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On first integrals of holomorphic foliations

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Contents

Introduction

Throughout this work, moving us in various scenarios different problems are boarded, some of them are solved, but even the unsuccessful ones are presented. All of them related with the existence of first integrals (meromorphic in one case, holomorphic in the remaining ones) for holomorphic foliations under different conditions.

In the first chapter we start the path that will drive us to our first main result. With the purpose of contextualize this chapter remember an important tool in the study of foliations (real and holomorphic), *the holonomy group*, two clear examples (among many others) of its importance are the stability theorems of Reeb (see Camacho and Lins Neto [10] chap. IV) and the theorem of existence and uniqueness of first integrals of Mattei and Moussu [30]. In the context of holomorphic foliations, the holonomy groups are finitely generated groups of germs of diffeomorphisms in $\mathbb{C}^n$ fixing the origin. Those groups have been highly studied for many authors and important results have been achieved both in dimension 1 and in general dimension (for a survey of results in this area see Abate [3], Bracci [7, 8], Raissy [38]). In particular in Chapter 1, aiming to find conditions for their periodicity, we analyze groups of germs of diffeomorphisms in dimension $n \geq 2$, finitely generated and having infinitely many invariant curves. The following is the first result we present, note that it is quite similar to Theorem 3.1 in Brochero Martínez [9] (we properly explain in Chapter 1 why we use one instead the other),

Theorem A. *Let $G \in \mathrm{Diff}(\mathbb{C}^n, 0)$. Then G generates a finite group if and only if, there exists a neighborhood U of 0 such that $|O_U(x, G)| < \infty$ for all $x \in U$ and G leaves invariant non-enumerable many analytic varieties at 0 of dimension $n-1$.*

To see the importance of Theorem A in this work, is necessary to jump to Chapter 4, where is obtained the following result, where "Gen $(\mathfrak{X}(\mathbb{C}^3, 0))$" stands for *generic vector field*, see Definition 3.1.1, and condition $(\star)$ means that the eigenvalues of the vector field can be rotated in a such way that one of them has positive real part and the others negative (see Definition 3.1.4).

Theorem B. *Let $\mathcal{F}(\mathcal{X})$ be the germ of a holomorphic foliation with $\mathcal{X} \in$ $\mathrm{Gen}\,(\mathfrak{X}(\mathbb{C}^3, 0))$ and satisfying condition ($\star$). Then $\mathcal{F}(\mathcal{X})$ has a holomorphic first integral if, and only if, the leaves of $\mathcal{F}(\mathcal{X})$ are closed off the singularity and there exist non-enumerable many $\mathcal{X}$-invariant analytic hypersurfaces passing through* 0 *and in general position.*

In Chapter 2 properties of formal series and diffeomorphisms, which are the machinery needed in Chapter 3, are presented. With the aim of motivate the next result, remember that in Mattei and Moussu [30] besides showing that the germ of a holomorphic foliation of codimension 1 in a neighborhood of $0 \in \mathbb{C}^n$ with an isolated singularity at the origin, closed leaves off the singularity and finitely many separatrices possesses a holomorphic first integral. The authors also manage to prove that having a formal first integral there is way to obtain from it a holomorphic one. A similar step is given in Malgrange [27] where some other results about the existence of a holomorphic first integral are found (see also Cerveau and Lins Neto [17]).

Our main motivation comes from Câmara and Scárdua [14] where conditions are given for the existence of a holomorphic first integral for a generic germ of foliation of dimension one in $(\mathbb{C}^3, 0)$ but it is missing the "formal to holomorphic step" letting open the question: *Does in this scenario the existence of a formal first integral implies the existence of a holomorphic one?*. It turns out that there is a positive answer that we resume in the next theorem:

Theorem C. *Let $\mathcal{F}(\mathcal{X})$ be a germ of holomorphic foliation with $\mathcal{X} \in \mathrm{Gen}\,(\mathfrak{X}(\mathbb{C}^3, 0))$. If $\mathcal{F}(\mathcal{X})$ has a formal first integral then it also has a holomorphic one.*

Up to here we were focused in dimension 3, though some results are valid in greater dimension. Changing to dimension two, in Chapter 5 is presented the following stability theorem:

Theorem 5.2.2. *Let $\mathcal{F}$ be a holomorphic foliation of codimension 1 on a compact, connected and complex surface M. Suppose that there is an invariant divisor $\mathcal{D} \subset M$ such that:*

(i) *The virtual holonomy of the components of $\mathcal{D}$ is finite.*

(ii) *The elements in $\mathcal{D} \cap \mathrm{sing}(\mathcal{F})$ are isolated singularities of $\mathcal{F}$.*

(iii) *If a singularity $p \in \mathcal{D} \cap \mathrm{sing}(\mathcal{F})$ is non dicritical then $\mathcal{D}$ contains all the separatrices of $\mathcal{F}$ through p.*

(iv) *If a singularity $q \in \mathcal{D} \cap \mathrm{sing}(\mathcal{F})$ is dicritical then for its separatrices L_q in $\mathcal{D}$ the closure of $\tilde{L}_q = E^{-1}(L_q \setminus \{q\})$, where $E : \tilde{M} \to M$ is* *the resolution map (finite composition of blow-ups), cuts a dicritical component of D_q (its exceptional divisor).*

Then $\mathcal{F}$ has a meromorphic first integral.

Finally, in Chapter 6 we unsuccessfully sought a way of proof the main result of Câmara and Scárdua [14] but with the technique of Moussu [31]. We explain there the technique and the limitations we found.

Maybe less important but possibly useful there are results obtained while working in the previous ones. For instance, the same methods used to prove Theorem A allowed us to show the following

Theorem 1.2.7. *Let $G \in \mathrm{Diff}(\mathbb{C}^n, 0)$. The group generated by G is finite if and only if, there exists a neighborhood U of 0 such that $|O_U(x, G)| < \infty$ for all $x \in U$, and G leaves invariant non-enumerable many analytic varieties of complex dimension 1, in general position, arbitrarily close to 0, and each one intersecting the set $C = C_{\mathbb{C}^n}$ defined as in Lemma 1.1.5.*

and

Theorem 1.3.2. *Let $G \in \mathrm{Diff}(\mathbb{C}^n, 0)$. The group generated by G is finite if and only if, it exists $m \in \mathbb{N}$ such that for an arbitrary neighborhood of 0, G leaves invariant infinitely many analytic varieties of complex dimension 1, in general position and each one having a convergent sequence of periodic points of order at most m.*

Also, as a first step in the proof of Theorem B we proof

Theorem 4.2.3. *Let $\mathcal{X}$ be a germ of homogeneous vector field in $0 \in \mathbb{C}^3$. Suppose that $\mathcal{X}$ leaves invariant infinitely many hypersurfaces passing through 0 and in general position. Then, there exists a rational map $f : \mathbb{C}P(2) \to \mathbb{C}P(1)$ that is $\mathcal{F}(\mathcal{X})$-invariant (i. e., $\mathcal{X}(f) \equiv 0$) this map is also call it a weak first integral of $\mathcal{F}(\mathcal{X})$.*

Obviously, we got propositions and lemmas along the way.

Chapter 1

Groups of germs diffeomorphisms

In this chapter, after introducing some definitions and notations, we will mention some results that although interesting on their own, the way how they intervene throughout this work is what transform them in a fundamental piece of this thesis.

Sections two and three are based on Theorem 3.1 in [9]. Theorem A is its generalization to dimension $n > 2$ (as the author points out in [9]) and Theorem 1.2.3 is its version for finite generated groups. In Theorems 1.3.1 and 1.3.2 we make a few changes in its hypothesis, maintaining valid the original conclusion, obtaining in this way two new versions of it.

It is worth to say that only small changes in the original proof in [9] are needed to demonstrate the previous theorems. Nevertheless, we will write down each one of the proofs in order to make easy to note the difference among them. After this, we present some recent results in this topic (see [39, 43])

We end this chapter with some comments on Theorem 3.1 in [9].

1.1 Preliminaries

Let $\mathrm{Diff}(\mathbb{C}^n, 0)$ be the group of germs of diffeomorphisms at $0 \in \mathbb{C}^n$. The germ $G \in \mathrm{Diff}(\mathbb{C}^n, 0)$ will be represented by the map G in a domain U where $G(U)$ and $G^{-1}(U)$ are well defined, and U is an open neighborhood of the origin with compact closure. We will use the following notation,

$$\begin{aligned} O_U(x, G) = \{&G^p(x) \,|\, G(x), \ldots, G^p(x) \in U\} \cup \\ &\{G^{-q}(x) \,|\, G^{-1}(x), \ldots, G^{-q}(x) \in U\} \cup \{x\} \end{aligned}$$

for the G-orbit of x in U, $|O_U(x,G)|$ for the number of elements in its G-orbit and

$$\begin{aligned}\mu_U(x,G) = \sup\{p>0 \,|\, G^p(x) \in O_U(x,G)\}+ \\ \sup\{p>0 \,|\, G^{-p}(x) \in O_U(x,G)\} + 1,\end{aligned}$$

for the number of iterates of x in U. If $\mu_U(x,G)=\infty$ and $|O_U(x,G)|<\infty$ we say that the point x is *periodic* in U, if $\mu_U(x,G)$ is finite then it is equal to $|O_U(x,G)|$. We say that G has *finite orbits* if $|O_U(x,G)|<\infty$ for all $x\in U$.

Regarding the finiteness of groups generated by germs of diffeomorphisms Mattei-Moussu gave in [30] p. 477 the following criteria for the one dimensional case.

Theorem 1.1.1. *An element $G \in \mathrm{Diff}(\mathbb{C},0)$ is periodic if and only if it has finite orbits.*

Another proof of this theorem (using Pérez-Marco's work) is given in [31].

It is easy to see that Theorem 1.1.1 is not true in dimension grater than one (consider for example the map $G(x,y)=(x+y^2,y)$ whose orbits are finite but is not periodic). However, with an additional hypothesis, Theorem 1.1.2 (which is Theorem 3.1 in [9]) attempts to generalize this criterion. The reason we say "attempts" is because the proof presented in [9] is inaccurate. We believe in the result but our attempt to prove it did not succeed. For this reason we put an additional hypothesis that allows us to prove it, as we do below in Theorem A.

Theorem 1.1.2 (Brochero)**.** *Let $G\in \mathrm{Diff}(\mathbb{C}^2,0)$. Then G generates a finite group if and only if there exists a neighborhood V of 0 such that $|O_V(x,G)|<\infty$ for all $x\in V$ and G leaves invariant infinitely many analytic varieties at 0.*

In fact, in the previous two theorems we can change the diffeomorphism G by a finite generated group $\mathcal{G}\subset \mathrm{Diff}(\mathbb{C},0)$ (or $\mathrm{Diff}(\mathbb{C}^2,0)$ respectively) taking into account the second affirmation of Lemma 3.3 in [42] that says:

Lemma 1.1.3. *Let $\mathcal{G}\subset \mathrm{Diff}(\mathbb{C}^k,0)$ be a finitely generated subgroup. Assume that there is an invariant connected neighborhood W of the origin in $\mathbb{C}^k$ such that each point x is periodic for each element $G\in\mathcal{G}$. Then $\mathcal{G}$ is a finite group.*

The following topological lemma is a modification of the Lewowicz's Lemma and plays an important role throughout this chapter.

Lemma 1.1.4. *Let M, be a complex analytic variety in $\mathbb{C}^n$, with $0 \in M$, and K be the connected component of 0 in $\overline{B}_r(0) \cap M$. Suppose that f is a homeomorphism from K to $f(K) \subset M$ such that $f(0) = 0$. Then there exists $x \in \partial K$ such that the number of iterations $f^m(x) \in K$ is infinite.*

Proof. Denote by $\overline{\mu} = \mu|_K$ and $\mu = \mu|_{\mathring{K}}$ the number of iteration in K and $\mathring{K}$. It is easy to see that $\overline{\mu}$ is upper semicontinuous, μ is lower semicontinuous and $\overline{\mu}(x) \geq \mu(x)$ for all $x \in \mathring{K}$. Suppose by contradiction that $\overline{\mu}(x) < \infty$ for all $x \in \partial K$, therefore there exists $n \in \mathbb{N}$ such that $\overline{\mu}(x) < n$ for all $x \in \partial K$. Let $A = \{x \in K \,|\, \overline{\mu}(x) < n\} \supset \partial K$ and $B = \{x \in \mathring{K} \,|\, \mu(x) \geq n\} \ni 0$. They are open sets satisfying and $A \cap B = \emptyset$ since $\overline{\mu}(x) \geq \mu(x)$.
Using the fact that K is a connected set, there exists $x_0 \in K \setminus (A \cup B)$ i.e $\overline{\mu}(x_0) \geq n > \mu(x_0)$. Then the orbit of x_0 intersects the border of K, which is a contradiction since $\partial K \subset A$ would give that $x_0 \in A$. To see clearer this last point, note that if $G^{m'}(x_0) \in \partial K$ for some $m' \in \mathbb{Z}$, then $G^{m'}(x_0) \in A$, i. e., $\overline{\mu}(G^{m'}(x_0)) < n$ but by definition $\overline{\mu}(G^{m'}(x_0)) = \overline{\mu}(x_0)$ thus $x_0 \in A$ ✓

In our framework Lemma 1.1.4 implies:

Lemma 1.1.5. *Let $G \in \mathrm{Diff}(\mathbb{C}^n, 0)$ and M be a G-invariant complex analytic variety passing through $0 \in \mathbb{C}^n$. There exist a compact, connected, and non-enumerable set C_M such that $0 \in C_M$ and, for all $x \in C_M$ and $n \in \mathbb{N}$, we have $G^n(x) \in M \cap U$ for a domain U where $G(U)$ and $G^{-1}(U)$ are well defined.*

Proof. Without loss of generality we suppose that $U = \overline{B}_r(0)$. Let M be a G-invariant complex analytic variety and $K = M \cap U$ be the connected component of $M \cap U$ in 0. Let $A_1 = K$, $A_{j+1} = K \cap G^{-1}(A_j)$ and C_n be the connected component of A_n in 0. It is clear, by construction, that A_n is the set of points of K with n or more G-iterates in K. Moreover, since A_n is compact and C_n is compact and connected, it follows that $C_M = \bigcap_n C_n$ is compact and connected too, and therefore either $C_M = \{0\}$ or C_M is non-enumerable.

We claim that $C_M \cap \partial K \neq \emptyset$ and then it is non-enumerate. In fact, if $C_M \cap \partial K = \emptyset$ then there would exist j such that $C_j \cap \partial K = \emptyset$. Let B be a compact connected neighborhood of C_j such that $(A_j \setminus C_j) \cap B = \emptyset$. Therefore for all $x \in \partial B$ we have $\mu_B(x, G) < j$, that is a contradiction by the Lemma 1.1.4. ✓

The previous lemma is part of the proof of Theorem 1.1.2 in [9], but due to its importance and constant use throughout this chapter, we decided to write it as an independent result.

1.2 Groups of diffeomorphisms in dimension n fixing 0

We start by presenting a proof of Theorem 1.1.2 in dimension n. In this proof we follow the original one, although adapting some arguments to our case and changing one of the hypothesis in order to avoid an imprecision found in the original proof (later on we will discuss this topic).

The following well known proposition is the analytic case of Proposition 3.1 in [9], it is also true in the formal case (the demonstration is the same) and it will be use in the proof of the Theorem A.

Proposition 1.2.1. *Let $\mathcal{G}$ be a finite subgroup of $\mathrm{Diff}(\mathbb{C}^n,0)$ then $\mathcal{G}$ is analytic linearizable, and it is isomorphic to a finite subgroup of $Gl(n,\mathbb{C})$.*

Proof. If $\mathcal{G}=\{G_1,\ldots,G_r\}$, let $h^{-1}(x)=\sum_{j=1}^{r}(\mathrm{d}G_j)_0^{-1}G_j(x)$, Note that h^{-1} is a diffeomorphism because $\mathrm{d}h^{-1}(0)=rI$ and

$$\begin{aligned} h^{-1}\big(G_i(x)\big) &= \sum_{j=1}^{r}(\mathrm{d}G_j)_0^{-1}G_j\big(G_i(x)\big) = (\mathrm{d}G_i)_0\sum_{j=1}^{r}(\mathrm{d}G_i)_0^{-1}(\mathrm{d}G_j)_0^{-1}G_j\big(G_i(x)\big) \\ &= (\mathrm{d}G_i)_0\sum_{j=1}^{r}\big((\mathrm{d}G_j)_0(\mathrm{d}G_i)_0\big)_0^{-1}G_j\big(G_i(x)\big) \\ &= (\mathrm{d}G_i)_0\sum_{j=1}^{r}\Big(\mathrm{d}\big(G_j\circ G_i\big)\Big)_0^{-1}G_j\big(G_i(x)\big) = (\mathrm{d}G_i)_0 h^{-1}(x). \end{aligned}$$

Thus $h^{-1}\circ G_i\circ h(x)=(\mathrm{d}G_i)_0(x)$. In fact, we obtain an injective group homomorphism

$$\begin{aligned} \Lambda:\mathcal{G} &\longrightarrow Gl(n,\mathbb{C}) \\ G &\longrightarrow (h^{-1}\circ G\circ h)'(0). \end{aligned}$$ ✓

Furthermore, in [9] it is proved (after the proposition above) that the group $\Lambda(\mathcal{G})\subset Gl(n,\mathbb{C})$ of linear parts of the diffeomorphisms in $\mathcal{G}$ is diagonalizable.

The following theorem is the generalization of Theorem 1.1.2 to dimension n but, as we mention above, it is necessary to change one of the hypothesis. To be precise, instead of "G leaves invariant infinitely many analytic varieties at 0" we put "G leaves invariant non-enumerable many analytic varieties at 0". In order to clarify this point, after Theorem 1.2.3 we write down the proof of Theorem 1.1.2 as spears in [9] and we explain why this change was necessary.

Theorem A. *Let $G \in \mathrm{Diff}(\mathbb{C}^n, 0)$. Then G generates a finite group if and only if there exists a neighborhood U of 0 such that $|O_U(x, G)| < \infty$ for all $x \in U$ and G leaves invariant non-enumerable many analytic varieties at 0 of dimension $n-1$.*

Proof. ($\Rightarrow$) If the group generated by G is $\mathcal{G} = \{G, G^2, \ldots, G^r\}$ obviously, for all x in a neighborhood U where G^i is defined for all i, we have that $O_U(x, \mathcal{G})$ is finite. In fact, $O_U(x, \mathcal{G}) = \{G(x), \ldots, G^r(x)\}$.

Now, consider, as in Proposition 1.2.1, $h^{-1}(x) = \sum_j^r (\mathrm{d}G^j)_0^{-1} G^j(x)$ which is such that $h^{-1} \circ G^i \circ h(x) = (\mathrm{d}G^i)_0(x)$ for all i, where $(\mathrm{d}G^i)_0^{n_i} = I$ for some n_i. This implies that $(\mathrm{d}G^i)_0$ is diagonalizable, we then suppose that $(\mathrm{d}G^i)_0$ is diagonal, in fact h can be defined as a diffeomorphism which also diagonalizes the group, since in our case the group is cyclic then the linear parts are simultaneously diagonalizable. In the definition of h^{-1} is sufficient to change $(\mathrm{d}G^j)_0$ to $P^{-1}(\mathrm{d}G^j)_0 P$, where P is the matrix that diagonalizes the group of linear parts, it is easy to see that the proof of Proposition 1.2.1 works. With this, we define

$$M_c = \big\{ y = h(x) \in U \,\big|\, c_1 x_1^m + \cdots + c_n x_n^m = 0 \big\}, \tag{1.1}$$

where $m = n_1 \cdots n_r$ and $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$. M_c is a $\mathcal{G}$-invariant complex analytic variety of dimension $n-1$ for each $c \in \mathbb{C}^n$. In order to see this, take $y \in M_c$ which, by definition, is equal to $h(x)$ for some $x \in U$ satisfying (1.1) then we have to prove that $G^i(y) \in M_c$ for $i = 1, \ldots, r$,

$$\begin{aligned} G^i(y) = G^i(h(x)) &= h\big(h^{-1} \circ G^i \circ h(x)\big), \\ &= h\big((\mathrm{d}G^i)_0 x\big), \end{aligned}$$

and using that $(\mathrm{d}G^i)_0$ is diagonal, we have (in multi index notation)

$$\big((\mathrm{d}G^i)_0 x\big)^m = (\mathrm{d}G^i)_0^m x^m = x^m.$$

Therefore, if $y = h(x) \in M_c$ then $G^i(y) = h\big((\mathrm{d}G^i)_0 x\big) \in M_c$.

($\Leftarrow$) Consider $M = \mathbb{C}^n$ in Lemma 1.1.5. Then $C = C_{\mathbb{C}^n}$ is the compact, connected and non-enumerable set of points in U such that $\mu_U(x, G) = \infty$ and therefore every point in C is periodic. If we denote $D_m = \bigcup\{x \in C \,|\, G^{m!}(x) = x\}$, it is clear that D_m is a closed set and $D_m \subset D_{m+1}$, moreover $C = \bigcup D_m$. Fix $m \in \mathbb{N}$ and consider $F = G^{m!}$, which is well defined in some neighborhood U' of $0 \in \mathbb{C}^n$, observe that C is in the domain U' of F and take $L = \{x \in U' |\, F(x) = x\}$. Since L is a complex analytic variety of U' then it can be written as a finite union of them, with dimensions ranging from 1 to n. Even if all of them were of dimension $n-1$, using Lemma

1.1.5 for every invariant analytic variety W we conclude that there are non-enumerable $C_W \subset C$ not contained in the decomposition of L, since m is arbitrary and C is the enumerable union of the D_m, it can be deduced that there exists an m such that L is of dimension n. It follows that $G^{m!}(x) = x$ for all $x \in U'$ by the identity theorem (see [22] pag 5), hence the group generated by G is finite. ✓

The version of the previous theorem for finitely generated groups of diffeomorphisms is immediate,

Theorem 1.2.3. *Let $\mathcal{G} = \langle\{G_1, \ldots, G_m\}\rangle \prec \mathrm{Diff}(\mathbb{C}^n, 0)$ be a finitely generated subgroup of diffeomorphisms. Then $\mathcal{G}$ is finite if and only if there exists a neighborhood U of 0 such that $|O_U(x, \mathcal{G})| < \infty$ for all $x \in U$ and each G_i leaves invariant non-enumerable many analytic varieties at 0 of dimension $n-1$.*

Proof. ($\Rightarrow$) This part is the same as the previous theorem. Note that in the hypothesis each generator of the group leaves invariant infinitely many analytic varieties, then we can apply the same construction for each one.

($\Leftarrow$) Using Theorem A we have that every element in $\mathcal{G}$ has finite order and, since $\mathcal{G}$ is finitely generated, we can apply Lemma 1.1.3 in order to conclude that $\mathcal{G}$ is finite. ✓

The following is the proof of Theorem 1.1.2 as can be seen in [9] page 7.

Proof of Theorem 1.1.2. ($\Rightarrow$) Let $N = \#\langle F\rangle$ and $h \in \mathrm{Diff}(\mathbb{C}^2, 0)$ such that $h \circ F \circ h^{-1}(x, y) = (\lambda_1 x, \lambda_2 y)$ where $\lambda_1^N = \lambda_2^N = 1$. It is clear than $|O(x, F)| \leq N$ for all x in the domain of F , and $M_c = \{h(x, y) \,|\, x^N - cy^N = 0\}$ is a complex analytic variety invariant by F for all $c \in \mathbb{C}$.

($\Leftarrow$) Consider Lemma 1.1.5 with $M = \mathbb{C}^2$, then $C = C_{\mathbb{C}^2}$ is a set of point with infinite orbits in a domain $U = \overline{B}_r(0)$ where F and F^{-1} are well defined and therefore every point in C is periodic. If we denote $D_m = \{x \in C \,|\, F^{m!}(x) = x\}$, it is clear that D_m is a closed set and $D_m \subset D_{m+1}$. Moreover, since $C = \cup_{m=1} D_m$, there exists $m \in \mathbb{N}$ such that $C = D_m$. Let $G = F^{m!}$, which is well defined in some neighborhood U of $0 \in \mathbb{C}^n$, observe that C is in the domain U of G and $C \subset \{x \in U \,|\, G(x) = x\} = L$. Since L is a complex analytic variety of U that contains C, its dimension is 1 or 2. The case $\dim L = 1$ is impossible because $C_M \subset C \subset L$ for all F-invariant analytic variety M, contradicting that fact that $\mathcal{O}_2$ is Noetherian ring. In the case $\dim L = 2$ follows that $F^{m!}(x) = x$ for all $x \in U$, therefore $\langle F\rangle$ is finite. ✓

The problem with the proof above is in the statement:

"..., there exists $m \in \mathbb{N}$ such that $C = D_m$".

which is not always true because the sets D_n may have empty interior. In fact if one of them happens to have interior the proof ends by the Identity Theorem. Another way of see the problem with this statement is to note that the increasing sequence of analytic sets $D_n \subset D_{n+1}$ generates a decreasing sequence of ideals, and even in Noetherian rings (as $\mathcal{O}_n$) decreasing sequences of ideals do not always stabilize. They do when they are prime which is equivalent to the D_n being irreducible (see [23] pag. 15). Now, if they are irreducible and of dimension 1 all of them are the same one and the set C consists of a single analytic curve which contradicts the hypothesis that there are infinitely many G-invariant analytic varieties at 0, and we are done. It would remain the case where the sequence of ideals does not stabilize.

We could not get a different proof of Theorem 1.1.2. We have advances in this direction though (see Section 1.5), but its importance in our work forces us to change the hypothesis to those of Theorem A.

We close this section by noting that Theorem A is valid, as the author [9] mentions, if we consider analytic varieties of complex dimension 1 in *general position* instead of analytic varieties of complex dimension $n-1$,

Definition 1.2.4. We say that infinitely many analytic varieties of complex dimension 1 are in *general position* if they are not contained in finitely many analytic varieties of complex dimension $n-1$.

The only change in the proof is in the "if" part, where it is necessary one more step. Note that choosing $n-1$ linearly independent constants c, the intersection of the corresponding M_c has a component of dimension 1 passing through 0. In this way we can obtain a non-enumerable set in general position. We state the theorem in terms of analytic varieties of complex dimension $n-1$ because it is more natural and it does not require to add more conditions. However, it can be useful to think in dimension one as we see next.

The following lemma follows from an analysis similar to the one made in the proof of the Theorem A, but in order to make further reference, it is more appropriate to state it on its own.

Lemma 1.2.5. *Let $G \in \mathrm{Diff}(\mathbb{C}^n, 0)$. The group generated by G is finite if and only if there exist $m \in \mathbb{N}$ such that for an arbitrary neighborhood of $0 \in \mathbb{C}^n$, G^m has infinitely many fixed analytic varieties of complex dimension* 1*, in general position.*

Proof. Consider $F = G^m$, which is well defined in some neighborhood U of $0 \in \mathbb{C}^n$ and take $L = \{x \in U |\, F(x) = x\}$. Since L is a complex analytic variety of U then it can be written as a finite union of them, with dimensions ranging from 1 to n. But, that finite union can not contain infinitely many analytic varieties of complex dimension 1 in general position, unless the dimension of L (i.e., the supreme of he dimensions of its connected components) be equal to n. It follows that $F(x) = x$ for all $x \in U$ by the identity theorem (see [22] pag 5), hence the group generated by G is finite. ✓

Remark 1.2.6. Observe that in Lemma 1.2.5 we are not asking for the analytic varieties that they contain zero. What we need is that infinitely many of them cut the neighborhood U where F is defined.

In Theorems 1.2.7 and 1.3.2 we basically find a way to obtain the infinitely many fixed analytic varieties of complex dimension 1 that Lemma 1.2.5 requires.

Theorem 1.2.7. *Let $G \in \mathrm{Diff}(\mathbb{C}^n, 0)$. The group generated by G is finite if and only if there exists a neighborhood U of 0 such that $|O_U(x, G)| < \infty$ for all $x \in U$, and G leaves invariant non-enumerable many analytic varieties of complex dimension 1, in general position, arbitrarily close to 0, with each one intersecting the set $C = C_{\mathbb{C}^n}$ defined in Lemma 1.1.5.*

Proof. Consider $M = \mathbb{C}^n$ in Lemma 1.1.5. Then $C = C_{\mathbb{C}^n}$ is the compact, connected and non-enumerable set of points in U such that $\mu_U(x, G) = \infty$ and therefore every point in C is periodic. If we denote $D_m = \bigcup\{x \in C \mid G^{m!}(x) = x\}$, it is clear that D_m is a closed set and $D_m \subset D_{m+1}$, moreover $C = \bigcup D_m$. Now, if some invariant analytic variety W intersects C in a periodic point $q \in U'$ of order k, then Lemma 1.1.5 can be applied to the map G^k in some neighborhood of q contained in W and we obtain a compact, connected and non-enumerable set $C_W \subset W$ which is fixed for some iterate of G^k (see Remark 1.2.8.), observe that C_W belongs to C. Therefore, there exist non-enumerable many curves C_W in C, this implies that infinitely many belongs to some D_m, because C is the enumerable union of them. We conclude that $G^{m!}$ has infinitely many fixed analytic varieties of complex dimension 1, in general position. Hence the group generated by G is finite by Lemma 1.2.5. ✓

Remark 1.2.8. $C_W \subset W$ is fixed for some iterate of G^k because, by Lemma 1.1.5 and the finiteness of the orbits of G, it is the set of G^k-periodic points. Since C_W is non-enumerable, there exist infinitely many periodic points of some order $k' = km$, for an $m \in \mathbb{N}$. The compactness of C_W implies that those periodic points have an accumulation point. The dimension of W is one hence the Identity Theorem implies that C_W is $G^{k'}$-fixed.

1.3 Conditions over the set of periodic points

The second part of the proof of Theorem A make us think that what we really need is a sufficient amount of periodic points, but even in dimension one infinitely many of them accumulating at the origin is not enough. To be precise, according to Perez-Marco in [33], it is possible to construct map germs in $\mathrm{Diff}(\mathbb{C}, 0)$ exhibiting a sequence of periodic points converging to $0 \in \mathbb{C}$ and not linearizable. Obviously the orders of points in that sequence goes to infinity because if some subsequence had bounded order by some m then after $m!$ iterates the function could have a sequence of fixed points accumulating $0 \in \mathbb{C}$. By the identity theorem that iteration could be identity and the map periodic. However, in dimension greater than 1, the existence of of a convergent sequence of fixed points is not enough to guarantee that a map is the identity. That is why we asked for a dense set of periodic points while keeping the bound over the order.

Proposition 1.3.1. *Let $G \in \mathrm{Diff}(\mathbb{C}^n, 0)$. The group generated by G is finite if and only if there exist $m \in \mathbb{N}$ such that for an arbitrary neighborhood of 0 the set of periodic orbits of period at most m is dense.*

Proof. ($\Rightarrow$) Suppose $\langle G \rangle = \{id, \ldots, G^{r-1}\}$ for $r \in \mathbb{N}$ and G well defined in a neighborhood U of 0. Consider U the connected component of $U \cap G^{-1}(U) \cap \cdots \cap G^{r-1}(U)$ at 0 then every point in U, which is an open set, is periodic.
($\Leftarrow$) Consider $F = G^{m!}$ defined in some neighborhood U of 0 and $L = \{x \in U \,|\, F(x) = x\}$. Since L is a complex analytic variety of U then it can be written as a finite union of analytic varieties with dimensions ranging from 1 to n. It can not be 0 because it contains infinitely many points accumulating $0 \in \mathbb{C}^n$. However, the union of finitely many analytic varieties, can not contain a dense set of points accumulating $0 \in \mathbb{C}^n$. Therefore $\dim L = n$ and we have that $G^{m!}(x) = x$ for all $x \in U$ and we are done. ✓

The following theorem shows that we do not need a dense set of periodic points as long as we have infinitely many, let us say, "well located" points.

Theorem 1.3.2. *Let $G \in \mathrm{Diff}(\mathbb{C}^n, 0)$. The group generated by G is finite if and only if there exist $m \in \mathbb{N}$ such that for an arbitrary neighborhood of $0 \in \mathbb{C}^n$, G leaves invariant infinitely many analytic varieties of complex dimension 1, in general position and each one having a convergent sequence of periodic points of order at most m.*

Proof. ($\Rightarrow$) The same as Theorem A. And we obtain infinitely many analytic varieties of complex dimension 1 passing through 0, and the periodicity of the group implies that every point on them is periodic of same order.

($\Leftarrow$) First, take $F = G^{m!}$ defined in some neighborhood U of 0, with m as in the statement. Take a G-invariant analytic variety M in U which, by hypothesis, has a convergence sequence of periodic points of order at most m converging to some point $q \in M$. We can apply Lemma 1.1.5 taking F as the map, M as the F-invariant complex analytic variety, q as the F-fixed point and K_q as the connected component of M containing q. Then there exists a C_M (compact, connected and non-enumerable) containing q and a sequence of F-fixed points converging to it, by the identity theorem (the one dimensional version because we are restricted to M) K_q is formed by F-fixed points. Therefore, there exist non-enumerable many F-fixed curves K_q in U. Hence the group generated by G is finite by Lemma 1.2.5. ✓

If in Theorem 1.3.2 we take the analytic varieties passing through $0 \in \mathbb{C}^n$, we get as a corollary a version of Theorem A changing the finite many orbits hypothesis to the existence of periodic points of bounded order accumulating 0.

Corollary 1.3.3. *Let $G \in \mathrm{Diff}(\mathbb{C}^n, 0)$. The group generated by G is finite if and only if there exist $m \in \mathbb{N}$ such that G leaves invariant infinitely many analytic varieties of complex dimension 1 in general position, each one having a sequence of periodic points of order at most m accumulating $0 \in \mathbb{C}^n$.*

1.4 Advances found in the literature

The final part of this chapter is devoted to present some recent generalizations of Theorem 1.1.1. Their proofs can be found in the referenced articles

The first one we mention is taken from [39] ,

Theorem 1.4.1. *Let $\mathcal{G} \subset \mathrm{Diff}(\mathbb{C}^n, 0)$ be a finitely generated pseudogroup on a small neighborhood of the origin in $\mathbb{C}^n$. Given $G \in \mathcal{G}$, let $Dom(G)$ denote the domain of definition of G as an element of the pseudogroup in question. Suppose that for every $G \in \mathcal{G}$ and $p \in \mathrm{Dom}(G)$ satisfying $G(p) = p$, one of the following holds: either p is an isolated fixed point of G or G coincides with the identity on a neighborhood of p. Then the pseudogroup $\mathcal{G}$ has finite orbits on a neighborhood of the origin if and only if $\mathcal{G}$ itself is finite.*

This theorem is consequence of the following proposition (Proposition 4. in [39]) and an argument like Lemma 1.1.3.

Proposition 1.4.2. *Suppose that $\mathcal{G} \subset \mathrm{Diff}(\mathbb{C}^n, 0)$ is a group satisfying the condition of isolated fixed points of Theorem 1.4.1. Let G be an element of $\mathcal{G}$ and assume that G has only finite orbits. Then G is periodic.*

As the authors observe, this proposition is obtained by repeating the proof of Theorem 1.1.1 in [30] p. 477, noting that the *isolated fixed points condition* replaces the argument that in dimension one is a consequence of the Identity Theorem.

The next generalization of Theorem 1.1.1 moves in another direction. Instead of changing the dimension it deals with the hypothesis of "all orbits be finite", analyzing the case where a diffeomorphism has a set of closed orbits of positive measure. This result can be found in [43] and in its proof is used the work of Perez-Marco ([33, 34, 35]).
We first introduce some notations:

Expand a germ of a complex diffeomorphism f at the origin $0 \in \mathbb{C}$ as

$$f(z) = e^{2\pi i\lambda} z + a_{k+1} z^{k+1} + \dots .$$

The multiplier $f'(0) = e^{2\pi i\lambda}$ does not depend on the coordinate system. We shall say that the germ $f \in \mathrm{Diff}(\mathbb{C}, 0)$ is *non-resonant* if $\lambda \in \mathbb{C} \setminus \mathbb{Q}$.

Definition 1.4.3. A map germ $f \in \mathrm{Diff}(\mathbb{C}, 0)$ is called a *Cremer map germ* if it is non-linearizable and non-resonant.

Cremer gave the first proof of the existence of such a map in [18].

Definition 1.4.4. We call *(PCO) Cremer map germ* a Cremer map germ. Whose representatives exhibit sets of closed orbits of positive measure, in arbitrarily small neighborhoods of the origin. We shall say that a subgroup $\mathcal{G} \subset \mathrm{Diff}(\mathbb{C}, 0)$ has the *(PCO) property* if for any sufficiently small neighborhood U of the origin $0 \in \mathbb{C}$, the set of points having closed pseudo-orbit has positive measure in U.

Lemma 1.4.5. *Let $\mathcal{G} \subset \mathrm{Diff}(\mathbb{C}, 0)$ be a finitely generated subgroup with the (PCO) property. Then either $\mathcal{G}$ is a cyclic finite (resonant) group or it is an abelian formally linearizable group, containing some (PCO) Cremer diffeomorphism.*

1.5 About a possible proof of theorem 1.1.2

In this section we present some partial results in the direction of 1.1.2. They do not form a proof of this statement but they may shed some light in the construction of a complete proof of it. In addition, in this section we start working in the "formal world", which plays and important role throughout this thesis.

Some of the definitions and notation we introduce next are taken from [2] (or [38]).
We are interested in the local dynamics of a germ of diffeomorphism $G \in \mathrm{Diff}(\mathbb{C}^n, 0)$ of the form

$$G(z) = z + P_\nu(z) + P_{\nu+1}(z) + \cdots , \tag{1.2}$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and P_ν is the first non-zero term in the homogeneous expansion of G. In this case (i.e., when $\mathrm{d}G_0 = \mathrm{id}$) we say that G is *tangent to the identity*

Definition 1.5.1. If $G \in \mathrm{Diff}(\mathbb{C}^n, 0)$ is of the form (1.2), the number $\nu \geq 2$ is *the order of* G and is denoted $\mathrm{ord}(G)$.

Note that we are always assuming that $G \not\equiv \mathrm{id}$.

Definition 1.5.2. Let $G \in \mathrm{Diff}(\mathbb{C}^n, 0)$ be tangent to the identity and of order ν. A *characteristic direction for* G is a non-zero vector $v \in \mathbb{C}^n \setminus \{0\}$ such that $P_\nu(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. If $P_\nu(v) = 0$ (that is, $\lambda = 0$) we shall say that v is a *degenerate characteristic direction*; otherwise, (that is, if $\lambda \neq 0$) we shall say that v is *non-degenerate.*

Definition 1.5.3. We shall say that an orbit $\{G^k(z_0)\}$ converges to the origin *tangentially* to a direction $[v] \in \mathbb{C}P(n-1)$ if $G^k(z_0) \to 0$ in $\mathbb{C}^n$ and $[G^k(z_0)] \to [v]$ in $\mathbb{C}P(n-1)$, where $[\cdot] : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}P(n-1)$ denotes the canonical projection.

Definition 1.5.4. A *parabolic curve* for $G \in \mathrm{Diff}(\mathbb{C}^n, O)$ tangent to the identity is an injective holomorphic map $\varphi : \Delta \to \mathbb{C}^n \setminus \{0\}$ satisfying the following properties:

(a) Δ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$;

(b) φ is continuous at the origin, and $\varphi(0) = 0$;

(c) $\varphi(\Delta)$ is G-invariant, and $(G|_{\varphi(\Delta)})^k \to 0$ uniformly on compact subsets as $k \to +\infty$.

Furthermore, if $[\varphi(\zeta)] \to [v]$ in $\mathbb{C}P(n-1)$ as $\zeta \to 0$ in Δ, we shall say that the parabolic curve φ is *tangent to the direction* $[v] \in \mathbb{C}P(n-1)$.

Theorem 1.5.5 (Abate [1])**.** *Let $G \in$ Diff$(\mathbb{C}^2, 0)$ be a germ of diffeomorphism tangent to the identity, with an isolated fixed point at* 0*. Then there exist* ord$(G) - 1$ *disjoint parabolic curves for G at the origin.*

Denote the ring of *formal series* on $(\mathbb{C}^n,0)$ by $\hat{\mathcal{O}}_n$, its maximal ideal, denoted by $\widehat{\mathcal{M}}_n$, corresponds to the elements in $\hat{\mathcal{O}}_n$ whose constant coefficient is zero. An element $\hat{F} \in \hat{\mathcal{O}}_2$ is written as $\hat{F}(x,y) = \sum_{i,j} F_{i,j} x^i y^j$, where $F_{i,j} \in \mathbb{C}$ for all $i,j \in \mathbb{Z}_{\geq 0}$.

A formal curve $\hat{\gamma}$ through zero is defined as the zero set of some $\hat{F} \in \widehat{\mathcal{M}}_2$ given in Puiseux parametrization, i.e.,

$$\hat{\gamma}(T) = \big(\sum_{k\geq k_1} \gamma_{1,k} T^k, \sum_{k\geq k_2} \gamma_{2,k} T^k\big) \text{ and } \hat{F}(\hat{\gamma}(T)) \equiv 0,$$

where the constants and the variable in γ are complex numbers. If a formal curve $\hat{\gamma}$ and a diffeomorphism $G \in \mathrm{Diff}(\mathbb{C}^2,0)$ satisfy the relationship $G \circ \hat{\gamma} = \hat{\gamma}$, where this notation means that $\mathrm{id}(G\circ\hat{\gamma}) = \mathrm{id}(\hat{\gamma}) := \{\hat{F} \in \hat{\mathcal{O}}_2 \,|\, \hat{F}(\hat{\gamma}) \equiv 0\}$; we say that $\hat{\gamma}$ is G-invariant. Note that this definition implies $\hat{F}(G^m \circ \hat{\gamma}) \equiv 0$ for $m \in \mathbb{Z}$.

Our analysis is divided in two parts according to whether the linear part of the diffeomorphism we are taking is a diagonal matrix or a Jordan block $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$.

Diagonal linear part. Let $G \in \mathrm{Diff}(\mathbb{C}^2,0)$ be a diffeomorphism whose linear part is diagonal i.e., $\mathrm{d}G_0(x,y) = (\lambda_1 x, \lambda_2 y)$ where $\lambda_1, \lambda_2 \in \mathbb{C}^*$, our objective is to show that λ_1 and λ_2 are roots of unity. If this is the case, we have that, for some integer m, the map G^m is tangent to the identity and Theorem 1.5.5 implies the existence of a parabolic curve. This, together with the hypothesis of finite orbits, would imply Theorem 1.1.2.

Suppose there exists a G-invariant formal curve $\hat{\gamma}$. Then the condition $\hat{F}(G \circ \hat{\gamma}(T)) \equiv 0$ in series form is

$$\begin{aligned} \hat{F}(G\circ\hat{\gamma}(T)) &= \sum_{i,j} F_{i,j}\big(\lambda_1 \sum_{k\geq k_1} \gamma_{1,k}T^k + \cdots\big)^i \big(\lambda_2 \sum_{k\geq k_2}\gamma_{2,k}T^k + \cdots\big)^j \\ &= \sum_{i,j} F_{i,j}\big(\lambda_1\gamma_{1,k_1} + T(\cdots)\big)^i \big(\lambda_2\gamma_{2,k_2} + T(\cdots)\big)^j T^{ik_1+jk_2} \equiv 0. \end{aligned}$$

Now, define the sets

$$\nu = \min\{ik_1 + jk_2 \,|\, F_{i,j} \neq 0\} \quad \text{and} \quad J = \{(i,j) \,|\, ik_1 + jk_2 = \nu\}.$$

Therefore, the first element of $\hat{F}(G\circ\hat{\gamma}(T)) \equiv 0$, in other words, the coefficient of T^ν, is

$$0 = \sum_{(i,j)\in J} F_{i,j}\gamma_{1,k_1}^i \gamma_{2,k_2}^j \lambda_1^i \lambda_2^j. \tag{1.3}$$

Moreover, since $\hat{F}(\hat{\gamma}(T)) \equiv 0$, we have $0 = \sum_{(i,j)\in J} F_{i,j}\gamma_{1,k_1}^{i}\gamma_{2,k_2}^{j}$. We can get a more general version of (1.3) using $G^m(x,y) = (\lambda_1^m x, \lambda_2^m x) + \text{h.o.t.}$ and observing that $\hat{F}(G^m \circ \hat{\gamma}(T)) \equiv 0$, which gives

$$0 = \sum_{(i,j)\in J} F_{i,j}\gamma_{1,k_1}^{i}\gamma_{2,k_2}^{j}\lambda_1^{mi}\lambda_2^{mj}. \tag{1.4}$$

Supposing that $\#J = K$, (1.4) is equal to

$$0 = F_{i_1,j_1}\gamma_{1,k_1}^{i_1}\gamma_{2,k_2}^{j_1}(\lambda_1^{i_1}\lambda_2^{j_1})^m + \cdots + F_{i_K,j_K}\gamma_{1,k_1}^{i_K}\gamma_{2,k_2}^{j_K}(\lambda_1^{i_K}\lambda_2^{j_K})^m. \tag{1.5}$$

if we set $A_r = F_{i_r,j_r}\gamma_{1,k_1}^{i_r}\gamma_{1,k_2}^{j_r}$ and $X_r = \lambda_1^{i_r}\lambda_2^{j_r}$, we can write (1.5) as

$$A_1X_1^m + \cdots + A_KX_K^m = 0.$$

Besides, this is true for $m \in \mathbb{Z}_{\geq 0}$, which allows us to construct the system

$$\begin{array}{lcccl} A_1 & + & \dots & + & A_K & = & 0, \\ A_1X_1 & + & \dots & + & A_KX_K & = & 0, \\ \vdots & & \vdots & & \vdots & & \vdots \\ A_1X_1^{K-1} & + & \dots & + & A_KX_K^{K-1} & = & 0, \end{array}$$

whose matrix form is

$$XA = \begin{bmatrix} 1 & \cdots & 1 \\ X_1 & \cdots & X_K \\ \vdots & \cdots & \vdots \\ X_1^{K-1} & \cdots & X_K^{K-1} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_K \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The matrix X is a Vandermonde matrix. It is clearly singular because $A_i \neq 0$ for all i. The determinant of X is the product $\prod_{1\leq i<j\leq k}(X_i - X_j)$. Thus at least one of the factors has to be zero. Suppose $X_r = X_s$ for $r, s \in \{1, \dots, K\}$ and $r \neq s$. Hence $\lambda_1^{i_r}\lambda_2^{j_r} = \lambda_1^{i_s}\lambda_2^{j_s}$. Because of the identities

$$\begin{array}{l} i_rk_1 + j_rk_2 = \nu \\ i_sk_1 + j_sk_2 = \nu \end{array} \implies (i_r - i_s)k_1 + (j_r - j_s)k_2 = 0$$

we know that $(i_r - i_s)$ and $(j_r - j_s)$ have different sign, suppose $i_r - i_s > 0$ and $j_r - j_s < 0$ with this $\lambda_1^{i_r - i_s} = \lambda_2^{j_s - j_r}$.

In order to conclude that λ_1 and λ_2 are roots of the unity, the only thing we need is another pair or couples $(i_\alpha, j_\alpha), (i_\beta, j_\beta)$ such that $\lambda_1^{i_\alpha - i_\beta} = \lambda_2^{j_\beta - j_\alpha}$ and at least one of the following occurs $i_r - i_s \neq i_\alpha - i_\beta$ or $j_s - j_r \neq j_\beta - j_\alpha$. It looks like something reasonable. In fact the same curve can give you the second pair of couples you need. But even if you say "there are infinitely many G-invariant formal curves" the precise condition needed to guarantee the existence of the second pair of couples required is not immediately satisfied.

Jordan block. Let $G \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be a diffeomorphism whose linear part is a Jordan block $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$ i.e., $\mathrm{d}G_0(x, y) = (\lambda x + y, \lambda y)$ where $\lambda \in \mathbb{C}^*$, suppose that there exists a G-invariant formal curve $\hat{\gamma}$. Our aim in this part is to show that this case does not happen. However, as before, we have to impose some conditions on $\hat{\gamma}$.
As in the previous paragraph, we start by writing $\hat{F}(G^m \circ \hat{\gamma}(T)) \equiv 0$ in series for. Observe that $G^m(x, y) = (\lambda^m x + m\lambda^{m-1} y, \lambda^m y) + \text{h.o.t.}$. The expression $\hat{F}(G^m \circ \hat{\gamma}(T)) \equiv 0$ gives

$$\sum_{i,j} F_{i,j} \big(\lambda^m \sum_{k \geq k_1} \gamma_{1,k} T^k + m\lambda^{m-1} \sum_{k \geq k_2} \gamma_{2,k} T^k + \cdots\big)^i \big(\lambda^m \sum_{k \geq k_2} \gamma_{2,k} T^k + \cdots\big)^j =$$
$$\sum_{i,j} F_{i,j} \big(\lambda^m \gamma_{1,k_1} T^{k_1} + m\lambda^{m-1} \gamma_{2,k_2} T^{k_2} + \cdots\big)^i \big(\lambda^m \gamma_{2,k_2} T^{k_2} + \cdots\big)^j = 0.$$

We consider three different cases:

Case 1: $k_1 < k_2$

We have

$$\begin{aligned} \hat{F}(G^m \circ \hat{\gamma}(T)) =& \sum_{i,j} F_{i,j} \big(\lambda^m \gamma_{1,k_1} + T(\cdots)\big)^i \big(\lambda^m \gamma_{2,k_2} + T(\cdots)\big)^j T^{ik_1 + jk_2} \\ \equiv& 0. \end{aligned}$$

Defining, as before,

$$\nu = \min\{ik_1 + jk_2 \,|\, F_{i,j} \neq 0\} \quad \text{and} \quad J = \{(i, j) \,|\, ik_1 + jk_2 = \nu\}.$$

the coefficient of T^ν is

$$0 = \sum_{(i,j) \in J} F_{i,j} \gamma_{1,k_1}^i \gamma_{1,k_2}^j \lambda^{(i+j)m} \quad \text{for } m \in \mathbb{Z}_{\geq 0}.$$

Take $\#J = K$, analogously to the previous paragraph, denote

$$A_r = F_{i_r,j_r} \gamma_{1,k_1}^{i_r} \gamma_{1,k_2}^{j_r}, \quad X_r = \lambda^{i_r + j_r}$$

and varying m from 0 to $K - 1$, we construct the system

$$XA = \begin{bmatrix} 1 & \cdots & 1 \\ X_1 & \cdots & X_K \\ \vdots & \cdots & \vdots \\ X_1^{K-1} & \cdots & X_K^{K-1} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_K \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

Again, X has to be singular. Therefore there exists $1 \leq r < s \leq K - 1$ such that $X_r = X_s$, which means $\lambda^{i_r + j_r} = \lambda^{i_s + j_s}$. As a conclusion, λ is a root of unity. We were expecting a contradiction in order to prove this case can not occurs, so for our purpose this is a case we have to eliminate.

Case 2: $k_1 = k_2 = k$**.**

We have

$$\begin{aligned}
\hat{F}(G^m \circ \hat{\gamma}(T)) =& \sum_{i,j} F_{i,j}\big(\lambda^m \gamma_{1,k} T^k + m\lambda^{m-1}\gamma_{2,k}T^k + \cdots\big)^i \big(\lambda^m \gamma_{2,k} T^k + \cdots\big)^j \\
=& \sum_{i,j} F_{i,j}\big(\lambda^m \gamma_{1,k} + m\lambda^{m-1}\gamma_{2,k} + T(\cdots)\big)^i \big(\lambda^m \gamma_{2,k} + T(\cdots)\big)^j T^{(i+j)k} \\
\equiv& 0
\end{aligned}$$

In this case,

$$\nu = \min\{i + j \,|\, F_{i,j} \neq 0\} \quad \text{and} \quad J = \{(i,j) \,|\, i + j = \nu\}.$$

The coefficient of $T^{\nu k}$ is

$$\begin{aligned}
& \sum_{(i,j)\in J} F_{i,j}(\lambda^m \gamma_{1,k} + m\lambda^{m-1}\gamma_{2,k})^i (\lambda^m \gamma_{2,k})^j \\
=& \sum_{(i,j)\in J} F_{i,j}(\lambda\gamma_{1,k} + m\gamma_{2,k})^i \gamma_{2,k}^j \lambda^{i(m-1)+jm} = 0.
\end{aligned}$$

As $i + j = \nu$ then $i(m-1) + jm = \nu m - i$, and the equation above can be written as

$$\sum_{i}^{\nu} F_{i,\nu-i}(\lambda\gamma_{1,k} + m\gamma_{2,k})^i \gamma_{2,k}^{\nu-i} \lambda^{\nu m - i} = 0.$$

Multiplying by $\lambda^{\nu - \nu m}$ this gives

$$F_{0,\nu}\gamma_{2,k}^{\nu}\lambda^{\nu} + F_{1,\nu-1}(\lambda\gamma_{1,k} + m\gamma_{2,k})\gamma_{2,k}^{\nu-1}\lambda^{\nu-1} + \cdots + F_{\nu,0}(\lambda\gamma_{1,k} + m\gamma_{2,k})^{\nu} = 0.$$

By varying m from 1 to $\nu + 1$, we form the system

$$\begin{bmatrix}
1 & \lambda\gamma_{1,k} + \gamma_{2,k} & \cdots & (\lambda\gamma_{1,k} + \gamma_{2,k})^{\nu} \\
1 & \lambda\gamma_{1,k} + 2\gamma_{2,k} & \cdots & (\lambda\gamma_{1,k} + 2\gamma_{2,k})^{\nu} \\
\vdots & \vdots & \cdots & \vdots \\
1 & \lambda\gamma_{1,k} + (\nu+1)\gamma_{2,k} & \cdots & (\lambda\gamma_{1,k} + (\nu+1)\gamma_{2,k})^{\nu}
\end{bmatrix}
\begin{bmatrix}
F_{0,\nu}\gamma_{2,k}^{\nu}\lambda^{\nu} \\
F_{1,\nu-1}\gamma_{2,k}^{\nu-1}\lambda^{\nu-1} \\
\vdots \\
F_{\nu,0}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ \vdots \\ 0
\end{bmatrix}$$

Since the left side matrix has to be singular, because not all the $F_{i,\nu-i}$ can be zero, then there exists $1 \leq r < s \leq (\nu + 1)$ such that $\lambda\gamma_{1,k} + r\gamma_{2,k} = \lambda\gamma_{1,k} + s\gamma_{2,k}$ hence $\gamma_{2,k} = 0$, which contradicts our hypothesis $k_2 = k$. We conclude that, for this case, there does not exist a G-invariant curve.

Case 3: $k_1 > k_2$**.**

We have

$$
\begin{aligned}
&\hat{F}(G^m \circ \hat{\gamma}(T)) = \\
&\qquad \sum_{i,j} F_{i,j}\big(\lambda^m \gamma_{1,k_1} T^{k_1} + m\lambda^{m-1}\gamma_{2,k_2}T^{k_2} + \cdots\big)^i \big(\lambda^m \gamma_{2,k_2} T^{k_2} + \cdots\big)^j \\
&\qquad = \sum_{i,j} F_{i,j}\big(m\lambda^{m-1}\gamma_{2,k_2} + T(\cdots)\big)^i \big(\lambda^m \gamma_{2,k_2} + T(\cdots)\big)^j T^{(i+j)k_2} \\
&\qquad \equiv 0.
\end{aligned}
$$

Defining ν and J, as in the previous case,

$$\nu = \min\{i + j \mid F_{i,j} \neq 0\} \quad \text{and} \quad J = \{(i,j) \mid i + j = \nu\}.$$

We find,

$$\sum_{(i,j)\in J} F_{i,j}\gamma_{2,k_2}^{\nu} m^i \lambda^{i(m-1)+jm} = 0 \quad \text{where} \quad \gamma_{2,k_2} \neq 0.$$

Taking $j = \nu - i$, we have

$$\sum_{i}^{\nu} F_{i,\nu-i} m^i \lambda^{\nu m - i} = 0$$

Multiplying by $\lambda^{\nu-\nu m}$ this gives

$$F_{0,\nu}\lambda^{\nu} + F_{1,\nu-1} m \lambda^{\nu-1} + \cdots + F_{\nu,0} m^{\nu} = 0.$$

By varying m from 1 to $\nu + 1$ once again we form the system

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2^{\nu} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \nu+1 & \cdots & (\nu+1)^{\nu} \end{bmatrix} \begin{bmatrix} F_{0,\nu}\lambda^{\nu} \\ F_{1,\nu-1}\lambda^{\nu-1} \\ \vdots \\ F_{\nu,0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since the matrix of coefficients is not singular, we get $F_{0,\nu} = F_{1,\nu-1} = \cdots = F_{\nu,0} = 0$ and, as in Case 2, we conclude that for this case there does not exist a G-invariant curve.

Summarizing, we need at least two pairs of couples of exponents (maybe given by the same curve) with the condition mentioned at the end of the diagonal case in order to conclude that the diffeomorphism with diagonal linear part is tangent to the identity, and a curve with $k_1 \geq k_2$ (in the notation used above) to discard the linear part in the form $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$.

The problem is, whether the existence of infinitely many (formal) invariant curves guarantees the existence of one or two with the properties required. Presumably the answer would be given in terms of a blow-up process, but this is precisely what we are trying to avoid. This technique is currently being explored by some mathematicians.

Chapter 2

Groups of formal diffeomorphisms and formal series

This chapter is devoted to the study of formal difeomorphisms and formal series. Here we obtain some useful properties for our upcoming work.

2.1 Preliminaries

Let us introduce (or recapitulate) some standard notation. Denote the ring of *formal series* on $(\mathbb{C}^n, 0)$ by $\hat{\mathcal{O}}_n$, its maximal ideal denoted by $\hat{\mathcal{M}}_n$ and the group of *formal diffeomorphisms* of $(\mathbb{C}^n, 0)$ by $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. The convergent versions of the previous sets are, the ring of *germs of holomorphic functions* on $(\mathbb{C}^n, 0)$ denoted by $\mathcal{O}_n$, its maximal ideal denoted by $\mathcal{M}_n$ and the group of *diffeomorphisms* of $(\mathbb{C}^n, 0)$ by $\mathrm{Diff}(\mathbb{C}^n, 0)$.

The first step is to study the properties we can get from the relationship $\hat{f} \circ \hat{G} = \hat{f}$, where $\hat{f} \in \hat{\mathcal{O}}_n$ and $\hat{G} \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. In this case we say that $\hat{G}$ *leaves* $\hat{f}$ *invariant*. As we state in propositions 2.2.1 and 2.2.3 this relationship characterizes both maps. Our work will guarantee that we only need to analyze the case where $\hat{G}$ is linearizable.

We start with the following definitions:

Definition 2.1.1. Let $\Lambda \in \mathbb{C}^n$. We say that a multi-index $Q = (q_1, \dots, q_n) \in \mathbb{N}^n$, with $|Q| = q_1 + \cdots + q_n \geq 1$, gives a *multiplicative resonant relation for* Λ if

$$\Lambda^Q := \lambda_1^{q_1} \cdots \lambda_n^{q_n} = 1.$$

If there exists a Q giving this property we say that Λ is *multiplicative resonant*.

Observe that this definition is a particular case of the usual definition of multiplicative resonant that can be seen for example in [4] pp. 192-193. There you can also see that the existence of these kinds of resonances is the obstruction to formal linearization. Recent results on this topic can be found in [37].

Definition 2.1.2. We shall say that a monomial $x^Q := x_1^{q_1} \cdots x_n^{q_n}$ is *resonant with respect to* $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ (or simply $(\lambda_1, \ldots, \lambda_n)$-*resonant*) if $|Q| \geq 1$ and $\Lambda^Q = 1$.

2.1.1 Formal chain rule

The aim of this paragraph is to show that the Chain Rule holds in the formal case.

Lemma 2.1.3. *Let $\hat{F} \in \hat{\mathcal{O}}_n$ and $\hat{G} \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ be given. Then*

$$\mathrm{d}(\hat{F} \circ \hat{G}) = \mathrm{d}\hat{F} \cdot \mathrm{d}\hat{G}.$$

Proof. We start with $n = 1$. Let $\hat{f} \in \hat{\mathcal{O}}_1$ given by $\hat{f}(x) = \sum_{i=1}^{\infty} a_i x^i$, define $f_n \in \mathcal{O}_1$ by $f_n(x) = \sum_{i=1}^{n} a_i x^i$ and take $g \in \mathcal{O}_1$. We want to show that $\mathrm{d}(\hat{f} \circ g) = \mathrm{d}\hat{f}_g \, \mathrm{d}g$.

We already have that $\mathrm{d}(f_n \circ g) = (\mathrm{d}f_n)_g \, \mathrm{d}g$, because they are holomorphic functions. Besides, by the definition of derivative of a formal series, we have $\lim_{n\to\infty} \mathrm{d}f_n = \mathrm{d}\hat{f}$. Therefore, what we need to justify is that $\lim_{n\to\infty} (\mathrm{d}f_n)_g = (\mathrm{d}\hat{f})_g$ and $\lim_{n\to\infty} \mathrm{d}(f_n \circ g) = \mathrm{d}(\hat{f} \circ g)$. Both are consequence of the equality $\lim_{n\to\infty} f_n \circ g = \hat{f} \circ g$ and for this, think in the coefficient c_k of x^k in $\hat{f} \circ g(x) = \sum_{i=1}^{\infty} c_i x^i = \sum_{i=1}^{\infty} a_i (\sum_{j=1}^{\infty} b_j x^j)^i$, where $g(x) = \sum_{j=1}^{\infty} b_j x^j$. This coefficient is formed after algebraic computation by some of the coefficients in $\sum_{i=1}^{k} a_i (\sum_{j=1}^{k} b_j x^j)^i$. Indeed after $i, j = k$ all the elements in $\sum_{i=1}^{\infty} a_i (\sum_{j=1}^{\infty} b_j x^j)^i$ are of order greater than k. Thus the same coefficients of x^k belongs to both sides of $\lim_{n\to\infty} f_n \circ g = \hat{f} \circ g$.

Hence

$$\mathrm{d}(\hat{f} \circ g) = \mathrm{d}\hat{f}_g \, \mathrm{d}g,$$

as we wanted.

Consider now $g \in \mathcal{O}_2$ and the same $\hat{f}$ as before. In this case the chain rule is consequence of the previous one, because if we fix one of the variables, for example $y = y_0$, then $g(\cdot, y_0) \in \mathcal{O}_1$ and $\frac{\partial}{\partial x}(\hat{f} \circ g) = \mathrm{d}\hat{f}_{g(x,y_0)} \frac{\partial}{\partial x} g|_{(x,y_0)}$ by the previous case.

The two dimensional case works in a similar way, by taking $\hat{F} \in \hat{\mathcal{O}}_2$ and $G(x,y) = (g_1(x,y), g_2(x,y))$ given by $\hat{F}(x) = \sum_I a_I x^i y^j$ and $g_1, g_2 \in \mathcal{O}_2$. Denoting $\hat{F}_i(x) = \sum_j a_{i,j} x^j$. Then we have

$$\begin{aligned}
\hat{F} \circ G(x,y) &= \sum_I a_I \big(g_1(x,y)\big)^i \big(g_2(x,y)\big)^j \\
&= \sum_i (g_1(x,y))^i \Big(\sum_j a_{i,j} \big(g_2(x,y)\big)^j \Big) \\
&= \sum_i (g_1(x,y))^i \hat{F}_i\big(g_2(x,y)\big).
\end{aligned}$$

So, $\hat{F} \circ G$ can be written as a sum of products of two formal series $(g_1(x,y))^i$ and $\hat{F}_i\big(g_2(x,y)\big)$, whose derivatives are known by the previous case. Now, note that if $\hat{F} \circ G$ is a formal series then is derivation is made term by term, and in the previous paragraph we only rearrange those terms. Thus

$$\begin{aligned}
\frac{\partial}{\partial x}\big(\hat{F} \circ G\big)(x,y) &= \sum_i \frac{\partial}{\partial x}\Big((g_1(x,y))^i \hat{F}_i\big(g_2(x,y)\big) \Big) \\
&= \sum_i \Big(i g_1^{i-1} \frac{\partial g_1}{\partial x} \hat{F}_i(g_2) + g_1^i \frac{\partial \hat{F}_i}{\partial x}\Big|_{g_2} \frac{\partial g_2}{\partial x} \Big)(x,y)
\end{aligned}$$

$$\begin{aligned}
&= \sum_i \Big(i g_1^{i-1} \frac{\partial g_1}{\partial x} \sum_j a_{i,j} g_2^j + g_1^i \big(\sum_j j a_{i,j} g_2^{j-1} \big) \frac{\partial g_2}{\partial x} \Big)(x,y) \\
&= \sum_{i,j} \Big(i a_{i,j} \big(g_1(x,y)\big)^{i-1} \big(g_2(x,y)\big)^j \frac{\partial g_1}{\partial x} + j a_{i,j} \big(g_1(x,y)\big)^i \big(g_2(x,y)\big)^{j-1} \Big) \frac{\partial g_2}{\partial x} \\
&= \frac{\partial \hat{F}}{\partial x}\Big|_G \frac{\partial G}{\partial x}(x,y).
\end{aligned}$$

Now consider $\hat{f}, \hat{g} \in \hat{\mathcal{O}}_1$, by the previous step $\mathrm{d}(\hat{f} \circ g_n) = \mathrm{d}\hat{f}_{g_n} \mathrm{d}g_n$ where g_n is the truncated series, and the chain rule is consequence of $\lim_{n\to\infty} \hat{f} \circ g_n = \hat{f} \circ g$, as before just note that the coefficient of x^r of $\hat{f} \circ g$ appear in $\hat{f} \circ g_n$ for all $n > N$ for some N. The case $\hat{f} \in \hat{\mathcal{O}}_2$, $\hat{G} \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ is the same as above.

As a conclusion, for the case $\hat{F} \in \hat{\mathcal{O}}_2$ and $\hat{G} \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ the chain rule, $\mathrm{d}(\hat{F} \circ \hat{G}) = \mathrm{d}\hat{F} \cdot \mathrm{d}\hat{G}$, holds and the process above is easily generalized to higher dimension. ✓

2.2 Invariance relationship

Take $G(x) = ax$ with $a \in \mathbb{C}\backslash_0$ and let $\hat{f}$ be the formal series $\hat{f}(x) = \sum_{i\geq 1} a_i x^i$. Suppose that $\hat{f} \circ G = \hat{f}$ and that $\hat{f}$ is not a power, meaning by this that if $\hat{f} = f_1^{p_1} \cdots f_r^{p_r}$, where $f_1, \dots f_r$ are r different irreducible factors of $\hat{f}$, then $\gcd(p_1, \dots, p_r) = 1$. We have

$$\hat{f}(x) = \sum_{i\geq 1} a_i x^i = \hat{f} \circ G(x) = \sum_{i\geq 1} a_i (ax)^i,$$

which implies $a_i a^i = a_i$ for all $i = 1, 2 \dots$. If $\hat{f} \not\equiv 0$ there is a coefficient $a_\nu \neq 0$ so, $a^\nu = 1$ (i.e. a is a root of unity). Besides, supposing that a is a ν-th root of unity, we find that $a_i = 0$ if $i \neq m\nu$ where $m \in \mathbb{Z}^+$. Then

$$G(x) = e^{2\pi i/\nu} x \text{ and } \hat{f}(x) = \hat{l}(x^\nu) \text{ where } \hat{l} \in \hat{\mathcal{O}}_1.$$

Moreover, $\hat{f}$ is not a power, then $\hat{l}$ is invertible i.e., $\hat{l}'(0) \neq 0$ and we have that $(\hat{l}^{-1} \circ \hat{f})(x) = x^\nu$. To see this, suppose that $\hat{l}$ is not invertible, write $\hat{l}(x) = a_p x^p + a_{p+1} x^{p+1} + \cdots$ where $p > 1$ and $a_p \neq 0$. Then

$$\begin{aligned} \hat{f}(x) &= \hat{l}(x^\nu) = a_p x^{p\nu} + a_{p+1} x^{(p+1)\nu} + \cdots \\ &= x^{p\nu}(a_p + a_{p+1} x^\nu + \cdots) \\ &= \big(g(x^\nu)\big)^p, \quad \text{where} \quad g(x) = x(a_p + a_{p+1}x + \cdots)^{1/p}. \end{aligned}$$

Since $a_p \neq 0$, g is well defined and this contradicts the fact that $\hat{f}$ is not a power. Therefore, if a formal series $\hat{f}$ is invariant by a rotation, there exists an invertible formal series $\hat{l}$ such that $\hat{l}^{-1} \circ \hat{f}$ is holomorphic.
The result described above is a portion of the Proposition 1.2. in [30] and our intention is to generalize it to arbitrary dimensions. In order to do that we start with the following,

Proposition 2.2.1. *Let $\hat{f} \in \hat{\mathcal{O}}_n$ and $\hat{G} \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ formally linearizable such that $\hat{G}$ leaves $\hat{f}$ invariant. If the linear part of $\hat{G}$ is a diagonal matrix,*

$$\mathrm{d}\hat{G}_0 = \mathrm{diag}(\lambda_1, \dots, \lambda_n),$$

then $(\lambda_1, \dots, \lambda_n)$ is multiplicative resonant and $\hat{f}$, after a formal change of coordinates, is the sum of $(\lambda_1, \dots, \lambda_n)$-resonant monomials.

Proof. We consider first the linear case taking $G(x) = Ax$ and $\hat{f}(x) = \sum_{|I|\geq 1} a_I x^I$, where A is a non-singular, diagonal $(n \times n)$-matrix and $x = (x_1 \dots, x_n)$,

$$G(x_1, \dots, x_n) = (\lambda_1 x_1, \dots, \lambda_n x_n).$$

Thus

$$\hat{f} \circ G(x_1, \ldots, x_n) = \sum_{|I| \geq 1} a_I (\lambda_1 x_1)^{i_1} \cdots (\lambda_n x_n)^{i_n} = \sum_{|I| \geq 1} a_I x_1^{i_1} \cdots x_n^{i_n},$$

which means that

$$\lambda_1^{i_1} \cdots \lambda_n^{i_n} = 1, \text{ for all } I \text{ such that } a_I \neq 0,$$

This is, $\Lambda = (\lambda_1, \ldots, \lambda_n)$ is multiplicative resonant. If $\hat{f} \not\equiv 0$ then it is formed only by resonant monomials. Furthermore there exist at most n independent (as vectors in $\mathbb{C}^n$) n-tuples $I = (i_1, \ldots, i_n) \in \mathbb{N}^n \backslash_0$ such that $\lambda_1^{i_1} \cdots \lambda_n^{i_n} = 1$. In case we have n independent n-tuples, all λ_i's are roots of the unity as we explain in the proposition below.

Finally, suppose that $\hat{G} \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ is formally diagonalizable, i.e., there is a formal change of coordinates such that $g^{-1} \circ \hat{G} \circ g(x) = \mathrm{d}\hat{G}(0)x$. We make the previous analysis on its linear part $G(x) = \mathrm{d}\hat{G}(0)x$, concluding that, it has to be a diagonal matrix with multiplicative resonant entries. ✓

Proposition 2.2.2. *Let $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ and $I_1, \ldots, I_n \in \mathbb{N}^n \backslash_0$ be n independent n-tuples such that $\Lambda^{I_j} = 1$, for $j = 1, \ldots, n$, then for each λ_j there exist a $n_j \in \mathbb{N} \backslash_0$ such that $\lambda_j^{n_j} = 1$.*

Proof. By hypothesis we have n equalities of the form $\lambda_1^{i_{1\,1}} \cdots \lambda_n^{i_{1,n}} = 1$. Taking logarithm in each one of them we can form the following linear system

$$\begin{bmatrix} i_{1\,1} & \ldots & i_{1,n} \\ \vdots & \ddots & \vdots \\ i_{n\,1} & \ldots & i_{n,n} \end{bmatrix} \begin{bmatrix} \log \lambda_1 \\ \vdots \\ \log \lambda_n \end{bmatrix} = \begin{bmatrix} 2\pi i k_1 \\ \vdots \\ 2\pi i k_n \end{bmatrix},$$

its real part is a homogeneous linear system whose solution implies that $\log |\lambda_j| = 0$ for all j and, from the imaginary part of the system we obtain that the argument of each λ_j is a rational factor of 2π. ✓

Proposition 2.2.3. *Let $\hat{f} \in \hat{\mathcal{O}}_n$ and $\hat{G} \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ be formally linearizable such that $\hat{G}$ leaves $\hat{f}$ invariant. If the linear part of $\hat{G}$ in its Jordan form has a block*

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

i.e, $d\hat{G}_0(x_1, \ldots, x_n) = (\ldots, \lambda x_j + x_{j+1}, \lambda x_{j+1}, \ldots)$ after a linear change of coordinates, then $\lambda^m = 1$ for some $m \in \mathbb{Z}^+$. Besides, after a formal change of coordinates, in the variables related to that block, $\hat{f}$ is a formal series in the m-th power of the second variable, that is

$$\hat{f}(0, \ldots, 0, x_j, x_{j+1}, 0, \ldots, 0) = l(x_{j+1}^m) \text{ for } l \in \hat{\mathcal{O}}_1.$$

Observe that, if the block is bigger, its upper sub matrix 2×2 is like the previous one, thus the proposition is true also in this case.

Proof. We only need to consider the two dimensional case. Let $G(x_1, x_2) = (\lambda x_1 + x_2, \lambda x_2)$ and $\hat{f}(x_1, x_2) = \sum_{|I| \geq 1} a_I x_1^i x_2^j$. Then the condition $\hat{f} \circ G = \hat{f}$ implies

$$\hat{f} \circ G(x_1, x_2) = \sum_{|I| \geq 1} a_I (\lambda x_1 + x_2)^i (\lambda x_2)^j = \sum_{|I| \geq 1} a_I x_1^i x_2^j.$$

Thus $a_{i,j} = \sum_{k=0}^{j} C_{i+k,k} \lambda^{i+j-k} a_{i+k,j-k}$ where $C_{l,m} = \binom{l}{m}$. If $\lambda^j \neq 1$ for all $j \in \mathbb{N}$ then $a_{i,0} = a_{i,0}\lambda^i$ implies $a_{i,0} = 0$ and $a_{i,1} = \lambda^{i+1} a_{i,1} + C_{i+1,1}\lambda^i a_{i+1,0}$ implies $a_{i,1} = 0$. Repeating this we get that $f \equiv 0$. Therefore, $\lambda^i = 1$ for some i such that $a_{i,0} \neq 0$. First consider the case $\lambda = 1$. We have

$$\begin{aligned}
&a_{i,0} = a_{i,0}, \\
&a_{i,1} = a_{i,1} + C_{i+1,1} a_{i+1,0} \Longrightarrow a_{i,0} = 0 \quad \text{for } i > 0, \\
&a_{i,2} = a_{i,2} + C_{i+1,1} a_{i+1,1} \Longrightarrow a_{i,1} = 0 \quad \text{for } i > 0,
\end{aligned}$$

by induction, suppose that $a_{i,j} = 0$ for $i > 0$ and $j \leq n$ then

$$a_{i,n+2} = a_{i,n+2} + C_{i+1,1} a_{i+1,n+1} \Longrightarrow a_{i,n+1} = 0 \quad \text{for } i > 0,$$

hence the only remaining terms are of the form $a_{0,j}$ and then $f(x_1, x_2) = l(x_2)$ as we wanted. In a similar way, if $\lambda^m = 1$ but $\lambda^n \neq 1$ for $0 < n < m$ with $m, n \in \mathbb{N}$. Since $a_{i,0} = a_{i,0}\lambda^i$ then $a_{i,0} = 0$ when $m \not| \, i$. The next term is calculated in the expansion $a_{i,1} = a_{i,1}\lambda^{i+1} + C_{i+1,1} a_{i+1,0} \lambda^i$.
If $m \mid i + 1$ we have that $a_{i+1,0} = 0$ and, using the previous step, $a_{i,0} = 0$ for all i. If $m \not| \, i + 1$ we have that $a_{i,1} = 0$, using the next expansion

$$a_{i,2} = a_{i,2}\lambda^{i+2} + C_{i+1,1} a_{i+1,1} \lambda^{i+1},$$

we can repeat the analysis. If $m \mid i + 2$ we have that $a_{i+1,1} = 0$ and using the previous step $a_{i,1} = 0$ for all i. We proceed by induction. Suppose that $a_{i,j}$ for $j \leq n$ and $i > 0$ then

$$a_{i,n+1} = a_{i,n+1}\lambda^{i+n+1}, \quad \text{if } m \not| \, (i + n + 1) \text{ then } a_{i,n+1} = 0.$$

As above consider the next term

$$a_{i,n+2} = a_{i,n+2}\lambda^{i+n+2} + C_{r+1,1} a_{i+1,n+1} \lambda^{i+n+1},$$

if $m \mid (i + n + 2)$ then $a_{i+1,n+1} = 0$ and using the previous step (where we show that if m does not divide the sum of the sub-indices of $a_{i,n+1}$ then

$a_{i,n+1} = 0$), we have $a_{i,n+1} = 0$ for all i. Finally, for the case $i = 0$ note that $a_{0,j} = a_{0,j}\lambda^j$ and we can not argue like above, therefore $\hat{f}(x_1, x_2) = \hat{l}(x_2^m)$ for $l \in \hat{\mathcal{O}}_1$.

The higher dimensional case works in the same way, because some part of $\hat{G}$ will be of the form $(\ldots, \lambda x_j + x_{j+1}, \ldots, \lambda x_{j+k-1} + x_{j+k}, \lambda x_{j+k}, \ldots)$, for a eigenvalue λ, and making all $x_i = 0$ except for x_{j+k-1} and x_{j+k} we can apply the same analysis. Then, $\hat{f}(0, \ldots, 0, x_{j+k-1}, x_{j+k}, 0, \ldots, 0) = \hat{l}(x_{j+k}^m)$ for $\hat{l} \in \hat{\mathcal{O}}_1$.

Finally, if $\hat{G} \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ is formally linearizable then, there is a formal change of coordinates such that $\hat{g}^{-1} \circ \hat{G} \circ \hat{g}(x) = \mathrm{d}\hat{G}(0)x$ and we make the previous analysis over its linear part $G(x) = \mathrm{d}\hat{G}(0)x$. ✓

Definition 2.2.4. Let $\hat{f}_1 \ldots, \hat{f}_n \in \hat{\mathcal{O}}_n$.

- We say that $\hat{f}_1 \ldots, \hat{f}_n$ are *generically transverse* if $\mathrm{d}\hat{f}_1 \wedge \cdots \wedge \mathrm{d}\hat{f}_n \not\equiv 0$.
- We say that $\hat{f}_1 \ldots, \hat{f}_n$ are *transversal at the origin* if $(\mathrm{d}\hat{f}_1 \wedge \cdots \wedge \mathrm{d}\hat{f}_n)_0 \neq 0$.

An immediate consequence of this definition is:

Proposition 2.2.5. *Let $\hat{f}_1, \ldots, \hat{f}_n \in \hat{O}_n$ be n generically transverse formal series, written in series form as $\hat{f}_j(x) = \sum_I a_{j,I}x^I$, then there exist n independent multi-indexes $I_1, \ldots, I_n \in \mathbb{N}^n$, i.e., there exist n multi-indexes $I_k = (i_{k,1}, \ldots, i_{k,2})$ such that the matrix $[i_{k,l}]$, where $1 \leq k, l \leq n$, is no singular and such that for each I_k at least one a_{j,I_k}, for $1 \leq j \leq n$, is not zero.*

Proof. By hypothesis we have that $\mathrm{d}\hat{f}_1 \wedge \cdots \wedge \mathrm{d}\hat{f}_n \not\equiv 0$. The associativity of the wedge product allows us to work in couples. Note that

$$\mathrm{d}\hat{f}_1 \wedge \cdots \wedge \mathrm{d}\hat{f}_n \not\equiv 0 \Longrightarrow \mathrm{d}\hat{f}_1 \wedge \mathrm{d}\hat{f}_2 \not\equiv 0$$

and $\mathrm{d}\hat{f}_j = \frac{\partial \hat{f}_j}{\partial x_1}\mathrm{d}x_1 + \cdots + \frac{\partial \hat{f}_j}{\partial x_n}\mathrm{d}x_n$ where

$$\frac{\partial \hat{f}_j}{\partial x_r} = \sum_{I_k} i_{k,r} a_{j,I_k} x_1^{i_{k,1}} \cdots x_r^{i_{k,r}-1} \cdots x_n^{i_{k,n}} = \sum_{I_k} i_{k,r} a_{j,I_k} x^{I_k - e_r}$$

using the notation $I_k - e_r = (i_{k,1}, \ldots, i_{k,r} - 1, \ldots, i_{k,n})$. We have

$$\begin{aligned}
\mathrm{d}\hat{f}_1 \wedge \mathrm{d}\hat{f}_2 &= \sum_{r<s} \Big(\frac{\partial \hat{f}_1}{\partial x_r} \frac{\partial \hat{f}_2}{\partial x_s} - \frac{\partial \hat{f}_1}{\partial x_s} \frac{\partial \hat{f}_2}{\partial x_r} \Big) \mathrm{d}x_r \wedge \mathrm{d}x_s \\
&= \sum_{r<s} \Big[\Big(\sum_{I_1} i_{1,r} a_{1,I_1} x^{I_1 - e_r} \Big) \Big(\sum_{I_2} i_{2,s} a_{2,I_2} x^{I_2 - e_s} \Big) \\
&\qquad - \Big(\sum_{I_1} i_{1,s} a_{1,I_1} x^{I_1 - e_s} \Big) \Big(\sum_{I_2} i_{2,r} a_{2,I_2} x^{I_2 - e_r} \Big) \Big] \mathrm{d}x_r \wedge \mathrm{d}x_s.
\end{aligned}$$

This becomes

$$\mathrm{d}\hat{f}_1 \wedge \mathrm{d}\hat{f}_2 = \sum_{r<s} \Big[\sum_{I_1\, I_2} a_{1,I_1} a_{2,I_2} (i_{1,r} i_{2,s} - i_{1,s} i_{2,r}) x^{I_1 + I_2 - (e_r + e_s)} \Big] \mathrm{d}x_r \wedge \mathrm{d}x_s. \quad (2.1)$$

Therefore, there exist r, s such that $i_{1,r} i_{2,s} - i_{1,s} i_{2,r} \neq 0$ with $a_{1,I_1} a_{2,I_2} \neq 0$ or $a_{1,I_2} a_{2,I_1} \neq 0$, i.e., $(i_{1,r}, i_{1,s})$ and $(i_{2,r}, i_{2,s})$ are independent multi-indexes. Hence

$$I_1 = (i_{1,1}, \ldots, i_{1,r}, \ldots, i_{1,s}, \ldots, i_{1,n}) \text{ and } I_2 = (i_{2,1}, \ldots, i_{2,r}, \ldots, i_{2,s}, \ldots, i_{2,n})$$

are linearly independent. We continue by doing the wedge product of (2.1) with $\mathrm{d}\hat{f}_3$.

$$\begin{aligned}
\mathrm{d}\hat{f}_1 \wedge \mathrm{d}\hat{f}_2 \wedge \mathrm{d}\hat{f}_3 &= \Big(\sum_{r<s} \Big[\sum_{I_1\, I_2} a_{1,I_1} a_{2,I_2} \begin{vmatrix} i_{1,r} & i_{1,s} \\ i_{2,r} & i_{2,s} \end{vmatrix} x^{I_1 + I_2 - (e_r + e_s)} \Big] \mathrm{d}x_r \wedge \mathrm{d}x_s \Big) \\
&\quad \wedge \Big(\sum_{I_3} i_{3,1} a_{3,I_3} x^{I_3 - e_1} \mathrm{d}x_1 + \cdots + \sum_{I_3} i_{3,n} a_{3,I_3} x^{I_3 - e_n} \mathrm{d}x_n \Big) \\
&= \Big(\sum_{r<s} \Big[\sum_{I_1\, I_2} a_{1,I_1} a_{2,I_2} \begin{vmatrix} i_{1,r} & i_{1,s} \\ i_{2,r} & i_{2,s} \end{vmatrix} x^{I_1 + I_2 - (e_r + e_s)} \Big] \mathrm{d}x_r \wedge \mathrm{d}x_s \Big) \\
&\quad \wedge \Big(\sum_{j} \sum_{I_3} i_{3,j} a_{3,I_3} x^{I_3 - e_j} \mathrm{d}x_j \Big).
\end{aligned}$$

This can be written as

$$\mathrm{d}\hat{f}_1 \wedge \mathrm{d}\hat{f}_2 \wedge \mathrm{d}\hat{f}_3 =$$
$$\sum_{j \neq r,s} \sum_{r<s} \sum_{I_1\, I_2\, I_3} a_{1,I_1} a_{2,I_2} a_{3,I_3} \begin{vmatrix} i_{1,r} & i_{1,s} \\ i_{2,r} & i_{2,s} \end{vmatrix} i_{3,j} x^{I_1 + I_2 + I_3 - (e_r + e_s + e_j)} \mathrm{d}x_r \wedge \mathrm{d}x_s \wedge \mathrm{d}x_j. \quad (2.2)$$

Fixing $r < s < j$. The following terms appear in (2.2)

$$a_{1,I_1}a_{2,I_2}a_{3,I_3}\Big[\begin{vmatrix} i_{1,r} & i_{1,s} \\ i_{2,r} & i_{2,s}\end{vmatrix} i_{3,j}\mathrm{d}x_r\wedge\mathrm{d}x_s\wedge\mathrm{d}x_j + \begin{vmatrix} i_{1,s} & i_{1,j} \\ i_{2,s} & i_{2,j}\end{vmatrix} i_{3,r}\mathrm{d}x_s\wedge\mathrm{d}x_j\wedge\mathrm{d}x_r$$
$$+\begin{vmatrix} i_{1,r} & i_{1,j} \\ i_{2,r} & i_{2,j}\end{vmatrix} i_{3,s}\mathrm{d}x_r\wedge\mathrm{d}x_j\wedge\mathrm{d}x_s\Big] =$$
$$a_{1,I_1}a_{2,I_2}a_{3,I_3}\Big[\begin{vmatrix} i_{1,r} & i_{1,s} \\ i_{2,r} & i_{2,s}\end{vmatrix} i_{3,j} + \begin{vmatrix} i_{1,s} & i_{1,j} \\ i_{2,s} & i_{2,j}\end{vmatrix} i_{3,r} - \begin{vmatrix} i_{1,r} & i_{1,j} \\ i_{2,r} & i_{2,j}\end{vmatrix} i_{3,s}\Big]\mathrm{d}x_r\wedge\mathrm{d}x_s\wedge\mathrm{d}x_j$$
$$= a_{1,I_1}a_{2,I_2}a_{3,I_3}\begin{vmatrix} i_{1,r} & i_{1,s} & i_{1,j} \\ i_{2,r} & i_{2,s} & i_{2,j} \\ i_{3,r} & i_{3,s} & i_{3,j}\end{vmatrix}\mathrm{d}x_r\wedge\mathrm{d}x_s\wedge\mathrm{d}x_j$$

Hence (2.2) can be written as

$$\mathrm{d}\hat{f}_1\wedge\mathrm{d}\hat{f}_2\wedge\mathrm{d}\hat{f}_3 =$$
$$\sum_{r<s<j}\sum_{I_1\, I_2\, I_3} a_{1,I_1}a_{2,I_2}a_{3,I_3}\begin{vmatrix} i_{1,r} & i_{1,s} & i_{1,j} \\ i_{2,r} & i_{2,s} & i_{2,j} \\ i_{3,r} & i_{3,s} & i_{3,j}\end{vmatrix} x^{I_1+I_2+I_3-(e_r+e_s+e_j)}\mathrm{d}x_r\wedge\mathrm{d}x_s\wedge\mathrm{d}x_j$$

Therefore, there exist r, s, j such that $\begin{vmatrix} i_{1,r} & i_{1,s} & i_{1,j} \\ i_{2,r} & i_{2,s} & i_{2,j} \\ i_{3,r} & i_{3,s} & i_{3,j}\end{vmatrix} \neq 0$ such that for each I_k at least one a_{j,I_k}, for $1 \leq j \leq 3$, is not zero, i.e., $(i_{1,r}, i_{1,s}, i_{1,j})$, $(i_{2,r}, i_{2,s}, i_{2,j})$ and $(i_{3,r}, i_{3,s}, i_{3,j})$ are independent multi-indexes. Hence

$$\begin{aligned} I_1 &= (i_{1,1},\ldots,i_{1,r},\ldots,i_{1,s},\ldots,i_{1,j},\ldots,i_{1,n}) \\ I_2 &= (i_{2,1},\ldots,i_{2,r},\ldots,i_{2,s},\ldots,i_{2,j},\ldots,i_{2,n}) \\ I_3 &= (i_{3,1},\ldots,i_{3,r},\ldots,i_{3,s},\ldots,i_{3,j},\ldots,i_{3,n}) \end{aligned}$$

are linearly independent. This process can be continued until we obtain n independent multi-indexes $I_1, \ldots, I_n \in \mathbb{N}^n$. ✓

Remark 2.2.6. Observe that in dimension 2 there can not exist $\hat{f}_1$ and $\hat{f}_2$ generically transverse such that $\hat{f}_i \circ \hat{G} = \hat{f}_i$ with $\hat{G}$ as in the Proposition 2.2.3. In a similar way for dimension n, there can not exist $\hat{f}_1, \ldots, \hat{f}_n$ transversal at the origin such that $\hat{f}_i \circ \hat{G} = \hat{f}_i$ with $\hat{G}$ as in the proposition above. This is because each one satisfies

$$\hat{f}_i(0,\ldots,0,x_j,x_{j+1},0,\ldots,0) = l_i(x_{j+1}^m) \text{ for some } l_i \in \hat{\mathcal{O}}_1,$$

and then $\mathrm{d}\hat{f}_1\wedge\cdots\wedge\mathrm{d}\hat{f}_n$ is 0 when restricted to the plane $\{x_j, x_{j+1}\}$. In particular $(\mathrm{d}\hat{f}_1\wedge\cdots\wedge\mathrm{d}\hat{f}_n)_0 = 0$.

We will apply the propositions above in order to study groups of formal diffeomorphisms leaving invariant a set of generically transverse formal series.

Definition 2.2.7. The *invariance group of* $\hat{f} \in \hat{\mathcal{O}}_n$ is defined as

$$H(\hat{f}) = \{\hat{G} \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0) \mid \hat{f} \circ \hat{G} = \hat{f}\}.$$

The *invariance group of* $\{\hat{f}_1 \ldots, \hat{f}_n\}$, where $\hat{f}_1, \ldots, \hat{f}_n \in \hat{\mathcal{O}}_n$, is

$$H(\hat{f}_1 \ldots, \hat{f}_n) = \{\hat{G} \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0) \mid \hat{f}_i \circ \hat{G} = \hat{f}_i \quad \text{for} \quad i = 1, \ldots, n\}.$$

The following proposition is one of the key parts of our work.

Proposition 2.2.8. *Let $\hat{f}_1 \ldots, \hat{f}_n \in \hat{\mathcal{O}}_n$ be generically transverse. Then the group $H(\hat{f}_1 \ldots, \hat{f}_n)$ is periodic (in particular linearizable and finite).*

The proof of Proposition 2.2.8 requires algebraic properties of groups of diffeomorphisms. In Appendix A we give part of the supporting material and a sketch of the proof. Using the theory we have built so far, we can give a proof of the following particular case:

Proposition 2.2.9. *Let $\hat{f}_1 \ldots, \hat{f}_n \in \hat{\mathcal{O}}_n$ be transversal at the origin. Then the group $H(\hat{f}_1 \ldots, \hat{f}_n)$ is periodic (in particular linearizable and finite).*

For the proof of Proposition 2.2.9 we need the following result from [9], whose demonstration we put here to emphasize that it is also valid in the formal case:

Proposition 2.2.10. *A group $\mathcal{G} \subset \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ is linearizable if and only if there exists a vector field $\mathcal{X} = \mathcal{R} + \cdots$, where $\mathcal{R}$ is a radial vector field, such that $\mathcal{X}$ is invariant for every $\hat{G} \in \mathcal{G}$, i.e. $\hat{G}^*\mathcal{X} = \mathcal{X}$.*

Proof.

($\Longrightarrow$) Suppose that $\mathcal{G}$ is linearizable, i.e. there exists $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $g \circ \mathcal{G} \circ g^{-1} = \{\mathrm{d}\hat{G}_0 \mid \hat{G} \in \mathcal{G}\}$. Since $(A(\cdot))^*\mathcal{R} = \mathcal{R}$ for all $A \in Gl(n, \mathbb{C})$ (by a direct calculation, $(A(\cdot))^*\mathcal{R}_z = \mathrm{d}A(\cdot)_{A^{-1}z}\mathcal{R}A^{-1}z = z$), in particular for every element $\hat{G} \in \mathcal{G}$ we have

$$\begin{aligned}
\mathcal{R}_z &= (g \circ \hat{G} \circ g^{-1})^*\mathcal{R}_z = \mathrm{d}(g \circ \hat{G} \circ g^{-1})_{(g \circ \hat{G}^{-1} \circ g^{-1})(z)}\mathcal{R}((g \circ \hat{G}^{-1} \circ g^{-1})(z)),\\
z &= \mathrm{d}g_{g^{-1}(z)}\mathrm{d}\hat{G}_{\hat{G}^{-1} \circ g^{-1}(z)}\mathrm{d}g^{-1}_{(g \circ \hat{G}^{-1} \circ g^{-1})(z)}(g \circ \hat{G}^{-1} \circ g^{-1})(z).
\end{aligned}$$

Taking $z = g(y)$ and multiplying by $\mathrm{d}g^{-1}_{g(y)}$, we have

$$\mathrm{d}g^{-1}_{g(y)}(g(y)) = \mathrm{d}\hat{G}_{\hat{G}^{-1}(y)}\mathrm{d}g^{-1}_{(g \circ \hat{G}^{-1}(y))}(g \circ \hat{G}^{-1}(y)).$$

Denoting $\mathcal{X} = \mathrm{d}g^{-1}_{g(\cdot)}(g(\cdot))$, we have $\hat{G}^*\mathcal{X} = \mathcal{X}$. It is easy to see that $\mathcal{X} = \mathcal{R} + \cdots$. For this, suppose that

$$\begin{aligned} g(z) &= Az + P_l(z) + P_{l+1}(z) + \cdots , \\ g^{-1}(z) &= A^{-1}z + Q_\nu(z) + Q_{\nu+1}(z) + \cdots , \end{aligned}$$

where $A \in \mathcal{M}_n(\mathbb{C})$ and P_l, Q_ν are polynomial vector fields of degrees l and ν, then

$$\begin{aligned} \mathrm{d}g^{-1}_z &= A^{-1} + \mathrm{d}Q_\nu(z) + \mathrm{d}Q_{\nu+1}(z) + \cdots , \\ \mathrm{d}g^{-1}_{g(z)} &= A^{-1} + \mathrm{d}Q_\nu(z)_{g(z)} + \mathrm{d}Q_{\nu+1}(z)_{g(z)} + \cdots , \end{aligned}$$

which gives

$$\begin{aligned} \mathcal{X}_z = \mathrm{d}g^{-1}_{g(z)}g(z) &= \big(A^{-1} + \mathrm{d}Q_\nu(z)_{g(z)} + \cdots\big)\big(Az + P_l(z) + \cdots\big) \\ &= z + A^{-1}\big(P_l(z) + P_{l+1}(z) + \cdots\big) + \\ &\qquad + \mathrm{d}Q_\nu(z)_{g(z)}\big(Az + P_l(z) + P_{l+1}(z) + \cdots\big) + \cdots \end{aligned}$$

The terms after z, if not 0, are of degree greater than one, thus $\mathcal{X} = \mathcal{R} + \cdots$ as we wanted.

($\Longleftarrow$) Since every eigenvalue of the linear part of $\mathcal{X}$ is 1, then $\mathcal{X}$ is in the Poincaré domain without resonances (additive resonances), therefore using Poincaré linearization theorem ([24] Theorem 4.3) there exists a formal diffeomorphism $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $g^*\mathcal{X} = \mathcal{R}$, i.e. $\mathcal{X} = (\mathrm{d}g(\cdot))^{-1}g(\cdot)$.

We claim that $g \circ \hat{G} \circ g^{-1}(y) = \mathrm{d}\hat{G}_0(y)$ for every $\hat{G} \in \mathcal{G}$. In fact, using the same procedure as before we can observe that

$$\mathcal{R}_z = \ (g \circ \hat{G} \circ g^{-1})^*\mathcal{R}_z.$$

For this, note that $\hat{G}^*\mathcal{X} = \mathcal{X}$ means that $\mathrm{d}\hat{G}_{\hat{G}^{-1}(y)}\mathcal{X}_{\hat{G}^{-1}(y)} = \mathcal{X}_z$, which gives $\mathrm{d}\hat{G}_{\hat{G}^{-1}(y)}\mathrm{d}g^{-1}_{g\circ\hat{G}^{-1}(y)}g \circ \hat{G}^{-1}(y) = \mathrm{d}g^{-1}_{g(y)}g(y)$. Taking $z = g(y)$, we have

$$\mathrm{d}\hat{G}_{\hat{G}^{-1}\circ g^{-1}(z)}\mathrm{d}g^{-1}_{g\circ\hat{G}^{-1}\circ g^{-1}(z)}g \circ \hat{G}^{-1} \circ g^{-1}(z) = \mathrm{d}g^{-1}_z(z).$$

Therefore,

$$\begin{aligned} (g \circ \hat{G} \circ g^{-1})^*\mathcal{R}_z &= \mathrm{d}(g \circ \hat{G} \circ g^{-1})_{g\circ\hat{G}^{-1}\circ g^{-1}(z)}\mathcal{R}(g \circ \hat{G}^{-1} \circ g^{-1}(z)) \\ &= \mathrm{d}g_{g^{-1}(z)}\mathrm{d}\hat{G}_{\hat{G}^{-1}\circ g^{-1}(z)}\mathrm{d}g^{-1}_{g\circ\hat{G}^{-1}\circ g^{-1}(z)}g \circ \hat{G}^{-1} \circ g^{-1}(z) \\ &= \mathrm{d}g_{g^{-1}(z)}\mathrm{d}g^{-1}_z(z) \quad \text{(by the the previous computation)} \\ &= z. \end{aligned}$$

Now, if we suppose that $g \circ \hat{G} \circ g^{-1}(z) = Az + P_l(z) + P_{l+1}(z) + \cdots$, where $P_j(z)$ is a polynomial vector field of degree j, then it is easy to prove that

$$(g \circ \hat{G} \circ g^{-1})^* \mathcal{R} = Az + lP_l(z) + (l+1)P_{l+1}(z) + \cdots,$$

In order to prove it, observe that $(g \circ \hat{G} \circ g^{-1})^* \mathcal{R}_z = \mathcal{R}_z$ implies

$$\mathrm{d}(g \circ \hat{G} \circ g^{-1})_{g \circ \hat{G}^{-1} \circ g^{-1}(y)} \mathcal{R}(g \circ \hat{G}^{-1} \circ g^{-1}(y)) = \mathcal{R}(y)$$

taking $y = g \circ \hat{G} \circ g^{-1}(z)$ then

$$\mathrm{d}(g \circ \hat{G} \circ g^{-1})_z \mathcal{R}(z) = \mathcal{R}(g \circ \hat{G} \circ g^{-1}(z)),$$

which implies $\mathrm{d}(g \circ \hat{G} \circ g^{-1})_z z = g \circ \hat{G} \circ g^{-1}(z)$. By hypothesis $\mathrm{d}(g \circ \hat{G} \circ g^{-1})_z = A + \mathrm{d}(P_l)_z + \mathrm{d}(P_{l+1})_z + \cdots$. Then we have

$$\begin{aligned} \mathrm{d}(g \circ \hat{G} \circ g^{-1})_z z &= Az + lP_l(z) + (l+1)P_{l+1}(z) + \cdots, \\ &= Az + P_l(z) + P_{l+1}(z) + \cdots. \end{aligned}$$

and therefore $P_j(z) \equiv 0$ for every $j \geq 2$. ✓

Proof of Proposition 2.2.9. The idea is to find an invariant vector field $\mathcal{X}$ and then use the above proposition. First, consider the formal map $H = (\hat{f}_1, \ldots, \hat{f}_n)$. For each $\hat{G} \in \mathcal{G}$ we have by hypothesis $\hat{f}_i \circ \hat{G} = f_i$ and then $H \circ \hat{G} = H$. Note that $H \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ because $(\mathrm{d}\hat{f}_1 \wedge \cdots \wedge \mathrm{d}\hat{f}_n)_0 \neq 0$. This implies that $H \circ \hat{G}^{-1} = H$, $\hat{G} \circ H^{-1} = H^{-1}$ and $\mathrm{d}\hat{G}_{H^{-1}(\cdot)} \mathrm{d}H^{-1}_{(\cdot)} = \mathrm{d}H^{-1}_{(\cdot)}$.

Let us define $\mathcal{X} = (\mathrm{d}H)^{-1} H = \mathrm{d}H^{-1}_{H(\cdot)} H(\cdot)$, which satisfies $\hat{G}^* \mathcal{X} = \mathcal{X}$, as shown below:

$$\begin{aligned} \hat{G}^* \mathcal{X}_z &= \mathrm{d}\hat{G}_{\hat{G}^{-1}(z)} \mathcal{X}_{\hat{G}^{-1}(z)} \\ &= \mathrm{d}\hat{G}_{\hat{G}^{-1}(z)} \mathrm{d}H^{-1}_{H(\hat{G}^{-1}(z))} H(G^{-1}(z)) \\ &= \mathrm{d}\hat{G}_{\hat{G}^{-1}(z)} \mathrm{d}H^{-1}_{H(z)} H(z) \\ &= \mathrm{d}H^{-1}_{H(z)} H(z), \text{ because } \mathrm{d}\hat{G}_{H^{-1}(H \circ \hat{G}^{-1}(z))} \mathrm{d}H^{-1}_{H \circ \hat{G}^{-1}(z)} = \mathrm{d}H^{-1}_{H \circ G^{-1}(z)}, \\ &= \mathcal{X}_z. \end{aligned}$$

Besides, by the proof of Proposition 2.2.10, we have that $\mathcal{X} = \mathcal{R} + \cdots$.

Then, by Proposition 2.2.10, we have that $\mathcal{G}$ is linearizable. Furthermore, this implies that $\mathcal{G}$ is in fact diagonalizable by Propositions 2.2.1 and 2.2.3, and Remark 2.2.6. In addition its diagonal form is made of roots of unity by Proposition 2.2.2, since the transversality condition of $\{\hat{f}_i\}$ implies the existence of n independent multi-indexes by Proposition 2.2.5.

Indeed, the previous analysis is more subtle, because we have to consider $(\hat{f}_i \circ \hat{G})(g) = (\hat{f}_i \circ g)(g^{-1} \circ \hat{G} \circ g) = (\hat{f}_i \circ g)(G) = (\hat{f}_i \circ g)$ where g is a formal diffeomorphism that diagonalizes $\mathcal{G}$. The result is the same because the $\hat{f}_i \circ g$ are generically transverse.

Therefore, there exists $N \in \mathbb{N}$ such that $G^N = I$ and then $\langle \hat{G} \rangle$ (i.e. the group generated by $\hat{G}$) is finite. It remains to note that $\mathcal{G}$ is commutative. Consider $\hat{G}_1, \hat{G}_2 \in \mathcal{G}$ and denote by G_1, G_2 their linear parts, then

$$\begin{aligned}
\hat{G}_1 \circ \hat{G}_2 &= g(g^{-1} \circ \hat{G}_1 \circ g)(g^{-1} \circ \hat{G}_2 \circ g)g^{-1} \\
&= g(G_1 \circ G_2)g^{-1} \\
&= g(G_2 \circ G_1)g^{-1} \\
&= \hat{G}_2 \circ \hat{G}_1.
\end{aligned}$$

Where we know that G_1 and G_2 commute because they are diagonal diffeomorphisms. ✓

Chapter 3

On formal first integrals

In this chapter we will show that the existence of a formal first integral in our framework, implies the existence of a holomorphic one. We use the notation and results of [14, 15]

3.1 Preliminaries

Given a germ of a holomorphic vector field $\mathcal{X} \in \mathfrak{X}(\mathbb{C}^n, 0)$ we shall denote by $\mathcal{F}(\mathcal{X})$ the germ of a one-dimensional holomorphic foliation on $(\mathbb{C}^n, 0)$ induced by $\mathcal{X}$.

Definition 3.1.1. We shall say that $\mathcal{F}(\mathcal{X})$ is *non-degenerate generic* if $\mathrm{d}\mathcal{X}(0)$ is non-singular, diagonalizable and, after some suitable change of coordinates, $\mathcal{X}$ leaves invariant the coordinate planes. Denote the set of germs of non-degenerate generic vector fields on $(\mathbb{C}^n, 0)$ by $\mathrm{Gen}\big(\mathfrak{X}(\mathbb{C}^n, 0)\big)$. Such vector fields, after a change of coordinates, can be written in the form

$$\mathcal{X}(x) = \lambda_1 x_1(1 + a_1(x))\frac{\partial}{\partial x_1} + \cdots + \lambda_n x_n(1 + a_n(x))\frac{\partial}{\partial x_n}, \tag{3.1}$$

where $a_i \in \mathcal{M}_n$ for $i = 1, \ldots, n$.

Definition 3.1.2. We say that a germ of one-dimensional holomorphic foliation $\mathcal{F}(\mathcal{X})$ has a *holomorphic first integral* if there is a germ of a holomorphic map $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-1}, 0)$ such that:

(a) F is a submersion outside some proper analytic subset. Equivalently if we write $F = (f_1, \ldots, f_{n-1})$ in coordinate functions, then the $(n-1)$-form $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{n-1}$ is non-identically zero.

(b) The leaves of $\mathcal{F}(\mathcal{X})$ are contained in level curves of F .

Further, a germ f of a meromorphic function at the origin $0 \in \mathbb{C}^n$ is called $\mathcal{F}(\mathcal{X})$-*invariant* if the leaves of $\mathcal{F}(\mathcal{X})$ are contained in the level sets of f. This can be precisely stated in terms of representatives for $\mathcal{F}(\mathcal{X})$ and f, but can also be written as $i_{\mathcal{X}}(\mathrm{d}f) = \mathcal{X}(f) \equiv 0$.

We start with the following definition inspired in the concept of *holomorphic fist integral* (Definition 3.1.2);

Definition 3.1.3 (formal first integral)**.** We say that a germ of a holomorphic foliation $\mathcal{F}(\mathcal{X})$, were $\mathcal{X} \in \mathfrak{X}(\mathbb{C}^n, 0)$, has a *formal first integral* if there is a formal map $\hat{F} = (\hat{f}_1, \ldots, \hat{f}_{n-1})$, with $\hat{f}_1, \ldots, \hat{f}_{n-1} \in \hat{\mathcal{O}}_n$, such that:

(a) The formal $(n-1)$-form $\mathrm{d}\hat{f}_1 \wedge \cdots \wedge \mathrm{d}\hat{f}_{n-1}$ is non-identicaly zero.

(b) $\mathcal{X}(\hat{F}) \equiv 0$, (i.e. the f_i are $\mathcal{F}(\mathcal{X})$-invariant, $\mathcal{X}(\hat{f}_i) \equiv 0$ for all $\hat{f}_i$, $i = 1, \ldots, n-1$).

Definition 3.1.4 (condition $(\star)$)**.** Let $\mathcal{X}$ be a germ of a holomorphic vector field at the origin such that $0 \in \mathbb{C}^m, m \geq 3$, is a non-degenerate singularity of $\mathcal{X}$ (i.e. $\mathrm{d}\mathcal{X}(0)$ is non-singular). We say that $\mathcal{X}$ satisfies *condition* $(\star)$ if there is a real line $L \subset \mathbb{C}$ through the origin separating some eigenvalue $\lambda(\mathcal{X})$ from the others. If $\mathcal{X}$ satisfies $(\star)$ we denote by $S_{\mathcal{X}}$ the smooth invariant curve associated to $\lambda(\mathcal{X})$.

Although the methods we use in this chapter are in general independent of the dimension, our work will imply directly condition $(\star)$ only when $n = 3$. In the remaining cases we have to include it as a hypothesis. This condition, together with the generic conditions of the vector field $\mathcal{X}$, is what allows to use the following well known result (see [20]) whose demonstration can also be found in [40].

Theorem 3.1.5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two vector fields in* $\mathrm{Gen}\,(\mathfrak{X}(\mathbb{C}^n, 0))$ *with an isolated singularity at the origin satisfying condition $(\star)$. Let $h_{\mathcal{X}}$ and $h_{\mathcal{Y}}$ be the holonomies of $\mathcal{X}$ and $\mathcal{Y}$ relatively to $S_{\mathcal{X}}$ and $S_{\mathcal{Y}}$, respectively. Then $\mathcal{X}$ and $\mathcal{Y}$ are analytically equivalent if and only if the holonomies $h_{\mathcal{X}}$ and $h_{\mathcal{Y}}$ are analytically conjugate.*

This theorem is basically the heart of the proof of the equivalence (3) $\Leftrightarrow$ (4) in Theorem 1 of [14], whose statement is:

Theorem 3.1.6. *Suppose that* $\mathcal{X} \in \mathrm{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ *satisfies condition* $(\star)$ *and let $S_{\mathcal{X}}$ be the axis associated to the separable eigenvalue of $\mathcal{X}$.*

Then, $\mathrm{Hol}(\mathcal{F}(\mathcal{X}), S_{\mathcal{X}}, \Sigma)$ *is periodic (in particular linearizable and finite) if and only if $\mathcal{F}(\mathcal{X})$ has a holomorphic first integral.*

We can state now the main result in this chapter.

Theorem C. *Let $\mathcal{F}(\mathcal{X})$ be a germ of holomorphic foliation with $\mathcal{X} \in \mathrm{Gen}\,(\mathfrak{X}(\mathbb{C}^3, 0))$. If $\mathcal{F}(\mathcal{X})$ has a formal first integral then it also has a holomorphic one.*

In order to prove this result result we show that having a formal first integral, gives enough properties to the vector field that Theorem 3.1.6 can be used.

3.2 Algebraic criterion

In this section we show that we can restrict ourselves to a vector fields written in a particular way.

The following lemma and proposition are, at first glance, essentially n dimensional versions of Lemma 1 and Proposition 1 in [14]. Nevertheless, there is a difference which turns out to be an important property, as we explain after the following lemma.

Lemma 3.2.1. *Let $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \setminus 0$ and $N_{n-1\times n}$ be a matrix with entries in $\mathbb{N}$ and linearly independent lines satisfying*

$$N\Lambda^t = 0 \in \mathbb{C}^{n-1}.$$

Then there are $k_1 \ldots, k_n \in \mathbb{Z}$ and $\lambda \in \mathbb{C}^*$ such that

$$(\lambda_1, \ldots, \lambda_n) = (k_1 \ldots, k_n)\lambda.$$

Proof. The proof consists in the solution of a linear system. Take

$$N = \begin{bmatrix} n_{1\,1} & \ldots & n_{1\,n-1} & n_{1\,n} \\ \vdots & \ddots & \vdots & \vdots \\ n_{n-1\,1} & \ldots & n_{n-1\,n-1} & n_{n-1\,n} \end{bmatrix} \text{ and } A = \begin{bmatrix} n_{1\,1} & \ldots & n_{1\,n-1} \\ \vdots & \ddots & \vdots \\ n_{n-1\,1} & \ldots & n_{n-1\,n-1} \end{bmatrix}.$$

The independence allows to take $n-1$ independent columns, which we suppose to be the first ones. Thus A is invertible and multiplying by A^{-1} the system $N\Lambda^t = 0$, we get

$$\begin{bmatrix} 1 & \ldots & 0 & \tilde{k}_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & \tilde{k}_{n-1} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{n-1\times 1}.$$

We have $n-1$ equations of the form $\lambda_i + \tilde{k}_i \lambda_n = 0$. Then

$$(\lambda_1, \ldots, \lambda_n) = (-\tilde{k}_1, \ldots, -\tilde{k}_{n-1}, 1)\lambda_n.$$

We know exactly who are the $\tilde{k}_i$'s, because they satisfy

$$\begin{bmatrix} n_{1\,1} & \ldots & n_{1\,n-1} \\ \vdots & \ddots & \vdots \\ n_{n-1\,1} & \ldots & n_{n-1\,n-1} \end{bmatrix} \begin{bmatrix} \tilde{k}_1 \\ \vdots \\ \tilde{k}_{n-1} \end{bmatrix} = \begin{bmatrix} n_{1\,n} \\ \vdots \\ n_{n-1\,n}, \end{bmatrix}$$

and, by the Cramer rule, $\tilde{k}_i = \frac{|A_i|}{|A|}$, where $|\cdot|$ means determinant and A_i is the matrix obtained from A by replacing the column i to $[n_{1\,n} \ldots n_{n-1\,n}]^t$. Finally, we get

$$(\lambda_1, \ldots, \lambda_n) = (|A_1|, \ldots, |A_{n-1}|, -|A|)\lambda,$$

with $\lambda = -\lambda_n/|A|$, $k_i = |A_i| \in \mathbb{Z}$ for $i = 1, \ldots, n-1$ and $k_n = -|A| \in \mathbb{Z}$ as we wanted. ✓

We know that the signs of the k_i cannot be all neither positive nor negative thanks to the condition $n_{1\,1}k_1 + \cdots + n_{1\,n}k_n = 0$. The three dimensional case is special because this implies that $k_1 \cdot k_2 \cdot k_3 < 0$. So we can make one of them negative and the other two positive by changing λ. However, in dimension $n > 3$ this is not necessarily true. Here we have an example in dimension 4 where $k_1 \cdot k_2 \cdot k_3 \cdot k_3 > 0$. Take

$$N = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

if we have $N\Lambda^t = 0$ for some $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ then,

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-1, 1, 1-1)\lambda.$$

With this example we can also see that a vector field of Siegel type may not necessarily satisfy condition $(\star)$, whereas the opposite is always true.

Proposition 3.2.2. *Suppose that $\mathcal{X} \in \mathrm{Gen}(\mathfrak{X}(\mathbb{C}^n, 0))$ has a formal first integral. Then $\mathcal{F}(\mathcal{X})$ can be given in local coordinates by a vector field of the form*

$$\mathcal{X}(x) = k_1 x_1 (1 + a_1(x)) \frac{\partial}{\partial x_1} + \cdots + k_n x_n (1 + a_n(x)) \frac{\partial}{\partial x_n} \tag{3.2}$$

where $k_1, \ldots k_n \in \mathbb{Z}$ and $a_1, \ldots, a_n \in \mathcal{M}_n$. In particular, if $n = 3$, $\mathcal{X}$ satisfies condition $(\star)$.

Proof. We are considering $\mathcal{X} \in \mathrm{Gen}(\mathfrak{X}(\mathbb{C}^n, 0))$. If $\hat{F} = (\hat{f}_1 \dots, \hat{f}_{n-1})$ is the formal first integral, then $\mathcal{X}(\hat{f}_i) \equiv 0$ for $i = 1, \dots, n-1$. If $\hat{f}_i(x) = \sum_{|I| \geq p_i} a_{i\,I} x^I$, then

$$\frac{\partial \hat{f}_i}{\partial x_r}(x) = \sum_{|I| \geq p_i} (i_r) a_{i\,I} x_1^{i_1} \cdots x_r^{i_r - 1} \cdots x_n^{i_n},$$

and

$$\begin{aligned}
\mathcal{X}(\hat{f}_i) &= \sum_{r=1}^{n} \lambda_r x_r (1 + a_r(x)) \big(\sum_{|I| \geq p_i} (i_r) a_{i\,I} x_1^{i_1} \cdots x_r^{i_r - 1} \cdots x_n^{i_n} \big) \\
&= \sum_{r=1}^{n} \sum_{|I| \geq p_i} i_r \lambda_r a_{i\,I} (1 + a_r(x)) x_1^{i_1} \cdots x_r^{i_r} \cdots x_n^{i_n} \\
&= \sum_{|I| \geq p_i} \sum_{r=1}^{n} i_r \lambda_r a_{i\,I} (1 + a_r(x)) x^I \\
&= \sum_{|I| \geq p_i} a_I \Big(\sum_{r=1}^{n} i_r \lambda_{ri} \Big) x^I + \sum_{|I| \geq p_i} a_I \Big(\sum_{r=1}^{n} i_r \lambda_{ri} a_r(x) \Big) x^I = 0.
\end{aligned}$$

Thus given

$$J^{p_i} \mathcal{X}(\hat{f}_i) = \sum_{|I| = p_i} a_I \Big(\sum_{r=1}^{n} i_r \lambda_{ri} \Big) x^I = 0,$$

which implies $\sum_{r=1}^{n} i_r \lambda_{ri} = 0$ for each $I = (i_1, \dots, i_n)$ such that $a_I \neq 0$. Now, by the same argument at the end of the proof of Proposition 2.2.9, there are $n-1$ linearly independent n-tuples satisfying this condition. With them we can form the matrix N of Lemma 3.2.1, and we are done.

✓

3.3 Holonomy and formal first integrals

We know that holonomy maps (by its construction) leave invariant the level sets of a holomorphic first integral. What we want to obtain is a similar invariance relation in the case of formal first integral. For simplicity, we work in dimension 3, but small changes are needed for the general case.

Consider the foliation with formal first integral given by the vector field $\mathcal{X}$ which, by Proposition 3.2.2, can be taken in the form (3.2). Note that the vector field obtained from (3.2) by multiplying by $\big(- k_3(1 + a_3(x)) \big)^{-1}$

defines the same foliation. Then we can write $\mathcal{X}$ as

$$\mathcal{X}(x_1,x_2,x_3) = px_1a_1(x)\frac{\partial}{\partial x_1} + qx_2a_2(x)\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3},$$

where $a_1, a_2 \in \mathcal{M}_3$ and $p, q \in \mathbb{Q}$, let $S := (x_1 = x_2 = 0)$ and $\Sigma := (x_3 = 1)$. Consider the closed loop $\gamma : [0,1] \mapsto S$ given by $\gamma(t) = (0,0,e^{2\pi it})$ and let $\overline{\Gamma}_{(x_1,x_2)}(t) = (\Gamma_1(x_1,x_2,t), \Gamma_2(x_1,x_2,t), e^{2\pi it})$ be its lifting along the leaves of $\mathcal{F}(\mathcal{X})$ starting at $(x_1,x_2,1) \in \Sigma$. In particular, the map $h \in \mathrm{Diff}(\mathbb{C}^2,0)$ given by $\overline{\Gamma}_{(x_1,x_2)}(1) = (h(x_1,x_2),1)$ is a generator of $\mathrm{Hol}(\mathcal{F}(\mathcal{X}),S,\Sigma)$. Since $\overline{\Gamma}_{(x_1,x_2)}(t)$ belongs to a leaf of $\mathcal{F}(\mathcal{X})$, then

$$\frac{\partial}{\partial t}\overline{\Gamma}_{(x_1,x_2)}(t) = \alpha\mathcal{X}(\Gamma_1(x_1,x_2,t), \Gamma_2(x_1,x_2,t), e^{2\pi it}).$$

From this vector equation one has $2\pi i e^{2\pi it} = \alpha e^{2\pi it}$, thus $\alpha = 2\pi i$. Furthermore,

$$\begin{aligned}\frac{\partial \Gamma_1}{\partial t} &= 2\pi i p\Gamma_1(x_1,x_2,t)a_1(\overline{\Gamma}),\\ \frac{\partial \Gamma_2}{\partial t} &= 2\pi i q\Gamma_2(x_1,x_2,t)a_2(\overline{\Gamma}).\end{aligned}$$

Proposition 3.3.1. *Let $\mathcal{F}(\mathcal{X})$ be a foliation induced by a vector field $\mathcal{X} \in \mathrm{Gen}\,(\mathfrak{X}(\mathbb{C}^3,0))$, h be as before and let $\hat{F} = (\hat{f}_1,\hat{f}_2)$, with $\hat{f}_1,\hat{f}_2 \in \hat{\mathcal{O}}_3$, be a formal first integral of $\mathcal{F}(\mathcal{X})$. Then*

$$\hat{f}_i(x_1,x_2,1) = \hat{f}_i(h(x_1,x_2),1) \tag{3.3}$$

Furthermore, there exist a coordinate change $\Phi(x_1,x_2,x_3) = (u_1,u_2,u_3)$ such that (3.3) *can be written as*

$$\Phi^*\hat{f}_i(u_1,u_2) = \Phi^*\hat{f}_i(\tilde{h}(u_1,u_2)), \tag{3.4}$$

where $\tilde{h}$ is given by $\Phi\circ\overline{\Gamma}\circ\Phi^{-1}(u_1,u_2,0) = (\tilde{h}(u_1,u_2,0)$.

Proof. Let U be a small neighborhood of $(0,0,1)$ such that the vector field $\mathcal{X}$ is not singular in U. Then $\mathcal{X}$ can be trivialized (using Rectification Theorem, see [5]) i.e., there exist a biholomorphisms $\Phi : \big(U,(0,0,1)\big) \to \big(\Phi(U) \subset \mathbb{C}^3, 0\big)$ such that $\Phi(x_1,x_2,1) = (u_1,u_2,0)$ and $\Phi^*\mathcal{X}(u) = \partial/\partial u_3$. Observe that $\mathcal{X}(\hat{f}_i) \equiv 0$, for $i = 1,2$, implies $\mathcal{X}(\hat{f}_i)(\Phi^{-1}(u)) = 0$ for $u \in \Phi(U)$ and

$$\begin{aligned}\mathcal{X}(\hat{f}_i)(\Phi^{-1}(u)) &= \mathcal{X}(\Phi^{-1}(u))\cdot(\nabla\hat{f}_i)_{\Phi^{-1}(u)}\\ &= \big(\mathrm{d}\Phi_{\Phi^{-1}(u)}\mathcal{X}(\Phi^{-1}(u))\big)\cdot\big(\nabla\hat{f}_{i\,\Phi^{-1}(u)}\cdot\mathrm{d}\Phi_u^{-1}\big)\\ &= \Phi^*\mathcal{X}(\Phi^*\hat{f}_i).\end{aligned}$$

Hence $\Phi^*\mathcal{X}(\Phi^*\hat{f}_i) \equiv 0$ in U, this means that $\Phi^*\hat{f}_i(u_1, u_2, u_3) = \Phi^*\hat{f}_i(u_1, u_2)$ in other words, $\Phi^*\hat{f}_i$ is a formal series in the variables (u_1, u_2). If $\overline{\Gamma}$ is as before we have $\mathcal{X}(\overline{\Gamma}) = 2\pi i\overline{\Gamma}'$ this implies $\mathcal{X}(\Phi^{-1} \circ \Phi \circ \overline{\Gamma}) = 2\pi i\overline{\Gamma}'$, then we have

$$\begin{aligned} \mathrm{d}\Phi_{\overline{\Gamma}}\mathcal{X}(\Phi^{-1} \circ \Phi \circ \overline{\Gamma}) &= 2\pi i \mathrm{d}\Phi_{\overline{\Gamma}}\overline{\Gamma}', \\ \Phi^*\mathcal{X}(\Phi \circ \overline{\Gamma}) &= 2\pi i(\Phi \circ \overline{\Gamma})', \\ \partial/\partial u_3 &= 2\pi i(\Phi \circ \overline{\Gamma})' \end{aligned}$$

thus $\Phi \circ \overline{\Gamma}(x_1, x_2, t)$ is a vertical line for $t \in [0, \epsilon)$ such that $\overline{\Gamma}(x_1, x_2, t) \in U$ and $\Phi^*\hat{f}_i$ is constant on it, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\Phi^*\hat{f}_i(\Phi \circ \overline{\Gamma}_{(x_1,x_2)}(t))\big) \equiv 0, \text{ for } t \in [0, \epsilon),$$

but $\Phi^*\hat{f}_i(\Phi \circ \overline{\Gamma}_{(x_1,x_2)}(t)) = \hat{f}_i(\overline{\Gamma}_{(x_1,x_2)}(t))$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\hat{f}_i(\overline{\Gamma}_{(x_1,x_2)}(t))\big) \equiv 0, \text{ for } t \in [0, \epsilon).$$

Fix a point $(x_1, x_2, 1) \in U$ and take a finite partition $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$ and open neighborhoods $\{U_i\}_{i=0}^m$ forming a covering of $\overline{\Gamma}_(x_1, x_2)$ such that $\overline{\Gamma}(x_1, x_2, t_i) \in U_i$ and $U_0 = U_1 = U$. Each U_i can be chosen sufficiently small that we can repeat the previous analysis. We conclude that $\hat{f}_i$ is constant along $\overline{\Gamma}(x_1, x_2, t)$ for $t \in [0, 1]$. Hence

$$\hat{f}_i(\overline{\Gamma}_{(x_1,x_2)}(t)) = \hat{f}_i(\overline{\Gamma}_{(x_1,x_2)}(0)) = \hat{f}_i(\overline{\Gamma}_{(x_1,x_2)}(1))$$

and

$$\begin{aligned} \hat{f}_i(\overline{\Gamma}_{(x_1,x_2)}(0)) &= \hat{f}_i(x_1, x_2, 1), \\ \hat{f}_i(\overline{\Gamma}_{(x_1,x_2)}(1)) &= \hat{f}_i(h(x_1, x_2), 1) \end{aligned}$$

then

$$\hat{f}_i(x_1, x_2, 1) = \hat{f}_i(h(x_1, x_2), 1)$$

Consider $H(x_1, x_2, x_3) = (h(x_1, x_2), x_3)$ defined in a small neighborhood of $(x_1, x_2, 1)$. Note that $H(x_1, x_2, 1) = \overline{\Gamma}_{(x_1,x_2)}(1)$ and that $\Phi \circ H \circ \Phi^{-1}$ is a diffeomorphism in an open set of $0 \in \Phi(U) \subset \mathbb{C}^3$. Denote $\Phi \circ H \circ \Phi^{-1}(u_1, u_2, 0) = (\tilde{h}(u_1, u_2), 0)$ then we have

$$\begin{aligned} \hat{f}_i(x_1, x_2, 1) &= \Phi^*\hat{f}_i(u_1, u_2, 0) = \Phi^*\hat{f}_i(u_1, u_2), \\ \hat{f}_i(h(x_1, x_2), 1) &= \hat{f}_i \circ H(x_1, x_2, 1) = \hat{f}_i \circ H \circ \Phi^{-1}(u_1, u_2, 0) \\ &= \hat{f}_i \circ \Phi^{-1}\big(\Phi \circ H \circ \Phi^{-1}(u_1, u_2, 0)\big) \\ &= \Phi^*\hat{f}_i(\tilde{h}(u_1, u_2), 0) = \Phi^*\hat{f}_i(\tilde{h}(u_1, u_2)). \end{aligned}$$

Using (3.3) we have

$$\Phi^* \hat{f}_i(u_1, u_2) = \Phi^* \hat{f}_i(\tilde{h}(u_1, u_2)).$$

In conclusion, after a variable change Φ, we obtain the relation we were looking for. ✓

Remark 3.3.2. One would think that is possible to make the following computation. Using the formal chain rule (Section 2.1.1) and that $\hat{f}_1$ and $\hat{f}_2$ are $\mathcal{F}(\mathcal{X})$-invariant. We have

$$px_1a_1(x_1, x_2, x_3)\frac{\partial \hat{f}_1}{\partial x_1} + qx_2a_2(x_1, x_2, x_3)\frac{\partial \hat{f}_1}{\partial x_2} + x_3\frac{\partial \hat{f}_1}{\partial x_3} = 0$$

Evaluating $\overline{\Gamma}$ and multiplying by $2\pi i$,

$$\begin{aligned}
0 &= 2\pi i p\Gamma_1 a_1(\overline{\Gamma})\frac{\partial \hat{f}_1}{\partial x_1}\Big|_{\overline{\Gamma}} + 2\pi i q\Gamma_2 a_2(\overline{\Gamma})\frac{\partial \hat{f}_1}{\partial x_2}\Big|_{\overline{\Gamma}} + 2\pi i e^{2\pi i t}\frac{\partial \hat{f}_1}{\partial x_3}\Big|_{\overline{\Gamma}} \\
&= \frac{\partial \Gamma_1}{\partial t}\frac{\partial \hat{f}_1}{\partial x_1}\Big|_{\overline{\Gamma}} + \frac{\partial \Gamma_2}{\partial t}\frac{\partial \hat{f}_1}{\partial x_2}\Big|_{\overline{\Gamma}} + \frac{\mathrm{d}}{\mathrm{d}t}(e^{2\pi i t})\frac{\partial \hat{f}_1}{\partial x_3}\Big|_{\overline{\Gamma}} \\
&= \frac{\partial}{\partial t}(\hat{f}_1 \circ \overline{\Gamma}).
\end{aligned}$$

The last line (note that the same holds for $\hat{f}_2$) implies that $\hat{f}_1 \circ \overline{\Gamma}$ is constant whit respect to t. Then,

$$\begin{aligned}
\hat{f}_1 \circ \overline{\Gamma}(x_1, x_2, 1) &= \hat{f}_1 \circ \overline{\Gamma}(x_1, x_2, 0), \\
\hat{f}_1(h(x_1, x_2), 1) &= \hat{f}_1(x_1, x_2, 1).
\end{aligned}$$

The previous is true but it is useless in somehow, unless $\hat{f}_1$ and $\hat{f}_2$ be formal series in two variables when $x_3 = 1$. This happens for example if they are *formal along* x_3; series of this kind but defined along submanifolds are used in [17] pag. 456. They also appear naturally in dimension 2 as a result of blow-ups of formal series at the origin, as can be seen in [30] pag. 487. In that setting the series can be defined in a neighborhood U of the divisor (projective line) and is said to be a *germ along U of a transversally formal holomorphic function*.

3.3.1 From formal to holomorphic first integral

Now we are in conditions to prove our first main result:

Theorem C. *Let $\mathcal{F}(\mathcal{X})$ be a germ of holomorphic foliation with $\mathcal{X} \in \mathrm{Gen}\,(\mathfrak{X}(\mathbb{C}^3, 0))$. If $\mathcal{F}(\mathcal{X})$ has a formal first integral then it also has a holomorphic one.*

Proof of Theorem C. Let $\hat{F} = (\hat{f}_1, \hat{f}_2)$ be a formal first integral of $\mathcal{F}(\mathcal{X})$. By definition of formal first integral $\mathrm{d}\hat{f}_1 \wedge \mathrm{d}\hat{f}_2 \neq 0$. By Proposition 3.2.2, the vector field $\mathcal{X}$ can be written the form (3.2):

$$\mathcal{X}(x) = mx_1(1 + a_1(x))\frac{\partial}{\partial x_1} + nx_2(1 + a_2(x))\frac{\partial}{\partial x_2} - kx_3(1 + a_3(x))\frac{\partial}{\partial x_3},$$

were $m, n, k \in \mathbb{Z}^+$ and $a_1, a_2, a_3 \in \mathcal{M}_3$. In particular $\mathcal{X}$ satisfies *condition* ($\star$). We just need the periodicity of the holonomy respect to the x_3 axis to satisfy the conditions of Theorem 3.1.6 and conclude the demonstration. For this we use the notation and result (Proposition 3.3.1) of the previous section. We want to show that the map $\tilde{h}$, which is a diffeomorphism leaving invariant $\Phi^*\hat{f}_1$ and $\Phi^*\hat{f}_2$, is periodic and then h the holonomy map is periodic.

But first, we have to guarantee that $\Phi^*\hat{f}_1$ and $\Phi^*\hat{f}_2$ are still generically transverse because in general $\mathrm{d}\hat{f}_1 \wedge \mathrm{d}\hat{f}_2 \not\equiv 0$ does not imply $\mathrm{d}(\hat{f}_1(x_1, x_2, 1)) \wedge \mathrm{d}(\hat{f}_2(x_1, x_2, 1)) \not\equiv 0$. In order to proof this, suppose that $(\mathrm{d}\hat{f}_1 \wedge \mathrm{d}\hat{f}_2)_{x_3=1} \equiv 0$ and observe that

$$\begin{aligned}\mathrm{d}(\Phi^*\hat{f}_1) \wedge \mathrm{d}(\Phi^*\hat{f}_2) &= (\mathrm{d}\hat{f}_{1\,\Phi^{-1}(\cdot)}\mathrm{d}\Phi^{-1}_{(\cdot)}) \wedge (\mathrm{d}\hat{f}_{2\,\Phi^{-1}(\cdot)}\mathrm{d}\Phi^{-1}_{(\cdot)}) \\ &= (\mathrm{d}\hat{f}_1 \wedge \mathrm{d}\hat{f}_2)_{\Phi^{-1}(\cdot)}|\mathrm{d}\Phi^{-1}_{(\cdot)}|\end{aligned}$$

thus in $(u_1, u_2, 0)$, $(\mathrm{d}(\Phi^*\hat{f}_1) \wedge \mathrm{d}(\Phi^*\hat{f}_2))_{(u_1,u_2,0)} \equiv 0$ which implies $(\mathrm{d}(\Phi^*\hat{f}_1) \wedge \mathrm{d}(\Phi^*\hat{f}_2))_{\Phi(U)} \equiv 0$ and then $(\mathrm{d}\hat{f}_1 \wedge \mathrm{d}\hat{f}_2)_U \equiv 0$, contradiction. We can now use the previous sections and Chapter 2.

With this in mind, by Proposition 2.2.8, we have that $\tilde{h}$ is periodic because it preserves $\Phi^*\hat{f}_1$, $\Phi^*\hat{f}_2$ and it is generated by one germ of diffeomorphism, therefore after a change of coordinates $\mathrm{Hol}(\mathcal{F}(\mathcal{X}), S, \Sigma)$ is periodic and Theorem 3.1.6 implies that $\mathcal{F}(\mathcal{X})$ has a holomorphic first integral. ✓

As for arbitrary dimension we have:

Theorem 3.3.4. *Let $\mathcal{F}(\mathcal{X})$ be a germ of holomorphic foliation with $\mathcal{X} \in \mathrm{Gen}\,(\mathfrak{X}(\mathbb{C}^n, 0))$ satisfying condition ($\star$). If $\mathcal{F}(\mathcal{X})$ has a formal first integral, then it also has a holomorphic one.*

Proof of Theorem 3.3.4. The proof goes on as the previous one but instead of Theorem 3.1.6 in the last part, we use the following theorem:

Theorem 3.3.5 (Theorem 5 in [39]). *Let $\mathcal{F}$ be a singular foliation on $(\mathbb{C}^n, 0)$ possessing n invariant pairwise transverse hyperplanes and denote by $\lambda_1, \ldots, \lambda_n$ its eigenvalues. Suppose also that $\lambda_n \in \mathbb{R}_+$ while $\lambda_1, \ldots, \lambda_{n-1}$ are all negative reals. Denote by h_n the local holonomy map associated to the axis x_n (corresponding to the eigenvalue λ_n) and suppose that h_n has isolated fixed points (in the sense of Theorem 1.4.1) and that it has finite orbits. Then $\mathcal{F}$ admits a holomorphic first integral.*

By Proposition 3.2.2, the vector field $\mathcal{X}$ can be written in the form (3.2):

$$\mathcal{X}(x) = k_1 x_1 (1 + a_1(x)) \frac{\partial}{\partial x_1} + \cdots + k_n x_n (1 + a_n(x)) \frac{\partial}{\partial x_n},$$

where $k_1, \ldots k_n \in \mathbb{Z}$ and $a_1, \ldots, a_n \in \mathcal{M}_n$. This vector field, by hypothesis, satisfies condition $(\star)$. Therefore, $\mathcal{X}$ is on the conditions of Theorem 3.3.5. The same method in the proof of Theorem C shows that the holonomy group, in this case associated to x_n, is periodic. Thus h_n has isolated fixed points (in the sense of Theorem 1.4.1) and has finite orbits. We can now apply Theorem 3.3.5. ✓

The unsuccessful part of this chapter is that unlike Mattei and Moussu [30] we did not manage to establish a relationship between the formal first integral and the holomorphic one. In dimension one the latter is the composition of a formal series with the former one. It is possible just a matter of computation but perhaps there is something deeper.

Chapter 4

Vector fields and Darboux's Theorem

In this chapter we suppose the existence of analytic hypersurfaces invariant by a vector field and we try to show the existence of an holomorphic first integral. Its existence would imply that all the leaves are closed off the singularity. This is a Darboux's like proceeding and this made us try to use Darboux's Theorem (Theorem 4.1.2) in our framework, however only in a particular case we mange to use it. In our main result we use Chapter 1.

4.1 Preliminaries

Let be $\mathcal{F}$ a foliation by curves in $\mathbb{C}P(n)$ and L a leaf of $\mathcal{F}$.

Definition 4.1.1. We say that L *is algebraic* if the closure $\overline{L}$ of L in $\mathbb{C}P(n)$, is an algebraic subset of dimension 1, i.e., an algebraic curve. In this case, we also say that $\overline{L}$ is *an algebraic solution of* $\mathcal{F}$.

Let be $\mathcal{F}$ a foliation in $\mathbb{C}P(n)$, whose singularities are isolated. Then, a leaf L of $\mathcal{F}$ is an algebraic solution, if and only if, $\overline{L}$ is obtained from L by the adjunction of the singularities of $\mathcal{F}$ to which L is adherent (see [32] pag. 103).

Theorem 4.1.2 (Darboux's Theorem [19, 26])**.** *Let $\mathcal{F}$ be a foliation in $\mathbb{C}P(2)$ having infinitely many algebraic solutions. Then $\mathcal{F}$ admits a rational first integral.*

4.2 Vector fields with infinitely many invariant hypersurfaces

4.2.1 Homogeneous case

Definition 4.2.1. Let $\mathcal{X} \in \mathcal{X}(\mathbb{C}^3, 0)$. We say that $\mathcal{X}$ is *homogeneous of degree* ν if $\mathcal{X}(x) = a_\nu(x)\frac{\partial}{\partial x_1} + b_\nu(x)\frac{\partial}{\partial x_2} + c_\nu(x)\frac{\partial}{\partial x_3}$ where a_ν, b_ν and c_ν are homogeneous polynomials of same degree ν and without common factors.

Note that, if $\mathcal{X}$ is homogeneous of degree ν, then $\mathcal{X}(\lambda x) = \lambda^\nu \mathcal{X}(x)$ for every $\lambda \in \mathbb{C}^*$. Intuitively this means that along the line λx the vector field $\mathcal{X}$ points in the same direction allowing us to define a vector field $\tilde{\mathcal{X}}$ in the projective plane $\mathbb{C}P(2)$ as follows.

Remember that the usual differential structure of $\mathbb{C}P(2)$ is given by the atlas $\{(U_i, \varphi_i)\}_{i=1}^3$ where $U_i = \{[x_1; x_2; x_3] \in \mathbb{C}P(2) \,|\, x_i \neq 0\}$ and

$$\begin{aligned}
\varphi_1([x_1; x_2; x_3]) &= \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right) =: (x, y),\\
\varphi_2([x_1; x_2; x_3]) &= \left(\frac{x_1}{x_2}, \frac{x_3}{x_2}\right) =: (u, v),\\
\varphi_3([x_1; x_2; x_3]) &= \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right) =: (s, r).
\end{aligned}$$

Consider the projection

$$\Pi : \mathbb{C}^3 \to \mathbb{C}P(2) : (x_1, x_2, x_3) \to [(x_1; x_2; x_3)] = \{\lambda(x_1, x_2, x_3) \,|\, \lambda \in \mathbb{C}^*\}$$

that in the first chart is written as $\Pi_1(x_1, x_2, x_3) = \varphi_1 \circ \Pi(x_1, x_2, x_3) = (x, y)$. Putting all of this together, $\tilde{\mathcal{X}}$ in the first chart is

$$\begin{aligned}
\tilde{\mathcal{X}}_1(x, y) = \Pi_1^* \mathcal{X}(x, y)\big|_{x_1=1} &= \left\{ \mathrm{d}\Pi_{1\,\Pi_1^{-1}(x,y)} \mathcal{X}\big(\Pi_1^{-1}(x, y)\big) \right\}_{x_1=1},\\
&= \begin{bmatrix} -x & 1 & 0 \\ -y & 0 & 1 \end{bmatrix} \mathcal{X}(1, x, y),\\
\tilde{\mathcal{X}}_1(x, y) = \big(b_\nu(1, x, y) - x a_\nu(1, x, y)\big)\frac{\partial}{\partial x} &+ \big(c_\nu(1, x, y) - y a_\nu(1, x, y)\big)\frac{\partial}{\partial y}.
\end{aligned}$$

We proceed in a similar way for the other two charts.

Definition 4.2.2. We say that two hypersurfaces S_1 and S_2 are *first jet different* if S_1 and S_2 are given by the zero set of irreducibles $g_1, g_2 \in \mathcal{M}_3$ whose first jets are different.

Theorem 4.2.3. *Let $\mathcal{X}$ be a germ of homogeneous vector field in $0 \in \mathbb{C}^3$. Suppose that $\mathcal{X}$ leaves invariant infinitely many first jet different hypersurfaces passing through* 0. *Then, there exists a rational map $f : \mathbb{C}P(2) \to \mathbb{C}P(1)$ that is $\mathcal{F}(\mathcal{X})$*-invariant *(i. e., $\mathcal{X}(f) \equiv 0$). This map is also called a weak first integral for $\mathcal{F}(\mathcal{X})$.*

Proof. The idea of the proof is to use the homogeneity of $\mathcal{X}$ to define a vector field $\tilde{\mathcal{X}}$ in the complex projective space $\mathbb{C}P(2)$ and show that the foliation $\mathcal{F}(\tilde{\mathcal{X}})$ has infinitely many algebraic leaves. Then we use Darboux's Theorem 4.1.2 to obtain a first integral for $\mathcal{F}(\tilde{\mathcal{X}})$. This first integral is a weak first integral for $\mathcal{F}(\mathcal{X})$.

Suppose that $S := \{g = 0\}$, for an irreducible $g \in \mathcal{M}_3$, is an $\mathcal{X}$-invariant hypersurface. This is equivalent to saying that g *divides* $\mathcal{X}(g)$, denoted as $g \,\big|\, \mathcal{X}(g)$. To see this if $x_0 \in S$ and $\phi(T)$ is the integral curve of the vector field $\mathcal{X}$ with $\phi(0) = x_0$ defined in a neighborhood of $0 \in \mathbb{C}$ then,

$$\begin{cases} g(\phi) = 0 \\ \mathcal{X}(\phi(T)) = \phi'(T), \end{cases}$$

together they imply that $\mathcal{X}(g)(\phi) = 0$. Therefore $\mathcal{X}(g)(\cdot)$ is a holomorphic function which is zero when restricted to S. Therefore, It can be written as

$$\mathcal{X}(g)(\cdot) = g(\cdot)h(\cdot), \tag{4.1}$$

where $h \in \mathcal{O}_3$.

Remember that if κ is the order of g then $g = g_\kappa + g_{\kappa+1} + \cdots$ where g_m is a homogeneous polynomial of degree m. Thus, by the linearity of $\mathcal{X}$ as a derivation operator, we have that

$$\mathcal{X}(g) = \mathcal{X}(g_\kappa) + \mathcal{X}(g_{\kappa+1}) + \cdots,$$

is also a sum of homogeneous polynomials. If $\mathcal{X}(g_\kappa) \equiv 0$, we have that g_κ is a weak first integral for $\mathcal{F}(\mathcal{X})$ and we are done. Suppose that $\mathcal{X}(g_\kappa) \not\equiv 0$, then $\mathcal{X}(g_\kappa)$ is homogeneous of order $\nu + \kappa - 1$, $\mathcal{X}(g_{\kappa+1})$ is homogeneous of order $\nu + \kappa$, etc., where ν is the order of $\mathcal{X}$ as before. Obviously, h in (4.1) can also be written as a sum of homogeneous polynomials and the degree of the first non zero of them (the order of h) necessarily is $\nu - 1$ by (4.1). Using this, (4.1) can be rewritten in the following way;

$$\begin{aligned} \mathcal{X}(g_\kappa) + \mathcal{X}(g_{\kappa+1}) + \cdots &= (g_\kappa + g_{\kappa+1} + \cdots)(h_{\nu-1} + h_\nu + \cdots) \\ &= g_\kappa h_{\nu-1} + \ldots. \end{aligned}$$

This implies, by comparing terms with same degree in both sides, that

$$\mathcal{X}(g_\kappa) = g_\kappa h_{\nu-1},$$

in other words $g_\kappa \,|\, \mathcal{X}(g_\kappa)$. Thereby $S_\kappa := \{g_\kappa = 0\}$ is an $\mathcal{X}$-invariant algebraic hypersurface.

Next, as we mention previously, the homogeneity of $\mathcal{X}$ can be used to define a vector field $\tilde{\mathcal{X}}$ in $\mathbb{C}P(2)$. The same can be done for g_κ, defining a function $\tilde{g}_\kappa$ in $\mathbb{C}P(2)$ as follows:

$$\begin{aligned}\tilde{g}_\kappa(x,y) &= \Pi_1^* g_\kappa|_{x_1=1}\\ &= g_\kappa(\Pi_1^{-1}(x,y))|_{x_1=1}\\ &= g_\kappa(1,x,y).\end{aligned}$$

We proceed analogously in the other two charts. Let us see that $\tilde{g}_\kappa \,|\, \tilde{\mathcal{X}}(\tilde{g}_\kappa)$, first we use the equality $g_\kappa(x_1,x_2,x_3) = x_1^\kappa g_\kappa(1, x_2/x_1, x_3/x_1) = x_1^\kappa g_\kappa(1,x,y)$ in order to calculate $\nabla g_\kappa(1,x,y)$ in terms of x_1, x_2 and x_3 as follows:

$$\begin{aligned}\frac{\partial g_\kappa}{\partial x_1} &= \kappa x_1^{\kappa-1} g_\kappa + x_1^\kappa\Big(\frac{\partial g_\kappa}{\partial x}\frac{dx}{dx_1} + \frac{\partial g_\kappa}{\partial y}\frac{dy}{dx_1}\Big)\\ &= \kappa x_1^{\kappa-1} g_\kappa + x_1^{\kappa-1}\Big(-x\frac{\partial g_\kappa}{\partial x} - y\frac{\partial g_\kappa}{\partial y}\Big),\\ \frac{\partial g_\kappa}{\partial x_2} &= x_1^\kappa\Big(\frac{\partial g_\kappa}{\partial x}\frac{dx}{dx_2} + \frac{\partial g_\kappa}{\partial y}\frac{dy}{dx_2}\Big)\\ &= x_1^{\kappa-1}\frac{\partial g_\kappa}{\partial x},\\ \frac{\partial g_\kappa}{\partial x_3} &= x_1^\kappa\Big(\frac{\partial g_\kappa}{\partial x}\frac{dx}{dx_3} + \frac{\partial g_\kappa}{\partial y}\frac{dy}{dx_3}\Big)\\ &= x_1^{\kappa-1}\frac{\partial g_\kappa}{\partial y},\end{aligned}$$

if we set $x_1 = 1$, they become

$$\frac{\partial g_\kappa}{\partial x_1} = \kappa\tilde{g}_\kappa + \Big(-x\frac{\partial g_\kappa}{\partial x} - y\frac{\partial g_\kappa}{\partial y}\Big), \quad \frac{\partial g_\kappa}{\partial x_2} = \frac{\partial g_\kappa}{\partial x}, \quad \frac{\partial g_\kappa}{\partial x_3} = \frac{\partial g_\kappa}{\partial y}.$$

Second, keep in mind that $\mathcal{X}(g_\kappa) = a_\nu\frac{\partial g_\kappa}{\partial x_1} + b_\nu\frac{\partial g_\kappa}{\partial x_2} + c_\nu\frac{\partial g_\kappa}{\partial x_3} = g_\kappa h_{\nu-1}$. In particular, for $x_1 = 1$. Thus, we have that $\tilde{g}_\kappa \,|\, \tilde{\mathcal{X}}(\tilde{g}_\kappa)$ is consequence of the previous considerations, as shown in the following calculations.

$$\begin{aligned}
\tilde{\mathcal{X}}(\tilde{g}_\kappa) &= \begin{bmatrix} -x & 1 & 0 \\ -y & 0 & 1 \end{bmatrix} \mathcal{X}(1,x,y) \cdot \nabla g_\kappa(1,x,y) \\
&= \big(-xa_\nu + b_\nu \big) \frac{\partial g_\kappa}{\partial x} + \big(-ya_\nu + c_\nu \big) \frac{\partial g_\kappa}{\partial y} \\
&= a_\nu \Big(-x \frac{\partial g_\kappa}{\partial x} - y \frac{\partial g_\kappa}{\partial y} \Big) + b_\nu \frac{\partial g_\kappa}{\partial x} + c_\nu \frac{\partial g_\kappa}{\partial y} \\
&= -\kappa a_\nu \tilde{g}_\kappa + \Big(a_\nu \frac{\partial g_\kappa}{\partial x_1} + b_\nu \frac{\partial g_\kappa}{\partial x_2} + c_\nu \frac{\partial g_\kappa}{\partial x_3} \Big) \\
&= -\kappa a_\nu \tilde{g}_\kappa + \tilde{g}_\kappa h_{\nu-1} \\
\tilde{\mathcal{X}}(\tilde{g}_\kappa) &= \tilde{g}_\kappa \big(-\kappa a_\nu + h_{\nu-1} \big),
\end{aligned}$$

where all functions are evaluated in $(1,x,y)$.

Thus $\{\tilde{g}_\kappa = 0\}$ is an $\tilde{\mathcal{X}}$-invariant algebraic curve. The same argument is valid with each of the infinitely many $\mathcal{X}$-invariant hypersurfaces. The fact that they are first jet different implies that there exists infinitely many $\tilde{\mathcal{X}}$-invariant algebraic curves. Then by Darboux's Theorem, $\tilde{\mathcal{X}}$ posseses a rational first integral $f : \mathbb{C}P(2) \to \mathbb{C}P(1)$. It only remains to see that f is $\mathcal{F}(\mathcal{X})$-invariant. This is equivalent to verify that $\mathcal{X}(f) \equiv 0$, which is the next and final step in the proof.

We can think f as a rational function in $\mathbb{C}^3$, constant along the directions $f(\lambda x) = f(x)$ in other words, homogeneous of order 0. So, as we did before with g_κ, f can be written as $f(x_1,x_2,x_3) = f(1, x_2/x_1, x_3/x_1) = f(1,x,y)$. Taking derivatives

$$\frac{\partial f}{\partial x_1} = -\frac{x}{x_1}\frac{\partial f}{\partial x} - \frac{y}{x_1}\frac{\partial f}{\partial y}, \qquad \frac{\partial f}{\partial x_2} = \frac{1}{x_1}\frac{\partial f}{\partial x}, \qquad \frac{\partial f}{\partial x_3} = \frac{1}{x_1}\frac{\partial f}{\partial y}$$

and using that $\tilde{\mathcal{X}}_1(f) = (-xa_\nu + b_\nu)\frac{\partial f}{\partial x} + (-ya_\nu + c_\nu)\frac{\partial f}{\partial y} \equiv 0$, where all the functions are evaluated in $(1,x,y)$, we can calculate

$$\begin{aligned}
\mathcal{X}(f) &= a_\nu(x_1,x_2,x_3)\frac{\partial f}{\partial x_1} + b_\nu(x_1,x_2,x_3)\frac{\partial f}{\partial x_2} + c_\nu(x_1,x_2,x_3)\frac{\partial f}{\partial x_3} \\
&= x_1^\nu \Big(a_\nu(1,x,y)\frac{\partial f}{\partial x_1} + b_\nu(1,x,y)\frac{\partial f}{\partial x_2} + c_\nu(1,x,y)\frac{\partial f}{\partial x_3} \Big) \\
&= x_1^{\nu-1} \Big(a_\nu(1,x,y)\big(-x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} \big) + b_\nu(1,x,y)\frac{\partial f}{\partial x} \\
&\qquad\qquad + c_\nu(1,x,y)\frac{\partial f}{\partial y} \Big) \\
&= x_1^{\nu-1} \Big((-xa_\nu + b_\nu)\frac{\partial f}{\partial x} + (-ya_\nu + c_\nu)\frac{\partial f}{\partial y} \Big) \equiv 0 \qquad \checkmark
\end{aligned}$$

In order to conclude the homogeneous case is important to note that the previous method does not produce two weak first integrals transversally independent, because both of them are first integrals of $\tilde{\mathcal{X}}$ then in $\mathbb{C}^3$ they have the same level sets.

4.2.2 Generalities on blow-ups.

Suppose that $\mathcal{X}(x) = a(x_1,x_2,x_3)\frac{\partial}{\partial x_1} + b(x_1,x_2,x_3)\frac{\partial}{\partial x_2} + c(x_1,x_2,x_3)\frac{\partial}{\partial x_3}$, where $a,b,c \in \mathcal{O}_3$ are given by

$$a(x) = \sum_{|I|\geq p_1} a_I x^I,\ b(x) = \sum_{|J|\geq p_2} b_J x^J \text{ and } c(x) = \sum_{|K|\geq p_3} c_K x^K.$$

If φ_1 is the first chart of the punctual blow-up at $0 \in \mathbb{C}^3$. We denote $E \circ \varphi_1^{-1}(z_1,z_2,z_3) = (z_1, z_1z_2, z_1z_3)$ simply by $E_1(z)$, $a(E_1(z))$ by $a(z)$ (in the same way $b(z)$ and $c(z)$) and the p_1-jet of $a(z)$ by $a_{p_1}(z)$ (in the same way $b_{p_2}(z)$ and $c_{p_3}(z)$). Observe that in this chart the divisor, $D := E^{-1}(0) = \mathbb{C}P(2)$, is given by $\{z_1 = 0\}$.

Using this notation we calculate $\tilde{\mathcal{X}}(z) = (\mathrm{d}E_1^{-1})_{E_1(z)}\mathcal{X}\big(E_1(z)\big)$,

$$\mathrm{d}E_1 = \begin{bmatrix} 1 & 0 & 0 \\ z_2 & z_1 & 0 \\ z_3 & 0 & z_1 \end{bmatrix}, \quad \mathrm{d}E_1^{-1} = \frac{1}{z_1^2}\begin{bmatrix} z_1^2 & 0 & 0 \\ -z_1z_2 & z_1 & 0 \\ -z_1z_3 & 0 & z_1 \end{bmatrix}.$$

Thus

$$\tilde{\mathcal{X}}(z) = \frac{1}{z_1^2}\begin{bmatrix} z_1^2 & 0 & 0 \\ -z_1z_2 & z_1 & 0 \\ -z_1z_3 & 0 & z_1 \end{bmatrix} \cdot \begin{bmatrix} a(z) \\ b(z) \\ c(z) \end{bmatrix},$$

$$\begin{aligned}
\tilde{\mathcal{X}}(z) &= a(z)\frac{\partial}{\partial z_1} + \frac{1}{z_1}(-z_2a(z) + b(z))\frac{\partial}{\partial z_2} + \frac{1}{z_1}(-z_3a(z) + c(z))\frac{\partial}{\partial z_3} \\
&= \big(z_1^\nu a_\nu(1,z_2,z_3) + z_1^{\nu+1}(\dots)\big)\frac{\partial}{\partial z_1} + \\
&\quad \big(-z_2z_1^{\nu-1}a_\nu(1,z_2,z_3) + z_1^{\nu-1}b_\nu(1,z_2,z_3) + z_1^\nu(\dots)\big)\frac{\partial}{\partial z_2} + \\
&\quad \big(-z_3z_1^{\nu-1}a_\nu(1,z_2,z_3) + z_1^{\nu-1}c_\nu(1,z_2,z_3) + z_1^\nu(\dots)\big)\frac{\partial}{\partial z_3},
\end{aligned}$$

which gives

$$\begin{aligned}\tilde{\mathcal{X}}(z) = {} & z_1^{\nu} a_\nu(1, z_2, z_3)\frac{\partial}{\partial z_1} + \\ & z_1^{\nu-1}\big(- z_2 a_\nu(1, z_2, z_3) + b_\nu(1, z_2, z_3)\big)\frac{\partial}{\partial z_2} + \\ & z_1^{\nu-1}\big(- z_3 z_1^{\nu-1} a_\nu(1, z_2, z_3) + c_\nu(1, z_2, z_3)\big)\frac{\partial}{\partial z_3} + z_1^{\nu}(\dots),\end{aligned}$$

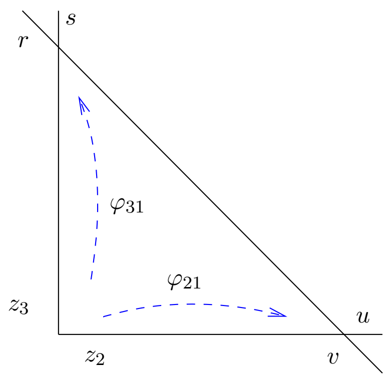
where $\nu = \min\{p_1, p_2, p_3\}$. Then, supposing that $x_2 a_\nu(x) \neq x_1 b_\nu(x)$ or $x_3 a_\nu(x) \neq x_1 c_\nu(x)$ (i.e., 0 is a not dicritical singularity, see [16]). In that case we can define in the first chart of D

$$\tilde{\mathcal{X}}_D(z_2, z_3) := \big((z_1^{\nu-1})^{-1}\tilde{\mathcal{X}}(z)\big)_{z_1=0}$$

and, we have that

$$\begin{aligned}\tilde{\mathcal{X}}_D(z_2, z_3) = {} & \big(- z_2 a_\nu(1, z_2, z_3) + b_\nu(1, z_2, z_3)\big)\frac{\partial}{\partial z_2} + \\ & \big(- z_3 a_\nu(1, z_2, z_3) + c_\nu(1, z_2, z_3)\big)\frac{\partial}{\partial z_3}.\end{aligned} \tag{4.2}$$

In order to write $\tilde{\mathcal{X}}_D$ in the other two charts, that we will denote $\tilde{\mathcal{X}}_D(s,t)$ and $\tilde{\mathcal{X}}_D(u, v)$ for simplicity, consider the following diagram:



where,

$$\begin{aligned} \varphi_{21}(z_2, z_3) &= (u, v) \\ u &= 1/z_2 \\ v &= z_3/z_2, \end{aligned} \qquad \text{and} \qquad \begin{aligned} \varphi_{31}(z_2, z_3) &= (r, s) \\ r &= z_2/z_3 \\ s &= 1/z_3, \end{aligned}$$

hence,

$$\begin{aligned}\tilde{\mathcal{X}}_D(u,v) &= u^{\nu-1}\mathrm{d}\varphi_{21}\tilde{\mathcal{X}}_D(\varphi_{21}^{-1}(u,v)) \quad \text{and}\\ \tilde{\mathcal{X}}_D(r,s) &= s^{\nu-1}\mathrm{d}\varphi_{31}\tilde{\mathcal{X}}_D(\varphi_{31}^{-1}(r,s)),\end{aligned}$$

using that

$$\mathrm{d}\varphi_{21} = \begin{bmatrix} -u^2 & 0 \\ -uv & u \end{bmatrix} \quad \text{and} \quad \mathrm{d}\varphi_{31} = \begin{bmatrix} s & -rs \\ 0 & -s^2 \end{bmatrix},$$

we have,

$$\begin{aligned}\tilde{\mathcal{X}}_D(u,v) &= u^{\nu}\Big(-ub_{\nu}\Big(1,\frac{1}{u},\frac{v}{u}\Big) + a_{\nu}\Big(1,\frac{1}{u},\frac{v}{u}\Big)\Big)\frac{\partial}{\partial u} + \\ &\qquad u^{\nu}\Big(-vb_{\nu}\Big(1,\frac{1}{u},\frac{v}{u}\Big) + c_{\nu}\Big(1,\frac{1}{u},\frac{v}{u}\Big)\Big)\frac{\partial}{\partial v},\\ \tilde{\mathcal{X}}_D(u,v) &= \big(-ub_{\nu}(u,1,v) + a_{\nu}(u,1,v)\big)\frac{\partial}{\partial u} + \\ &\qquad \big(-vb_{\nu}(u,1,v) + c_{\nu}(u,1,v)\big)\frac{\partial}{\partial v}\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{X}}_D(r,s) &= s^{\nu}\Big(-rc_{\nu}\Big(1,\frac{r}{s},\frac{1}{s}\Big) + b_{\nu}\Big(1,\frac{r}{s},\frac{1}{s}\Big)\Big)\frac{\partial}{\partial r} + \\ &\qquad s^{\nu}\Big(-sc_{\nu}\Big(1,\frac{r}{s},\frac{1}{s}\Big) + a_{\nu}\Big(1,\frac{r}{s},\frac{1}{s}\Big)\Big)\frac{\partial}{\partial s},\\ \tilde{\mathcal{X}}_D(r,s) &= \Big(-rc_{\nu}(s,r,1) + b_{\nu}(s,r,1)\Big)\frac{\partial}{\partial r} + \\ &\qquad \Big(-sc_{\nu}(s,r,1) + a_{\nu}(s,r,1)\Big)\frac{\partial}{\partial s}.\end{aligned}$$

Observe that $\tilde{\mathcal{X}}_D(z_2,z_3)$ is a polynomial vector field of degree $\leq \nu+1$ leaving D invariant.

Lemma 4.2.4. *If a vector field $\mathcal{X} \in \mathcal{X}(\mathbb{C}^3,0)$ leaves invariant a hypersurface passing through* 0*, then its first jet $\mathcal{X}_\nu$ leaves invariant a homogeneous algebraic hypersurface passing through* 0*.*

Proof. The argument is similar to the one in the first part of the proof of Theorem 4.2.3. Let be $S = \{g=0\}$, for $g \in \mathcal{M}_3$ irreducible, a $\mathcal{X}$-invariant hypersurface. Then there exists $h \in \mathcal{M}_3$ such that $\mathcal{X}(g) = gh$. The three of them, $\mathcal{X}, g$ and h can be written as a sum of homogeneous terms:

$$\begin{aligned}\mathcal{X} &= \mathcal{X}_\nu + \mathcal{X}_{\nu+1} + \cdots,\\ g &= g_\kappa + g_{\kappa+1} + \cdots,\\ h &= h_{\nu-1} + h_\nu + \cdots.\end{aligned}$$

The equality $\mathcal{X}(g) = gh$ implies that the order of h is $\nu - 1$. By comparing both sides of

$$\mathcal{X}_\nu(g_\kappa + g_{\kappa+1} + \cdots) + \mathcal{X}_{\nu+1}(g_\kappa + \cdots) + \cdots = (g_\kappa + \cdots)(h_{\nu-1} + \cdots),$$

we get that $\mathcal{X}_\nu(g_\kappa) = g_\kappa h_{\nu-1}$. Therefore, $g_k = 0$ is a $\mathcal{X}_\nu$-invariant a homogeneous algebraic hypersurface passing through 0 ✓

In what follows, we denote by $\tilde{\mathcal{X}}$ the pull-back of the vector field $\mathcal{X}$ by the blow-up $E : \tilde{\mathbb{C}}^3 \to \mathbb{C}^3$ at the origin and by $\tilde{\mathcal{X}}_D$ its restriction to the divisor. We have,

Proposition 4.2.5. *Let $\mathcal{F}(\mathcal{X})$ be a germ of holomorphic foliation, where $\mathcal{X} \in \mathfrak{X}(\mathbb{C}^3, 0)$, having an isolated non dicritical singularity at $0 \in \mathbb{C}^3$. If there exist infinitely many $\mathcal{X}$-invariant analytic hypersurfaces passing through $0 \in \mathbb{C}^3$ and in general position then $\tilde{\mathcal{X}}_D$ possesses a rational first integral.*

Proof. The previous lemma, together with Theorem 4.2.3 implies that $\mathcal{X}_\nu$ possesses a weak first integral. Now, $\tilde{\mathcal{X}}$ in the first chart of the blow-up can be written in the form (4.2).

Now, as we mention before, there exists $f : \mathbb{C}P(2) \to \mathbb{C}P(1)$ such that $\mathcal{X}_\nu(f) \equiv 0$, i.e., $a_\nu(x)\frac{\partial f}{\partial x_1} + b_\nu(x)\frac{\partial f}{\partial x_2} + c_\nu(x)\frac{\partial f}{\partial x_3} \equiv 0$. We then proceed as in the end of the proof of Theorem 4.2.3,

$$\begin{aligned}
\tilde{\mathcal{X}}_D(f) &= \big(-z_2a_\nu(z) + b_\nu(z)\big)\frac{\partial f}{\partial z_2} + \big(-z_3a_\nu(z) + c_\nu(z)\big)\frac{\partial f}{\partial z_3}\\
&= -z_2a_\nu(z)\frac{\partial f}{\partial z_2} - z_3a_\nu(z)\frac{\partial f}{\partial z_3} + b_\nu(z)\frac{\partial f}{\partial z_2} + c_\nu(z)\frac{\partial f}{\partial z_3}\\
&= x_1\Big(a_\nu(z)\frac{\partial f}{\partial x_1} + b_\nu(z)\frac{\partial f}{\partial x_2} + c_\nu(z)\frac{\partial f}{\partial x_3}\Big)\\
\tilde{\mathcal{X}}_D(f) &\equiv 0.
\end{aligned}$$

In the part above we use the following notation $a_\nu(x) = a_\nu(x_1, x_2, x_3) = z_1^\nu a_\nu(1, z_2, z_3) = z_1^\nu a_\nu(z)$, and that $f(x_1, x_2, x_3) = f(1, z_2, z_3)$ which implies by derivation,

$$\frac{\partial f}{\partial x_1} = -\frac{z_2}{x_1}\frac{\partial f}{\partial z_2} - \frac{z_3}{x_1}\frac{\partial f}{\partial z_3}, \quad \frac{\partial f}{\partial x_2} = \frac{1}{x_1}\frac{\partial f}{\partial z_2}, \quad \frac{\partial f}{\partial x_3} = \frac{1}{x_1}\frac{\partial f}{\partial z_3}.$$

✓

4.2.3 General case

The condition $(\star)$ introduced in Definition 3.1.4 can be understood in the following way: Let $\{\lambda_i\}_1^n \subset \mathbb{C}^*$ be the eigenvalues of $(\mathrm{d}\mathcal{X})_0$, if $\mathcal{X}$ satisfies condition $(\star)$, then it is possible to choose a complex vector $v \in \mathbb{C}^*$ such that, for one of the eigenvalues λ_i, $\mathrm{Re}\left(\frac{\lambda_i}{v}\right)$ has different sign. Then this eigenvalue can be separated (see Fig. 4.1).

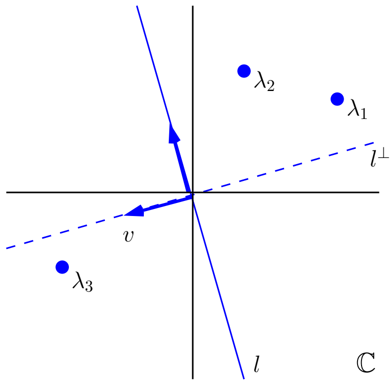


Figure 4.1: Condition $(\star)$ with l the line separating λ_3.

Theorem B. *Let $\mathcal{F}(\mathcal{X})$ be the germ of a holomorphic foliation with $\mathcal{X} \in$ $\mathrm{Gen}\,(\mathfrak{X}(\mathbb{C}^3,0))$ satisfying condition $(\star)$. Then $\mathcal{F}(\mathcal{X})$ has a holomorphic first integral if, and only if, the leaves of $\mathcal{F}(\mathcal{X})$ are closed off the singularity and there exist non-enumerable many $\mathcal{X}$-invariant analytic hypersurfaces passing through 0 in general position.*

Proof. $(\Longrightarrow)$ This implication is obvious.
$(\Longleftarrow)$ To see this, consider $\mathcal{X} \in \mathrm{Gen}(\mathfrak{X}(\mathbb{C}^n,0))$, and according to Definition (3.1.1) after a change of coordinates, it can be written in the form

$$\mathcal{X}(x) = \lambda_1 x_1\big(1+a_1(x)\big)\frac{\partial}{\partial x_1}+\lambda_2 x_2\big(1+a_2(x)\big)\frac{\partial}{\partial x_2}+\lambda_3 x_3\big(1+a_3(x)\big)\frac{\partial}{\partial x_3}. \quad (4.3)$$

This vector field is in the conditions of Theorem 3.1.6, just remaining to prove that the holonomy respect to the distinguished axis of $\mathcal{X}$ (denoted $S_{\mathcal{X}}$ as before) is periodic. Remember that $S_{\mathcal{X}}$ is the invariant manifold associated to the eigenvalue that can be separated, assumed to be λ_3. We can calculate $\mathrm{Hol}(\mathcal{F}(\mathcal{X}), S_{\mathcal{X}}, \Sigma)$ taking a small transversal section Σ to $S_{\mathcal{X}}$, diffeomorphic to a ball in $\mathbb{C}^2$, at some point z_0 close to the origin.

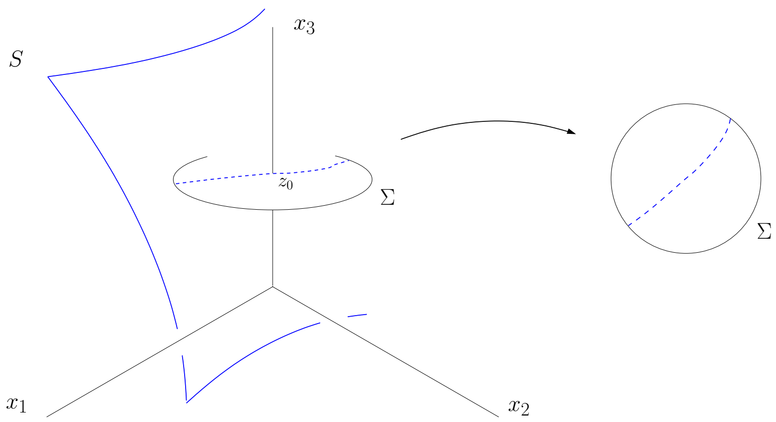


Figure 4.2: Holonomy of $S_{\mathcal{X}}$

Observe first that if z_0 is close enough to the origin, the saturate of Σ together with the hyperplane $\{x_3 = 0\}$ contains a neighborhood of the origin (see Proposition 1 [40]). This means that every $\mathcal{X}$-invariant hypersurfaces different from $\{x_3 = 0\}$ necessarily cuts Σ, since it contains 0 and then cuts the saturate of Σ, containing the leaves coming through Σ by its $\mathcal{X}$-invariance. Furthermore, we can guarantee that not only cut Σ

Assertion 4.2.1. Non-enumerable many $\mathcal{X}$-invariant analytic hypersurfaces contain the x_3-axis.

In order to see this, take $S = \{g = 0\}$ an $\mathcal{X}$-invariant hypersurface given by the zero set of $g(x) = \Sigma_{|I|\geq\nu} b_I x^I$. Then $\mathcal{X}(g)(x) = g(x)h(x)$, where $h(x) = \Sigma_I c_I x^I$. Using (4.3), we write this equation in terms of series, getting

$$\Sigma_{|I|\geq\nu}\big[\lambda_1 i\big(1 + a_1(x)\big) + \lambda_2 j\big(1 + a_2(x)\big) + \lambda_3 k\big(1 + a_3(x)\big)\big] b_I x^I = \Big(\Sigma_{|I|\geq\nu} b_I x^I\Big)\Big(\Sigma_I c_I x^I\Big).$$

Making $x_2 = x_3 = 0$, we get

$$\Sigma_{i\geq i_0} \lambda_1 i\big(1 + a_1(x_1, 0, 0)\big) b_{i,0,0} x_1^i = \Big(\Sigma_{i\geq i_0} b_{i,0,0} x_1^i\Big)\Big(\Sigma_i c_{i,0,0} x_1^i\Big).$$

We proceed in a similar way for $x_1 = x_2 = 0$ and $x_1 = x_3 = 0$. Comparing the first terms in both sides,

$$\begin{aligned} \lambda_1 i_0 b_{i_0,0,0} &= b_{i_0,0,0} c_{0,0,0}, \\ \lambda_2 j_0 b_{0,j_0,0} &= b_{0,j_0,0} c_{0,0,0}, \\ \lambda_3 k_0 b_{0,0,k_0} &= b_{0,0,k_0} c_{0,0,0}. \end{aligned}$$

Remember that our intention is to show that $g(0,0,x_3) = \Sigma_{k \geq k_0} b_{0,0,k} x_3^k \equiv 0$ (because this implies that the x_3 axis belongs to S). For this it is enough to show that $b_{0,0,k_0}$=0 because in principle it is the first not zero term. If $c_{0,0,0} = 0$, then $b_{0,0,k_0} = 0$ given that $k_0 > 0$ because $g(0) = 0$ (by hypothesis $0 \in S$), then we are done. If $c_0 \neq 0$, suppose first that the three coefficients $b_{i_0,0,0}$, $b_{0,j_0,0}$ and $b_{0,0,k_0}$ are non zero. Then

$$\lambda_1 i_0 = \lambda_2 j_0 = \lambda_3 k_0,$$

dividing by the vector v as in fig. 4.1 and comparing the real parts we have

$$\mathrm{Re}\Big(\frac{\lambda_1}{v}\Big) i_0 = \mathrm{Re}\Big(\frac{\lambda_2}{v}\Big) j_0 = \mathrm{Re}\Big(\frac{\lambda_3}{v}\Big) k_0.$$

This is a contradiction because v can be chosen so that $\mathrm{Re}\big(\frac{\lambda_3}{v}\big) > 0$, implying that the other two are negative. Hence, at least one of the coefficients $b_{i_0,0,0}$, $b_{0,j_0,0}$ or $b_{0,0,k_0}$ has to be zero. The same analysis shows that $b_{i_0,0,0} \cdot b_{0,0,k_0} \neq 0$ or $b_{0,j_0,0} \cdot b_{0,0,k_0} \neq 0$ cannot happen. Thus any hypersurface not containing the axis x_1 and x_2 necessarily contains the axis x_3. This proves the assertion .

The previous assertion implies that infinitely many $\mathcal{X}$-invariant hypersurfaces cut Σ forming G-invariant analytic curves (calling G the holonomy map) as in fig. 4.2 in a such way that if we think Σ as a ball in $\mathbb{C}^2$, each of these G-invariants curves contains $0 \in \mathbb{C}^2$. Therefore G generates a finite group according to Theorem A, and this implies the existence of a holomorphic first integral for $\mathcal{F}(\tilde{\mathcal{X}})$ in some neighborhood of 0. ✓

It remains open to see is only finite many $\mathcal{X}$-invariant analytic hypersurfaces are necessary, also if it is possible to conclude something taking "formal" instead of "analytic" in Theorem B.

Chapter 5

Complete stability theorem for foliations with singularities

In this chapter we present a stability theorem (Theorem 5.2.2) for a holomorphic foliation $\mathcal{F}$ of codimension 1 on a compact, connected and complex surface M. For this, we use the following result of Jouanolou about closed leaves of holomorphic foliations:

Theorem 5.0.1 ([21, 25])**.** *Let $\mathcal{F}$ be a holomorphic foliation (possibly singular) of codimension* 1 *in a compact and connected complex manifold. Then $\mathcal{F}$ has a finite number of closed leaves unless it possesses a meromorphic first integral, in which case all the leaves are closed.*

In our result, which will be stated properly in Section 5.2, we suppose the existence of $\mathcal{F}$-invariant irreducible hypersurfaces and find conditions on them that guarantees the existence of infinitely many closed leaves. Then we use the theorem above.

In the prove of our result we also need the following well known theorem of Mattei and Moussu.

Theorem 5.0.2 (Mattei-Moussu [30])**.** *Let $\mathcal{F}$ be a germ at $0 \in \mathbb{C}^2$ of holomorphic foliation. Suppose that:*

1. $\mathrm{Sing}\,(\mathcal{F}) = \{0\}$.
2. *There are only finite many separatices S_k.*
3. *The leaves are closed off the origin.*

Then, there exist a neighborhood V of 0*, such that $\mathcal{F}|_V$ has a holomorphic first integral.*

5.1 Holonomy and virtual holonomy groups

Let $\mathcal{F}$ be a holomorphic foliation with isolated singularities on a complex surface M. Denote by $\mathrm{Sing}(\mathcal{F})$ the singular set of $\mathcal{F}$. Given a leaf L_0 of $\mathcal{F}$, choose any base point $p \in L_0 \subset M \setminus \mathrm{Sing}(\mathcal{F})$ and a transverse disc $\Sigma_p \subset M$ to $\mathcal{F}$ centered at p. The holonomy group of the leaf L_0 with respect to the disc Σ_p and to the base point p is the image of the representation $\mathrm{Hol}\colon \pi_1(L_0, p) \to \mathrm{Diff}(\Sigma_p, p)$ obtained by lifting closed paths in L_0 with base point p to paths in the leaves of $\mathcal{F}$, starting at points $z \in \Sigma_p$, by means of a transverse fibration to $\mathcal{F}$ containing the disc Σ_p ([10]). Given a point $z \in \Sigma_p$ we denote the leaf through z by L_z. Given a closed path $\gamma \in \pi_1(L_0, p)$ we denote by $\tilde{\gamma}_z$ its lift to the leaf L_z starting (the lifted path) at the point z. Then the image of the corresponding holonomy map is $h_{[\gamma]}(z) = \tilde{\gamma}_z(1)$, i.e., the final point of the lifted path $\tilde{\gamma}_z$. This defines a germ of diffeomorphism $h_{[\gamma]}\colon (\Sigma_p, p) \to (\Sigma_p, p)$ and also a group homomorphism $\mathrm{Hol}\colon \pi_1(L_0, p) \to \mathrm{Diff}(\Sigma_p, p)$. The image $\mathrm{Hol}(\mathcal{F}, L_0, \Sigma_p, p) \subset \mathrm{Diff}(\Sigma_p, p)$ of such homomorphism is called the *holonomy group* of the leaf L_0 with respect to Σ_p and p. By considering any parametrization $z\colon (\Sigma_p, p) \to (\mathbb{D}, 0)$ we may identify (in a non-canonical way) the holonomy group with a subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$. It is clear from the construction that the maps in the holonomy group preserve the leaves of the foliation. Nevertheless, this property can be shared by a larger group that may therefore contain more information about the foliation in a neighborhood of the leaf. The *virtual holonomy group* of the leaf with respect to the transverse section Σ_p and base point p is defined as ([13], [12])

$$\mathrm{Hol}^{\mathrm{virt}}(\mathcal{F}, \Sigma_p, p) = \{f \in \mathrm{Diff}(\Sigma_p, p) \big| L_z = L_{f(z)}, \forall z \in (\Sigma_p, p)\}.$$

The virtual holonomy group contains the holonomy group and consists of all map germs that preserve the leaves of the foliation. Fix now a germ of holomorphic foliation with a singularity at the origin $0 \in \mathbb{C}^2$, with a representative $\mathcal{F}(U)$. Let Γ be a separatrix of $\mathcal{F}$. By Newton-Puiseaux parametrization theorem, the topology of Γ is the one of a disc. Further, $\Gamma \setminus \{0\}$ is biholomorphic to a punctured disc $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. In particular, we may choose a loop $\gamma \in \Gamma \setminus \{0\}$ generating the (local) fundamental group $\pi_1(\Gamma \setminus \{0\})$. The corresponding holonomy map h_γ is defined in terms of a germ of complex diffeomorphism at the origin of a local disc Σ transverse to $\mathcal{F}$ centered at a non-singular point $q \in \Gamma \setminus \{0\}$. This map is well-defined up to conjugacy by germs of holomorphic diffeomorphisms and is generically referred to as *local holonomy* of the separatrix Γ.

5.2 Main result

Definition 5.2.1 ([6, 23])**.** A *divisor* $\mathcal{D}$ on a compact complex manifold M, is a formal sum $\mathfrak{d}_p = \sum_j k_j \mathbf{V}_j$ where $k_j \in \mathbb{Z}$ and $\{\mathbf{V}_j\}_j$ is a locally finite sequence of irreducible hypersurfaces on M, where locally finite means that every point has a neighborhood which meets only finitely many $\mathbf{V}_j$'s.

Consider $\mathcal{F}$ a holomorphic foliation of codimension 1 on a compact, connected and complex surface M having an invariant divisor $\mathcal{D} \subset M$. We denote by $\tilde{M}$ the surface obtained from M after the resolution of the dicritical singularities in $\mathcal{D}$. Let $E : \tilde{M} \to M$ be the resolution map (finite composition of blow-ups) and $\tilde{\mathcal{F}}$ the foliation induced by E; there is a divisor D_q for each dicritical $q \in \mathcal{D} \cap \operatorname{sing}(\mathcal{F})$ consisting of a finite union of projective lines, moreover we can suppose that there are no singularities in the dicritical components of D_q; note that the foliation $\tilde{\mathcal{F}}$ is tranverse to the dicritical components of D_q.
Remember that $\mathcal{F}$ off the dicritical singularities in $\mathcal{D}$ and $\tilde{\mathcal{F}} \setminus D$ are biholomorphic, where D is the union of the divisors D_q one for each dicritical $q \in \mathcal{D} \cap \operatorname{sing}(\mathcal{F})$.

Finally, denote $\tilde{\mathcal{D}}^* = E^{-1}(\mathcal{D} \setminus \operatorname{sing}(\mathcal{F}))$. We are now in conditions to state the main result of this chapter.

Theorem 5.2.2. *Let $\mathcal{F}$ be a holomorphic foliation of codimension 1 on a compact, connected and complex surface M. Suppose that there is an invariant divisor $\mathcal{D} \subset M$ such that:*

(i) *The virtual holonomy of the components of $\mathcal{D}$ is finite.*

(ii) *The elements in $\mathcal{D} \cap \operatorname{sing}(\mathcal{F})$ are isolated singularities of $\mathcal{F}$.*

(iii) *If a singularity $p \in \mathcal{D} \cap \operatorname{sing}(\mathcal{F})$ is non dicritical then $\mathcal{D}$ contains all the separatrices of $\mathcal{F}$ through p.*

(iv) *If a singularity $q \in \mathcal{D} \cap \operatorname{sing}(\mathcal{F})$ is dicritical then for its separatrices L_q in $\mathcal{D}$ the closure of $\tilde{L}_q = E^{-1}(L_q \setminus \{q\})$ cuts a dicritical component of D_q.*

Then $\mathcal{F}$ has a meromorphic first integral.

Proof. If $p \in \mathcal{D} \cap \operatorname{sing}(\mathcal{F})$ is non dicritical, note that it has finitely many separatrices due to the locally finiteness of $\mathcal{D}$ and condition (iii), furthermore, they have finite holonomy by (i), then by Mattei-Moussu's Theorem 5.0.2 there exist a neighborhood U_p of p such that the foliation $\mathcal{F}_{U_p}$ has a holomorphic first integral, then the leaves of $\mathcal{F}_{U_p}$ are closed off p. Avoiding the introduction of unnecessary notation, in what follows we are going to consider the elements in this paragraph as their pre-images by E; they are biholomorphic because we are blowing-up only the dicritical singularities in $\mathcal{D}$.

If $q \in \mathcal{D} \cap \operatorname{sing}(\mathcal{F})$ is dicritical, by (iv) we have that for its separatrices $L_q \in \mathcal{D}$, $\overline{\tilde{L}}_q$ cuts some dicritical component of D_q, suppose in a point $\tilde{q}$. Consider now a foliation chart $U_{\tilde{q}}$ for the foliation $\tilde{\mathcal{F}}$ centered in $\tilde{q}$; the foliation induced by $\tilde{\mathcal{F}}$ in $U_{\tilde{q}}$, denoted by $\tilde{\mathcal{F}}_{U_{\tilde{q}}}$, is not singular and its leaves ($\overline{\tilde{L}}_q \cap U_{\tilde{q}}$ is one of them) are plaques transverse to the dicritical component of D_q.

Suppose that $\mathcal{D} \cap \operatorname{sing}(\mathcal{F}) = \{p_1, \ldots, p_r, q_1, \ldots, q_s\}$ where $\{p_i\}_1^r$ are non dicritical and $\{q_i\}_1^s$ are dicritical singularities, observe that $r, s < \infty$ by (ii). Consider neighborhoods $\{U_{p_i}\}_1^r$ and $\{U_{\tilde{q}_i}\}_1^s$ as in the previous paragraphs, and take small ones $\{U'_{p_i}\}_1^r$ and $\{U'_{\tilde{q}_i}\}_1^s$ such that $p_i \in U'_{p_i} \subset \overline{U'}_{p_i} \subset U_{p_i}$, analogously for $\tilde{q}_i$. Note that $\tilde{\mathcal{D}}^* \setminus (\cup_{i=1}^r U'_{p_i} \bigcup \cup_{i=1}^s U'_{\tilde{q}_i})$ is a compact set, then for each leaf in $\tilde{\mathcal{F}} \cap \tilde{\mathcal{D}}^* \setminus (\cup_{i=1}^r U'_{p_i} \bigcup \cup_{i=1}^s U'_{\tilde{q}_i})$ we can apply a stability argument, as Reeb's Local Stability Theorem (see for example [11] pag. 71) or Theorem 4.15 in [44] pag. 71, and find a fundamental system of neighborhoods $\mathcal{W}$ of $\tilde{\mathcal{D}}^* \setminus (\cup_{i=1}^r U'_{p_i} \bigcup \cup_{i=1}^s U'_{\tilde{q}_i})$ where all the leaves are compact.
Therefore, adjusting the sizes of the sets $\{U'_{p_i}\}_1^r$, $\{U'_{\tilde{q}_i}\}_1^s$ and choosing $W \in \mathcal{W}$, we can create (with the union of all of them) a neighborhood V of $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}^* \cup \{p_i, \tilde{q}_i\}$ invariant by $\tilde{\mathcal{F}}$. The leaves of $\tilde{\mathcal{F}}_V$ are closed because each leaf $L \in \tilde{\mathcal{F}}_V$ can be written as a finite union of closed sets $L = (\cup_i^r L \cap U'_{p_i}) \cup (\cup_i^s L \cap U'_{\tilde{q}_i}) \cup (L \cap W)$, note that if some element in that union (for instance $L \cap U'_{p_i}$) consists of infinitely many plaques cumulating $\mathcal{D}$ then the same is true for all the nonempty sets in $\{L \cap U'_{p_i}, L \cap U'_{\tilde{q}_j}, L \cap W\}_{i,j}^{r,s}$ hence L acumulates $\tilde{\mathcal{D}}$, this implies that one element of the group of holonomy of some leaf L' in $\tilde{\mathcal{D}}$ is not periodic and this contradicts (i) because that element can be seen as the composition of elements in the groups of holonomy of $L'_W \in \tilde{\mathcal{F}}_W$, $L'_{U'_{p_i}} \in \tilde{\mathcal{F}}_{U'_{p_i}}$ and $L'_{U'_{q_j}} \in \tilde{\mathcal{F}}_{U'_{q_j}}$ for some i, j, and all of them are periodic, thereby there exist infinitely many compact leaves and according to Theorem 5.0.1 this implies that $\tilde{\mathcal{F}}$ has a meromorphic first integral $\tilde{f}$, hence $\tilde{f} \circ E^{-1}$ is a meromorphic first integral of $\mathcal{F}$.

✓

Chapter 6

First integrals around the separatrix set

6.1 Introduction

One of the key stones in the theory of holomorphic foliations is the article of Mattei and Moussu [30] where the important result 5.0.2 about the existence of holomorphic first integrals is presented.

Years later the second author (in [31]) revisited this result in order to create a new proof, simpler and more geometric. In this chapter we present two minor results products of unsuccessful attempts to give a proof of Theorem 3.1.6 repeating Moussu's technique [31].

6.2 Generic vector fields in dimension n

This section is dedicated to show our attempt to prove Theorem 3.1.6 following the proof in [31].

Theorem 3.1.6. *Suppose that* $\mathcal{X} \in \mathrm{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ *satisfies condition* $(\star)$ *and let* $S_{\mathcal{X}}$ *be the axis associated to the separable eigenvalue of* $\mathcal{X}$.

Then, $\mathrm{Hol}(\mathcal{F}(\mathcal{X}), S_{\mathcal{X}}, \Sigma)$ *is periodic (in particular linearizable and finite) if and only if* $\mathcal{F}(\mathcal{X})$ *has a holomorphic first integral.*

The following definitions is inspired by [36] where it is shown that the existence of a holomorphic first integral for foliations by curves on $(\mathbb{C}^3, 0)$ is not a topological invariant. More precisely, it is provide an example of two topologically equivalent foliations such that only one of them admits a holomorphic first integral.

Definition 6.2.1. Let $\mathcal{F}$ be a dimension one foliation on $(\mathbb{C}^3, 0)$, a (possibly singular) $\mathcal{F}$-invariant surface will be called *dicritical* (or *dicritical invariant surface*) if the restriction of $\mathcal{F}$ to it possesses infinitely many separatrices. The particular case of a hyperplane will be called *dicritical hyperplane.*

6.2.1 Attempt to a geometric proof of Theorem 3.1.6

We divided the proof in two parts:

a. The construction of a neighborhood V of the origin.

b. The study of the quotient space $V/\mathcal{F}_V$.

We succeeded to prove the first part, i.e., we built an invariant neighborhood V of the separatrices (in this case the distinguished axis and the dicritical hyperplane Proposition 6.2.2) that can be seen as the saturated of a transverse section to the distinguished axis. It is important to mention that this was already done in [40] (Proposition 1.) but, in our case, we use the hypothesis about the periodicity of the holonomy of $S_{\mathcal{X}}$. Our proof is more geometric though, except by the used of the Proposition 6.2.2.

Fix a sufficiently small ball $B = B_r^{2n}$ centered at $0 \in \mathbb{C}^n (\cong \mathbb{R}^{2n})$ contained in an open set U where the germ of generic vector field $\mathcal{X} \in \mathrm{Gen}(\mathfrak{X}(\mathbb{C}^n, 0))$ is defined.

Proposition 6.2.2. *Suppose that $\mathcal{X} \in \mathrm{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ satisfies condition $(\star)$ and let $S_{\mathcal{X}}$ be the axis associated to the separable eigenvalue of $\mathcal{X}$.*

Then, the separatrices of $\mathcal{F}(\mathcal{X})$ are $S_{\mathcal{X}}$ and the leaves contained in the dicritical hyperplane.

Proof. Remember that a generic vector field can be written in the form (3.1)

$$\mathcal{X}(x) = \lambda_1 x_1(1 + a_1(x))\frac{\partial}{\partial x_1} + \lambda_2 x_2(1 + a_2(x))\frac{\partial}{\partial x_2} + \lambda_3 x_3(1 + a_3(x))\frac{\partial}{\partial x_3},$$

where $a_i \in \mathcal{M}_3$ for $i = 1, 2, 3$. We can also choose v such that $\mathrm{Re}(\lambda_1/v)$, $\mathrm{Re}(\lambda_2/v) < 0$ e $\mathrm{Re}(\lambda_3/v) > 0$. Also, as $a_3(0) = 0$ we know that for $|x|$ small $|a_3(x)| < \epsilon$ thus $|1 + a_3(x)| \geq |1 - |a_3(x)|| \geq 1 - |a_3(x)| > 1 - \epsilon$ and the function $\frac{1+a_i(x)}{1+a_3(x)}$ is holomorphic, take $1 + \tilde{a}_i(x) = \frac{1+a_i(x)}{1+a_3(x)}$, suppose that $|\tilde{a}_i(x)| \leq |\frac{\mathrm{Re}(\lambda_i/v)}{2\lambda_i/v}|$ and write $\mathcal{X}$ as

$$\mathcal{X}(x) = \frac{\lambda_1}{v} x_1(1 + \tilde{a}_1(x))\frac{\partial}{\partial x_1} + \frac{\lambda_2}{v} x_2(1 + \tilde{a}_2(x))\frac{\partial}{\partial x_2} + \frac{\lambda_3}{v} x_3 \frac{\partial}{\partial x_3},$$

Now, let $\gamma(T) = (x_1(T), x_2(T), x_3(T))$ be a separatrix of $\mathcal{F}(\mathcal{X})$ not contained in the hyperplane $x_3 = 0$ or in the x_3 axis. We know that γ is $\mathcal{F}(\mathcal{X})$-invariant then $\mathcal{X}(\gamma) = \gamma'$ which is equivalent to $x_i'(T) = (\lambda_i/v)x_i(T)(1 + \tilde{a}_i(\gamma(T)))$ for $i = 1, 2$ and $x_3'(T) = (\lambda_3/v)x_3(T)$. Consider the case where $T = t \in \mathbb{R}$. Hence $\gamma(t)$ is a curve with real dimension one. Reparametrize γ such that $\gamma(0) \neq 0$ and $\lim_{t\to\infty} \gamma(t) = 0$. Therefore

$$x_i(t) = x_i(0)e^{\frac{\lambda_i}{v}t + \frac{\lambda_i}{v}\int_0^t \tilde{a}_i(\gamma(t))\mathrm{d}t}$$

for $i = 1, 2$ and $x_3(t) = x_3(0)e^{\frac{\lambda_3}{v}t}$. Now, taking modulus

$$|x_i(t)| = |x_i(0)|e^{\mathrm{Re}\left(\frac{\lambda_i}{v}\right)t + \mathrm{Re}\left(\frac{\lambda_i}{v}\int_0^t \tilde{a}_i(\gamma(t))\mathrm{d}t\right)}.$$

Considering the upper quotes

$$\begin{aligned}\mathrm{Re}\Big(\frac{\lambda_i}{v}\int_0^t \tilde{a}_i(\gamma(t))\mathrm{d}t\Big) &\leq \big|\frac{\lambda_i}{v}\big|\int_0^t |\tilde{a}_i(\gamma(t))|\mathrm{d}t,\\ &\leq \frac{1}{2}|\mathrm{Re}(\lambda_i/v)|t,\end{aligned}$$

we have that $|x_i(t)| \leq |x_i(0)|e^{\frac{1}{2}\mathrm{Re}(\lambda_i/v)t}$ and this goes to 0 when $t \to \infty$. On the other hand $|x_3(t)| = |x_3(0)|e^{\mathrm{Re}(\frac{\lambda_3}{v})t}$ and goes to ∞ because $\mathrm{Re}(\lambda_3/v) > 0$. As a conclusion, γ cannot be as we supposed and it has to be contained in the hyperplane $x_3 = 0$ or in the x_3 axis. ✓

We will denote by S the union of $S_{\mathcal{X}}$ with the dicritical hyperplane, i.e., the set of separatrices of $\mathcal{F}(\mathcal{X})$.

Lemma a. *Suppose that $\mathcal{X} \in \mathrm{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ satisfies condition $(\star)$ and let $S_{\mathcal{X}}$ be the axis associated to the separable eigenvalue of $\mathcal{X}$.*

If $\mathrm{Hol}(\mathcal{F}(\mathcal{X}), S_{\mathcal{X}}, \Sigma)$ is periodic, then there exists a $\overline{\mathcal{F}}$-invariant neighborhood V of S in $\overline{B}$ such that the leaves in V cut ∂B transversally. Furtheremore, V is the union of the saturate of a small transversal section of $S_{\mathcal{X}}$ and the dicritical hyperplane.

Proof of Lemma a. In this paragraph we use some of the arguments of [[10], Lemma 2 pag. 66]. First observe that if $L \in \overline{\mathcal{F}}$ is a closed leaf transverse to ∂B, then $\partial L = L \cap \partial B$ is a closed set of real dimension one, and each connected component in ∂L is diffeomorphic to the circle S^1. Suppose that $K \subset \partial L$ is one of this connected components. Consider neighborhoods $U_k \supset W_k$ of K, U_K open in $\mathbb{C}^n$ and W_K open in L, where W_K can be taken as a finite union of plates because $K \subset L$ is a compact subset of a leaf. As ∂B

intersects W_K transversally, we can choose U_K small enough such that for every $x \in U_K$ the leaf of $\mathcal{F}|_{U_K}$ through x meets ∂B transversally.

Continuing with this argument, if there exist K_1 and K_2 as above, we can use global trivialization to show the existence of a homeomorphism between transversal sections to W_{K_1} and W_{K_2} contained respectively in U_{K_1} and U_{K_2} (in fact is just to repeat the technique used in the construction of the holonomy map). This homeomorphism shows that we can find an invariant neighborhood of L of leaves transversal to ∂B in $\partial B \cap U_{K_1}$ and $\partial B \cap U_{K_2}$.

In what follows we will use the notation $K_1 = S_{\mathcal{X}} \cap \partial B$ where $S_{\mathcal{X}}$ is the distinguished axis of the generic vector field $\mathcal{X}$; K_1 is compact with periodic holonomy, then it possesses a neighborhood where $\partial\mathcal{F}$ is a transversally holomorphic foliation without singularities. Applying Reeb's in $(\partial S_k, \partial\mathcal{F})$, we have

Assertion 6.2.1. The leaf K_1 of $\partial\mathcal{F}$ possesses a tubular neighborhood $T_1(\epsilon)$ in ∂B

$$J_1 : (\mathbb{D}_\epsilon \times \mathbb{D}_\epsilon) \times S^1 \to T_1(\epsilon),$$

such that $J_1^{-1}(\partial\mathcal{F})$ is the suspension of a periodic rotation in $\mathbb{D}_\epsilon \times \mathbb{D}_\epsilon$.

The neighborhood $T_1(\epsilon)$ is $\partial\mathcal{F}$-invariant and $T_1(\epsilon') = J_1\big((\mathbb{D}_{\epsilon'} \times \mathbb{D}_{\epsilon'}) \times S^1\big)$ with $0 < \epsilon' < \epsilon$ forms a fundamental system of neighborhoods of K_1 in ∂B. In addition $T_1(\epsilon)$ is transverse to $\mathcal{F}$.

Consider also the following set

$$T_2(\epsilon_2) = \{x \in \mathbb{C}^3 \,\big|\, |x_1|^2 + |x_2|^2 = 1,\ |x_3| \leq \epsilon_2\}$$

Assertion 6.2.2. There exist $0 < \epsilon' < \epsilon$ such that the intersection of ∂B with the $\overline{\mathcal{F}}$-saturated $V(\epsilon')$ of $T_1(\epsilon')$ is contained in $T(\epsilon) = T_1(\epsilon) \cap T_2(\epsilon)$.

Proof of Assertion 6.2.2. By contradiction, take a sequence $\{a_k\}_k$ of points in $T_1(\epsilon)$ such that $a_k \to a \in K_1$ and satisfying $L_{a_k} \cap \partial B \not\subset T(\epsilon)$ where L_{a_k} is the leaf in $\overline{\mathcal{F}}$ through a_k. Take b_k a point in $(L_{a_k} \cap \partial B) \setminus T(\epsilon)$, then $\{b_k\}_k$ is a sequence in a compact set thus $b_k \to b \in \partial B$ (using the same notation for a subsequence). If L_b is transverse to ∂B, then we can use the previous paragraph supposing that b belongs to some K_2, then there exist an invariant neighborhood of L_b of leaves transversal to ∂B in $\partial B \cap U_{K_1}$ and $\partial B \cap U_{K_2}$, this implies that L_b is far from $S_{\mathcal{X}}$.

If L_b is not transverse to ∂B we can take a sphere of radius $1 + \delta$, $\delta > 0$, and proceed as above.

Assertion 6.2.3. There exist $0 < \epsilon_1 < \epsilon'$ such that $V(\epsilon_1) = V$, the $\overline{\mathcal{F}}$-saturate of $T_1(\epsilon_1)$ together with the dicritic hyperplane is a neighborhood of 0 in $\overline{B}$.

Proof of Assertion 6.2.3. The periodicity of the holonomy group $\mathrm{Hol}(\mathcal{F}(\mathcal{X}), S_{\mathcal{X}}, \Sigma)$ implies that it can be conjugated to a cyclic group generated by a diagonal diffeomorphism. Therefore, there are not fixed points other than the origin in a neighborhood of it, so we can choose ϵ_1 such that $0 < \epsilon_1 < \epsilon'$ and the leaves cutting $J_1(\partial B^4_{\epsilon_1} \times \{1\}) = C_{\epsilon_1}$ have trivial holonomy and the compactness of the leaves allows to apply Reeb stability theorem. For all $a \in C_{\epsilon_1}$ the leaf L_a in $\overline{\mathcal{F}}$ through a possesses a $\overline{\mathcal{F}}$-saturated tubular neighborhood:

$$J_a : \tau_a \times L_a \to T(L_a),$$

such that $J^{-1}(\overline{\mathcal{F}})$ is foliated by fibers $z \times L_a$, where τ_a (whose complex dimension is two) is a small transverse section to $\overline{\mathcal{F}}$ through a contained in $T_1(\epsilon')$. In particular the $\overline{\mathcal{F}}$-saturate of $\nu_a = \tau_a \cap C_{\epsilon_1}$ is C^∞-diffeomorphic to the product $\nu_a \times L_a$ and the saturated of C_{ϵ_1} is a C^∞-hypersurface (whose boundary is contained in ∂B) fibered over S^1. By construction, is the boundary of $V = V(\epsilon_1)$ the $\overline{\mathcal{F}}$-saturated of $T_1(\epsilon_1)$. ✓

We would like to have the analogous of Lemma 2 in [31], something like:

"There exists a homeomorphism

$$h : V^*/\mathcal{F}_{V^*} \to B^*(= B \setminus \{0\})$$

such that $h \circ q_{V^*} = p_{V^*}$ is holomorphic."

In order to proof such lemma, it would be necessary to understand the topology of the space of leaves $q(V^*) = V^*/\mathcal{F}$. We know that $q(V^*) = q\big(J_1(B_{\epsilon'} \times \{1\})\big)$ is a Hausdorff space (because the leaves we are considering are closed) but the major difference is that in dimension two it can be shown, using machinery like the Riemann map and fundamental group, that the $q(V^*)$ is biholomorphic to $\mathbb{D}^*$. In our case, what we need is to find a biholomorphism between $q(V^*)$ and B^{4*} (where $q(V^*)$ has a differentiable structure possibly defined as in dimension two) but the machinery used in dimension two do not exist (or are not as useful) in higher dimensions. Our intention of repeat Moussu's proof in dimension three was unsuccessful but it helped us to achieve a better understanding of our problem.

Appendices

Appendix A

Algebraic properties of groups of diffeomorphisms

Here we give a sketch of the proof of Proposition 2.2.8. We start by introducing some notations, definitions and results needed for this purpose, they mainly come from [41], we also recommend [28, 29].

A.1 Preliminaries

Let $\phi \in \mathrm{Diff}(\mathbb{C}^n, 0)$. We consider its action in the space of k-jets. More precisely we consider the element $\phi_k \in \mathrm{GL}(\mathfrak{m}/\mathfrak{m}^{k+1})$ defined by

$$\mathfrak{m}/\mathfrak{m}^{k+1} \overset{\phi_k}{\to} \mathfrak{m}/\mathfrak{m}^{k+1}$$
$$g + \mathfrak{m}^{k+1} \mapsto g \circ \phi + \mathfrak{m}^{k+1}$$

where $\mathfrak{m}/\mathfrak{m}^{k+1}$ can be interpreted as a finite dimensional complex vector space. In this point of view, diffeomorphisms are interpreted as operators acting on function spaces.

Definition A.1.1. We define $D_k = \{\phi_k : \phi \in \mathrm{Diff}(\mathbb{C}^n, 0)\}$.

The natural projections $\pi_{k,l} : D_k \to D_l$ for $k \geq l$ define a projective system and hence we can consider the projective limit $\varprojlim D_k$. It is the so called group of formal diffeomorphisms $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$.

Definition A.1.2. Let G be a subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. We define G_k as the smallest algebraic subgroup of D_k containing $\{\varphi_k : \varphi \in G\}$.

Definition A.1.3. Let G be a subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n,0)$. We define $\overline{G}^z$ as $\varprojlim_{k\in\mathbb{N}} G_k$, more precisely $\overline{G}^z$ is the subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n,0)$ defined by

$$\overline{G}^z = \{\varphi \in \widehat{\mathrm{Diff}}(\mathbb{C}^n,0) : \ \varphi_k \in G_k \ \forall\, k \in \mathbb{N}\}$$

We say that $\overline{G}^z$ is the pro-algebraic closure of G. We say that G is *pro-algebraic* if $G = \overline{G}^z$

Proposition A.1.4. *Let $\phi \in \widehat{\mathrm{Diff}}(\mathbb{C}^n,0)$. Then ϕ is unipotent if and only if $j^1\phi$ is unipotent.*

Lemma A.1.5. *Let H_k be an algebraic subgroup of D_k for $k \in \mathbb{N}$. Suppose that $\pi_{l,k}(H_l) \subset H_k$ for all $l \geq k \geq 1$. Then $\varprojlim_{k\in\mathbb{N}} H_k$ is a pro-algebraic subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n,0)$. Moreover the natural map $\varprojlim H_j \to H_k$ is surjective for any $k \in \mathbb{N}$ if $\pi_{l,k}(H_l) = H_k$ for all $l \geq k \geq 1$.*

The group G is a projective limit of algebraic groups and closed in the Krull topology by definition. Since G_k is an algebraic group of matrices and in particular a Lie group, we can define the connected component $G_{k,0}$ of the identity in G_k. We also consider the set $G_{k,u}$ of unipotent elements of G_k.

Proposition A.1.6. *Let G be a subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n,0)$. Then we have $\overline{G}_0^z = \{\varphi \in \overline{G}^z : \varphi_1 \in G_{1,0}\}$. Moreover $\overline{G}_0^z$ is pro-algebraic.*

Remark A.1.7. Let G be a solvable subgroup of $\mathrm{Diff}(\mathbb{C}^n,0)$. Since membership in $\overline{G}_0^z$ and $\overline{G}_u^z$ can be checked out in the first jet, these groups have finite codimension in $\overline{G}^z$. Indeed the kernels of the natural maps

$$\overline{G}^z \to G_1/G_{1,u} \text{ and } \overline{G}^z \to G_1/G_{1,0}$$

are equal to $\overline{G}_u^z$ and $\overline{G}_0^z$ respectively by Propositions A.1.4 and A.1.6. In particular $\overline{G}^z/\overline{G}_0^z$ is a finite group.

Proposition A.1.8 (Proposition 2. [28])**.** *Let $G \subset \widehat{\mathrm{Diff}}(\mathbb{C}^n,0)$ be a group. Then $\mathfrak{g}$ is equal to $\{\mathcal{X} \in \hat{\mathfrak{X}}(\mathbb{C}^n,0) : \exp(t\mathcal{X}) \in \overline{G}^z \ \forall t \in \mathbb{C}\}$ and $\overline{G}_0^z$ is generated by the set $\{\exp(\mathcal{X}) : \mathcal{X} \in \mathfrak{g}\}$. Moreover if G is unipotent then the map*

$$\exp : \mathfrak{g} \to \overline{G}^z$$

is a bijection and $\mathfrak{g}$ is a Lie algebra of nilpotent formal vector fields.

Remark A.1.9. Invariance properties typically define pro-algebraic groups. Let us present an example. Consider $f_1, \ldots, f_n \in \hat{\mathcal{O}}_n$ and

$$G = \{\varphi \in \widehat{\mathrm{Diff}}(\mathbb{C}^n,0) \mid \quad f_j \circ \varphi \equiv f_j \quad \forall\, 1 \leq j \leq n\}$$

We define

$$H_k = \{A \in D_k : A(f_j + \mathfrak{m}^{k+1}) = f_j + \mathfrak{m}^{k+1} \ \forall 1 \leq j \leq p\}$$

for $k \in \mathbb{N}$. It is clear that H_k is an algebraic subgroup of D_k for $k \in \mathbb{N}$. Moreover we have $\pi_{l,k}(H_l) \subset H_k$ for $l \geq k \geq 1$. Since $f \circ \phi - f = 0$ is equivalent to $f \circ \phi - f \in \mathfrak{m}^k$ for any $k \in \mathbb{N}$, the group $\varprojlim H_k$ is equal to G. Moreover G is pro-algebraic by Lemma A.1.5.

A.2 Finiteness of a invariance group

Now, using the previous theory we have the necessary tools to give a sketch of the proof of Proposition 2.2.8.

Proposition A.2.1. *Let us consider n elements $f_1, \ldots, f_n$ of the field of fractions of $\hat{\mathcal{O}}_n$. Suppose $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \not\equiv 0$. Then the group*

$$G = \{\varphi \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0) \mid \quad f_j \circ \varphi \equiv f_j \quad \forall\, 1 \leq j \leq n\}$$

is finite.

Proof. We have that G is pro-algebraic by Remark A.1.9. Consider an element $\mathcal{X} = \sum_{j=1}^n a_j \partial/\partial x_j$ in the Lie algebra $L(G)$ of G. By definition we have

$$f_j \circ \exp(t\mathcal{X}) \equiv f_j \ \forall t \in \mathbb{C} \Longrightarrow \mathcal{X}(f_j) = \lim_{t \to 0} \frac{f_j \circ \exp(t\mathcal{X}) - f_j}{t} \equiv 0$$

for any $1 \leq j \leq n$. The property $\mathcal{X}(f_j) = 0$ for any $1 \leq j \leq n$ is equivalent to

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \not\equiv 0$, the $n \times n$ matrix in the previous equation has a non-vanishing determinant and then $\mathcal{X} \equiv 0$. Hence $L(G)$ is trivial and $\overline{G_0}^z$ is the trivial group by Proposition A.1.8. Since $G/\overline{G_0}^z$ is finite by Remark A.1.7, G is finite. ✓

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