Spectral Gap for Contracting Fiber Systems and Applications

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Thesis presented to Post-graduate Program at Institute of Mathematics, of the Universidade Federal do Rio de Janeiro, as partial fulfilment of requirements for the degree of Doctor in Mathematics.

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Rio de Janeiro

September 2015

CIP - Catalogação na Publicação

L931s

Lucena, Rafael Nóbrega de Oliveira
Spectral Gap for Contracting Fiber Systems and Applications / Rafael Nóbrega de Oliveira Lucena. - Rio de Janeiro, 2015.
80 f.

Orientadora: Maria José Pacífico.
Coorientador: Stefano Galatolo.
Tese (doutorado) - Universidade Federal do Rio de Janeiro, Instituto de Matemática, Programa de Pós-Graduação em Matemática, 2015.

1. Spectral Gap. 2. Lorenz. 3. Estabilidade. 4. Teoria Ergódica. 5. Sistemas Dinâmicos. I. Pacífico, Maria José, orient. II. Galatolo, Stefano, coorient. III. Título.

Elaborado pelo Sistema de Geração Automática da UFRJ com os dados fornecidos pelo(a) autor(a).

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15 de Setembro de 2015

Acknowledgments

To God.

To my mother Maria das Neves Lucena, my father Eudes Lucena and to my wife Cibelle Lucena.

To my advisers Maria José Pacífico and Stefano Galatolo.

To all my friends, specially Davi Lima, Rizwan Khan, Daniel Reis, Andrés Lopes, Midory Komatsudani, Victor Arturo, Abraham Muñoz, Marcio Cavalcante and Isaia Nisoli.

This thesis was partially supported by BREUDS with reference EU Marie-Curie IRSES Brazilian-European partnership in Dynamical Systems (FP7-PEOPLE-2012-IRSES 318999 BREUDS). It was also supported by CNPq and CAPES.

“ São essenciais, na vida cristã, a oração, a humildade, a caridade para com todos: este é o caminho para a santidade.”

Papa Francisco

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Resumo

Consideramos um sistema que preserva e contrai uma folheação vertical. Provamos que o operador de transferência associado, agindo sobre um espaço vetorial adequado, satisfaz uma propriedade chamada "spectral gap".

Como aplicação consideramos sistemas tipo Lorenz bi-dimensionais (hiperbólicos por partição com contração e expansão possivelmente ilimitadas): provamos que estes sistemas possuem "spectral gap" e obtemos uma estimativa quantitativa para sua estabilidade estatística. Sob certas perturbações determinísticas do sistema, a medida física varia continuamente, com módulo de continuidade igual a $O(\delta \log \delta)$.

Abstract

We consider transformations preserving a contracting foliation, such that the associated quotient map satisfies a Lasota Yorke inequality.

We prove that the associated transfer operator, acting on suitable normed spaces, has spectral gap.

As an application we consider Lorenz-Like two dimensional maps (piecewise hyperbolic with unbounded contraction and expansion rate): we prove that those systems have spectral gap and we show a quantitative estimation for their statistical stability. Under deterministic perturbations of the system, the physical measure varies continuously, with a modulus of continuity $O(\delta \log \delta)$.

Contents

Chapter 1

Introduction

The study of the behaviour of the transfer operator restricted to a suitable functional space has proven to be a powerful tool for the understanding of the statistical properties of a dynamical system. This approach gave first results in the study of the dynamics of piecewise expanding maps where the involved spaces are made of regular, absolutely continuous measures (see [6], [18], [8] for some introductory text). In recent years this approach was extended to piecewise hyperbolic systems by the use of suitable anisotropic norms (the expanding and contracting direction are treated differently), leading to suitable distribution spaces on which the transfer operator has good spectral properties (see e.g. [7], [5], [10], [14]). From these properties, several limit theorems or stability statements can be deduced. This approach has proven to be successful in non-trivial classes of systems like geodesic flows (see [18],[9]) or billiard maps (see e.g. [11] [12] where a relatively simple and unified approach to many limit and perturbative results is given for the Lorentz gas).

In this thesis, we consider maps preserving an uniformly contracting foliation. We show how it is possible, in a simple way, to define suitable spaces of signed measures (with an anisotropic norm) such that, under very weak regularity assumptions, the transfer operator associated to the dynamics has a spectral gap (in the sense given in Theorem 3.3.1). This shows an exponential convergence in a certain norm, for the iteration of a large class of measures by the transfer operator. Therefore, we present the construction and give some properties of such spaces of signed measures in chapter (2).

The main part of this work is presented in chapter (3). There we deal with a skew product of the type $F : \Sigma \longrightarrow \Sigma$, $F(x, y) = (T(x), G(x, y))$, where $T : N_1 \longrightarrow N_1$ and $G : \Sigma \longrightarrow N_2$ are mensurable maps satisfying some conditions and Σ is defined by $(\Sigma =) N_1 \times N_2$, where N_1 and N_2 are manifolds endowed with a Riemannian metric. Assuming certain assumptions given in the beginning of section 3.1 (**G1**, **T1**,...,**T3.4**) we prove a spectral gap for its transfer operator acting on a suitable space of sign measure (denoted by S^1). More precisely

Theorem 1.0.1 (Spectral gap on S^1) *If F satisfies **G1**, **T1**,...,**T3.4**, then the operator $\mathrm{F}^* : S^1 \longrightarrow S^1$ can be written as*

$$\mathrm{F}^* = \mathrm{P} + \mathrm{N}$$

where

a) P *is a projection, i.e.* $\mathrm{P}^2 = \mathrm{P}$*, and* $\dim Im(\mathrm{P}) = 1$*;*

b) there are $0 < \xi < 1$ *and* $C > 0$ *such that* $|| \mathrm{N}^n ||_{S^1 \rightarrow S^1} \leq \xi^n C$ *for all* $n \geq 1$*;*

c) $\mathrm{P}\,\mathrm{N} = \mathrm{N}\,\mathrm{P} = 0$*.*

Also in chapter (3), section 3.4, we present an application of this approach, showing spectral gap for 2-dimensional Lorenz like maps (piecewise hyperbolic maps with unbounded expansion and contraction rates) and a quantitative estimation for their statistical stability. We remark that a qualitative estimation for a class of similar maps was given in [1].

We also present two additional appendixes. In the first Chapter (4) we give a proof of the Proposition (3.5.3), which is used to get stability for the invariant measure and to prove that the invariant measure of such systems has a strong regularity property: bounded variation. In appendix presented in chapter (5) we present an alternative approach to obtain spectral gap for Lorenz Like systems with, like we did in chapter (3), with a stronger norm and convergence to equilibrium properties.

Chapter 2

Fundamental Results

In this chapter we formalize the construction of all spaces we are going to work.

2.0.1 The $BV_{1,\frac{1}{p}}$ $(1 \leq p \leq \infty)$ space.

In this section we briefly introduce the space of functions $BV_{1,\frac{1}{p}}$. For more details and a more general approach see [4].

Set $I = [0,1]$ and let m be the Lebesgue measure on it.

Definition 2.0.1 *For an arbitrary function $h : I \longrightarrow \mathbb{C}$ and $\epsilon > 0$ define $\operatorname{osc}(h, B_\epsilon(x)) : I \longrightarrow [0,\infty]$ by ($B_\epsilon(x)$ denotes the open ball of center x and radius ϵ)*

$$\operatorname{osc}(h, B_\epsilon(x)) = \text{ess sup}\{|h(y_1) - h(y_2)|; y_1, y_2 \in B_\epsilon(x)\}, \tag{2.1}$$

where the essential supremum is taken with respect to the product measure m^2 on $I \times I$.

Also define the real function $\operatorname{osc}_1(h,\epsilon)$, on the variable ϵ, by $\epsilon \longmapsto \operatorname{osc}_1(h,\epsilon) := \int \operatorname{osc}(h, B_\epsilon(x))dm(x)$.

Definition 2.0.2 *Fix $A_1 > 0$ and denote by Φ the class of all isotonic maps $\phi : (0, A_1] \longrightarrow [0,\infty]$ ($x \leq y \Longrightarrow \phi(x) \leq \phi(y)$ such that $\phi(x) \longrightarrow 0$ if $x \longrightarrow 0$). Set*

(a) $R_1 = \{h : I \longrightarrow \mathbb{C}; \operatorname{osc}_1(h, .) \in \Phi\}$;

(b) For $n \in \mathbb{N}$, define $R_{1,n\cdot\frac{1}{p}} = \{h \in R_1; \operatorname{osc}_1(h,\epsilon) \leq n \cdot \epsilon^{\frac{1}{p}} \quad \forall \epsilon \in (0, A_1]\}$;

(c) And set $S_{1,\frac{1}{p}} = \bigcup_{n\in\mathbb{N}} R_{1,n\cdot\frac{1}{p}}$.

Definition 2.0.3

(a) $BV_{1,\frac{1}{p}}$ *is the space of* m*-equivalence classes of functions in* $S_{1,\frac{1}{p}}$*;*

(b) For $h : I \longrightarrow \mathbb{C}$ *set*

$$\mathrm{var}_{1,\frac{1}{p}}(h) = \sup_{0\le\epsilon\le A_1}\left(\frac{1}{\epsilon^{\frac{1}{p}}}\,\mathrm{osc}_1(h,\epsilon)\right); \tag{2.2}$$

(c) For $h \in BV_{1,\frac{1}{p}}$ *set*

$$||h||_{1,\frac{1}{p}} = \mathrm{var}_{1,\frac{1}{p}} + ||h||_1 \tag{2.3}$$

The proof of the following result can be found in [4].

Theorem 2.0.1 $\left(BV_{1,\frac{1}{p}}, ||\ \ ||_{1,\frac{1}{p}}\right)$ *is a Banach space.*

Definition 2.0.4 *For a function* $f : [0,1] \longrightarrow \mathbb{C}$ *define the universal* p*-variation (*$1 \le p < \infty$*) by*

$$\mathrm{var}_p(f) = \sup_{0\le x_0<\cdots x_n\le 1}\left(\sum_{i=1}^{n}|f(x_i) - f(x_{i-1})|^p\right)^{\frac{1}{p}}.$$

Define space of universally bounded p*-variation functions by*

$$UBV_p := \{f : [0,1] \longrightarrow \mathbb{C}; \mathrm{var}_p(f) < \infty\}. \tag{2.4}$$

Lemma 2.0.1 $UBV_p \subset \bigcap_{n\in\mathbb{N}} BV_{p,n\cdot\frac{1}{p}}$ *for all* $1 \le p < \infty$*, where the intersection ranges over all spaces* $BV_{p,\frac{1}{p}}$ *which stem from any atom-free finite Borel measure* m *on* $[0,1]$ *and its associated pseudo-distance* d*. In particular, if* m *is a probability measure, then* $\mathrm{var}_{p,\frac{1}{p}}(f) \le 2^{\frac{1}{p}} \cdot \mathrm{var}_p(f)$*.*

2.0.2 The $\mathcal{L}^1$ Space

Let Σ be defined by $N_1 \times N_2$, where N_1 and N_2 are compact manifolds endowed with a Riemannian metric. Denote by m_1, m_2 and $m = m_1 \times m_2$ the corresponding Riemannian volume, normalized so that $m(\Sigma) = m_1(N_1) = m_2(N_2) = 1$.

Define the set $b1 - Lip(\Sigma) := 1 - Lip(\Sigma) \cap \{g \in L^\infty(m); ||g||_\infty \leq 1\}$, where $1 - Lip(\Sigma) = \{g \in Lip(\Sigma); L(g) \leq 1\}$ and $L(g)$ is the best Lipschitz constant of g i.e.,

$$L(g) = \sup_{x \neq y \in \Sigma} \left\{ \frac{|g(x) - g(y)|}{|x - y|} \right\}.$$

In the same way we define the sets $b1 - Lip(N_2)$ and $b1 - Lip(\gamma)$, for some leaf γ.

Definition 2.0.5 *Given two signed measures μ and ν on Σ we define the* ***Wasserstein like distance*** *between μ and ν as the real number*

$$W_1^0(\mu, \nu) = \sup_{g \in b1 - Lip(\Sigma)} \left| \int g d\mu - \int g d\nu \right|. \tag{2.5}$$

In the same way we define $W_1^0(\mu, \nu)$ when μ and ν are signed measures on any other compact metric space.

Remark 2.0.1 *From now, we are going to denote $||\mu||_W := W_1^0(0, \mu)$. As a matter of fact, $|| \ \ ||_W$ defines a norm on the vector space of signed measures, $\mathcal{SM}(M)$ defined on a compact metric space (M, d).*

Let $\mathcal{SB}(\Sigma)$ be the space of signed measures on $\Sigma = N_1 \times N_2$. Moreover, given a signed measure $\mu \in \mathcal{SB}(\Sigma)$ denote by μ^+ and μ^- the positive and the negative parts of it. It means $\mu = \mu^+ - \mu^-$. Now define the set $\mathcal{AB}$ as

$$\mathcal{AB} = \{\mu \in \mathcal{SB}(\Sigma) : \pi_x^* \mu^+ \ll m_1 \ \text{ and } \ \pi_x^* \mu^- \ll m_1\}, \tag{2.6}$$

where $\pi_x : \Sigma \longrightarrow N_1$ is the projection defined by $\pi(x, y) = x$.

In order to define the norms, we need to invoke the Rokhlin Disintegration Theorem. Let $(\Sigma_d, \mathcal{B})$ be a mensurable space, where $\Sigma_d := (\Sigma, d)$ is a compact metric space and $\mathcal{B}$ is its Borel's σ-algebra.

Definition 2.0.6 *A **disintegration** of μ with respect the partition $\mathcal{P}$ is a family of probabilities $\{\mu_P\}_{P\in\mathcal{P}}$ on Σ such that for every mensurable set $A \subset \Sigma$ we have*

a) $\mu_P(P) = 1$ *for* $\hat{\mu}$*-ae.* $P \in \mathcal{P}$*;*

b) The function $\mathcal{P} \longrightarrow \mathbb{R}$ *defined by* $P \longmapsto \mu_P(A)$ *is mensurable;*

c) $\mu(A) = \int \mu_P(A) d\hat{\mu}(P)$*.*

Now let us state the Rokhlin Theorem which will be used as a basis for the construction of the normed spaces we are going to work with.

Theorem 2.0.2 *Let Σ be a compact and separable metric space and let $\mathcal{P}$ be a mensurable partition. Then every probability μ admits a disintegration with respect to $\mathcal{P}$.*

Moreover

Proposition 2.0.1 *Suppose that the σ-algebra $\mathcal{B}$ admits an enumerable generator. If $\{\mu_P : P \in \mathcal{P}\}$ and $\{\mu'_P : P \in \mathcal{P}\}$ are disintegrations for μ with respect to $\mathcal{P}$ then $\mu_P = \mu'_P$ $\hat{\mu}$-ae.*

In our case the compact metric space and the mensurable partition are $\Sigma = N_1 \times N_2$ and $\mathcal{F}^s := \{\gamma_x\}_{x\in N_1}$ respectively, where $\gamma_x = \{x\} \times N_2$ for all $x \in N_1$. When there is no risk of confusion we denote γ_x just by γ. However given a probability $\mu \in \mathcal{AB}$, the theorem (2.0.2) gives its disintegration $(\{\mu_\gamma\}_\gamma, \mu_x = \phi_x m_1)$ along the stable leaves $\mathcal{F}^s$, where $\phi_x : N_1 \longrightarrow \overline{\mathbb{R}}$ is an extended real function (see equation (2.6) for the definition of $\mathcal{AB}$).

Now consider a finite measure μ on Σ, then $\overline{\mu} := \dfrac{\mu}{\mu(\Sigma)}$ is a probability measure on Σ and we can find its disintegration with respect to $\mathcal{F}^s$, i.e. a pair $\left(\{\overline{\mu}_\gamma\}_\gamma, \overline{\mu}_x = \overline{\phi}_x m_1\right)$ which satisfies the definition (2.0.6). The disintegration of $\overline{\mu}$ induces a natural disintegration for μ along the stable leaves $\mathcal{F}^s$ as the pair $(\{\mu_\gamma\}_\gamma, \mu_x)$, where

$$\mu_\gamma = \overline{\mu}_\gamma \quad \text{and} \quad \mu_x = \mu(\Sigma)\overline{\mu}_x = \pi_x^* \mu = \left(\mu(\Sigma)\overline{\phi}_x\right) m_1.$$

Indeed, for an arbitrary mensurable set $A \in \Sigma$, denoting $\overline{\mu}_\gamma(A) := \overline{\mu}_\gamma(A \cap \gamma)$, we have

$$\begin{aligned}\mu(A) &= \mu(\Sigma)\frac{\mu}{\mu(\Sigma)}(A)\\ &= \mu(\Sigma)\overline{\mu}(A)\\ &= \mu(\Sigma)\int \overline{\mu}_\gamma(A)d\overline{\mu}_x\\ &= \int \overline{\mu}_\gamma(A)d\left(\mu(\Sigma)\overline{\mu}_x\right).\end{aligned}$$

Moreover, we have

Corollary 2.0.1 *If μ is a finite Borel measure on Σ, then it has a unique disintegration along $\mathcal{F}^s$. In the sense that, if $\left(\{\overline{\mu}'_\gamma\}_\gamma, \mu_x\right)$ is another disintegration for the finite measure μ, then $\overline{\mu}'_\gamma = \overline{\mu}_\gamma$ μ_x-a.e.*

Proof. Suppose there is another disintegration, $\left(\{\overline{\mu}'_\gamma\}_\gamma, \mu_x\right)$, for μ. Let us show that $\left(\{\overline{\mu}'_\gamma\}_\gamma, \overline{\mu}_x\right)$ is a disintegration for $\overline{\mu}$.

Indeed,

$$\begin{aligned}\mu(A) &= \int \overline{\mu}'_\gamma(A)d\mu_x\\ &= \int \overline{\mu}'_\gamma(A)d\mu(\Sigma)\overline{\mu}_x\\ &= \mu(\Sigma)\int \overline{\mu}'_\gamma(A)d\overline{\mu}_x..\end{aligned}$$

And so

$$\frac{\mu}{\mu(\Sigma)}(A) = \int \overline{\mu}'_\gamma(A)d\overline{\mu}_x.$$

Thus

$$\overline{\mu}(A) = \int \overline{\mu}'_\gamma(A)d\overline{\mu}_x.$$

It implies that, the pair $\left(\{\overline{\mu}'_\gamma\}_\gamma, \overline{\mu}_x\right)$ is a disintegration for $\overline{\mu}$. Since, by proposition (2.0.1), $\overline{\mu}$ has only one disintegration, we get $\overline{\mu}'_\gamma = \overline{\mu}_\gamma$ $\overline{\mu}_x$-a.e. $\gamma \in N_1$ and also μ_x-a.e. $\gamma \in N_1$ (since they are equivalent). □

Proposition 2.0.2 *Let $\mu, \nu \in \mathcal{AB}$ be to finite measures and denote their marginal measures by $\mu_x = \phi_x m_1$ and $\nu_x = \psi_x m_1$, where $\phi_x, \psi_x \in L^1(m_1)$. Then the disintegration of $\mu + \nu$ is given by the pair*

$$\left(\frac{\phi_x(\gamma)}{\phi_x(\gamma) + \psi_x(\gamma)} \mu_\gamma + \frac{\psi_x(\gamma)}{\phi_x(\gamma) + \psi_x(\gamma)} \nu_\gamma, (\phi_x + \psi_x) m_1 \right).$$

In other words

$$(\mu + \nu)_\gamma = \frac{\phi_x(\gamma)}{\phi_x(\gamma) + \psi_x(\gamma)} \mu_\gamma + \frac{\psi_x(\gamma)}{\phi_x(\gamma) + \psi_x(\gamma)} \nu_\gamma \quad \text{and} \quad (\mu + \nu)_x = (\phi_x + \psi_x) m_1.$$

Proof. First we observe that the expression is well defined, in the sense that if $\phi_x(\gamma) + \psi_x(\gamma) = 0$ on γ, then $\phi_x(\gamma) = 0$ and $\psi_x(\gamma) = 0$. Therefore we can consider $(\mu + \nu)_\gamma \equiv 0$ on each leaf γ where it happens.

Thus, for a given mensurable set $A \subset \Sigma$ we have

$$\begin{aligned}
(\mu + \nu)(A) &= \mu(A) + \nu(A) \\
&= \int \mu_\gamma(A) \phi_x(\gamma) dm_1(\gamma) + \int \nu_\gamma(A) \psi_x(\gamma)_x dm_1(\gamma) \\
&= \int \mu_\gamma(A) \phi_x(\gamma) + \nu_\gamma(A) \psi_x(\gamma)_x \frac{\phi_x(\gamma) + \psi_x(\gamma)}{\phi_x(\gamma) + \psi_x(\gamma)} dm_1(\gamma) \\
&= \int \mu_\gamma(A) \frac{\phi_x(\gamma)}{\phi_x(\gamma) + \psi_x(\gamma)} + \nu_\gamma(A) \frac{\psi_x(\gamma)}{\phi_x(\gamma) + \psi_x(\gamma)} \left(\phi_x(\gamma) + \psi_x(\gamma) \right) dm_1(\gamma).
\end{aligned}$$

And we are done. □

Definition 2.0.7 *Let $\pi_{\gamma,y} : \gamma \longrightarrow N_2$ be the restriction $\pi_y|_\gamma$, where $\pi_y : \Sigma \longrightarrow N_2$ is the projection defined by $\pi_y(x, y) = y$ (however $\pi_{\gamma,y}$ is a bijection), where $(x, y) \in N_1 \times N_2$. For a given positive measure $\mu \in \mathcal{AB}$ and its disintegration along the stable leaves $\mathcal{F}^s$, $(\{\mu_\gamma\}_\gamma, \mu_x = \phi_x m_1)$, we define the* ***restriction of*** *μ* ***on*** *γ as the positive measure $\mu|_\gamma$ on N_2 (not on γ) defined, for a given mensurable set $E \subset N_2$, as*

$$\mu|_\gamma(E) = \pi^*_{\gamma,y}(\phi_x(\gamma) \mu_\gamma)(E).$$

For a given signed measure $\mu \in \mathcal{AB}$ and its decomposition $\mu = \mu^+ - \mu^-$, define the ***restriction of*** *μ* ***on*** *γ by $\mu|_\gamma = \mu^+|_\gamma - \mu^-|_\gamma$.*

Definition 2.0.8 *Let $\mathcal{L}^1$ be the space of signed measures defined as*

$$\mathcal{L}^1 = \left\{ \mu \in \mathcal{AB} : \int W_1^0(\mu^+|_\gamma, \mu^-|_\gamma) dm_1(\gamma) < \infty \right\} \tag{2.7}$$

and define the application $|| \ \ ||_1 : \mathcal{L}^1 \longrightarrow \mathbb{R}$ as

$$||\mu||_1 = \int W_1^0(\mu^+|_\gamma, \mu^-|_\gamma) dm_1(\gamma). \tag{2.8}$$

Remark 2.0.2 $||\mu||_1 = \int W_1^0(0, \mu|_\gamma) dm_1(\gamma) = \int ||\mu|_\gamma||_W dm_1(\gamma)$.

Lemma 2.0.2 *If μ_1, μ_2, ν_1 and μ_2 are measures on Σ (or on any other compact metric space) then*

$$W_1^0(\mu_1 + \mu_2, \nu_1 + \nu_2) \leq W_1^0(\mu_1, \nu_1) + W_1^0(\mu_2, \nu_2).$$

PROOF.

$$\begin{aligned} W_1^0(\mu_1 + \mu_2, \nu_1 + \nu_2) &= \sup_{g \in b1-Lip(\Sigma)} \left| \int g d(\mu_1 + \mu_2) - \int g d(\nu_1 + \nu_2) \right| \\ &\leq \sup_{g \in b1-Lip(\Sigma)} \left| \int g d\mu_1 - \int g d\nu_1 \right| + \sup_{g \in b1-Lip(\Sigma)} \left| \int g d\mu_2 - \int g \nu_2 \right| \\ &= W_1^0(\mu_1, \nu_1) + W_1^0(\mu_2, \nu_2). \end{aligned}$$

As desired.

□

Proposition 2.0.3 $(\mathcal{L}^1, || \ \ ||_1)$ *is a normed space.*

PROOF.

We divide the proof into several lemmas.

Lemma 2.0.3 *(Triangular Inequality) Consider $\mu, \nu \in \mathcal{L}^1$ two signed measures. Then*

$$||\mu + \nu||_1 \leq ||\mu||_1 + ||\nu||_1.$$

Proof.

By proposition (2.0.2) it holds

$$\int W_1^0((\mu^+ + \nu^+)|_\gamma, (\mu^- + \nu^-)|_\gamma) dm_1 = \int W_1^0\left(\pi^*_{\gamma,y}\left(\phi^+_x \mu^+_\gamma + \psi^+_x \nu^+_\gamma\right), \pi^*_{\gamma,y}\left(\phi^-_x \mu^-_\gamma + \psi^-_x \nu^-_\gamma\right)\right) dm_1.$$

Moreover, using the above relation and lemma (2.0.2) we have

$$\begin{aligned}
||\mu + \nu||_1 &= ||(\mu^+ - \mu^-) + (\nu^+ - \nu^-)||_1 \\
&= ||(\mu^+ + \nu^+) - (\mu^- + \nu^-)||_1 \\
&= \int W_1^0((\mu^+ + \nu^+)|_\gamma, (\mu^- + \nu^-)|_\gamma) dm_1 \\
&\leq \int W_1^0\left(\pi^*_{\gamma,y}\left(\phi^+_x \mu^+_\gamma\right), \pi^*_{\gamma,y}\left(\phi^-_x \mu^-_\gamma\right)\right) + W_1^0\left(\pi^*_{\gamma,y}\left(\psi^+_x \nu^+_\gamma\right), +\pi^*_{\gamma,y}\left(\psi^-_x \nu^-_\gamma\right)\right) dm_1 \\
&\leq \int W_1^0\left(\pi_{\gamma,y} * \left(\phi^+_x \mu^+_\gamma\right), \pi_{\gamma,y} * \left(\phi^-_x \mu^-_\gamma\right)\right) dm_1 \\
&+ \int W_1^0\left(\pi_{\gamma,y} * \left(\psi^+_x \nu^+_\gamma\right), +\pi_{\gamma,y} * \left(\psi^-_x \nu^-_\gamma\right)\right) dm_1 \\
&= \int W_1^0\left(\mu^+|_\gamma, \mu^-|_\gamma\right) dm_1 + \int W_1^0\left(\nu^+|_\gamma, \nu^-|_\gamma\right) dm_1 \\
&= ||\mu||_1 + ||\nu||_1.
\end{aligned}$$

As we wished. □

Lemma 2.0.4 *For a given signed measure $\mu \in \mathcal{AB}$, it holds $||\mu||_1 = 0$ if and only if $\mu \equiv 0$.*

Proof. If $\mu \equiv 0$, then $||\mu||_1 = 0$ immediately. Reciprocally, suppose we have $||\mu||_1 = 0$ then

$$\begin{aligned}
\int W_1^0\left(\mu^+|_\gamma, \mu^-_\gamma\right) dm_1 = 0 &\Longrightarrow W_1^0\left(\mu^+|_\gamma, \mu^-_\gamma\right) = 0, \quad m_1 - \text{ae.} \;\; \gamma \in N_1 \\
&\Longrightarrow \mu^+|_\gamma = \mu^-_\gamma, \quad m_1 - \text{ae.} \;\; \gamma \in N_1 \\
&\Longrightarrow \pi_{\gamma,y} * (\phi^+ \mu^+_\gamma) = \pi_{\gamma,y} * (\phi^- \mu^-_\gamma), \quad m_1 - \text{ae.} \;\; \gamma \in N_1 \\
&\Longrightarrow \phi^+ \mu^+_\gamma = \phi^- \mu^-_\gamma, \quad m_1 - \text{ae.} \;\; \gamma \in N_1, \quad \text{since} \;\; \pi^*_{\gamma,y} \;\; \text{is a bijection.}
\end{aligned}$$

Thus, for a given mensurable set $A \subset \Sigma$, we have

$$\begin{aligned}
\mu(A) &= \mu^+(A) - \mu^-(A) \\
&= \int \mu^+_\gamma(A)\phi^+_x(\gamma)dm_1(\gamma) - \int \mu^-_\gamma(A)\phi^-_x(\gamma)dm_1(\gamma) \\
&= \int \mu^+_\gamma(A)\phi^+_x(\gamma) - \mu^-_\gamma(A)\phi^-_x(\gamma)dm_1(\gamma) \\
&= 0.
\end{aligned}$$

As we desired. □

Lemma 2.0.5 *For every signed measure $\mu \in \mathcal{L}^1$ and every real number α it holds*

$$||\alpha\mu||_1 = |\alpha|||\mu||_1.$$

Proof. In fact, by definition we have

$$\begin{aligned}
||\alpha\mu||_1 &= ||\alpha\mu^+ - \alpha\mu^-||_1 \\
&= \int W_1^0(\alpha\mu^+|_\gamma, \alpha\mu^-|_\gamma)dm_1 \\
&= \int \sup_{g\in b1-Lip(I)} \left|\int gd\alpha\mu^+|_\gamma - \int gd\alpha\mu^-|_\gamma\right| dm_1 \\
&= |\alpha| \int \sup_{g\in b1-Lip(I)} \left|\int gd\mu^+|_\gamma - \int gd\mu^-|_\gamma\right| dm_1 \\
&= |\alpha| \int W_1^0(\mu^+|_\gamma, \mu^-|_\gamma)dm_1 \\
&= |\alpha|||\mu||_1.
\end{aligned}$$

□

With the above lemmas we finish the proof of the proposition (2.0.3). □

2.0.3 The $\mathcal{L}^\infty$ Space

Definition 2.0.9 *Let $\mathcal{L}^\infty \subseteq \mathcal{AB}(\Sigma)$ be defined as*

$$\mathcal{L}^\infty = \left\{\mu \in \mathcal{AB} : \text{ess sup}_{\gamma\in N_1}(W_1^0(\mu^+|_\gamma, \mu^-|_\gamma)) < \infty\right\} \tag{2.9}$$

where the essential supremum is taken over N_1, with respect to m_1. Define $||\ \ ||_\infty : \mathcal{L}^\infty \longrightarrow \mathbb{R}$ as

$$||\mu||_\infty = \text{ess sup}_{\gamma\in N_1} W_1^0(\mu^+|_\gamma, \mu^-|_\gamma) = \text{ess sup}_{\gamma\in N_1} W_1^0(0, \mu|_\gamma). \tag{2.10}$$

Proposition 2.0.4 *($\mathcal{L}^\infty, ||\ \ ||_\infty$) is a normed vector space.*

PROOF. Is straightforward to prove that $||\mu||_\infty = 0$ if and only if $\mu = 0$. Thus let us prove the remaining part. Consider $\mu_1, \mu_2, \mu \in \mathcal{L}^\infty$ and $\alpha \in \mathbb{R}$. Note that, for every signed measure μ and $\alpha \geq 0$ it holds $(\alpha\mu)^+ = \alpha\mu^+$ and $(\alpha\mu)^- = \alpha\mu^-$. And, for $\alpha < 0$ we have $(\alpha\mu)^+ = |\alpha|\mu^-$ and $(\alpha\mu)^- = |\alpha|\mu^+$. Then, for every Lipschitz function g s.t. $||g||_\infty \leq 1$ and $L(g) \leq 1$ and for all $\alpha \in \mathbb{R}$ we have

$$\left|\int gd(\alpha\mu)^+|_\gamma - \int gd(\alpha\mu)^-|_\gamma\right| = |\alpha| \left|\int gd\mu^+|_\gamma - \int gd\mu^-|_\gamma\right|.$$

Then we get $||\alpha\mu||_\infty = |\alpha|||\mu||_\infty$.

In order to prove the triangular inequality note that $(\mu_1 + \mu_2)|_\gamma = \mu_1|_\gamma + \mu_2|_\gamma$. So we have, by definition of W_1^0, that

$$\begin{aligned} ||\mu_1 + \mu_2||_\infty &= \text{ess sup}\, W_1^0(0, (\mu_1 + \mu_2)|_\gamma) \\ &\leq \text{ess sup}\, W_1^0(0, \mu_1|_\gamma) + \text{ess sup}\, W_1^0(0, \mu_2|_\gamma) \\ &= ||\mu_1||_\infty + ||\mu_2||_\infty. \end{aligned}$$

□

2.0.4 The $\mathcal{BV}$ Space

In this section, we set $N_1 = N_2 = I = [0,1]$ and so $\Sigma = I^2$.

Definition 2.0.10 *Consider a pair* $(\{\mu_\gamma\}_{\gamma \in I}, \phi_x)$*, where* $\{\mu_\gamma\}_\gamma$ *is a family of probabilities on* γ *defined m-a.e.* $\gamma \in I$ *and* $\phi_x : I \longrightarrow \overline{R}$ *is a non-negative extended real function. Given such pair,* $(\{\mu_\gamma\}_\gamma, \phi_x)$*, denote by* G_μ *the path (of positive measures on* I*)* $G_\mu : I \longrightarrow \mathcal{B}(I)$ *defined m-a.e. by* $G_\mu(\gamma) = \pi^*_{\gamma,y}\phi_x(\gamma)\mu_\gamma$*. Call the set on which* G_μ *is defined by* I_{G_μ}*. Let* $\mathcal{P} = \mathcal{P}(G_\mu)$ *be a finite sequence* $\mathcal{P} = \{x_i\}_{i=1}^n \subset I_{G_\mu}$ *such that* $G_\mu(\gamma_{x_i})$ *is well defined for all* $i = 0, \cdots, n$*. Define the* ***variation of*** G_μ ***with respect to*** $\mathcal{P}$ *as (denote* $\gamma_i := \gamma_{x_i}$*)*

$$Var(G_\mu, \mathcal{P}) = \sum_{j=1}^{n} ||G_\mu(\gamma_j) - G_\mu(\gamma_{j-1})||_W,$$

where $||G_\mu(\gamma_j) - G_\mu(\gamma_{j-1})||_W = W_1^0(G_\mu(\gamma_j), G_\mu(\gamma_{j-1}))$*. Finally we define the* ***variation of*** G_μ *as*

$$Var(G_\mu) := \sup_{\mathcal{P}} Var(G_\mu, \mathcal{P}).$$

Remark 2.0.3 *For an interval* $\eta \subset I$ *we define*

$$Var_{\overline{\eta}}(G_\mu) := Var(G_\mu|_\eta).$$

Remark 2.0.4 *When there is no risk of confusion, to simplify the notation, we denote* $G_\mu(\gamma)$ *just by* $\mu|_\gamma$*.*

We say that a pair $(\{\mu_\gamma\}_\gamma, \phi_x)$ (or its path G_μ) represents a positive measure μ if, for every mensurable set $A \subset \Sigma$, holds

$$\mu(A) = \int \mu_\gamma(A \cap \gamma)\phi_x(\gamma)dm(\gamma).$$

Denote by $[[\mu]]$ the set of all pairs $(\{\mu_\gamma\}_\gamma, \phi_x)$ which represents μ. The Rokhlin Disintegration Theorem ensures that $[[\mu]] \neq \emptyset$.

Definition 2.0.11 *Define the variation of a positive measure* μ *by*

$$Var(\mu) = \inf_{G_\mu \in [[\mu]]} \{Var(G_\mu)\}.$$

Remark 2.0.5 *Note that*

$$||\mu||_1 = \int W_0(0, G_\mu(\gamma))dm(\gamma)$$

for any $G_\mu \in [[\mu]]$.

Definition 2.0.12 *From the definition (3.5.2) we define the set of positive measures* $\mathcal{BV}$ *as*

$$\mathcal{BV} = \{\mu \in \mathcal{AB} : Var(\mu) < \infty\}. \tag{2.11}$$

Define the real function $|| \ \ ||_{\mathcal{BV}}$ on $\mathcal{BV}$ by $||\mu||_{\mathcal{BV}} = Var(\mu) + ||\mu||_1$. The proof of the next proposition is equivalent of the Proposition 2.0.6 below, so we omit it.

Proposition 2.0.5 $(\mathcal{BV}, || \ \ ||_{\mathcal{BV}})$ *is a normed space.*

2.0.5 The $\mathcal{BV}_2$ Space

In this subsection, we deal with the particular case when $\Sigma = I^2$, where $I = [0, 1]$. Let us define the variation of a signed measure μ. To do it, let us consider the set B_{C^2} defined by

$$B_{C^2} = \{g \in C^2(I); ||g||_\infty \leq 1, ||g'||_\infty \leq \Omega, ||g''||_\infty \leq 1\},$$

where $0 \leq \Omega \leq 1$(the constant Ω depends of the application, see (5.7) in section 5). Using the above sets we define a new norm on the space $\mathcal{SB}(I)$ of signed measures on I by

$$||\mu||_{C^{2\prime}} = \sup_{g \in B_{C^2}} \left\{ \left| \int g d\mu \right| \right\} \quad \forall\ \mu \in I. \tag{2.12}$$

Definition 2.0.13 *Consider a pair* $(\{\mu_\gamma\}_{\gamma \in I}, \phi_x)$*, where* $\{\mu_\gamma\}_\gamma$ *is a family of probabilities on* γ *defined* m*-a.e.* $\gamma \in I$ *and* $\phi_x : I \longrightarrow \overline{R}$ *is a non-negative extended real function. Given such pair,* $(\{\mu_\gamma\}_\gamma, \phi_x)$*, denote by* G_μ *the path (of positive measures on* I*)* $G_\mu : I \longrightarrow \mathcal{B}(I)$ *defined* m*-a.e. by* $G_\mu(\gamma) = \pi^*_{\gamma,y}\phi_x(\gamma)\mu_\gamma$*. Call the set on which* G_μ *is defined by* I_{G_μ}*. Let* $\mathcal{P} = \mathcal{P}(G_\mu)$ *be a finite sequence* $\mathcal{P} = \{x_i\}_{i=1}^n \subset I_{G_\mu}$ *such that* $G_\mu(\gamma_{x_i})$ *is well defined for all* $i = 0, \cdots, n$*.*

We say that a pair $(\{\mu_\gamma\}_\gamma, \phi_x)$ (or its path G_μ) represents a positive measure μ if, for every mensurable set $A \subset \Sigma$, holds

$$\mu(A) = \int \mu_\gamma(A \cap \gamma)\phi_x(\gamma)dm(\gamma).$$

Denote by $[\mu]$ the set of all pairs $(\{\mu_\gamma\}_\gamma, \phi_x)$ which represents μ. The Rokhlin Disintegration Theorem ensures that $[\mu] \neq \emptyset$.

In case μ is a signed measure, we say that a path of sing measures $G_\mu : I \longrightarrow \mathcal{SM}(I)$ represents the signed measure μ on Σ (i.e. $\mu = \mu^+ - \mu^-$ where $\mu^\pm$ are positive measures) if $G_\mu = G_{\mu^+} - G_{\mu^-}$ for $G_{\mu^+} \in [\mu^+]$ and $G_{\mu^-} \in [\mu^-]$. Denote by $[[\mu]]$ the set of all paths, $G_\mu : I \longrightarrow \mathcal{SM}(I)$, which represents μ. The Rokhlin Disintegration Theorem ensures that $[[\mu]] \neq \emptyset$ for all $\mu \in \mathcal{AB}$.

Definition 2.0.14 *Given a signed measure* $\mu \in \mathcal{AB}$ *define the* ***variation of*** G_μ ***with***

respect to $\mathcal{P}$ *as (denote* $\gamma_i := \gamma_{x_i}$*)*

$$Var(G_\mu, \mathcal{P}) = \sum_{j=1}^{n} ||G_\mu(\gamma_j) - G_\mu(\gamma_{j-1})||_{C^{2'}},$$

where $||G_\mu(\gamma_j) - G_\mu(\gamma_{j-1})||_{C^{2'}}$. *Finally we define the* ***variation of*** G_μ *as*

$$Var(G_\mu) := \sup_{\mathcal{P}} Var(G_\mu, \mathcal{P}).$$

Remark 2.0.6 *For an interval* $\eta \subset I$ *we define*

$$Var_{\overline{\eta}}(G_\mu) := Var(G_\mu|_\eta).$$

Definition 2.0.15 *Define the variation of a signed measure* $\mu \in \mathcal{L}^1$ *by*

$$Var(\mu) = \inf_{G_\mu \in [[\mu]]} \{Var(G_\mu)\}.$$

Remark 2.0.7 *Note that*

$$||\mu||_1 = \int W_0(0, G_\mu(\gamma))dm(\gamma), \quad \text{for any} \quad G_\mu \in [[\mu]].$$

Definition 2.0.16 *From the definition (2.0.14) we define the set of signed measures* $\mathcal{BV}_2(m)$ *(we'll denote it just by* $\mathcal{BV}_2$*) as*

$$\mathcal{BV}_2 = \{\mu \in \mathcal{L}^1 : Var(\mu) < \infty\}. \tag{2.13}$$

Define the real function $||\ \ ||_{\mathcal{BV}}$ on $\mathcal{BV}_2$ by $||\mu||_{\mathcal{BV}} = Var(\mu) + ||\mu||_1$. Thus, $\mathcal{BV}_2$ provided with $||\ \ ||_{\mathcal{BV}}$ is a normed space.

Proposition 2.0.6 $(\mathcal{BV}_2, ||\ \ ||_{BV})$ *is a normed space.*

Proof.

Consider two paths G_{μ_1} and G_{μ_2} defined on the same full measure set $\widehat{I} \subset I$ which represents the signed measures $\mu_1, \mu_2 \in \mathcal{L}^1$. Since we have proposition (2.0.3) and $Var(G_{\mu_1} + G_{\mu_2}) \leq Var(G_{\mu_1}) + Var(G_{\mu_2})$ it'll be a straightforward computation. Indeed, holds

$$Var(G_{\mu_1} + G_{\mu_2}) + ||\mu_1 + \mu_2||_1 \quad = \quad Var(G_{\mu_1}) + Var(G_{\mu_2}) + ||\mu_1||_1 + ||\mu_2||_1.$$

Taking the infimum we get $||\mu_1 + \mu_2||_{BV} \leq ||\mu_1||_{BV} + ||\mu_2||_{BV}$. Besides that, is easy to see that $||\alpha\mu||_{BV} = |\alpha|||\mu||_{BV}$, for every scalar α. And since $||\mu||_1 = 0$ if and only if $\mu = 0$ we get $||\mu||_{BV} = 0$ if only if $\mu = 0$. □

Chapter 3

Spectral Gap for Lorenz Systems

3.1 Contracting Fiber Maps

In this section we continue to consider the same setting as in subsection 2.0.2 i.e. let Σ be defined by $N_1 \times N_2$, where N_1 and N_2 are compact manifolds endowed with a Riemannian metric. Denote by m_1 and m_2 their corresponding Riemannian volume, normalized so that $m_1(N_1) = m_2(N_2) = 1$ and $m = m_1 \times m_1$. Consider a dynamical system $F : \Sigma \longrightarrow \Sigma$, $F(x, y) = (T(x), G(x, y))$, where $T : N_1 \longrightarrow N_1$ and $G : \Sigma \longrightarrow N_2$ are mensurable maps satisfying some conditions stated below. Moreover, the spaces $\mathcal{L}^1$ and $\mathcal{L}^\infty$ were defined in subsections (2.0.2) and (2.0.3).

Properties of G

G1 Consider the F-invariant foliation $\mathcal{F}^s := \{\{x\} \times N_2\}_{x \in N_1}$. Suppose there exists $0 < \alpha < 1$ such that for all $x \in N_1$ holds

$$|G(x, y_1) - G(x, y_2)| \leq \alpha |y_1 - y_2| \quad for\ all \quad y_1, y_2 \in N_2. \tag{3.1}$$

Properties of T and of its associated transfer operator.

We suppose that:

T1 T is non-singular with respect to m_1 ($m_1(A) = 0 \Rightarrow m_1(T^{-1}(A)) = 0$).

T2 There exists a collection of open sets $\mathcal{P} = \{P_1, \cdots, P_q\}$ of N_1, such that $m_1\left(\bigcup_{i=1}^q P_i\right) = 1$ and $T_i := T|_{P_i}$ is a diffeomorphism, with $\det T_i'(x) \neq 0$ $\forall x \in P_i$ and for all i, where T_i' is the Jacobian of T_i with respect to the Riemannian metric of N_1.

T3 Let us consider the Perron-Frobenius Operator associated to T, P_T [1].

We will now make some assumption on the existence of a suitable functional analytic setting adapted to P_T. Let us hence denote the $L^1_{m_1}$ norm[2] by $|\ \ |_1$ and suppose that there exists a normed space $(S_{\text{-}}, |\ \ |_s)$ such that

T3.1 $S_{\text{-}} \subset L^1_{m_1}$ is P_T-invariant and $|\ \ |_1 \leq |\ \ |_s$;

T3.2 The unit ball of $(S_{\text{-}}, |\ \ |_s)$ is relatively compact in $(L^1_{m_1}, |\ \ |_1)$;

T3.3 (Lasota Yorke inequality) There exists $k \in \mathbb{N}$, $0 < \beta_0 < 1$ and $C > 0$ such that, for all $f \in S_{\text{-}}$, holds

$$|\,\mathrm{P}_T^k f|_s \leq \beta_0 |f|_s + C|f|_1. \tag{3.2}$$

T3.4 Suppose there is a unique $\psi_x \in S_{\text{-}}$ with $\psi_x \geq 0$ and $|\psi_x|_1 = 1$ such that $\mathrm{P}_T(\psi_x) = \psi_x$, and if $\psi \in S_{\text{-}}$ is another density for a probability measure, then $P_T^k(\psi_x - \psi) \to 0$ in $S_{\text{-}}$.

By the Ionescu-Tulcea and Marinescu theorem (see [16]) the following result holds.

Theorem 3.1.1 *If T satisfies $T3.1, ..., T3.4$ then there exist $0 < r < 1$ and $D > 0$ such that for all*

$$\phi \in V := \{\phi \in S_{\text{-}}; \int \phi \ dm = 0\}$$

and for all $n \geq 0$, it holds

$$|\,\mathrm{P}_T^n(\phi)|_s \leq Dr^n |\phi|_s. \tag{3.3}$$

The following property on $|\ |_s$ will be supposed, sometimes in the future, to obtain spectral gap on L^∞ like spaces.

N1 $|\ |_s \geq |\ |_\infty$ (where $|\ |_\infty$ is the usual L^∞ norm on N_1)

[1]The unique operator $\mathrm{P}_T : L^1_{m_1} \longrightarrow L^1_{m_1}$ such that

$$\forall \phi \in L^1_{m_1} \ \ and \ \ \forall \psi \in L^\infty_{m_1} \quad \int \psi \cdot \mathrm{P}_T(\phi) \ dm = \int (\psi \circ T) \cdot \phi \ dm.$$

[2]**Notation:** In the following we use $|\ |$ to indicate the usual absolute value or norms for signed measures on the basis space N_1. We will use $||\ ||$ for norms defined for signed measures on Σ.

Now define the following set of signed measures on Σ

$$S^1 = \left\{\mu \in \mathcal{L}^1; \phi_x \in S_{\text{-}}\right\}. \tag{3.4}$$

Consider $|| \ ||_{S^1} : S^1 \longrightarrow \mathbb{R}$, defined by

$$||\mu||_{S^1} = |\phi_x|_s + ||\mu||_1 \tag{3.5}$$

where we recall that ϕ_x is the marginal density of the disintegration of μ.

Analogous to the previous, define

$$S^\infty = \{\mu \in \mathcal{L}^\infty; \phi_x \in S_{\text{-}}\}. \tag{3.6}$$

Consider $|| \ ||_{s^\infty} : S^\infty \longrightarrow \mathbb{R}$ defined by

$$||\mu||_{s^\infty} = |\phi_x|_s + ||\mu||_\infty. \tag{3.7}$$

Proposition 3.1.1 *$(S^1, || \ ||_{S^1})$ and $(S^\infty, || \ ||_{s^\infty})$ are normed vector spaces.*

PROOF. Consider $\mu_1, \mu_2 \in S^\infty$ and $\alpha \in \mathbb{R}$. Remember that the restriction of a measure is a linear operation, in the sense that $(\mu_1 + \alpha\mu_2)|_\gamma = \mu_1|_\gamma + \alpha\mu_2|_\gamma$. Moreover $\mu|_\gamma(N_1) = \phi_x(\gamma)$. Denote by $\phi_x^{\mu_1+\alpha\mu_2}, \phi_x^1, \phi_x^2$ the densities of $\mu_1 + \alpha\mu_2$, μ_1 and μ_2 respectively. So we have

$$\begin{aligned} \phi_x^{\mu_1+\alpha\mu_2}(\gamma) &= (\mu_1 + \alpha\mu_2)|_\gamma(N_1) \\ &= \mu_1|_\gamma(N_1) + \alpha\mu_2|_\gamma(N_1) \\ &= \phi_x^1(\gamma) + \alpha\phi_x^2(\gamma). \end{aligned}$$

Then $\mu_1 + \alpha\mu_2 \in S^\infty$. The same argument tells us that, if $\mu_1, \mu_2 \in S^1$ and $\alpha \in \mathbb{R}$ then $\mu_1 + \alpha\mu_2 \in S^1$.

To see that $|| \ ||_{s^1}$ and $|| \ ||_{s^1}$ are norms is a straightforward computation.

First of all, is immediate to see that $||\mu||_{s^1} = 0$ iff $\mu = 0$ and $||\mu||_{s^\infty} = 0$ iff $\mu = 0$. So we omit this part. So let's prove the rest.

$$\begin{aligned} ||\mu_1 + \mu_2||_{s^1} &= |\phi_x^1 + \alpha\phi_x^2|_s + ||\mu_1 + \mu_2||_1 \\ &\leq |\phi_x^1|_s + |\phi_x^2|_s + ||\mu_1||_1 + ||\mu_2||_1 \\ &= ||\mu_1||_{s^1} + ||\mu_2||_{s^1}. \end{aligned}$$

The same argument shows us that $||\mu_1 + \mu_2||_{s^\infty} \leq ||\mu_1||_{s^\infty} + ||\mu_2||_{s^\infty}$ for every $\mu_1, \mu_2 \in S^\infty$.

Besides that

$$\begin{aligned} ||\alpha\mu_1||_{s^1} &= |\alpha\phi_x^1|_s + ||\alpha\mu_1||_1 \\ &= |\alpha|||\mu_1||_{s^1}. \end{aligned}$$

The same argument shows us that $||\alpha\mu_1||_{s^\infty} = |\alpha|||\mu_1||_{s^\infty}$.

□

3.2 Basic properties of the norms and convergence to equilibrium

In this section we will get some properties of the actions $\mathrm{F}^* : \mathcal{L}^1 \longrightarrow \mathcal{L}^1$ and $\mathrm{F}^* : \mathcal{L}^\infty \longrightarrow \mathcal{L}^\infty$, where $\mathcal{L}^1$ and $\mathcal{L}^\infty$ are from subsections (2.0.2) and (2.0.3) and F^*, is the transfer operator associated with F, i.e.,

$$[\mathrm{F}^* \mu](E) = \mu(F^{-1}(E))$$

for all signed measure μ on Σ and for all mensurable set $E \subset \Sigma$.

Lemma 3.2.1 *For all probability $\mu \in \mathcal{AB}$ disintegrated by $(\{\mu_\gamma\}_\gamma, \phi_x)$, the disintegration $((\mathrm{F}^* \mu)_\gamma, (\mathrm{F}^* \mu)_x)$ of $\mathrm{F}^* \mu$ is given by*

$$(\mathrm{F}^* \mu)_x = \mathrm{P}_T(\phi_x) m_1 \tag{3.8}$$

and

$$(\mathrm{F}^* \mu)_\gamma = \sum_{i=1}^{q} \frac{\phi_x}{|\det DT|} \circ T_i^{-1}(\gamma) \cdot \frac{\chi_{T_i(P_i)}(\gamma)}{\mathrm{P}_T(\phi_x)(\gamma)} \cdot \mathrm{F}^* \mu_{T_i^{-1}(\gamma)} \tag{3.9}$$

when $\mathrm{P}_T(\phi_x)(\gamma) \neq 0$. Otherwise, if $\mathrm{P}_T(\phi_x)(\gamma) = 0$, then $(F^*\mu)_\gamma = \nu_\gamma$, where ν_γ is the Lebesgue measure on γ (the expression $\frac{\phi_x}{|\det DT|} \circ T_i^{-1}(\gamma) \cdot \frac{\chi_{T_i(P_i)}(\gamma)}{\mathrm{P}_T(\phi_x)(\gamma)} \cdot F^*\mu_{T_i^{-1}(\gamma)}$ is understood to be zero outside $T_i(P_i)$ for all $i = 1, \cdots, q$).

Proof.

To prove the proposition is enough to prove the following equation

$$\mathrm{F}^* \mu(E) = \int_{N_1} (\mathrm{F}^* \mu)_\gamma (E \cap \gamma) d(\mathrm{F}^* \mu)_x(\gamma) \tag{3.10}$$

where $(\mathrm{F}^* \mu)_\gamma$ and $(\mathrm{F}^* \mu)_x$ are given by (3.8) and (3.9), for a measurable set $E \in \Sigma$. And by the uniqueness of the disintegration (see [?], proposition 5.1.7) the result will be established.

To do it, let us define the sets $B_1 = \{\gamma \in N_1; T^{-1}(\gamma) = \emptyset\}$, $B_2 = \{\gamma \in B_1^c; \mathrm{P}_T(\phi_x)(\gamma) = 0\}$ and $B_3 = (B_1 \cup B_2)^c$. The following properties can be easily proven.

1. $B_i \cap B_j = \emptyset$, $T^{-1}(B_i) \cap T^{-1}(B_j) = \emptyset$ for all $1 \leq i, j \leq 3$ such that $i \neq j$ and $\bigcup_{i=1}^3 B_i = \bigcup_{i=1}^3 T^{-1}(B_i) = N_1$;
2. $m_1(T^{-1}(B_1)) = m_1(T^{-1}(B_2)) = 0$;

Using the change of variables $\gamma = T_i(\beta)$, we have

$$\begin{aligned}
\int_{N_1} (\mathrm{F}^* \mu)_\gamma (E \cap \gamma) d(\mathrm{F}^* \mu)_x(\gamma) &= \int_{B_3} (\mathrm{F}^* \mu)_\gamma (E \cap \gamma) d(\mathrm{F}^* \mu)_x(\gamma) \\
&= \sum_{i=1}^r \int_{T_i(P_i) \cap B_3} \frac{\phi_x}{|\det DT|} \circ T_i^{-1}(\gamma)\, \mathrm{F}^* \mu_{T_i^{-1}(\gamma)}(E) dm_1(\gamma) \\
&= \sum_{i=1}^r \int_{P_i \cap T_i^{-1}(B_3)} \phi_x(\beta) \mu_\beta (F^{-1}(E)) dm_1(\beta) \\
&= \int_{\bigcup_{i=1}^3 T^{-1}(B_i)} \mu_\beta (F^{-1}(E)) d\phi_x m_1(\beta) \\
&= \int_{N_1} \mu_\beta (F^{-1}(E)) d\phi_x m_1(\beta) \\
&= \mu(F^{-1}(E)) \\
&= \mathrm{F}^* \mu(E).
\end{aligned}$$

□

Remark 3.2.1 *For a given leaf $\gamma \in \mathcal{F}^s$, define the map $F_\gamma : N_2 \longrightarrow N_2$ by $F_\gamma := \pi_y \circ F|_\gamma \circ \pi_{\gamma,y}^{-1}$. We remark that, by the previous lemma, for all $\mu \in \mathcal{L}^1$ and for almost all $\gamma \in N_1$ holds*

$$(\mathrm{F}^* \mu)|_\gamma = \sum_{i=1}^q \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)} \mu|_{T_i^{-1}(\gamma)}}{|T_i'^{-1}(\gamma))|} \chi_{T(\eta_i)}(\gamma) \quad \text{for almost all} \quad \gamma \in N_1. \tag{3.11}$$

3.2.1 Basic properties of the $\mathcal{L}^1$ norm

Remark 3.2.2 *For every $\mu \in \mathcal{AB}$, holds*

$$||\,\mathrm{F}^*_\gamma\,\mu|_\gamma||_W \leq ||\mu|_\gamma||_W \tag{3.12}$$

where $F_\gamma : N_2 \longrightarrow N_2$ was defined in remark (3.2.1). Indeed, since F_γ is a contraction, if $|g|_\infty \leq 1$ and $Lip(g) \leq 1$ we have that the same holds for $g \circ F_\gamma$. Then

$$\begin{aligned} \left|\int g \; d\,\mathrm{F}^*_\gamma\,\mu\right| &= \left|\int g(F_\gamma)\; d\mu\right| \\ &\leq ||\mu||_W\,. \end{aligned}$$

Taking the supremum over $|g|_\infty \leq 1$ and $Lip(g) \leq 1$ we finish the proof of the inequality. We also remark that, if μ is a probability measure, then $||\mu||_W = 1$. Hence

$$||\,\mathrm{F}^{*\,n}\mu||_W = ||\mu||_W = 1 \;\;\forall\;\; n \geq 1. \tag{3.13}$$

Proposition 3.2.1 (The weak norm is weakly contracted by F^*) *For all $\mu \in \mathcal{L}^1$ the following weak contraction holds*

$$||\,\mathrm{F}^*\,\mu||_1 \leq ||\mu||_1. \tag{3.14}$$

Proof. By Lemma 3.2.1 (remark (3.2.1)), for every signed measure $\mu \in \mathcal{L}^1_0$ and for a.e. $\gamma \in N_1$ holds

$$(\mathrm{F}^*\,\mu)|_\gamma = \sum_{i=1}^{q} \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)}\,\mu|_{T_i^{-1}(\gamma)}}{|T_i'^{-1}(\gamma))|}\chi_{T(\eta_i)}(\gamma). \tag{3.15}$$

Then for all i, making the change of variable $\gamma = T_i(\alpha)$ and by remark (3.2.2), we have that

$$\begin{aligned}
||\,\mathrm{F}^*\,\mu||_1 &= \int_{N_1} ||(\mathrm{F}^*\,\mu)|_\gamma||_W dm_1(\gamma) \\
&\leq \sum_{i=1}^{q} \int_{T(\eta_i)} \left|\left| \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)}\,\mu|_{T_i^{-1}(\gamma)}}{|T_i'(T_i^{-1}(\gamma))|} \right|\right|_W dm_1(\gamma) \\
&= \sum_{i=1}^{q} \int_{\eta_i} ||\mathrm{F}^*_\alpha\,\mu|_\alpha||_W dm_1(\alpha) \\
&= \sum_{i=1}^{q} \int_{\eta_i} ||\mu|_\alpha||_W dm_1(\alpha) \\
&= ||\mu||_1.
\end{aligned}$$

□

Proposition 3.2.2 *There are $\overline{C} > 0$ and $k \in \mathbb{N}$ (see **T3.3**) such that for all $\mu \in S$, it holds*

$$||\,\mathrm{F}^{*\,k}\mu||_{S^1} \leq \beta_0||\mu||_{S^1} + \overline{C}||\mu||_1. \tag{3.16}$$

Proof. Set $\overline{C} = 1 - \beta_0 + C$ where C and β_0 are from equation (3.2). Thus it holds (note that $|\phi_x|_1 \leq ||\mu||_1$)

$$\begin{aligned}
||\,\mathrm{F}^{*\,k}\mu||_{S^1} &= |\,\mathrm{P}_T^k\,\phi_x|_s + ||\,\mathrm{F}^{*\,k}\mu||_1 \\
&\leq \beta_0|\phi_x|_s + C|\phi_x|_1 + ||\mu||_1 \\
&= \beta_0(|\phi_x|_s + ||\mu||_1) - \beta_0||\mu||_1 + C||\mu||_1 + ||\mu||_1 \\
&\leq \beta_0||\mu||_{S^1} + \overline{C}||\mu||_1.
\end{aligned}$$

□

Corollary 3.2.1 (Lasota Yorke inequality for S^1) *There exist $A,\ B_2 \in \mathbb{R}, \lambda < 1$ s.t. for all $\mu \in S^1$ holds*

$$||\,\mathrm{F}^{*\,n}\mu||_{S^1} \leq A\lambda^n||\mu||_{S^1} + B_2||\mu||_1 \ \ \forall n \geq 1. \tag{3.17}$$

Proof. Iterating the relation of the proposition (3.2.2), one will find the following inequality

$$||\,\mathrm{F}^{*\,nk}\mu||_{S^1} \leq \beta_0^n||\mu||_{S^1} + \overline{C}\sum_{i=0}^{\infty}\beta_0^i||\mu||_1. \tag{3.18}$$

from which one easily gets (3.17). □

Now we prove that F has exponential convergence to equilibrium. This is weaker with respect to spectral gap. However the spectral gap follows from the above Lasota Yorke inequality and the convergence to equilibrium. First we need some preliminary lemma.

Lemma 3.2.2 *For all signed measure μ on N_2 and for all $\gamma \in N_1$ holds*

$$|| \mathrm{F}_\gamma^* \mu||_W \leq \alpha||\mu||_W + \mu(N_2).$$

*(where α is the one given in **G1**). In particular, if $\mu(N_2) = 0$ then*

$$|| \mathrm{F}_\gamma^* \mu||_W \leq \alpha||\mu||_W.$$

Proof. If $Lip(g) \leq 1$ and $||g||_\infty \leq 1$, then $g \circ F_\gamma$ is α-Lipschitz. Moreover since $||g||_\infty \leq 1$ then $||g \circ F_\gamma - \theta||_\infty \leq \alpha$ for some $\theta \leq 1$. This implies

$$\begin{aligned}\left|\int g d\,\mathrm{F}_\gamma^* \mu\right| &= \left|\int g \circ F_\gamma d\mu\right| \\ &= \left|\int g \circ F_\gamma - \theta d\mu\right| + \left|\int \theta d\mu\right| \\ &= \alpha \left|\int \frac{g \circ F_\gamma - \theta}{\alpha} d\mu\right| + \theta\mu(N_2) \\ &= \alpha\, ||\mu||_W + \mu(N_2).\end{aligned}$$

And taking the supremum over $|g|_\infty \leq 1$ and $Lip(g) \leq 1$ we have $|| \mathrm{F}_\gamma^* \mu||_W \leq \alpha||\mu||_W + \mu(N_2)$. In particular, if $\mu(N_2) = 0$ we get the second part. □

Proposition 3.2.3 *For all signed measure $\mu \in \mathcal{L}^1$ holds*

$$|| \mathrm{F}^* \mu||_1 \leq \alpha||\mu||_1 + (\alpha + 1)|\phi_x|_1. \tag{3.19}$$

Proof. Consider a signed measure $\mu \in \mathcal{L}^1$ and its restriction on the leaf γ, $\mu|_\gamma = \pi_{\gamma,y}^*(\phi_x(\gamma)\mu_\gamma)$. Set

$$\overline{\mu}|_\gamma = \pi_{\gamma,y}^*\mu_\gamma.$$

If μ is a positive measure then $\overline{\mu}|_\gamma$ is a probability on N_2. Moreover $\mu|_\gamma = \phi_x(\gamma)\overline{\mu}|_\gamma$.

By the above comments and the expression given by remark 3.2.1 we have

$$\begin{aligned}
|| \mathrm{F}^* \mu||_1 &\leq \sum_{i=1}^{q} \int_{T(I_i)} \left|\left| \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)} \overline{\mu^+}|_{T_i^{-1}(\gamma)} \phi_x^+(T_i^{-1}(\gamma))}{|T_i'| \circ T_i^{-1}(\gamma)} - \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)} \overline{\mu^+}|_{T_i^{-1}(\gamma)} \phi_x^-(T_i^{-1}(\gamma))}{|T_i'| \circ T_i^{-1}(\gamma)} \right|\right|_W dm_1(\gamma) \\
&\leq \sum_{i=1}^{q} \int_{T(I_i)} \left|\left| \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)} \overline{\mu^+}|_{T_i^{-1}(\gamma)} \phi_x^+(T_i^{-1}(\gamma))}{|T_i'| \circ T_i^{-1}(\gamma)} - \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)} \overline{\mu^+}|_{T_i^{-1}(\gamma)} \phi_x^-(T_i^{-1}(\gamma))}{|T_i'| \circ T_i^{-1}(\gamma)} \right|\right|_W dm_1(\gamma) \\
&+ \sum_{i=1}^{q} \int_{T(I_i)} \left|\left| \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)} \overline{\mu^+}|_{T_i^{-1}(\gamma)} \phi_x^-(T_i^{-1}(\gamma))}{|T_i'| \circ T_i^{-1}(\gamma)} - \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)} \overline{\mu^-}|_{T_i^{-1}(\gamma)} \phi_x^-(T_i^{-1}(\gamma))}{|T_i'| \circ T_i^{-1}(\gamma)} \right|\right|_W dm_1(\gamma) \\
&= \mathrm{I}_1 + \mathrm{I}_2
\end{aligned}$$

where

$$\mathrm{I}_1 = \sum_{i=1}^{q} \int_{T(I_i)} \left|\left| \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)} \overline{\mu^+}|_{T_i^{-1}(\gamma)} \phi_x^+(T_i^{-1}(\gamma))}{|T_i'| \circ T_i^{-1}(\gamma)} - \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)} \overline{\mu^+}|_{T_i^{-1}(\gamma)} \phi_x^-(T_i^{-1}(\gamma))}{|T_i'| \circ T_i^{-1}(\gamma)} \right|\right|_W dm_1(\gamma)$$

and

$$\mathrm{I}_2 = \sum_{i=1}^{q} \int_{T(I_i)} \left|\left| \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)} \overline{\mu^+}|_{T_i^{-1}(\gamma)} \phi_x^-(T_i^{-1}(\gamma))}{|T_i'| \circ T_i^{-1}(\gamma)} - \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)} \overline{\mu^-}|_{T_i^{-1}(\gamma)} \phi_x^-(T_i^{-1}(\gamma))}{|T_i'| \circ T_i^{-1}(\gamma)} \right|\right|_W dm_1(\gamma).$$

Let us estimate I_1 and I_2.

By remark 3.2.2 and a change of variable we have

$$\begin{aligned}
\mathrm{I}_1 &= \sum_{i=1}^{q} \int_{T(I_i)} \left|\left| \mathrm{F}^*_{T_i^{-1}(\gamma)} \overline{\mu^+}|_{T_i^{-1}(\gamma)} \right|\right|_W \frac{|\phi_x^+ - \phi_x^-|}{|T_i'|} \circ T_i^{-1}(\gamma) dm_1(\gamma) \\
&= \int_I \left|\left| \mathrm{F}^*_\beta \overline{\mu^+}|_\beta \right|\right|_W |\phi_x^+ - \phi_x^-|(\beta) dm_1(\beta) \\
&= \int_I |\phi_x^+ - \phi_x^-|(\beta) dm_1(\beta) \\
&= |\phi_x|_1
\end{aligned}$$

and by lemma 3.2.2 we have

$$\begin{aligned}
\mathrm{I}_2 &= \sum_{i=1}^{q}\int_{T(I_i)} \left|\left|\mathrm{F}^*_{T_i^{-1}(\gamma)}\left(\overline{\mu^+}|_{T_i^{-1}(\gamma)} - \overline{\mu^-}|_{T_i^{-1}(\gamma)}\right)\right|\right|_W \frac{\phi_x^-}{|T_i'|}\circ T_i^{-1}(\gamma)dm_1(\gamma) \\
&\leq \sum_{i=1}^{q}\int_{I_i} \left|\left|\mathrm{F}^*_{\beta}\left(\overline{\mu^+}|_{\beta} - \overline{\mu^-}|_{\beta}\right)\right|\right|_W \phi_x^-(\beta)dm_1(\beta) \\
&\leq \alpha\int_I \left|\left|\overline{\mu^+}|_{\beta} - \overline{\mu^-}|_{\beta}\right|\right|_W \phi_x^-(\beta)dm_1(\beta) \\
&\leq \alpha\int_I \left|\left|\overline{\mu^+}|_{\beta}\phi_x^-(\beta) - \overline{\mu^+}|_{\beta}\phi_x^+(\beta)\right|\right|_W dm_1(\beta) \\
&\leq \alpha\int_I \left|\left|\overline{\mu^+}|_{\beta}\phi_x^-(\beta) - \overline{\mu^+}|_{\beta}\phi_x^+(\beta)\right|\right|_W dm_1(\beta) + \alpha\int_I \left|\left|\overline{\mu^+}|_{\beta}\phi_x^+(\beta) - \overline{\mu^-}|_{\beta}\phi_x^-(\beta)\right|\right|_W dm_1(\beta) \\
&= \alpha|\phi_x|_1 + \alpha||\mu||_1.
\end{aligned}$$

Summing the above estimates we finish the proof.

□

Iterating (3.19) we get.

Corollary 3.2.2 *For all signed measure $\mu \in \mathcal{L}^1$ holds*

$$||\,\mathrm{F}^{*\,n}\mu||_1 \leq \alpha^n||\mu||_1 + \overline{\alpha}|\phi_x|_1,$$

where $\overline{\alpha} = \frac{1+\alpha}{1-\alpha}$.

Now let us consider the set of zero average measures

$$\mathcal{V} = \{\mu \in S^1 : \mu(\Sigma) = 0\}. \tag{3.20}$$

Note that for all $\mu \in \mathcal{V}$ we have $\pi_x^*\mu(I) = 0$. Moreover, since $\pi_x^*\mu = \phi_x m_1$ ($\phi_x := \phi_x^+ - \phi_x^-$) we have $\int \phi_x dm_1 = 0$.

Proposition 3.2.4 (Exponential convergence to equilibrium) *There exist $D \in \mathbb{R}$ and $0 < \beta_1 < 1$ such that, for every signed measure $\mu \in \mathcal{V}$, holds*

$$||\,\mathrm{F}^{*\,n}\mu||_1 \leq D_2\beta_1^n||\mu||_{S^1}$$

for all $n \geq 1$.

Proof. Given $\mu \in \mathcal{V}$ and denoting $\phi_x = \phi_x^+ - \phi_x^-$, holds that $\int \phi_x dm_1 = 0$. Moreover, from (3.3) we have $|\,\mathrm{P}_T^n(\phi_x)|_s \leq Dr^n|\phi_x|_s$ for all $n \geq 1$, then $|\,\mathrm{P}_T^n(\phi_x)|_s \leq Dr^n||\mu||_{S^1}$ for all $n \geq 1$.

Let l and $0 \leq d \leq 1$ be the coefficients of the division of n by 2, i.e. $n = 2l+d$. Thus $l = \frac{n-d}{2}$ (remember $||\,\mathrm{F}^{*\,s}\mu||_1 \leq ||\mu||_1$ for all s and $||\mu||_1 \leq ||\mu||_{S^1}$) and by corollary (3.2.2) holds (below, set $\overline{r} = \sup\{r, \alpha\}$)

$$\begin{aligned}
||\,\mathrm{F}^{*\,n}\mu||_1 &\leq ||\,\mathrm{F}^{*\,2l+d}\mu||_1 \\
&\leq \alpha^l||\,\mathrm{F}^{*\,l+d}\mu||_1 + \overline{\alpha}||\pi_x * (\mathrm{F}^{*\,l+d}\mu)||_1 \\
&\leq \alpha^l||\mu||_1 + \overline{\alpha}|\,\mathrm{P}_T^l(\phi_x)|_1 \\
&\leq \alpha^l||\mu||_s + \overline{\alpha}|\,\mathrm{P}_T^l(\phi_x)|_s \\
&\leq \alpha^l||\mu||_s + \overline{\alpha}r^l D|\phi_x|_s \\
&\leq \left(\alpha^l + \overline{\alpha}r^l D\right)||\mu||_{S^1} \\
&\leq (1+\overline{\alpha}D)\overline{r}^l||\mu||_{S^1} \\
&= (1+\overline{\alpha}D)\overline{r}^{\frac{n-d}{2}}||\mu||_{S^1}
\end{aligned}$$

from which the statement follows directly. □

Now recall that we denoted by ψ_x the unique invariant density in $S_{\_}$ for T. Consider the measure $\nu_0 = \psi_x m_1 \times m_2$, and the iterates $\mathrm{F}^{*n}(\nu_0)$. By what was just proved, this define a Cauchy sequence for the weak norm. The existence of a limit in S^1 is not trivial, because such a space is not complete.

Proposition 3.2.5 *Define $\mu_0 = \lim_{n\to\infty} \mathrm{F}^{*n}(\nu_0)$. Such limit exists and μ_0 is the unique invariant measure of the system in S^1.*

PROOF. We prove the existence of the limit: by Proposition 5.3.3 the sequence $\nu_n = \mathrm{F}^{*n}(\nu_0)$ is a Cauchy sequence in $\mathcal{L}^1$, then ν_n is also a Cauchy sequence for the Wasserstein distance on the square. Since this is a sequence of probability measures it has a limit μ_0 which is a signed measure. We now prove that $\mu_0 \in \mathcal{L}^1$.

Since a sequence converging in $L^1_{m_1}$ has a subsequence which converges almost everywhere, then there is n_k such that for almost each γ, $\nu_{n_k}|_\gamma \to \mu_0|_\gamma$ in W_1^0(on N_1). Hence $\mu_0 \in \mathcal{L}^1$.

Since $\pi_x\mu_0 = \psi_x m$, $\mu_0 \in S^1$. For the uniqueness, if μ_0, $\mu_1 \in S^1$ are invariant, then $\mu_0 - \mu_1 \in V$ and then $\mathrm{F}^{*n}(\mu_0 - \mu_1) \to 0$ in S^1. Contradicting invariance.

□

Remark 3.2.3 *By **N1** we have* $|\ |_\infty \leq |\ |_s$. *Since, for all* $g : I \longrightarrow \mathbb{R}$ *such that* $|g|_\infty \leq 1$ *and* $L(g) \leq 1$, *it holds* $\left|\int gd(\mu_0|_\gamma)\right| \leq |g|_\infty \psi_x(\gamma) \leq |\psi_x|_s$, *we get that* $\mu_0 \in S^\infty$.

3.2.2 Properties of the $\mathcal{L}^\infty$ norm

Lemma 3.2.3 *Under the assumptions* $G1$, $T1, ..., T3.4$, *for all signed measure* $\mu \in S^\infty$ *with marginal density* ϕ_x *it holds*

$$||\,\mathrm{F}^*\,\mu||_\infty \leq \alpha|\,\mathrm{P}_T\,1|_\infty ||\mu||_\infty + |\,\mathrm{P}_T\,\phi_x|_\infty.$$

Proof. Let T_i be the branches of T and

$$\begin{aligned}
||(\mathrm{F}^*\,\mu)|_\gamma||_W &= \left|\left|\sum_{i=1}^n \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)}\,\mu|_{T_i^{-1}(\gamma)}}{|T_i'(T_i^{-1}(\gamma))|}\chi_{T(\eta_i)}\right|\right|_W \\
&\leq \sum_{i=1}^n \frac{||\,\mathrm{F}^*_{T_i^{-1}(\gamma)}\,\mu|_{T_i^{-1}(\gamma)}||_W}{|T_i'(T_i^{-1}(\gamma))|}\chi_{T(\eta_i)} \\
&\leq \sum_{i=1}^n \frac{\alpha||\mu|_{T_i^{-1}(\gamma)}||_W + \phi_x(T_i^{-1}(\gamma))}{|T_i'(T_i^{-1}(\gamma))|}\chi_{T(\eta_i)} \\
&\leq \alpha||\mu||_\infty \sum_{i=1}^n \frac{\chi_{T(\eta_i)}(\gamma)}{|T_i'(T^{-1}(\gamma))|} + \sum_{i=1}^n \frac{\phi_x(T_i^{-1}(\gamma))}{|T_i'(T^{-1}(\gamma))|}\chi_{T(\eta_i)}.
\end{aligned}$$

hence taking the supremum on γ we get the statement. □

Applying the last lemma to F^n instead of F one obtains.

Lemma 3.2.4 *Under the assumptions* $G1$, $T1, ..., T3.4$, *for all signed measure* $\mu \in S^\infty$ *and* ϕ_x *its marginal, it holds*

$$||\,\mathrm{F}^n\,\mu||_\infty \leq \alpha^n|\,\mathrm{P}_T^n\,1|_\infty ||\mu||_\infty + |\,\mathrm{P}_T^n\,\phi_x|_\infty.$$

Proposition 3.2.6 (Lasota Yorke inequality for S^∞) *Suppose* F *satisfies the assumptions* $G1$, $T1, ..., T3.4$ *and* $N1$. *Then there is* $0 < \alpha_1 < 1$ *and* $A_1, B_4 \in \mathbb{R}$ *such that*

$$||\,\mathrm{F}^n\,\mu||_{S^\infty} \leq A_1\alpha_1^n ||\mu||_{S^\infty} + B_4||\mu||_1.$$

Proof. We remark that by the Lasota Yorke inequality (T3) and (N1) it follows

$|\,\mathrm{P}_T^n\,1|_\infty \le 1 + C$, for each n. Then

$$\begin{aligned} ||\,\mathrm{F}^n\,\mu||_{S^\infty} &= |\,\mathrm{P}_T^n\,\phi_x|_s + ||\,\mathrm{P}_T^n\,\mu||_\infty \\ &\le [\beta_0^n|\phi_x|_s + C|\phi_x|_1] + [\alpha^n|\,\mathrm{P}_T^n\,1|_\infty||\mu||_\infty + |\,\mathrm{P}_T^n\,\phi_x|_\infty] \\ &\le [\beta_0^n|\phi_x|_s + C|\phi_x|_1] + [\alpha^n(C+1)||\mu||_\infty + \beta_0^n|\phi_x|_s + C|\phi_x|_1]. \\ &\le \alpha_1^n(C+2)||\mu||_s + 2C||\mu||_1. \end{aligned}$$

where we set $\alpha_1 = \max(\alpha, \beta_0)$ and recall that $|\phi_x|_1 \le ||\mu||_1$. □

3.3 Spectral gap.

Theorem 3.3.1 (Spectral gap on S^1) *If F satisfies **G1**, **T1**,...,**T3.4** given at beginning of Section 3.1, then the operator $\mathrm{F}^* : S^1 \longrightarrow S^1$ can be written as*

$$\mathrm{F}^* = \mathrm{P} + \mathrm{N}$$

where

- *a) P is a projection i.e. $\mathrm{P}^2 = \mathrm{P}$ and $\dim Im(\mathrm{P}) = 1$;*
- *b) there are $0 < \xi < 1$ and $C > 0$ such that [3] $||\,\mathrm{N}^n(\mu)||_{S^1\to S^1} \le \xi^n C$;*
- *c) $\mathrm{P}\,\mathrm{N} = \mathrm{N}\,\mathrm{P} = 0$.*

Proof. First let us show there exist $0 < \xi < 1$ and $K > 0$ such that, for all $n \ge 1$ holds

$$||(\mathrm{F}^*\,|_{\mathcal{V}})^n||_{S^1\to S^1} \le \xi^n K. \qquad (3.21)$$

Indeed, consider $\mu \in \mathcal{V}$ (see equation (3.20)) s.t. $||\mu||_{S^1} \le 1$ and for a given $n \in \mathbb{N}$ let m and $0 \le d \le 1$ be the coefficients of the division of n by 2, i.e. $n = 2m + d$. Thus $m = \frac{n-d}{2}$. By the Lasota Yorke inequality (corollary 3.2.1) we have the uniform bound $||\,\mathrm{F}^{*\,n}\mu||_{S^1} \le B_2 + 1$ for all $n \ge 1$. Moreover, by propositions 5.3.3 and 3.2.1 there is

[3] We remark that by this, the spectral radius of $\overline{\mathrm{N}}$ satisfies $\rho(\overline{\mathrm{N}}) < 1$, where $\overline{\mathrm{N}}$ is the extension of N to $\overline{S^1}$ (the completion of S_1). This gives us spectral gap, in the usual sense, for the operator $\overline{\mathrm{F}} : \overline{S_1} \longrightarrow \overline{S_1}$. The same remark holds for theorem (3.3.2).

some D_2 such that it holds (below, let λ_0 be defined by $\lambda_0 = \max\{\beta_1, \lambda\}$)

$$\begin{aligned}
||\,\mathrm{F}^{*\,n}\mu||_{S^1} &\leq A\lambda^m||\,\mathrm{F}^{*\,(m+d)}\mu||_{S^1} + B_2||\,\mathrm{F}^{*\,(m+d)}\mu||_1 \\
&\leq \lambda^m A(A+B_2) + B_2||F^{*m}\mu||_1 \\
&\leq \lambda^m A(A+B_2) + B_2 D_2 \beta_1^m \\
&\leq \lambda_0^m \left[A(A+B_2) + B_2 D_2\right] \\
&\leq \lambda_0^{\frac{n-d}{2}} \left[A(A+B_2) + B_2 D_2\right] \\
&\leq \left(\sqrt{\lambda_0}\right)^n \left(\frac{1}{\lambda_0}\right)^{\frac{d}{2}} \left[A(A+B_2) + B_2 D_2\right] \\
&= \xi^n K
\end{aligned}$$

Where and $\xi = \sqrt{\lambda_0}$. Hence, defining $K = \left(\frac{1}{\lambda_0}\right)^{\frac{d}{2}} [A(A+B_2) + B_2D_2]$, we arrive at

$$||(\mathrm{F}^*\,|_{\mathcal{V}})^n||_{S^1 \to S^1} \leq \xi^n K. \tag{3.22}$$

Now recall that $\mathrm{F}^* : S^1 \longrightarrow S^1$ has a unique fixed point μ_0. Consider the operator $\mathrm{P} : S^1 \longrightarrow [\mu_0]$ ($[\mu_0]$ is the space spanned by μ_0), defined by $\mathrm{P}(\mu) = \mu(1)\mu_0$. By definition P is a projection. Now define the operator

$$\mathrm{S} : S^1 \longrightarrow \mathcal{V},$$

by

$$\mathrm{S}(\mu) = \mu - \mathrm{P}(\mu) \quad \forall \;\; \mu \in S.$$

Thus define $\mathrm{N} = \mathrm{F}^* \circ \mathrm{S}$ and observe that, by definition $\mathrm{P}\,\mathrm{N} = \mathrm{N}\,\mathrm{P} = 0$ and $\mathrm{F}^* = \mathrm{P} + \mathrm{N}$. Moreover, $\mathrm{N}^n(\mu) = \mathrm{F}^{*\,n}(\mathrm{S}(\mu))$. Since S is bounded and $\mathrm{S}(\mu) \in \mathcal{V}$ we get, by (3.22) $||\,\mathrm{N}^n(\mu)||_{S^1} \leq \xi^n K||\,\mathrm{S}\,||_{S^1\to S^1}||\mu||_{S^1}$. □

In the same way, using the $\mathcal{L}^\infty$ Lasota Yorke inequality of proposition 3.2.6, it is possible to obtain spectral gap on the L^∞ like space, we omit the proof which is essentially the same as above:

Theorem 3.3.2 (Spectral gap on S^∞) *If F satisfies the assumptions $G1, T1, ..., T3.4$ and $N1$, then the operator $\mathrm{F}^* : S^\infty \longrightarrow S^\infty$ can be written as*

$$\mathrm{F}^* = \mathrm{P} + \mathrm{N}$$

where

a) P *is a projection i.e.* $\mathrm{P}^2 = \mathrm{P}$ *and* $\dim Im(\mathrm{P}) = 1$*;*

b) there are $0 < \xi < 1$ *and* $C > 0$ *such that* $||\,\mathrm{N}^n(\mu)||_{S^\infty \to S^\infty} \leq \xi^n C$*;*

c) $\mathrm{P}\,\mathrm{N} = \mathrm{N}\,\mathrm{P} = 0$.

3.4 Application to Lorenz like maps

In this section we apply the theorem (3.3.2) to a large class of Lorenz-Like flows, and more precisely to its Poincaré maps for suitable secions. In these systems (see e.g [4]), it can be proved that there is a Poincaré section Σ, whose return map has the form $F_L(x, y) = (T_L(x), G_L(x, y))$ after a suitable change of coordinates, with the properties given at beginning of Section 3.1. The map T_L, in this case, can be supposed to be piecewise expanding with $C^{1+\alpha}$ branches.

More precisely, we consider a class of maps satisfying $(G1)$ and the following additional properties on T_L :

Properties of T_L in Lorenz like systems

(P'1) $\dfrac{1}{|T_L'|}$ is of universally bounded p-variation ($1 \leq p < \infty$), i.e.

$$\sup_{0 \leq x_0 < \cdots < x_n \leq 1} \left(\sum_{i=0}^{n} |\frac{1}{|T_L'(x_i)|} - \frac{1}{|T_L'(x_{i-1})|}|^p \right)^{\frac{1}{p}} < \infty;$$

(P'2) $\inf |T_L^{n\prime}| > 1$ for some n.

From these properties it follows ([15]) that we can define a suitable strong space for the transfer operator associated to such a T_L, in a way that it satisfies the assumptions $T1, ..., T3.4$ and $N1$. And then we can apply our results.

For this, let us introduce the set $BV_{1,\frac{1}{p}}$ of real valued functions (for more details and results see [15]).

Definition 3.4.1 *For an arbitrary function* $h : I \longrightarrow \mathbb{C}$ *and* $\epsilon > 0$ *define* $\mathrm{osc}(h, B_\epsilon(x)) : I \longrightarrow [0, \infty]$ *by (*$B_\epsilon(x)$ *denotes the open ball of center* x *and radius* ϵ*)*

$$\mathrm{osc}(h, B_\epsilon(x)) = \mathrm{ess\ sup}\{|h(y_1) - h(y_2)|; y_1, y_2 \in B_\epsilon(x)\}, \tag{3.23}$$

where the essential supremum is taken with respect to the product measure m^2 on $I \times I$.

Also define the the real function $\mathrm{osc}_1(h, \epsilon)$, *on the variable* ϵ, *by* $\mathrm{osc}_1(h, \epsilon) := \int \mathrm{osc}(h, B_\epsilon(x)) dm(x)$.

Definition 3.4.2 *Fix* $A_1 > 0$ *and denote by* Φ *the class of all isotonic maps* $\phi : (0, A_1] \longrightarrow [0, \infty]$ $(x \leq y \Longrightarrow \phi(x) \leq \phi(y))$ *such that* $\phi(x) \longrightarrow 0$ *if* $x \longrightarrow 0$. *Set*

- $R_1 = \{h : I \longrightarrow \mathbb{C}; \mathrm{osc}_1(h, .) \in \Phi\}$;
- *For* $n \in \mathbb{N}$, *define* $R_{1,n \cdot p} = \{h \in R_1; \mathrm{osc}_1(h, \epsilon) \leq n \cdot \epsilon^{\frac{1}{p}} \quad \forall \epsilon \in (0, A_1]\}$;
- *And set* $S_{1,p} = \bigcup_{n \in \mathbb{N}} R_{1,n \cdot p}$.

Definition 3.4.3

1. $BV_{1,\frac{1}{p}}$ *is the space of* m*-equivalence classes of functions in* $S_{1,p}$;
2. *For* $h : I \longrightarrow \mathbb{C}$ *set*

$$\mathrm{var}_{1,\frac{1}{p}}(h) = \sup_{0 \leq \epsilon \leq A_1} \left(\frac{1}{\epsilon^{\frac{1}{p}}} \mathrm{osc}_1(h, \epsilon) \right). \tag{3.24}$$

Considering the real function $|\ \ |_{1,\frac{1}{p}} : BV_{1,\frac{1}{p}} \longrightarrow \mathbb{R}$ defined by

$$|f|_{1,\frac{1}{p}} = \mathrm{var}_{1,\frac{1}{p}}(f) + |f|_1, \tag{3.25}$$

it holds the following

Proposition 3.4.1 $\left(BV_{1,\frac{1}{p}}, |\ \ |_{1,\frac{1}{p}}\right)$ *is a Banach space.*

Under those above settings G. Keller has shown (see [15]) that there is an $A_1 > 0$ (we recall that definition (3.4.2) depends on A_1) such that:

(a) $BV_{1,\frac{1}{p}} \subset L^1$ is P_T-invariant and holds $|\ \ |_1 \leq |\ \ |_{1,\frac{1}{p}}$;

(b) The unit ball of $(BV_{1,\frac{1}{p}}, |\ \ |_{1,\frac{1}{p}})$ is relatively compact in $(L^1, |\ \ |_1)$;

(c) There exists $k \in \mathbb{N}$, $0 < \beta_0 < 1$ and $C > 0$ such that

$$|\,\mathrm{P}_T^k f|_{1,\frac{1}{p}} \leq \beta_0 |f|_{1,\frac{1}{p}} + C|f|_1. \tag{3.26}$$

Moreover, in [2] (Lemma 2) it is shown that

(d) $| \ |_\infty \leq A_1^{\frac{1}{p}-1} | \ |_{1,\frac{1}{p}}$.

By this it follows that the properties $T1, T2, T3.1, .., T3.3, N1.$ of section 3.1 are satisfied with $S_- = BV_{1,\frac{1}{p}}$ and we can apply our construction to such maps.

We hence define the following strong set of signed measures on Σ

$$\mathcal{BV}_{1,\frac{1}{p}} := \left\{ \mu \in \mathcal{L}^1; \mathrm{var}_{1,\frac{1}{p}}(\phi_x) < \infty \right\}. \tag{3.27}$$

Consider $|| \ ||_{1,\frac{1}{p}} : \mathcal{BV}_{1,\frac{1}{p}} \longrightarrow \mathbb{R}$ defined by

$$||\mu||_{1,\frac{1}{p}} = |\phi_x|_{1,\frac{1}{p}} + ||\mu||_1. \tag{3.28}$$

Clearly, $\left(\mathcal{BV}_{1,\frac{1}{p}}, || \ ||_{1,\frac{1}{p}}\right)$ is a normed space. If we suppose that the system, $T_L : I \longrightarrow I$, satisfies $T3.4$, then the system then has a unique invariant probability measure with density $\varphi_x \in BV_{1,\frac{1}{p}}$

Now, directly from the above construction and from theorem (3.3.2) it follows the spectral gap for these kind of maps.

Theorem 3.4.1 *If* F_L^* *satisfies the above assumptions it satisfies assumptions* $G1$, $T3.4$, $P'1$ *and* $P'2$, *then the operator* $\mathrm{F}_L^* : \mathcal{BV}_{1,\frac{1}{p}} \longrightarrow \mathcal{BV}_{1,\frac{1}{p}}$ *can be written as*

$$\mathrm{F}_L^* = \mathrm{P} + \mathrm{N}$$

where

a) P *is a projection i.e.* $\mathrm{P}^2 = \mathrm{P}$ *and* $\dim Im(\mathrm{P}) = 1$;

b) the spectral radius of N *satisfies* $\rho(\mathrm{N}) < 1$;

c) $\mathrm{P}\,\mathrm{N} = \mathrm{N}\,\mathrm{P} = 0$.

In other words $\mathrm{F}_L^* : \mathcal{BV}_{1,\frac{1}{p}} \longrightarrow \mathcal{BV}_{1,\frac{1}{p}}$ *has spectral gap.*

3.5 Quantitative Stability

In this section we consider small perturbations of the transfer operator of a given system and try to study the dependence of the physical invariant measure with respect to the perturbation. We give a general result relating the stability of the invariant measure for a uniform family of operators and the convergence to equilibrium.

Let L be the transfer operator for a map acting on two vector subspaces of signed measures on X, $\mathrm{L} : (B_s, ||\cdot||_s) \longrightarrow (B_s, ||\cdot||_s)$ and $\mathrm{L} : (B_w, ||\cdot||_w) \longrightarrow (B_w, ||\cdot||_w)$ endowed with two norms, the strong norm $||\cdot||_s$ on B_s, and the weak norm $||\cdot||_w$ on B_w, such that $||\cdot||_s \geq ||\cdot||_w$. Suppose that

$$B_s \subseteq B_w \subseteq \mathcal{SB}(X),$$

where $\mathcal{SB}(X)$ denotes the space of signed measures on X.

We say that the a transfer operator L has convergence to equilibrium with at least speed Φ with respect to norms $||\cdot||_s, ||\cdot||_w$, if

$$\text{for any } \ f \in \mathcal{V}_s = \{f \in B_s, f(X) = 0\} \ \text{ it holds}$$

$$||\,\mathrm{L}\, f||_w \leq \Phi(n)||f||_s. \tag{3.29}$$

Definition 3.5.1 *A one parameter family of operators* $\{\mathrm{L}_\delta\}_{\delta\in[0,1)}$ *is said to be a* ***uniform family of operators*** *if*

UF1 $||f_\delta||_s \leq M$ *for all* δ, *where* $f_\delta \in B_s$ *is a fixed probability measure of the operator* L_δ *for all* δ;

UF2 L_δ *approximates* L_0 *when* δ *is small in the following sense: there is* $C \in \mathbb{R}^+$ *such that:*

$$||(\mathrm{L}_0 - \mathrm{L}_\delta) f_\delta||_w \leq \delta C; \tag{3.30}$$

UF3 L_0 *has exponential convergence to equilibrium with respect to the norms* $||\cdot||_s$ *and* $||\cdot||_w$*: there exists* $0 < \rho_2 < 1$ *and* $C_2 > 0$ *such that for all* $f \in \mathcal{V}_s$ *it holds*

$$||\,\mathrm{L}_0^n\, f||_w \leq \rho_2^n C_2 ||f||_s;$$

UF4 *The iterates of the operators are uniformly bounded for the weak norm: there exists* $M_2 > 0$ *such that*

$$\forall \delta, n, g \in B_s \text{ it holds } ||\,\mathrm{L}_\delta^n\, g||_w \leq M_2 ||g||_w.$$

We will see that under these assumptions we can ensure that the invariant measure of the system varies continuously (in the weak norm) when L_0 is perturbed to L_δ for small values of δ. Let us state a general result on the stability of fixed points satisfying certain assumptions.

Let us consider two operators L_0 and L_δ preserving a normed space of signed measures $\mathcal{B} \subseteq \mathcal{SB}(X)$ with norm $||\cdot||_{\mathcal{B}}$. Suppose that f_0, $f_\delta \in \mathcal{B}$ are fixed points, respectively of L_0 and L_δ.

Proposition 3.5.1 *Suppose that:*

a) $||\,\mathrm{L}_\delta f_\delta - \mathrm{L}_0 f_\delta||_{\mathcal{B}} < \infty$*;*

b) L_0^i *is continuous on* $\mathcal{B}$*;* $\exists\, C_i$ *s.t.* $\forall g \in \mathcal{B}$, $||\,\mathrm{L}_0^i\, g||_{\mathcal{B}} \leq C_i ||g||_{\mathcal{B}}$.

Then for each N

$$||f_\delta - f_0||_{\mathcal{B}} \leq ||\,\mathrm{L}_0^N (f_\delta - f_0)||_{\mathcal{B}} + ||\,\mathrm{L}_\delta f_\delta - \mathrm{L}_0 f_\delta||_{\mathcal{B}} \sum_{i \in [0, N-1]} C_i. \tag{3.31}$$

Proof. The proof is a direct computation

$$\begin{aligned} ||f_\delta - f_0||_{\mathcal{B}} &\leq ||\,\mathrm{L}_\delta^N f_\delta - \mathrm{L}_0^N f_0||_{\mathcal{B}} \\ &\leq ||\,\mathrm{L}_0^N f_0 - \mathrm{L}_0^N f_\delta||_{\mathcal{B}} + ||\,\mathrm{L}_0^N f_\delta - \mathrm{L}_\delta^N f_\delta||_{\mathcal{B}} \\ &\leq ||\,\mathrm{L}_0^N (f_0 - f_\delta)||_{\mathcal{B}} + ||\,\mathrm{L}_0^N f_\delta - \mathrm{L}_\delta^N f_\delta||_{\mathcal{B}} \end{aligned}$$

(applying item b)). Hence

$$||f_0 - f_\delta||_{\mathcal{B}} \leq ||\,\mathrm{L}_0^N (f_0 - f_\delta)||_{\mathcal{B}} + ||\,\mathrm{L}_0^N f_\delta - \mathrm{L}_\delta^N f_\delta||_{\mathcal{B}}$$

but

$$\mathrm{L}_0^N - \mathrm{L}_\delta^N = \sum_{k=1}^{N} \mathrm{L}_0^{(N-k)} (\mathrm{L}_0 - \mathrm{L}_\delta)\, \mathrm{L}_\delta^{(k-1)}$$

hence

$$\begin{aligned} (\mathrm{L}_0^N - \mathrm{L}_\delta^N) f &= \sum_{k=1}^{N} \mathrm{L}_0^{(N-k)} (\mathrm{L}_0 - \mathrm{L}_\delta)\, \mathrm{L}_\delta^{*(k-1)} f_\delta \\ &= \sum_{k=1}^{N} \mathrm{L}_0^{(N-k)} (\mathrm{L}_0 - \mathrm{L}_\delta) f_\delta \end{aligned}$$

by item c), hence

$$\begin{aligned} ||(\mathrm{L}_0^N - \mathrm{L}_\delta^N)f_\delta||_{\mathcal{B}} &\leq \sum_{k=1}^{N} C_{N-k}||(\mathrm{L}_0 - \mathrm{L}_\delta)f_\delta||_{\mathcal{B}} \\ &\leq ||(\mathrm{L}_0 - \mathrm{L}_\delta)f_\delta||_{\mathcal{B}} \sum_{i\in[0,N-1]} C_i \end{aligned}$$

by item a), and then

$$||f_\delta - f_0||_{\mathcal{B}} \leq ||\,\mathrm{L}_0^N(f_0 - f_\delta)||_{\mathcal{B}} + ||(\mathrm{L}_0 - \mathrm{L}_\delta)f_\delta||_{\mathcal{B}} \sum_{i\in[0,N-1]} C_i.$$

□

Now, let us apply the statement to our family of operators satisfying assumptions UF 1,...,4, supposing $B_w = \mathcal{B}$. By this we have the following

Proposition 3.5.2 *Suppose* $\{\mathrm{L}_\delta\}_{\delta\in[0,1)}$ *is a uniform family of operators where* f_0 *is the unique invariant measure of* L_0 *and* f_δ *is an invariant measure of* L_δ*. Then*

$$||f_\delta - f_0||_w = O(\delta \log \delta).$$

Proof.Let us apply proposition 3.5.1. By UF2,

$$||\,\mathrm{L}_\delta\, f_\delta - \mathrm{L}_0\, f_\delta||_w \leq \delta C$$

(see proposition 3.5.1, item a)). Moreover by UF4, $C_i \leq M_2$.

Hence,

$$||f_\delta - f_0||_w \leq \delta C M_2 N + ||\,\mathrm{L}_0^N(f_0 - f_\delta)||_w.$$

Now by the exponential convergence to equilibrium of L_0 (UF3), there exists $0 < \rho_2 < 1$ and $C_2 > 0$ such that (recalling that by UF1 $||(f_\delta - f_0)||_s \leq 2M$)

$$\begin{aligned} ||\,\mathrm{L}_0^N(f_\delta - f_0)||_w &\leq C_2\rho_2^N||(f_\delta - f_0)||_s \\ &\leq 2C_2\rho_2^N M \end{aligned}$$

hence

$$||f_\delta - f_0||_{\mathcal{B}} \leq \delta C M_2 N + 2C_2\rho_2^N M$$

choosing $N = \left\lfloor \frac{\log\delta}{\log\rho_2} \right\rfloor$

$$\begin{aligned} ||f_\delta - f_0||_{\mathcal{B}} &\leq \delta C M_2 \left\lfloor \frac{\log\delta}{\log\rho_2} \right\rfloor + 2C_2\rho_2^{\left\lfloor \frac{\log\delta}{\log\rho_2} \right\rfloor} M \\ &\leq \delta \log\delta C M_2 \frac{1}{\log\rho_2} + 2C_2\delta M. \end{aligned} \tag{3.32}$$

□

3.5.1 Quantitative stability of Lorenz like maps

Here we apply the general techniques of the previous section to Lorenz-like maps. We will show a set of assumptions on the family of maps, such that the related transfer operators satisfy UF1,...UF4. We remark that:

UF1 easily follows by a uniform Lasota-Yorke inequality;

UF3 depends only on the first element L_0 of the family, and it is proved above for transfer operators associated to Lorenz-like maps;

UF4 depends on the weak norm, and is also proved above.

Some work is necessary for the property UF2. To find a reasonable set of assumptions implying it we need to prove some further regularity of the invariant measure.

For this we introduce a space of measures having bounded variation in some sense, and prove that the invariant measure of a Lorenz-like map is in it. We need some preliminary notations.

We have seen that a positive measure on the square, $[0,1]^2$, can be disintegrated along the stable leaves $\mathcal{F}^s$ in a way that we can see it as a family of positive measures on the interval, $\{\mu|_\gamma\}_{\gamma\in\mathcal{F}^s}$. Since $\mathcal{F}^s$ is identified with $[0,1]$, this defines a path in the space of positive measures, $[0,1] \longmapsto \mathcal{SB}(I)$. It will be convenient to use a functional notation and denote such a path by G_μ. It means that $G_\mu : I \longrightarrow \mathcal{SB}(I)$ is the path defined by $G_\mu(\gamma) = \mu|_\gamma$, where $(\{\mu_\gamma\}_{\gamma\in I}, \phi_x)$ is some disintegration for μ. However, since such a disintegration is defined μ_x-a.e. $\gamma \in [0,1]$, the path G_μ is not unique.

Definition 3.5.2 *Consider a disintegrated measure $(\{\mu_\gamma\}_{\gamma\in I}, \phi_x)$, where $\{\mu_\gamma\}_{\gamma\in I}$ is a family of probabilities on Σ defined μ_x-a.e. $\gamma \in I$ (where $\mu_x = \phi_x m$) and $\phi_x : I \longrightarrow \overline{R}$ is a non-negative marginal density, as before. Denote by G_μ the path (of positive measures on I) $G_\mu : I \longrightarrow \mathcal{SB}(I)$ defined μ_x-a.e. $\gamma \in I$ by*

$$G_\mu(\gamma) = \mu|_\gamma = \pi^*_{\gamma,y}\phi_x(\gamma)\mu_\gamma.$$

Call the set on which G_μ is well defined by I_{G_μ}[4]*. Let $\mathcal{P} = \mathcal{P}(G_\mu)$ be a finite sequence $\mathcal{P} = \{x_i\}_{i=1}^n \subset I_{G_\mu}$ and define the **variation of G_μ with respect to $\mathcal{P}$** as (denote $\gamma_i := \gamma_{x_i}$)*

$$\mathrm{Var}(G_\mu, \mathcal{P}) = \sum_{j=1}^{n} ||G_\mu(\gamma_j) - G_\mu(\gamma_{j-1})||_W,$$

[4]Remark that to a measure many different paths and sets I_{G_μ} may be associated, but they coincide almost everywhere.

where $||G_\mu(\gamma_j) - G_\mu(\gamma_{j-1})||_W = W_1^0(G_\mu(\gamma_j), G_\mu(\gamma_{j-1}))$. *Finally we define the* ***variation of*** G_μ *taking the supremum over the sequences, as*

$$\mathrm{Var}(G_\mu) := \sup_{\mathcal{P}} \mathrm{Var}(G_\mu, \mathcal{P}).$$

Remark 3.5.1 *For an interval* $\eta \subset I$ *we define*

$$\mathrm{Var}_{\overline{\eta}}(G_\mu) := \mathrm{Var}(G_\mu|_{\overline{\eta}}).$$

Remark 3.5.2 *When no confusion can be done, to simplify the notation, we denote* $G_\mu(\gamma)$ *just by* $\mu|_\gamma$.

Definition 3.5.3 *Denote by* $[[\mu]]$ *the set of all paths* $G_\mu : I \longrightarrow \mathcal{SB}(I)$ *which represents* μ. [5]

The Rokhlin Disintegration Theorem ensures that $[[\mu]] \neq \emptyset$. *Define the* ***variation of a positive measure*** μ *by*

$$\mathrm{Var}(\mu) = \inf_{G_\mu \in [[\mu]]} \{\mathrm{Var}(G_\mu)\}$$

we recall that

$$||\mu||_1 = \int W_0^1(0, G_\mu(\gamma)) dm(\gamma), \quad for\ any \ \ G_\mu \in [\mu].$$

Definition 3.5.4 *From the definition 3.5.2 we define the set of bounded variation positive measures* $\mathcal{BV}^+$ *as*

$$\mathcal{BV}^+ = \{\mu \in \mathcal{AB} : \mu \geq 0, \mathrm{Var}(\mu) < \infty\}. \tag{3.33}$$

Now we are ready to state a lemma estimating the regularity of the iterates $\mathrm{F}^{*n}(m)$. We will explicit the assumptions we need on F. The following definition characterizes a class of piecewise expanding maps of the interval with bounded variation derivative $T_L : I \longrightarrow I$ which is a subclass of the ones considered in section 3.4.

[5] We say that a pair $(\{\mu_\gamma\}_\gamma, \phi_x)$, or its path G_μ, represents the positive measure μ if, for every mensurable set $E \subset \Sigma$, $\gamma \longmapsto \mu_\gamma(E \cap \gamma)\phi_x(\gamma)$ is mensurable and it holds

$$\mu(E) = \int \mu_\gamma(E \cap \gamma)\phi_x(\gamma) dm(\gamma).$$

Definition 3.5.5 (Piecewise expanding functions with BV inverse of the derivative) *Suppose there exists a partition* $\mathcal{P} = \{\eta_i := (a_{i-1}, a_i), i = 1, \cdots, q\}$ *of* I *s.t.* $T_L : I \longrightarrow I$ *satisfies the following conditions. For all* i

1) $T_{L_i} := T_L|_{\eta_i}$ *is of class* C^1 *and* $g_{\eta_i} = \dfrac{1}{|T_{L_i}'|_{\eta_i}|}$ *has* bounded variation for all $i = 1, \cdots, q$;

2) $\inf |T_{L_i}^{k}{}'| \geq \dfrac{1}{\lambda_1} > 1$ *for some* $k \in \mathbb{N}$ *and* $0 < \lambda_1 < 1$*;*

To ensure T3.4 is satisfied, we assume that the system $T_{L_i}|_{I_*}$ *is topological mixing.*

3) (Topological Mixing) There is an interval $I_* \subset I$ *such that* $T_{L_i}(I_*) = I_*$*, every orbit* $T_{L_i}^n$ *eventually enters* I_**, and* $T_{L_i}|I_*$ *is topologically mixing: for each interval* $J \subset I_*$ *there exists* $n \geq 1$ *such that* $T_{L_i}^n(J) = I_*$*.*

In particular $T_{L_i}|_{\eta_i}$ and g_{η_i} admit a continuous extension to $\overline{\eta_i} = [a_{i-1}, a_i]$ for all $i = 1, \cdots, q$.

Remark 3.5.3 *The definition 3.5.5 allows infinite derivative for* T_L *at the extreme points of its regularity intervals. For instance, see [1] section 2.4.*

Henceforth we consider a particular class of the Lorenz-like systems.

Definition 3.5.6 *A map* $F_L : [0,1]^2 \longrightarrow [0,1]^2$*,* $F_L(x,y) = (T_L(x), G_L(x,y))$*, is said to be a* ***BV Lorenz-like map*** *if it satisfies*

1. *There are* $H \geq 0$ *and a partition* $\mathcal{P}' = \{J_i := (b_{i-1}, b_i), i = 1, \cdots, d\}$ *of* I *such that for all* $x_1, x_2 \in J_i$ *and for all* $y \in I$ *: the following inequality holds*

$$|G_L(x_1, y) - G_L(x_2, y)| \leq H \cdot |x_1 - x_2|;$$

2. F_L *satisfy property* $G1$ *(hence is uniformly contracting) on each leaf* γ *with rate of contraction* α*;*

3. $T_L : I \to I$ *satisfies the definition 3.5.5.*

Remark 3.5.4 *Without loss of generality we can suppose that the regularity intervals of* T_L *and* G_L *are equal,* $\mathcal{P}' = \mathcal{P}$ *(see definition 3.5.5).*

When F_L is a BV Lorenz-like map, it satisfies G1, T3.4, P'1 and P'2. Thus we apply the results obtained in section 3.4 (for p=1) by setting $BV_{1,1}$ with norm $|\psi|_{1,1} = \mathrm{var}_{1,1}(\psi) + |\psi|_1$, $\mathcal{BV}_{1,1}$ with norm $||\mu||_{1,1}$ and finally $\mathcal{BV}_{1,1}^{\infty}$ with norm $||\mu||_{1,1}^{\infty}$. Under these settings we have all results of section 3.4. This gives sense for the following definition.

Definition 3.5.7 *A family of maps $\{F_\delta\}_\delta$ is said to be a* ***Lorenz-like family*** *if F_δ is a BV Lorenz-like map for all δ and there exist $0 < \lambda < 1$ and $D > 0$ s.t. for all $\mu \in S^1$ and for all δ it holds*

$$|| \mathrm{F}_\delta^{*n} \mu||_{1,1} \leq D\lambda^n ||\mu||_{1,1} + D||\mu||_1 \quad \text{for all} \quad n \geq 1. \tag{3.34}$$

The proof of the next result which is our main tool to estimate the regularity of the invariant measure is too long. So it will be postponed to the appendix.

Proposition 3.5.3 *Let $F_L(x, y) = (T_L(x), G_L(x, y))$ be a BV Lorenz-like map and consider $\mu \in \mathcal{BV}^+$. Then, there are C_0 and $0 < \lambda_0 < 1$ such that for all $n \geq 1$ it holds*

$$\mathrm{Var}(\mathrm{F_L^{*n}}\, \mu) \leq C_0 \lambda_0^n \,\mathrm{Var}(\mu) + C_0||\mu||_1. \tag{3.35}$$

A precise estimate for C_0 will be found in theorem 4.0.1 and corollary 5.2.1. Remember that, by proposition 5.3.1, we have $\mu_0 \in S^\infty$.

Proposition 3.5.4 *Let $F_L(x, y) = (T_L(x), G_L(x, y))$ be BV Lorenz-like map and suppose that F_L has an unique invariant probability measure $\mu_0 \in S^\infty$. Then $\mu_0 \in \mathcal{BV}^+$ and*

$$\mathrm{Var}(\mu_0) \leq C_0.$$

Proof.

First of all, it is not hard to prove that if $G_n : \widehat{I} \longrightarrow \mathcal{SB}([0, 1])$ is a sequence of paths which converges to $G_{\mu_0} : \widehat{I} \longrightarrow \mathcal{SB}([0, 1])$ pointwise on a full measure set $\widehat{I} \subset I$, then for every fixed partition $\mathcal{P} = \{x_0, \cdots, x_n\} \subset \widehat{I}$ it holds

$$\lim_{n \longrightarrow \infty} \mathrm{Var}(G_n, \mathcal{P}) = \mathrm{Var}(G_{\mu_0}, \mathcal{P}).$$

Consider the Lebesgue measure m and the iterates $\mathrm{F_L^{*n}}(m)$. By theorem **??**, these iterates converge to μ_0 in $\mathcal{L}^\infty$. It means that the sequence $\{G_{\mathrm{F_L^{*n}}(m)}\}_n$ converges m-a.e. to

$G_{\mu_0} \in [[\mu_0]]$, where G_{μ_0} is a path given by the Rokhlin Disintegration Theorem. Moreover, by proposition 3.5.3, $\mathrm{Var}(G_{\mathrm{F}_{\mathrm{L}}^{*n}}(m)) \leq C_0$ for all n. Thus there is a full measure set $\widehat{I} \subset I$ such that $\{\widehat{G}_{\mathrm{F}_{\mathrm{L}}^{*n}(m)}\}_n$ converges to $\widehat{G}_{\mu_0}$ pointwise, where $\widehat{G}_{\mathrm{F}_{\mathrm{L}}^{*n}(m)} = G_{\mathrm{F}_{\mathrm{L}}^{*n}(m)}|_{\widehat{I}}$ and $\widehat{G}_{\mu_0} = G_{\mu_0}|_{\widehat{I}}$. Then it holds $\mathrm{Var}(\widehat{G}_{\mathrm{F}_{\mathrm{L}}^{*n}}(m)) \leq \mathrm{Var}(G_{\mathrm{F}_{\mathrm{L}}^{*n}(m)}) \leq C_0$ for all n. Since $\widehat{G}_{\mu_0}$ still represents μ_0, $\{\widehat{G}_{\mathrm{F}_{\mathrm{L}}^{*n}(m)}\}_n$ converges to $\widehat{G}_{\mu_0}$ everywhere and $\mathrm{Var}(G_{\mathrm{F}_{\mathrm{L}}^{*n}(m)}) \leq C_0$ (because $||m||_1 = 1$ and $\mathrm{Var}(m) = 0$) for all n, it holds $\mathrm{Var}(\widehat{G}_{\mu_0}, \mathcal{P}) \leq C_0$ for every partition $\mathcal{P}$ of $\widehat{I}$. Then, $\mathrm{Var}(\widehat{G}_{\mu_0}) \leq C_0$ which implies $\mathrm{Var}(\mu_0) \leq C_0$.

□

Remark 3.5.5 *We remark that proposition 3.5.4 is an estimation of the regularity of the disintegration of μ_0. Similar estimations are presented in [13] and [**?**].*

In the following proposition we see a family of deterministic perturbations allowed on our maps (implying property UF2).

Proposition 3.5.5 (assumptions to obtain UF2) *Let $F_0 = (T_0, G_0)$ and $F_\delta = (T_\delta, G_\delta)$ be two BV Lorenz-like maps and denote by F_0^* and F_δ^* their transfer operators with f_0, f_δ as their fixed points. Let $\mathrm{P}_{T_0}, \mathrm{P}_{T_\delta}$ be the Perron-Frobenius operators of T_0 and T_δ respectively.*

Suppose that when δ is small enough there is an $\epsilon = O(\delta)$ such that

1. *$|\,\mathrm{P}_{T_0} - \mathrm{P}_{T_\delta}\,|_{BV \to L^1} \leq \epsilon$ (assumption on T)*
2. *The branches $T_{i,\delta}$ are such that when δ is small enough $T_{\delta,i}^{-1} \circ T_{0,i}$ is well defined on a set A_1 with $m(A_1) \geq 1-\epsilon$ and $|T_\delta^{-1} \circ T_0 - Id|_\infty \leq \epsilon$ on A_1.*
3. *there is a set A_2 such that $m(A_2) \geq 1-\epsilon$ such that for all $x \in A_2, y \in I$: $|G_0(x,y) - G_\delta(x,y)| \leq \epsilon$.*

Then there is K such that

$$||(\mathrm{F}_0^* - \mathrm{F}_\delta^*)f_\delta||_1 \leq K\epsilon.$$

Proof. Set $\mu = f_\delta$ and let us estimate the integral

$$\int ||(\mathrm{F}_0^*\,\mu - \mathrm{F}_\delta^*\,\mu)|_\gamma||_W dm(\gamma) = \int_{A_1} ||(\mathrm{F}_0^*\,\mu - \mathrm{F}_\delta^*\,\mu)|_\gamma||_W dm(\gamma) + \int_{A_1^c} ||(\mathrm{F}_0^*\,\mu - \mathrm{F}_\delta^*\,\mu)|_\gamma||_W dm(\gamma).$$

Since

$$(\mathrm{F}_0^*\,\mu-\mathrm{F}_\delta^*\,\mu)|_\gamma=\sum_{i=1}^{q}\frac{\mathrm{F}^*_{0,\mathrm{T}^{-1}_{0,\mathrm{i}}(\gamma)}\,\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{T_0(\eta_i)}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|}-\sum_{i=1}^{q}\frac{\mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\,\mu|_{T_{\delta,i}^{-1}(\gamma)}\chi_{T_\delta(\eta_i)}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|}\ \ \mu_x\text{−}a.e.\ \gamma\in I,$$

there exists $K_1>0$ such that

$$\int_{A_1^c}||(\mathrm{F}_0^*\,\mu-\mathrm{F}_\delta^*\,\mu)|_\gamma||_W dm(\gamma)\le K_1\epsilon.$$

Let us estimate the remaining term

$$\int_{A_1}||(\mathrm{F}_0^*\,\mu-\mathrm{F}_\delta^*\,\mu)|_\gamma||_W dm(\gamma).$$

$$\begin{aligned}
&\int_{A_1}||(\mathrm{F}_0^*\,\mu-\mathrm{F}_\delta^*\,\mu)|_\gamma||_W dm(\gamma)\\
=&\int_{A_1}\left|\left|\sum_{i=1}^{q}\frac{\mathrm{F}^*_{0,\mathrm{T}^{-1}_{0,\mathrm{i}}(\gamma)}\,\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{T_0(\eta_i)}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|}-\sum_{i=1}^{q}\frac{\mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\,\mu|_{T_{\delta,i}^{-1}(\gamma)}\chi_{T_\delta(\eta_i)}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|}\right|\right|_W dm\\
\le&\int_{A_1}\left|\left|\sum_{i=1}^{q}\frac{\mathrm{F}^*_{0,\mathrm{T}^{-1}_{0,\mathrm{i}}(\gamma)}\,\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{T_0(\eta_i)}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|}-\sum_{i=1}^{q}\frac{\mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\,\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{T_\delta(\eta_i)}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|}\right|\right|_W dm\\
+&\int_{A_1}\left|\left|\sum_{i=1}^{q}\frac{\mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\,\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{T_\delta(\eta_i)}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|}-\sum_{i=1}^{q}\frac{\mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\,\mu|_{T_{\delta,i}^{-1}(\gamma)}\chi_{T_\delta(\eta_i)}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|}\right|\right|_W dm\\
=&\int_{A_1}I(\gamma)\ dm(\gamma)+\int_{A_1}II(\gamma)\ dm(\gamma)
\end{aligned}$$

The two summands will be treated separately. Moreover let's denote $\overline{\mu}|_\gamma=\pi^*_{\gamma,y}\mu_\gamma$ (note that $\mu|_\gamma=\phi_x(\gamma)\overline{\mu}|_\gamma$ and $\overline{\mu}|_\gamma$ is a probability measure).

$$\begin{aligned}
I(\gamma)\ =&\ \left|\left|\sum_{i=1}^{q}\frac{\mathrm{F}^*_{0,\mathrm{T}^{-1}_{0,\mathrm{i}}(\gamma)}\,\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{T_0(\eta_i)}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|}-\sum_{i=1}^{q}\frac{\mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\,\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{T_\delta(\eta_i)}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|}\right|\right|_W\\
\le&\ \left|\left|\sum_{i=1}^{q}\frac{\mathrm{F}^*_{0,\mathrm{T}^{-1}_{0,\mathrm{i}}(\gamma)}\,\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{T_0(\eta_i)}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|}-\sum_{i=1}^{q}\frac{\mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\,\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{T_\delta(\eta_i)}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|}\right|\right|_W\\
+&\ \left|\left|\sum_{i=1}^{q}\frac{\mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\,\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{T_0(\eta_i)}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|}-\sum_{i=1}^{q}\frac{\mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\,\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{T_\delta(\eta_i)}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|}\right|\right|_W\\
=&\ I_a(\gamma)+I_b(\gamma)
\end{aligned}$$

Note that, since f_δ is a probability measure, for all $\gamma\in A_1\cap A_2$ $\left(\chi_{T_0(\eta_i)}=\chi_{T_\delta(\eta_i)}\right.$ on $\left.A_1\right)$ it holds

$$\begin{aligned}
I_a(\gamma) &= \left|\left|\sum_{i=1}^{q} \frac{\mathrm{F}^*_{0,\mathrm{T}^{-1}_{0,\mathrm{i}}(\gamma)}\, \mu|_{T^{-1}_{0,i}(\gamma)}\chi_{T_0(\eta_i)}}{|T'_{0,i}(T^{-1}_{0,i}(\gamma))|} - \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\, \mu|_{T^{-1}_{0,i}(\gamma)}\chi_{T_\delta(\eta_i)}}{|T'_{0,i}(T^{-1}_{0,i}(\gamma))|}\right|\right|_W \\
&\leq \sum_{i=1}^{q}\left|\left| \frac{\mathrm{F}^*_{0,\mathrm{T}^{-1}_{0,\mathrm{i}}(\gamma)}\, \mu|_{T^{-1}_{0,i}(\gamma)}\chi_{T_0(\eta_i)}}{|T'_{0,i}(T^{-1}_{0,i}(\gamma))|} - \frac{\mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\, \mu|_{T^{-1}_{0,i}(\gamma)}\chi_{T_\delta(\eta_i)}}{|T'_{0,i}(T^{-1}_{0,i}(\gamma))|}\right|\right|_W \\
&\leq \left|\frac{\phi_{x,0}(T^{-1}_{0,i}(\gamma))}{T'_{0,i}(T^{-1}_{0,i}(\gamma))}\right| \sum_{i=1}^{q}\left|\left|\left(\mathrm{F}^*_{0,\mathrm{T}^{-1}_{0,\mathrm{i}}(\gamma)} - \mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\, \overline{\mu}|_{T^{-1}_{0,i}(\gamma)}\right)\right|\right|_W \\
&\leq q\epsilon \sup\left|\frac{\phi_{x,0}}{T'_{0,i}}\right|
\end{aligned}$$

by Item (3).

Doing the same computation as above we get, for all $\gamma \in A_2^c \cap A_1$ (remember that $m(A_2^c) \leq \epsilon$)

$$I_a(\gamma) \leq q \sup\left|\frac{\phi_{x,0}}{T'_{0,i}}\right|$$

Then (remember that $m(A_2^c) \leq \epsilon$)

$$\int_{A_1} I_a(\gamma)dm(\gamma) = \int_{A_1\cap A_2} I_a(\gamma)dm(\gamma) + \int_{A_1\cap A_2^c} I_a(\gamma)dm(\gamma) \leq 2q\epsilon \sup\left|\frac{\phi_{x,0}}{T'_{0,i}}\right|. \quad (3.36)$$

To estimate $I_b(\gamma)$ we have:

$$\begin{aligned}
I_b(\gamma) &= \left|\left|\sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\, \mu|_{T^{-1}_{0,i}(\gamma)}\chi_{T_0(\eta_i)}}{|T'_{0,i}(T^{-1}_{0,i}(\gamma))|} - \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\, \mu|_{T^{-1}_{0,i}(\gamma)}\chi_{T_\delta(\eta_i)}}{|T'_{\delta,i}(T^{-1}_{\delta,i}(\gamma))|}\right|\right|_W \\
&= \left|\sum_{i=1}^{q} \frac{\chi_{T_0(\eta_i)}(\gamma)}{|T'_{0,i}(T^{-1}_{0,i}(\gamma))|} - \sum_{i=1}^{q} \frac{\chi_{T_\delta(\eta_i)}(\gamma)}{|T'_{\delta,i}(T^{-1}_{\delta,i}(\gamma))|}\right| \left|\left|\mathrm{F}^*_{\delta,T^{-1}_{\delta,i}(\gamma)}\, \mu|_{T^{-1}_{0,i}(\gamma)}\right|\right|_W \\
&= |(\mathrm{P}_{T_0} - \mathrm{P}_{T_\delta})(1)| \sup|\phi_{x,0}|
\end{aligned}$$

by Item (1). Thus

$$\int_{A_1} I_b(\gamma)\ dm(\gamma) \leq \sup|\phi_{x,0}||(\mathrm{P}_{T_0} - \mathrm{P}_{T_\delta})1|_1 \leq \epsilon \sup|\phi_{x,0}|.$$

Let us estimate the integral of the second summand

$$II(\gamma) = \left|\left|\sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\, \mu|_{T^{-1}_{0,i}(\gamma)}\chi_{T_\delta(\eta_i)}}{|T'_{\delta,i}(T^{-1}_{\delta,i}(\gamma))|} - \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\, \mu|_{T^{-1}_{\delta,i}(\gamma)}\chi_{T_\delta(\eta_i)}}{|T'_{\delta,i}(T^{-1}_{\delta,i}(\gamma))|}\right|\right|_W$$

on the set A_1.

In what follows, let's make the change of variable $\gamma = T_{\delta,i}(\beta)$.

$$\begin{aligned}
\int_{A_1} II(\gamma)\, dm(\gamma) &= \int_{A_1} \left|\left| \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\, \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{T_\delta(\eta_i)}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} - \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)}\, \mu|_{T_{\delta,i}^{-1}(\gamma)} \chi_{T_\delta(\eta_i)}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right|\right|_W dm(\gamma) \\
&\leq \sum_{i=1}^{q} \int_{A_1 \cap T_{\delta,i}(\eta_i)} \frac{1}{|T'_{\delta,i}|} \circ T_{\delta,i}^{-1}(\gamma) \left|\left| \mathrm{F}^*_{\delta,\mathrm{T}^{-1}_{\delta,\mathrm{i}}(\gamma)} \left(\mu|_{T_{0,i}^{-1}(\gamma)} - \mu|_{T_{\delta,i}^{-1}(\gamma)} \right) \right|\right|_W dm(\gamma) \\
&\leq \sum_{i=1}^{q} \int_{A_1 \cap T_{\delta,i}(\eta_i)} \frac{1}{|T'_{\delta,i}|} \circ T_{\delta,i}^{-1}(\gamma) \left|\left| \mu|_{T_{0,i}^{-1}(\gamma)} - \mu|_{T_{\delta,i}^{-1}(\gamma)} \right|\right|_W dm(\gamma) \\
&\leq \sum_{i=1}^{q} \int_{T_{\delta,i}^{-1}(A_1) \cap A_1} \left|\left| \mu|_{T_{0,i}^{-1} \circ T_{\delta,i}(\beta)} - \mu|_\beta \right|\right|_W dm(\beta) \\
&\leq 2q\epsilon\, \mathrm{Var}(\mu).
\end{aligned}$$

Summing all, the statement is proved. □

Once this is done we have all the ingredients to apply proposition 3.5.2 and obtain the quantitative estimation.

Corollary 3.5.1 (Quantitative stability for deterministic perturbations) *Let $\{F_\delta\}_\delta$ be a Lorenz-Like family satisfying the assumptions of proposition 3.5.5 and denote by f_δ the fixed point of F_δ, for all δ. Then*

$$||f_\delta - f_0||_w = O(\delta \log \delta).$$

Chapter 4

Appendix 1: Semi Lasota-Yorke Inequality

In this section we give a proof for the proposition 3.5.3. Thus, let's consider a BV Lorenz-Like map (see definition 3.5.6) $F_L : [0,1]^2 \longrightarrow [0,1]^2$, $F_L = (T_L, G_L)$, and its transfer operator $\mathrm{F_L}^*$ restricted to the space $\mathcal{BV}^+$.

For all $n \geq 1$, let $\mathcal{P}^{(n)}$ be the partition of I s.t. $\mathcal{P}^{(n)}(x) = \mathcal{P}^{(n)}(y)$ if and only if $\mathcal{P}^{(1)}(T_L^j(x)) = \mathcal{P}^{(1)}(T_L^j(y))$ for all $j = 1, \cdots, n$. Given $\eta \in \mathcal{P}^{(n)}$ denote $g_\eta^{(n)} = \frac{1}{|T_L^{n\prime}|_\eta|}$. Then there exists $C_1 > 0$ s.t. $\sup\{g_\eta^{(n)}\} \leq C_1 \lambda_1^n$ for all $\eta \in \mathcal{P}^{(n)}$ and all $n \geq 1$. Moreover, there exists $\lambda_2 \in (\lambda_1, 1)$ and $C_2 > 0$ such that (see [21], section 3, equation (3.1))

$$\mathrm{var}(g_\eta^{(n)}) \leq C_2 \lambda_2^n \; for \; all \; \eta \in \mathcal{P}^{(n)} \; and \; n \geq 1. \tag{4.1}$$

Recall that we denote by $F_\gamma : I \longrightarrow I$ the function defined by

$$F_\gamma = \pi_{\gamma,y} \circ F_L|_\gamma \circ \pi_{\gamma,y}^{-1}, \tag{4.2}$$

where $\pi_{\gamma,y}$ is the restriction on γ of the projection $\pi(x,y) = y$.

4.0.1 Semi Lasota-Yorke Inequality

Henceforth, we fix a positive measure $\mu \in \mathcal{AB}$ and a path $G_\mu : I \longrightarrow \mathcal{SB}(I)$ which represents μ (i.e. a pair $(\{\mu_\gamma\}_\gamma, \phi_x)$ s.t. $G_\mu(\gamma) = \mu|_\gamma$).

For all $n \geq 1$ set

$$G_{\mu_{\mathrm{F}}^n}(\gamma) := \left(\pi_y \circ F_L^n|_\gamma \circ \pi_{\gamma,y}^{-1}\right)^* G_\mu(\gamma). \tag{4.3}$$

With the above notation and following the strategy of the proof of lemma 3.2.1 we have that the path $G_{\mathrm{F}_{\mathrm{L}}^{*\mathrm{n}}\mu}$ defined, on a full measure set, by

$$G_{\mathrm{F}_{\mathrm{L}}^{*\mathrm{n}}\mu}(\gamma) = \sum_{\eta \in \mathcal{P}^{(1)}} \left(g_\eta{}^{(n)} \cdot G_{\mu_{\mathrm{F}}^n}\right) \circ T_L|_\eta{}^{-1}(\gamma) \cdot \chi_{T_L(\eta)}(\gamma), \tag{4.4}$$

represents the measure $\mathrm{F}_{\mathrm{L}}^{*\mathrm{n}}\,\mu$.

Note that, by remark 3.2.2 and equation (4.2) it holds

$$||\,\mathrm{F}_\gamma^*{}^n G_\mu(\gamma)||_W \leq ||G_\mu(\gamma)||_W,$$

for all $n \geq 1$ and for m-a.e. $\gamma \in I$.

Lemma 4.0.1 *Let γ_1 and γ_2 be two leaves such that $\gamma_1, \gamma_2 \in J_i$ for some i (see definition 3.5.6). Then for every path $G_\mu \in [[\mu]]$, where $\mu \in \mathcal{AB}$, holds*

$$||\,\mathrm{F}_{\gamma_1}^*\,\mu|_{\gamma_1} - \mathrm{F}_{\gamma_2}^*\,\mu|_{\gamma_2}||_W \leq ||\mu|_{\gamma_1} - \mu|_{\gamma_2}||_W + H|\gamma_1 - \gamma_2|||\mu|_{\gamma_2}||_W, \tag{4.5}$$

where H is from definition 3.5.6.

Proof. Consider g such that $|g|_\infty \leq 1$ and $Lip(g) \leq 1$, and observe that by equation (3.12) it holds

$$\begin{aligned}
\left|\int gd\,\mathrm{F}_{\gamma_1}^*\,\mu|_{\gamma_1} - \int gd\,\mathrm{F}_{\gamma_2}^*\,\mu|_{\gamma_2}\right| &\leq \left|\int gd\,\mathrm{F}_{\gamma_1}^*\,\mu|_{\gamma_1} - \int gd\,\mathrm{F}_{\gamma_1}^*\,\mu|_{\gamma_2}\right| \\
&+ \left|\int g(\mathrm{F}_{\gamma_1}) - g(\mathrm{F}_{\gamma_2})d\mu|_{\gamma_2}\right| \\
&\leq \left|\left|\mathrm{F}_{\gamma_1}^*(\mu|_{\gamma_1} - \mu|_{\gamma_2})\right|\right|_W \\
&+ \int |g(\mathrm{F}_{\gamma_1}) - g(\mathrm{F}_{\gamma_2})|d\mu|_{\gamma_2} \\
&\leq ||\mu|_{\gamma_1} - \mu|_{\gamma_2}||_W + H|\gamma_1 - \gamma_2|\int 1d\mu|_{\gamma_2} \\
&\leq ||\mu|_{\gamma_1} - \mu|_{\gamma_2}||_W + H|\gamma_1 - \gamma_2|||\mu|_{\gamma_2}||_W.
\end{aligned}$$

Taking the supremum over g such that $||g||_\infty \leq 1$ and $L(g) \leq 1$, we finish the proof. □

The proofs of the next two lemmas are given on the next section. So we omit them.

Lemma 4.0.2 *Given paths G_{μ_0}, G_{μ_1} and G_{μ_2} (where $G_{\mu_i}(\gamma) = \mu_i|_\gamma$) representing the positive measures $\mu_0, \mu_1, \mu_2 \in \mathcal{BV}^+$ respectively, a function $\varphi : I \longrightarrow \mathbb{R}$, an homeomorphism $h : \eta \subset I \longrightarrow h(\eta) \subset I$ and a subinterval $\eta \subset I$, then the following properties hold*

P1) *If $\mathcal{P}$ is a partition of I by intervals η, then*

$$\mathrm{Var}(G_{\mu_0}) = \sum_{\eta} \mathrm{Var}_{\overline{\eta}}(G_{\mu_0});$$

P2) $\mathrm{Var}_{\overline{\eta}}(G_{\mu_1} + G_{\mu_2}) \leq \mathrm{Var}_{\overline{\eta}}(G_{\mu_1}) + \mathrm{Var}_{\overline{\eta}}(G_{\mu_2})$

P3) $\mathrm{Var}(\varphi \cdot G_{\mu_0}) \leq \left(\sup_{\overline{\eta}} |\varphi|\right) \cdot (\mathrm{Var}_{\overline{\eta}}(G_{\mu_0})) + \left(\sup_{\gamma \in \overline{\eta}} ||G_{\mu_0}(\gamma)||_W\right) \cdot \mathrm{var}_{\overline{\eta}}(\varphi)$

P4) $\mathrm{Var}_{\overline{\eta}}(G_{\mu_0} \circ h) = \mathrm{Var}_{\overline{h(\eta)}}(G_{\mu_0})$.

Remark 4.0.1 *For every path $G_\mu \in [[\mu]]$, where $\mu \in \mathcal{AB}$, it holds*

$$\sup_{\gamma \in \overline{\eta}} ||G_\mu(\gamma)||_W \quad \leq \quad \mathrm{Var}_{\overline{\eta}}(G_\mu) + \frac{1}{m(\overline{\eta})} \int_{\overline{\eta}} ||G_\mu(\gamma)||_W dm(\gamma).$$

Lemma 4.0.3 *For all path $G_\mu \in [[\mu]]$, where $\mu \in \mathcal{BV}^+$, it holds*

$$\mathrm{Var}(G_{\mathrm{F}_{\mathrm{L}}^{*n}\mu}) \leq \sum_{\eta \in \mathcal{P}^{(n)}} \left[\mathrm{var}_{\overline{\eta}}(g_\eta^{(n)}) + 2 \sup g_\eta^{(n)}\right] \cdot \sup_{\gamma \in \overline{\eta}} ||G_\mu(\gamma)||_W + \sup g_\eta^{(n)} \cdot \mathrm{Var}_{\overline{\eta}}(G_{\mu_{\mathrm{F}}^n}). \tag{4.6}$$

Proof. Using the properties P1, P2, P3, $\sup_{\gamma \in \overline{\eta}} ||G_{\mu_{\mathrm{F}}^n}(\gamma)||_W \leq \sup_{\gamma \in \overline{\eta}} ||\mu|_\gamma||_W$ and

$\sup|g_\eta^{(n)}| = \sup g_\eta^{(n)}$, we have

$$\begin{aligned}
\mathrm{Var}(G_{\mathrm{F}_{\mathrm{L}}^{*n}\mu}) &\leq \sum_{\eta\in\mathcal{P}^{(n)}} \mathrm{Var}_{\overline{T^n|_\eta(\eta)}}\left[\left(g_\eta^{(n)}\cdot G_{\mu_{\mathrm{F}}^n}\right)\circ(T^n|_\eta)^{-1}\cdot\chi_{T^n(\eta)}\right]\\
&\leq \sum_{\eta\in\mathcal{P}^{(n)}} \mathrm{Var}_{\overline{T^n|_\eta(\eta)}}\left[\left(g_\eta^{(n)}\cdot G_{\mu_{\mathrm{F}}^n}\right)\circ(T^n|_\eta)^{-1}\right]\cdot\sup|\chi_{T^n(\eta)}|\\
&+ \sum_{\eta\in\mathcal{P}^{(n)}} \sup_{\overline{T^n|_\eta(\eta)}}|\left(g_\eta^{(n)}\cdot G_{\mu_{\mathrm{F}}^n}\right)\circ(T^n|_\eta)^{-1}|\cdot\mathrm{var}(\chi_{T^n(\eta)})\\
&\leq \sum_{\eta\in\mathcal{P}^{(n)}} \mathrm{var}_{\overline{\eta}}\left(g_\eta^{(n)}\right)\cdot\sup_{\gamma\in\overline{\eta}}||\mu|_\gamma||_W + \mathrm{Var}_{\overline{\eta}}(G_{\mu_{\mathrm{F}}^n})\cdot\sup_{\overline{\eta}}|g_\eta^{(n)}|\\
&+ 2\cdot\sum_{\eta\in\mathcal{P}^{(n)}} \sup_{\overline{\eta}}||\left(g_\eta^{(n)}\cdot G_{\mu_{\mathrm{F}}^n}\right)||_W\\
&\leq \sum_{\eta\in\mathcal{P}^{(n)}} \mathrm{var}_{\overline{\eta}}\left(g_\eta^{(n)}\right)\cdot\sup_{\gamma\in\overline{\eta}}||\mu|_\gamma||_W + \mathrm{Var}_{\overline{\eta}}(G_{\mu_{\mathrm{F}}^n})\cdot\sup_{\overline{\eta}}|g_\eta^{(n)}|\\
&+ 2\cdot\sum_{\eta\in\mathcal{P}^{(n)}} \sup||G_{\mu_{\mathrm{F}}^n}||_W\cdot\sup_{\overline{\eta}}|g_\eta^{(n)}|\\
&\leq \sum_{\eta\in\mathcal{P}^{(n)}} \mathrm{var}_{\overline{\eta}}\left(g_\eta^{(n)}\right)\cdot\sup_{\gamma\in\overline{\eta}}||\mu|_\gamma||_W + \mathrm{Var}_{\overline{\eta}}(G_{\mu_{\mathrm{F}}^n})\cdot\sup_{\overline{\eta}}|g_\eta^{(n)}|\\
&+ 2\cdot\sum_{\eta\in\mathcal{P}^{(n)}} \sup_{\gamma\in\overline{\eta}}||\mu|_\gamma||_W\cdot\sup_{\overline{\eta}}|g_\eta^{(n)}|\\
&\leq \sum_{\eta\in\mathcal{P}^{(n)}} \left[\mathrm{var}_{\overline{\eta}}(g_\eta^{(n)}) + 2\sup_{\overline{\eta}} g_\eta^{(n)}\right]\cdot\sup_{\gamma\in\overline{\eta}}||\mu|_\gamma||_W + \sup_{\overline{\eta}} g_\eta^{(n)}\cdot\mathrm{Var}_{\overline{\eta}}(G_{\mu_{\mathrm{F}}^n})
\end{aligned}$$

□

Lemma 4.0.4 *For all $G_\mu \in [[\mu]]$, where $\mu \in \mathcal{BV}^+$, it holds*

$$\mathrm{Var}(G_{\mu_{\mathrm{F}}^n}) \leq \mathrm{Var}(G_\mu) + nH\int ||G_\mu(\gamma)||_W dm(\gamma).$$

Proof. By lemma 4.0.1 we have

$$\mathrm{Var}(G_{\mu_{\mathrm{F}}^1}) \leq \mathrm{Var}(G_\mu) + H\int ||G_\mu(\gamma)||_W dm(\gamma).$$

Iterating this relation and using equation (3.12) we arrive at the desired inequality. □

Lemma 4.0.5 *For all path $G_\mu \in [[\mu]]$, where $\mu \in \mathcal{BV}^+$, it holds*

$$\mathrm{Var}(G_{\mathrm{F}_{\mathrm{L}}^{*n}\mu}) \leq C_3\lambda_3^n\,\mathrm{Var}(G_\mu) + K_3(n)\int ||G_\mu(\gamma)||_W dm(\gamma) \tag{4.7}$$

where $\lambda_3 := \lambda_2$, $C_3 = 4C_2$ (λ_2 and C_2 comes from equation (4.1)) and $K_3(n) = 3C_2\lambda_2^n\sup\{\frac{1}{m(\overline{\eta})}; \eta\in\mathcal{P}^{(n)}\} + nHC_2\lambda_2^n$.

Proof. Replacing equation (4.1), lemma 4.0.1 and the definition 3.5.5 on the inequality given by the lemma 5.2.8 we get

$$
\begin{aligned}
\mathrm{Var}(G_{\mathrm{F}_{\mathrm{L}}^{*n}\mu}) &\leq \sum_{\eta \in \mathcal{P}^{(n)}} \left[\mathrm{var}_{\overline{\eta}}(g_{\eta}^{(n)}) + 2\sup g_{\eta}^{(n)}\right] \sup_{\gamma \in \overline{\eta}} ||\mu|_{\gamma}||_{W} + \sup g_{\eta}^{(n)} \mathrm{Var}_{\overline{\eta}}(G_{\mu_{\mathrm{F}}^{n}}) \\
&\leq \sum_{\eta \in \mathcal{P}^{(n)}} (C_1\lambda_1^n + 2C_2\lambda_2^n)\left(\mathrm{Var}_{\overline{\eta}}(G_{\mu}) + \frac{1}{m(\overline{\eta})}\int_{\overline{\eta}} ||\mu|_{\gamma}||_{W} dm(\gamma)\right) \\
&+ \sum_{\eta \in \mathcal{P}^{(n)}} C_1\lambda_1^n \left(\mathrm{Var}_{\overline{\eta}}(G_{\mu}) + nH\int_{\overline{\eta}} ||\mu|_{\gamma}||_{W} dm(\gamma)\right) \\
&\leq \sum_{\eta \in \mathcal{P}^{(n)}} (3C_2\lambda_2^n)\left(\mathrm{Var}_{\overline{\eta}}(G_{\mu}) + \frac{1}{m(\overline{\eta})}\int_{\overline{\eta}} ||\mu|_{\gamma}||_{W} dm(\gamma)\right) \\
&+ \sum_{\eta \in \mathcal{P}^{(n)}} C_2\lambda_2^n \mathrm{Var}_{\overline{\eta}}(G_{\mu}) + nHC_2\lambda_2^n \sum_{\eta \in \mathcal{P}^{(n)}} \int_{\overline{\eta}} ||\mu|_{\gamma}||_{W} dm(\gamma) \\
&\leq 4C_2\lambda_2^n \sum_{\eta \in \mathcal{P}^{(n)}} \mathrm{Var}_{\overline{\eta}}(G_{\mu}) \\
&+ 3C_2\lambda_2^n \sum_{\eta \in \mathcal{P}^{(n)}} \frac{1}{m(\overline{\eta})}\int_{\overline{\eta}} ||\mu|_{\gamma}||_{W} dm(\gamma) + nHC_2\lambda_2^n \sum_{\eta \in \mathcal{P}^{(n)}} \int_{\overline{\eta}} ||\mu|_{\gamma}||_{W} dm(\gamma) \\
&\leq 4C_2\lambda_2^n \sum_{\eta \in \mathcal{P}^{(n)}} \mathrm{Var}_{\overline{\eta}}(G_{\mu}) \\
&+ \left(3C_2\lambda_2^n \max\{\frac{1}{m(\overline{\eta})}; \eta \in \mathcal{P}^{(n)}\} + nHC_2\lambda_2^n\right) \sum_{\eta \in \mathcal{P}^{(n)}} \int_{\overline{\eta}} ||\mu|_{\gamma}||_{W} dm(\gamma) \\
&\leq C_3\lambda_3^n \mathrm{Var}_{\overline{\eta}}(G_{\mu}) + K_3(n)\int ||\mu|_{\gamma}||_{W} dm(\gamma)
\end{aligned}
$$

□

In order to be a Lasota-Yorke inequality $K_3(n) = 3C_2\lambda_2^n \sup\{\frac{1}{m(\overline{\eta})}; \eta \in \mathcal{P}^{(n)}\} + nHC_2\lambda_2^n$ can't depend on n. Let us remove this dependence on the next theorem.

Theorem 4.0.1 *There are C_0 and $0 \leq \lambda_0 \leq 1$ such that for all $n \geq 1$ and for all $\mu \in \mathcal{BV}^+$ it holds*

$$\mathrm{Var}(G_{\mathrm{F}_{\mathrm{L}}^{*n}\mu}) \leq C_0\lambda_0^n \mathrm{Var}(G_{\mu}) + C_0\int ||G_{\mu}(\gamma)||_{W} dm(\gamma) \tag{4.8}$$

Proof. First let us fix $N \geq 1$ such that $C_3\lambda_3^N < \frac{1}{2}$ and denote

$$\overline{K} := \max\{K_3(n); 1 \leq n \leq N\}.$$

Given $n \geq 1$, write it as $n = qN + r$, with $q \geq 0$ and $0 \leq r \leq N$. Then using lemma 5.2.9

$$\begin{aligned}
\mathrm{Var}(G_{\mathrm{F_L^{*n}}\mu}) &= \mathrm{Var}(G_{\mathrm{F_L^{*N}}(\mathrm{F_L^{*n\text{-}N}}\mu)}) \\
&\leq C_3\lambda_3^N \,\mathrm{Var}(G_{\mathrm{F_L^{*n\text{-}N}}\mu}) + K_3(N)\int ||G_{\mathrm{F_L^{*n\text{-}N}}\mu}(\gamma)||_W dm(\gamma) \\
&\leq \frac{1}{2}\mathrm{Var}(G_{\mathrm{F_L^{*n\text{-}N}}\mu}) + \overline{K}\int ||G_\mu(\gamma)||_W dm(\gamma).
\end{aligned}$$

Then we got the inequality

$$\mathrm{Var}(G_{\mathrm{F_L^{*n}}\mu}) \leq \frac{1}{2}\mathrm{Var}(G_{\mathrm{F_L^{*n\text{-}N}}\mu}) + \overline{K}\int ||G_\mu(\gamma)||_W dm(\gamma). \tag{4.9}$$

Doing the same computation as above, but now with $\mathrm{Var}(G_{\mathrm{F_L^{*n\text{-}N}}\mu})$ instead of $\mathrm{Var}(G_{\mathrm{F_L^{*n}}\mu})$ we have

$$\begin{aligned}
\mathrm{Var}(G_{\mathrm{F_L^{*n\text{-}N}}\mu}) &= \mathrm{Var}(G_{\mathrm{F_L^{*N}}(\mathrm{F_L^{*n\text{-}2N}}\mu)}) \\
&\leq \frac{1}{2}\mathrm{Var}(G_{\mathrm{F_L^{*n\text{-}2N}}\mu}) + \overline{K}\int ||G_\mu(\gamma)||_W dm(\gamma).
\end{aligned}$$

Using what is written above and joining with the inequality (4.9) we have

$$\begin{aligned}
\mathrm{Var}(G_{\mathrm{F_L^{*n}}\mu}) &\leq \frac{1}{2}\mathrm{Var}(G_{\mathrm{F_L^{*n\text{-}N}}\mu}) + \overline{K}\int ||G_\mu(\gamma)||_W dm(\gamma) \\
&\leq \frac{1}{2}\left(\frac{1}{2}\mathrm{Var}(G_{\mathrm{F_L^{*n\text{-}2N}}\mu}) + \overline{K}\int ||G_\mu(\gamma)||_W dm(\gamma)\right) + \overline{K}\int ||G_\mu(\gamma)||_W dm(\gamma) \\
&\leq \frac{1}{2^2}\mathrm{Var}(G_{\mathrm{F_L^{*n\text{-}2N}}\mu}) + \left(1 + \frac{1}{2}\right)\overline{K}\int ||G_\mu(\gamma)||_W dm(\gamma).
\end{aligned}$$

Thus

$$\mathrm{Var}(G_{\mathrm{F_L^{*n}}\mu}) \leq \frac{1}{2^2}\mathrm{Var}(G_{\mathrm{F_L^{*n\text{-}2N}}\mu}) + \left(1 + \frac{1}{2}\right)\overline{K}\int ||G_\mu(\gamma)||_W dm(\gamma). \tag{4.10}$$

Repeating the same process q times we arrive at

$$\mathrm{Var}(G_{\mathrm{F_L^{*n}}\mu}) \leq \frac{1}{2^q}\mathrm{Var}(G_{\mathrm{F_L^{*n\text{-}qN}}\mu}) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{q-1}}\right)\overline{K}\int ||G_\mu(\gamma)||_W dm(\gamma). \tag{4.11}$$

Since $r = n - qN$ the above inequality become

$$\operatorname{Var}(G_{\mathrm{F}_{\mathrm{L}}^{*n}\mu}) \leq \frac{1}{2^q}\operatorname{Var}(G_{\mathrm{F}_{\mathrm{L}}^{*r}\mu}) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{q-1}}\right)\overline{K}\int ||\mu|_\gamma||_W dm(\gamma). \qquad (4.12)$$

Now apply the lemma 5.2.9, with $n = r$ to get

$$\begin{aligned}\operatorname{Var}(G_{\mathrm{F}_{\mathrm{L}}^{*r}\mu}) &\leq C_3\lambda_3^r \operatorname{Var}(G_\mu) + K_3(r)\int ||G_\mu(\gamma)||_W dm(\gamma)\\ &\leq C_3\lambda_3^r \operatorname{Var}(G_\mu) + \overline{K}\int ||G_\mu(\gamma)||_W dm(\gamma).\end{aligned}$$

Replace the above inequality in inequality (4.12) to obtain

$$\begin{aligned}\operatorname{Var}(G_{\mathrm{F}_{\mathrm{L}}^{*n}\mu}) &\leq \frac{1}{2^q}\left(\lambda_3^r C_3 \operatorname{Var}(G_\mu) + \overline{K}\int ||G_\mu(\gamma)||_W dm(\gamma)\right) + \left(\sum_{i=0}^{q-1}\frac{1}{2^i}\right)\overline{K}\int ||G_\mu(\gamma)||_W dm(\gamma)\\ &\leq \frac{1}{2^q}C_3\lambda_3^r \operatorname{Var}(G_\mu) + \left(\sum_{i=0}^{q}\frac{1}{2^i}\right)\overline{K}\int ||G_\mu(\gamma)||_W dm(\gamma)\\ &\leq \frac{1}{2^q}C_3\lambda_3^r \operatorname{Var}(G_\mu) + \left(\sum_{i=0}^{\infty}\frac{1}{2^i}\right)\overline{K}\int ||G_\mu(\gamma)||_W dm(\gamma)\\ &\leq \frac{1}{2^q}C_3\lambda_3^r \operatorname{Var}(G_\mu) + 2\overline{K}\int ||G_\mu(\gamma)||_W dm(\gamma)\end{aligned}$$

and we arrive at

$$\operatorname{Var}(G_{\mathrm{F}_{\mathrm{L}}^{*n}\mu}) \leq \frac{1}{2^q}\lambda_3^r C_3 \operatorname{Var}(G_\mu) + 2\overline{K}\int ||G_\mu(\gamma)||_W dm(\gamma).$$

In order to finish the proof, choose $C_0 := \max\{2\overline{K}, C_3\}$ e $\lambda_0 := \max\{2^{-\frac{1}{N}}, \lambda_3\}$ to get

$$
\begin{aligned}
\mathrm{Var}(G_{\mathrm{F_L^{*n}}\mu}) &\leq \frac{1}{2^q}\lambda_3^r C_3 \,\mathrm{Var}(G_\mu) + 2\overline{K}\int ||G_\mu(\gamma)||_W dm(\gamma) \\
&\leq \frac{1}{2^q}\lambda_3^r C_0 \,\mathrm{Var}(G_\mu) + C_0\int ||G_\mu(\gamma)||_W dm(\gamma) \\
&\leq \left(2^{-q}\right)^{\frac{N}{N}} \lambda_3^r C_0 \,\mathrm{Var}(G_\mu) + C_0\int ||G_\mu(\gamma)||_W dm(\gamma) \\
&\leq \left(2^{-\frac{1}{N}}\right)^{Nq} \lambda_3^r C_0 \,\mathrm{Var}(G_\mu) + C_0\int ||G_\mu(\gamma)||_W dm(\gamma) \\
&\leq \lambda_0^{Nq}\lambda_0^r C_0 \,\mathrm{Var}(G_\mu) + C_0\int ||G_\mu(\gamma)||_W dm(\gamma) \\
&\leq \lambda_0^{Nq+r} C_0 \,\mathrm{Var}(G_\mu) + C_0\int ||G_\mu(\gamma)||_W dm(\gamma) \\
&\leq \lambda_0^{n} C_0 \,\mathrm{Var}(G_\mu) + C_0\int ||G_\mu(\gamma)||_W dm(\gamma)
\end{aligned}
$$

which is what we were looking for. □

Corollary 4.0.1 *For every $\mu \in \mathcal{BV}^+$ it holds*

$$
\mathrm{Var}(\mathrm{F_L^{*n}}\mu) \leq C_0\lambda_0^n \,\mathrm{Var}(\mu) + C_0\int ||G_\mu(\gamma)||_W dm(\gamma). \tag{4.13}
$$

And the proof of proposition 3.5.3 is complete.

Chapter 5

Appendix 2: An Alternative Approach with a Stronger Norm

In this chapter we are going to prove spectral gap for the transfer operator associated to a certain class of skew products $F = (T, G)$, on the square Σ (described below), with a stronger notion of variation of signed measures. To do it, we add some restrictions on the derivative of G.

5.1 Contracting Fibers Maps

Let Σ be the set defined by $I \times I$, where $I = [0, 1]$ and m is the Lebesgue measure on I. Consider the dynamical system $F : \Sigma \longrightarrow \Sigma$, where $F(x, y) = (T(x), G(x, y))$, $T : I \longrightarrow I$ is a **piecewise expanding C^1 function** (definition (3.5.5)) and $G : \Sigma \longrightarrow I$ having the following properties:

1) $\dfrac{\partial G}{\partial x}(x, .) : I \longrightarrow \mathbb{R}$ is of class C^1 for all $x \in I$;

2) there is a partition $\mathcal{P} = \{0 = x_0, \cdots, x_q\}$ of I, $I_i = [x_{i-1}, x_i]$ $i = 1, \cdots, q$ and $0 < \alpha < 1$ such that

$$\max_{0 \leq i \leq q} \sup_{(x,y) \in I_i \times I} \left| \frac{\partial G}{\partial y}(x, y) \right| \leq \alpha < 1;$$

3)

$$H_2 = \max_{0 \leq i \leq q} \sup_{(x,y) \in I_i \times I} \left| \frac{\partial G}{\partial x}(x, y) \right| + \max_{0 \leq i \leq q} \sup_{(x,y) \in I_i \times I} \left| \frac{\partial^2 G}{\partial y \partial x}(x, y) \right| < \infty.$$

5.2 Lasota-Yorke inequality and convergence to equilibrium

Let's work on the space $\mathcal{BV}_2$ defined in the subsection (2.0.5). Henceforth, we fix a positive measure $\mu \in \mathcal{BV}_2$ and a path $G_\mu : I \longrightarrow \mathcal{B}(I)$ which represents μ (i.e. a pair $(\{\mu_\gamma\}_\gamma, \phi_x)$).

For all $n \geq 1$ set

$$\mu_{\mathrm{F}}^n(\gamma) := {\mathrm{F}^n_\gamma}^* \mu|_\gamma = \left(\pi_y \circ \mathrm{F}^n \circ \pi_{\gamma,y}^{-1}\right)^* \mu|_\gamma. \tag{5.1}$$

With the above notation define

$$(\mathrm{F}^* \mu)|_\gamma = \sum_{\eta \in \mathcal{P}^{(1)}} \left({g_\eta}^{(1)} \cdot {\mu^1}_{\mathrm{F}}\right) \circ T|_\eta{}^{-1}(\gamma) \cdot \chi_{T(\eta)}(\gamma), \tag{5.2}$$

and by induction

$$(\mathrm{F}^{n*} \mu)|_\gamma = \sum_{\eta \in \mathcal{P}^{(n)}} \left(g_\eta^{(n)} \cdot \mu_{\mathrm{F}}^n\right) \circ (T^n|_\eta)^{-1}(\gamma) \cdot \chi_{T^n(\eta)}(\gamma), \tag{5.3}$$

where $g^{(n)}$ was defined in definition (3.5.5).

Remark 5.2.1 *The equations (4.3), (4.4) and (**??**) lead us to define the expression*

$$G_{\mathrm{F}^{n*}\mu}(\gamma) := \sum_{\eta \in \mathcal{P}^{(n)}} g_\eta^{(n)}(T^n|_\eta{}^{-1}(\gamma)) \cdot {\mathrm{F}^n_{T^n|_\eta{}^{-1}(\gamma)}}^* G_\mu(T^n|_\eta{}^{-1}(\gamma)) \cdot \chi_{T^n(\eta)}(\gamma). \tag{5.4}$$

Then, given a path G_μ we denote by $G_{\mathrm{F}^{n*}\mu}$ the path given by the above expression (5.4), defined on a full measure set which contains $T^n(I_{G_\mu})$. The path $G_{\mathrm{F}^{*n}\mu}$ represents the positive measure $\mathrm{F}^{*n}\mu$. This gives us an association $[\mu] \longmapsto [\mathrm{F}^{*n}\mu]$ which allows to estimate the variation of the measure $\mathrm{F}^{*n}\mu$ by estimating the variation of the path $G_{\mathrm{F}^{n*}\mu}$. In the sense that, once we prove an estimation like (see theorem (4.0.1))

$$\mathrm{Var}(G_{\mathrm{F}^{*n}\mu}) \leq C_0 \lambda_0^n \,\mathrm{Var}(G_\mu) + C_0 \int ||\mu|_\gamma||_W dm(\gamma), \tag{5.5}$$

we can take the infimum on both sides to obtain

$$\mathrm{Var}(\mathrm{F}^{*n}\mu) \leq C_0 \lambda_0^n \,\mathrm{Var}(\mu) + C_0 \int ||\mu|_\gamma||_W dm(\gamma). \tag{5.6}$$

Moreover, we set (see the definition of $\mathcal{BV}_2$)

$$\Omega = \min\left\{1, \frac{\alpha - \alpha^2}{\max_{1\leq i\leq q}\sup_{(x,y)\in I_i\times I}\left|\frac{\partial^2 G}{\partial y^2}(x,y)\right|}\right\}. \tag{5.7}$$

The following three results; lemma (5.2.1), proposition (5.2.1) and proposition (5.2.2), have the same proofs as lemma (3.12), proposition (3.13) and proposition (3.2.1) respectively. Hence we'll omit them.

Lemma 5.2.1 *For every leaf γ, the operator $\mathrm{F}_\gamma^* : \mathcal{SB}(I) \longrightarrow \mathcal{SB}(I)$ is a weak contraction under the $|| \ ||_W$ norm, where $F_\gamma := \pi_y \circ F|_\gamma \circ \pi_{\gamma,y}^{-1} : I \longrightarrow I$. In particular $||\mathrm{F}_\gamma^* \mu|_\gamma||_W \leq ||\mu|_\gamma||_W$ for every $\mu \in \mathcal{AB}$.*

Proposition 5.2.1 *If μ is a probability measure, then $||\mu||_W = 1$. Hence*

$$||\mathrm{F}^{*\,n}\mu||_W = ||\mu||_W = 1$$

for all $n \geq 1$.

Proposition 5.2.2 *If $\mu \in \mathcal{L}^1$ then*

$$||\mathrm{F}^*\mu||_1 \leq ||\mu||_1. \tag{5.8}$$

Lemma 5.2.2 *For every leaf γ, the operator $\mathrm{F}_\gamma^* : \mathcal{SB}(I) \longrightarrow \mathcal{SB}(I)$ is a weak contraction with the $|| \ ||_{C^{2'}}$ norm, where $F_\gamma := \pi_y \circ F|_\gamma \circ \pi_{\gamma,y}^{-1} : I \longrightarrow I$. In particular $||\mathrm{F}_\gamma^* \mu|_\gamma||_{C^{2'}} \leq ||\mu|_\gamma||_{C^{2'}}$ for every $\mu \in \mathcal{AB}$.*

Proof. For a given $g \in B_{C^2}$, by 1), 2), 3) given in the beginning of section (5.1) and by definition of Ω (equation (5.7)) is straightforward to see that $g(F_\gamma) \in B_{C^2}$. Hence

$$\begin{aligned}\left|\int g d\,\mathrm{F}_\gamma^*\mu\right| &= \left|\int g(F_\gamma) d\mu\right| \\ &\leq ||\mu||_{C^{2'}}.\end{aligned}$$

Taking the supremum over $g \in B_{C^2}$ the proof is complete. □

Lemma 5.2.3 *Let μ and ν be two signed measures, where $\nu = \sum_{i=1}^{n} \alpha_i \delta_{y_i}$ with $\alpha_i \in \mathbb{R}$ for all $i = 1, \cdots, n$. Let γ_1 and γ_2 be two leaves such that $\gamma_1, \gamma_2 \in I_j$ for some $j = 1, \cdots, q$. Then there exists $H_2 > 0$ such that*

$$|| \mathrm{F}^*_{\gamma_1} \mu - \mathrm{F}^*_{\gamma_2} \nu ||_{C^{2\prime}} \leq ||\mu - \nu||_{C^{2\prime}} + H_2 |\gamma_1 - \gamma_2| ||\nu||_W .$$

Proof.

For a given $g \in B_{C^2}$ we have

$$\begin{aligned}
\left| \int g d \, \mathrm{F}^*_{\gamma_1} \mu - \int g d \, \mathrm{F}^*_{\gamma_2} \nu \right| &\leq \left| \int g d \, \mathrm{F}^*_{\gamma_1} \mu - \int g d \, \mathrm{F}^*_{\gamma_1} \nu \right| \\
&+ \left| \int g d \, \mathrm{F}^*_{\gamma_1} \nu - \int g d \, \mathrm{F}^*_{\gamma_2} \nu \right| \\
&\leq \left|\left| \mathrm{F}^*_{\gamma_1} \mu - \mathrm{F}^*_{\gamma_1} \nu \right|\right|_{C^{2\prime}} \\
&+ \left| \int g d \, \mathrm{F}^*_{\gamma_1} \nu - \int g d \, \mathrm{F}^*_{\gamma_2} \nu \right| \\
&\leq H_1 \, ||\mu - \nu||_{C^{2\prime}} \\
&+ \left| \int g d \, \mathrm{F}^*_{\gamma_1} \nu - \int g d \, \mathrm{F}^*_{\gamma_2} \nu \right| .
\end{aligned}$$

So we've found

$$\left| \int g d \, \mathrm{F}^*_{\gamma_1} \mu - \int g d \, \mathrm{F}^*_{\gamma_2} \nu \right| \leq ||\mu - \nu||_{C^{2\prime}} + \left| \int g d \, \mathrm{F}^*_{\gamma_1} \nu - \int g d \, \mathrm{F}^*_{\gamma_2} \nu \right| . \tag{5.9}$$

Let us estimate the second term of the right hand side of equation (5.9).

$$
\begin{aligned}
\left|\int gd\,\mathrm{F}_{\gamma_1}^*\,\nu - \int gd\,\mathrm{F}_{\gamma_2}^*\,\nu\right| &= \left|\int g\circ G(\gamma_1,z)d\nu(z) - \int g\circ G(\gamma_2,z)d\nu(z)\right| \\
&= \left|\sum_{i=1}^{n}\alpha_i g\circ G(\gamma_1,y_i) - \sum_{i=1}^{n}\alpha_i g\circ G(\gamma_2,y_i)\right| \\
&= \left|\sum_{i=1}^{n}\alpha_i\left(g\circ G(\gamma_1,y_i) - g\circ G(\gamma_2,y_i)\right)\right| \\
&= \left|\sum_{i=1}^{n}\alpha_i\left(\int_{[\gamma_2,\gamma_1]}\frac{\partial}{\partial x}(g\circ G)(x,y_i)dm(x)\right)\right| \\
&= \left|\left(\int_{[\gamma_2,\gamma_1]}\sum_{i=1}^{n}\alpha_i\frac{\partial}{\partial x}(g\circ G)(x,y_i)dm(x)\right)\right| \\
&\leq \left(\int_{[\gamma_2,\gamma_1]}\left|\sum_{i=1}^{n}\alpha_i\frac{\partial}{\partial x}(g\circ G)(x,y_i)\right|dm(x)\right) \\
&= \int_{[\gamma_2,\gamma_1]}\left|\int\frac{\partial}{\partial x}(g\circ G)(x,z)d\nu(z)\right|dm(x).
\end{aligned}
$$

So we get

$$
\left|\int gd\,\mathrm{F}_{\gamma_1}^*\,\nu - \int gd\,\mathrm{F}_{\gamma_2}^*\,\nu\right| \leq \int_{[\gamma_2,\gamma_1]}\left|\int\frac{\partial}{\partial x}(g\circ G)(x,z)d\nu(z)\right|dm(x). \tag{5.10}
$$

Set

$$
H_2 = \max_{0\leq i\leq q}\sup_{(x,y)\in I_i\times I}\left|\frac{\partial G}{\partial x}(x,y)\right| + \max_{0\leq i\leq q}\sup_{(x,y)\in I_i\times I}\left|\frac{\partial^2 G}{\partial y\partial x}(x,y)\right|. \tag{5.11}
$$

Defining in this way we have that the real function,h_x, defined by $h_x := \frac{1}{H_2}\cdot\frac{\partial}{\partial x}(g\circ G)(x,.) : I \longrightarrow \mathbb{R}$ satisfies $||h_x||_\infty \leq 1$ and $L(h_x)\leq 1$ for all $x\in I$.

Indeed, given a $x\in I$ and for all $y\in I$ we have

$$
\begin{aligned}
\left|\frac{1}{H_2}\cdot\frac{\partial}{\partial x}(g\circ G)(x,y)\right| &= \left|\frac{1}{H_2}g'(G(x,y))\cdot\frac{\partial G}{\partial x}(x,y)\right| \\
&\leq \frac{1}{H_2}\left|\frac{\partial G}{\partial_x}(x,y)\right| \\
&\leq 1.
\end{aligned}
$$

Hence

$$
\left|\left|\frac{1}{H_2}\cdot\frac{\partial}{\partial x}(g\circ G)(x,.)\right|\right|_\infty \leq 1.
$$

Moreover

$$\begin{aligned}
\left|\left(\frac{1}{H_2}\frac{\partial}{\partial x}(g\circ G)(x,y)\right)'\right| &= \frac{1}{H_2}\left|g''(G(x,y))\cdot\frac{\partial G}{\partial y}(x,y)\cdot\frac{\partial G}{\partial x}(x,y)+g'(G(x,y))\cdot\frac{\partial^2 G}{\partial y\partial x}(x,y)\right| \\
&\leq \frac{\alpha}{H_2}\left|\frac{\partial G}{\partial x}(x,y)\right|+\left|\frac{\partial^2 G}{\partial y\partial x}(x,y)\right| \\
&\leq 1.
\end{aligned}$$

Hence the inequality (5.10) gives us

$$\begin{aligned}
\left|\int g d\,\mathrm{F}_{\gamma_1}^*\nu-\int g d\,\mathrm{F}_{\gamma_2}^*\nu\right| &\leq \int_{[\gamma_2,\gamma_1]}\left|\int\frac{\partial}{\partial x}(g\circ G)(x,z)d\nu(z)\right|dm(x) \\
&= \int_{[\gamma_2,\gamma_1]}H_2\left|\int\frac{1}{H_2}\frac{\partial}{\partial x}(g\circ G)(x,z)d\nu(z)\right|dm(x) \\
&\leq \int_{[\gamma_2,\gamma_1]}H_2||\nu||_W d\nu(z)dm(x) \\
&= (\gamma_1-\gamma_2)H_2||\nu||_W.
\end{aligned}$$

And we get

$$\left|\int g d\,\mathrm{F}_{\gamma_1}^*\nu-\int g d\,\mathrm{F}_{\gamma_2}^*\nu\right|\leq(\gamma_1-\gamma_2)H_2||\nu||_W. \tag{5.12}$$

Joining the above inequality (5.12) with (5.9) we arrive

$$\left|\int g d\,\mathrm{F}_{\gamma_1}^*\mu-\int g d\,\mathrm{F}_{\gamma_2}^*\nu\right|\leq||\mu-\nu||_{C^{2\prime}}+(\gamma_1-\gamma_2)H_2||\nu||_W. \tag{5.13}$$

We finish the proof taking the supremum over $g\in B_{C^2}$.

□

Lemma 5.2.4 *Let μ be a signed measure on I. Given $s\in\mathbb{N}$, consider a partition of I given by $\mathcal{P}=\{I_1,\cdots,I_s\}$, where $m(I_i)=\dfrac{1}{s}$ for all i. Denote $\epsilon_s=\dfrac{1}{s}$ and consider the signed measure ν_s, on I, defined by $\nu_s=\sum_{i=1}^s\mu(I_i)\delta_{x_i}$ where $x_i\in I_i$ for all i. Then $||\mu-\nu_s||_W\leq\epsilon_s|\mu|(I)$.*

Proof. Given a $g\in b1-Lip(I)$ we have $L(g)=1$. Then, for any $y\in I_i$, holds $|g(y)-g(x_i)|\leq m(I_i)$ for all i. Hence

$$\begin{aligned}
\left|\int g d\mu - \int g d\nu_s\right| &= \left|\sum_{i=1}^{s}\int_{I_i} g d\mu - \sum_{i=1}^{s} g(x_i)\mu(I_i)\right| \\
&= \left|\int_{I_i}\sum_{i=1}^{s} g(y)\chi_{I_i}(y)d\mu(y) - \int\sum_{i=1}^{s} g(x_i)\chi_{I_i}(y)d\mu(y)\right| \\
&= \left|\int_{I_i}\sum_{i=1}^{s}(g(y)-g(x_i))\chi_{I_i}(y)d\mu(y)\right| \\
&\leq \int_{I_i}\sum_{i=1}^{s}|g(y)-g(x_i)|\,\chi_{I_i}(y)d|\mu|(y) \\
&\leq \int_{I_i}\sum_{i=1}^{s}\epsilon_s\chi_{I_i}(y)d|\mu|(y) \\
&\leq \epsilon_s|\mu|(I).
\end{aligned}$$

We finish the proof taking the supremum over $g \in b1 - Lip(I)$.

□

Proposition 5.2.3 *Let μ and ν be two signed measures and let γ_1 and γ_2 be two leaves such that $\gamma_1, \gamma_2 \in I_i$ for some $i = 1, \cdots, q$. Then*

$$||\,\mathrm{F}^*_{\gamma_1}\mu - \mathrm{F}^*_{\gamma_2}\nu||_{C^{2'}} \leq ||\mu - \nu||_{C^{2'}} + H_2|\gamma_1 - \gamma_2|||\nu||_W.$$

PROOF. By lemma (5.2.4), consider a sequence of signed measures $\{\nu_n\}_n$ of the type $\nu_n = \sum_{i=1}^{k(n)}\alpha_i(n)\delta_{y_i(n)}$, where $\{y_i(n)\}_n \subset I$ for all i and $\{\alpha_i(n)\}_n \subset \mathbb{R}$ for all i, such that $\lim_{n\to\infty}\nu_n = \nu$ with the $||\quad||_W$ norm. Since $||\quad||_{C^{2'}} \leq ||\quad||_W$ we also have that $\lim_{n\to\infty}\nu_n = \nu$ with the $||\quad||_{C^{2'}}$ norm. By lemma (5.2.3) and since, for all γ, the operator F^*_γ is continuous with respect to both norms $||\quad||_W$ and $||\quad||_{C^{2'}}$, we finish the proof. □

The proofs of the next two lemmas equals to the proofs of the lemmas (**??**) and (4.0.1). So, we omit them.

Lemma 5.2.5 *Given paths G_μ, G_{μ_1}* and *G_{μ_2} (where $G_\mu(\gamma) = \mu|_\gamma, G_{\mu_1}(\gamma) = \mu_1|_\gamma$* and *$G_{\mu_2}(\gamma) = \mu_2|_\gamma$) representing the positive measures measures $\mu, \mu_1, \mu_2 \in \mathcal{BV}_2$ respectively, a function $\varphi : I \longrightarrow \mathbb{R}$, an homomorphism $h : \eta \subset I \longrightarrow h(\eta) \subset I$ and a subinterval $\eta \subset I$, then the following properties hold*

P1) If $\mathcal{P}$ is a partition of I by intervals η, then

$$\mathrm{Var}(G_\mu) = \sum_\eta \mathrm{Var}_{\overline{\eta}}(G_\mu);$$

P2) $\mathrm{Var}_{\overline{\eta}}(G_{\mu_1} + G_{\mu_2}) \leq \mathrm{Var}_{\overline{\eta}}(G_{\mu_1}) + \mathrm{Var}_{\overline{\eta}}(G_{\mu_2})$

P3) $\mathrm{Var}(\varphi \cdot G_\mu) \leq \left(\sup_{\overline{\eta}} |\varphi|\right) \cdot (\mathrm{Var}_{\overline{\eta}}(G_\mu)) + \left(\sup_{\gamma \in \overline{\eta}} ||\mu|_\gamma||_{C^{2'}}\right) \cdot (\mathrm{Var}_{\overline{\eta}}(\varphi))$

P4) $\mathrm{Var}_{\overline{\eta}}(G_\mu \circ h) = \mathrm{Var}_{\overline{h(\eta)}}(G_\mu)$.

Remark 5.2.2 *As remarked in (4.1) There exists $\lambda_2 \in (\lambda_1, 1)$ e $C_2 > 0$ such that $\mathrm{var}(g_\eta^{(n)}) \leq C_2 \lambda_2^n$ for all $\eta \in \mathcal{P}^{(n)}$ and $n \geq 1$.*

Lemma 5.2.6 *For a given measure μ on Σ we have*

$$\begin{aligned} \sup_{\gamma \in \eta} ||\mu|_\gamma||_{C^{2'}} &\leq \mathrm{var}_\eta(||\mu||_{C^{2'}}) + \frac{1}{m(\eta)} \int_\eta ||\mu|_\gamma||_{C^{2'}} dm(\gamma) \\ &\leq \mathrm{Var}_\eta(\mu) + \frac{1}{m(\eta)} \int_\eta ||\mu|_\gamma||_{C^{2'}} dm(\gamma) \\ &\leq \mathrm{Var}_\eta(\mu) + \frac{1}{m(\eta)} \int_\eta ||\mu|_\gamma||_W dm(\gamma) \end{aligned}$$

Lemma 5.2.7 *For all $G_\mu \in [\mu]$, where $\mu \in \mathcal{BV}_2$, it holds*

$$\mathrm{Var}(\mu_{\mathrm{F}}^n) \leq \mathrm{Var}(G_\mu) + nH_2 \int ||G_\mu(\gamma)||_W dm(\gamma).$$

Proof. By lemma (5.2.3) we have

$$\mathrm{Var}(\mu_F^1) \leq \mathrm{Var}(G_\mu) + H_2 \int ||G_\mu(\gamma)||_W dm(\gamma).$$

Iterating this relation and using lemma (3.12) we arrive at the desired inequality. □

Lemma 5.2.8 *For all path $G_\mu \in [\mu]$, where $\mu \in \mathcal{BV}_2$, it holds*

$$\mathrm{Var}(G_{\mathrm{F}^{*n}\mu}) \leq \sum_{\eta \in \mathcal{P}^{(n)}} \left[\mathrm{var}_{\overline{\eta}}(g_\eta^{(n)}) + 2 \sup g_\eta^{(n)}\right] \cdot \sup_{\gamma \in \overline{\eta}} ||G_\mu(\gamma)||_{C^{2'}} + \sup g_\eta^{(n)} \cdot \mathrm{Var}_{\overline{\eta}}(\mu_F^n). \tag{5.14}$$

Proof. Using the properties P1, P2, P3, $\sup_{\gamma\in\overline{\eta}}||\mu^n{}_{\mathrm{F}}(\gamma)||_{C^{2'}} \leq \sup_{\gamma\in\overline{\eta}}||\mu|_\gamma||_{C^{2'}}$ and $\sup|g_\eta^{(n)}| = \sup g_\eta^{(n)}$, we have

$$
\begin{aligned}
\mathrm{Var}(G_{\mathrm{F}^{n*}\mu}) &\leq \sum_{\eta\in\mathcal{P}^{(n)}} \mathrm{Var}_{\overline{T^n|_\eta(\eta)}}\left[\left(g_\eta^{(n)}\cdot\mu_{\mathrm{F}}^n\right)\circ(T^n|_\eta)^{-1}\cdot\chi_{T^n(\eta)}\right] \\
&\leq \sum_{\eta\in\mathcal{P}^{(n)}} \mathrm{Var}_{\overline{T^n|_\eta(\eta)}}\left[\left(g_\eta^{(n)}\cdot\mu_{\mathrm{F}}^n\right)\circ(T^n|_\eta)^{-1}\right]\cdot\sup|\chi_{T^n(\eta)}| \\
&+ \sum_{\eta\in\mathcal{P}^{(n)}} \sup_{\overline{T^n|_\eta(\eta)}} |\left(g_\eta^{(n)}\cdot\mu_{\mathrm{F}}^n\right)\circ(T^n|_\eta)^{-1}|\cdot\mathrm{var}(\chi_{T^n(\eta)}) \\
&\leq \sum_{\eta\in\mathcal{P}^{(n)}} \mathrm{Var}_{\overline{\eta}}\left(g_\eta^{(n)}\cdot\mu_{\mathrm{F}}^n\right) + 2\cdot\sup_{T^n|_\eta(\eta)}||\left({g_\eta}^{(n)}\cdot\mu_{\mathrm{F}}^n\right)\circ(T^n|_\eta)^{-1}||_{C^{2'}} \\
&\leq \sum_{\eta\in\mathcal{P}^{(n)}} \mathrm{var}_{\overline{\eta}}\left(g_\eta^{(n)}\right)\cdot\sup_{\overline{\eta}}||\mu_F^n||_{C^{2'}} + \mathrm{Var}_{\overline{\eta}}(\mu_{\mathrm{F}}^n)\cdot\sup_{\overline{\eta}}|g_\eta^{(n)}| \\
&+ 2\cdot\sum_{\eta\in\mathcal{P}^{(n)}} \sup_{\overline{\eta}}||\left(g_\eta^{(n)}\cdot\mu_{\mathrm{F}}^n\right)\circ(T^n|_\eta)^{-1}||_{C^{2'}} \\
&\leq \sum_{\eta\in\mathcal{P}^{(n)}} \mathrm{var}_{\overline{\eta}}\left(g_\eta^{(n)}\right)\cdot\sup_{\gamma\in\overline{\eta}}||\mu|_\gamma||_{C^{2'}} + \mathrm{Var}_{\overline{\eta}}(\mu_{\mathrm{F}}^n)\cdot\sup_{\overline{\eta}}|g_\eta^{(n)}| \\
&+ 2\cdot\sum_{\eta\in\mathcal{P}^{(n)}} \sup_{\overline{\eta}}||\left(g_\eta^{(n)}\cdot\mu_{\mathrm{F}}^n\right)||_{C^{2'}} \\
&\leq \sum_{\eta\in\mathcal{P}^{(n)}} \mathrm{var}_{\overline{\eta}}\left(g_\eta^{(n)}\right)\cdot\sup_{\gamma\in\overline{\eta}}||\mu|_\gamma||_{C^{2'}} + \mathrm{Var}_{\overline{\eta}}(\mu_{\mathrm{F}}^n)\cdot\sup_{\overline{\eta}}|g_\eta^{(n)}| \\
&+ 2\cdot\sum_{\eta\in\mathcal{P}^{(n)}} \sup||\left(\mu_{\mathrm{F}}^n\right)||_{C^{2'}}\cdot\sup_{\overline{\eta}}|g_\eta^{(n)}| \\
&\leq \sum_{\eta\in\mathcal{P}^{(n)}} \mathrm{var}_{\overline{\eta}}\left(g_\eta^{(n)}\right)\cdot\sup_{\gamma\in\overline{\eta}}||\mu|_\gamma||_{C^{2'}} + \mathrm{Var}_{\overline{\eta}}(\mu_{\mathrm{F}}^n)\cdot\sup_{\overline{\eta}}|g_\eta^{(n)}| \\
&+ 2\cdot\sum_{\eta\in\mathcal{P}^{(n)}} \sup_{\gamma\in\overline{\eta}}||\mu|_\gamma||_{C^{2'}}\cdot\sup_{\overline{\eta}}|g_\eta^{(n)}| \\
&\leq \sum_{\eta\in\mathcal{P}^{(n)}} \left[\mathrm{var}_{\overline{\eta}}(g_\eta^{(n)}) + 2\sup_{\overline{\eta}} g_\eta^{(n)}\right]\cdot\sup_{\gamma\in\overline{\eta}}||\mu|_\gamma||_{C^{2'}} + \sup_{\overline{\eta}} g_\eta^{(n)}\cdot\mathrm{Var}_{\overline{\eta}}(\mu_{\mathrm{F}}^n)
\end{aligned}
$$

□

Lemma 5.2.9 *For all path $G_\mu \in [\mu]$, where $\mu \in \mathcal{BV}_2$, it holds*

$$\mathrm{Var}(G_{\mathrm{F}^{*n}\mu}) \leq C_3\lambda_3^n\,\mathrm{Var}(G_\mu) + K_3(n)\int ||G_\mu(\gamma)||_W dm(\gamma) \tag{5.15}$$

where $\lambda_3 := \lambda_2$, $C_3 = 4C_2$ (λ_2 and C_2 comes from equation (4.1)) and $K_3(n) = 3C_2\lambda_2^n\sup\{\frac{1}{m(\overline{\eta})};\eta\in\mathcal{P}^{(n)}\} + nH_2C_2\lambda_2^n$.

PROOF. Replacing equation (4.1), lemma (4.0.1) and the definition (3.5.5) on the inequality given by the lemma (5.2.8) we get

$$
\begin{aligned}
\mathrm{Var}(G_{\mathrm{F}^{*n}\mu}) &\leq \sum_{\eta\in\mathcal{P}^{(n)}} \left[\mathrm{var}_{\overline{\eta}}(g_\eta^{(n)}) + 2\sup g_\eta^{(n)}\right] \sup_{\gamma\in\overline{\eta}} ||\mu|_\gamma||_W + \sup g_\eta^{(n)} \mathrm{Var}_{\overline{\eta}}(\mu_{\mathrm{F}}^n) \\
&\leq \sum_{\eta\in\mathcal{P}^{(n)}} (C_1\lambda_1^n + 2C_2\lambda_2^n)\left(\mathrm{Var}_{\overline{\eta}}(G_\mu) + \frac{1}{m(\overline{\eta})}\int_{\overline{\eta}} ||\mu|_\gamma||_W dm(\gamma)\right) \\
&+ \sum_{\eta\in\mathcal{P}^{(n)}} C_1\lambda_1^n \left(\mathrm{Var}_{\overline{\eta}}(G_\mu) + nH_2\int_{\overline{\eta}} ||\mu|_\gamma||_W dm(\gamma)\right) \\
&\leq \sum_{\eta\in\mathcal{P}^{(n)}} (3C_2\lambda_2^n)\left(\mathrm{Var}_{\overline{\eta}}(G_\mu) + \frac{1}{m(\overline{\eta})}\int_{\overline{\eta}} ||\mu|_\gamma||_W dm(\gamma)\right) \\
&+ \sum_{\eta\in\mathcal{P}^{(n)}} C_2\lambda_2^n \mathrm{Var}_{\overline{\eta}}(G_\mu) + nH_2C_2\lambda_2^n \sum_{\eta\in\mathcal{P}^{(n)}} \int_{\overline{\eta}} ||\mu|_\gamma||_W dm(\gamma) \\
&\leq 4C_2\lambda_2^n \sum_{\eta\in\mathcal{P}^{(n)}} \mathrm{Var}_{\overline{\eta}}(G_\mu) \\
&+ 3C_2\lambda_2^n \sum_{\eta\in\mathcal{P}^{(n)}} \frac{1}{m(\overline{\eta})}\int_{\overline{\eta}} ||\mu|_\gamma||_W dm(\gamma) + nH_2C_2\lambda_2^n \sum_{\eta\in\mathcal{P}^{(n)}} \int_{\overline{\eta}} ||\mu|_\gamma||_W dm(\gamma) \\
&\leq 4C_2\lambda_2^n \sum_{\eta\in\mathcal{P}^{(n)}} \mathrm{Var}_{\overline{\eta}}(G_\mu) \\
&+ \left(3C_2\lambda_2^n \max\{\frac{1}{m(\overline{\eta})}; \eta\in\mathcal{P}^{(n)}\} + nH_2C_2\lambda_2^n\right) \sum_{\eta\in\mathcal{P}^{(n)}} \int_{\overline{\eta}} ||\mu|_\gamma||_W dm(\gamma) \\
&\leq C_3\lambda_3^n \mathrm{Var}_{\overline{\eta}}(G_\mu) + K_3(n)\int ||\mu|_\gamma||_W dm(\gamma)
\end{aligned}
$$

□

In order to be a Lasota-Yorke inequality $K_3(n) = 3C_2\lambda_2^n \sup\{\frac{1}{m(\overline{\eta})}; \eta \in \mathcal{P}^{(n)}\} + nH_2C_2\lambda_2^n$ can't depend on n. The next theorem removes this dependence. The argument of its proof is analogous of the proof of theorem (4.0.1), so we skip it.

Theorem 5.2.1 *There are C_0 and $0 \leq \lambda_0 \leq 1$ such that for all $n \geq 1$ and for all $\mu \in \mathcal{BV}_2$ it holds*

$$\mathrm{Var}(G_{\mathrm{F}^{*n}\mu}) \leq C_0\lambda_0^n \mathrm{Var}(G_\mu) + C_0\int ||G_\mu(\gamma)||_W dm(\gamma) \tag{5.16}$$

By the same argument of remark (5.2.1) we get

Corollary 5.2.1 *For every* $\mu \in \mathcal{BV}_2$ *it holds*

$$\operatorname{Var}(\mathrm{F}^{*n}\mu) \leq C_0\lambda_0^n \operatorname{Var}(\mu) + C_0 \int ||G_\mu(\gamma)||_W dm(\gamma). \tag{5.17}$$

Corollary 5.2.2 *For all* $\mu \in \mathcal{BV}_2$ *and* $n \geq 1$

$$||\,\mathrm{F}^{*n}\mu||_{BV} \leq C_0\lambda_0^n||\mu||_{BV} + (C_0+1)||\mu||_1.$$

PROOF. Since F^* is a weak $||\ \ ||_1$-contraction (proposition (**??**)), to get the result, we add $||\,\mathrm{F}^{*\,n}\mu||_1$ on both sides of the inequality (4.8). □

5.3 Spectral Gap

For a given $\mu \in \mathcal{BV}_2$ and its restriction on the leaf γ, $\mu|_\gamma := \pi^*_{\gamma,y}(\phi_x(\gamma)\mu_\gamma)$, define

$$\overline{\mu}|_\gamma := \pi^*_{\gamma,y}\mu_\gamma.$$

Hence $\overline{\mu}|_\gamma$ is a probability on I. Moreover $\mu|_\gamma = \phi_x(\gamma)\overline{\mu}|_\gamma$.

Proposition 5.3.1 *There exists a real number θ, such that for all signed measure μ on I and for all $\gamma \in I$, holds $|| \mathrm{F}^*_\gamma \mu||_W \leq \alpha||\mu||_W + \theta\mu(I)$. In particular, if $\mu(I) = 0$ then $|| \mathrm{F}^*_\gamma \mu||_W \leq \alpha||\mu||_W$.*

Proof. If $g \in b1 - Lip(I)$ then $g \circ F_\gamma$ is α-Lipschitz. Moreover since $||g||_\infty \leq 1$ then $||g \circ F_\gamma - \theta||_\infty \leq \alpha$ for some θ. This implies that

$$\begin{aligned}
\left|\int g d\, \mathrm{F}^*_\gamma \mu\right| &= \left|\int g \circ F_\gamma d\mu\right| \\
&= \left|\int g \circ F_\gamma - \theta d\mu\right| + \left|\int \theta d\mu\right| \\
&= \alpha \left|\int \frac{g \circ F_\gamma - \theta}{\alpha} d\mu\right| + \theta\mu(I) \\
&= \alpha\, ||\mu||_W + \theta\mu(I).
\end{aligned}$$

And taking the supremum over $g \in b1 - Lip(I)$ we have $|| \mathrm{F}^*_\gamma \mu||_W \leq \alpha||\mu||_W + \theta\mu(I)$. In particular, if $\mu(I) = 0$ we get the second part. □

The proof of the following proposition equals to the proof of the proposition (3.19). Hence we omit it.

Proposition 5.3.2 *For all signed measure μ on Σ, holds $|| \mathrm{F}^* \mu||_1 \leq \alpha||\mu||_1 + (\alpha + 1)||\phi_x||_1$.*

Iterating the relation of the above proposition we get.

Corollary 5.3.1 *For all signed measure μ on Σ, we have $|| \mathrm{F}^{*\,n}\mu||_1 \leq \alpha^n||\mu||_1 + \overline{\alpha}||\phi_x||_1$, where $\overline{\alpha} = \frac{1+\alpha}{1-\alpha}$.*

From [2] and [3] we get the following.

Lemma 5.3.1 *Let $\mathrm{P}_T : BV(I) \longrightarrow BV(I)$ be the Peron-Frobenius operator associated with a piecewise expanding C^1 map, $T : I \longrightarrow I$. Then $\mathrm{P}_T : BV(I) \longrightarrow BV(I)$ has spectral gap. Moreover there exists $0 < r < 1$ and $D > 0$ such that for all $\phi \in V = \{\phi \in BV(I), \int \phi dm = 0\}$ and for all $n \geq 0$ we have $|\mathrm{P}_T^n(\phi)|_{BV} \leq Dr^n|\phi|_{BV}$.*

Denote by $\mathcal{V}$, the following set of zero average measures

$$\mathcal{V} = \{\mu \in \mathcal{BV}_2 : \mu(\Sigma) = 0\}.$$

Note that for all $\mu \in \mathcal{V}$ holds $\pi_x^*\mu(I) = 0$. Moreover, since $\pi_x^*\mu = \phi m$ (denoting $\phi = \phi^+ - \phi^-$) we have $\int \phi dm = 0$. Indeed

$$\begin{aligned}
\mu(\Sigma) = 0 &\implies \mu^+(\Sigma) - \mu^-(\Sigma) = 0 \\
&\implies \int \mu_\gamma^+(\Sigma \cap \gamma) d\phi_x^+ m - \int \mu_\gamma^-(\Sigma \cap \gamma) d\phi_x^- m = 0 \\
&\implies \int \phi_x^+ - \phi_x^- dm = 0 \\
&\implies \int \phi dm = 0.
\end{aligned}$$

Proposition 5.3.3 *There exist $0 < \overline{r} < 1$ and $0 < \beta < 1$ such that, for every signed measure $\mu \in \mathcal{L}^1$ with $\mu(\Sigma) = 0$, $||\mu||_1 \leq 1$, $\pi_x^*\mu = \phi_x m$, $\phi_x \in BV(I)$ and $|\phi_x|_{BV} \leq 1$, holds $||\,\mathrm{F}^{*\,n}\mu||_1 \leq (1 + \overline{\alpha}D)\left(\frac{1}{\overline{r}}\right)^{\frac{1}{2}}\beta^n$, for all $n \geq 1$. In particular, for every signed measure $\mu \in \mathcal{V}$ such that $||\mu||_{BV} \leq 1$ holds the same estimation, $||\,\mathrm{F}^{*\,n}\mu||_1 \leq (1 + \overline{\alpha}D)\left(\frac{1}{\overline{r}}\right)^{\frac{1}{2}}\beta^n$, for all $n \geq 1$.*

Proof. Let $\mu \in \mathcal{L}^1$ be a signed measure such that $\mu(\Sigma) = 0$, $||\mu||_1 \leq 1$, $\pi_x^*\mu = \phi_x m$ with $\phi_x \in BV(I)$ and $|\phi_x|_{BV} \leq 1$. Denoting $\phi = \phi_x^+ - \phi_x^-$, we have $\int \phi dm = 0$ and $|\phi|_{BV}, |\phi|_1 \leq 1$. Moreover, from lemma (5.3.1) we have $|\mathrm{P}_T(\phi)|_{BV} \leq Dr^n|\phi|_{BV}$.

Let m and $0 \leq d \leq 1$ be the coefficients of the division of n by 2, i.e. $n = 2m + d$. Thus $m = \frac{n-d}{2}$ (remember $||\,\mathrm{F}^{*\,s}\mu||_1 \leq ||\mu||_1$ for all s). And so, we have (below, let $\overline{r}$ be defined by $\overline{r} = \max\{r, \alpha\}$ and $\beta = \sqrt{\overline{r}}$, $0 < \beta < 1$)

$$\begin{aligned}
||\,\mathrm{F}^{*\,n}\mu||_1 &\leq ||\,\mathrm{F}^{*\,2m+1}\mu||_1 \\
&\leq \alpha^m||\,\mathrm{F}^{*\,m+1}\mu||_1 + \overline{\alpha}||\pi_x^*(\mathrm{F}^{*\,m+1}\mu)||_1 \\
&\leq \alpha^m||\mu||_1 + \overline{\alpha}|\,\mathrm{P}_T^{m+1}(\phi)|_1 \\
&\leq \alpha^m||\mu||_1 + \overline{\alpha}|\,\mathrm{P}_T^{m+1}(\phi)|_{BV} \\
&\leq \alpha^m + \overline{\alpha}r^{m+1}D|\phi|_{BV} \\
&\leq \alpha^m + \overline{\alpha}r^m D \\
&\leq (1+\overline{\alpha}D)\max\{r,\alpha\}^m \\
&\leq (1+\overline{\alpha}D)\overline{r}^m \\
&= (1+\overline{\alpha}D)\overline{r}^{\frac{n-d}{2}} \\
&= (1+\overline{\alpha}D)\left(\frac{1}{\overline{r}}\right)^{\frac{d}{2}}\sqrt{\overline{r}}^n \\
&= (1+\overline{\alpha}D)\left(\frac{1}{\overline{r}}\right)^{\frac{d}{2}}\beta^n \\
&\leq (1+\overline{\alpha}D)\left(\frac{1}{\overline{r}}\right)^{\frac{1}{2}}\beta^n.
\end{aligned}$$

□

Proposition 5.3.4 *There exist* $0 < \xi < 1$ *and* $K > 0$ *such that, for all* $n \geq 1$ *holds*

$$||(\mathrm{F}^*\,|_{\mathcal{V}})^n||_{BV} \leq \xi^n K.$$

Proof.

Let $\mu \in \mathcal{V}$ be a signed measure such that $\mu(\Sigma) = 0$ and $||\mu||_{BV} \leq 1$ (and so $||\mu||_1 \leq 1$).

Let m and $0 \leq d \leq 1$ be the coefficients of the division of n by 2, i.e. $n = 2m + d$. Thus $m = \frac{n-d}{2}$. And so, we have (below, let $\overline{\lambda_0}$ be defined by $\overline{\lambda_0} = \max\{\lambda_0, \beta\}$ and $\xi = \sqrt{\overline{\lambda_0}}$).

It holds that $||\,\mathrm{F}^{*\,n}\mu||_{BV} \leq C\lambda_0^n||\mu||_{BV} + C||\mu||_1$ (where $C = C_0+1$) for all $\mu \in \mathcal{BV}_2$. Moreover $||\ \ ||_1 \leq ||\ \ ||_{BV}$ and $||\,\mathrm{F}^{*\,n}||_{BV} \leq 2C$ for all $n \geq 1$. Besides that, using proposition (5.3.3) and (5.2.2), we have

$$
\begin{aligned}
||(\mathrm{F}^*|_{\mathcal{V}})^n\mu||_{BV} &\leq \lambda_0^m C||(\mathrm{F}^*|_{\mathcal{V}})^{m+1}\mu||_{BV} + C||(\mathrm{F}^*|_{\mathcal{V}})^{m+1}\mu||_1 \\
&\leq \lambda_0^m 2C^2 + C(1+\overline{\lambda}D)\left(\frac{1}{\overline{r}}\right)^{\frac{1}{2}} \beta^{m+1} \\
&\leq \lambda_0^m 2C^2 + C(1+\overline{\lambda}D)\left(\frac{1}{\overline{r}}\right)^{\frac{1}{2}} \beta^{m} \\
&\leq \overline{\lambda_0}^m \left[2C^2 + C(1+\overline{\lambda}D)\left(\frac{1}{\overline{r}}\right)^{\frac{1}{2}}\right] \\
&\leq \overline{\lambda_0}^{\frac{n-d}{2}} \left[2C^2 + C(1+\overline{\lambda}D)\left(\frac{1}{\overline{r}}\right)^{\frac{1}{2}}\right] \\
&\leq (\sqrt{\overline{\lambda_0}})^n \left(\frac{1}{\overline{r}}\right)^{\frac{d}{2}} \left[2C^2 + C(1+\overline{\lambda}D)\left(\frac{1}{\overline{r}}\right)^{\frac{1}{2}}\right] \\
&= \xi^n \left(\frac{1}{\overline{r}}\right)^{\frac{d}{2}} \left[2C^2 + C(1+\overline{\lambda}D)\left(\frac{1}{\overline{r}}\right)^{\frac{1}{2}}\right]
\end{aligned}
$$

However

$$||(\mathrm{F}^*|_{\mathcal{V}})^n||_{BV} \leq \xi^n K.$$

□

Theorem 5.3.1 *Let μ_0 be the invariant measure for 2-dimensional Lorenz transformation $F : \Sigma \longrightarrow \Sigma$. Then $\mu_0 \in \mathcal{BV}_2$.*

PROOF. Let $\mu_x = \phi_x m$ be the marginal measure of the disintegration of μ_0. Since $||\mu||_1 = ||\phi_x||_1 = 1$ we have that $\mu \in \mathcal{L}^1$. Since $(m - \mu_0)(\Sigma) = 0$ and $\pi_x^*(m - \mu_0) = (1 - \phi_x)m$, by the first part of proposition (5.3.3) and the F-invariance of μ_0 we get

$$\mathrm{F}^{*\,n} m \longrightarrow \mu_0 \ \text{ as } \ n \longrightarrow \infty, \ \text{ in the } \ \mathcal{L}^1 - \text{norm}.$$

Besides that, by the theorem (5.2.1) it holds $(\mathrm{F}^{*\,n} m)_n \subset \mathcal{BV}_2$. Moreover by proposition (5.3.4) for every $p \in \mathbb{N}$, $||\,\mathrm{F}^{*\,n}(m - \mathrm{F}^{*\,p} m)||_{BV} \longrightarrow 0$ as $n \longrightarrow \infty$. Then $(\mathrm{F}^{*\,n}(m))_n$ is a $||\ \ ||_{BV}$-Cauchy sequence.

Since $\overline{\mathcal{BV}_2}$ is a Banach space there exist a measure $\widetilde{\mu} \in \overline{\mathcal{BV}_2}$ such that

$$\mathrm{F}^{*\,n} m \longrightarrow \mu_0 \ \text{ as } \ n \longrightarrow \infty, \ \text{ in the } \ \mathcal{BV}_2 - \text{norm}.$$

But it means

$$\mathrm{F}^{*\,n} m \longrightarrow \widetilde{\mu} \ \text{ as } \ n \longrightarrow \infty, \ \text{ in the } \ \mathcal{L}^1 - \text{norm}.$$

By the uniqueness of the limit $\widetilde{\mu} = \mu_0$ and the proof is complete. □

Theorem 5.3.2 (Spectral gap on $\mathcal{BV}_2$) *If $F : \Sigma \longrightarrow \Sigma$, $F = (T, G)$, where T is a piecewise expanding C^1 map and G satisfies **1)**, **2)** and **3)** given at beginning of Section 5.1, then the operator $\mathrm{F}^* : \mathcal{BV}_2 \longrightarrow \mathcal{BV}_2$ can be written as*

$$\mathrm{F}^* = \mathrm{P} + \mathrm{N}$$

where

a) P *is a projection i.e.* $\mathrm{P}^2 = \mathrm{P}$ *and* $\dim Im(\mathrm{P}) = 1$*;*

b) there are $0 < \xi < 1$ *and* $K > 0$ *such that* [1] $||\,\mathrm{N}^n(\mu)||_{\mathcal{BV}_2} \leq \xi^n K$*;*

c) $\mathrm{P}\,\mathrm{N} = \mathrm{N}\,\mathrm{P} = 0$.

Proof.

By theorem (5.3.1) we have that $\mathrm{F}^* : \mathcal{BV}_2 \longrightarrow \mathcal{BV}_2$ has a fixed point μ_0.

Define the projection $\mathrm{P} : \mathcal{BV}_2 \longrightarrow [\mu_0]$ ($[\mu_0]$ is the space spanned by μ_0), by $\mathrm{P}(\mu) = \mu(\Sigma)\mu_0$. Now define the operator

$$\mathrm{S} : \mathcal{BV}_2 \longrightarrow \mathcal{V},$$

by

$$\mathrm{S}(\mu) = \mu - \mathrm{P}(\mu) \quad \text{for every} \quad \mu \in \mathcal{BV}_2.$$

Thus define $\mathrm{N} = \mathrm{F}^* \circ \mathrm{S}$ and observe that $\mathrm{N}^n(\mu) = \mathrm{F}^{*\,n}(\mathrm{S}(\mu))$. Since S is bounded and $\mathrm{S}(\mu)$ has zero average we get, by proposition (5.3.4), $||\,\mathrm{N}^n(\mu)||_{BV} \leq \xi^n K ||S||_{BV} ||\mu||_{BV}$. Note that $\mathrm{F}^* = \mathrm{P} + \mathrm{N}$. We finish the proof observing that, since μ_0 is mixing (see [13]) it holds $\dim Im(\mathrm{P}) = \dim([\mu_0]) = 1$. □

[1] We remark that by this, the spectral radius of $\overline{\mathrm{N}}$ satisfies $\rho(\overline{\mathrm{N}}) < 1$, where $\overline{\mathrm{N}}$ is the extension of N to $\overline{\mathcal{BV}_2}$ (the completion of $\mathcal{BV}_2$). This gives us spectral gap, in the usual sense, for the operator $\overline{\mathrm{F}} : \overline{\mathcal{BV}_2} \longrightarrow \overline{\mathcal{BV}_2}$.

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