))} \leq CT^{1/2}s^4\|f_1\|^2_{L^2((0,T)\times(-L,L))}. \tag{5.48}$$

Then, by (5.42), (5.48) for s_0 large enough, we have

$$\int_0^T e^{-2s\hat{\varphi}}\check{\varphi}^{-9}\|u(\cdot,t)\|^2_{H^{8/3}(\omega)}dt \leq CT^{1/2}s^2 \int_Q s^5\check{\varphi}^5 e^{-2s\hat{\varphi}}|u|^2 dxdt,$$

for some positive constant C and for all $s \geq s_0$. Then, picking $\varepsilon = \frac{1}{4CT^{1/2}}$ in (5.40), the proof is completed. □

Proof of Proposition 5.2.3. After the change of variables $v(x,t) = u(T-t, L-x)$, we have

$$\begin{cases} -v_t - v_{xx} - v_{xxx} = 0 & \text{in } (0,2L)\times(0,T), \\ v(0,t) = v(2L,t) = v_x(0,t) = 0 & \text{in } (0,T), \\ v(x,0) = u_0(L-x) & \text{in } (0,2L). \end{cases} \tag{5.49}$$

Scaling in (5.49) by v and integrating over $(0,2L)$, it follows that

$$-\frac{1}{2}\frac{\partial}{\partial t}\int_0^{2L}|v(t)|^2dx + \int_0^{2L}|v_x(t)|^2dx + \frac{1}{2}v_x^2(2L) = 0.$$

Integrating over $[0,\tau]$, with $\tau \in [T/3, 2T/3]$, we get

$$\|v(0)\|^2_{L^2(0,2L)} \le \|v(\tau)\|^2_{L^2(0,2L)},$$

Integrating again over $[T/3, 2T/3]$, the following estimate holds

$$\|v(0)\|^2_{L^2(0,2L)} \le \frac{3}{T}\int_{\frac{T}{3}}^{\frac{2T}{3}}\|v(\tau)\|^2_{L^2(0,2L)}d\tau.$$

Pick any s large enough. Thus we obtain

$$\int_{\frac{T}{3}}^{\frac{2T}{3}}\|v(\tau)\|^2_{L^2(0,2L)}d\tau = \int_{\frac{T}{3}}^{\frac{2T}{3}}\|u(t)\|^2_{L^2(-L,L)}dt \le C_1\int_0^T\int_{-L}^{L}s^5\check{\varphi}^5|u|^2e^{-2s\hat{\varphi}}dxdt$$

where $C_1 = \left[\min_{t\in[T/3,2T,3]}\{s^5\check{\varphi}^5e^{-2s\hat{\varphi}}\}\right]^{-1} > 0$. By Lemma 5.2.6, it follows that

$$\|v(0)\|^2_{L^2(0,2L)} \le Cs^{10}\int_0^T e^{s(6\hat{\varphi}-8\check{\varphi})}\check{\varphi}^{31}\|u(\cdot,t)\|^2_{L^2(\omega)}dt.$$

Noting that $\hat{\varphi} < \frac{4}{3}\check{\varphi}$, it easy to see that the maximum of the function

$$\chi(t) = e^{s(6\hat{\varphi}(t)-8\check{\varphi}(t))}\check{\varphi}^{31}(t)$$

is attained in $T/2$ for s large enough. Thus, we have that

$$\|v(0)\|^2_{L^2(0,2L)} \le C\int_0^T\|u(\cdot,t)\|^2_{L^2(\omega)}dt, \tag{5.50}$$

where $C = C(s,T) > 0$. Finally, by a simple change of variable in (5.50), it follows that

$$\|u\|^2_{L^2((0,T)\times(-L,L))} \le CT\|u_0\|^2_{L^2(-L,L)} = \|v(0)\|^2_{L^2(0,2L)} \le CT\|u\|^2_{L^2((0,T)\times\omega)}.$$

This concludes the proof. □

5.3 Proof of the Main Result

The next Proposition is carried out as in [79, Proposition 4.1] and its proof uses the internal observability (5.6) and an Approximation theorem. The proof is sketched in the Appendix.

Proposition 5.3.1. *Let t_1, t_2, T such that $0 < t_1 < t_2 < T$ and let $f = f(x,t)$ be any function such that*

$$f \in L^2_{loc}(\mathbb{R}^2) \qquad \textit{and} \qquad \operatorname{supp} f \subset \mathbb{R}\times[t_1,t_2].$$

Let $\varepsilon > 0$ such that

$$\varepsilon < \min(t_1, T - t_2).$$

Then, there exists $u \in L^2_{loc}(\mathbb{R}^2)$ such that

$$u_t - u_{xx} + u_{xxx} = f \quad \text{in } \mathcal{D}'(\mathbb{R}^2) \tag{5.51}$$

and

$$\operatorname{supp} u \subset \mathbb{R} \times [t_1 - \varepsilon, t_2 + \varepsilon]. \tag{5.52}$$

Proof of the Main Result, Theorem 5.1.1. Let $u_0, u_T \in L^2(\mathbb{R})$, and consider the differential operator $A = \partial_x^2 - \partial_x^3$ with domain $D(A) = H^3(\mathbb{R})$. It is well known that A generates a C_0 semigroup of contraction $S(\cdot)$ on $L^2(\mathbb{R})$. Thus, if $u_0, u_T \in L^2(\mathbb{R})$, $u_1(t) = S(t)u_0$ and $u_2(t) = S(t)u_T$ are the solutions of

$$\begin{cases} \partial_t u_1 - \partial_x^2 u_1 + \partial_x^3 u_1 = 0 & \text{in } \mathbb{R} \times (0,T) \\ u_1(0,x) = u_0(x) & \text{in } \mathbb{R} \end{cases} \quad \text{and} \quad \begin{cases} \partial_t u_2 - \partial_x^2 u_2 + \partial_x^3 u_2 = 0 & \text{in } \mathbb{R} \times (0,T) \\ u_2(0,x) = u_T(x) & \text{in } \mathbb{R}. \end{cases}$$

Respectively. For any $\varepsilon' \in (\varepsilon, T/2)$, consider the function $\varphi \in C^\infty([0,T])$ given by

$$\varphi(t) = \begin{cases} 1 & \text{if } t \le \varepsilon' \\ 0 & \text{if } t \ge T - \varepsilon'. \end{cases} \tag{5.53}$$

Note that the change of variable

$$u(t) = \varphi(t)u_1(t) + (1 - \varphi(t))u_2(t) + w(t)$$

transforms (5.2) in

$$\begin{cases} w_t - w_{xx} + w_{xxx} = \frac{d\varphi}{dt}(u_2 - u_1) & \text{in } \mathcal{D}'(\mathbb{R} \times (0,T)) \\ w(x,0) = w(x,T) = 0 & \text{in } \mathbb{R} \end{cases} \tag{5.54}$$

we finish the proof by applying the Proposition 5.3.1 written $f(x,t) = \frac{d\varphi}{dt}(t)(u_2(x,t) - u_1(x,t))$. □

The Theorem 5.1.2 can be obtained with some minor changes of the proof of the Theorem 5.1.1. Indeed, it is easy to sse that the operators $A = \partial_x^2 - \partial_x^3$ and $B = -\partial_x^2 - \partial_x^3$ generate a semigroups of contraction on $L^2(0,\infty)$ and $L^2((0,\infty), e^{2bx}dx)$ for $b \ge \frac{1}{3}$, respectively, for instance see [76, Lemma 2.1]. Then, taking u_0 in $L^2(0,\infty)$ and u_T in $L^2((0,\infty), e^{-2bx}dx)$, there exist mild solutions u_1 and u_2 of the problems,

$$\partial_t u_1 - \partial_x^2 u_1 - \partial_x^3 u_1 = 0, \quad \text{and} \quad \partial_t u_2 + \partial_x^2 u_2 + \partial_x^3 u_2 = 0, \quad \text{in } \mathbb{R} \times (0,T),$$

with initials data

$$u_1(x,0) = \begin{cases} u_0(x) & \text{for a.e } x > 0, \\ 0 & \text{for a.e } x < 0, \end{cases} \quad \text{and} \quad u_2(x,0) = \begin{cases} 0 & \text{for a.e } x > 0, \\ u_T(-x) & \text{for a.e } x < 0. \end{cases}$$

With this solutions in hand, we proceed as in proof of [79, Theorem 1.3]. Thus, consider the change of function $\widetilde{u}_2(x,t) = u_2(-x, T-t)$. Clearly, $\partial_t \widetilde{u}_2 - \partial_x^2 \widetilde{u}_2 + \partial_x^3 \widetilde{u}_2 = 0$ with $\widetilde{u}_2(x,T) = u_T(x)$ on $(0,\infty)$. In order to obtain the result desired is sufficient consider the change of variable

$$u(t) = \varphi(t)u_1(t) + (1 - \varphi(t))\widetilde{u}_2(t) + \widetilde{w}(t),$$

where φ is the cut off function defined by (5.53) and $\widetilde{w}$ is the solution of the Cauchy problem given by the Proposition 5.3.1 with $f(x,t) = \frac{d\varphi}{dt}(t)(\widetilde{u}_2(x,t) - u_1(x,t))$.

APPENDIX

A

APPENDIX

A.1 Proof of Lemma 2.2.2

In this section, we prove the Lemma 2.2.2 used in the proof of the Proposition 2.2.1. Without loss of generality we can consider the following linear non-homogeneous boundary value problem,

$$\begin{cases} w_t + w_{xxx} = 0, \quad w(x,0) = 0 & x \in (0,L), t > 0, \\ w_{xx}(0,t) = h_1(t), \; w_x(L,t) = h_2(t), \; w_{xx}(L,t) = h_3(t) & t > 0. \end{cases} \tag{A.1}$$

Proof of Lemma 2.2.2. Applying the Laplace transform with respect to t, (A.1) is converted to

$$\begin{cases} s\hat{w} + \hat{w}_{xxx} = 0, \\ \hat{w}_{xx}(0,s) = \hat{h}_1(s), \; \hat{w}_x(L,s) = \hat{h}_2(s), \; \hat{w}_{xx}(L,s) = \hat{h}_3(s), \end{cases} \tag{A.2}$$

where

$$\hat{w}(x,s) = \int_0^{+\infty} e^{-st} w(x,t)dt$$

and

$$\hat{h}_j(s) = \int_0^{+\infty} e^{-st} h(t)dt, \quad j = 1,2,3.$$

The solution $\hat{w}(x,s)$ can be written in the form

$$\hat{w}(x,s) = \sum_{j=1}^{3} c_j(s) e^{\lambda_j(s)x},$$

where $\lambda_j(s)$, $j = 1,2,3$, are the solutions of the characteristic equation

$$s + \lambda^3 = 0$$

and $c_j(s)$, $j = 1,2,3$, solve the linear system

$$\underbrace{\begin{pmatrix} \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1 e^{\lambda_1 L} & \lambda_2 e^{\lambda_2 L} & \lambda_3 e^{\lambda_3 L} \\ \lambda_1^2 e^{\lambda_1 L} & \lambda_2^2 e^{\lambda_2 L} & \lambda_3^2 e^{\lambda_3 L} \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \hat{h}_1 \\ \hat{h}_2 \\ \hat{h}_3 \end{pmatrix}}_{\vec{h}}. \tag{A.3}$$

Let $\Delta(s)$ be the determinant of the coefficient matriz A and $\Delta_j(s)$, $j = 1, 2, 3$, the determinant of the matrix A with the jth-column replaced by $\vec{h}$. By Cramer's rule,

$$c_j = \frac{\Delta_j(s)}{\Delta(s)}, \quad j = 1, 2, 3,$$

provided that $\Delta(s) \neq 0$.

Claim: $\Delta(s) \neq 0$, for any $Re(s) \geq 0$.

Indeed, if otherwise, suppose $\Delta(s) = 0$, for some s with $Re(s) \geq 0$. Then, there exists a nontrivial $f \in H^3(0, L)$ satisfying

$$\begin{cases} sf(x) + f'''(x) = 0, & x \in (0, L), \\ f''(0) = 0, \quad f'(L) = 0, \quad f''(L) = 0. \end{cases} \tag{A.4}$$

Consider now the conjugate of (A.4), that is, the following system

$$\begin{cases} s\overline{f(x)} + \overline{f'''(x)} = 0, & x \in (0, L), \\ \overline{f''(0)} = 0, \quad \overline{f'(L)} = 0, \quad \overline{f''(L)} = 0. \end{cases} \tag{A.5}$$

Multiplying both sides of (A.4) by $\overline{f}$ and integrating over $(0, L)$, we get

$$\int_0^L sf\overline{f}dx + \int_0^L f'''\overline{f}dx = 0. \tag{A.6}$$

Then, if we multiply both sides of (A.5) by f and integrate over $(0, L)$, it follows that

$$\int_0^L s\overline{f}fdx + \int_0^L \overline{f'''}fdx = 0. \tag{A.7}$$

Integrating by parts (A.6) and (A.7) and adding the two resulting identities together yields that

$$2Re(s)\int_0^L |f(x)|^2dx = -|f'(0)|^2.$$

Consequently, we must have $Re(s) < 0$, as $||f||_{L^2(0,L)} \neq 0$ by the assumption. This is a contradiction. Thus, we conclude that $\Delta(s) \neq 0$, for any $Re(s) \geq 0$.

Note that the solution $w(x, t)$ for (A.1) can be written in the form

$$w(x, t) = \sum_{m=1}^{3} w_m(x, t), \tag{A.8}$$

where $w_m(x, t)$ solves (A.1) with $h_j \equiv 0$ when $j \neq m$, $j, m = 1, 2, 3$. Using the inverse Laplace transform yields

$$w(x, t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\hat{w}(x, s)ds = \sum_{j=1}^{3} \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{\Delta_j(s)}{\Delta(s)}e^{\lambda_j(s)x}ds,$$

for $r > 0$. Combining this formula and (A.8) we can write the values of w_m as follows, for $m = 1, 2, 3$,

$$w_m(x, t) = \sum_{j=1}^{3} \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_m(s)ds \equiv [W_{m,j}(t)h_m](x).$$

In the last two formulas, the right-hand sides are continuous with respect to r for $r \geq 0$. As the left-hand sides do not depend on r, we can take $r = 0$ in these formulas. Moreover,

$$w_{j,m}(x,t) = w^+_{j,m}(x,t) + w^-_{j.m}(x,t)$$

where

$$w^+_{j,m}(x,t) = \frac{1}{2\pi i}\int_0^{+i\infty} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}\hat{h}_m(s)e^{\lambda_j(s)x}ds$$

and

$$w^-_{j,m}(x,t) = \frac{1}{2\pi i}\int_{-i\infty}^{0} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}\hat{h}_m(s)e^{\lambda_j(s)x}ds,$$

for $j, m = 1, 2, 3$. Here, $\Delta_{j,m}(s)$ is obtained from $\Delta_j(s)$ by letting $\hat{h}_m(s) = 1$ and $\hat{h}_k(s) = 0$ for $k \neq m$, $k, m = 1, 2, 3$. Making the substitution $s = i\rho^3$ with $\rho \geq 0$ in the characteristic equation

$$s + \lambda^3 = 0,$$

the three roots are given in terms of ρ by

$$\lambda_1(\rho) = i\rho, \quad \lambda_2(\rho) = -i\rho\left(\frac{1+i\sqrt{3}}{2}\right), \quad \lambda_3(\rho) = -i\rho\left(\frac{1-i\sqrt{3}}{2}\right), \tag{A.9}$$

thus $w^+_{j,m}$ has the following form

$$w^+_{j,m}(x,t) = \frac{1}{2\pi i}\int_0^{+\infty} e^{i\rho^3 t}\frac{\Delta^+_{j,m}(\rho)}{\Delta^+(\rho)}\hat{h}^+_m(\rho)e^{\lambda^+_j(\rho)x}3i\rho^2 d\rho$$

and

$$w^-_{j,m}(x,t) = \overline{w^+_{j,m}(x,t)},$$

where $\hat{h}^+_m(\rho) = \hat{h}_m(i\rho^3)$, $\Delta^+(\rho) = \Delta(i\rho^3)$, $\Delta^+_{j,m}(\rho) = \Delta_{j,m}(i\rho^3)$ and $\lambda^+_j(\rho) = \lambda_j(i\rho^3)$.

Therefore, we have that the solution of the IBVP (A.1) has the representation in the form (2.8)-(2.11) as required. Thus the proof is finished. □

A.2 Proof Lemma 4.3.5

Proof Lemma 4.3.5. We follow the steps of [79]. Let $q = q(x,t)$ satisfying (4.100) and (4.101) and $\varphi(t,x) = \frac{\psi(x)}{t(T-t)}$, where ψ is a positive function (to be specified later). Consider $u := e^{-s\varphi}q$ and $w := e^{-s\varphi}P(e^{s\varphi}u)$, where P is a differential operator given by

$$P = \partial_t - \partial_x^2 + \partial_x^3.$$

Note that

$$\begin{aligned}
\partial_t(e^{s\varphi}u) &= e^{s\varphi}\left\{s\varphi_t u + u_t\right\}\\
\partial_x(e^{s\varphi}u) &= e^{s\varphi}\left\{s\varphi_x u + u_x\right\}\\
\partial_x^2(e^{s\varphi}u) &= e^{s\varphi}\left\{s\varphi_{xx}u + s^2\varphi_x^2 u + 2s\varphi_x u_x + u_{xx}\right\}\\
\partial_x^3(e^{s\varphi}u) &= e^{s\varphi}\left\{s\varphi_{xxx}u + 3s^2\varphi_x\varphi_{xx}u + 3s\varphi_{xx}u_x + s^3\varphi_x^3 u + 3s^2\varphi_x^2 u_x + 3s\varphi_x u_{xx} + u_{xxx}\right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
P(e^{s\varphi}u) = e^{s\varphi}\big\{&\left(s\varphi_t + s\varphi_{xxx} + 3s^2\varphi_x\varphi_{xx} + s^3\varphi_x^3 - s\varphi_{xx} - s^2\varphi_x^2\right)u\\
&+ \left(3s\varphi_{xx} + 3s^2\varphi_x^2 - 2s\varphi_x\right)u_x + \left(3s\varphi_x - 1\right)u_{xx} + u_{xxx} + u_t\big\}
\end{aligned}$$

and

$$w = Au + Bu_x + Cu_{xx} + u_{xxx} + u_t, \tag{A.10}$$

where

$$\begin{aligned}
A &= s(\varphi_t - \varphi_{xx} + \varphi_{xxx}) + 3s^2\varphi_x\varphi_{xx} + s^3\varphi_x^3 - s^2\varphi_x^2 \\
B &= 3s\varphi_{xx} + 3s^2\varphi_x^2 - 2s\varphi_x \\
C &= 3s\varphi_x - 1.
\end{aligned}$$

Set $L_1 u := u_t + u_{xxx} + Bu_x$ and $L_2 u := Au + Cu_{xx}$. Thus, we have

$$2\int_0^T\int_{-L}^L L_1(u)L_2(u)dxdt \le \int_0^T\int_{-L}^L (L_1(u)+L_2(u))^2\,dxdt \;= \int_0^T\int_{-L}^L w^2dxdt \tag{A.11}$$

Next, we compute the double product in (A.11). Let us denote by $(L_i u)_j$ the j-th term of $L_i u$ and $Q = (0,T)\times(-L,L)$. Then, to compute the integrals on the right hand side of (A.11), we perform integrations by part in x or t:

$$\begin{aligned}
((L_1u)_1,(L_2u)_1)_{L^2(Q)} &= -\frac{1}{2}\int_Q A_t u^2 dxdt \\
((L_1u)_2,(L_2u)_1)_{L^2(Q)} &= -\frac{1}{2}\int_Q A_{xxx} u^2 dxdt + \frac{3}{2}\int_Q A_x u_x^2 dxdt \\
((L_1u)_3,(L_2u)_1)_{L^2(Q)} &- \frac{1}{2}\int_Q (AB)_x u^2 dxdt \\
((L_1u)_2,(L_2u)_2)_{L^2(Q)} &= -\frac{1}{2}\int_Q C_x u^2 dxdt \\
((L_1u)_3,(L_2u)_2)_{L^2(Q)} &= -\frac{1}{2}\int_Q (BC)_x u_x^2 dxdt
\end{aligned}$$

By (A.10) and Young inequality, it follows that

$$\begin{aligned}
((L_1u)_1,(L_2u)_2)_{L^2(Q)} &= \frac{1}{2}\int_Q C_t u_x^2 dxdt + \frac{1}{2}\int_Q AC_x\partial_x(u^2)dxdt + \int_Q BCu_x^2dxdt \\
&\quad + \frac{1}{2}\int_Q CC_x\partial_x(u_x^2)dxdt + \int_Q C_x u_x u_{xxx}dxdt - \int_Q C_x u_x w dxdt \\
&\ge \frac{1}{2}\int_Q \{C_t + 2BC_x - (CC_x)_x + C_{xxx}\}\, u_x^2 dxdt - \frac{1}{2}\int_Q (AC_x)_x u^2 dxdt \\
&\quad - \int_Q C_x u_{xx}^2 dxdt - \varepsilon\int_Q C_x^2 u_x^2 dxdt - C(\varepsilon)\int_Q w^2 dxdt,
\end{aligned}$$

where ε is any number in $(0,1)$. Putting together the computations above, we obtain

$$\int_Q \left\{Du^2 + Eu_x^2 + Fu_{xx}^2\right\} dxdt = 2\left((L_1u, L_2u)_{L^2(Q)} \le C(\varepsilon)\int_Q w^2dxdt \tag{A.12}$$

with, D, E and F given by

$$\begin{aligned}
D &= -\left(A_t + A_{xxx} + (AB)_x + (C_xA)_x\right) \\
E &= 3A_x + BC_x - B_xC - (CC_x)_x + C_xxx + C_t - \varepsilon C_x^2 \\
F &= -3C_x.
\end{aligned} \tag{A.13}$$

The identities above allow us to conclude that

$$\begin{aligned}
D &= -15s^5\varphi_x^4\varphi_{xx} + \frac{O(s^4)}{t^4(T-t)^4}, \qquad (\text{as } s\to\infty), \\
&= -15\frac{s^5}{t^5(T-t)^5}\psi_x^4(x)\psi_{xx}(x) + \frac{O(s^4)}{t^4(T-t)^4}.
\end{aligned}$$

If we consider

$$|\psi_x(x)| > 0 \text{ and } \psi_{xx}(x) < 0, \text{ for all } x \in [-L, L], \tag{A.14}$$

taking s large enough, we obtain a constant $C_1 > 0$, such that

$$D \geq C_1 \frac{s^5}{t^5(T-t)^5}. \tag{A.15}$$

On the other hand, note that

$$\begin{aligned}
BC_x =& 9s^2\varphi_{xx}^2 + 9s^3\varphi_x^2\varphi_{xx} - 6s^2\varphi_x\varphi_{xx}\\
B_xC =& 9s^2\varphi_x\varphi_{xxx} + 18s^3\varphi_x^2\varphi_{xx} - 12s^2\varphi_x\varphi_{xx} - 3s\varphi_{xxx} + 2s\varphi_{xx}\\
3A_x =& 3s\varphi_{xt} + 3s\varphi_{4x} - 3s\varphi_{3x} + 9s^2\varphi_{xx}^2 + 9s^2\varphi_x\varphi_{xxx} + 9s^3\varphi_x^2\varphi_{xx} - 6s^2\varphi_x\varphi_{xx}\\
CC_x =& 9s^2\varphi_x\varphi_{xx} - 3s\varphi_{xx}\\
(CC_x)_x =& 9s^2\varphi_{xx}^2 + 9s^2\varphi_x\varphi_{xxx} - 3s\varphi_{xxx}\\
C_{xxx} + C_t - \varepsilon C_x^2 =& 3s\varphi_{4x} + 3s\varphi_{xt} - 9\varepsilon s^2\varphi_{xx}^2
\end{aligned}$$

Putting together these expressions, we have

$$E = 6s\varphi_{xt} + 6s\varphi_{4x} + 9(1-\varepsilon)s^2\varphi_{xx}^2 - 9s^2\varphi_x\varphi_{xxx} + 3s\varphi_{xxx} - 2s\varphi_{xx},$$

for E defined in (A.13). Hence,

$$\begin{aligned}
E =& 9s^2\left\{(1-\varepsilon)\varphi_{xx}^2 - \varphi_x\varphi_{xxx}\right\} + \frac{O(s)}{t^2(T-t)^2}, \qquad \text{as } s \to \infty,\\
=& 9\frac{s^2}{t^2(T-t)^2}\left\{(1-\varepsilon)\psi_{xx}^2(x) - \psi_x(x)\psi_{xxx}(x)\right\} + \frac{O(s)}{t^2(T-t)^2}.
\end{aligned}$$

For s large enough and ψ satisfying

$$\psi_x(x)\psi_{xxx}(x) < (1-\varepsilon)\psi_{xx}^2(x), \quad \text{for all } x \in [-L, L], \tag{A.16}$$

we obtain a constant $C_2 > 0$, such that

$$E \geq C_2 \frac{s^2}{t^2(T-t)^2}. \tag{A.17}$$

Finally, since

$$F = -9s\varphi_{xx} = -\frac{9s\psi_{xx}(x)}{t(T-t)},$$

(A.14) guarantees the existence of a constant $C_3 > 0$, such that

$$F \geq C_3 \frac{s}{t(T-t)}. \tag{A.18}$$

Combining (A.15), (A.17), (A.18) and (A.12), we obtain

$$\int_Q \left\{\frac{s^5}{t^5(T-t)^5}|u|^2 + \frac{s^2}{t^2(T-t)^2}|u_x|^2 + \frac{s}{t(T-t)}|u_{xx}|^2\right\} dxdt \leq C_4 \int_Q w^2 dxdt,$$

for some $C_4 > 0$. On the other hand, note that

$$\begin{aligned}
\int_Q \frac{s^3}{t^3(T-t)^3}u_x^2 dxdt =& -\int_Q \frac{s^3}{t^3(T-t)^3}uu_{xx}dxdt\\
\leq& \int_Q \frac{s^5}{2t^5(T-t)^5}|u|^2dxdt + \int_Q \frac{s}{2t(T-t)}|u_{xx}|^2dxdt\\
\leq& \frac{C_4}{2}\int_Q w^2 dxdt.
\end{aligned}$$

Then,

$$\int_Q \left\{ \frac{s^5}{t^5(T-t)^5}|u|^2 + \frac{s^3}{t^3(T-t)^3}|u_x|^2 + \frac{s}{t(T-t)}|u_{xx}|^2 \right\} dxdt \le C_5 \int_Q w^2 dxdt$$

where $C_5 > 0$, provided that (A.14) and (A.16) hold. Returning to the original variable $u = e^{-s\varphi}q$, we conclude the proof of Lemma 4.3.5. □

A.3 Proof of Proposition 5.3.1

In this section, we present the proof of the Proposition 5.3.1. It is based on an approximation theorem which has some resemblance to a result obtained in [79, Lemma 4.4].

In order to obtain our goal in this section, we establish some results. The first of them is a global Carleman inequality for the operator $-P^* = \partial_t + \partial_x^2 + \partial_x^3$ proved in [43, Lemma 3.5].

Lemma A.3.1. *Let T and L be positive numbers. Then, there exist a smooth positive function ψ on $[-L, L]$ (which depends on L) and positive constants $s_0 = s_0(L, T)$ and $C = C(L, T)$, such that, for all $s \ge s_0$ and any*

$$q \in C^3([-L, L] \times [0, T]) \tag{A.19}$$

satisfying

$$q(\pm L, t) = q_x(\pm L, t) = q_{xx}(\pm L, t) = 0, \text{ for } 0 \le t \le T, \tag{A.20}$$

we have that

$$\int_0^T \int_{-L}^L \left\{ \frac{s^5}{t^5(T-t)^5}|q|^2 + \frac{s^3}{t^3(T-t)^3}|q_x|^2 + \frac{s}{t(T-t)}|q_{xx}|^2 \right\} e^{-\frac{2s\psi(x)}{t(T-t)}} dxdt$$
$$\le C \int_0^T \int_{-L}^L |q_t + q_{xx} + q_{xxx}|^2 e^{-\frac{2s\psi(x)}{t(T-t)}} dxdt. \tag{A.21}$$

The following result uses the global Carleman inequality to obtain solutions to the KdV-Burgers equation, posed on $\mathbb{R}$, in the distribution sense.

Proposition A.3.2. *Let $L > 0$ and let $f = f(t, x)$ be any function such that*

$$f \in L^2(\mathbb{R} \times (-L, -L)), \qquad \operatorname{supp} f \subset [t_1, t_2] \times (-L, L),$$

where $-\infty < t_1 < t_2 < +\infty$. Then, for any $\varepsilon > 0$, there exists a positive constant C and a function $v \in L^2(\mathbb{R} \times (-L, L))$ such that

$$v_t - v_{xx} + v_{xxx} = f \quad \text{in } \mathcal{D}'(\mathbb{R} \times (-L, L)), \tag{A.22}$$

$$\operatorname{supp} v \subset [t_1 - \varepsilon, t_2 + \varepsilon] \times (-L, L), \tag{A.23}$$

$$\|v\|_{L^2(\mathbb{R}\times(-L,L))} \le C\|f\|_{L^2(\mathbb{R}\times(-L,L))}. \tag{A.24}$$

Proof. With the Global Carleman estimate (A.21) in hands, we use the same approach as in [79, Corollary 3.2]) with minor changes. □

Next, we state a lemma, which may be seen as a preliminary version of the approximation Theorem A.3.4 (below). Since the characteristic hyperplanes of the linear KdV-Burgers equation take the form $\{t = Const\}$, by using the Holmgren's uniqueness theorem, the proof of the lemma is word for word the same as the one given for the KdV equation [79, Lemma 4.2], hence we omit it.

Lemma A.3.3. *Let l_1, l_2, L, t_1, t_2, T be numbers, such that $0 < l_1 < l_2 < L$ and $0 < t_1 < t_2 < T$. Let $u \in L^2((0,T) \times (-l_2, l_2))$ be such that*

$$Pu = 0 \quad \text{in } (0,T) \times (-l_2, l_2) \quad \text{and} \quad \operatorname{supp} u \subset [t_1, t_2] \times (-l_2, l_2).$$

Then, for any $0 < 2\delta < \min(t_1, T - t_2)$ and $\eta > 0$, there exist $v_1, v_2 \in L^2(-L, L)$ and $v \in L^2((0,T) \times (-L, L))$ satisfying

$$Pv = 0 \quad \text{in } (0,T) \times (-L, L), \tag{A.25}$$

$$v(t) = S_L(t - t_1 + 2\delta)v_1 \quad \text{for} \quad t_1 - 2\delta < t < t_1 - \delta, \tag{A.26}$$

$$v(t) = S_L(t - t_2 - \delta)v_2 \quad \text{for} \quad t_2 + \delta < t < t_2 + 2\delta, \tag{A.27}$$

and

$$\|v - u\|_{L^2((t_1 - 2\delta, t_2 + 2\delta) \times (-l_1, l_1))} < \eta, \tag{A.28}$$

where P is the differential operator given by $P = \partial_t - \partial_x^2 + \partial_x^3$ and $S_L(\cdot)$ is the C_0 semigroup of contraction in $L^2(-L, L)$ generated by (5.3).

Now, we can establish the Approximation theorem, which differs from the approximation theorem in [83] by an additional property on the support of the solution.

Theorem A.3.4 (Approximation Theorem)**.** *Let $n \in \mathbb{N} \setminus \{0, 1\}$, t_1, t_2, T be numbers, such that $0 < t_1 < t_2 < T$, and let $u \in L^2((0,T) \times (-n, n))$ be such that*

$$u_t - u_{xx} + u_{xxx} = 0 \quad \text{in } (0,T) \times (-n, n) \quad \text{and} \quad \operatorname{supp} u \subset [t_1, t_2] \times (-n, n).$$

Then, for any $0 < \varepsilon < \min(t_1, T - t_2)$, there exists $v \in L^2((0,T) \times (-n-1, n+1))$ satisfying

$$v_t - v_{xx} + v_{xxx} = 0 \quad \text{in } (0,T) \times (-n-1, n+1), \tag{A.29}$$

$$\operatorname{supp} v \subset [t_1 - \varepsilon, t_2 + \varepsilon] \times (-n-1, n+1), \tag{A.30}$$

and

$$\|v - u\|_{L^2((0,T) \times (-n+1, n-1))} < \varepsilon. \tag{A.31}$$

Proof. The proof combines Lemma A.3.3, Proposition A.3.2 and the observability inequality (5.6) given in the Proposition 5.2.3. With these three ingredients, the proof is obtained following the same approach used for the KdV equation [79, Lemma 4.4]. □

Proof of Proposition 5.3.1. We begin with a claim that give us a sequence of functions, which limit will be the desired function.

Claim A.3.5. *There exist a sequence of numbers $\{t_1^n\}_{n \geq 2}$ and $\{t_2^n\}_{n \geq 2}$ such that*

$$t_1 - \varepsilon < t_1^{n+1} < t_1^n < t_1^2 < t_1 < t_2 < t_2^2 < t_2^n < t_2^{n+1} < t_2 + \varepsilon, \quad \forall n \geq 2,$$

with

$$\lim_{n \to \infty} t_1^n = t_1 - \varepsilon, \qquad \lim_{n \to \infty} t_2^n = t_2 + \varepsilon$$

and sequence of functions $\{u_n\}_{n \geq 2}$, such that

$$u_n \in L^2((0,T) \times (-n, n)), \tag{A.32}$$

$$\partial_t u_n - \partial_x^2 u_n + \partial_x^3 u_n = f, \quad \text{in} \quad (0,T) \times (-n, n), \tag{A.33}$$

$$\operatorname{supp} u_n \subset [t_1^n, t_2^n] \times (-n, n), \tag{A.34}$$

and, if $n > 2$,

$$\|u_n - u_{n-1}\|_{L^2((0,T) \times (-n+2, n-2))} < 2^{-n}. \tag{A.35}$$

Proof of Claim A.3.5. We will construct the sequence $\{t_i^n\}_{n\geq 2}$ and $\{u_n\}_{n\geq 2}$ by induction on n. Indeed, u_2 is given by Proposition A.3.2. Suppose that $u_2, ..., u_n$ have been construct satisfying (A.32)-(A.35). Again, applying the Proposition A.3.2 with $L = n+1$, there exists $w \in L^2((0,T) \times (-n-1, n+1))$ such that

$$\begin{gathered} Pw = f, \quad \text{in} \quad \mathcal{D}'((0,T) \times (-n-1, n+1)), \\ \operatorname{supp} w \subset [t_1^2, t_2^2] \times (-n-1, n+1), \end{gathered}$$

where $P = \partial_t - \partial_x^2 + \partial_x^3$. Note that $P(u_n - w) = 0$ in $(0,T) \times (-n, n)$ and

$$\operatorname{supp}(u_n - w|_{(0,T)\times(-n,n)}) \subset [t_1^n, t_2^n] \times (-n, n).$$

From the Approximation Theorem A.3.4, there exist $v \in L^2((0,T) \times (-n-1, n+1))$ such that

$$\begin{gathered} Pw = 0, \quad \text{in} \quad (0,T) \times (-n-1, n+1), \\ \operatorname{supp} v \subset [t_1^{n+1}, t_2^{n+1}] \times (-n-1, n+1), \\ \|v - (u_n - w)\|_{L^2((0,T)\times(-n+1,n-1))} < 2^{-n-1}. \end{gathered}$$

with $t_1^{n+1} < t_1^n < t_2^n < t_2^{n+1}$. Now, define $u_{n+1} = v - w$, thus (A.32)-(A.35) are fulfilled. □

Consider the extension

$$\widetilde{u}_n = \begin{cases} u_n & \text{in } (0,T) \times (-n, n) \\ 0 & \text{in } \mathbb{R}^2 \setminus (0,T) \times (-n, n). \end{cases}$$

Thus, by (A.35) in Claim A.3.5, $\{\widetilde{u}_n\}_{n\geq 2}$ is a Cauchy sequence in $L^2_{loc}(\mathbb{R}^2)$, hence there exists a function $u \in L^2_{loc}(\mathbb{R}^2)$, such that

$$\widetilde{u}_n \longrightarrow u, \quad \text{in} \quad L^2_{loc}(\mathbb{R}^2).$$

Then (A.33) and (A.34) imply (5.51) and (5.52), respectively. □

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