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Some aspects of the isoperimetric problem in non-compact Riemannian manifolds and applications

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Sob a orientação do

Prof. Stefano Nardulli

Tese apresentada ao Programa de Pós-Graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio de Janeiro como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo : Para uma variedade Riemanniana completa não compacta com geometria limitada, provamos um resultado de compacidade por sequências de conjuntos de perímetro finito com volume e perímetro uniformemente limitado em um espaço maior obtido por adição de variedades limites no infinito. Estendemos resultados previous contidos em [Nar14a], de tal modo que o teorema de existência generalizada, Teorema 1 de [Nar14a] seja realmente um teorema de compacidade generalizada. As modificações necessárias para os argumentos e afirmações dos resultados em [Nar14a] são não triviais. Como consequência damos uma versão multi-pontos do Teorema 1.1 de [LW11], e uma prova bem fácil da continuidade da função perfil isoperimétrico. Mostramos existência e caracterização das regiones isoperimétricas por grandes volumes, em variedades Riemannianas C^2-localmente assintóticamente Euclidianas com um número finito de fins C^0 assintoticamente Schwarzchild. Estendendo os resultados prévios de [EM13b], [EM13a] e [BE13]. Tais resultados são de interesse para a relatividade geral. Para uma variedade Riemanniana completa conexa não compacta com geometria limitada M^n, provamos que a função perfil isoperimétrico I_{M^n} é localmente $\left(1-\frac{1}{n}\right)$-Hölder contínua e em particular contínua. Neste ponto por geometria limitada entendemos que M tem curvatura de Ricci limitada inferiormente e o volume das bolas de raio 1 uniformemente limitadas inferiormente com respeito aos centros. Provamos também a equivalência da formulação fraca e forte da função perfil isoperimétrico numa variedade Riemanniana completa. Finalmente mostramos a semicontinuidade por todas métricas completas.

Palavras chaves : Teoria geométrica da medida, Conjuntos de perímetro finito, Calculo das variações , Geometria Riemanniana, Geometria métrica, convergência de Gromov-Hausdorff, isoperimetria Riemanniana, Relatividade Geral, métricas de Schwarzschild.

Abstract For a complete noncompact connected Riemannian manifold with bounded geometry, we prove a compactness result for sequences of finite perimeter sets with uniformly bounded volume and perimeter in a larger space obtained by adding limit manifolds at infinity. We extends previous results contained in [Nar14a], in such a way that the generalized existence theorem, Theorem 1 of [Nar14a] is actually a generalized compactness theorem. The suitable modifications to the arguments and statements of the results in [Nar14a] are non-trivial. As a consequence we give a multipointed version of Theorem 1.1 of [LW11], and a simple proof of the continuity of the isoperimetric profile function. We show existence and characterization of isoperimetric regions for large volumes, in C^2-locally asymptotically Euclidean Riemannian manifolds with a finite number of C^0-asymptotically Schwarzschild ends. Extending previous results contained in [EM13b], [EM13a], and [BE13]. Such a results are of interest for mathematical general relativity. For a complete noncompact connected Riemannian manifold with bounded geometry M^n, we prove that the isoperimetric profile function I_{M^n} is a locally $\left(1-\frac{1}{n}\right)$-Hölder continuous function and so in particular it is continuous. Here for bounded geometry we mean that M have Ricci curvature bounded below and volume of balls of radius 1, uniformly bounded below with respect to its centers. We prove also the equivalence of the weak and strong formulation of the isoperimetric profile function in complete Riemannian manifolds which is based on a lemma having its own interest about the approximation of finite perimeter sets with finite volume by open bounded with smooth boundary ones of the same volume. Finally the upper semicontinuity of the isoperimetric profile for every complete metric is shown.

Key Words : Geometric measure theory, Finite perimeter sets, Calculus of variations, Riemannian geometry , Metric geometry, Gromov-Hausdorff's convergence, Riemannian isoperimetry, General Relativity, asymptotically Schwarzschild metrics.

Contents

1 Introduction

In this thesis we study various aspect of the Riemannian isoperimetric problem, including compactness of the set of finite perimeter sets with a bounded volume and perimeter in a complete Riemannian manifold of infinite volume, assuming some bounded geometry conditions. We study also existence and characterization of isoperimetric regions in asymptotically Schwarzschild Riemannian manifolds possesing a finite number of ends, which are of interest in general relativity. Finally we treat the problem of continuity, Hölder continuity of the isoperimetric profile and its differentiability properties. This whole work gave rise to 4 articles, one of these namely [MFN16] is accepted in Geometriae Dedicata and the other 3 are submitted to international reviews and already posted in arXiv [FN15b], [FN14], and [FN15a]. We start by the problem of compactness which is treated in [FN15b]. The difficulty here is that for a sequence of regions with uniformly bounded perimeter and volume, some volume may disappear to infinity. We show that such a sequence splits into an at most countable number of pieces which carry a positive fraction of the volume, one of them possibly staying at finite distance and the others concentrating along divergent directions. Moreover, each of these pieces will converge to a finite perimeter set lying in some pointed limit manifold, possibly different from the original. So a limit finite perimeter set exists in a generalized multipointed L^1_{loc} convergence sense. The range of applications of these results is wide. The vague notions invoked in this introductory paragraph will be made clear and rigorous in the sequel.
In the second part of the thesis we will discuss the isoperimetric problem for this, we consider the isoperimetric profile function $I_M(v)$ equal to infimun area required for a region with smooth boundary in M enclosing a volume v. The isoperimetric problem consist studying, among the compact hypersurfaces $\Sigma \subset M$ enclosing a region Ω of volume $V(\Omega) = v$, those which minimize area $A(\Sigma)$, the region Ω is called an isoperimetric region. If M is the Euclidean space $\mathbb{R}^n$, the sphere $\mathbb{S}^n$ or hyperbolic space $\mathbb{H}^n$, then the isoperimetric regions are metric balls. If M is compact, classical compactness arguments of geometric measure theory see Theorem 3.1 combined with the direct method of the calculus of variations provide existence of isoperimetric regions in any dimension n. Finally, if M is complete, non-compact, and $V(M) < +\infty$, an easy consequence of Theorem 2.1 in [RR04] yields existence of isoperimetric regions for every volume. This is not the case for general complete infinite-volume manifolds M. In fact it is easy to construct simple ex-

amples of 2-dimensional Riemannian manifolds M with $V(M) = +\infty$ obtained as the rotation about the y-axis in $\mathbb{R}^3 = \{x, y, z\}$ of a graph in the $\{y, z\}$ plane of a smooth function, e.g.: $y \mapsto \frac{1}{y^\alpha}$, for some $0 < \alpha \leq 1$. In this example $I_M(v) = 0$ for every $v \in]0, V(M)[$. Another example for which there are no isoperimetric regions is the familiar hyperbolic parabolid, i.e., the quadric surface $S \subset \mathbb{R}^3$ defined, by the equation $z = x^2 - y^2$. In This latter example the isoperimetric is eqaul to the Euclidean isoperimetric profile and in particular is strictly bigger than zero. For completeness we remind the reader that if $n \leq 7$, then the boundary $\partial\Omega$ of an isoperimetric region is smooth. More generally, the support of the boundary of an isoperimetric region is the disjoint union of a regular part R and a singular part S. R is smooth at each of its points and has constant mean curvature, while S has Hausdorff-codimension at least 7 in $\partial\Omega$. For more details on regularity theory see [Mor03] or [Mor09] Sect. 8.5, Theorem 12.2.

In general, we don't have compactness. The reason for this behavior appears in the proof of Theorem 2.1 in [RR04], which illustrates very clearly that the lack of compactness in the variational problem is due to the fact that a part $v_2 > 0$ of the total volume $v = v_1 + v_2$ goes to infinity. In the first part of this thesis, we describe exactly what happens to the divergent part in the case of bounded geometry, which in this context means that both the Ricci curvature and the volume of geodesic balls of a fixed radius are bounded below. For any fixed volume v, we can build an example of a 2-dimensional Riemannian manifold obtained by altering an Euclidean plane with an infinite number (or under stronger assumptions, a finite number) of sequences of caps, diverging in a possibly infinite number of directions. Our main result concerning compactness is Theorem 1 contained in Section 3, which shows that it is essentially all that can occur, in bounded geometry and that the limit isoperimetric region could live in a larger space which includes the limits at infinity. Before to start to treat the problem of compactness, in Section 2 we want to make clear a point left a little bit obscure in the literature about a weak and strong definition of the isoperimetric profile function. In Section 4 we discuss local Hölder continuity of the isoperimetric profile function joint with its continuity and differentiability properties. In Section 5 we treat the problem of existence and characterization of isoperimetric regions in a C^0-asymptotically Schwarzschild manifold with a finite number of ends. The remaining sections are devoted to appendices which recall some known fact from the literature used all along this text.

2 Equivalence of the weak and strong formulation

2.1 Finite perimeter sets: some known results

In the remaining part of this thesis we always assume that all the Riemannian manifolds M considered are smooth with smooth Riemannian metric g. We denote by V the canonical Riemannian measure induced on M by g, and by A the $(n-1)$-Hausdorff measure associated to the canonical Riemannian length space metric d of M. When it is already clear from the context, explicit mention of the metric g will be suppressed in what follows.

Definition 2.1. *Let M be a Riemannian manifold of dimension n, $U \subseteq M$ an open subset, $\mathfrak{X}_c(U)$ the set of smooth vector fields with compact support on U. Given a function $u \in L^1(M)$, define the variation of u by*

$$|Du|(M) := sup\left\{\int_M udiv_g(X)dV_g : X \in \mathfrak{X}_c(M), ||X||_\infty \leq 1\right\}, \quad (1)$$

*where $||X||_\infty := \sup\left\{|X_p|_{g_p} : p \in M\right\}$ and $|X_p|_{g_p}$ is the norm of the vector X_p in the metric g_p on T_pM. We say that a function $u \in L^1(M)$, has **bounded variation**, if $|Du|(M) < \infty$ and we define the set of all functions of bounded variations on M by $BV(M) := \{u \in L^1(M) : |Du|(M) < +\infty\}$.*

Definition 2.2. *Let M be a Riemannian manifold of dimension n, $U \subseteq M$ an open subset, $\mathfrak{X}_c(U)$ the set of smooth vector fields with compact support on U. Given $E \subset M$ measurable with respect to the Riemannian measure, the **perimeter of** E **in** U, $\mathcal{P}(E,U) \in [0,+\infty]$, is*

$$\mathcal{P}(E,U) := sup\left\{\int_U \chi_E div_g(X)dV_g : X \in \mathfrak{X}_c(U), ||X||_\infty \leq 1\right\}, \quad (2)$$

*where $||X||_\infty := \sup\left\{|X_p|_{g_p} : p \in M\right\}$ and $|X_p|_{g_p}$ is the norm of the vector X_p in the metric g_p on T_pM. If $\mathcal{P}(E,U) < +\infty$ for every open set U, we call E a **locally finite perimeter set**. Let us set $\mathcal{P}(E) := \mathcal{P}(E,M)$. Finally, if $\mathcal{P}(E) < +\infty$ we say that E **is a set of finite perimeter**.*

When dealing with finite perimeter sets or locally finite perimeter sets we will denote the reduced boundary $\partial^*\Omega$, by $\partial\Omega$ when no confusion may arise. For this reason we will denote $\mathcal{P}(\Omega) = A(\partial^*\Omega) = A(\partial\Omega)$ and for every finite perimeter set Ω' we always choose a representative Ω (i.e., that differs from Ω' by a set of Riemannian measure 0), such that $\partial_{top}\Omega = \overline{\partial^*\Omega}$, where $\partial_{top}\Omega$ is the topological boundary of Ω.

Theorem 2.1 (Fleming-Rishel)**.** *let $u \in BV(M)$. Then the function $t \longrightarrow \mathcal{P}_M(\{x \in M : u(x) > t\})$ is Lebesgue measurable on $\mathbb{R}$ and the following formula holds:*

$$|Du|(M) = \int_{-\infty}^{+\infty} \mathcal{P}_M(\{x \in M : u(x) > t\}) \tag{3}$$

Proof: See Theorem 4.3 of [AMP04]. q.e.d.

Theorem 2.2 (Proposition 1.4 of [JPPP07])**.** *For every $u \in BV(M)$ there exists a sequence $(u_j)_j \in C_c^\infty(M)$ such that $u_j \to u$ in $L^1_{loc}(M)$ and*

$$|Du|(M) = \lim_{j\to\infty} \int_M |\nabla u_j| dV_g. \tag{4}$$

Remark 1. *As a consequence of Theorem 2.2 we have*

$$\lim_{j\to\infty} |\{x \in M : |u_j(x) - u(x)| \geq \eta\}| = 0, \quad \forall \eta > 0. \tag{5}$$

We state here a well known result.

Lemma 2.1 (Morse-Sard's Lemma)**.** *If $u \in C^\infty(M)$ and $E = \{x \in M : \nabla u(x) = 0\}$, then $|u(E)| = 0$. In particular, $\{u = t\} = \{x \in M : u(x) = t\}$ is a smooth hypersurface in M for a.e. $t \in \mathbb{R}$.*

Definition 2.3. *We say that a sequence of finite perimeter sets E_j **converges in** $L^1_{loc}(M)$ to another finite perimeter set E, and we denote this by writing $E_j \to E$ in $L^1_{loc}(M)$, if $\chi_{E_j} \to \chi_E$ in $L^1_{loc}(M)$, i.e., if $V((E_j \Delta E) \cap U) \to 0$ $\forall U \subset\subset M$. Here χ_E means the characteristic function of the set E and the notation $U \subset\subset M$ means that $U \subseteq M$ is open and $\overline{U}$ (the topological closure of U) is compact in M.*

Definition 2.4. *We say that a sequence of finite perimeter sets E_j **converge in the sense of finite perimeter sets** to another finite perimeter set E if $E_j \to E$ in $L^1_{loc}(M)$, and*

$$\lim_{j\to+\infty} \mathcal{P}(E_j) = \mathcal{P}(E).$$

For a more detailed discussion on locally finite perimeter sets and functions of bounded variation on a Riemannian manifold, one can consult [JPPP07], for the more classical theory in $\mathbb{R}^n$ we refer the reader to [AFP00], [Mag12].

2.2 Weak and strong formulation of the isoperimetric profile

At this point we give the definition of the isoperimetric profile function which is the main object of study in this paper.

Definition 2.5. *Typically in the literature, the* ***isoperimetric profile function of*** M *(or briefly, the isoperimetric profile)* $I_M : [0, V(M)[\to [0, +\infty[$*, is defined by*

$$I_M(v) := \inf\{A(\partial\Omega) : \Omega \in \tau_M, V(\Omega) = v\},$$

where τ_M *denotes the set of relatively compact open subsets of* M *with smooth boundary.*

However there is a more general context in which to consider this notion that will be better suited to our purposes. Namely, we can give a weak formulation of the preceding variational problem replacing the set τ_M with the family of subsets of finite perimeter of M.

Definition 2.6. *Let* M *be a Riemannian manifold of dimension* n *(possibly with infinite volume). We denote by* $\tilde{\tau}_M$ *the set of finite perimeter subsets of* M*. The function* $\tilde{I}_M : [0, V(M)[\to [0, +\infty[$ *defined by*

$$\tilde{I}_M(v) := \inf\{\mathcal{P}(\Omega) = A(\partial\Omega) : \Omega \in \tilde{\tau}_M, V(\Omega) = v\},$$

is called the ***isoperimetric profile function*** *(or shortly the* ***isoperimetric profile****) of the manifold* M*. If there exists a finite perimeter set* $\Omega \in \tilde{\tau}_M$ *satisfying* $V(\Omega) = v$, $\tilde{I}_M(V(\Omega)) = A(\partial\Omega) = \mathcal{P}(\Omega)$ *such an* Ω *will be called an* ***isoperimetric region****, and we say that* $\tilde{I}_M(v)$ *is* ***achieved****.*

There are are many others possible definitions of isoperimetric profile corresponding to the minimization over various different admissible sets, as stated in the following definition.

Definition 2.7.

$$I^*_M(v) := inf\{A(\partial_{top}\Omega) : \Omega \subset M, \partial_{top}\Omega \ is \ C^\infty, V(\Omega) = v\},$$
$$\tilde{I}^*_M(v) := inf\{\mathcal{P}_M(\Omega) : \Omega \subset M, \Omega \in \tilde{\tau}_M, V(\Omega) = v, diam(\Omega) < +\infty\},$$

where $diam(\Omega) := \sup\{d(x, y) : x, y \in \Omega\}$ *denotes the diameter of* Ω*.*

Remark 2. *Trivially one have* $I_M \geq I^*_M \geq \tilde{I}_M$ *and* $I_M \geq \tilde{I}^*_M \geq \tilde{I}_M$*.*

Theorem 2.3. *If* M^n *is complete then* $I_M(v) = \tilde{I}^*_M(v) = \tilde{I}_M(v) = I^*_M(v)$*.*

2.3 Weak and strong formulations are equivalents, i.e., proof of Theorem 2.3

Roughly speaking to prove Theorem 2.3 we make a construction which replace a finite perimeter set by one of the same volume with a small ball inside and one outside, by adding a small geodesic ball (with smooth boundary) to a point of density 0 and subtracting a small geodesic ball to a point of density 1 taking care of not altering the volume. This enables us to obtain again a finite perimeter set of the same volume with a perimeter that is a small perturbation of the original one and that in addition has the property that we can put inside and outside a small ball. This construction legitimate us to apply mutatis mutandis the arguments of the proof of Lemma 1 of [Mod87] to conclude the proof of Theorem 2.3. Our adapted version of Lemma 1 of [Mod87] is the following lemma. In what follows we denote by $\mathring{E}$ or by $Interior(E)$ the interior of E for every set E and $X^c := M \setminus X$.

Lemma 2.2. *Let $\Omega_1 \in \tilde{\tau}_M$ with $V(\Omega_1) < +\infty$, such that there exists two geodesic balls satisfying $B(x_1, r_1) \subset \Omega_1$ and $B(x_2, r_2) \cap \Omega_1 = \emptyset$, with $0 < r_1 < inj_M(x_1)$ and $0 < r_2 < inj_M(x_2)$. We set $v^* := \min\left\{V(B(x_1, \frac{r_1}{2})), V(B(x_2, \frac{r_2}{2}))\right\}$. For any $v \in [0, v^*]$ we denote by $R_{i,v}$ a radius such that $V(B(x_i, R_{i,v})) = v$ and by $S(x,r)$ the sphere of radius r and center x. Let us define*

$$f_{\Omega_1}(v) := \max\left\{\sup_{0\leq t\leq R_{1,v}} A(S(x_1,t)), \sup_{0\leq t\leq R_{2,v}} A(S(x_2,t))\right\}. \qquad (6)$$

Then for any $\varepsilon > 0$ and any $v \in]V(\Omega_1) - v^*, V(\Omega_1) + v^*[$, there exists $\Omega_2 \in \tau_M$ such that $V(\Omega_2) = v$ and

$$\mathcal{P}(\Omega_2) \leq \mathcal{P}(\Omega_1) + f_{\Omega_1}(|v - V(\Omega_1)|) + \frac{\varepsilon}{4}.$$

Remark 3. *We observe that if M is noncompact and Ω bounded, then we always have $Interior(\Omega^c) \neq \emptyset$.*

Proof:[of Lemma 2.2] By the proof of the claim p. 105 of [JPPP07], there exists a sequence of BV-functions (u_l) on M such that $\lim_l ||u_l - \chi_{\Omega_1}||_1 = 0$, $|Du_l|(M) = \mathcal{P}(\Omega_1)$ and each u_l has compact support K_l. Note that we can assume that $B(x_1, r_1) \subset K_l$. Moreover, construction the u_l satisfy $0 \leq u_l \leq \chi_{\Omega_1}$, which gives $K_l \subset \Omega_1$. Considering a smooth positive kernel ρ with compact support the mollified functions $u_{j,l} = u_l * \rho_{\frac{1}{j}}$ satisfy $0 \leq u_{j,l} \leq 1$, $\lim_{j\to+\infty} ||u_{j,l} - u_l||_1 = 0$,

$\lim_l |Du_{j,l}|(M) = |Du_l|(M)$ and for j large enough the support $K_{j,l}$ of $u_{j,l}$ satisfies $B(x_1, \frac{r}{2}) \cap K_{j,l} = \emptyset$.

Remark 4. *As explained in [JPPP07] to perform a convolution on a manifold one have just to use a partition of unity associated to finite sets of local charts covering the compact support of u_l and then mollify in each local chart.*

By a diagonal argument we extract a subsequence $v_l = u_{j,l}$, satisfying $0 \leq v_l \leq 1$, $\lim_l ||v_l - \chi_{\Omega_1}||_1 = 0$, $\lim_l |Dv_l|(M) = \mathcal{P}(\Omega_1)$, and for l large enough the support C_l of v_l satisfies $B(x_1, \frac{r_1}{2}) \subset C_l$ and $B(x_2, \frac{r_2}{2}) \cap C_l = \emptyset$. Putting $F_t^l := \{x \in M : v_l(x) > t\}$ and using the Fleming-Rishel Theorem (compare Theorem 4.3 of [AMP04]) we have

$$\mathcal{P}(\Omega_1) = \lim_l |Dv_l| = \lim_l \int_0^1 \mathcal{P}(F_t^l) dt \geq \int_0^1 \varliminf_l \mathcal{P}(F_t^l) dt.$$

An application of Sard's Theorem ensures that the sets F_t^l are smooth for almost every $t \in]0, 1[$. Thus for every l we can choose a $t \in]0, 1[$ (depending on l), such that $\varliminf_l \mathcal{P}(F_t^l) \leq \mathcal{P}(\Omega_1)$. Moreover, we have $|V(F_t^l) - V(\Omega_1)| \leq V(F_t^l \setminus \Omega_1) + V(\Omega_1 \setminus F_t^l)$ and

$$V(F_t^l \setminus \Omega_1) \leq \frac{1}{t} ||v_l - \chi_{\Omega_1}||_1,$$

$$V(\Omega_1 \setminus F_t^l) \leq \frac{1}{1-t} ||v_l - \chi_{\Omega_1}||_1.$$

Since we have $|v - V(\Omega_1)| < v_0$, we can choose l large enough to get

$$|v - V(\Omega_1)| + \frac{||v_l - \chi_{\Omega_1}||_1}{t(1-t)} < v^*,$$

which yields for l large enough $|V(F_t^l) - v| < v^*$. Hence by subtracting $B(x_1, R_{1,V(F_t^l)-v})$ or adding $B(x_2, R_{2,v-V(F_t^l)})$ to F_t^l, we obtain a bounded open set with smooth topological boundary and volume v and perimeter equal to

$$\mathcal{P}(F_t^l) + A(S(x_{i,l}, R_{i,l}) \leq \mathcal{P}(F_t^l) + f_{\Omega_1}(|v - V(F_t^l)|),$$

where $R_{i,l} := R_{2,v-V(F_t^l)}$ if $V(F_t^l) < v$ and $R_{i,l} := R_{1,V(F_t^l)-v}$, if $V(F_t^l) < v$ and $R_{i,l} = 0$ if $V(F_t^l) = v$ otherwise. We finally get Ω_2 for any l large enough and we conclude the proof. q.e.d.

Lemma 2.3. *Let $\Omega \in \tilde{\tau}_M$, bounded, $\Omega^o \neq \emptyset$, and $Interior(\Omega^c) \neq \emptyset$. Then there exists a sequence $\Omega_k \in \tau_M$ with $V(\Omega_k) = V(\Omega)$ which converges to Ω in the sense of finite perimeter sets.*

Remark 5. *We observe that if M is noncompact and Ω bounded, then we always have $Interior(\Omega^c) \neq \emptyset$.*

In connection with the original paper [Mod87], we want just to point out two things. First, Lemma 1 of [Mod87] is stated and proved in $\mathbb{R}^n$ but the proof generalizes immediately to complete Riemannian manifolds. The technical theorems needed to make this generalization are provided or are easily deducible from the paper [JPPP07] which extends the theory of BV-functions from $\mathbb{R}^n$ to the setting of Riemannian manifolds. Second the assumption that Ω and Ω^c have nonvoid interior cannot be dropped to make the proof of Lemma 1 of [Mod87] (and also Lemma 2.3) works. This is just a technical problem that we will solve in Lemma 2.6. Although the proof of Lemma 2.3 goes along the same lines of Lemma 1 of [Mod87], to make the paper self contained we write it here. But before let us define a crucial concept in Riemannian geometry that will constantly used, i.e., the injectivity radius.

Definition 2.8. *Let M be a Riemannian manifold, the* ***injectivity radius*** *of M, noted inj_M, is defined as follow*

$$inj_M := \inf_{p \in M}\{inj_{p,M}\},$$

where for every point $p \in M$, $inj_{p,M}$ is the injectivity radius at p of M, i.e., the largest radius r for which the exponential map $exp_p : B(0,r) \to B_M(p,r)$ is a diffeomorphism.

Proof:[Lemma 2.3] Take a bounded finite perimeter set Ω such that there exist $x_1 \in \Omega$, $x_2 \in M \setminus \Omega$, and $0 < r_0 < Min\{inj_{x_1}(M), inj_{x_2}(M)\}$ where for every $p \in M$, $inj_p(M)$ denotes the injectivity radius of M at p, with $B_1 := B_M(x_1, r_0) \subseteq \Omega$ and $B_2 := B_M(x_2, r_0) \subseteq M \setminus \Omega$ and consider its characteristic function χ_Ω. Consider the usual sequence of approximating functions $u_\varepsilon := \chi_\Omega \in C_c^\infty(M)$ as defined in Proposition 1.4 of [JPPP07]. By Proposition 1.4 of [JPPP07] we have that $u_\varepsilon \to \chi_\Omega$ in $L^1(M)$ topology, when $\varepsilon \to 0^+$, in particular we have

$$\lim_{\varepsilon \to 0^+} V\left(\{x \in M : |u_\varepsilon(x) - \chi_E(x)| \geq \eta\}\right) = 0, \forall \eta \geq 0. \tag{7}$$

Moreover, $u_\varepsilon \in \mathcal{C}_c^\infty(M)$ and $\mathcal{P}_M(\Omega) = \lim_{\varepsilon \to 0^+} \int_M |\nabla u_\varepsilon| dV$. For every $\eta > 0$ we can choose $0 < \varepsilon_\eta < Min\{\eta, \frac{r_0}{2}\}$ such that

$$V\left(\left\{x \in M : |u_{\varepsilon_\eta}(x) - \chi_E(x)| \geq \eta\right\}\right) \leq \eta.$$

Since $t \mapsto \mathcal{P}_M\left(\left\{x \in M : u_{\varepsilon_\eta}(x) > t\right\}\right)$ is a Lebesgue measurable function, we can define

$$A := \{\mathcal{P}_M\left(\left\{x \in M : u_{\varepsilon_\eta}(x) > t\right\}\right) : \eta \leq t \leq 1-\eta\},$$

and

$$\nu_\eta := essinf_{\eta \leq t \leq 1-\eta} \mathcal{P}_M\left(\left\{x \in M : u_{\varepsilon_\eta}(x) > t\right\}\right). \tag{8}$$

By the very definition of ν_η we know that for every $\eta' > 0$ we have $|[\nu_\eta, \nu_\eta + \eta'] \cap A| > 0$, thus using the Morse-Sard's theorem, i.e., Lemma 2.1 and (8) we get the existence of $t_\eta \in]\eta; 1-\eta[$ such that

$$\mathcal{P}_M\left(\left\{x \in M : u_{\varepsilon_\eta}(x) > t_\eta\right\}\right) < \nu_\eta + \eta, \tag{9}$$

t_η is a regular value of u_{ε_η}, i.e.,

$$\nabla u_{\varepsilon_\eta}(x) \neq 0,\ \forall x \in M : u_{\varepsilon_\eta} = t_\eta.$$

In view of this we can define $\Omega'_{\varepsilon_\eta} \in \tau_M$, $\Omega'_{\varepsilon_\eta} := u^{-1}_{\varepsilon_\eta}(]t_{\varepsilon_\eta}, +\infty[)$, this ensures that $\Omega'_{\varepsilon_\eta}$ is bounded if Ω is bounded, furthermore we have also that $\partial\Omega'_\varepsilon = \left\{x \in M : u_{\varepsilon_\eta}(x) = t_\eta\right\}$ is smooth, again by Lemma 2.1, and $\Omega'_{\varepsilon_\eta} \triangle \Omega \subset \left\{x \in M : |u_{\varepsilon_\eta}(x) - \chi_\Omega(x)| \geq \eta\right\}$. This last property joint with (7) imply

$$V(\Omega'_{\varepsilon_\eta} \triangle \Omega) \to 0, \tag{10}$$

which by the lower semicontinuity of the perimeter, gives

$$\mathcal{P}_M(\Omega) \leqslant \varliminf_{\eta \to 0^+} \mathcal{P}_M(\Omega'_{\varepsilon_\eta}).$$

For the converse inequality, we deduce from (9) that

$$\mathcal{P}_M(\Omega'_{\varepsilon_\eta}) \leqslant \eta + \nu_\eta \leqslant \eta + \mathcal{P}_M(\{x \in M : u_{\varepsilon_\eta} > t\}),$$

for every $\eta > 0$, and for almost all $t \in [\eta, 1-\eta]$, so, integrating over the interval $[\eta, 1-\eta]$ and applying the Fleming-Rishel formula (compare Theorem 2.1), we obtain

$$(1-2\eta)\mathcal{P}_M(\Omega_{\varepsilon_\eta}) \leqslant \eta(1-2\eta) + \int_M |\nabla u_{\varepsilon_\eta}| dV, \tag{11}$$

which combined with Proposition 2.2 yields

$$\overrightarrow{\lim}_{\eta \to 0^+} \mathcal{P}_M(\Omega_{\varepsilon_\eta}) \leqslant \mathcal{P}_M(\Omega).$$

Therefore we have proved that corresponding to every sequence $\eta_k \to 0^+$, there exists a sequence $\Omega'_k \in \tau_M$ such that Ω'_k converges to Ω in the sense of finite perimeter sets. As it is easy to check for every k large enough $B_1 \subseteq \Omega'_k$ and $B_2 \subseteq M \setminus \Omega'_k$. Set $\delta_k := V(\Omega) - V(\Omega'_k)$ and take k large enough to ensure that $Min\{V(B_1), V(B_2)\} > |\delta_k|$. Now we choose $\Omega_k := \Omega'_k \mathring{\cup} B(x_1, r_k)$, where $V(B(x_1, r_k)) = |\delta_k|$, if $\delta_k > 0$, and $\Omega_k := \Omega'_k \setminus B(x_2, r_k)$, where $V(B(x_2, r_k)) = |\delta_k|$, if $\delta_k < 0$, and finally $\Omega_k := \Omega'_k$, if $\delta_k = 0$. Using the fact that $\delta_k \to 0$ we see that also $A(\partial B(x_i, r_k)) \to 0$. It is straightforward to verify that $V(\Omega_k) = V(\Omega)$, $\partial\Omega_k$ is C^∞, Ω_k is still bounded and

$$V(\Omega_k \Delta \Omega'_k) \leq |\delta_k| \to 0,\ k \to +\infty,$$

$$|\mathcal{P}(\Omega_k) - \mathcal{P}(\Omega'_k)| \leq A(\partial B(x_i, r_k)) \to 0,\ k \to +\infty.$$

From the last properties it follows easily that the sequence (Ω_k) converges to Ω in the sense of finite perimeter sets, and the lemma follows. q.e.d.

We list here some lemmas that will be used in the proof of Lemma 2.6.

Lemma 2.4. *Let $\Omega \subset M$ be any measurable set , then for all $J_k := (k, 2k+1) \subset (0, +\infty)$, $k \in \mathbb{N}$, there exists $r_k \in (k, 2k+1)$ such that*

$$\mathcal{H}^{n-1}(\Omega \cap \partial B_M(x, r_k)) \leqslant \frac{V(\Omega)}{k},$$

where $x \in M$ is being taken fixed.

Proof: By coarea formula we know that

$$V(\Omega) = \int_0^\infty \mathcal{H}^{n-1}(\Omega \cap \partial B(x, r))dr,$$

where x is any fixed point in M.

We affirm that given $k \in \mathbb{N}$, there exists $r_k \in (k, 2k+1)$ such that

$$\mathcal{H}^{n-1}(\Omega \cap \partial B(x, r_k)) \leqslant \frac{V(\Omega)}{k},$$

because otherwise we would have

$$V(\Omega) \geqslant \int_k^{2k+1} \mathcal{H}^{n-1}(\Omega \cap \partial B(x, r))dr > \frac{(k+1)V(\Omega)}{k},$$

which is a contradiction. q.e.d.

Remark 6. *See that when $r_k \to \infty$, it holds*

$$V(\Omega \cap B(x, r_k)) \to V(\Omega), \quad k \to \infty.$$

Lemma 2.5. *Let M be a Riemannian manifold and $\Omega \in \tilde{\tau}_M$ be a set of finite perimeter with finite volume $V(\Omega) \in]0, V(M)[$. For any $\varepsilon > 0$, there exists a set of finite perimeter $\tilde{\Omega} \subseteq M$ and two geodesic balls $B(x_1, r_1)$, and $B(x_2, r_2)$ such that $V(\Omega) = V(\tilde{\Omega})$, $B(x_1, r_1) \subset \Omega_1$, $B(x_2, r_2) \cap \tilde{\Omega} = \emptyset$, and*

$$\mathcal{P}(\tilde{\Omega}) \leq \mathcal{P}(\Omega) + \frac{\varepsilon}{4}. \tag{12}$$

Proof: Consider an arbitrary set $\Omega \in \tilde{\tau}_M$ and take two distinct points $x_1 \in \Omega$ and $x_2 \in \Omega^c$ of density $\Theta(x_1, V \llcorner \Omega) = 1$ and $\Theta(x_2, V \llcorner \Omega) = 0$, where $\Theta(p, V \llcorner \Omega) := \lim_{r \to 0^+} \frac{V(\Omega \cap B(p,r))}{\omega_n r^n}$, for every $p \in M$. By ω_n we denote the volume of the ball of radius 1 in $\mathbb{R}^n$. Consider the two continuous functions $f_1, f_2 : I \to \mathbb{R}$, where $I := [0, r_0[$ such that $f_1(r) := V(\Omega \cap B_M(x_1, r))$, $f_2(r) := V(\Omega^c \cap B_M(x_2, r))$. The radius r_0 could be chosen small enough to have $B_M(x_1, r_1) \cap B_M(x_2, r_2) = \emptyset$ for every $r_1, r_2 \in I$ and such that there exist $r_1, r_2 \in I$ satisfying the property $f_1(r_1) = f_2(r_2)$ and $\partial B_M(x_1, r_1), \partial B_M(x_2, r_2)$ smooths (for this last property it is enough to take r_0 less than the injectivity radius at x_1 and x_2). Then we set

$$\tilde{\Omega} := [\Omega \setminus B_M(x_1, r_1)] \mathring{\cup} [\Omega^c \cap B_M(x_2, r_2)] = [\Omega \setminus B_M(x_1, r_1)] \cup B_M(x_2, r_2).$$

As it is easy to see $V(\tilde{\Omega}) = V(\Omega)$,

$$|\mathcal{P}(\tilde{\Omega}) - \mathcal{P}(\Omega)| \leq \sum_{i=1}^{2} [A(\partial B_M(x_i, r_i)) + \mathcal{P}(\Omega, B_M(x_i, r_i))], \tag{13}$$

$$V(\Omega \Delta \tilde{\Omega}) = f_1(r_1) + f_2(r_2), \tag{14}$$

$\mathring{\tilde{\Omega}} \neq \emptyset$, and $Interior(\tilde{\Omega}^c) \neq \emptyset$. It is straightforward to verify that the right hand sides of (13) and (14) converge to zero when the radii r_1 and r_2 go to zero and the theorem easily follows. q.e.d.

As an easy consequence of Lemmas 2.2 and 2.5 we have the following isovolumic approximation lemma.

Lemma 2.6. *Let $\Omega \in \tilde{\tau}_M$ be a finite perimeter set with $V(\Omega) < +\infty$, $V(\Omega), V(\Omega^c) > 0$, where $\Omega^c := M \setminus \Omega$. Then there exists a sequence $\Omega_k \in \tau_M$ such that $V(\Omega_k) = V(\Omega)$ and Ω_k converges to Ω in the sense of finite perimeter sets.*

In the proof of this Lemma we really differ from the paper [Mod87], even if we make a crucial use of Lemma 1 of that paper.

Proof:

We prove the lemma first for bounded sets $\Omega \in \tilde{\tau}_M$, and then we pass to the general case by observing that one can always approximate an unbounded $\Omega \in \tilde{\tau}_M$ in the sense of finite perimeter sets by a sequence of bounded ones. Let us assume that $\Omega \in \tilde{\tau}_M$ is bounded, then for any arbitrary $\varepsilon > 0$, the Riemannian version of Lemma 1 of [Mod87], namely Lemma 2.3 applied to $\tilde{\Omega}$ permits to find $\tilde{\Omega}_\varepsilon \in \tau_M$ such that $V(\tilde{\Omega}_\varepsilon) = V(\tilde{\Omega}) = V(\Omega)$ and

$$V(\tilde{\Omega}_\varepsilon \Delta \tilde{\Omega}) \leq \frac{\varepsilon}{2},$$

$$|\mathcal{P}(\tilde{\Omega}_\varepsilon) - \mathcal{P}(\tilde{\Omega})| \leq \frac{\varepsilon}{2}.$$

These last two inequalities combined with (13) and (14) imply that

$$V(\tilde{\Omega}_\varepsilon \Delta \Omega) \leq \varepsilon, \tag{15}$$

$$|\mathcal{P}(\tilde{\Omega}_\varepsilon) - \mathcal{P}(\Omega)| \leq \varepsilon. \tag{16}$$

To finish the proof we consider now the case of an unbounded $\Omega \in \tilde{\tau}_M$, with $V(\Omega) = v < +\infty$. Fix a point $p \in M$, a fine use of the coarea formula as explained in Lemma 2.4 gives a sequence of radio $r_k \to \infty$, $r_k \geqslant k$, such that whenever $B(p, r_k) \cap \Omega =: \Omega_k$ we have

$$\lim_{k \to \infty} \mathcal{P}(\Omega_k) = \mathcal{P}(\Omega),$$

because

$$\mathcal{P}(\Omega_k) \leqslant \mathcal{P}(\Omega, B(p, r_k)) + \frac{V(\Omega)}{k},$$

which after taking limits leads to

$$\overrightarrow{\lim} \mathcal{P}(\Omega_k) \leqslant \mathcal{P}(\Omega).$$

Furthermore from $V(\Omega \bigtriangleup \Omega_k) \to 0$ and the lower semicontinuity of the perimeter we get $\underrightarrow{\lim} \mathcal{P}(\Omega_k) \geqslant \mathcal{P}(\Omega)$. It remains to readjust the volumes of these Ω_k's and we do it by perturbing Ω_k in adding a small geodesic ball $B_M(p_1, r'_k)$ such that $V(B_M(p_1, r'_k) \cap \Omega^c) = v - v_k$, with $v_k = V(\Omega_k)$, centered at a fixed point p_1 of density $\Theta(p_1, V \llcorner \Omega) = 0$, with k sufficiently large. It is worth to note that as above (see Lemma 2.3) $r'_k \to 0$, when $k \to \infty$. This construction gives sets $\Omega'_k \in \tilde{\tau}_M$, such that $V(\Omega'_k) = v = V(\Omega)$, Ω'_k is bounded,

$$V(\Omega_k \Delta \Omega'_k) = v - v_k, \tag{17}$$

$$|\mathcal{P}(\Omega_k^{'}) - \mathcal{P}(\Omega_k)| \leq \mathcal{P}(\Omega, B_M(p_1, r'_k)) + A(\partial B_M(p_1, r'_k)). \tag{18}$$

Therefore $(\Omega'_k)_{k\in\mathbb{N}}$ converges to Ω in the sense of sets of finite perimeter, because the right hand sides of (17) and (18) go to 0, when $k \to +\infty$. Unfortunately the sets Ω'_k are still not open with smooth boundary. Hence we have to continue our construction to achieve the proof. To do so consider any given sequence of positive numbers $\varepsilon_k \to 0$ the fact that Ω'_k is bounded allow us, as above in Lemma 2.6, to find $\Omega''_k \in \tau_M$ such that $V(\Omega''_k) = V(\Omega'_k) = V(\Omega) = v$

$$V(\Omega'_k \Delta \Omega''_k) \leq \varepsilon_k, \tag{19}$$

$$|\mathcal{P}(\Omega'_k) - \mathcal{P}(\Omega''_k)| \leq \varepsilon_k. \tag{20}$$

Since the sequence $(\Omega'_k)_{k\in\mathbb{N}}$ converges to Ω in the sense of sets of finite perimeter, the last two equations ensures that the sequence $\Omega''_k \in \tau_M$ converges to Ω in the sense of finite perimeter sets too, which is our claim. q.e.d.

Now we are ready to prove easily Theorem 2.3.

Proof:[of Theorem 2.3] Taking into account Remark 2, to show the theorem, it is enough to prove the nontrivial inequality $I_M(v) \leq \tilde{I}_M(v)$ for every $v \in [0, +\infty[$. To this aim, let us consider $\varepsilon > 0$ and $\Omega \in \tilde{\tau}_M$, with $V(\Omega) = v$. By Lemma 2.6 there is a sequence $\Omega_k \in \tau_M$ such that $V(\Omega_k) = v$, (Ω_k) converging to Ω in the sense of finite perimeter sets. In particular we have that $\lim_{k\to+\infty} \mathcal{P}(\Omega_k) = \mathcal{P}(\Omega)$. On the other hand by definition we have that $I_M(v) \leq \mathcal{P}(\Omega_k)$ for every $k \in \mathbb{N}$. Passing to limits leads to have

$$I_M(v) \leq \mathcal{P}(\Omega), \tag{21}$$

for every $\Omega \in \tilde{\tau}_M$ with $V(\Omega) = v$. Taking the infimum in (21) when Ω runs over $\tilde{\tau}_M$ keeping $V(\Omega)$ fixed and equal to v, we infer that $I_M(v) \leq \tilde{I}_M(v)$. This completes the proof. q.e.d.

As a result of this we can get another proof of the theorem 4.1.

Proof:[of Theorem 4.1] In view of Theorem 2.3 we actually prove that $\tilde{I}_M$ is upper semicontinuous. For any $v \in]0, V(M)[$ and any $\varepsilon > 0$, consider a finite perimeter set Ω such that $V(\Omega) = v$ and $\mathcal{P}(\Omega) \leq \frac{\varepsilon}{4}$. We then apply Lemma 2.5 to it, which gives us Ω_1 such that $V(\Omega_1) = v$, $\mathcal{P}(\Omega_1) \leq \tilde{I}_M(v) + \frac{\varepsilon}{2}$, and a $\bar{v} = \bar{v}_{\Omega_{1,\varepsilon}}$ such that for any $w \in]v - \bar{v}, v + \bar{v}[$ there exists $\Omega_2 \in \tilde{\tau}_M$ satisfying $V(\Omega_2) = w$ and $\mathcal{P}(\Omega_2) \leq I_M(v) + f(|w - v|) + \frac{3\varepsilon}{4}$, where f is given by (6). By the very definition of isoperimetric profile we have immediately that

$$I_M(w) \leq I_M(v) + f(|w - v|) + \frac{3\varepsilon}{4}.$$

Now, the function f depends only on Ω_1, satisfies $f(0) = 0$ and is continuous at 0. So there exists $v_1 \in]0, \bar{v}[$ such that $f(|w - v|) \leq \frac{\varepsilon}{4}$ for every $w \in]v - v_1, v + v_1[$, which gives the upper semicontinuity in v. By the arbitrariness of v the corollary readily follows. q.e.d.

Remark 7. *When $\tilde{I}_M$ is continuous, is a trivial task to show that $I_M = \tilde{I}_M$ using just the classical approximation given by Proposition* 1.4 *of [JPPP07].*

3 Generalized Compactness

The first result that we present in this section is Theorem 1, which provides a generalized compactness result for isoperimetric regions in a noncompact Riemannian manifold satisfying the condition of bounded geometry. In the general case, limit finite perimeter sets do not exist in the original ambient manifold, but rather in the disjoint union of an at most countable family of pointed limit manifolds $M_{1,\infty}, \ldots, M_{N,\infty}$, obtained as limit of N sequences of pointed manifolds $(M, p_{ij}, g)_j$, $i \in \{1, \ldots, N\}$, here N is allowed to be ∞. When we start with a minimizing sequence, then $N < +\infty$ and we recover Theorem 1 of [Nar14a]. To prove Theorem 1, we get a decomposition lemma (Lemma 2) for the thick part of a subsequence of an arbitrary sequence with uniformly bounded volume and area, but that is not contained in a fixed ball. Lemma 2 is interesting for himself.

Now, let us recall the basic definitions from the theory of convergence of manifolds, as exposed in [Pet06]. This will help us to state the main result in a precise way.

Definition 3.1. *For any $m \in \mathbb{N}$, $\alpha \in [0,1]$, a sequence of pointed smooth complete Riemannian manifolds is said to* ***converge in the pointed $C^{m,\alpha}$, respectively C^m topology to a smooth manifold*** *M (denoted $(M_i, p_i, g_i) \to (M, p, g)$), if for every $R > 0$ we can find a domain Ω_R with $B(p, R) \subseteq \Omega_R \subseteq M$, a natural number $\nu_R \in \mathbb{N}$, and C^{m+1} embeddings $F_{i,R} : \Omega_R \to M_i$, for large $i \geq \nu_R$ such that $B(p_i, R) \subseteq F_{i,R}(\Omega_R)$ and $F^*_{i,R}(g_i) \to g$ on Ω_R in the $C^{m,\alpha}$, respectively C^m topology.*

It is easy to see that this type of convergence implies pointed Gromov-Hausdorff convergence. When all manifolds in question are closed, the maps F_i are global C^{m+1} diffeomorphisms. So for closed manifolds we can speak about unpointed convergence. What follows is the precise definition of **$C^{m,\alpha}$-norm at scale** r, which can be taken as a possible definition of bounded geometry.

Definition 3.2 ([Pet06])**.** *A subset A of a Riemannian n-manifold M has* ***bounded $C^{m,\alpha}$ norm on the scale of*** *r, $||A||_{C^{m,\alpha},r} \leq Q$, if every point p of M lies in an open set U with a chart ψ from the Euclidean r-ball into U such that*

- *(i): For all $p \in A$ there exists U such that $B(p, \frac{1}{10}e^{-Q}r) \subseteq U$.*
- *(ii): $|D\psi| \leq e^Q$ on $B(0, r)$ and $|D\psi^{-1}| \leq e^Q$ on U.*
- *(iii): $r^{|j|+\alpha}||D^j g||_\alpha \leq Q$ for all multi indices j with $0 \leq |j| \leq m$, where g is the matrix of functions of metric coefficients in the ψ coordinates regarded as a matrix on $B(0, r)$.*

We write that $(M, g, p) \in \mathcal{M}^{m,\alpha}(n, Q, r)$, if $||M||_{C^{m,\alpha},r} \leq Q$.

In the sequel, unless otherwise specified, we will make use of the technical assumption on $(M, g, p) \in \mathcal{M}^{m,\alpha}(n, Q, r)$ that $n \geq 2$, $r, Q > 0$, $m \geq 1$, $\alpha \in]0, 1]$. Roughly speaking, $r > 0$ is a positive lower bound on the injectivity radius of M, i.e., $inj_M > C(n, Q, \alpha, r) > 0$.

Definition 3.3. *A complete Riemannian manifold (M, g), is said to have* ***bounded geometry*** *if there exists a constant $k \in \mathbb{R}$, such that $Ric_M \geq k(n-1)$ (i.e., $Ric_M \geq k(n-1)g$ in the sense of quadratic forms) and $V(B_{(M,g)}(p, 1)) \geq v_0$ for some positive constant v_0, where $B_{(M,g)}(p, r)$ is the geodesic ball (or equivalently the metric ball) of M centered at p and of radius $r > 0$.*

Remark 8. *In general, a lower bound on Ric_M and on the volume of unit balls does not ensure that the pointed limit metric spaces at infinity are still manifolds.*

Remark 9. *We observe here that Definition 3.4 is weaker than Definition 3.2. In fact, using Theorem* 72 *of [Pet06], one can show that if a manifold M has bounded $C^{m,\alpha}$ norm on the scale of r for $\alpha > 0$ in the sense of Definition 3.2 then M has $C^{m,\beta}$-locally asymptotic bounded geometry in the sense of Definition 3.4, for every $0 < \beta < \alpha$, while in general the converse is not true.*

This motivates the following definition, that is suitable for most applications to general relativity see for example [FN15a].

Definition 3.4. *We say that a smooth Riemannian manifold (M^n, g) has $C^{m,\alpha}$-**locally asymptotic bounded geometry** if it is of bounded geometry and if for every diverging sequence of points (p_j), there exist a subsequence (p_{j_l}) and a pointed smooth manifold $(M_\infty, g_\infty, p_\infty)$ with g_∞ of class $C^{m,\alpha}$ such that the sequence of pointed manifolds $(M, p_{j_l}, g) \to (M_\infty, g_\infty, p_\infty)$, in $C^{m,\alpha}$-topology.*

For a more detailed discussion about this point the reader could find useful to consult [Nar14a].

Definition 3.5 (Multipointed Gromov-Hausdorff convergence)**.** *We say that a sequence of multipointed proper metric spaces $(X_i, d_i, p_{i,1}, ..., p_{i,l}, ...)$ converges to the multipointed metric space $(X_\infty, d_\infty, p_{\infty,1}, ..., p_{\infty,l}, ...)$, in the **multipointed Gromov-Hausdorff topology**, if for every j we have*

$$(X_i, d_i, p_{i,j}) \to (X_\infty, d_\infty, p_{\infty,j}),$$

in the pointed Gromov-Hausdorff topology.

Definition 3.6 (Multipointed C^0-convergence)**.** *We say that a sequence of multipointed Riemannian manifolds $(M_i, g_i, p_{i,1}, ..., p_{i,l}, ...)$ converges to the multipointed Riemannian manifold $(M_\infty, g_\infty, p_{\infty,1}, ..., p_{\infty,l}, ...)$, in the **multipointed C^0-topology**, if for every j we have*

$$(M_i, d_i, p_{i,j}) \to (M_\infty, d_\infty, p_{\infty,j}),$$

in the pointed C^0-pointed topology.

Remark 10. *Multipointed C^0-convergence is stronger than Multipointed Gromov-Hausdorff convergence.*

Remark 11. *The perimeter is lower semicontinuous with respect to multipointed C^0-topology. The volume is continuous with respect to multipointed Gromov-Hausdorff topology. This last assertion about volumes is a deep result due to Tobias Colding [Col97].*

At this point we recall the classical compactness theorem of the theory of finite perimeter sets that will be generalized in Theorem 1.

Theorem 3.1. *Let M be a compact Riemannian manifold. Let (Ω_i) be a sequence of finite perimeter sets such that there exists a positive constant $C > 0$, satisfying $V(\Omega_i) + \mathcal{P}(\partial\Omega_i) \leq C$, $\Omega_i \subseteq B$, where B is a fixed large ball. Then there exists Ω such that Ω_i tends to Ω, in $L^1(M)$ topology.*

Theorem 1 (Main: Generalized Compactness)**.** *Let M^n be a complete Riemannian manifold with C^0-bounded geometry. Let (Ω_j) be a sequence of finite perimeter sets with $\mathcal{P}(\Omega_j) \leq A$ and $V(\Omega_j) \leq v$. Then there exist a subsequence (Ω_{j_k}) that we rename by (Ω_k), $k \in S \subseteq \mathbb{N}$, a double sequence of points $p_{ik} \in M^n$, a finite perimeter set $\Omega \subseteq \tilde{M}$ and a sequence $(p_{\infty,i})$ of points of $\tilde{M}$ such that $(\Omega_k, (p_{ik})) \to (\Omega, (p_{\infty,i}))$ in the multipointed C^0-topology.*

Corollary 1 (Generalized existence [Nar14a])**.** *Let M have C^0-locally asymptotically bounded geometry. Given a positive volume $0 < v < V(M)$, there are a finite number N, of limit manifolds at infinity such that their disjoint union with M contains an isoperimetric region of volume v and perimeter $I_M(v)$. Moreover, the number of limit manifolds is at worst linear in v. Indeed $N \leq \left[\frac{v}{v^*}\right] + 1 = l(n, k, v_0, v)$, where v^* is as in Lemma 3.2 of [Heb00].*

Remark 12. *Observe that if $(M, g, p) \in \mathcal{M}^{m,\alpha}(n, Q, r)$ for every $p \in M$, then M have C^0-bounded geometry. So Theorem 1 and of course Corollary 1 applies to pointed manifolds $M \in \mathcal{M}^{m,\alpha}(n, Q, r)$, hence a posteriori also to manifolds with bounded (from above and below at the same time) sectional curvature and positive injectivity radius.*

Now we come back to the main interest of our theory, i.e., to extend arguments valid for compact manifolds to noncompact ones. To this aim let us introduce the following definition suggested by Theorem **??**.

Definition 3.7. *Let M be a C^0-locally asymptotic bounded geometry Riemannian manifolds. We call $D_\infty = \bigcup_i D_{\infty,i}$ a finite perimeter set in $\tilde{M}$ a **generalized set of finite perimeter of** $\tilde{M}$ and an isoperimetric region of $\tilde{M}$ a **generalized isoperimetric region**, where $\tilde{M} := \{(\mathcal{N}, q, g_{\mathcal{N}}) : \exists (p_j), p_j \in M, p_j \to +\infty, (M, p_j, g) \to (\mathcal{N}, q, g_{\mathcal{N}})\}$.*

Remark 13. *We remark that D_∞ is a finite perimeter set in volume v in $\mathring{\bigcup}_i M_{\infty,i}$.*

Remark 14. *If D is a genuine isoperimetric region contained in M, then D is also a generalized isoperimetric region with $N = 1$ and*

$$(M_{\infty,1}, g_{\infty,1}) = (M, g).$$

This does not prevent the existence of another generalized isoperimetric region of the same volume having more than one piece at infinity.

3.1 Proof of Theorem 1

The general strategy used in calculus of variations to understand the structure of solutions of a variational problem in a noncompact ambient manifold is the Concentration-Compactness principle of P.L. Lions. This principle suggests an investigation of regions in the manifold where volume concentrates. For the aims of the proof, this point of view it is not strictly necessary. But we prefer this language because it points the way to further applications of the theory developed here for more general geometric variational problems, PDE's, and the Calculus of Variations.

Lemma 3.1. *(Concentration-Compactness Lemma, [Lio84] Lemma I.1) Let M be a complete Riemannian manifold. Let μ_j be a sequence of Borel measures on M with $\mu_j(M) \to v$. Then there is a subsequence (μ_j) such that only one of the following three conditions holds*

(I): (Concentration) There exists a sequence $p_j \in M$ such that for any $\frac{v}{2} > \varepsilon > 0$ there is a radius $R > 0$ with the property that

$$\mu_j(B(p_j, R)) > v - \varepsilon, \tag{22}$$

(II): (Vanishing) For all $R > 0$ there holds

$$\lim_{j\to+\infty} Sup_{p\in M} \{\mu_j(B(p, R))\} = 0, \tag{23}$$

(III): (Dichotomy) There exists a number v_1, $0 < v_1 < v$ and a sequence of points (p_j) such that for any $0 < \varepsilon < \frac{v_1}{4}$ there is a number $R = R_{\varepsilon,v_1} > 0$ and two non-negative measures $\mu^1_{j,\varepsilon}$, $\mu^2_{j,\varepsilon}$ with the property that for every $R' > R$ and every strictly increasing sequence (K_j) tending to $+\infty$ there exists $j_{R'}$ s.t. for all $j \geq j_{R'}$,

$$0 \leq \mu^1_j + \mu^2_j \leq \mu_j, \tag{24}$$
$$Supp(\mu^1_j) \subseteq B(p_j, R) for\ all\ j, \tag{25}$$
$$Supp(\mu^2_j) \subseteq M - B(p_j, R'), \tag{26}$$
$$|\mu_j(B(p_j, R)) - v_1| \leq \varepsilon, \tag{27}$$
$$|\mu^2_j(M) - (v - v_1)| \leq \varepsilon, \tag{28}$$
$$dist(Supp(\mu^2_j), Supp(\mu^1_j)) \geq K_j. \tag{29}$$

Proof: Let us define below, the functions of concentration Q_j of Paul Lévy are defined below. This notion serves to locate points at which volumes $v_1, ..., v_N$ concentrate which are optimal in a certain sense. We define $Q_j : [0, +\infty[\to [0, v]$ by

$$Q_j(R) := Sup_{p\in M}\{\mu_j(B(p, R))\}.$$

(Q_j) is uniformly bounded in $L^1_{loc}([0, +\infty[)$ with respect to j, so there exists $Q \in L^1_{loc}$ such that there is a subsequence (Q_j) having j within $S_1 \subseteq \mathbb{N}$ such that $Q_j \to Q(R)$ pointwise a.e. $[0, +\infty[$. Since the functions Q_j are monotone increasing, so is Q. This ensures that the set of points of discontinuity of Q is a countable set. Completing Q by continuity from the left, we indeed obtain a lower semicontinuous function $Q : [0, +\infty[\to [0, v]$. It is easy to check by the theorem of existence of limit of monotone functions that there exists $v_1 \in [0, v]$ such that

$$\lim_{R\to+\infty} Q(R) = v_1 \in [0, v]. \tag{30}$$

Now, only three cases are possible: evanescence, dichotomy, and concentration. If $v_1 = 0$ we have evanescence, if $v_1 = v$ we have concentration, and if $v_1 \in]0, v[$ we have dichotomy. Let us to explain how one can deduce (I)-(III) from (30). The cases $v_1 = 0$ and $v_1 = v$ are treated exactly in the same manner as in [Lio84], and we improve slightly the conclusion in the case of dichotomy.

If $v_1 \in]0, v[$ then (30) is equivalent to saying that for every $\varepsilon > 0$ there is $R_\varepsilon > 0$ such that for all $R' > R_\varepsilon$ we have

$$v_1 - \varepsilon < Q(R') < v_1 + \varepsilon, \tag{31}$$

$$v_1 - \varepsilon < Q_j(R') < v_1 + \varepsilon, \tag{32}$$

for large j. From (31) for every fixed $R > R_\varepsilon$ we get the existence of a sequence of points p_{1j} (depending on ε and R) with the property that

$$v_1 - \varepsilon < \mu_j(B(p_{1j}, R)) < v_1 + \varepsilon, \tag{33}$$

for every $j \geq j_{\varepsilon,R}$. Equation (33) is not quite what is needed for our arguments, and we must improve it to obtain exactly (III). This can be done by observing that if ε is sufficiently small (e.g. smaller than a constant depending on v_1), then we can make the sequence p_{1j} independent of ε. Following this heuristic argument, taking $\varepsilon < b_1 := \frac{1}{4}v_1$, $R_0 > 0$ such that $Q(R_0) > \frac{3}{4}v_1$, there exist $p_{1j} \in M$ for which (33) holds with R replaced by R_0. Next, take $R > 0$ such that $Q(R) > v_1 - \varepsilon$, so that for sufficiently large j there exists a second sequence of points $p'_{1j} \in M$ for which (33) holds, and hence

$$Q_j(R) + Q_j(R_0) \geq \frac{3}{4}v_1 + v_1 - \frac{1}{4}v_1 = \frac{3}{2}v_1 > \frac{5}{4}v_1 > v_1 + \varepsilon.$$

This implies that $B(p_{1j}, R_0) \cap B(p'_{1j}, R) \neq \emptyset$ for sufficiently large j. Thus we have

$$v_1 - \varepsilon < \mu_j(B(p_{1j}, R_0 + 2R)) \leq Q_j(R_0 + 2R) < v_1 + \varepsilon,$$

where the last inequality becomes obvious after replacing R' with $R_0 + 2R$ in (32). This proves (III) with $R_{1,v_1,\varepsilon} = R_0 + 2R$. q.e.d.

This lemma will be used in our problem, taking measures μ_j having densities χ_{D_j}, where χ_{D_j} is the characteristic function of D_j for an almost minimizing sequence (D_j) defined below.

Definition 3.8. *We say that $(D_j)_j \subseteq \tilde{\tau}_M$ (see Defn. 2.6) is an **almost minimizing sequence** in volume $v > 0$ if*

(i): $V(D_j) \to v$,

(ii): $A(\partial D_j) \to I_M(v)$.

The following two Lemmas, 3.5 and 3.6, are inspired by [Mor94] Lemma 4.2 and [LR03] Lemma 3.1. By virtue of these we can avoid the evanescence case of Concentration-Compactness Lemma 3.1. The difference in our treatment here is essentially in two minor changes: bounding the number of overlapping balls (which we called the multiplicity m of the covering used in the proofs), and the Riemannian relative isoperimetric inequality. Both arguments use only our bounded geometry assumption, as it appears in Definition 3.3.

Lemma 3.2. *[Doubling property][Heb00] Let (M, g) be a complete Riemannian manifold with $Ric \geq kg$. Then for all $0 < r < R$ we have*

$$V(B(p,R)) \leq e^{\sqrt{(n-1)|k|}R} \left(\frac{R}{r}\right)^n V(B(p,r)). \tag{34}$$

Proof: The proof follows easily from the strong form of the Bishop-Gromov theorem, the fact that $V_{\lambda^2 g}(B_M(x,R))) = \lambda^2 V_g(B_M(x, \frac{R}{\lambda}))$, and the following inequalities

$$\alpha_n r^{(n-1)} \leq V(B_{\mathbb{M}^n_k}) = \alpha_n \int_0^r sinh(s)^{(n-1)} ds \leq \alpha_n r^{(n-1)} e^{r(n-1)},$$

via a conformal change of the metric. See [Heb00]. q.e.d.

Corollary 3.1. *Let M^n be a complete Riemannian manifold with with $Ric \geq kg$ and $V(B(p,1) \geq v_0$. Then for each $r > 0$ there exist $c_1 = c_1(n,k,r) > 0$ such that $V(B(p,r)) > c_1(n,k,r)v_0$.*

Proof: If $r \geq 1$ then $V(B(p,r)) \geq V(B(p,1)) \geq v_0$. If $r < 1$ then (34) holds with $R = 1$, hence

$$v_0 \leq V(B(p,1)) \leq e^{\sqrt{(n-1)|k|}} \left(\frac{1}{r}\right)^n V(B(p,r)). \tag{35}$$

Therefore

$$V(B(p,r)) \geq c_1(n,k,r)v_0, \tag{36}$$

where $c_1(n,k,r) = Min\left\{\frac{r^n}{e^{\sqrt{(n-1)|k|}}}, 1\right\}$. q.e.d.

Lemma 3.3. *(Covering Lemma [Heb00], Lemma 1.1) Let (M,g) be a complete Riemannian manifold with Ricci $\geq kg$, $k \leq 0$ and let $\rho > 0$ be given. There exists a sequence of points $(x_j) \in M$ such that for any $r \geq \rho$ the following three conditions are satisfied.*

(i): $M \subset \cup_j B(x_j, r)$;

(ii): $B(x_j, \frac{\rho}{2}) \cap B(x_i, \frac{\rho}{2}) = \emptyset$;

(iii): if $N \in \mathbb{N} \setminus \{0\}$ is defined as

$$N := Max\{l | \exists\, j_1, ..., j_l, \, satisfying \, \cap_{i=1}^{l} B(x_{j_i}, \frac{\rho}{2}) \neq \emptyset\},$$

then there exists a constant $N_1 = N_1(n,k,\rho,r) > 0$, s.t. $N \leq N_1$.

Proof: See Hebey [Heb00] Lemma 1.1 q.e.d.

Lemma 3.4. *(Relative isoperimetric inequality [MC95] corollary 1.2) Let M^n be a connected complete Riemannian manifold with Ricci $\geq kg$, $k \leq 0$. Then for every geodesic ball $B = B(p,r)$ and domain $D \subset M$ with smooth boundary ∂D such that $V(D \cap B) \leq \frac{V(B)}{2}$, there exists $c(n)$ depending only on n such that*

$$V(D \cap B)^{\frac{n-1}{n}} \leq e^{c(n)(1+\sqrt{|k|}r)} r V(B(p,r))^{-\frac{1}{n}} A(\partial D \cap B). \tag{37}$$

for all $p \in M$ and $r > 0$. In particular, if $r = 1$ and M has bounded geometry, then

$$V(D \cap B)^{\frac{n-1}{n}} \leq e^{c(n)(1+\sqrt{|k|})} v_0^{-\frac{1}{n}} A(\partial D \cap B). \tag{38}$$

Lemma 3.5. *(Non-evanescence I) Let M^n be a Riemannian manifold with bounded geometry (Defn. 3.3). Given a radius $r > 0$, there are positive constants $c_3 = c_3(n, k, v_0, r) > 0$ and $w = w(n, k, v_0, r) > 0$ such that for any set D of finite perimeter A and of volume v, there is a point $p \in M$ such that*

$$V(B(p, r) \cap D) \geq m_0' = m_0'(n, k, r, v_0, v), \tag{39}$$

where $m_0' := Min\left\{w, c_3 \frac{v^n}{A^n+1}\right\}$.

Proof: Fix $r > 0$. If for some point $p \in M$ one

$$V(D \cap B(p, r)) \geq \frac{1}{2} V(B(p, r)) \geq c(n, k, r) v_0,$$

with $c(n, k, r) = \frac{1}{2} c_1$ and c_1 given in Corollary 3.1, then we can take $m_0' = w(n, k, v_0, r) = c(n, k, r) v_0$. So assume that for all points x in M, one has

$$V(D \cap B(x, r)) < \frac{1}{2} V(B(x, r)).$$

Let $\mathcal{A}$ be a maximal family of points in M such that $d(x, x') \geq \frac{r}{2}$ for all $x, x' \in \mathcal{A}$ with $x \neq x'$, and $V(D \cap B(x, \frac{r}{2})) > 0$ for all $x \in \mathcal{A}$. Then $V(D - \bigcup_{x \in \mathcal{A}} B(x, r))) = 0$, since otherwise there would exist a point $y \in M$ such that

$$V\left(\left(D - \bigcup_{x \in \mathcal{A}} B(x, r)\right) \cap B(y, \frac{r}{2})\right) > 0, \tag{40}$$

and maximality of $\mathcal{A}$ would imply that $y \in B(x, \frac{r}{2})$ for some $x \in \mathcal{A}$, so $B(y, \frac{r}{2}) \subset B(x, r)$, which contradicts (40). Putting

$$C = Max_{x \in \mathcal{A}} \{V(D \cap B(x, r))^{\frac{1}{n}}\},$$

we have

$$\begin{array}{rcl} V(D) & \leq & \sum_{x \in \mathcal{A}} V(D \cap B(x, r)) \\ & \leq & C \sum_{x \in \mathcal{A}} V(D \cap B(x, r))^{\frac{n-1}{n}}. \end{array} \tag{41}$$

A relative isoperimetric inequality for balls of radius r in a Riemannian manifold with bounded geometry as stated in Lemma 3.4 (see [MC95] corollary 1.2) will give a constant $\gamma = \gamma(n, k, v_0, r) > 0$ such that

$$V(D \cap B(x, r))^{\frac{n-1}{n}} \leq \gamma A((\partial D) \cap B(x, r)). \tag{42}$$

Now

$$V(D) \leq C\sum_{x\in\mathcal{A}}\gamma A((\partial D)\cap B(x,r)) \leq C\gamma m A(\partial D). \tag{43}$$

where m is a constant which bounds the multiplicity of the current $\sum_{x\in\mathcal{A}}((\partial D)\cap B(x,r))$ and which depends only the ratio of the volumes of balls of radio $2r$ and $\frac{1}{4}r$ of the comparison manifolds. We can take m equal to $N_1(n,k,r,2r)$, where N_1 is the constant computed in Lemma 3.3. Another estimate for this number is given as follows. Setting $\mathcal{A}(z) := \{x\in\mathcal{A}|\ z\in B(x,r)\}$, we observe that the balls of the family $\left\{B(x,\frac{1}{4}r)\right\}_{x\in\mathcal{A}}$ are disjoint, and moreover

$$\cup_{x\in\mathcal{A}(z)}B(x,\frac{1}{4}r)\subseteq B(z,2r).$$

$$\begin{aligned} card(\mathcal{A}(z))V(B(x,\frac{1}{4}r)) &\leq V(\cup_{x\in\mathcal{A}(z)}B(x,\frac{1}{4}r)) \\ &\leq V(B(z,2r)) \\ &\leq V(B_k(2r)), \end{aligned}$$

and finally

$$card(\mathcal{A}(z))\leq\frac{V(B_k(2r))}{V(B_k(\frac{r}{4}))}\leq c_2(n,k,r). \tag{44}$$

Observe that to $c_2(n,k,r)$ could be given a very explicit expression just using (34). Setting $m(z) := card(\mathcal{A}(z))$, the function $m: z\to m(z)$ is exactly the multiplicity of the current $\sum_{x\in\mathcal{A}}(\partial D)\cap B(x,r)$, and for this reason

$$\sum_{x\in\mathcal{A}}A\left((\partial D)\cap B(x,r)\right)=\int_{\partial D}m(z)d\mathcal{H}^{n-1}(z)\leq c_2(n,k,v_0,r)A(\partial D).$$

It follows that for some $p\in M$,

$$V(D\cap B(p,r))^{\frac{1}{n}}\geq\frac{V(D)}{c_2(n,k,v_0,r)\gamma(n,k,v_0,r)A(\partial D)}>0. \tag{45}$$

Taking

$$m_0' := Min\{\frac{V(D)^n}{c_2(n,k,v_0,r)^nA(\partial D)^n+1},w(n,k,v_0,r)\}, \tag{46}$$

the theorem is proved. q.e.d.

The lemma that follows is used to avoid the evanescence case among the concentration-compactness principle alternatives for the isoperimetric problem in bounded geometry.

Lemma 3.6. *(Non-evanescence II) Let M be a Riemannian manifold with bounded geometry. Given a positive radius $r>0$, then there exist two constants $c_4 = c_4(n,k,v_0,r)>0$ and $w=w(n,k,v_0,r)>0$ with the following properties. Assume that D_j are domains such that $V(D_j)\to v>0$ and $I_M(v)=\lim_{j\to+\infty}A(\partial D_j)$. Then there exists a sequence of points $p_j\in M$ such that*

$$V(B(p_j,r)\cap D_j)\geq m_1 = m_1(n,k,v_0,r,v). \tag{47}$$

Moreover there is an $m_2\leq m_1$ that can be choosen such that $v\mapsto m_2(n,k,r,v_0,v)$ is continuous.

Remark: The importance of this lemma is that the constant m_1 it produces is independent of the minimizing sequence. This will guarantee that all the mass is recovered according to Morgan's scheme of the proof of the main theorem in [Mor94]. The use of this lemma in the proof of the main theorem is done taking a fixed radius $r=1$. The centers p_j depend on the minimizing sequence, but thankfully this is irrelevant for what follows.

Proof: For sufficiently large j, (45) implies that

$$\begin{aligned} V(D_j\cap B(p,r))^{\frac{1}{n}} \geq \frac{V(D_j)}{c_2A(\partial D_j)} &\geq \frac{V(D_j)}{2c_2I_M(v)+1} \\ &\geq \frac{v}{4c_2A(\partial B_{M_k^n}(\rho))+1}>0. \end{aligned} \tag{48}$$

Here $B_{\mathbb{M}_k^n}(\rho_j)$ is the ball of volume v in the simply connected comparison space $\mathbb{M}_k^n$ of constant sectional curvature k. We recall here that $A(\partial B_{M_k^n}(\rho)) = A(n,k,v)$ depends only on n,k,v, and $v\mapsto A(n,k,v)$ for each n,k fixed is continuous. Put

$$m_2 := Min\{\frac{v^n}{4^nc_2(n,k,v_0,r)^nA(\partial B_{M_k^n}(\rho))^n+1}, w(n,k,v_0,r)\}. \tag{49}$$

Although $v\mapsto m_1(n,k,r,v_0,v)$ is not necessarily continuous, where

$$m_1(n,k,r,v_0,v) := Min\{\frac{v^n}{4^nc_2^nI_M(v)^n+1}, w(n,k,v_0,r)\}, \tag{50}$$

we observe the crucial fact that $v\mapsto m_2(n,k,r,v_0,v)$ is continuous. q.e.d.

3.1.1 Existence of a minimizer in limit manifolds

Some known preliminary results

Here we want to apply the theory of convergence of manifolds to the isoperimetric problem when there is a lack of compactness due to a divergence to infinity of a non-neglegible part of volume in a almost minimizing sequence.

Let us recall the basic compactness result from the theory of convergence of manifolds, as exposed in [Pet06].

Theorem 3.2. *(Fundamental Theorem of Convergence Theory) [[Pet06] theorem 72]. For given $Q > 0$, $n \geq 2$, $m \geq 0$, $\alpha \in]0,1]$, and $r > 0$, consider the class $\mathcal{M}^{m,\alpha}(n, Q, r)$ of complete, smooth pointed Riemannian n-manifolds (M, p, g) with $||(M, g)||_{C^{m,\alpha},r} \leq Q$. $\mathcal{M}^{m,\alpha}(n, Q, r)$ is compact in the pointed $C^{m,\beta}$ topology for all $\beta < \alpha$.*

In subsequent arguments, we will need a regularity theorem in the context of variable metrics.

Theorem 3.3. *[Nar09b] Let M^n be a compact Riemannian manifold, g_j a sequence of Riemannian metrics of class C^∞ that converges to a fixed metric g_∞ in the C^4 topology. Assume that B is a domain of M with smooth boundary ∂B, and T_j is a sequence of currents minimizing area under volume contraints in (M^n, g_j) satisfying*

$$V_{g_\infty}(B \Delta T_j) \to 0. \tag{51}$$

Then in normal exponential coordinates ∂T_j is the graph of a function u_j on ∂B. Furthermore, for all $\alpha \in]0,1[$, $u_j \in C^{2,\alpha}(\partial B)$ and $||u_j||_{C^{2,\alpha}(\partial B)} \to 0$ as $j \to +\infty$.

Remark: Loosely speaking, Theorem 3.3 says that if an integral rectifiable current T is minimizing and sufficiently close in the flat norm to a smooth current, then ∂T is also smooth and ∂T can be represented as a normal graph over ∂B. In [Nar09b], the proof of the theorem includes a precise computation of the constants involved.

Remark: Theorems 3.2 and 3.3 are the main reasons for the C^4 bounded geometry assumptions in this paper, that are inherent just to the estimation of the number of components N in terms of the bounds of curvature.

In the sequel we use often the following classical isoperimetric inequality due to Pierre Bérard and Daniel Meyer.

Theorem 3.4. *([BM82] Appendix C). Let M^n be a smooth, complete Riemannian manifold, possibly with boundary, of bounded sectional curvature and positive injectivity radius. Then given $0 < \delta < 1$, there exists $v_0 > 0$ such that any open set U of volume $0 < v < v_0$ satisfies*

$$A(\partial U) \geq \delta c_n v^{\frac{n-1}{n}}. \tag{52}$$

Theorem 3.5. *([Heb00], Lemma 3.2) If M is a smooth, complete Riemannian manifold with bounded geometry, then there exist a positive constant $c = c(n, k, v_0)$ and a volume $\bar{v} = \bar{v}(n, k, v_0)$ such that any open set U with smooth boundary and satisfying $0 \leq V(U) \leq \bar{v}$ also satisfies*

$$V(U)^{\frac{n-1}{n}} \leq c(n, k, v_0) A(\partial U). \tag{53}$$

Remark: The same conclusion of the preceding theorem can be obtained if one replaces U by any finite perimeter set, using a customary approximation theorem in the sense of finite perimeter sets.

Remark: The preceding theorem implies in particular that for a complete Riemannian manifold with Ricci curvature bounded below and strictly positive injectivity radius, we have $I_M(v) \sim c_n v^{\frac{n-1}{n}} = I_{\mathbb{R}^n}(v)$ as $v \to 0$. For what follows it will be useful to give the definitions below.

Definition 3.9. *Let $\phi : M \to N$ a diffeomorphism between two Riemannian manifolds and $\varepsilon > 0$. We say that ϕ is a $(1+\varepsilon)$**-isometry** if for every $x, y \in M$, $(1-\varepsilon) d_M(x, y) \leq d_N(\phi(x), \phi(y)) \leq (1+\varepsilon) d_M(x, y)$.*

For the reader's convenience, we have divided the proof of Theorem 1 into a sequence of lemmas that in our opinion have their own inherent interest.

Lemma 1 (Ritoré's Lemma)**.** *Let (M, g) be a Riemannian manifold and $E \subset M$ a finite perimeter set with non-empty interior and smooth boundary. Then there exists a small deformation $E_\varepsilon \supset E$, $0 < \varepsilon < \varepsilon_0$, such that:*

$$P(\partial (E_\varepsilon - E)) \leq C V(E_\varepsilon \setminus E).$$

Proof. To prove the Lemma it is enough to look at the Taylor expansion of the variation of the area

$$A(\epsilon) = A(0) + A^{'}(0)\epsilon + o(\epsilon). \tag{54}$$

With respect to the following variation

$$\begin{array}{llll}\Psi : & (0,\epsilon_0)\times\partial E & \longrightarrow & M \\ & (\epsilon, p) & \longrightarrow & \Psi(\epsilon,p)=\exp_p(\epsilon\nu_p),\end{array}$$

where ν denote the exterior unit normal vector field along ∂E at p. Denote the variation of volume of E by

$$V(\epsilon)=V(E_\epsilon)=\int_{\Psi_\epsilon(E)} dV,$$

then, the standard variational formulae reads as:

$$V^{'}(\epsilon)=A(\epsilon),\ A^{'}(\epsilon)=\int_{\partial E_\epsilon} H_\epsilon dA \tag{55}$$

$$A^{'}(0)=\int_{\partial E} HdA \leq \parallel H_{\partial E}\parallel_\infty A(0). \tag{56}$$

Replacing (56) in equation (54) and using (55) yields

$$\begin{aligned}|A(\epsilon)-A(0)| &\leq \epsilon \parallel H_{\partial E}\parallel_\infty A(0)+o(\epsilon)\\ &\leq \epsilon \parallel H_{\partial E}\parallel_\infty V^{'}(0)+o(\epsilon)\\ &\leq \frac{3}{2}\epsilon \parallel H_{\partial E}\parallel_\infty V^{'}(0),\end{aligned}$$

for every $0<\epsilon<\epsilon_1$, for some $\epsilon_1>0$, and

$$|A(\epsilon)-A(0)|\leq \frac{3}{2}\epsilon \parallel H_{\partial E}\parallel_\infty \left(\frac{V(\epsilon)-V(0)}{\epsilon}+\frac{o(\epsilon)}{\epsilon}\right).$$

$\epsilon_1>0$ such that $o(\epsilon)\leqslant \frac{1}{2}\epsilon \parallel H_{\partial E}\parallel_\infty V^{'}(0)$.
Moreover,

$$|A(\epsilon)-A(0)|\leq \frac{\epsilon \parallel H_{\partial E}\parallel_\infty}{h}(V(h)-V(0))+\parallel H_{\partial E}\parallel_\infty \frac{o(h)}{h}.$$

Hence for all $0<h<\epsilon_2$ we have,

$$|A(\epsilon)-A(0)|\leq \epsilon\frac{\parallel H_{\partial E}\parallel_\infty}{h}(V(h)-V(0))$$

Finally consider $\epsilon_0<min\{\epsilon_1;\epsilon_2\}$, thus we obtain

$$|A(\epsilon)-A(0)|\leq \parallel H_{\partial E}\parallel_\infty (V(\epsilon)-V(0))$$

for every $0<\epsilon<\epsilon_0$ $\square$

Lemma 3.7. *If $(M, p_j, g) \to (M_\infty, p_\infty, g_\infty)$ in C^0-topology, then*

$$I_{M_\infty} \geq I_M. \tag{57}$$

Proof: We rewrite the proof appearing in [Nar10] Lemma 3.4 for the convenience of the reader. Fix $0 < v < V(M)$. Let $D_\infty \subseteq M_\infty$ be an arbitrary domain of volume $v = V_{g_\infty}(D_\infty)$. Put $r := d_H(D_\infty, p_\infty)$, where d_H denotes the Hausdorff distance. Consider the sequence $\varphi_j : B(p_\infty, r+1) \to M$ of $(1+\varepsilon_j)$-isometries given by the convergence of pointed manifolds, for some sequence $\varepsilon_j \searrow 0$. Setting $D_j := \varphi_j(D_\infty)$ and $v_j := V(D_j)$, it is easy to see that

(i): $v_j \to v$,

(ii): $A_g(\partial D_j) \to A_{g_\infty}(\partial D_\infty)$.

Moreover, (i)-(ii) hold because φ_j is a $(1+\varepsilon_j)$-isometry.

We now proceed with the proof of (57) by contradiction. Suppose that there exist a volume $0 < v < V(M)$ satisfying

$$I_{M_\infty}(v) < I_M(v). \tag{58}$$

Then there is a domain $D_\infty \subseteq M_\infty$ such that

$$I_{M_\infty}(v) \leq A_{g_\infty}(\partial D_\infty) < I_M(v).$$

As above we can find domains $D_j \subset M$ satisfying (i)-(ii). Unfortunately the volumes v_j in general are not exactly equal to v. So we have to readjust the domains D_j to get $v_j = v$, for every j, preserving the property $A_g(\partial D_j) \to A_{g_\infty}(\partial D_\infty)$ as $j \to +\infty$, to get the desired contradiction. This can be done using the following construction that will be used in many places in the sequel. Examining the proofs of the deformation lemma of 1 and the compensation lemma of [Nar09b], one can construct domains $D_j^\infty \subseteq B(p_\infty, r+1) \subseteq M_\infty$ as small perturbations of D_∞ such that

$$A_{g_\infty}(\partial D_j^\infty) \leq A_{g_\infty}(\partial D_\infty) + c\tilde{v}_j, \tag{59}$$

$$\tilde{v}_j \searrow 0, \tag{60}$$

and

$$V_g(\varphi_j(D_j^\infty)) = v. \tag{61}$$

The preceeding discussion shows the existence of bounded finite perimeter sets (in fact, smooth domains) $D_j := \varphi_j(D_j^\infty) \subset M$ satisfying

$$V_g(D_j) = v, \tag{62}$$

$$|A_g(\partial D_j) - A_{g_\infty}(\partial D_j^\infty)| \to 0, \tag{63}$$

again using the fact that φ_j is a $(1+\varepsilon_j)$ isometry. Thus we have a sequence of domains D_j of equal volume such that

$$A_g(\partial D_j) \to A_{g_\infty}(\partial D_\infty) < I_M(v), \tag{64}$$

which is the desired contradiction. The theorem follows from the fact that v is arbitrary. q.e.d.

Lemma 3.8. *Let $\tilde{M} := M \bigcup_{i=1}^N M_{\infty,i}$ be a disjoint union of finitely many limit manifolds $(M_{\infty,i}, g_{\infty,i}) = \lim_j (M, p_{i,j}, g)$. Then $I_{\tilde{M}} = I_M$.*

Proof: It is a trivial to check that $I_M \geq I_{\tilde{M}}$. Observe that when $d_M(p_{ij}, p_{lj}) \leq K$ for some constant $K > 0$, we have $(M_{i,\infty}, g_{i,\infty}) = (M_{l,j}, g_{l,\infty})$. Thus we can restrict ourselves to the case of sequences diverging in different directions, i.e., $d_M(p_{i,j}, p_{l,j}) \to +\infty$ for every $i \neq j$. Then mimicking the proof of the preceding lemma, one can obtain $I_M \leq I_{\tilde{M}}$. q.e.d.

The proof of the next lemma contains a construction of a decomposition of the ε-thick part of a subsequence of a minimizing sequence D_j into a finite number of pieces. These are obtained cutting with geodesic balls centered at concentration points, whose radius is determined by a coarea formula argument. The proof is inspired by [RR04].

Lemma 3.9. *Let $D_j \subseteq M$ be a minimizing sequence of volume $0 < v < V(M)$. Suppose that there are $N \geq 1$ sequences of points $(p_{ij})_j$, $i \in \{1, \dots, N\}$, N pointed limit manifolds $(M_{\infty,i}, p_{i,\infty})$, and N volumes v_i such that*

- *(i):* $0 < \sum_i^N v_i \leq v$ *(possibly* $\sum_i^N v_i < v$*),*
- *(ii):* $(M, p_{i,j}) \to (M_{\infty,i}, p_{i,\infty})$ *in the* $C^{m,\beta}$ *topology for every* $\beta < \alpha$*,*
- *(iii):* $d_M(p_{hj}, p_{lj}) \to +\infty$*, for every* $h \neq l$*,*
- *(iv): for every* $\varepsilon > 0$ *there exists* R_ε *such that for all* $R \geq R_\varepsilon$*, there is a* $j_{\varepsilon,R}$ *satisfying* $V(D_j \cap B(p_{ij}, R_\varepsilon)) \in [v_i - \varepsilon, v_i + \varepsilon]$ *for all* $j \geq j_{\varepsilon,R}$*.*

Then there exist finite perimeter sets $D_{\infty,i} \subseteq M_{\infty,i}$ such that

- *(I):* $V(D_{\infty,i}) = v_i$*,*

(II): $I_{M_{\infty,i}}(v_i) = A_{g_{i,\infty}}(\partial D_{\infty,i}), \forall i \in \{1, ..., N\}$,

(III): $I_{M_\infty^{(N)}}(v_1 + \cdots + v_N) = \sum_{i=1}^N A(\partial D_{\infty,i}) = A(\partial D_\infty^{(N)})$,

(IV): $I_M(v) = \sum_{i=1}^N A(\partial D_{\infty,i}) + \underrightarrow{\lim}_{j\to+\infty} A(\partial D''_{N,j})$,

(V): $V(D''_{N,j}) \to v - \sum_{i=1}^N v_i$,

(VI):

$$\begin{aligned} I_M(v) &\geq \sum_i^N I_{M_{i,\infty}}(v_i) + I_M(v - \sum_i^N v_i) \\ &= I_{M_\infty^{(N)}}(\sum_i^N v_i) + I_M(v - \sum_i^N v_i), \end{aligned}$$

for some $D''_{N,j} \subseteq D_j$*, and* $D_\infty^{(N)} = \bigcup_{i=1}^N D_{\infty,i}$ *and* $M_\infty^{(N)} := \bigcup_{i=1}^N M_{i,\infty}$ *are disjoint unions.*

Remark: (iv) is equivalent to (iv'): for every $\varepsilon > 0$ there exists R_ε such that for all $R \geq R_\varepsilon$ we have $v_i - \varepsilon \leq \underrightarrow{\lim}_{j\to+\infty} V(D_j \cap B(p_{ij}, R)) \leq \overrightarrow{\lim}_{j\to+\infty} V(D_j \cap B(p_{ij}, R)) \leq v_i + \varepsilon$.

Lemma 2. *[Bounded Volume and Area Sequence's Structure] Let* M^n *be a complete Riemannian manifold with bounded geometry. Let* (Ω_j) *be a sequence of finite perimeter sets with* $\mathcal{P}(\Omega_j) \leq A$ *and* $V(\Omega_j) \leq v$*. Then there exist a subsequence* (Ω_{j_k}) *that we rename by* (Ω_k)*, a set* $S \subseteq N$*, with* $k \in S$*, a natural number* $l \in (N \setminus \{0\}) \mathring{\cup} \{+\infty\}$*,* l *sequences of points* $(p_{ik}) \in M^n$*,* l *sequences of radio* $R_{ik} \to +\infty$*,* $i \in \{1, ..., l\}$*, a sequence of volumes* $v_i \in]0, +\infty[$*, and areas* $A_i \in [0, +\infty[$ *such that*

(I): $v_i = \lim_{k\to+\infty} V(\Omega_{ik})$*, where* $\Omega_{ik} := \Omega_k \cap B(p_{ik}, R_{ik})$*,*

(II): $0 < \sum_{i=1}^\infty v_i = \bar{v} \leq v$*,*

(III): If $V(\Omega_k) \to v$*, then* $\bar{v} = v$*,*

(IV): Set $A_i = \lim_{k\to+\infty} \mathcal{P}(\Omega_{ik})$*, we get*

$$\lim_{k\to+\infty} \mathcal{P}(\mathring{\cup}_{i=1}^\infty \Omega_{ik}) = \sum_{i=1}^\infty A_i = \bar{A} \leq A.$$

Remark 15. *In (IV): $\bar{A}$, could be strictly less than A even if $\mathcal{P}(\Omega_k) \to A$. However there are cases in which $\bar{A}$ is equal to A, as showed in Corollary 3.2.*

Proof:[of Lemma 2] This proof consist in finding a subsequence Ω_k of an arbitrary sequence Ω_j of bounded volume and area that could be decomposed as an at most countable union $\Omega_k = \bigcup_i^N \Omega_{ik}$, with N possibly equal to $+\infty$. To this aim, we consider the concentration functions

$$Q_{1,j}(R) := Sup_{p\in M}\{V(\Omega_j \cap B_M(p,R))\}.$$

Since Lemma 2.5 of [Nar14a] prevents evanescence in bounded geometry, the concentration-compactness argument of Lemma I.1 of [Lio84] or Lemma 2.5 of [Nar14a], provides a concentration volume $0 < v_1 \in [m_0(n, r = 1, k, v_0, v), v]$.

Suppose the concentration of volumes occurs at points p_{1j}, a ... then there exists $E \subseteq R$, with $|R \setminus E| = 0$, $S \subseteq N$ such that for every $\epsilon_k \to 0$ there exists R_{ϵ_k} such that for every $R \in E \cap [R_{\epsilon_k}, +\infty[$ there exists $j_{k,R}$ satisfying the property that if $j \geq j_{k,R}$ then

$$|V(B(p_{k,R}, R) \cap \Omega_j) - v_1| \leq \varepsilon_k. \tag{65}$$

By the fact that the preceeding equation is true for every $R \in E \cap [R_{\epsilon_k}, +\infty[$, we can associate to every k a radius R'_{1k} satisfying

$$R'_{1,k+1} \geq R'_{1,k} + k, \tag{66}$$

then by the coarea formula we get radio $R_{1k} \in [R'_{1,k}, R'_{1,k+1}]$ such that

$$\mathcal{P}(B(p_{k,R_{1k}}, R_{1k}) \cap \Omega_j) \leq A + \frac{v}{j}. \tag{67}$$

We can set now $\Omega_{1k} := B(p_{k,R_{1k}}, R_{1k}) \cap \Omega_k$ for k sufficiently large inside S. If $v_1 = v$ then the theorem is proved with $l = 1$. If $v_1 < v_2$, we apply the same procedure to the domains $\Omega'_{1k} := \Omega_k \setminus B(p_{k,R_{1k}}, R_{1k})$, observing that $V(\Omega'_{1k}) \to v - v_1$, we obtain R_{2k}, p_{2k}, v_2, A_2. If $v_1 + v_2 = v$, then we finish the construction and the theorem is proved in a finite number of steps. If this does not happen then we obtain at the i-th step a decomposition $\Omega_k = \Omega'_{ik} \mathring{\cup}_{l=1}^i \Omega_{ik}$. As k goes to infinity we have $V(\Omega_{ik}) \to v_i$ and $V(\Omega'_{ik}) \to v - \sum_{l=1}^i v_l$, if $V(\Omega_k) \to v$, and $\mathcal{P}(\partial\Omega'_{ik}) \to A - \sum_{l=1}^i A_l$ where,

$$A_l := \lim_{k\to+\infty} \mathcal{P}((\partial\Omega_k) \cap B(p_{lk}, R_{lk})),$$

if $\mathcal{P}(\Omega_k) \to A$. In any case $\mathcal{P}(\Omega'_{ik}) \leq A + i\frac{v}{k}$, so for k big enough $\mathcal{P}(\Omega'_{ik}) \leq 2A$. At this stage we found a sequence (v_i) of positive volumes giving rise to a convergent series $\sum_{i=1}^{\infty} v_i =: \bar{v} \leq v$. To prove that if $V(\Omega_j) \to v$ then $\bar{v} = v$ we use the non evanescence Lemma 2.5 [Nar14a]. Indeed $v_i \to 0$ because $\sum_{i=1}^{\infty} v_i$ is a convergent series, on the other end an application of Lemma 2.5 of [Nar14a] to Ω'_{ik} yields to

$$V(\Omega'_{ik} \cap B(p_{i,k+1}, R_{i,k+1})) =: v_{i+1,k} \geq \frac{V(\Omega'_{ik})^n}{\mathcal{P}(\Omega'_{ik})^n + 1}, \tag{68}$$

but $v_{i+1} \geq v_{i+1,k}$ so we obtain

$$\lim_{i\to+\infty} v_i = \lim_{i\to+\infty} v_{i+1} \geq \lim_{i\to+\infty} \overrightarrow{\lim}_{k\to+\infty} v_{i+1,k} \geq C(n,k,v_0,v)\frac{(v-\bar{v})^n}{A^n+1} > 0.$$

This last inequality contradicts the fact that $v_i \to 0$, if $\bar{v} < v$. It remains to prove (V), but this is a simple consequence of the definitions. q.e.d.

Corollary 3.2. *Let M^n be a complete Riemannian manifold with bounded geometry. If (Ω_j) is an almost minimizing sequence of volume v then there exist a finite number $N = N(n, v_0, k, v) \in N \setminus \{0\}$, a subsequence (Ω_{j_k}) that we rename by (Ω_k), $k \in S \subseteq N$, N sequences of points $(p_{ik})_{i\in\{1,...,N\}}$, N sequences of radio $(R_{ik})_{i\in\{1,...,N\}} \to +\infty$, when $k \to +\infty$, N volumes $v_i \in]0, +\infty[$, with $i \in \{1, ..., N\}$, N areas $A_i \in [0, +\infty[$, such that*

- *(I): $v_i = \lim_{k\to+\infty} V(\Omega_{ik})$, where $\Omega_{ik} := \Omega_k \cap B(p_{ik}, R_{ik})$,*
- *(II): $0 < \sum_{i=1}^{N} v_i = \bar{v} \leq v$,*
- *(III): If $V(\Omega_k) \to v$, then $\bar{v} = v$,*
- *(IV): $A_i = \lim_{k\to+\infty} \mathcal{P}(\Omega_{ik})$, moreover $\sum_{i=1}^{N} A_i = \bar{A} \leq A$*
- *(V): If $\mathcal{P}(\Omega_k) \to A$, then $\bar{A} = A$.*
- *(VI): $N \leq \left[\frac{v}{v^*}\right] + 1$, where $v^* = v^*(n, k, v_0)$ can be taken equal to $\bar{v}$ of Lemma 3.2 of [Heb00]. In particular if $v < v^*$ then $N = 1$.*

Proof: The proof goes along the same lines of Theorem 1 of [Nar14a]. We prove just that $\bar{A} = A$. In view of Theorem 1 of [FN14] the isoperimetric profile in bounded geometry is continuous so it is enough to prove (V) in the case of a minimizing sequence in volume v. Suppose, now, that $V(\Omega_k) = v$ and $A = I_M(v)$. From Lemma 2 we get a decomposition of $\Omega_k = \Omega'_k \mathring{\cup} \tilde{\Omega}_k$, where $\tilde{\Omega}_k := \mathring{\cup}_{i=1}^{\infty} \Omega_{ik}$ (this infinite union in the case of a minimizing sequence is in fact a finite union) and $\Omega'_k := \Omega_k \setminus \tilde{\Omega}_k$. It is easy to see that $V(\Omega'_k) =: v'_k \to 0$ and $V(\tilde{\Omega}_k) =: \tilde{v}_k \to 0$ as $k \to 0$, which implies

$$I_M(v) = A = \lim_{k \to +\infty} I_M(\tilde{v}_k) \leq \lim_{k \to +\infty} \mathcal{P}(\tilde{\Omega}_k) = \bar{A}.$$

The second equality is due to the continuity of the isoperimetric profile, the remaining are easy consequences of the definitions. To obtain an estimate on N, one way to proceed is to show that there is no dichotomy for volumes less than some fixed $v^* = v^*(n, k, v_0) > 0$. Assuming the existence of such a v^*, we observe that the algorithm produces $v_1 \geq v_2 \geq \cdots \geq v_N$. Furthermore, $v_N \leq v^* \leq v_{N-1}$ because it is the first time that dichotomy cannot appear, which yields

$$v^*(N-1) \leq v_{N-1}(N-1) \leq \sum_{i=1}^{N-1} v_i \leq \sum_{i=1}^{N} v_i = v. \tag{69}$$

Consequently

$$N \leq \left[\frac{v}{v^*} + 1\right] = \left[\frac{v}{v^*}\right] + 1, \tag{70}$$

where $v^* = v^*(n, k, v_0)$ can be taken equal to $\bar{v}$ of Lemma 3.2 of [Heb00]. On the other hand, one can construct examples such that for every v, there are exactly $N = \left[\frac{v}{v^*}\right] + 1$ pieces. So in this sense, the estimate (70) is sharp and it concludes the proof of the theorem. q.e.d.

Now we prove Theorem 1.

Proof:[of Theorem 1] By Gromov's compactness theorem and a cumbersome diagonalization process there is a subsequence converging in the multipointed Gromov-Hausdorff topology, by the results of [Col97] volumes converges. Just assuming Gromov-Hausdorff convergence we don't know if perimeter converges, but the hypothesis of C^0-bounded geometry ensures the convergence of perimeters. q.e.d.

As a corollary to Theorem 1 we have the following result.

Theorem 3.6. *If M^n have C^0 asymptotically bounded geometry, then the isoperimetric profile I_M is continuous on $[0, V(M)[$.*

Proof: Take a sequence of isoperimetric regions $\Omega_i \subset M$, such that $V(\Omega_i) = v_i \to v$. Then apply generalized compactness to extract a subsequence Ω_k converging in the multipointed C^0 topology to a generalized isoperimetric region Ω of volume v. From the existence of the generalized isoperimetric region Ω we deduce that I_M is upper semicontinuous as in Theorem 4.1 and from lower semicontinuity of the perimeter with respect to the multipointed C^0 topology we get lower semicontinuity of $I_{\tilde{M}} = I_M$. q.e.d.

Remark 16. *Using Theorem 3.6, we can prove easily Corollary* 1 *of [Nar14a] and Corollary* 1 *of [FN14] without using Theorem* 1 *of [FN14]. Of course Theorem* 1 *of [FN14] is stronger than Theorem 3.6 and have its own interest independently of the use that we did in proving Corollary* 1 *of [FN14].*

4 Local Hölder Continuity and differentiability properties of isoperimetric profile in complete noncompact Riemannian manifolds with bounded geometry

Theorem 2 (Local $\left(1-\frac{1}{n}\right)$-Hölder continuity of the isoperimetric profile)**.** *Let M^n be a complete smooth Riemannian manifold with bounded geometry. Then there exists a positive constant $C = C(n,k)$ such that for every $v, v' \in]0, V(M)[$ satisfying $|v-v'| \leq \frac{1}{C(n,k)}\min\left(v_0, \left(\frac{v}{I_M(v)+C(n,k)}\right)^n\right)$, we have*

$$\left|I_M(v)-I_M(v')\right| \leq C(n,k)\left(\frac{|v-v'|}{v_0}\right)^{\frac{n-1}{n}}. \tag{71}$$

In particular I_M is continuous on $[0, V(M)[$.

Definition 4.1. *Let $f:(X,d)\to\mathbb{R}$ and $\alpha\in[0,1]$, we say that f is **locally α-Hölder continuous on** X, for every $z\in X$ there exist $\delta_z, C_z>0$ such that for every $x,y\in X$ satisfying $|x-z|,|y-z|\leq\delta_z$ we have $|f(x)-f(y)|\leq C_z|x-y|^\alpha$. We say that f is **uniformly locally α-Hölder continuous on** X, if there exist two constants $\delta, C>0$ such that for every $x,y\in X$ satisfying $d(x,y)\leq\delta$ we have $|f(x)-f(y)|\leq C|x-y|^\alpha$. We say that f is **(globally) α-Hölder continuous on** X, if there exists $C>0$ such that $|f(x)-f(y)|\leq C|x-y|^\alpha$ for every $x,y\in X$. We call the various constants C_z, C appearing in this definition the **Hölder constants of** f.*

Corollary 2 (Local $\frac{n-1}{n}$-Hölder continuity of the isoperimetric profile)**.** *Let M^n be a complete smooth Riemannian manifold with bounded geometry and $v\in]0,V(M)[$. Then there exists positive constants $\delta = \delta(n,k,v_0,v)>0$, if $k\leq 0$, $\delta=\delta(n,k,v_0,,v,V(M))$, if $k>0$, and $C=C(n,k)>0$, such that for every $v_1,v_2\in[v-\delta,v+\delta]$ we have*

$$|I_M(v_1)-I_M(v_2)| \leq C(n,k)\left(\frac{|v_1-v_2|}{v_0}\right)^{\frac{n-1}{n}}. \tag{72}$$

Moreover, if $V(M)=+\infty$ then I_M is uniformly locally $\frac{n-1}{n}$-Hölder continuous on $[\bar{v},+\infty[$, for every $\bar{v}>0$. If $V(M)=+\infty$ then I_M is globally $\frac{n-1}{n}$-Hölder continuous on every interval $[a,b]\subset]0,+\infty[$ with Hölder constant $\bar{C}$ depending on n,k,v_0,a,b. If $V(M)<+\infty$, then I_M is globally $\frac{n-1}{n}$-Hölder continuous on $[\bar{v}, V(M)-\bar{v}]$, for every $\bar{v}\in]0,\frac{V(M)}{2}[$.

Remark 17. *Unfortunately* $\lim_{a\to 0^+}\bar{C}(n,k,v,a,b)=+\infty$ *and* $\lim_{b\to 0^+}\bar{C}(n,k,v,a,b)=+\infty$.

Remark 18. *Observe that in the statement of the preceding Corollary the Hölder constant* C *does not depend on* v_0 *and* v*, but just* δ *depends on them.*

Remark 19. *At our actual knowledge, it is still an open question wether or not we can prove global* $\frac{n-1}{n}$*-Hölder continuity of* I_M *on an arbitrary proper interval* $[0,b]\subset[0,V(M)[$ *or on the entire interval* $[0,V(M)[$*, or at least unifom local* $\frac{n-1}{n}$*-Hölder continuity on* $[0,V(M)[$*, when we assume the manifold* M *to be with bounded geometry and with* $V(M)=+\infty$.

Corollary 3 (Bavard-Pansu-Morgan-Johnson in bounded geometry)**.** *Let* M *have* C^0*-locally asymptotic bounded geometry in the sense of Definition 3.4. Suppose that all the limit manifolds have a metric at least of class* C^2*. Then* I_M *is absolutely continuous and twice differentiable almost everywhere. The left and right derivatives* $I_M^-\geq I_M^+$ *exist everywhere and their singular parts are non-increasing. If* $k>0$ *then* I_M *is strictly concave on* $]0,V(M)[$*. If* $k=0$*, then* I_M *is just concave on* $]0,V(M)[$*. If* $k<0$*, then* $I_M(v)+C(a,b)v^2$ *is concave, (*I_M *could not be concave). Moreover, we have for every* $k\in\mathbb{R}$ *and almost everywhere*

$$I_M I_M^{''}\leq-\frac{{I'}_M^2}{n-1}-(n-1)k, \tag{73}$$

with equality in the case of the simply connected space form of constant sectional curvature k*. In this case, a generalized isoperimetric region is totally umbilic.*

Proof: Show the continuity of the isoperimetric profile function, then make a second variation formula for manifolds that are a little bit singular then adapt the proof of Morgan-Johnson to this context. q.e.d.

Remark 20. *We observe that if* $(M_i,g_i,p_i)\to(M,g,p)$ *in the pointed Gromov-Hausdorff topology and* M_i *satisfy* $Ric_{g_i}\geq(n-1)k_0g_i$*, it is not true, in general, that* $Ric_g\geq(n-1)k_0g$*. Instead, if* $(M_i,g_i,p_i)\to(M,g,p)$ *in the pointed* C^0*-topology then* $(M_i,g_i,V_i,p_i)\to(M,g,V,p)$ *converge in the measured pointed Gromov-Hausdorff topology. Therefore, if all the Riemannian* n*-manifolds* (M_i,g_i) *satisfy* $Ric_{g_i}\geq(n-$

$1)k_0 g_i$ then also the limit Riemannian manifold (M, g) satisfies $Ric_g \geq (n-1)k_0 g$ (see Section 7 in [AG09]). Notice that for the convergence of the Ricci curvature one should need a stronger convergence of the (M_i, g_i, p_i) to (M, g, p), say in C^2-topology; here we just need the convergence of a lower bound. Some details of this are given in the appendix.

Corollary 4 (Morgan-Johnson isoperimetric inequality in bounded geometry)**.** *Let M have $C^{2,\alpha}$-bounded geometry, sectional curvature K and Gauss-Bonnet-Chern integrand G. Suppose that*

- *$K < K_0$, or*
- *$K \leq K_0$, and $G \leq G_0$,*

where G_0 is the Gauss-Bonnet-Chern integrand of the model space form of constant curvature K_0. Then for small prescribed volume, the area of a region R of volume v is at least as great as $A(\partial B_v)$, where B_v is a geodesic ball of volume v in the model space, with equality only if R is isometric to B_v.

The proofs of Corollaries 3 and 4 run along the same lines as the corresponding proofs of theorems 3.3 and 4.4 of [MJ00].

4.1 Local Hölder continuity in bounded geometry

To illustrate the proof of Theorem 2 we start this section with the easy part of the proof resumed in the next lemma that is straightforward, compare [AMN13] Proposition 1. As the example 3.53 of [AFP00] shows, in general we can have finite perimeter sets with positive perimeter and void interior that are not equivalent to any other set of finite perimeter with non void interior. So the question of putting a ball inside or outside a set of finite perimeter is a genuine technical problem, on the other hand, following [GMT83] Theorem 1, it is always possible to put a small ball inside and outside an isoperimetric region, which justify the constructions performed in this proof. As a general remark a result of Federer (the reader could consult [AFP00] Theorem 3.61) states that for a given set of finite perimeter E the density is either 0 or $\frac{1}{2}$ or 1, $\mathcal{H}^{n-1}$-a.e. $x \in M$, moreover points of density 1 always exist V-a.e. inside D, because of the Lebesgue's points Theorem applied to the characteristic function of any V-measurable set of M. About this topic the reader could consult the book [Mag12] Example 5.17. Thus $V(D) > 0$ ensures the existence of at least one point p belonging to D of density 1, which is enough for the aims of our proofs.

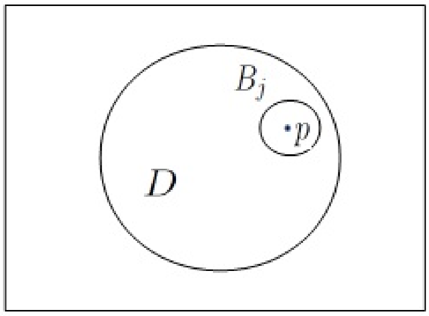

Fig. 1: $v' < v$ Upper Semicontinuity

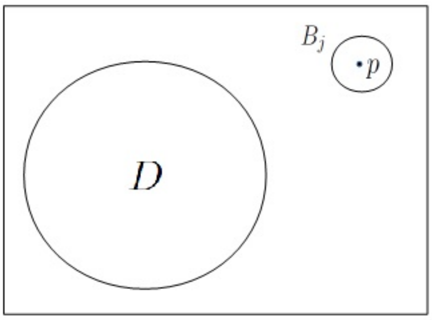

Fig. 2: $v' > v$ Upper Semicontinuity

Theorem 4.1. *Let M be a Riemannian manifold (possibly incomplete, or possibly complete not necessarily with bounded geometry). If there exists an isoperimetric region in volume $v \in]0, V(M)[$ then I_M is upper semicontinuous in v.*

Proof: To prove the theorem it is enough to prove the next two inequalities.

$$\overrightarrow{\lim}_{v' \to v^-} I_M(v') \leq I_M(v). \tag{74}$$

$$\overrightarrow{\lim}_{v' \to v^+} I_M(v') \leq I_M(v). \tag{75}$$

In first we prove (74). If $v_j \nearrow v$, consider an isoperimetric region D in volume $V(D) = v$,

$$I_M(v) = A(\partial D).$$

Then for j sufficiently large one can subtract a small geodesic ball (i.e. of small radius) $B_j = B(p, r'_j)$ of volume $v - v_j$ from D, centered to a point of density 1, to obtain $D'_j := D \setminus B(p, r'_j)$ of volume $V(D'_j) = v_j$ and $A(\partial D'_j) \leq A(\partial D) + A(\partial B_j)$. Observe here that the center p of B_j is fixed with respect to j. Moreover $r'_j \to 0$, and this is always possible to obtain in any Riemannian manifold. So by definition of $I_M(v_j)$, holds

$$I_M(v_j) \leq A(\partial D'_j) \leq A(\partial D) + A(\partial B_j) = I_M(v) + A(\partial B_j),$$

which implies that

$$\overrightarrow{\lim} I_M(v_j) \leq \overrightarrow{\lim} A(\partial D) + A(\partial B_j) \leq I_M(v),$$

since the sequence v_j is arbitrary we get (74). In second, we prove (75). If $v_j \searrow v$, then take an isoperimetric region of voume v, i.e., $V(D) = v$, $A(\partial D) = I_M(v)$ and then add a small ball $B_j := B(p, r_j)$ of volume $v_j - v$ to D outside D to obtain $D'_j := D \mathring{\cup} B_j$ of volume $V(D'_j) = v_j$ and $A(\partial D'_j) = A(\partial D) + A(\partial B_j)$. Observe again that the center p of B_j here is fixed with respect to j and $r_j \to 0$, this is always possible in any Riemannian manifold. By definition of $I_M(v_j)$ we get

$$I_M(v_j) \leq A(\partial D'_j) = A(\partial D) + A(\partial B_j) = I_M(v) + A(\partial B_j),$$

now taking the $\overrightarrow{\lim}$ it follows

$$\overrightarrow{\lim} I_M(v_j) \leq \overrightarrow{\lim}[A(\partial D) + A(\partial B_j)] = I_M(v) + \overrightarrow{\lim} A(\partial B_j) = I_M(v),$$

since the sequence v_j is arbitrary we get (75), which completes the proof. q.e.d.

At this point, we may finish the proof of the main Theorem 2.

We will prove separately the following four inequalities that together will give the proof of our theorem 2. For the needs of the proof of Theorem 2 we restate here a version of Lemma 2.5 of [Nar14a] that we will use in the sequel.

Lemma 4.1 (Lemma 2.5 of [Nar14a])**.** *There is a constant $c = c(n, k)$, with $0 < c < 1$ such that for any Riemannian manifold M^n with bounded geometry, any radius $0 < r \leq 1$, any set $D \in \tilde{\tau}_M$ with $V(D) < +\infty$, there is a point $p \in M$ such that*

$$V(B(p, r) \cap D) \geq c \min\{v_0 r^n, \left(\frac{V(D)}{\mathcal{P}(D)}\right)^n\}. \tag{76}$$

The proof of the preceding Lemma is essentially the same as in Lemma 2.5 of [Nar14a].

Now we can start the proof of Theorem 2.

Proof:[of Theorem 2] As a preliminary remark we observe that it is enough to prove the theorem thinking to the definition of $\tilde{I}_M$ when it is more useful for our reasoning. Let $\varepsilon \in]0,1]$. By Theorem 2.3 we can get $\Omega \in \tau_M$ with $V(\Omega) = w$ and $\mathcal{P}(\Omega) \leq I_M(w) + \varepsilon$. When M is not compact, there exists a ball $B(x_2, 1)$ not intersecting Ω (that could be chosen compact). Then for every $v' \in]w, w + v_0[$ there exists $r_{v'} \leq 1$ such that $\Omega_1 = \Omega \mathring{\cup} B_M(x_2, r_{v'})$ satisfies $V(\Omega_1) = v'$ and

$$I_M(v') \leq \mathcal{P}(\Omega_1) \leq \mathcal{P}(\Omega) + \mathcal{P}(B_M(x_2, r_{v'})) \leq I_M(w) + \varepsilon + C(n,k) r_{v'}^{n-1}, \quad (77)$$

where the last inequality comes from the spherical Bishop-Gromov's theorem (which asserts that when $Ric_g \geq (n-1)kg$ the area of spheres are less than the area of corresponding spheres in space form of constant curvature k) and from the value of the area of the spheres in constant curvature. Since by Bishop-Gromov's Theorem we have $\frac{v_0 r_{v'}^n}{C_1(n,k)} \leq V(B(x_2, r_{v'}) = v' - v$, Inequality (77) gives us

$$I_M(v') \leq I_M(v) + \varepsilon + C_2(n,k) \left(\frac{v' - v}{v_0} \right)^{\frac{n-1}{n}}. \quad (78)$$

The case $v' \leq v$ needs more work. Let us apply Lemma 4.1 to Ω, we get for any $v' \in]v - v_1, v[$, where $v_1 = c \min \left\{ v_0, \left(\frac{v}{I_M(v)+\varepsilon} \right)^n \right\}$, then we have

$$V\left(\Omega \cap B\left(p, \left(\frac{v - v'}{c v_0} \right)^{\frac{1}{n}} \right) \right) \geq \min \left\{ v - v', c \left(\frac{v}{I_M(v) + \varepsilon} \right)^n \right\} = v - v', \quad (79)$$

and so there exists a $r_{v'} \leq \left(\frac{v - v'}{c v_0} \right)^{\frac{1}{n}}$ such that $\Omega_2 := \Omega \setminus B(p, r_{v'})$ has volume v' and so, by the spherical Bishop-Gromov's Theorem, we get

$$I_M(v') \leq \mathcal{P}(\Omega_2) \leq \mathcal{P}(\Omega) + \mathcal{P}(B_M(p, r_{v'})) \leq I_M(v) + \varepsilon + C_2(n,k) \left(\frac{v' - v}{v_0} \right)^{\frac{n-1}{n}}. \quad (80)$$

Now, we can let ε tends to 0 in (78) and (80). If we have $v' \leq v$, then we get the result combining (80) and (78) where we exchange v and v'. If $v \leq v$, we first control $I_M(v')$ by $I_M(v)$ using (78) and then apply (80) with v and v' exchanged. Combined with (78) we conclude the proof in the case $V(M) = +\infty$. If $V(M) < +\infty$ we can just take as Ω an isoperimetric region of volume v (which exists always), then apply the arguments leading to (80) to $M \setminus \Omega$ and consider as a competitor the finite perimeter set $\Omega' := \Omega \cup B_M(p, r_{v'})$, then it is straightforward to adapt the preceding arguments to conclude the proof. q.e.d.

4.2 Differentiability of I_M in bounded geometry

Lemma 3 (Lemma 3.2 of [MJ00] improved)**.** *Let $f :]a, b[\to \mathbb{R}$ be a continuous function. Then f is concave (resp. convex) if and only if for every $x_0 \in]a, b[$ there exists an open interval $I_{x_0} \subseteq]a, b[$ of x_0 and a concave (resp. convex) C^2 function $g_{x_0} : I_{x_0} \to \mathbb{R}$ such that $g_{x_0} = f(x_0)$ and $f(x) \leq g_{x_0}(x)$ (resp. $f(x) \geq g_{x_0}(x)$) for every $x \in I_{x_0}$.*

Remark 21. *The preceding Theorem is just a rephrasing of the supporting hyperplanes theorem for closed convex sets of $\mathbb{R}^n$. To apply it the hypothesis of continuity is crucial, we cannot assume f just lower or upper semicontinuous. In fact take as a counterexample a function that is strictly monotone increasing on $[a, b]$, right continuous in an interior point x_0 but not continuous at x_0 with a strictly positive jump in x_0, concave at the left of x_0 and to the right of x_0. This function is not concave on the entire interval $[a, b]$, is upper semicontinuous and satisfies the other hypothesis of Lemma 3.*

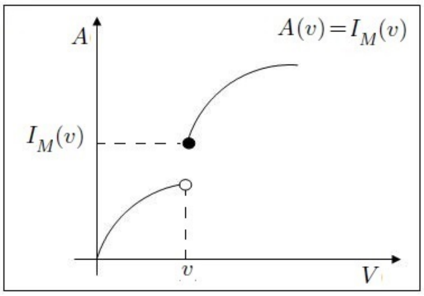

Fig. 3: An example of an upper semicontinuous function that satisfies all the assumptions of Lemma 3 but the continuity.

We recall here the generalized existence Theorem 1 of [Nar14a] stated under more general assumptions to check why this is legitimate one can see Remark 2.9 of [MN16], or Remarks 22, 20.

Theorem 4.2 (Generalized existence)**.** *Let M have C^0-locally asymptotically bounded geometry in the sense of Definition 3.4. Given a positive volume $0 < v < V(M)$, there are a finite number of limit manifolds at infinity such that their disjoint union with M contains an isoperimetric region of volume v and perimeter $I_M(v)$. Moreover, the number of limit manifolds is at worst linear in v.*

Remark 22. *The regularity discussion made there in Remark* 2.2 *of [MN16], is necessary in the proof of Corollary 3, where we need to do analysis on the limit manifolds, applying a (by now classical) formula for the second variation of the area functional on those isoperimetric regions which eventually lie in a limit manifold of possibly non-smooth boundary. The assumption of C^0 convergence of the metric tensor in the preceding lemma is due to the necessity of transporting volumes and perimeters in the limit manifold.*

Remark 23. *One possible application is to simplify part of the proof of different papers about existence and caracterisation of isoperimetric regions in non compact Riemannian manifolds and prove new theorems of the same kind. We can finish now the proof of Corollary 3.*

Proof: Using the generalized existence theorem of [Nar14a] and evaluating the second variation formula for the area functional on a generalized isoperimetric region $\Omega_{\bar{v}}$ in volume $V(\Omega_{\bar{v}}) = \bar{v}$ we can construct a smooth function $f_{\bar{v}}$ defined in a small neighborhood of $\bar{v}$, that we can compare locally with I_M. Consider the equidistant domains $\Omega_t := \{x \in M : d(x, \Omega_{\bar{v}}) \leq t\}$, if $r_{\bar{v}} \geq t \geq 0$, and $\Omega_t := M \setminus \{x \in M : d(x, M \setminus \Omega_{\bar{v}}) \leq t\}$, if $-r_{\bar{v}} \leq t < 0$, where $r_{\bar{v}} > 0$ is the normal injectivity radius of $\partial\Omega_{\bar{v}}$. Consider the inverse function of $t \mapsto V(\Omega_t)$ as a function of the volume, $v \mapsto t(v)$, and finally set $f_{\bar{v}}(v) := A(\partial\Omega_{t(v)})$ for v belonging to a small neighbourhood $I_{\bar{v}} = [\bar{v} - \varepsilon_{\bar{v}}, \bar{v} + \varepsilon_{\bar{v}}]$. To be rigorous in this construction we have to take care of the singular part the boundaries of domains $\partial\Omega_t$. This is done, carefully, in Proposition 2.1 and 2.3 of [Bay04]. Here we just ignore this technical complication, to make the exposition simpler to read. We just observe that the proof that we give here works mutatis mutandis also if we consider the case in which Ω is allowed to have a nonvoid singular part. Hence, for every $\bar{v} \in]0, V(M)[$, $f_{\bar{v}}$ gives smooth function $f_{\bar{v}} : [\bar{v} - \varepsilon_{\bar{v}}, \bar{v} + \varepsilon_{\bar{v}}] \to [0, +\infty[$, such that $f_{\bar{v}}(\bar{v}) = I_M(\bar{v})$ and $f_{\bar{v}} \geq I_M$. A standard application of the second variation formula see (V.4.3) [Cha06], or [BP86], shows that

$$f''_{\bar{v}}(v) = -\frac{1}{f^2_{\bar{v}}(v)} \left\{ \int_{\partial\Omega_{t(v)}} (|II|^2 + Ricci(\nu)) d\mathcal{H}^{n-1} \right\}. \tag{81}$$

From an elementary fact of linear algebra we know that $|II|^2 \geq \frac{h^2}{n-1}$. So substituting in the preceding inequality, we get

$$f''_{\bar{v}}(v) \leq -\frac{(n-1)k}{f_{\bar{v}}(v)}. \tag{82}$$

If $k \geq 0$, then $f_{\bar{v}}$ is concave and a straightforward application of Lemma 3 implies that I_M is concave in all $]0, V(M)[$. If $k < 0$ then

$$f''_{\bar{v}}(v) \leq -\frac{(n-1)k}{I_M(v)}, \tag{83}$$

because $|II|^2 \geq \frac{h^2}{n-1}$, where $h = f'_{\bar{v}}(\bar{v})$ by the first variation formula. To Suppose, now $k < 0$. Put

$$C = C(n, k, a, b) := \frac{(n-1)k}{2\delta_{M,a,b}}, \tag{84}$$

where $\delta_{M,a,b} := \inf\{I_M(v) : v \in [a, b]\}$ is strictly positive because by Theorem 3.6, I_M is continuous. For every $\bar{v} \in]a, b[$ it is easily seen that

$$I_M(v) + C(a, b)v^2 \leq f_{\bar{v}}(v) + C(a, b)v^2,$$

with

$$(f_{\bar{v}}(v) + C(a, b)v^2)'' \leq 0,$$

for every $v \in]a, b[\cap I_{\bar{v}}$. By Lemma 3, for $a, b \in]0, V(M)[$, $I_M(v) + C(a, b)v^2$ is concave in $[a, b]$. Hence, $I_M(v) + C(a, b)v^2$ is locally Lipschitz and it is straightforward to see that I_M is locally Lipschitz too, with $I'^+ \leq f'_{\bar{v}} \leq I'^-$, with equality holding at all but a countable set of points, which are the only points of discontinuity of I'^+ and I'^-. Moreover I'^+ and I'^- are nonincreasing so the set of points at which I_M is nonderivable is at most countable, moreover I'_M or $I'_M + 2Cv$ are respectively monotone nonincreasing see for this standard convexity arguments Corollary 2, page 29 of [Bou04] this implies that they are special cases of absolutely continuous functions and for this reason differentiable almost everywhere. So exists $I''_M(v)$ almost everywhere. Now, following [Bay04], for an arbitrary function f, set

$$\overline{D^2 f}(x_0) := \overrightarrow{\lim}_{\delta \to 0} \frac{f(x_0 + \delta) + f(x_0 - \delta) - 2f(x_0)}{\delta^2}. \tag{85}$$

When f is differentiable two times at x_0 it is straightforward to see that $f''(x_0) = \overline{D^2 f}(x_0)$. In a point $\bar{v}$ at which I_M is twice differentiable we observe that

$$I''_M(\bar{v}) = \overline{D^2 I_M}(\bar{v}) \leq f''_{\bar{v}}(\bar{v}).$$

Hence, (81) yields

$$I_M(\bar{v})I''{}_M(\bar{v}) \leq I_M(\bar{v})f''_{\bar{v}}(\bar{v}) \leq -I_M(\bar{v})\left(\frac{I'^2_M(\bar{v})}{n-1} - (n-1)k\right),$$

which is exactly (73), because $|II|^2 \geq \dfrac{h^2}{n-1}$, where $h = f'_{\bar{v}}(\bar{v})$ by the first variation formula, if equality holds in (73), then $|II|^2 = \frac{h^2}{n-1}$, which is equivalent to say that the regular part of $\partial\Omega_{\bar{v}}$ is totally umbilic. q.e.d.

4.3 Bavard-Pansu in bounded geometry

We rewrite for completeness the details of a Theorem that could be immediately deduced from the proof of (i) of [BP86] pp. 482, even if that theorem is stated for compact manifolds some of the arguments are still valid for a noncompact manifolds satisfying the hypothesis of the theorem below.

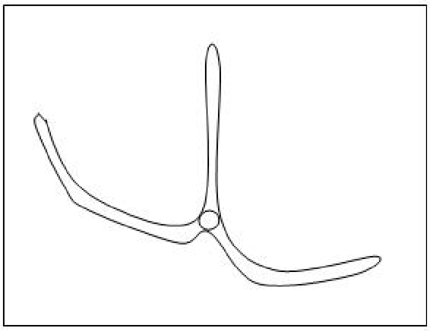

Fig. 4: Bavard-Pansu

Theorem 4.3. *[BP86] Let M^n be a complete Riemannian manifold with bounded geometry such that for every volume $v \in]0, V(M)[$ there exists an isoperimetric region Ω of volume v. Then I_M is continuous. Moreover ${I'_M}^+(v), {I_M}^-(v) \leq h = h(v, A(\partial B), n, k)$, where $B \subseteq M$ is any geodesic ball enclosing a volume v.*

Proof: Let $v \in]0, V(M)[$ be fixed. Consider a sequence of volumes $v_j \to v$. By the very definition of the isoperimetric profile we know that

$I_M(v_j) \leq A(\partial B_j)$ where $B_j := B(p, r_j)$ is any geodesic ball inclosing volume v_j and centered at a fixed point p. Now Take a sequence Ω_j of isoperimetric regions with $V(\Omega_j) = v_j$, this sequence exists by hypothesis. Theorem 2.1 of [HK78] ensures that the isoperimetric regions have length of mean curvature vector $|H_{\partial T_j}| =: h_j \leq h$, where h is a positive constant that does not depend on j but only on $\frac{v}{A(\partial B)}$ where B could be taken as the geodesic ball of center p and enclosing volume v in the comparison manifold $\mathbb{M}^n_k$. Again Theorem 2.1 of [HK78] shows that the inradius $\rho_j := \sup\{d(x, \partial\Omega_j) : x \in \Omega_j\} \geq \frac{v}{A(\partial B)}$, if $H_{\partial\Omega_j}$ points inside Ω_j. Observe here that $H_{\partial\Omega_j}$ cannot point outside in the noncompact part if $|H_{\partial\Omega_j}| > 1$. If $h_j = |H_{\partial\Omega_j}| \leq 1$ and points outside Ω_j then $V(\Omega_j) \leq A(\partial\Omega_j) \int_0^{\rho_j} (c_k(s) + h_j s_k(s))^{n-1} ds$ which implies again that $\rho_j \geq \rho = \rho(n, k, v, A(\partial B)) > 0$. This shows that Ω_j always contains a geodesic ball of radius ρ centered at a point p_j. Now by Theorem 4.1 I_M is upper semicontinuous. It remains to show lower semicontinuity. We know that $V(q, \rho) \geq \bar{v} > 0$ for every $q \in M$, by the noncollapsing hypothesis. Look at the case $v_j \geq v$ then if $v_j - v$ is small enough we can always pick a radius $0 < r_j < \rho$ such that $V(B(p_j, r_j)) = v_j - v$ again by the noncollapsing hypothesis. Put $\Omega'_j := \Omega_j \setminus B(p_j, r_j)$, we have $V(\Omega'_j) = v$, thus $I_M(v) \leq A(\partial\Omega') = A(\partial\Omega_j) + A(\partial B(p_j, r_j))$ and finally passing to the limit we obtain $I_M(v) \leq \varliminf I_M(v_j)$. If $v_j \leq v$ then the proof is easier and consists in just adding a small ball outside Ω_j to finish the proof. q.e.d.

Remark 24. *Applying the proof of Theorem 4.3 to generalized isoperimetric regions we see easily that the conclusions of Theorem 4.3 holds if we assume that M has C^0-locally bounded geometry.*

Remark 25. *It is not too hard to see that Corollary 3 could be seen also as a corollary of Theorem 4.3, without using the proof of Theorem 3.6, because we could argue the continuity of I_M from the proof of Theorem 4.3 applied to generalized isoperimetric regions and continue unchanged the proof of Corollary 3.*

The argument of the proof of [BP86] that cannot be extended easily to the noncompact case with collapsing, concerns the proof of the concavity of the isoperimetric function plus a quadratic function, without passing previously from a proof of the continuity of I_M. We don't know if this is possible but a priori the proof seems quite more involved and for the moment we are not able to do it. We present in the following theorem another extension of the arguments of [BP86] that permits to argue weaker conclusion on the isoperimetric profile but still not the continuity or concavity.

Theorem 4.4. *Let M^n be a complete Riemannian manifold with Ricci $\geq k$ such that for every volume $v \in]0, V(M)[$ there exists an isoperimetric region Ω of volume v. Then for every $[a,b] \subset]0, V(M)[$ there exists a constant $C = C(a,b,n,k,M)$ such that $v \mapsto I_M - C(a,b,n,k,M)v^2$ have nonpositive second derivatives in the sense of distribution.*

Proof: If $k < 0$ then

$$\begin{aligned} f''_{\bar{v}}(v) &\leq -\frac{(n-1)k}{I_M(v)} \\ &\leq -\frac{(n-1)k}{a}\sup\left\{\frac{\bar{v}}{I_M(\bar{v})}|\bar{v}\in[a,b]\right\} \\ &\leq -\frac{(n-1)k}{a}\sup\left\{J(h,\rho)|\bar{v}\in[a,b]\right\} \\ &= -\frac{(n-1)k}{a}\delta\left(n,k,a,b\right), \end{aligned}$$

where $J(h,\rho) := \int_0^\rho \left((c_k(s) + |h|s_k(s)\right)^{n-1} ds$, h is an upper bound on the length of the mean curvature of the isoperimetric regions in the interval $[a,b]$ and $\rho = \rho(n,k,v,A(\partial B)))$, where B is any geodesic ball enclosing a volume v in $\mathbb{M}^n_k$. q.e.d.

Remark 26. *In our opinion, remains still an open question wether Ricci bounded below and existence of isoperimetric regions for every volume implies continuity of the isoperimetric profile in presence of collapsing. We are not able to extend to this setting the arguments of [BP86]. The examples of discontinuous isoperimetric profile constructed in [NP14] have Ricci curvature tending to $-\infty$.*

5 The isoperimetric problem of a complete Riemannian manifold with a finite number of C^0-asymptotically Schwarzchild ends

The main result of this section is the following theorem which is a nontrivial consequence of the theory developed in [Nar14b], [Nar14a], [FN14], [FN15b], combined with the work done in [EM13b]. This gives answers to some mathematical problems arising naturally in general relativity.

Theorem 3. *Let (M^n, g) be an $n \geq 3$ dimensional complete boundaryless Riemannian manifold. Assume that there exists an open relatively compact set $U \subset\subset M$ such that $M \setminus U = \mathring{\bigcup}_{i \in \mathcal{I}} E_i$, where $\mathcal{I} := \{1, ..., l\}$, $l \in \mathbb{N} \setminus \{0\}$, and each E_i is an end which is C^0-asymptotic to Schwarzschild of mass $m > 0$ at rate γ, see Definition 5.4. Then there exists $V_0 = V_0(M, g) > 0$ such that for every $v \geq V_0$ there exists at least one isoperimetric region Ω_v enclosing volume v. Moreover Ω_v satisfies the conclusions of Lemma 5.1.*

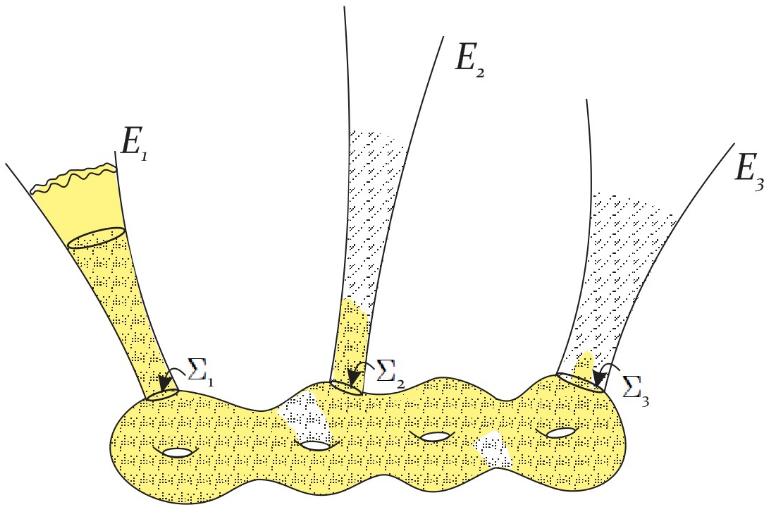

Fig. 5: The isoperimetric region Ω is in yellow and B is dotted.

Remark 27. *The characterization of isoperimetric regions in Theorem 3 is achieved using Theorem* 4.1 *of [EM13b] applied to the part of an isoperimetric region that have a sufficiently big volume in an end, the* *details and suitable modifications of the proof are presented in Lemma 5.1.*

Remark 28. *Since the ends are like in [EM13b] it follows trivially that, if it happens that an end E is C^2-asymptotic to Schwarzschild, then the volume V_0 can be chosen in such a manner that there exists a unique smooth isoperimetric (relatively to E) foliations of $E \setminus B$. Moreover, if E is asymptotically even (see Definition 2.1 of [EM13b]) then the centers of mass of $\partial\Omega_v$ converge to the center of mass of E, as V goes to $+\infty$, compare section 5 of [EM13b].*

Corollary 5. *If we allow M in the preceding theorem to have each end E_i, with mass $m_i > 0$. Then there exists a volume $V_0 = V_0(M, g) > 0$ and a subset $\mathcal{I} \subseteq \{1, ..., l\}$, defined as $\mathcal{I} := \{i : m_i = \max\{m_1, \dots, m_l\}\}$ such that for every volume $v \in [V_0, +\infty[$ there exist an isoperimetric region Ω_v that satisfies the conclusion of Lemma 5.1 in which the preferred end $E_{\Omega_v} \in \{E_i\}_{i \in \mathcal{I}}$. In particular, if $m_i \neq m_j$ for all $i \neq j$, then $\mathcal{I} = \{i\}$ is reduced to a singleton and this means that there exists exactly one end E_i in which the isoperimetric regions for large volumes prefer to stay with a large amount of volume.*

In the next theorem paying the price of strengthening the rate of convergence to the Scwarzschild metric inside each end, we can show existence of isoperimetric regions in every volumes. The proof uses the generalized existence theorem of [Nar14a] and a slight modification of the fine estimates for the area of balls that goes to infinity of Proposition 12 of [BE13].

Theorem 4. *Let (M^n, g) be an $n \geq 3$ dimensional complete boundaryless Riemannian manifold. Assume that there exists a relatively compact open set $U \subset\subset M$ such that $M \setminus U = \mathring{\bigcup}_{i \in \mathcal{I}} E_i$, where $\mathcal{I} := \{1, ..., l\}$, $l \in \mathbb{N} \setminus \{0\}$, and each E_i is a C^0-strongly asymptotic to Schwarzschild of mass $m > 0$ end, see Definition 5.5. Then for every volume $0 < v < V(M)$ there exists at least one isoperimetric region Ω_v enclosing volume v.*

Corollary 6. *The conclusion of the preceding theorem still holds if we allow to the ends E_i of M to have different masses $m_i > 0$.*

5.1 Definition, notations and some basic facts

Now we come back to the main interest in our theory, i.e., to extend arguments valid for compact manifolds to noncompact ones. To this aim let us introduce the following definition suggested by Theorem 1.

Definition 5.1. *We call* $D_\infty = \bigcup_i D_{\infty,i}$ *a finite perimeter sets in* $\tilde{M}$ *a* ***generalized set of finite perimeter of*** M *and an isoperimetric region of* $\tilde{M}$ *a* ***generalized isoperimetric region****, where* $\tilde{M}$ *is ....*

Remark 29. *We remark that* D_∞ *is a finite perimeter set in volume* v *in* $\mathring{\bigcup}_i M_{\infty,i}$.

Remark 30. *If* D *is a genuine isoperimetric region contained in* M*, then* D *is also a generalized isoperimetric region with* $N = 1$ *and*

$$(M_{\infty,1}, g_{\infty,1}) = (M, g).$$

This does not prevent the existence of another generalized isoperimetric region of the same volume having more than one piece at infinity.

Definition 5.2. *Let* $m \in \mathbb{N}$ *and* $\alpha \in [0, 1]$ *be given. We say that a complete Riemannian* n*-manifold* (M, g) *is* $C^{m,\alpha}$***-locally asymptotically flat*** *or equivalently* $C^{m,\alpha}$***-locally asymptotically Euclidean*** *if for every diverging sequence of points* $(p_j)_{j\in\mathbb{N}}$ *there exists a subsequence* $(p_{j_l})_{l\in\mathbb{N}}$ *such that the sequence of pointed manifolds*
$(M, g, p_{j_l}) \to (\mathbb{R}^n, \delta, 0)$ *in the pointed* $C^{m,\alpha}$*-topology, where* δ *is the canonical Euclidean metric of* $\mathbb{R}^n$.

Remark 31. *Observe that an* $C^{m,\alpha}$*-locally asymptotically Euclidean manifold in the sense of Definition 5.2 is of bounded geometry in the sense of definition 5.2.*

Definition 5.3. *An* ***initial data set*** (M, g) *is a connected complete boundaryless* n*-dimensional Riemannian manifold such that there exists a positive constant* $C > 0$*, a bounded open set* $U \subset M$*, a positive natural number* $\tilde{N}$*, such that* $M \setminus U = \mathring{\bigcup}_{i=1}^{\tilde{N}} E_i$ *each* E_i *is an end, and* $E_i \cong_{x_i} \mathbb{R}^n \setminus B_1(0)$*, in the coordinates induced by* $x_i = (x_i^1, ..., x_i^n)$ *satisfying*

$$r|g_{ij} - \delta_{ij}| + r^2|\partial_k g_{ij}| + r^3|\partial^2_{kl} g_{ij}| \leq C, \tag{86}$$

for all $r \geq 2$*, where* $r := |x| = \sqrt{\delta_{ij} x^i x^j}$*, (Einstein convention). We will use also the notations* $B_r := \{x \in \mathbb{R}^n : |x| < r\}$*, and* $S_r := \{x \in \mathbb{R}^n : |x| = r\}$*, for a* ***centered coordinate ball of radius*** r *and a* ***centered coordinate sphere of radius*** r*, respectively.*

Remark 32. *Observe that an initial data set in the sense of Definition 5.3 is* C^2*-locally asymptotically Euclidean in the sense of definition 5.2.*

In what follow we always assume that $n \geq 3$.

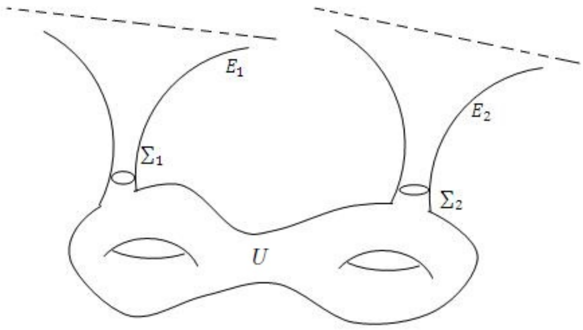

Fig. 6: Schwarzschild Multiend, with Σ_1 and Σ_2 boundaries of the ends E_1 and E_2.

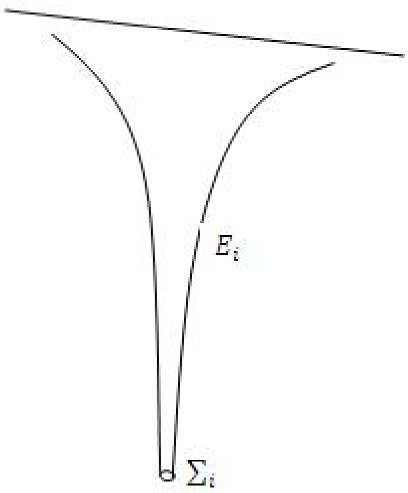

Fig. 7: Σ_i boundary of the end E_i.

Definition 5.4. *For any $m > 0$, $\gamma \in (0, 1]$, and $k \in N$, we say that an initial data set is C^k-**asymptotic to Schwarzschild of mass** $m > 0$ **at rate** γ, if*

$$\sum_{l=0}^{k} r^{n-2+\gamma+l} |\partial^l (g - g_m)_{ij}| \leq C, \tag{87}$$

for all $r \geq 2$, in each coordinate chart $x_i : E_i \cong \mathbb{R}^n \setminus B_{\mathbb{R}^n}(0, 1)$, where

$(g_m)_{ij} = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta_{ij}$ *is the usual Schwarzschild metric on* $(\mathbb{R}^n \setminus \{0\})$.

Definition 5.5. *For any* $m > 0$, $\gamma \in]0, +\infty[$, *we say that an initial data set is* C^0***-strongly asymptotic to Schwarzschild of mass*** $m > 0$ ***at rate*** γ, *if*

$$r^{2n+\gamma} | (g - g_m)_{ij} | \leq C, \tag{88}$$

for all $r \geq 2$, *in each coordinate chart* $x_i : E_i \cong \mathbb{R}^n \setminus B_{\mathbb{R}^n}(0,1)$, *where* $(g_m)_{ij} = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta_{ij}$ *is the usual Schwarzschild metric on* $(\mathbb{R}^n \setminus \{0\})$.

5.2 Proof of Theorems 3, 4, and of Corollaries 5, 6

Lemma 5.1. *Under the same assumptions of Theorem 3, there exists* $V_0 = V_0(M, g) > 0$, *and a large ball* B *such that if* $\Omega \subseteq M$ *is an isoperimetric region with* $V(\Omega) = v \geq V_0$, *then there exists an end* $E_i = E_\Omega$ *such that* $\Omega \cap E_\Omega$ *is the region below a normal graph based on* $\tilde{S}^i_r$ *where* $V_g(\Omega \cap E_\Omega) = V_g(\tilde{B}_r)$, *i.e.,* $\Omega = x_i^{-1}(\varphi(B_r \setminus B_1))\mathring{\cup}\Omega^*$, *with* $\Omega^* \subseteq B$ *and* $\varphi(B_r \setminus B_1) \subseteq \mathbb{R}^n \setminus B_{\mathbb{R}^n}(0,1)$ *is a suitable perturbation of* $B_r \setminus B_1$. $\Omega \cap E_\Omega$ *contains* Σ_i *and is an isoperimetric region as in Theorem 4.1 of [EM13b],* $\Omega \setminus E_\Omega$ *contains* Σ_i *and* $\Omega \setminus E_\Omega$ *has least relative perimeter with respect to all domains in* $B \setminus E$ *containing* Σ_i *and having volume equal to* $V(\Omega \setminus E_\Omega)$.

Remark 33. *In general* B *contains* U *and is much larger than* U, *see figure 8.* B *could be chosen in such a way that* $M \setminus B$ *is a union of ends that are foliated by the boundary of isoperimetric regions of that end, provided this foliation exists. Furthermore* B *contain* U *and all the* $\tilde{B}^i_r$, *with* r *large enough to enclose a volume bigger than the volume* V_0 *given by Theorem* 4.1 *of [EM13b].*

Remark 34. *Corollary* 16 *of [BE13] is a particular instance of Lemma 5.1 when the number of ends is two. Of course, in Corollary* 16 *of [BE13] more accurate geometrical informations are given due to the very special features of the double Schwarzschild manifolds considered there. See figure 10 in which the same notation of Corollary* 16 *of [BE13] are used.*

Proof: In first observe that the existence of a geodesic ball B satisfying the conclusions of the Lemma is essentially equivalent to the

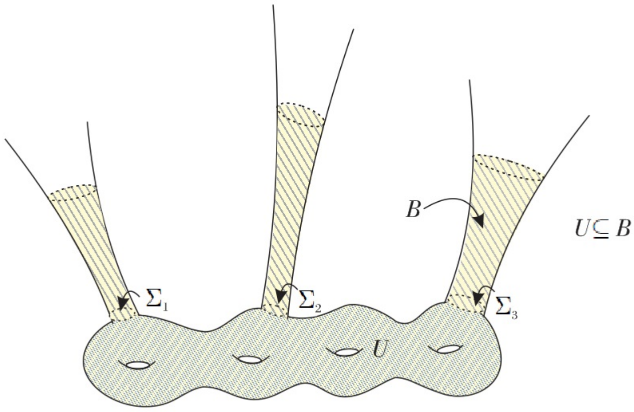

Fig. 8: M with $U \subseteq B$, $\Sigma_i = \partial E_i$ and E_1, E_2, E_3 ends.

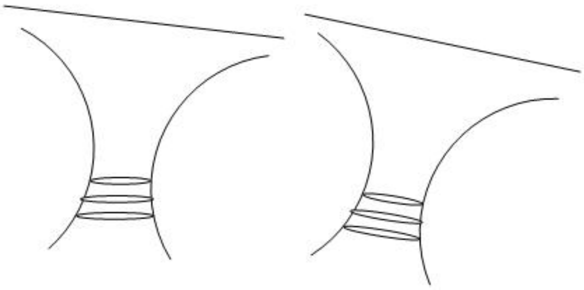

Fig. 9: $M \setminus B$

weaker assumption that the same geodesic ball B contain just all the volume of $\Omega \setminus E$. To see this is trivial because we know that we can modify a finite perimeter set on a set of measure zero and stay always in the same equivalence class. Now, assume that a geodesic ball B satisfying the conclusions of Lemma 5.1 does not exists. Let (Ω_j) be a sequence of isoperimetric regions in M, such that $V(\Omega_j) \to +\infty$. It follows easily using the fact that the number of ends is finite that there exists at least one end $E_{i_j} =: E_j$, such $V(\Omega_j \cap E_j) \to +\infty$. The crucial point is to show that this end E_{i_j} is unique. To show this we observe that the proof of Theorem 4.1 of [EM13b], applies exactly in the same way to our sequence (Ω_j) and our manifold M. This application of Theorem 4.1 of

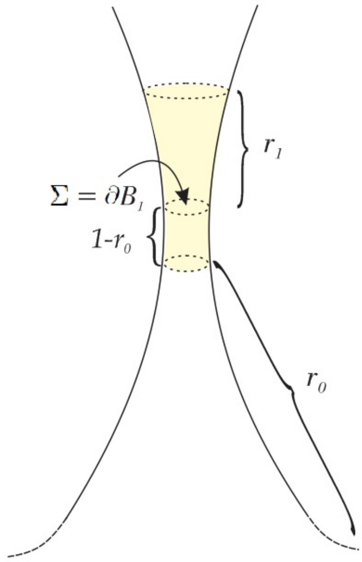

Fig. 10: In Corollary 16, of [BE13], (M, g) is the double Schwarzschild manifold of positive mass $m > 0$, $(\mathbb{R}^n \setminus 0, g_m)$. The isoperimetric region Ω is in yellow, B is dotted, E_1 is the preferred end. Just in this example Σ is outermost minimal, w.r.t. any end.

[EM13b] gives us a volume V_0 that depends only on the geometric data of the ends such that if Ω is an isoperimetric region of volume $v \geq V_0$ then there exist an end E such that $\Omega \cap E$ contains a large centered ball $\tilde{B}_{r/2}$ with $V(\tilde{B}_r) = V(\Omega \cap E)$. In particular, this discussion shows that for large values of the enclosed volume $v \geq V_0$ an isoperimetric region Ω is such that

$$\Sigma \subseteq \Omega, \tag{89}$$

and finally that $\Omega \cap E = x^{-1}(\varphi(B_r \setminus B_1))$, for larges values of $V(\Omega)$, where Σ is the boundary of E. Now we show that there is no infinite volume in more than one end. Roughly speaking, this follows quickly from the estimates (101) (a particular case of which is (3) of [EM13a]) because the dominant term in the expansion of the area with respect to volume is as the isoperimetric profile of the Euclidean space which shows that two big different coordinate balls each in one different end do worst than one coordinate ball in just one end enclosing a volume that is the sum of the other two balls. To show this rigorously we will argue by contradiction. Assume that for every j, there exist two distinct ends $E_{j1} \neq E_{j2}$ such that $V(\Omega_j \cap E_{j1}) =: v_{j1} \to +\infty$ and $V(\Omega_j \cap E_{j2}) =: v_{j2} \to +\infty$. Again an application of Theorem 4.1 of [EM13b] permits us to say that $\Omega_j \cap E_{j1}$ and $\Omega_j \cap E_{j2}$ are perturbations of large coordinates balls whose expansion of the area with respect to the enclosed volume is given as in (101) by

$$I_{\mathbb{R}^n}(v) - m^* v^{\frac{1}{n}} + o(v^{\frac{1}{n}}), \tag{90}$$

where v is the volume enclosed by the isoperimetric region inside an end. Now put $\Omega_j' := \Omega_j \setminus E_{j2} \cup \Omega_j''$, in such a way $\Omega_j' \cap E_{j1}$ is isoperimetric in E_{j1} and $V(\Omega_j') = V(\Omega_j) = v$, we have that

$$A(\partial\Omega_j') = I_{\mathbb{R}^n}(v_{j1} + v_{j2}) - m^* (v_{j1} + v_{j2})^{\frac{1}{n}} + A(\Sigma_2) + A_1, \tag{91}$$

$$A(\partial\Omega_j) = I_{\mathbb{R}^n}(v_{j1}) + I_{\mathbb{R}^n}(v_{j2}) - m^* v_{j1}^{\frac{1}{n}} - m^* v_{j2}^{\frac{1}{n}} + A_2, \tag{92}$$

where the quantities A_1 and A_2 could go at infinity because of the contribution of the other ends, but in fact does not contributes to the asymptotic expansion of the difference of the areas Ω_j' and Ω_j because Ω_j' and Ω_j coincides on $M \setminus E_{j1} \mathring{\cup} E_{j2}$. Thus we have

$$\begin{aligned} A(\partial\Omega_j') - A(\partial\Omega_j) &= I_{\mathbb{R}^n}(v_{j1} + v_{j2}) - I_{\mathbb{R}^n}(v_{j1}) - I_{\mathbb{R}^n}(v_{j2}) && (93)\\ &- m^* (v_{j1} + v_{j2})^{\frac{1}{n}} + m^* v_{j1}^{\frac{1}{n}} + m^* v_{j2}^{\frac{1}{n}} + \cdots && (94)\\ &\to -\infty, && (95) \end{aligned}$$

when $v_{j1}, v_{j2} \to +\infty$. In particular we obtain that for large j that $\mathcal{P}(\Omega_j') < \mathcal{P}(\Omega_j)$, which contradicts the hypothesis that Ω_j is a sequence of isoperimetric regions. At this point we have to show that the diameters of $\Omega_j \cap (M \setminus E_j)$ are uniformly bounded w.r.t. j. We start this argument by noticing that $V(\Omega_j \cap (M \setminus E_j)) \leq K$ are uniformly bounded, by the same fixed positive constant K and so we can pick a representative of Ω_j such that $\Omega_j \cap (M \setminus E_j)$ is entirely contained in B. Analogously to what is done in the proof of Theorem 3 of [Nar14a] or Lemma 3.8 of [Nar09b] (these proofs were inspired by preceding works of Frank Morgan [Mor94] proving boundedness of isoperimetric regions in the Euclidean setting and Manuel Ritoré and Cesar Rosales in Euclidean cones [RR04]) we can easily conclude that

$$diam(\Omega_j \cap (M \setminus E_j)) \leq C(n, k, v_0) V(\Omega_j \cap (M \setminus E_j))^{\frac{1}{n}} \leq C K^{\frac{1}{n}}, \tag{96}$$

which ensures the existence of our big geodesic ball B. According to (89) we have $\Sigma \subset \Omega \setminus E_\Omega$, to finish the proof we prove that $\Omega \setminus E_\Omega \subseteq B \setminus E_\Omega$ is such that $A(\partial\Omega \cap (B \setminus E_\Omega))$ is equal to

$$\inf \left\{ A(\partial D \cap (B \setminus E_\Omega)) : D \subseteq B \setminus E_\Omega, \Sigma \subseteq D, V(D) = V(\Omega \setminus E_\Omega) \right\}. \tag{97}$$

Now we proceed to the detailed verification of (97). In fact if (97) was not true we can find a finite perimeter set Ω' (that can be chosen open and bounded with smooth boundary, but this does not matter here) inside $B \setminus E$ such that $\Sigma \subseteq \Omega'$, $V(\Omega') = V(\Omega \setminus E_\Omega)$, and

$$A(\partial\Omega') < A(\partial\Omega \cap (B \setminus E_\Omega)),$$

but if this is the case we argue that $V((\Omega \cap E_\Omega) \cup \Omega') = v$ and

$$A\left(\partial\left[(\Omega \cap E_\Omega) \cup \Omega'\right]\right) < A(\partial\Omega),$$

which contradicts the fact that Ω is an isoperimetric region of volume v. q.e.d.

Now we prove Theorem 3.

Proof: Take a sequence of volumes $v_i \to +\infty$. Applying the generalized existence Theorem 1 of [Nar14a], we get that there exists $\Omega_i \subset M$, (Ω_i is eventually empty) isoperimetric region with $V(\Omega_i) = v_{i1}$ and $B_{\mathbb{R}^n}(0, r_i) \subset \mathbb{R}^n$ with $V(B_{\mathbb{R}^n}(0, r_i)) = v_{i2}$, satisfying $v_{i1} + v_{i2} = v_i$, and $I_M(v_i) = I_M(v_{i1}) + I_{\mathbb{R}^n}(v_{i2})$. We observe that $I_M(v_{i1}) = A(\partial\Omega_i)$ and that we have just one piece at infinity because two balls do worst than one in Euclidean space. Note that this argument was already used in the proof of Theorem 1 of [MN16]. If $v_{i2} = 0$ there is nothing to prove, the existence of isoperimetric regions follows immediately. If $v_{i2} > 0$ one can have three cases

1. $v_{i1} \to +\infty$,
2. there exist a constant $K > 0$ such that $0 < v_{i1} \leq K$ for every $i \in \mathbb{N}$,
3. $v_{i1} = 0$, for i large enough.

We will show, in first, that we can rule out case 2) and 3). To do this, suppose by contradiction that $0 < v_{i1} \leq K < +\infty$ then remember that by Theorem 1 of [FN14] the isoperimetric profile function I_M is continuous so $V(\Omega_i) + A(\partial\Omega_i) \leq K_1$ where $K_1 > 0$ is another positive constant. We can extract from the sequence of volumes v_{i1} a convergent subsequence named again $v_{i1} \to \bar{v} \geq 0$. By generalized existence we obtain a generalized isoperimetric region $D \subset \tilde{M}$ such that $V(D) = \bar{v}$, $I_M(\bar{v}) = A(\partial D)$. Again $D = D_1 \mathring{\cup} D_\infty$, with $D_1 \subset M$ and $D_\infty \subset \mathbb{R}^n$ isoperimetric regions in their respective volumes and in their respective ambient manifolds. Hence D_∞ is an Euclidean ball. But also by the continuity of $I_M = I_{\tilde{M}}$ we have that $D \mathring{\cup} B_{\mathbb{R}^n}(0, r_i)$ is a generalized isoperimetric region of volume $\bar{v} + v_{i2}$, it follows that $H_{\partial D_\infty} = \frac{n-1}{r_i}$ for every $i \in \mathbb{N}$. As a consequence of the fact that $v_i \to +\infty$ and (v_{i1}) is a bounded sequence we must have $v_{i2} \to +\infty$, hence $r_i \to +\infty$, and we get $H_{\partial D_\infty} = \lim_{r_i \to +\infty} \frac{n-1}{r_i} = 0$. As it is easy to see it is impossible to have an Euclidean ball with finite positive volume and zero mean curvature. This implies that $D_\infty = \emptyset$, for v_i large enough. As a consequence of the proof of Theorem 2.1 of [RR04] or Theorem 1 of [Nar14a] and the last fact we have $\Omega_i \to D_1$ in the sense of finite perimeter sets of M. This last assertion implies that $V(D_1) = \bar{v} = \lim_{i\to+\infty} v_{i1}$. By Lemma 2.7 of [Nar14a] we get $I_M \leq I_{\mathbb{R}^n}$. It follows that

$$0 \leq I_M(v_{i1}) \leq I_{\mathbb{R}^n}(v_i) - I_{\mathbb{R}^n}(v_{i2}) \to 0, \tag{98}$$

because $v_i - v_{i2} \to \bar{v}$ and $I_{\mathbb{R}^n}$ is the function $v \mapsto v^{\frac{n-1}{n}}$, with fractional exponent $0 < \frac{n-1}{n} < 1$. By (98) $\lim_{i\to+\infty} I_M(v_{i1}) = 0$, since I_M is continuous we obtain

$$\lim_{i\to+\infty} I_M(v_{i1}) = I_M(\bar{v}) = 0 = A(\partial D_1),$$

which implies that $V(D_1) = \bar{v} = 0$. Now for small nonzero volumes, isoperimetric regions are psedobubbles with small diameter and big mean curvature $H_{\partial\Omega_i} \to +\infty$, because M is C^2-locally asymptotically Euclidean, compare [Nar14b] (for earlier results in the compact case compare [Nar09a]), but this is a contradiction because by first variation of area $H_{\partial\Omega_i} = H_{\partial B_{\mathbb{R}^n}(0,r_i)} = \frac{n-1}{r_i}$, with $r_i \to +\infty$. We have just showed that $v_{i1} = 0$ for i large enough provided v_{i1} is bounded, that is case 2) is simply impossible.

Remark 35. *The argument just given here shows that v_{i1} uniformly bounded implies $v_{i1} = 0$ is a well formed formula valid in an arbitrary C^2-asymptotically Euclidean manifold, always for the same reason that in an Euclidean isoperimetric context two balls do worst then one.*

Consider, now, the case 3), i.e., $v_{i1} = 0$. To rule out this case we compare a large Euclidean ball of enclosed volume v_{i2} with $\Omega_v :=$

$x_i^{-1}(B_r \setminus B_1)$ choosing r such that $V(\Omega_{v_{i2}}) = v_{i2}$, by (100), we get $A(\partial\Omega_{v_{i2}}) \leq c_n v_{i2}^{\frac{n-1}{n}}$. If $v_{i1} = 0$, for large i then we have that all the mass stays in a manifold at infinity and so if we want to have existence we need an isoperimetric comparison for large volumes between $I_M(v)$ and $I_{M_\infty}(v) = I_{\mathbb{R}^n}(v)$. This isoperimetric comparison is a consequence of (100) which gives that there exists a volume $v_0 = v_0(C, m)$ (where C is as in Definition 5.4) such that

$$I_M(v) < I_{\mathbb{R}^n}(v), \tag{99}$$

for every $v \geq v_0$. To see this we look for finite perimeter sets $\Omega'_v \subset M$ which are not necessarily isoperimetric regions, which have volume $V(\Omega'_v) = v$ and $A(\partial\Omega'_v) < I_{\mathbb{R}^n}(v)$. A candidate for this kind of domains are coordinate balls inside a end $\tilde{B}_r := x_i^{-1}(B_r \setminus B_1(0))$, with r such that $V(\tilde{B}_r) = v$, because after straightforward calculations

$$A(\partial\tilde{B}_{r(v)}) = I_{\mathbb{R}^n}(v) - m^* v^{\frac{1}{n}} + o(v^{\frac{1}{n}}) = c_n v^{\frac{n-1}{n}} - m^* v^{\frac{1}{n}} + o(v^{\frac{1}{n}}), \tag{100}$$

where $m^* > 0$ is the same coefficient that appears in the asymptotic expansion of

$$A_{g_m}(\partial\Omega_{v_m}) = I_{\mathbb{R}^n}(v_m) - m^* v_m^{\frac{1}{n}} + o(v^{\frac{1}{n}}),$$

$v_m := V_{g_m}(\Omega_v)$. Namely $m^* = c'_n m > 0$, where c'_n is a dimensional constant that depends only on the dimension n of M. The calculation of m^* is straightforward and we omit here the details, in the case of $n = 3$ it comes immediately from (3) of [EM13a]. It is worth to note here that the assumption (87) in Definition 5.4, is crucial to have the remainder in (100) of order of infinity strictly less than $v^{\frac{1}{n}}$. If the rate of convergence of g to g_m was of the order $r^{-\alpha}$ with $0 < \alpha \leq n-2$ then this could add some extra term to m^* in the asymptotic expansion (100) that we could not control necessarily. This discussion permits to exclude case 3).

Remark 36. *As was pointed out to us by an anonymous referee that we aknowledge, there is a simpler way to show that when $v_i \to +\infty$ then $v_{i1} \to +\infty$ too. To see this we observe that asymptotically for large v_i it holds, by the asymptotic expansion of the area of a centered coordinate sphere with respect to the enclosed volume, that*

$$\begin{aligned} I_{\mathbb{R}^n}(v_i - v_{i1}) \leq I_M(v_{i1}) + I_{\mathbb{R}^n}(v_i - v_{i1}) &= I_M(v_i) \leq A(\partial\Omega_i^*) \\ &= I_{\mathbb{R}^n}(v_i) - m^* v_i^{\frac{1}{n}} + o(v_i^{\frac{1}{n}}), \end{aligned}$$

where Ω_i^* could be chosen as a domain $\Omega_i^* := U \cup \tilde{B}_{r_i}^1$ such that $V(\Omega_i^*) = v_i$. Now dividing by v_i the preceding inequality, the fact of assuming $v_i \to +\infty$ and v_{i1} uniformly bounded with respect to i, leads to a contradiction.

So we are reduced just to the case 1). We will show that the only possible phenomenon that can happen is $v_{i1} \to +\infty$ and $v_{i2} = 0$. With this aim in mind we will show in first that it is not possible to have $v_{i1} \to +\infty$ and also $v_{i2} \to +\infty$ at the same time. A way to see this fact is to consider equation (100) and observe that the leading term is Euclidean, now we take all the mass v_{i2} and from infinity and we transport a volume v_{i2} adding it to the part in the end E_i, in this way we construct a competitor set (as in the proof of Lemma 5.1) $\tilde{\Omega}_{v_i}$ which is isoperimetric in the preferred end E_i and such that $\tilde{\Omega}_{v_i} \setminus \Omega_{v_{i1}} = x_i^{-1}(\varphi_i(B_{\tilde{r}_i} \setminus B_{r_i}))$, where E_i is some fixed end satisfying $V(E_i \cap \Omega_{v_{i1}}) \to +\infty$, for suitable $\tilde{r}_i > r_i > 1$, φ_i diffeomorphism such that $V(x_i^{-1}(\varphi_i(B_{r_i} \setminus B_{r_i}))) = v_{i2}$ in such a way that $\tilde{\Omega}_{v_i} \cap E_i$ is an isoperimetric region containing Σ_i the boundary of E_i, i.e., a pertubation of a large coordinate ball as prescribed by Theorem 4.1 of [EM13b], $\tilde{\Omega}_{v_i} \cap (B \setminus E) = \Omega_{v_{i1}} \cap (B \setminus E) =: \bar{\Omega}_{\bar{v}_{i1}}$ and $V(\tilde{\Omega}_{v_i}) = v_i$. Hence by virtue of (100) we get for large v_{i1}

$$I_M(v_{i1}) = A(\partial B_{r(v_{i1})}) + c(\bar{v}_{i1}) + \varepsilon(v_{i1}) = I_{\mathbb{R}^n}(v_{i1}) - m^* v_{i1}^{\frac{1}{n}} + o(v_{i1}^{\frac{1}{n}}), \quad (101)$$

where $\varepsilon(v_{i1}) \to 0$ when $v_{i1} \to +\infty$, $c(\bar{v}_{i1})$ is the relative area of the locally isoperimetric region $\bar{\Omega}_{\bar{v}_{i1}}$ of volume $\bar{v}_{i1}$ inside $B \setminus E$ where B is the fixed big ball of Lemma 5.1.

$$I_M(v_i) = I_M(v_{i1}) + I_{\mathbb{R}^n}(v_{i2}) \leqslant A(\partial \tilde{\Omega}_{v_i})$$

and

$$A(\partial \tilde{\Omega}_{v_i}) = I_{\mathbb{R}^n}(v_{i1} + v_{i2} - \bar{v}_{i1}) - m^*(v_{i1} + v_{i2} - \bar{v}_{i1})^{\frac{1}{n}} + o((v_{i1} + v_{i2} - \bar{v}_{i1})^{\frac{1}{n}})$$

see that $I_{\mathbb{R}^n}(v_{i1}) + I_{\mathbb{R}^n}(v_{i2}) \gtrsim I_{\mathbb{R}^n}(v_{i1} + v_{i2} - \bar{v}_{i1})$ for $v_{i1} \to \infty$ and $v_{i2} \to \infty$.
Therefore, we have $I_M(v_i) \gtrsim A(\partial \tilde{\Omega}_{v_i})$ which is a contradiction.
It is easy to see that $\Omega_{v_{i1}} \cap (B \setminus E)$ could be caracterized as the isoperimetric region for the relative isoperimetric problem in $B \setminus E$ which contain the boundary Σ of E. Such a relative isoperimetric region Ω' exists by standard compactness arguments of geometric measure theory, and regularity theory as in [EM13a], (compare also Theorem 1.5 of [DS92]), in particular $A(\partial \Omega' \cap (B \setminus E))$ is equal to

$$\inf \left\{ A(\partial D \cap (B \setminus E)) : D \subseteq B, \Sigma \subseteq D, V(D) = V(\Omega' \setminus E) \right\}. \quad (102)$$

Again by compactness arguments it is easy to show that the relative isoperimetric profile $I_{B\setminus E}:[0,V(B\setminus E)]\to[0,+\infty[$ is continuous (one can see this using the proof Theorem 1 of [FN14] that applies because we are in bounded geometry), and so $||I_{B\setminus E}||_\infty = c < +\infty$. If one prefer could rephrase this in terms of a relative Cheeger constant. This shows that $c(v)\leq c$ for every $v\in[0,V(B\setminus E)]$. This last fact legitimate the second equality in equation (101). Thus readily follows

$$A(\partial\tilde{\Omega}_{v_i}) < I_M(v_{i1}) + I_{\mathbb{R}^n}(v_{i2}) = I_M(v_i), \tag{103}$$

for large volumes $v_i\to+\infty$, which is the desired contradiction. We remark that the use Lemma 5.1 is crucial to have the right shape of $\Omega_{v_{i1}}$ inside the preferred end E. To finish the proof, the only case that remains to rule out is when $v_{i1}\to+\infty$ and $0<v_{i2}\leq const.$ for every i. By the generalized compactness Theorem 1 of [FN15b] there exists $v_2\geq 0$ such that $v_{i2}\to v_2$. If $v_2>0$ then comparing the mean curvatures like already did in this proof, to avoid case 2) we obtain a contradiction, because the mean curvature of a large coordinate sphere tends to zero but the curvature of an Euclidean ball of positive volume v_2 is not zero. A simpler way to see this is again to look at formula (100), since the leading term is $I_{\mathbb{R}^n}$ that is strictly subadditive, we can consider again a competing domain $\tilde{\Omega}_{v_i}$ such that $\tilde{\Omega}_{v_i}\setminus\Omega_{v_{i1}} = x_i^{-1}(\varphi_i(B_{\tilde{r}_i}\setminus B_{r_i}))$, with E_i is such that $V(E_i\cap\Omega_{v_{i1}})\to+\infty$, for suitable $\tilde{r}_i>r_i>1$, φ_i diffeomorphism satisfying $V(x_i^{-1}(\varphi_i(B_{\tilde{r}_i}\setminus B_{r_i}))) = v_{i2}$. Now it is easily seen that (103) implies the claim. If $v_2=0$ the situation is even worst because the mean curvature of Euclidean balls of volumes going to zero goes to $+\infty$, again because isoperimetric regions for small volumes are nearly round ball, i.e., pseudobubbles as showed in [Nar14b], whose theorems apply here since M is C^2-locally asymptotically Euclidean. Hence we have necessarily that for v_i large enough $v_{i2}=0$, which implies existence of isoperimetric regions of volumes v_i, provided v_i is large enough. Since the sequence v_i is arbitrary the first part of the theorem is proved. Now that we have established existence of isoperimetric regions for large volumes the second claim in the statement of Theorem 3 follows readily from Lemma 5.1. q.e.d.

Remark 37. *If we allow to each end E_i of M to have a mass $m_i>0$ that possibly is different from the masses of the others ends, then we can guess in which end the isoperimetric regions for big volumes concentrates with "infinite volume". In fact the big volumes isoperimetric regions will* *prefer to stay in the end that for big volumes do better isoperimetrically and by (101) we conclude that the preferred end is to be found among the ones with bigger mass, because as it is easy to see an end with more positive mass do better than an end of less mass when we are considering large volumes. So from this perspective the worst case is the one considered in Theorem 3 in which all the masses m_i are equal to their common value m and in which we cannot say a priori which is the end that the isoperimetric regions for large volumes will prefer. However, Theorem 3 says that also in case of equal masses the number of ends in which the isoperimetric regions for large volumes concentrates is exactly one, but this end could vary from an isoperimetric region to another. An example of this behavior is given by Corollary* 16 *of [BE13], in which there are two ends and exactly two isoperimetric regions for the same large volume and they are obtained one from each other by reflection across the horizon, and each one of these isoperimetric regions chooses to have the biggest amount of mass in one end or in the other.*

After this informal presentation of the proof of Corollary 5, we are ready to go into its details.

Proof: Here we treat the case in which the masses are not all equals, the case of equal masse being already treated in Theorem 3. Without loss of generality we can assume that $1 \in \mathcal{I}$, i.e.,

$$m_1 = \max\{m_1, \dots, m_{\tilde{N}}\}.$$

We will prove the corollary by contradiction. To this aim, suppose that the conclusion of Corollary 5 is false, then there exists a sequence of isoperimetric regions Ω_j such that $V(\Omega_j) = v_j \to +\infty$, and

$$E_{\Omega_j} \notin \{E_i\}_{i \in \mathcal{I}}.$$

Now we construct a competitor $\Omega'_j := (\Omega_j \setminus E_{\Omega_j}) \mathring{\cup} \tilde{B}^1_{r_j}$, such that $V(\tilde{B}^1_{r_j}) = v'_j + v''_j$, with $v''_j := V(\Omega_j \cap E_1)$ and $v'_j := V(\Omega_j \setminus E_{\Omega_j})$. Roughly speaking it is like subtract the volume of Ω_j inside E_{Ω_j} and to put it inside the end E_1. As in the proof of Lemma 5.1, also in case of different masses we have that v''_j is uniformly bounded and $v'_j \to +\infty$. By construction $V(\Omega'_j) = V(\Omega_j) = v_j$. Furthermore, it is not too hard to prove that we have the following estimates

$$A(\partial\Omega'_j) - A(\partial\Omega_j) \leq -\left(m^*_1 - m^*_{E_{\Omega_j}}\right) {v'_j}^{\frac{1}{n}} + o({v'_j}^{\frac{1}{n}}). \tag{104}$$

This last estimate follows from an application of an analog of Lemma 5.1 in case of different masses which goes mutatis mutandis and uses in a crucial way Theorem 4.1 of [EM13b]. This cannot be avoided because again we need to control what happens to the area $A(\partial\Omega_j \cap E_{\Omega_j})$. The right hand side of (104), becomes strictly negative for $j \to +\infty$, since we have assumed $m_1^* - m_{E_{\Omega_j}}^* > 0$. This yields to the desired contradiction. q.e.d.

Here we prove Theorem 4.

Proof: By Proposition 12 of [BE13] and equation (88) we get by a direct calculation that for a given $0 < v < V(M)$ and any compact set $K \subseteq M$ there exists a smooth region $D \subset M \setminus K$ such that $V(D) = v$ and

$$A(\partial D) < c_n v^{\frac{n-1}{n}} = I_{\mathbb{R}^n}(v). \tag{105}$$

D is obtained by perturbing the closed balls $B := \{x : |x - a| \leq r\}$, for bounded radius r and big $|a|$. The remaining part of the proof follows exactly the same scheme of Theorem 13 of [BE13], that was previously employed in another context in the proof of Theorem 1.1 of [MN16]. Now, using Theorem 1 of [Nar14a], reported here in Theorem 1 we get that there exists a generalized isoperimetric region $\Omega = \Omega_1 \mathring{\cup} \Omega_\infty$, both $\Omega_1 \subseteq M$ and $\Omega_\infty \subseteq \mathbb{R}^n$ are isoperimetric regions in their own volumes in their respective ambient manifolds, with $V(\Omega) = v$, $V(\Omega_1) = v_1$, $V(\Omega_\infty) = v_\infty$, $v = v_1 + v_\infty$, moreover by Theorem 3 of [Nar14a] Ω_1 is bounded. If $\Omega_\infty = \emptyset$, the theorem follows promptly. Suppose, now that $\Omega_\infty \neq \emptyset$, one can chose as before a domain $D \subseteq M \setminus \Omega_1$ such that $V(D) = v_\infty$, $A(\partial D) < c_n {v_\infty}^{\frac{n-1}{n}} = I_{\mathbb{R}^n}(v_\infty)$. This yields to the construction of a competitor $\Omega' := \Omega_1 \mathring{\cup} D \subseteq M$ such that $V(\Omega') = v$ and $A(\partial\Omega') = A(\partial\Omega_1) + A(\partial D) < I_M(v) = A(\partial\Omega)$, this leads to a contradiction, hence $D_\infty = \emptyset$ and the theorem follows. q.e.d.

Remark 38. *As a final remark we observe that the hypothesis of convergence of the metric tensor stated in (88) are necessary for the proof of Theorem 4, because a weaker rate of convergence could destroy the estimate (105), when passing from the model Schwarzschild metric to a C^0-asymptotically one.*

To prove Corollary 6 it is enough to observe that the same proof of Theorem 4 applies mutatis mutandis.

5.3 Appendix: Blowdown

To make the paper self contained we will recall here the details of the arguments of Theorem 5.1 of [EM13a] and Theorem 4.1 of [EM13b] in our setting. We start with the following lemma that is the analog of Lemma 4.3 of [EM13a] in our context.

Lemma 5.2. *Let M a C^0-asymptotically Schwarzschild manifold having $N \geq 1$ ends, with each end E_i, with mass m_i. For every fixed $\Theta > 1$ there exists a volume $V_0 = V_0(\Theta, m_1, \ldots, m_N, C, n, k, v_0) > 0$, such that for every isoperimetric region Ω of the entire M, having $V_g(\Omega \cap E) \geq V_0$ it holds*

$$A_g\left((\partial\Omega) \cap \tilde{B}_r\right) \leq \Theta r^{n-1}, \ \forall r > 1, \tag{106}$$

$$A_g\left((\partial\Omega) \cap E\right) \leq \Theta V(\Omega \cap E)^{\frac{n-1}{n}}. \tag{107}$$

Proof: It is easily seen that,

$$\frac{A(\partial\Omega \cap E)}{V(\Omega \cap E)^{\frac{n-1}{n}}} \leqslant \frac{A(\tilde{S}_r) + A(\Sigma_1)}{V(\Omega \cap E)^{\frac{n-1}{n}}} \sim \frac{c_n v^{\frac{n-1}{n}} + \ldots + A(\Sigma_1)}{V(\Omega \cap E)^{\frac{n-1}{n}}} \leqslant \Theta.$$

q.e.d.

Take a sequence of isoperimetric regions $\Omega_i \subseteq M$ with $V_g(\Omega_i) \to +\infty$. We use the homothety $\mu_\lambda : \mathbb{R}^n \to \mathbb{R}^n$, $\mu_\lambda : x \mapsto \lambda x$, with scale factor $\lambda_i := \left(\frac{V_g(\Omega_i \cap E_i)}{\omega_n}\right)^{\frac{1}{n}}$ to obtain sets $\hat{\Omega}_i \subseteq \mathbb{R}^n \setminus B_{\mathbb{R}^n}(0, \lambda_i^{-1})$, $\hat{\Omega}_i := \mu_{\lambda_i}(x_i(\Omega_i \cap E_i))$ that are locally isoperimetric w.r.t. the metric $g_i := \frac{1}{\lambda_i^2}\mu_{\lambda_i}^* g$ and such that $V_{g_i}(\hat{\Omega}_i) = \omega_n$. As it is easy to check $\left(\mathbb{R}^n \setminus B(0, \frac{1}{\lambda_i}), g_i\right) \to (\mathbb{R}^n \setminus \{0\}, \delta)$ in the C^2_{loc} topology. Now we observe that $V_\delta(\hat{\Omega}_i) \sim V_{g_i}(\hat{\Omega}_i) = V_\delta(B_{\mathbb{R}^n}(0,1))$ and that for large volumes Lemma 5.2 implies $A_\delta(\partial\hat{\Omega}_i) \sim A_{g_i}(\partial\hat{\Omega}_i) \leq \Theta V_{g_i}(\hat{\Omega}_i)^{\frac{n-1}{n}} \leq K_3 = \Theta\omega_n^{\frac{n-1}{n}}$, where K_3 is a constant. It follows that the sequence $\hat{\Omega}_i$ has volumes and boundaries uniformly bounded. This implies the existence of a finite perimeter set $\Omega \subset \mathbb{R}^n \setminus \{0\}$ such that $\chi_{\hat{\Omega}_i} \to \chi_\Omega$ in $L^1_{loc}(\mathbb{R}^n)$ topology. About this point the reader could consult the beggining of the proof of Theorem 2.1 of [RR04]. In particular $V_\delta(\Omega) \leq \omega_n = \lim_{i \to +\infty} V_\delta(\hat{\Omega}_i)$ and the inequality could be strict. If we show that

$$V_\delta(\hat{\Omega}_i \setminus B_{\mathbb{R}^n}(0, 1+\theta)) = 0, \tag{108}$$

for every fixed $\theta > 0$, then it is straightforward to see that Ω must coincide with the unit ball of $\mathbb{R}^n$, i.e.,

$$\Omega = B_{\mathbb{R}^n}(0,1). \tag{109}$$

It is to prove (108) that we need in a crucial way that our initial data set M is indeed C^0-asymptotically Schwarzschild. The arguments used here does not works in a general initial data set, but only in C^0-asymptotically Schwarzschild, because we can use the effective comparison Theorem 3.5 of [EM13b] that is a special feature of the Schwarzschild geometry and it is not a consequences of effective Euclidean isoperimetric inequality as explained very well in [EM13b]. We will prove (109) by contradiction. To this aim, assume that there exist $\frac{4\alpha_{n-1}}{c_n} > \varepsilon > 0$, $\theta > 0$, such that $V_{g_i}(\hat{\Omega}_i \setminus B_{\mathbb{R}^n}(0,1+\theta)) \geq \varepsilon > 0$, for every $i \in \mathbb{N}$. By a relative isoperimetric inequality

$$\begin{aligned} A_{g_i}((\partial\hat{\Omega}_i) \setminus B_{\mathbb{R}^n}(0,1+\theta)) &\geq c_n V_{g_i}(\hat{\Omega}_i \setminus B_{\mathbb{R}^n}(0,1+\theta)) && (110)\\ &\geq c_n\varepsilon \geq 2\eta A_{g_i}(\partial B_{\mathbb{R}^n}(0,1)). && (111) \end{aligned}$$

We conclude that each $\Omega_i \cap E_i$ is $\left(1+\frac{\theta}{2},\eta\right)$-off-center. At this point we can apply Theorem 3.5 of [EM13b] to $\Omega_i \cap E_i$ and deduce that $\Omega_i' := \tilde{B}_r \mathring{\cup} \Omega_i \setminus E_i$ satisfies

$$A_g(\partial\Omega_i') - A_g(\partial\Omega_i) \leq A_g(\Sigma_i) - c\eta m_i \left(\frac{\theta}{2+\theta}\right)^2 r(V_g(\Omega_i \cap E_i)), \tag{112}$$

where $\Sigma_i := \partial E_i$, $c = c(n) > 0$ is a dimensional constant, and $r(V_g(\Omega_i \cap E_i))$ is such that $V_g(\tilde{B}_r \setminus \tilde{B}_1) = V_g(\Omega_i \cap E_i)$. By the fact that $r(V_g(\Omega_i \cap E_i)) \to +\infty$, when $i \to +\infty$, inequality (112) immediately shows that for large volumes Ω_i is not isoperimetric, which is the desired contradiction to our assumptions. To finish the proof at this point we follow a somewhat little bit different argument from the proof of Theorem 5.1 of [EM13a]. At this point we can use Theorem 1 of [Nar09b] complemented with Remark 4.1 of [Nar09b] to show that the boundary of Ω_i is the graph of a function based on a centered coordinate sphere. This is still not enough to guarantees that $\Sigma_i \subseteq \Omega_i$ for large volumes, but the arguments of the proof of Theorem 4.1 of [EM13b] show that $\tilde{B}_{r_i/2} \subseteq \Omega_i$ for large volumes. This finish easily the proof that $\Sigma_{j_i} \subseteq \Omega_i$ for large volumes.

6 Appendices

6.1 The Schwarzschild metric

We detail in section some properties of the Schwarzchild metric and remarks used throughout our work.

Remark 39. *The Schwarzchild metric of mass $m > 0$, can also be isometrically embedded into four-dimensional Euclidean space as the tree-dimensional set of point in $\mathbb{R}^4 = \{(x, y, z, w)\}$ satisfying $|(x, y, z)| = \frac{w^2}{8m} + 2m$.*

In fact using spherical coordinates we have a parametrization $\chi(R, \theta, \phi) = (R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta, \sqrt{8m(R-2m)})$, then the metric in this coordinate system written

$$g = \frac{1}{1-\frac{2m}{R}}dR^2 + R^2 g_{\mathbb{S}^2}, \tag{113}$$

which coincides with the metric of Schwarzchild. Making a change of variable in (113) for $R = (1+\frac{m}{2r})^2 r$, we have

$$g = \phi_m^4 dr^2 + \phi_m^4 r^2 g_{\mathbb{S}^2}$$

where $\phi_m = 1 + \frac{m}{2r}$. On the other hand using this facts and that

the Schwarzchild metric is conformal to the Euclidean metric, intuitively, as $|x| \to \infty$, the metric becomes flat. Now let S_r be a 2 sphere in Schwarzschild with radius r and the metric. The mean curvature of S_r is

$$H_r = \frac{2}{r}\phi_m^{-3}\left(1 - \frac{m}{2r}\right).$$

In fact,

$$g_m = \phi_m^4|dx|^2 = \phi_m^4 dr^2 + \phi_m^4 r^2 d\theta^2,$$

therefore, the mean curvature with respect to the normal unit vector: $\phi_m^{-2}\partial_r$ is given by

$$div_{g_m}(\phi_m^{-2}\partial_r) = \frac{1}{\sqrt{detg_m}}\partial_r\left(\phi_m^{-2}.\sqrt{detg_m}\right). \tag{114}$$

See that,

$$detg_m = \phi_m^4 \phi_m^8 r^4 detg_{\mathbb{S}^2}$$

therefore, $\sqrt{detg_m} = \phi_m^6 r^2 \sqrt{detg_{\mathbb{S}^2}}$ this imply

$$\begin{aligned} H_R = div_{g_m}(\phi_m^{-2}\partial_r) &= \frac{1}{\phi_m^6 r^2 \sqrt{detg_{\mathbb{S}^2}}} \partial_r \left(\phi_m^{-2} \phi_m^6 r^2 \sqrt{detg_{\mathbb{S}^2}} \right) \\ &= \frac{1}{\phi_m^6 r^2} \partial_r \left(\phi_m^4 . r^2 \right) \\ &= \frac{1}{\phi_m^6 r^2} \left(\partial_r(\phi_m^4) r^2 + \phi_m^4 2r \right) \\ &= \frac{1}{\phi_m^3 r} \left(4r\partial_r \phi_m + 2\phi_m \right) \\ &= \frac{2}{r} \phi_m^{-3} \left(1 - \frac{m}{2r} \right) \end{aligned}$$

i.e.,

$$H_r(S_r) = \left(2 - \frac{m}{r} \right) \frac{\phi_m^{-3}}{r}.$$

Notice that $H_r \to 0$ when $r \to \infty$. Calculation shows that the sphere $S_{\frac{(2+\sqrt{3})m}{2}}$ has the largest mean curvature. Moreover, H_r is increasing for $\frac{m}{2} \leq r \leq \frac{(2+\sqrt{3})m}{2}$ and H_r is decreasing for $r \geq \frac{(2+\sqrt{3})m}{2}$ to see this, observe that

$$\frac{\partial}{\partial r} H_r = \left(-\frac{2}{r^2} + \frac{2m}{r^3} \right) \phi_m^{-3} + \left(-\frac{2}{r} - \frac{m}{r^2} \right) \left(-3\phi_m^{-4} \frac{\partial \phi_m}{\partial r} \right),$$

and consider

$$\frac{\partial}{\partial r} H_r > 0$$

i.e.,

$$\begin{aligned} \left(-\frac{2}{r^2} + \frac{2m}{r^3} \right) \phi_m^{-3} + \left(-\frac{2}{r} - \frac{m}{r^2} \right) \frac{3}{2} \frac{m\phi_m^{-4}}{r^2} &> 0 \\ \left(-\frac{2}{r^2} + \frac{2m}{r^3} \right) + \left(-\frac{2}{r} - \frac{m}{r^2} \right) \frac{3}{2} m\phi_m^{-1} &> 0 \end{aligned}$$

finally,

$$4r^2 - 8rm + m^2 < 0$$

with this we get the desired result. The question is: are those spheres stable? To answer this, we compute the stability operator for S_r

$$L_{S_r} = - \triangle_{\mathbb{S}_r} - \big(|h|^2 + Ric(\nu;\nu)\big)$$

Where $\triangle_{\mathbb{S}_r}$ Laplacian for induced metric on S_r and $|h|$ is norm the Hilbert-Schmidt the second fundamental form. As S_r is umbilical, then

$$|h|^2 = \frac{2}{r^2}(1 - \frac{m}{2r})^2 \phi_m^{-6}.$$

Notice also that

$$Ric(x)(\partial_\alpha, \partial_\beta) := R_{\alpha\beta} = \frac{m}{r^3}\phi_m^{-2}(\delta_{\alpha\beta} - 3\frac{x^\alpha x^\beta}{r^2}).$$

Therefore

$$Ric(\nu, \nu) = \lambda^\alpha \lambda^\beta R_{\alpha\beta} = \frac{m}{r^3}\phi_m^{-2}(\lambda^\alpha\lambda^\beta\delta_{\alpha\beta} - 3\frac{x^\alpha x^\beta \lambda^\alpha \lambda^\beta}{r^2})$$

where $\nu = \lambda^\alpha \partial_\alpha$ and $\lambda^\alpha\lambda^\beta g_{\alpha\beta} = 1$, this implies $g^{\alpha\beta} = \lambda^\alpha\lambda^\beta$. Furthermore we have

$$\begin{aligned} \lambda^\alpha\lambda^\beta R_{\alpha\beta} &= \frac{m}{r^3}\phi_m^{-2}(\phi_m^{-4} - 3\frac{x^\alpha x^\beta \phi_m^{-4}\delta^{\alpha\beta}}{r^2}) \\ &= \frac{m}{r^3}\phi_m^{-2}(-2\phi_m^{-4}) \\ &= -2\frac{m}{r^3}\phi_m^{-6} \end{aligned}$$

Finally

$$L_{\mathbb{S}_r} = -\phi_m^{-4} r^{-2} \triangle_{\mathbb{S}^2} + \frac{-4r^2 + 8rm - m^2}{2r^4\phi_m^6}$$

where $\triangle_{\mathbb{S}^2}$ is the Laplacian of the standard round unit sphere. Hence S_r is stable for all $r \geqslant \frac{m}{2}$.

6.2 Weak Ricci curvature bound

There are several viewpoints from which one can see the necessity of a reference measure (which can certainly be the Hausdorff measure of appropiate dimension, if available). A first (cheap)one is the fact that in most of identities/inequalities where the Ricci curvature appears, also the reference measures appears. A more subtle point of view comes from studying stability issues: consider a sequence (M_n, g_n) of Riemannian manifolds and assume that it converges to a smooth Riemannian manifold (M, g) in the Gromov-Hausdorff sense. Assume that the Ricci curvature of (M_n, g_n) is uniformly bounded below by $K \in \mathbb{R}$. Can we deduce that the Ricci of (M, g) is bounded below by K? The answer is no (while the same question with sectional curvature in place of Ricci one has affirmative answer). It is possible to see that when Ricci bounds are not preserved in the limiting process, it happens that the volume measures of the approximating are not converging to the volume measure of the limit one. In the next section we recall some basic concepts concerning convergence of metric measure spaces (wich are key to discuss the stability issue), while in the following one we give the definition of curvature-dimension condition and analyse its properties.

6.3 Preliminares

Definition 6.1. *We say that (X, d) is a Polish space, provided that it is a complete and separable metric space.*

We will denote by $\mathcal{P}(X)$ the set of Borel probability measures on X. If X, Y are two Polish spaces, $T : X \to Y$ is Borel map, and $\mu \in \mathcal{P}(X)$ a measure, the measure $T\#\mu \in \mathcal{P}(Y)$, is called the **push forward** of μ throught T and is defined by

$$T\#\mu(E) = \mu(T^{-1}(E)), \quad \forall E \subset Y,$$

E Borel set.

Definition 6.2. *The set all **transport plans** $\gamma \in \mathcal{P}(X \times Y)$ from μ to ν is defined for the set of Borel Probability measures on $X \times Y$ such that*

$$\gamma(A \times Y) = \mu(A) \quad \forall A \in \mathcal{B}(X), \quad \gamma(X \times B) = \nu(B), \quad \forall B \in \mathcal{B}(Y),$$

and denoted $Adm(\mu, \nu)$.

We recall some basic notions concernig analysis over a Polish space. We say that a sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ narrowly converges to μ provided
$$\int \phi d\mu_n \to \int \phi d\mu, \quad \forall \phi \in C_b(X),$$
$C_b(X)$ being the space of continuous and bounded functions on X. It can be shown that the topology of narrow convergence is metrizable. A set $\mathcal{K} \subset \mathcal{P}(X)$ is called tight provided for every $\epsilon > 0$ there is a compact set $K_\epsilon \subset X$ such that

$$\mu(X - K_\epsilon) \leq \epsilon, \quad \forall \mu \in \mathcal{K}.$$

Theorem 6.1. *(Prokhorov) Let (X, d) be a Polish space. Then a family $\mathcal{K} \subset \mathcal{P}(X)$ is relatively compact w.r.t. the narrow topology if and only if it is tight.*

6.3.1 The Wasserstein distance W_2

Let (X, d) be a complete and separable metric space. The distance W_2 is defined as

$$\begin{aligned} W_2(\mu, \nu) : &= \sqrt{\inf_{\gamma \in Adm(\mu,\nu)} \int d^2(x, y) d\gamma(x, y)} \\ &= \sqrt{\int d^2(x, y) d\gamma(x, y)}, \qquad \forall \gamma \in Opt(\mu, \nu). \end{aligned}$$

Where,

$$Opt(\mu, \nu) := \{\gamma \in Adm(\mu, \nu) : \inf_{\tilde{\gamma} \in Adm(\mu,\nu)} \int d^2(x, y) d\tilde{\gamma}(x, y) = \int d^2(x, y) d\gamma(x, y)\}.$$

The natural space to endow with the Wasserstein distance W_2 is the space $\mathcal{P}(X)$ of Borel Probability measures with finite second moment:

$$\mathcal{P}_2(X) = \{\mu \in \mathcal{P}(X) : \int d^2(x, x_0) d\mu(x) < \infty \quad for\ \ some,\ and\ thus\ any, \ \ x_0 \in X\}$$

Theorem 6.2. *(W_2 is a distance) W_2 is a distance on $\mathcal{P}_2(X)$.*

Proof: See Theorem 2.2 of [AG09] q.e.d.

Definition 6.3. *(2-uniform integrability) Let $\mathcal{K} \subset \mathcal{P}_2(X)$, $\mathcal{K}$ called 2-uniformly integrable provided for any $\epsilon > 0$ and $x_0 \in X$ there exist $R_\epsilon > 0$ such that*

$$\sup_{\mu \in \mathcal{K}} \int_{X \setminus B_{R_\epsilon}} d^2(x, x_0) d\mu \leqslant \epsilon.$$

Proposition 1. *Let $(\mu_n) \subset \mathcal{P}_2(X)$ be a sequence narrowly converging to some μ. Then the following properties are equivalent*

- *$\{\mu_n\}_{n \in \mathbb{N}}$ is 2-uniformly integrable,*
- *$\int d^2(., x_0) d\mu_n \to \int d^2(., x_0) d\mu$ for some $x_0 \in X$.*

Proof: See Proposition 2.4 of [AG09] q.e.d.

Proposition 2. *(Stability of optimality) The distance W_2 is lower semicontinuous w.r.t narrow convergence of measures. Furthermore, if $(\gamma_n) \subset \mathcal{P}_2(X^2)$ is a sequence of optimal plans which narrowly converges to $\gamma \in \mathcal{P}_2(X^2)$, then γ is optimal as well.*

Proof: See Proposition 2.5 of [AG09] q.e.d.

Theorem 6.3. *Let (X, d) be complete and separable. then*

$$W_2(\mu_n, \mu) \to 0 \Leftrightarrow \left\{ \begin{array}{ccc} \mu_n & \to & \mu \qquad \textit{narrowly} \\ \int d^2(., x_0) d\mu_n & \to & \int d^2(., x_0) d\mu \quad \textit{for some } x_0 \in X \end{array} \right. \tag{115}$$

Furthermore, the space $(\mathcal{P}_2(X), W_2)$ is complete and separable. Finally, $\mathcal{K} \subset \mathcal{P}_2(X)$ is relatively compact w.r.t the topology induced by W_2 if and only if it is tight and 2-uniform integrable.

Proof: See Theorem 2.7 of [AG09] q.e.d.

6.3.2 X geodesic space

Definition 6.4. *A curve $\gamma : [0, 1] \to X$ is called a constant speed geodesic, if*

$$d(\gamma_t, \gamma_s) = \mid t - s \mid d(\gamma_0, \gamma_1), \quad \forall t, s \in [0, 1].$$

Definition 6.5. *(Geodesic space) A metric space (X, d) is called geodesic if for every $x, y \in X$ there exist a constant speed geodesic connecting them, i.e a constant speed geodesic such that $\gamma_0 = x$ and $\gamma_1 = y$.*

See that $x \to \delta_x \in \mathcal{P}(X)$ is an isometry. Therefore if $t \to \gamma_t$ is a constant speed geodesic on X connecting x to y, the curve $t \to \delta_t$ is a constant speed geodesic on $\mathcal{P}_2(X)$ connecting δ_x to δ_y.

Definition 6.6. *Let ζ be a metric space then, we say that a functional $\xi : \zeta \to [0, +\infty)$ is K-geodesically convex if for any $\mu_0, \mu_1 \in \zeta$ there exists a const speed geodesic γ with $\gamma_0 = \mu_0$, $\gamma_1 = \mu_1$ such that*

$$\xi(\gamma_t) \leqslant (1-t)\xi(\gamma_0) + t\xi(\gamma_1) - \frac{1}{2}Kt(1-t)d^2_\zeta(\gamma_0, \gamma_1), \quad , \ \forall t \in [0,1]$$

and we say that ξ is geodesically convex, if

$$\xi(\gamma_t) \leqslant (1-t)\xi(\gamma_0) + t\xi(\gamma_1), \quad , \ \forall t \in [0,1].$$

6.3.3 Convergence of metric measure spaces

We say that two metric measure spaces (X, d_X, m_X) and (Y, d_Y, m_Y) are isomorphic provided there exist a bijective isometry $f : supp(m_X) \to supp(m_Y)$ such that $f\#m_X = m_Y$.

Definition 6.7. *(Coupling between metric measure spaces) Given two metric measure spaces (X, d_X, m_X), (Y, d_Y, m_Y), we consider the product space $(X \times Y, D_{XY})$ where D_{XY} is distance defined by*

$$D_{XY}((x_1, y_1), (x_2, y_2)) = \sqrt{d^2_X(x_1, x_2) + d^2_Y(y_1, y_2)}$$

We say that couple (d, γ) is an admissible couple between (X, d_X, m_X) and (Y, d_Y, m_Y) and we write $(d, \gamma) \in Adm((d_X, m_X), (d_Y, m_Y))$ if

- *d is a pseudo distance on $supp(m_X) \sqcup supp(m_Y)$ (i.e. it may be zero on two different point) which coincides with d_X (resp. d_Y) when restricted to $supp(m_X) \times supp(m_X)$ (resp. $supp(m_Y) \times supp(m_Y)$).*
- *a Borel (w.r.t. the Polish structure given by D_{XY}) measure γ on $supp(m_X) \times supp(m_Y)$ such that $\pi^1\#\gamma = m_X$ and $\pi^2\#\gamma = m_Y$.*

Where for $\pi^i : X^2 \to X$ we intend the natural projection onto the i-th coordinate, $i = 1, 2$.

The cost $C(d, \gamma)$ of a coupling is given by

$$C(d, \gamma) := \int_{supp(m_X) \times supp(m_Y)} d^2(x, y) d\gamma(x, y)$$

The distance $\mathbb{D}((X, d_X, m_X), (Y, d_Y, m_Y))$ is then defined as

$$\mathbb{D}((X, d_X, m_X), (Y, d_Y, m_Y)) := \inf \sqrt{C(d, \gamma)} \tag{116}$$

the infimun being taken among all coupling (d, γ) of (X, d_X, m_X) and (Y, d_Y, m_Y). A consequence of the definition is that if (X, d_X, m_X) and (X_0, d_{X_0}, m_{X_0}) (resp. (Y, d_Y, m_Y) and (Y_0, d_{Y_0}, m_{Y_0})) are isomorphic, then

$$\mathbb{D}((X, d_X, m_X), (Y, d_Y, m_Y)) = \mathbb{D}((X_0, d_{X_0}, m_{X_0}), (Y_0, d_{Y_0}, m_{Y_0}))$$

so that $\mathbb{D}$ actually defined on the equivalence classes of isomorphic of metric measures spaces. In the next proposition we collect, without proof, the main properties of $\mathbb{D}$.

Lemma 6.1. *Let $\mathbb{X}$ be the set of isomorphic classes of Polish metric measures spaces, with measure of probability on $\mathcal{P}_2(X)$. Then $\mathbb{D}$ is distance on $\mathbb{X}$ and in particular $\mathbb{D}$ is* 0 *only on couples of isomorphic metric measure spaces. Finally, the space $(\mathbb{D}, \mathbb{X})$ is complete, separable and geodesic.*

Proof: See Proposition 7.3 of[AG09] and Section 3.1 of [AGS08] q.e.d.

We will denote by $Opt((d_X), m_X), (d_Y, m_Y))$ the set of optimal couplings between (X, d_X, m_X) and (Y, d_Y, m_Y), i.e., the set of couplings where the inf in (116) is realized. Given a metric measure space (X, d_X, m) we will denoted by $\mathcal{P}_2^a(X) \subset \mathcal{P}(X)$ the set of measures which are absolutely continuous w.r.t. m.

To any coupling (d, γ) of two metric measure spaces (X, d_X, m_X) and (Y, d_Y, m_Y), it is naturally associated a map $\gamma\# : \mathcal{P}_2^a(X) \to \mathcal{P}(X)$ defined as follows

$$\mu = \rho m_X \to \gamma\#\mu := \eta m_Y, \tag{117}$$

$$\text{where } \eta \text{ is defined by } \quad \eta(y) := \int \rho d\gamma_y(x)$$

Where γ_y is the disintegration of γ w.r.t the projection on Y. Similarly, there is natural map $\gamma\#^{-1} : \mathcal{P}_2^a(Y) \to \mathcal{P}_2^a(X)$ given by:

$$\nu = \eta m_Y \to \gamma\#^{-1} m_Y := \rho m_X, \;\; \text{where } \; \rho$$

$$\text{where } \rho \text{ is defined by } \; \rho(x) := \int \nu(y) d\gamma_x(y),$$

Where, obviously, γ_x is the disintegration of γ w.r.t the projection on X. Notice that $\gamma\# m_X = m_Y$ and $\gamma\#^{-1} m_Y = m_X$ and that in general $\gamma\#^{-1}\mu \neq \mu$.

Our goal now is to show that if $(X_n, d_n, m_n) \xrightarrow{\mathbb{D}} (X, d, m)$ of the internal energy kind on $(\mathcal{P}_2^a(X_n), W_2)$ Mosco-convergence to the corresponding functional on $(\mathcal{P}_2^a(X), W_2)$. Thus, fix a convex and continuous function $u : [0, +\infty) \to \mathbb{R}$, define

$$u^{'}(\infty) := \lim_{z \to +\infty} \frac{u(z)}{z}$$

and, for every compact metric space (X, d), define the functional $\xi : [\mathcal{P}(X)]^2 \to \mathbb{R} \cup +\infty$ by

$$\xi(\mu|\nu) := \int u(\rho) d\nu + u^{'}(\infty)\mu^s(X) \tag{118}$$

where $\mu = \rho\nu + \mu^s$ is the decomposition of μ in absolutely continuous $\rho\nu$ and singular part μ^s w.r.t to ν (Radom Nykodin).

Theorem 6.4. *(Approximation by continuous densities) Let $\mu \in \mathcal{P}_2^a(X)$ satisfy $\xi_\nu(\mu) < \infty$ where ξ_ν is defined as above. Then there is a sequence $\{\rho_k\}_{k \in \mathbb{N}}$ em $C(X)$ such that $\lim_{k \to \infty} \rho_k \nu = \mu$ in narrowly topology and $\lim_{k\infty} \xi(\rho_k \nu) = \xi(\mu)$.*

Proof: As in the Theorem C.5 of [LV09], using partitions of unity and mollifiers arguments. q.e.d.

Lemma 6.2. *(ξ decreases under $\gamma\#$)*
Let (X, d_X, m_X) and (Y, d_Y, m_Y) be two metric measure space and (d, γ) a coupling between them. Then if holds

$$\begin{aligned} \xi(\gamma\#\mu|m_Y) &\leqslant \xi(\mu|m_X), \quad \forall \mu \in \mathcal{P}_2^a(X), \\ \xi(\gamma\#^{-1}\nu|m_X) &\leqslant \nu(\mu|m_Y), \quad \forall \mu \in \mathcal{P}_2^a(Y) \end{aligned}$$

Proof: Clearly it is sufficient to prove the first inequality. Let $\mu = \rho m_X$ and $\gamma\#\mu = \eta m_Y$, with η given by 117. By Jensen's inequality and we have

$$\begin{aligned}
\xi(\gamma\#\mu|m_Y) &= \int u(\nu(y))dm_Y(y) = \int u(\int \rho(x)d\gamma_y(x))dm_Y(y) \\
&\leqslant \int\int u(\rho(x))d\gamma_y(x)dm_Y = \int u(\rho(x))d\gamma(x,y) \\
&= \int u(\rho(x))dm_X(x) = \xi(\mu|m_X).
\end{aligned}$$

q.e.d.

Lemma 6.3. *Assume that $(X_n, d_n, m_n) \stackrel{\mathbb{D}}{\to} (X, d, m)$, let $\mu \in \mathcal{P}_2^a(X)$ with bounded density, then for the sequence $\mu_n \in \mathcal{P}_2^a(X)$ defined by $\mu_n := (\gamma_n)\#\sigma_n$ where $\sigma_n = (\gamma_n^{-1})\#\mu$ we have that $W_2(\mu_n, \mu) \to 0$.*

Proof: Choose $(\tilde{d}_n, \gamma_n) \in Opt((d_n, m_n), (d, m))$, defined $\mu_n := (\gamma_n)\#\sigma_n$ where $\sigma_n = (\gamma_n^{-1})\#\mu = (\int \rho d\gamma_n)m_n$, now suppose that $\mu := \rho m_X$, therefore notice que a density of σ_n is $\int \rho d\gamma_n$, from here if we define the plan $\tilde{\gamma_n} \in \mathcal{P}(X_n \times X)$ by $d\tilde{\gamma_n}(y,x) := \rho(x)d\gamma_n(y,x)$ then $\tilde{\gamma_n} \in Adm(\sigma_n, \mu)$ thus

$$\begin{aligned}
W_2(\sigma_n, \mu) \leqslant \sqrt{\int\int d_n^2(x,y)d\tilde{\gamma_n}(y,x)} &\leqslant \sqrt{\int\int d_n^2(x,y)\rho(x)d\tilde{\gamma_n}(y,x)} \\
&\leqslant \sqrt{M}\sqrt{C(d_n,\gamma_n)},
\end{aligned}$$

where $\|\rho\|_\infty = M$ and as $\eta_n(y) := \int \rho(x)(d\gamma_n)_y(x)$ is density of σ_n therefore is also bounded above by M, now introduce the plan $\overline{\gamma}_n$ by $d\overline{\gamma}_n(y,x) := \eta_n(y)d\gamma_{y,x}$ and notice that $\overline{\gamma}_n \in Adm(\sigma_n, \mu_n)$, so that, before, we have

$$\begin{aligned}
W_2(\sigma_n, \mu_n) \leqslant \sqrt{\int\int d_n^2(x,y)d\overline{\gamma}_n(y,x)} &\leqslant \sqrt{\int\int d_n^2(x,y)\eta_n(x)d\overline{\gamma}_n(y,x)} \\
&\leqslant \sqrt{M}\sqrt{C(d_n,\gamma_n)}.
\end{aligned}$$

Using the triangle inequality we have

$$W_2(\mu, \mu_n) \leqslant W_2(\sigma_n, \mu_n) + W_2(\sigma_n, \mu) \leqslant 2\sqrt{M}\sqrt{C(d_n,\gamma_n)}.$$

see that we used metrics of Wassertein for the metric space $(X_n \sqcup X, d_n)$ q.e.d.

Proposition 3. *(Mosco' convergence of internal energy functionals) Let* $(X_n, d_n, m_n) \overset{\mathbb{D}}{\to} (X, d, m)$ *and* $(d_n, \gamma_n) \in Opt((d_n, m_n), (d, m))$. *Then the following two are true:*
Weak Γ-$\underline{\lim}$ *For any sequence* $n \to \mu_n \in \mathcal{P}_2^a(X_n)$ *such that* $n \to (\gamma_n)\#\mu_n$ *narrowly convergences to some* $\mu \in \mathcal{P}(X)$ *it holds*

$$\underline{\lim}_{n\to\infty}\xi(\mu_n|m_n) \geqslant \xi(\mu|m).$$

Strong Γ $-\ \overline{\lim}$. *For any* $\mu \in \mathcal{P}_2^a(X)$ *with bounded density there exist a sequence* $n \to \mu_n \in \mathcal{P}_2^a(X_n)$ *such that* $W_2((\gamma_n)\#\mu_n, \mu) \to 0$ *and*

$$\overline{\lim}_{n\to\infty}\xi(\mu_n, m_n) \leqslant \xi(\mu|m)$$

.

Proof: For the first statement we just notice that by Lemma 6.2, we have

$$\xi(\mu_n|m_n) \geqslant \xi((\gamma_n)\#\mu_n|m),$$

and conclusion follows from the narrrow lower semicontinuity of $\xi(.|m)$. For second one we defined $\mu_n := (\gamma_n^{-1})\#\mu$. Then applying Lemma 6.2 twice we get

$$\xi(\mu|m) \geqslant \xi(\mu_n|m_n) \geqslant \xi((\gamma_n)\#\mu_n|m),$$

the second part of the proof continues as in Lemma 6.3. q.e.d.

6.3.4 Weak Ricci curvature bound: definition and properties

Define the functions u_N, $N > 1$, and u_∞ on $[0, +\infty)$ as

$$u_N := N(z - z^{1-\frac{1}{N}}),$$

and,

$$u_\infty := z \log(z).$$

Then given a metric measure space (X, d, m) we define the functionals ξ_N , $\xi_\infty : \mathcal{P}(X) \to \mathbb{R} \cup \{+\infty\}$ by

$$\xi_N(\mu) := \xi(\mu|m)$$

, where $\xi(.|.)$ is given by formula 118 with $u := u_N$ similarly for ξ_∞ the definitions of weak Ricci curvature bounds are following:

Definition 6.8. *(Curvature $\geqslant K$ and no bound on dimension-$CD(K, \infty)$)*
We say that a metric measure space (X, d, m) has Ricci curvature bounded from below by $K \in \mathbb{R}$ provided the functional

$$\xi_\infty : \mathcal{P}(X) \to \mathbb{R} \cup \{+\infty\}$$

is K-geodesically convex on $(\mathcal{P}_2^a(X), W_2)$. In this case we say that (X, d, m) satisfies the curvature dimension condition $CD(K, \infty)$ or that (X, d, m) is a $CD(K, \infty)$ space.

Definition 6.9. *(Curvature $\geqslant 0$ and dimension $\leqslant N$ - $CD(0, N)$)*
We say that a metric measure space (X, d, m) has nonnegative Ricci curvature and dimension bounded from above by N provided the functional

$$\xi_{N'} : \mathcal{P}(X) \to \mathbb{R} \cup \{+\infty\}$$

is geodesically convex on $(\mathcal{P}_2^a(X), W_2)$ for every $N^{'} \geqslant N$. In this case we say that (X, d, m) satisfies the curvature dimension condition $CD(0, N)$ or that (X, d, m) is a $CD(0, N)$ space. Note that $N > 1$ is not necessarily an integer.

Lemma 6.4. *(Second derivate of the internal energy) Let M be a compact and smooth Riemannian manifold, $m = \dfrac{dvol_M}{vol(M)}$, $u : [0, +\infty)$ be convex, and continuos and C^2 on $(0, +\infty)$ with $u(0) = 0$ and define the "pressure" $p : [0, +\infty) \to \mathbb{R}$ by*

$$p(z) := zu^{'}(z) - u(z), \quad \forall z > 0$$

and $p(0) := 0$. *Also, let* $\mu = \rho m \in \mathcal{P}_2^a(M)$ *with* $\rho \in C^\infty$, *pick* $\phi \in C_c^\infty(M)$, *and define*
$T_t : M \to M$ *by* $T_t(x) := \exp_x(t\nabla\phi(x))$. *Then it holds:*

$$\frac{d^2}{dt^2}|_{t=0}\xi((T_t)\#\mu) = \int p^{'}(\rho)\rho(\triangle\phi)^2 - p(\rho)((\triangle\phi)^2 - |\nabla^2\phi|^2 - Ricc(\nabla\phi, \nabla\phi))dm,$$

where by $|\nabla^2\phi(x)|^2$ *we mean the trace of the linear map* $(\nabla^2\phi(x))^2 : T_xM \to T_xM$ *(in coordinates, this reads as* $\sum_{i,j}(\partial_{ij}\phi(x))^2$*).*

Proof: See the Lemma 7.10 of [AG09]. q.e.d.

Theorem 6.5. *(Compatibility of weak Ricci curvature bounds) Let M be a compact Riemannian manifold, d its Riemannian distance and m its normalized volume measure. Then*

- *the functional ξ_∞ is K-geodesically convex on $(\mathcal{P}_2(M), W_2)$ if and only if M has Ricci curvature uniformely bounded from below by K.*
- *the functional ξ_N is geodesically convex on $(\mathcal{P}_2(M), W_2)$ if only if M has non negative Ricci curvature and $dim(M) \leqslant N$.*

Proof: See the theorem 7.11 of [AG09]. q.e.d.

Lemma 6.5. *The sublevels $\{\mu \in \mathcal{P}_2^a(X) : \xi_\infty(\mu) \leqslant C\}$ of ξ_∞ are tight*

Proof: Given $E \subset X$ Borel, using that $z\log(z) \geqslant -\dfrac{1}{e}$, we have

$$\begin{aligned}\int_E \rho\log(\rho)dm &= \xi_\infty(\mu) - \int_{X\setminus E}\rho\log(\rho)dm \\ &\leqslant \xi_\infty(\mu) + \frac{m(X\setminus E)}{e} \\ &\leqslant C + \frac{1}{e}\end{aligned}$$

now as $z\log(z)$ is convex and $\mu = \rho m$, using the Jensen's inequality we get

$$\int_E \rho\log(\rho)dm \geqslant \mu(E)\log(\frac{\mu(E)}{m(E)})$$

therefore

$$\mu(E)\log(\frac{\mu(E)}{m(E)}) \leqslant \frac{1}{e} + C$$

now note that the left side is uniformly bounded for all μ in the set of level bounded by C, from here given $\{E_n\}_{n\in\mathbb{N}}$ a sequencia of sets Borel in X, with $m(E_n) \to 0$ then $\mu(E_n) \to 0$ for all μ such that $\xi_\infty(\mu) \leqslant C$. Finally see that $X = \cup_{n=1}^\infty K_n$ where K_n is compact, here we $\lim_{n\to\infty} m(X \setminus \cup_{l=1}^n K_l) = 0$, then just simply consider $E_n = X \setminus \cup_{l=1}^n K_l$ to get our result. q.e.d.

Lemma 6.6. *Assume that* $(X_n, d_n, m_n) \xrightarrow{\mathbb{D}} (X, d, m)$*, let* $\mu \in \mathcal{P}_2^a(X)$ *with density bounded, then there exist a sequencia* $\mu_n \in \mathcal{P}_2^a(X_n)$ *such that* $W_2(\mu_n, \mu) \to 0$

Theorem 6.6. *(Stability of weak Ricci curvature bound) Assume that* $(X_n, d_n, m_n) \xrightarrow{\mathbb{D}} (X, d, m)$ *and that for every* $n \in \mathbb{N}$ *the space* (X_n, d_n, m_n) *is* $CD(K, \infty)$*(resp.* $CD(0, N)$*. Then* (X, d, m) *is a* $CD(K, \infty)$ *(resp.* $CD(0, N)$ *space as well.*

Proof: Pick $\mu_0, \mu_1 \in \mathcal{P}_2^a(X)$, for the Theorem 6.4 we can assume they are both absolutely continuous with bounded densities, say $\mu_i = \rho_i m$, $i = 0, 1$. Choose $(\tilde{d_n}, \gamma_n) \in Opt((d_n, m_n), (d, m))$. Define $\mu_i^n := (\gamma_n^{-1})\#\mu_i \in \mathcal{P}_2^a(X)$, $i = 0, 1$. Then by assumption there is a geodesic $(\mu_t^n) \subset \mathcal{P}_2^a(X)$ such that

$$\xi_\infty(\mu_t^n) \leqslant (1-t)\xi_\infty(\mu_0^n) + t\xi_\infty(\mu_1^n) - \frac{K}{2}t(1-t)W_2^2(\mu_0^n, \mu_1^n). \quad (119)$$

Now let $\sigma_t := (\gamma_n)\#\mu_t^n \in \mathcal{P}_2^a(X)$, $t \in [0, 1]$. From Lemma 6.3 we have $W_2(\mu_i, \sigma_i^n)$ as $n \to \infty$

$$W_2(\mu_i^l, \mu_j^n) \leqslant \sqrt{M}\{\mathbb{D}((X_n, d_n, m_n), (X, d, m)) + \mathbb{D}((X_l, d_l, m_l), (X, d, m))\} \quad (120)$$

for $i, j = 0, 1$ and where M it is the highest among the limitations of densities μ_i. Using equations 119, 6.3.4 and Lemma 6.2, we know that $\xi_\infty(\sigma_t^n)$ is uniformly bounded in n, t. Thus of the Lemma 6.5 for every fixed t the sequence $n \to \sigma_t^n$ is tight, and we can extract a subsequence, not relabeled, such that σ_t^n narrowly converges to some $\sigma_t \in \mathcal{P}_2(supp(m))$ for every rational t. q.e.d.

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