SECTIONAL HYPERBOLIC SETS IN HIGHER DIMENSIONS

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Rio de Janeiro, __________ de __________ de *2015*.

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Dedication

A DIOS y

a toda mi familia.

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Abstract

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Consider a compact Riemannian manifold M of dimension $n \geq 3$ (a *compact n-manifold* for short). We denote by ∂M the boundary of M (with $\partial M \neq \emptyset$). Let $\mathcal{X}^1(M)$ be the space of C^1 vector fields in M endowed with the C^1 topology. Fix $X \in \mathcal{X}^1(M)$, inwardly transverse to the boundary ∂M and denote by X_t the flow of X, $t \in I\!R$.

The α-limit set and ω-limit set of $p \in M$ is the set $\alpha_X(p)$ and $\omega_X(p)$ formed by those points where the orbit is born and dies respectively, i.e., formed by those $q \in M$ such that $q = \lim_{n\to\infty} X_{t_n}(p)$ for some sequence $t_n \to -\infty$ and $t_n \to \infty$ respectively.

Given $\Lambda \subset M$ compact, we say that Λ is *invariant* if $X_t(\Lambda) = \Lambda$ for all $t \in I\!R$. We also say that Λ is *transitive* if $\Lambda = \omega_X(p)$ for some $p \in \Lambda$; *singular* if it contains a singularity and *attracting* if it is a set to which all nearby positive orbits converge, i.e., if $\Lambda = \cap_{t>0} X_t(U)$ for some compact neighborhood U of it. This neighborhood is often called *isolating block*. It is well known that the isolating block U can be chosen to be positively invariant, i.e., $X_t(U) \subset U$ for all $t > 0$. An *attractor* is a transitive attracting set. A *sink* is a trivial attractor of X, namely it reduces to a single orbit. A *source* is a trivial attactor of $-X$. A closed orbit is a compact orbit (singularity or periodic orbit). Thus, an attractor is *nontrivial* if it is not a closed orbit. A *repelling* is an

attracting for the time reversed vector field $-X$ and a *repeller* is a transitive repelling set.

The *maximal invariant* set of X is defined by $M(X) = \bigcap_{t\geq 0} X_t(M)$.

A *sectional hyperbolic set* is a partially hyperbolic set whose singularities are hyperbolic and whose central subbundle is *sectionally-expanding*.

Recall that a singularity of a vector field is hyperbolic if the eigenvalues of its linear part have non zero real part.

We say that X is a *sectional Anosov flow* if $M(X)$ is a sectional hyperbolic set. [31]

In this work we prove the following results:

1. Every attractor of every vector field C^1 close to a transitive sectional Anosov flow with singularities on a compact manifold has a singularity. This extends the three-dimensional result obtained in [28].

2. On small perturbations of a sectional hyperbolic set of a vector field on a compact manifold, we obtain an upper bound for the number of attractors and repellers that can arise from these perturbations. Moreover, no repeller can arise if the unperturbed set has singularities, is connected and consists of nonwandering points.

3. Every sectional Anosov flow (or, equivalently, every sectional-hyperbolic attracting set of a flow) on a compact manifold has a periodic orbit. This extends the previous three-dimensional result obtained in [8].

4. The existence of *venice masks* (i.e. nontransitive sectional Anosov flows with dense periodic orbits, [10], [36], [35],[7]) containing two equilibria on certain compact 3-manifolds. Indeed, we present two type of examples in which the homoclinic classes composing their maximal invariant set intersect in a very different way.

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Chapter 1

Introduction

Dynamical systems describe different properties about the evolution of *initial states*, *asymptotic behavior*, *stability*, *relationships* between system's elements and its properties. These concepts are fundamentals to the theory of dynamical systems. In fact, the starting motivation of this theory was the study of time behavior of classical mechanical systems, which were modeled by initial value problems describing systems of ordinary differential equations.

However, most of these system's behavior might be very complex, therefore, finding the link between them becomes a difficult task. But, if it is possible to identify enough information of the system, by choosing only one point, also will be possible to determine all its future positions, or a collection of points known such as a trajectory or orbit, stability and special sets.

There are two standard types of dynamical systems: *discrete dynamical system* and *continuous dynamical system*. The discrete dynamical system works on a manifold locally diffeomorphic to a Banach space, with a map (in general an homeomorphism or diffeomorphism function) and a time parameter that belongs to the set of integers. The continuous dynamical system works on a manifold locally diffeomorphic to a Banach space, with continuous function (called in general *flow*) and a time parameter that belongs to an open interval in the real numbers.

There exists different sets or elements of the system that provide a good information about the dynamic of this one, such as critical points (a fixed point for the discrete case and singular point for the continuous case), and the set of periodic points. In these particular points, the dynamics of the system can be represented by the eigenvalues associated with the critical point. It is well known that when these eigenvalues are not in the unit circle, the dynamics near the critical point is called *hyperbolic* and represents a *hyperbolic dynamical system*.

Hyperbolic systems have been extensively studied and have many properties that provide very important information about the dynamics. These systems are characterized by a continuous tangent bundle invariant decomposition of its tangent space in the critical point. This decomposition is associated to the set of points that converge towards the orbit of the critical point and another to the set of points that diverge from the orbit of the critical point (called the stable and unstable manifold respectively). Also, hyperbolic systems are structurally stable, exhibit the property of shadowing, the property of transitivity is robust and exhibit the properties ascribed to chaotic systems between others. In general, in the theory it is considered that the systems and sets with hyperbolic property must be connected, and therefore the hyperbolic systems are considered without singularities (i.e., a non-trivial hyperbolic set can have singularities, but these ones are isolated).

With the purpose of extending the notion of hyperbolicity, arise definitions and a new theory, such as partial hyperbolicity, singular hyperbolicity for 3-dimensional case and later definitions of singular hyperbolicity and sectional hyperbolicity for the higher dimensional case.

Many important results in hyperbolic dynamical systems are limited by dimension hypothesis, i.e., the environment space is a manifold of dimension three. An important question of interest is whether these results are valid in higher dimension. This is the main motivation, and our goal has been to attack this type of limitation for certain specific problems.

1.1 Thesis contributions

Our approach has different works talking about likeness between hyperbolic case and sectional hyperbolic sets (in dimension three and higher dimensions). Firstly, for this purpose is customary to introduce certain hypotheses on the system under consideration. Consider a compact manifold M of dimension $n \geq 3$ with a Riemannian structure $\|\cdot\|$ (a *compact n-manifold* for short). We denote by ∂M the boundary of M. Let $\mathcal{X}^1(M)$ be the space of C^1 vector fields in M endowed with the C^1 topology. Fix $X \in \mathcal{X}^1(M)$, inwardly transverse to the boundary ∂M, and denote by X_t the flow of X, $t \in I\!R$.

We begin by considering the relationship between robustly transitivity and partial hyperbolicity in the singular case, worked in [37] (three dimensional case). Here, some important properties are identified on a C^1 robustly transitive singular set, such as partial hyperbolicity, volume expanding in the central direction and with all its hyperbolic singularities. In fact, there appeared a very important theory three dimensional used in later works [38], [34], [50].

In [27] was introduced the concept of *sectional hyperbolic set*, that is a definition characterized by the exponential expansion of area elements along the central subbundle and that extends to the properties of dimension three in [38].

The *sectional Anosov flows* were introduced in [31] as a generalization of the *Anosov flows*. These also includes the *saddle-type hyperbolic attracting sets*, the *geometric and multidimensional Lorenz attractors* [1], [12], [20]. Some properties of these flows have been shown in the literature [2], [7].

Later, it was observed that a transitive sectional Anosov flow on a compact 3-manifold is not robustly transitive in general, motivating the study of the perturbations of such flows [40].

Particularly, [28] proved that every attractor of every vector field C^1 close to a transitive sectional Anosov flow with singularities on a compact 3-manifold has a singularity. Moreover, [4] generalized this result from transitive to nonwandering ones. Thus, our first contribution is to extend [28] but now to higher dimensions. More precisely, we prove that every attractor of every vector field C^1 close to a transitive sectional Anosov flow with singularities of a compact manifold has a singularity. Here, part of this first contribution includes a new definition of singular cross-section, which is outstanding for different results. Also, its definition provides useful tools for the next contributions of the thesis.

It is well known that many of the dynamical systems properties come from physics phenomena. In the sixties some definitions appeared that tried to explain these behaviors and properties, such as *attractors* and *repellers*. These concepts are well known and play a fundamental role in the dynamical systems theory. They have received some mathematical interpretations, such as *turbulence* that appears in the classical paper [49] which, simultaneously, provides existence of attractors for particular vector fields. Since a repeller is an attractor for the reverse flow, it is clear that this result provides existence for repellers too. Thereby, stressing the importance of attractors, we highlight the classical construction of the geometric Lorenz models [1], [20]. They provide a wide range of results and research in the theory of dynamical systems, particularly hyperbolic and sectional hyperbolic theories on three-dimensional manifolds. The study of sectional hyperbolic attractors for higher dimensional flows is, however, mostly open.

Thus, the second contribution aims to work on sectional hyperbolic sets of vector fields on compact higher dimensional manifolds, and to research two very important related problems, namely, how many attractors and repellers can arise from small perturbations and, also, the possible appearance of repellers from small perturbations. Motivations come from the previous result [4], [28], providing beside transitivity or nonwandering points, an upper bound in terms of the number of singularities in dimension three, its subsequently generalization to higher dimensions [25], and also the well known examples of sectional hyperbolic sets containing repellers (e.g. *Anomalous Anosov flow* [18] or [14]). We can also mention the recent paper [13] studying the similar problem but for one-dimensional maps with discontinuities.

Indeed, we remove both the transitivity and nonwandering hypotheses in order to obtain robust finiteness of attractors and repellers. Here, we obtain an upper bound for the number of attractors and repellers that can appear from small perturbations of vector field (this improves [25], [28]). Furthermore, we prove a robustly non-existence of repellers on a connected sectional hyperbolic set which both has singularities and consists of nonwandering points.

A well known problem in dynamics is to investigate the existence of periodic orbits for flows on compact manifolds. This problem has a satisfactory solution under certain circumstances. In fact, every Anosov flow of a compact manifold has not only one but infinitely many periodic orbits instead. In this paper we shall investigate this problem not for Anosov but for the sectional Anosov flows introduced in [31]. It is known for instance that every sectional Anosov flow of a compact 3-manifold has a periodic orbit, this was proved in [8]. In the transitive case (i.e. with a dense orbit in the maximal invariant set) it is known that the maximal invariant set consists of a homoclinic class and, therefore, the flow has infinitely many periodic orbits [2]. Another relevant result by Reis [47] proves the existence of infinitely many periodic orbits under certain conditions. As in the first contribution, our goal here is to extend [8] to the higher-dimensional setting and this provides a third contribution. More precisely, we shall prove that every sectional Anosov flow (or, equivalently, every sectional hyperbolic attracting set of a flow) on a compact manifold has a periodic orbit. It should be noted, that this contribution provides a wealth of new definitions for the higher dimensional case, which are necessary for this one.

We talk about the relationship between the hyperbolic and sectional hyperbolic theory. The natural motivation is to observe the properties that are preserved or which are not in the new scenario. Particularly, we mention two important properties related to hyperbolic sets which are not satisfied by all sectional hyperbolic sets. The first is the spectral decomposition theorem [52]. It says that an attracting hyperbolic set $\Lambda = Cl(Per(X))$ is a finite disjoint union of homoclinic classes, where $Per(X)$ is the set of periodic points of X. The second says that an Anosov flow on a closed manifold is transitive if and only if it has dense periodic orbits.

This results are false for sectional Anosov flows, i.e., sets whose maximal invariant is a sectional-hyperbolic set [35]. Specifically, it is proved that there exists a sectional Anosov flow such that it is supported on a compact 3-manifold, it has dense periodic orbits, is the union non disjoint of two homoclinic classes but is not transitive. So, a sectional Anosov flow is said a *Venice mask* if it has dense periodic orbits which is not transitive. The only known examples of venice masks have one or three singularities, and they are characterized by having two properties: are the union non disjoint of two homoclinic classes and the intersection of its homoclinic classes is the closure of the unstable manifold of a singularity.

Then, our fourth contribution provides two examples of venice masks with two singularities. Here, each one is the union of two different homoclinic classes. However, for the first, the intersection of homoclinic classes is the closure of the unstable manifold of two singularities. Whereas for the second, the intersection of homoclinic classes is just a hyperbolic periodic orbit.

1.2 Thesis outline

In Chapter 2 and Chapter 3 we introduce the preliminary theory, such as the basic concepts and proper definitions of our interest.

In Chapter 4, we prove that every attractor of every vector field C^1 close to a transitive sectional Anosov flow with singularities on a compact manifold has a singularity. This extends the three-dimensional result obtained in [28].
The only known examples of venice masks have one or three singularities, and they are characterized by to be the union non disjoint of two homoclinic classes and the intersection of its homoclinic classes is the closure of the unstable manifold of a singularity.

In Chapter 5, we study small perturbations of a sectional hyperbolic set of a vector field on a compact manifold. Indeed, we obtain an upper bound for the number of attractors and repellers that can arise from these perturbations. Moreover, no repeller can arise if the unperturbed set has singularities, is connected and consists of nonwandering points.

In Chapter 6, we prove that every sectional Anosov flow (or, equivalently, every sectional-hyperbolic attracting set of a flow) on a compact manifold has a periodic orbit. This extends the previous three-dimensional result obtained in [8].

In Chapter 7, we show the existence of venice masks (i.e. nontransitive sectional Anosov flows with dense periodic orbits, [10], [36], [35],[7]) containing two equilibria on certain compact 3-manifolds. Indeed, we present two type of examples in which the homoclinic classes composing their maximal invariant set intersect in a very different way.

Chapter 2

Preliminaries

This chapter provides some preliminaries to study of the hyperbolic theory, specifically, hyperbolic theory on sectional hyperbolic sets. Here, definitions are introduced, concepts and basic facts needed for understanding later chapters.

Some topics which appear in a number of research and which we consider as basic for several branches of dynamics, are presented in some detail. We shall define the hyperbolic and sectional-hyperbolic sets. We shall study the relationship between their basic properties and the dynamics of the system.

2.1 Definitions and basic concepts

Hereafter, we shall consider a compact Riemannian manifold M of dimension $n \geq 3$ (a *compact n-manifold* for short). We shall denote by ∂M the boundary of M. Let $\mathcal{X}^1(M)$ be the space of C^1 vector fields in M endowed with the C^1 topology. Let X be a fixed vector field in $\mathcal{X}^1(M)$ and inwardly transverse to the boundary ∂M. We shall denote by X_t the flow generated by X, $t \in I\!R$. Recall that the flow is set as the action $X : \mathbb{R} \times M \to M$, i.e., $X(t, \cdot) \equiv X_t(\cdot)$ and it satisfies $X_0(\cdot) = Id_M$ and $X_t(\cdot) \circ X_s(\cdot) = X_{t+s}(\cdot)$ for all $s, t \in \mathbb{R}$.

It is established that an *orbit* of X through p is $O = O_X(p) = \{X_t(p) : t \in \mathbb{R}\}$, for any point $p \in M$. A point σ is said to be a *singularity* (or *fixed point* or *singular point*) of X if $X(\sigma) = 0$ (or equivalently $O(\sigma) = \sigma$). The positive orbit of p is defined by $O^+(p) = X_t(p) : t > 0$ and its negative orbit by $O^-(p) = X_t(p) : t < 0$.

Given a point $p \in M$, the orbit $O = O_X(p) = O_p$ of X is said to be a *periodic orbit* of X, if for some positive number $T > 0$ one has that $X_T(p) = p$ (equivalently one has that O_p is compact and $O_p \neq \{p\}$). The minimun number $T > 0$ satisfiying this property is called the period of p. An orbit is said to be a *closed orbit* if this one is a singularity or a periodic orbit of X. Hereafter, we shall denote by $Per(X)$ the set of periodic points and by $Sing(X)$ the set of singularities of X.

The *ω-limit set* of $p \in M$ is the set $\omega_X(p)$ formed by those $q \in M$ such that $q = \lim_{n\to\infty} X_{t_n}(p)$ for some sequence $t_n \to \infty$. The *α-limit set* of $p \in M$ is the set $\alpha_X(p)$ formed by those $q \in M$ such that $q = \lim_{n\to\infty} X_{t_n}(p)$ for some sequence $t_n \to -\infty$.

We say that $p \in M$ is a *nonwandering point* of X if for all neighborhood U of p and all $T > 0$ there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. The set of nonwandering points of X is denoted by $\Omega(X)$ (or $\Omega(X_t)$).

We say that p is a *recurrent point* of X if $p \in \omega_X(p)$. Singularities and periodic points are examples of recurrent points but not conversely.

Given a compact subset $\Lambda \subset M$, is said to be:

- *Invariant* if $X_t(\Lambda) = \Lambda$, $\forall t \in \mathbb{R}$;
- *Positively Invariant* if $X_t(\Lambda) \subset \Lambda$, $\forall t \in \mathbb{R}$;
- *Transitive* if $\Lambda = \omega_X(p)$, for some $p \in \Lambda$;
- *Non-trivial* if Λ is not a closed orbit of X;
- *Singular* if Λ has a singularity;

- *Isolated* if there is a compact neighborhood U of Λ such that

$$\Lambda = \bigcap_{t \in I\!R} X_t(U)$$

(U is called *isolating block*) ;

- *Attracting* if it is an isolated set and has a positively invariant isolating block U, i.e.,

$$X_t(U) \subset U, \quad \forall t \geq 0;$$

- *Repelling* if it is an attracting for the time reversed vector field $-X$;

- *Attractor* if it is a transitive attracting set;

- *Repeller* if it is a transitive repelling set.

A *sink* is a trivial attractor of X, namely it reduces to a single orbit. A *source* is a trivial attractor of $-X$.

The *maximal invariant* set of X is defined by

$$M(X) = \bigcap_{t \geq 0} X_t(M).$$

We denote by $m(L)$ the minimum norm of a linear operator L, i.e.,

$$m(L) = inf_{v \neq 0} \frac{\|Lv\|}{\|v\|}.$$

Remark 2.1.1.

– *Note that the α-limit set and ω-limit set of $p \in M$ is the set formed by those points where the orbit is born and dies, respectively.*

– *Note that $Per(X) \cup Sing(X) \subset \Omega(X)$.*

– *Note that* $\omega_X(p) \cup \alpha_X(p) \subset \Omega(X)$.

– *Note that the set formed by a unique singularity is a transitive set and in this case such transitive set is trivial.*

– *An attracting is a set to which all nearby positive orbits converge.*

– *The attracting sets are isolated, but not conversely. In fact, a saddle type singularity is isolated but is not attracting.*

– *Note that by definition the maximal invariant is an attracting set.*

2.2 Sectional hyperbolic theory

Definition 2.2.1. *A compact invariant set* Λ *of* X *is* hyperbolic *if there are a continuous invariant splitting of the tangent bundle* $T_\Lambda M = E^s \oplus E^X \oplus E^u$ *and positive constants* C, λ *such that*

1– E^X *is the subespace generated by* $X(x)$ *in* T_xM, *for every* $x \in \Lambda$.

2– E^s *is* contracting, *i.e.,*

$$|| DX_t(x)v_x^s || \leq Ce^{-\lambda t} || v_x^s ||,$$

for all $x \in \Lambda$, $v_x^s \in E_x^s$ *and* $t \geq 0$.

3– E^u *is* expanding, *i.e.,*

$$|| DX_t(x)v_x^u || \leq C^{-1}e^{\lambda t} || v_x^u ||,$$

for all $x \in \Lambda$, $v_x^u \in E_x^u$ *and* $t \geq 0$.

Recall that a singularity of a vector field is hyperbolic if the eigenvalues of its linear part have non zero real part.

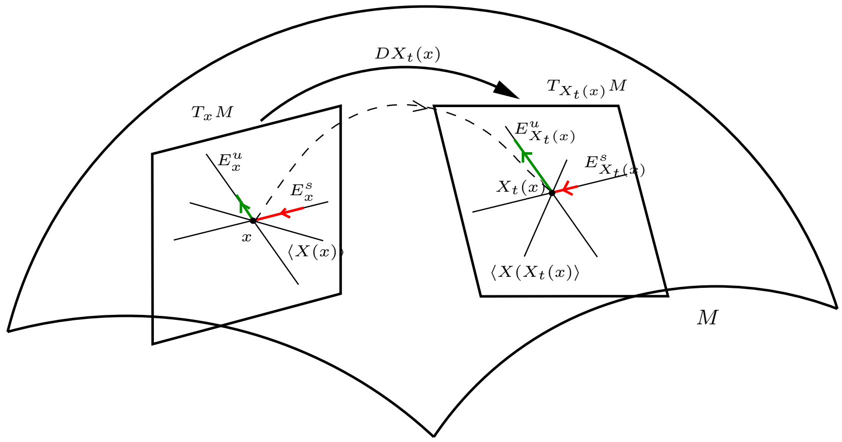


Figure 2.1: Hyperbolicity property.

Definition 2.2.2. *A compact invariant set* Λ *of* X *has a* dominated splitting with respect to the tangent flow *if there is a continuous invariant splitting* $T_\Lambda M = E \oplus F$ *such that the following property holds for some positive constants* C, λ*:*

$$|| \, DX_t(x)e_x \, || \cdot || \, f_x \, || \leq Ce^{-\lambda t} \, || \, DX_t(x)f_x \, || \cdot || \, e_x \, ||,$$

for all $x \in \Lambda$*,* $(e_x, f_x) \in E_x \times F_x$ *and* $t \geq 0$*.*

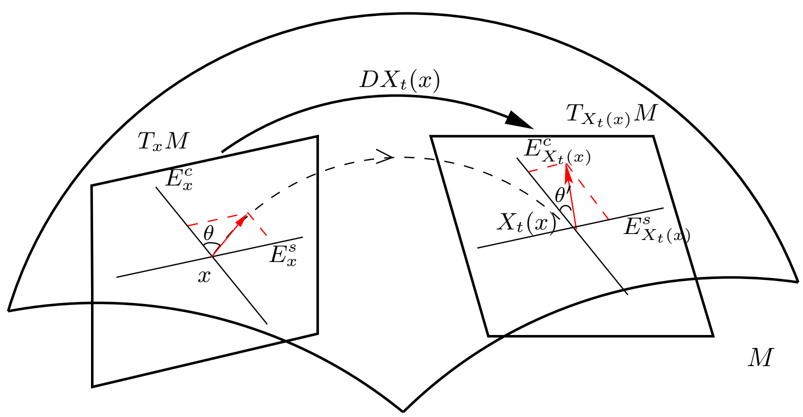


Figure 2.2: Domination property.

In this case, we say that E *dominates* F.

Definition 2.2.3. *A compact invariant set* Λ *of* X *is* partially hyperbolic *if there is a continuous invariant splitting* $T_\Lambda M = E^s \oplus E^c$ *such that the following properties hold for some positive constants* C, λ*:*

1− E^s *is* contracting, *i.e.,*

$$\| DX_t(x)v_x^s \| \leq Ce^{-\lambda t} \| v_x^s \|,$$

for all $x \in \Lambda$, $v_x^s \in E_x^s$ *and* $t \geq 0$.

2− E^s dominates E^c.

Given the Riemannian metric $\langle *, * \rangle$ on M, this one induces a 2−Riemannian metric [41],

$$\langle u, v/w \rangle_x = \langle u, v \rangle_x \cdot \langle w, w \rangle_x - \langle u, w \rangle_x \cdot \langle v, w \rangle_x,$$

for every point $x \in M$ and $u, v, w \in T_xM$.
This in turns induces a 2-norm $\|*, *\|$ [19] (or areal metric [23]) such that for every point $x \in M$ and $u, v, \in T_xM$ it is defined by

$$\|*, *\|_x : T_xM \times T_xM \to \mathbb{R}^+ \cup \{0\}$$

$$(u, v) \mapsto \|u, v\|_x = \sqrt{\langle u, u/v \rangle_x} = \sqrt{\langle u, u \rangle_x \cdot \langle v, v \rangle_x - \langle u, v \rangle^2}$$

which geometrically, represents the area of the parallelogram spanned by the vectors u and v on T_xM.

Definition 2.2.4. *Let* Λ *be a compact invariant set of* X *with dominated splitting with respect to the tangent flow* $T_\Lambda M = E \oplus F$*. We say that its central subbundle* F *is* sectional expanding (resp. sectionally contracting) *if the following property holds for some positive constants* C, λ*:*

$$\|DX_t(x)u, DX_t(x)v\|_{X_t(x)} \geq C^{-1}e^{\lambda t} \|u, v\|_x, \ \forall x \in \Lambda, \forall u, v \in F_x \quad and \quad t \geq 0.$$

$$(resp. \quad \|DX_t(x)u, DX_t(x)v\|_{X_t(x)} \geq Ke^{\lambda - t} \|u, v\|_x, \ \forall x \in \Lambda, \forall u, v \in F_x \quad and \quad t \geq 0.)$$

The standard definition of a sectional hyperbolic set requires partial hyperbolicity, hyperbolic singularities and a central subbundle sectionally expanding [27].

The following definition modifies slightly the standard definition of a sectional hyperbolic set by using the 2-norm above with essentially the same properties [9]. Nevertheless, it provides a dominated splitting with respect to the tangent flow on trivial sets, i.e., it adds the cases where the dominated splitting have a trivial subbundle. ($E_x = 0$ and $F_x = T_xM$ or $E_x = T_xM$ and $F_x = 0$ for $x \in \Lambda$).

Definition 2.2.5. *Let Λ be a compact invariant set of X whose singularities (if any) are hyperbolic. We say that Λ is a* sectional hyperbolic set *if there are a continuous tangent bundle invariant decomposition $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$, positive constants K, λ and Riemannian metric $\langle *, * \rangle$ such that for each x in Λ and every $t \geq 0$:*

1− $\| DX_t(x)v^s_x \| \leq Ce^{-\lambda t} \| v^s_x \|$, *for all $x \in \Lambda$, $v^s_x \in E^s_x$ and $t \geq 0$;*

2− $\| DX_t(x)v^s_x \| \cdot \| v^u_x \| \leq Ce^{-\lambda t} \| DX_t(x)v^u_x \| \cdot \| v^s_x \|$, *for all $x \in \Lambda$, $(v^s_x, v^u_x) \in E^s_x \times E^u_x$ and $t \geq 0$;*

3− $\|DX_t(x)u, DX_t(x)v\|_{X_t(x)} \geq K^{-1}e^{\lambda t} \|u, v\|_x$, *for all $u, v \in E^c_x$.*

Definition 2.2.6. *We say that X is a* sectional Anosov flow *if $M(X)$ is a sectional hyperbolic set [31].*

Definition 2.2.7. *An isolated transitive set Λ of $X \in \mathcal{X}^1(M)$ is a C^1* robust transitive set*, if there is an isolating block U of Λ so that*

$$\Lambda_Y = \Lambda(Y) = \bigcap_{t \in \mathbb{R}} Y_t(U)$$

is a transitive set for all Y C^1 nearby X.

If O is a closed orbit of X, then we denote by $O(Y)$ the continuation of O for Y C^r-close to X [44].

In [37] was proved for Λ a C^1 robust singular transitive set of $X \in \mathcal{X}^1(M)$ that, either, for X or $-X$, Λ is a sectional hyperbolic set which is an attractor. Also, any of its singularities is Lorenz-like (see Section 3).

2.3 Topics of hyperbolicity

2.3.1 Hyperbolic properties

By using the standard definition of hyperbolic set (Definition 2.2.1), it follows from the stable manifold theory [21] that if p belongs to a hyperbolic set Λ, there are submanifolds associated to each point p that provide a very important information about the dynamic of points close to p. Firstly, the Stable Manifold Theorem [21] provides C^r-immersed submanifolds of M for $p \in \Lambda$, specifically the following sets

$$W_X^{ss}(p) = \{x : d(X_t(x), X_t(p)) \to 0, t \to \infty\},$$

$$W_X^{uu}(p) = \{x : d(X_t(x), X_t(p)) \to 0, t \to -\infty\}$$

are C^1 immersed submanifolds of M which are tangent at p to the subspaces E_p^s and E_p^u of T_pM respectively. In this case $W_X^{ss}(p)$ and $W_X^{uu}(p)$ are called respectively the *strong stable manifold* and the *strong unstable manifold* of p. Note that $dim(W_X^{ss}(p)) = dim(E^s)$ and $dim(W_X^{uu}(p) = dim(E^u)$.

Similarly,

$$W_X^s(p) = \bigcup_{t \in \mathbb{R}} W_X^{ss}(X_t(p)),$$

$$W_X^u(p) = \bigcup_{t \in \mathbb{R}} W_X^{uu}(X_t(p))$$

are also C^1 immersed submanifolds tangent to $E^s_p \oplus E^X_p$ and $E^X_p \oplus E^u_p$ at p respectively. So, $W^s_X(p)$ and $W^u_X(p)$ are called respectively the *stable manifold* and the *unstable manifold* of p.

Moreover, for every $\epsilon > 0$ we have that the sets

$$W^{ss}_X(p,\epsilon) = \{x : d(X_t(x), X_t(p)) \leq \epsilon, \forall t \geq 0\}$$

$$W^{uu}_X(p,\epsilon) = \{x : d(X_t(x), X_t(p)) \leq \epsilon, \forall t \leq 0\}$$

are closed neighborhoods of p in $W^{ss}_X(p)$ and $W^{uu}_X(p)$ respectively. Also, they are can denote by $W^{ss}_{loc}(p)$ and $W^{uu}_{loc}(p)$ respectively.

It is well known from stability theory for hyperbolic sets, that we can fix a neighborhood $U \subset M$ of Λ, a neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X and $\epsilon > 0$ such that every hyperbolic set H in U of every vector field Y in $\mathcal{U}$ satisfies that its local stable and instable manifold

$$W^{ss}_Y(x,\epsilon) \text{ and } W^{uu}_Y(x,\epsilon) \text{ have uniform size } \epsilon \text{ for all } x \in H. \tag{2.1}$$

Let $O = \{X_t(x) : t \in \mathbb{R}\}$ be the orbit of X through x, then the stable and unstable manifolds of O defined by

$$W^s(O) = \bigcup_{x \in O} W^{ss}(x), \qquad \textit{and} \qquad W^u(O) = \bigcup_{x \in O} W^{uu}(x)$$

are C^1 submanifolds tangent to the subbundles $E^s_\Lambda \oplus E^X_\Lambda$ and $E^X_\Lambda \oplus E^u_\Lambda$ respectively.

A *homoclinic orbit* of a hyperbolic periodic orbit O is an orbit in $\gamma \subset W^s(O) \cap W^u(O)$. If additionally $T_qM = T_qW^s(O) + T_qW^u(O)$ for some (and hence all) point $q \in \gamma$, then we say that γ is a *transverse homoclinic orbit* of O.

Definition 2.3.1. *The* homoclinic class $H(O)$ *of a hyperbolic periodic orbit* O *is the closure of the union of the transverse homoclinic orbits of* O*. We say that an invariant set* L *is a* homoclinic class *if* $L = H(O)$ *for some hyperbolic periodic orbit* O*.*

2.3.2 Classical example

In the theory of dynamical systems there are two classical examples, namely the Smale's horseshoe and the GLA Geometric Lorenz attractor . In fact, we can take the geometric Lorenz attractor [20] as the most representative example of a three dimensional sectional hyperbolic set. These classical examples of a hyperbolic set and a sectional hyperbolic set were a main motivation to build up the hyperbolic theory [45].

Edward Lorenz in the sixties studies the dynamics of atmospheric motion, and provides a simplified model of ordinary differential equations (called the Lorenz equations) . This system of equations for certain parameter values and initial conditions has an interesting behavior with a special dynamic.

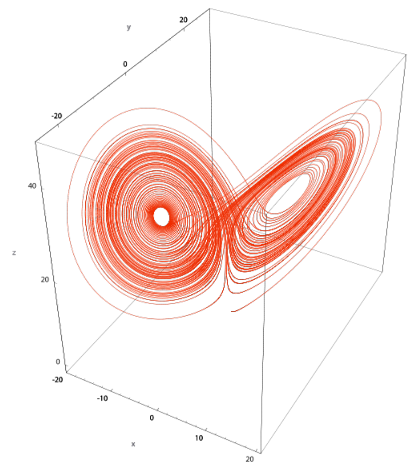
Naturally, the (GLA) geometric Lorenz attractor is motivated by the Lorenz study with its simplified model of ordinary differential equations. In fact, Guckenheimer (1976) by using this model, provides the following parameter values: $\sigma = 10$, $r = 28$ and $b = \frac{8}{3}$. He observes that the numeric simulation of these equations exhibit a similar behavior to the original equations and arises the called geometric Lorez attractor.

The following equations with parameter values $\sigma = 10$, $r = 28$ and $b = \frac{8}{3}$ are the known Lorenz equations, and the system represents the interesting dynamics.

LORENZ'S SIMPLIFIED MODEL

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = -bz + xy.$$

$\Longrightarrow$



Parameter values $\sigma = 10$, $r = 28$ and $b = \frac{8}{3}$.

The geometric Lorenz attractor is a nontrivial example, is singular (with hyperbolic singularity), is the closure of its periodic orbits, is a transitive set, has sensitivity with respect to the initial conditions and is a homoclinic class [45],[6].

It arises a lot of examples resembling at (GLA), modifying its topology and geometry, but with different properties. Particularly, the expanding geometric Lorenz attractor and the singular horseshoe (either expanding or contracting) are not C^1 robust singular transitive sets. Nevertheless, the contracting geometric Lorenz attractor is a C^1 robust singular transitive set.

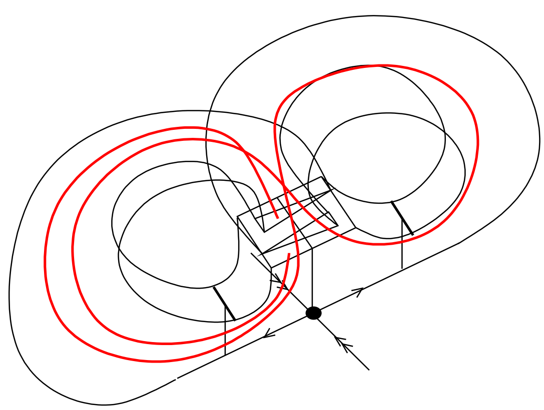


Figure 2.3: Geometric Lorenz attractor GLA.

Chapter 3

Lorenz-like singularity and singular cross-section

3.1 Lorenz-Like singularities on sectional hyperbolic sets

Given Λ a hyperbolic set, the stable manifold theorem provides the existence of contracting and expanding foliations for each one of its points. These are immersed, invariant and differentiable submanifolds (see Subsection 2.3.1).

There is also a stable manifold theorem in the case when Λ is a sectional hyperbolic set. Indeed, if we denote by $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ the corresponding sectional hyperbolic splitting over Λ, we assert that there exists such contracting foliation on a small neighborhood U of Λ [21]. Note that this extended foliation is not necessarily invariant, and we can only ensure the invariance if this one, at least, is an attracting set [3, Section 2.1]. This extension is carried out as follows: first, we can choose cone fields on U and we consider the space of tangent foliations to the cone fields. Given a point $x \in U$, whenever the positive orbit remain within to U, for example $t = 1$, we can use the map $DX_{-1}(x)$. This map sends the leaf at $X_{-1}(x)$ inside of cone $C_{X_{-1}(x)}$ to the leaf at x inside of cone C_x, contracting the angle and stretching the tangent vectors to the initial foliation. Then, we can apply fiber contraction [21, Page 30,31,80], [15, Theorem 1.243, Page 127]. Now, by using the Fiber Contraction Theorem [21] the foliation arises. Thus, we have from [21] that the contracting subbundle E^s_Λ can be extended to a contracting subbundle E^s_U in M (not necessarily invariant).

Moreover, such an extension by construction is tangent to a continuous foliation denoted by W^{ss} (or W^{ss}_X to indicate dependence on X). By adding the flow direction to W^{ss} we obtain a continuous foliation W^s (or W^s_X) now tangent to $E^s_U \oplus E^X_U$. Unlike the hyperbolic case W^s may have singularities, all of which being the leaves $W^{ss}(\sigma)$ passing through the singularities σ of X.

Note that W^s is transverse to ∂M because it contains the flow direction (which is transverse to ∂M by definition). Thus, note the following remark:

> It turns out that every singularity σ of a sectional hyperbolic set Λ satisfies $W^{ss}_X(\sigma) \subset W^s_X(\sigma)$.
> Furthermore, there are two possibilities for such a singularity, namely, (3.1)
> either $dim(W^{ss}_X(\sigma)) = dim(W^s_X(\sigma))$ (and so $W^{ss}_X(\sigma) = W^s_X(\sigma)$)
> or $dim(W^s_X(\sigma)) = dim(W^{ss}_X(\sigma)) + 1$.

In the later case we call it *Lorenz-like* according to the following definition.

Definition 3.1.1. *Let Λ be a sectional hyperbolic set of a C^1 vector field X of M. We say that a singularity σ of Λ is* Lorenz-like *if*

$$dim(W^s(\sigma)) = dim(W^{ss}(\sigma)) + 1.$$

By construction, the above definition is very useful and comfortable for a sectional hyperbolic scenario. Moreover, this definition provides a direct relation between the manifolds associates to the singularity. It is important to note that this definition is equivalent to the well known three dimensional Lorenz-like definition and definitions associated for the higher dimensional case in others sceneries.

Recall that a singularity σ of a three dimensional sectional hyperbolic set is Lorenz-like if it has three real eigenvalues $\lambda_3, \lambda_2, \lambda_1$ with $\lambda_3 < \lambda_2 < 0 < -\lambda_2 < \lambda_1$ (See Figure 3.1 *a*)). Here, its eigenvalues provide the force of contraction or expansion close to the singularity and they are directly associated to the strong stable, stable and unstable manifolds (see Figure 3.1 *b*)).

For the higher dimensional case, a singularity σ of a sectional hyperbolic set is Lorenz-like if it has three real eigenvalues $\lambda^{ss}, \lambda^s, \lambda^u$ with $\lambda^{ss} < \lambda^s < 0 < -\lambda^s < \lambda^u$, such that the real part of the remainder of eigenvalues are outside the compact interval $[\lambda^s, \lambda^u]$. $W^s(\sigma)$ is a manifold associated to eigenvalues λ with $Re(\lambda) < 0$, and $W^{ss}(\sigma)$ is a manifold associated to eigenvalues $Re(\lambda) < \lambda^s$.

The Figure 3.1 provides a geometric idea for the three dimensional case.

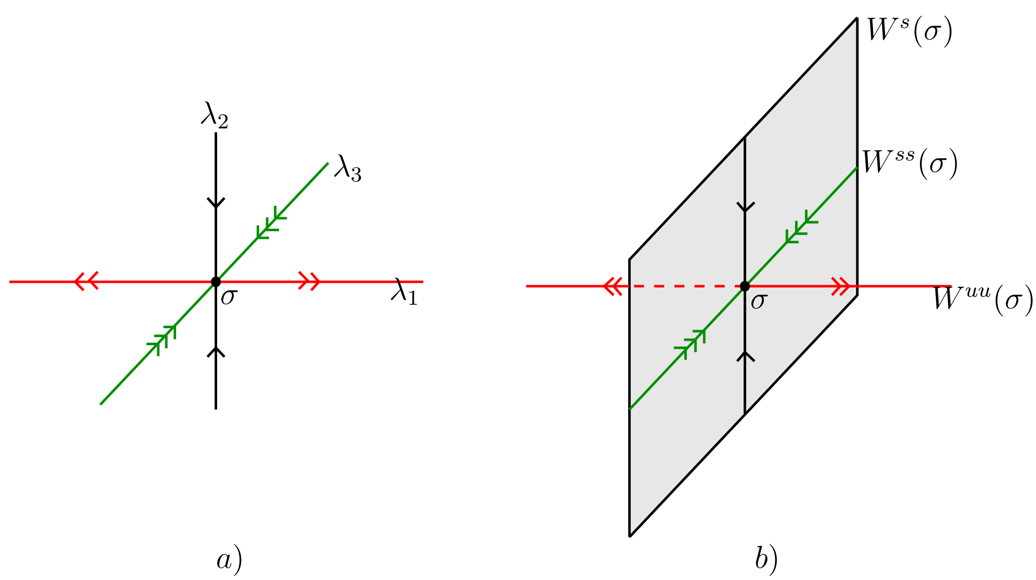


Figure 3.1: Three dimensional Lorenz-Like singularity.

3.2 Singular cross-section in higher dimension

In this section, we shall define *singular cross-section* in the higher dimensional context. First, we shall denote a cross-section by Σ and its boundary by $\partial\Sigma$.

Recall, in our context, the singular cross-sections are strongly associated with the Lorenz-like singularities. For this purpose, we start by presenting the cross-section for the three dimensional case. In this case, it will be a pair of simple submanifolds diffeomorphic to $[-1,1]\times[-1,1]$ characterized by an intersection with the strong stable manifold, as indicated in Figure 3.2.

Thus, the hypercube $I^k = [-1,1]^k$ will be submanifold of dimension k with $k \in \mathbb{N}$. Let σ be a Lorenz-like singularity. Hereafter, we shall denote $dim(W_X^{ss}(\sigma)) = s$, $dim(W_X^u(\sigma)) = u$ and therefore $dim(W_X^s(\sigma)) = s+1$ by definition. Moreover $W_X^{ss}(\sigma)$ separates $W_{loc}^s(\sigma)$ in two connected components denoted by $W_{loc}^{s,t}(\sigma)$ and $W_{loc}^{s,b}(\sigma)$ respectively.

Then, we begin by considering $B^u[0,1] \approx I^u$ and $B^{ss}[0,1] \approx I^s$ where $B^{ss}[0,1]$ is the ball centered at zero and radius 1 contained in $\mathbb{R}^{dim(W^{ss}(\sigma))} = \mathbb{R}^s$ and $B^u[0,1]$ is the ball centered at zero and radius 1 contained in $\mathbb{R}^{dim(W^u(\sigma))} = \mathbb{R}^{n-s-1} = \mathbb{R}^u$.

Definition 3.2.1. *A* singular cross-section *of a Lorenz-like singularity σ consists of a pair of submanifolds Σ^t, Σ^b, where Σ^t, Σ^b are cross-sections such that*

$$\Sigma^t \text{ is transversal to } W_{loc}^{s,t}(\sigma) \text{ and } \Sigma^b \text{ is transversal to } W_{loc}^{s,b}(\sigma).$$

Note that every singular cross-section contains a pair of *singular submanifolds* l^t, l^b defined as the intersection of the local stable manifold of σ with Σ^t, Σ^b respectively and, additionally, $dim(l^*) = dim(W^{ss}(\sigma))$ $(* = t, b)$. The positive orbit starting at $l^t \cup l^b$ goes directly to σ.

Thus, a singular cross-section Σ^* will be a *hypercube of dimension* $(n-1)$, i.e., diffeomorphic to $B^u[0,1] \times B^{ss}[0,1]$. Let $f : B^u[0,1] \times B^{ss}[0,1] \longrightarrow \Sigma^*$ be the diffeomorphism, such that

$$f(\{0\} \times B^{ss}[0,1]) = l^*$$

and where $\{0\}$ denotes the zero vector on $\mathbb{R}^u$. Define

$$\partial\Sigma^* = \partial^h\Sigma^* \cup \partial^v\Sigma^*$$

by:

$\partial^h\Sigma^* = \{$union of the boundary submanifolds which are transverse to l^* $\}$ and
$\partial^v\Sigma^* = \{$union of the boundary submanifolds which are parallel to l^* $\}$.

In fact, we can observe as the above definitions are extending the concept of singular cross-section for the higher dimensional case. These definitions are outstanding for the next results. Firstly, the Figure 3.2 shows the well known three dimensional case, with a 2-dimensional singular cross section that allows only one 1-dimensional stable foliation.

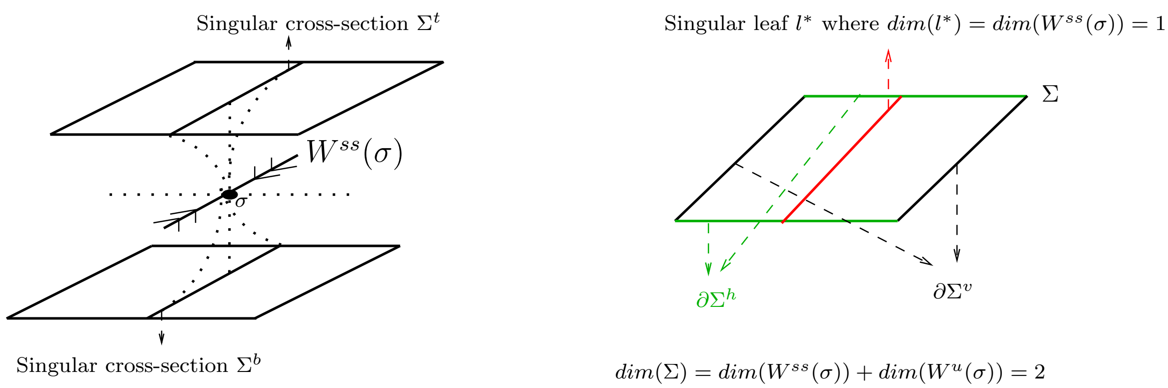


Figure 3.2: Three dimensional case. Cross-section.

Secondly, the Figure 3.3 shows the 4-dimensional case, where it is exhibiting two different cases for the stable foliation of a 3-dimensional singular cross section. Then, the new of singular cross section definition allows different stable foliations and this in turns allows to construct different singular cross-sections of Lorenz-like singularities.

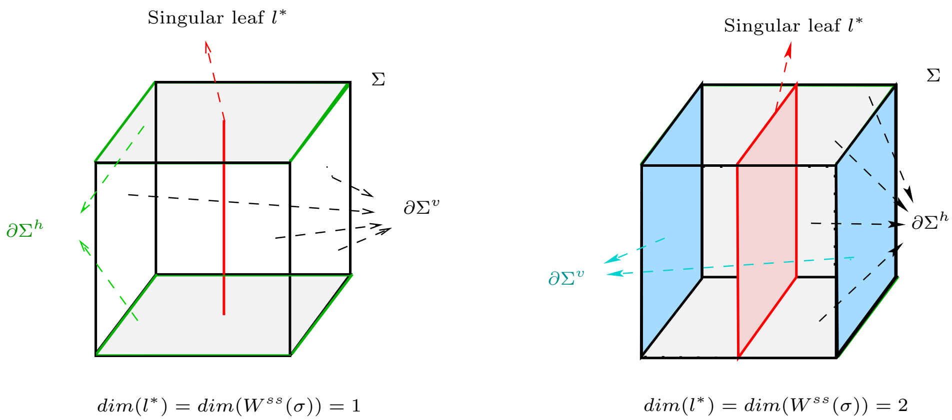


Figure 3.3: Four dimensional case. Cross-sections.

From this decomposition we obtain that

$$\partial^h\Sigma^* = (I^u \times [\cup_{j=0}^{s-1}(I^j \times \{-1\} \times I^{s-j-1})]) \bigcup (I^u \times [\cup_{j=0}^{s-1}(I^j \times \{1\} \times I^{s-j-1})])$$

$$\partial^v\Sigma^* = ([\cup_{j=0}^{u-1}(I^j \times \{-1\} \times I^{u-j-1})] \times I^s) \bigcup ([\cup_{j=0}^{u-1}(I^j \times \{1\} \times I^{u-j-1})] \times I^s),$$

where $I^0 \times I = I$.

Hereafter, we denote $\Sigma^* = B^u[0,1] \times B^{ss}[0,1]$.

Chapter 4

Sectional Anosov flows in higher dimensions

4.1 Introduction

We have talked about the fundamental role in the theory of dynamical systems that plays the hyperbolic theory. Recall that in this scenario the study of these systems (diffeomorphisms and flows) is done without singularities. A lot of these systems are well understood after the work of Smale [52] and others authors, such as the so called *Axiom A systems* or *Uniform Hyperbolic systems*. Also, it is important to recall that these systems are structurally stable, the set of periodic orbits is dense in the nonwandering set, and they have a spectral decomposition on a finite union of homoclinic classes.

Naturally, these features motivate the study of the dynamical properties of new sets containing singularities, taking the hyperbolic dynamical systems as a model, such as the sectional hyperbolic sets.

Thus, the *sectional Anosov flows* were introduced in [31] as a generalization of the *Anosov flows*. These also includes the *saddle-type hyperbolic attracting sets*, the *geometric and multidimensional Lorenz attractors* [1], [12], [20]. Some properties of these flows have been shown in the literature [2], [7]. Particularly, [28] proved that every attractor of every vector field C^1 close to a transitive sectional Anosov flow with singularities on a compact 3-manifold has a singularity. Moreover, [4] generalized this result from transitive to nonwandering ones. In this chapter we further extend [28] but now to higher dimensions. More precisely, we prove that every attractor of every vector field C^1 close to a transitive sectional Anosov flow with singularities of a compact manifold has a singularity.

Let us state our result in a precise way.

Theorem A. *Let X be a transitive sectional Anosov flow with singularities of a compact n-manifold. Then, every attractor of every vector field C^1 close to X has a singularity.*

The proof of this theorem follows closely that of [28]. More precisely, we assume by contradiction that there exists a sequence X^n of vector fields C^1 close to X each one exhibiting a non-singular attractor A^n. We then prove that A^n accumulates on a singularity of X and, consequently, for n large, we will prove that the corresponding attractor A^n contains a singularity. This give us the desired contradiction. To prove such assertions we will extend some tools in [28] including the definitions of both Lorenz-like singularity and singular cross-section.

4.2 Sectional hyperbolic sets in higher dimension

4.2.1 Preliminaries

In this section we exhibit and prove preliminaries results for transitive sectional Anosov flows that provide useful properties.

These following results examining the sectional hyperbolic splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ of a sectional hyperbolic set Λ of $X \in \mathcal{X}^1(M)$ appear in [38, Lemma 3,Page 5], [38, Theorem

A, Page 3] for three dimensional case and [7, Corollary 2.7, Page 65], [7, Lemma 2.7, Page 67] for the higher dimensional case.

The following lemma is well known and frequently called the *Hyperbolic lemma.* Also, we show the proof of the next result by its useful properties.

Lemma 4.2.1 (Hyperbolic lemma). *Let X be a sectional Anosov flow, X a C^1 vector field in M. If Y is C^1 close to X, then every nonempty, compact, non singular, invariant set H of Y is hyperbolic* saddle-type *(i.e. $E^s \neq 0$ and $E^u \neq 0$).*

Theorem 4.2.2. *Let X be a transitive sectional Anosov flow C^1 for M. Then, every $\sigma \in Sing(X) \cap M(X)$ is Lorenz-like and satisfies*

$$M(X) \cap W^{ss}_X(\sigma) = \{\sigma\}.$$

Proof. We begin by proving two claims.

Claim 1:

If $x \in (M(X) \setminus Sing(X))$, then $X(x) \notin E^s_x$.

Proof. Suppose by contradiction that there is $x_0 \in (M(X) \setminus Sing(X))$ such that $X(x_0) \in E^s_{x_0}$. Then, $X(x) \in E^s_x$ for every x in the orbit of x_0 since $E^s_{M(X)}$ is invariant.

$$\text{So } X(x) \in E^s_x \text{ for every } x \in \alpha(x_0) \text{ by continuity.} \tag{4.1}$$

It follows that $\omega(x)$ is a singularity for all $x \in \alpha(x_0)$. In particular, $\alpha(x_0)$ contains a singularity σ which is necessarily hyperbolic of saddle-type.

Now we have two cases: $\alpha(x_0) = \{\sigma\}$ or not.

If $\alpha(x_0) = \{\sigma\}$ then $x_0 \in W^u(\sigma)$. For all $t \in \mathbb{R}$ define the unitary vector

$$v^t = \frac{DX_t(x_0)(X(x_0))}{||DX_t(x_0)(X(x_0))||}.$$

It follows that

$$v^t \in T_{X_t(x_0)} W^u(\sigma) \cap E^s_{X_t(x_0)}, \quad \forall t \in \mathbb{R}.$$

Take a sequence $t_n \to \infty$ such that the sequence v^{-tn} converges to v^∞ (say). Clearly v^∞ is an unitary vector and, since $X_{-t_n}(x_0) \to \sigma$ and E^s is continuous we obtain

$$v^\infty \in T_\sigma W^u(\sigma) \cap E^s_\sigma.$$

Therefore v^∞ is an unitary vector which is simultaneously expanded and contracted by $DX_t(\sigma)$ a contradiction. This contradiction shows the result in the first case.

For the second, we assume $\alpha(x_0) \neq \{\sigma\}$. Then, $(W^u(\sigma) \setminus \{\sigma\}) \cap \alpha(x_0) \neq \emptyset$. Pick $x_1 \in (W^u(\sigma) \setminus \{\sigma\}) \cap \alpha(x_0)$. It follows from (4.1) that $X(x_1) \in E^s_{x_1}$ and then we get a contradiction as in the first case replacing x_0 by x_1. This contradiction proves the claim. □

Claim 2:
If $\sigma \in Sing(X)$, then $M(X) \cap W^{ss}(\sigma) = \{\sigma\}$.

Proof. Take $x \in W^{ss}(\sigma) \setminus \{\sigma\}$. Then, $E^s_x = T_x W^{ss}(\sigma)$. Moreover, since $W^{ss}(\sigma)$ is an invariant, we obtain $X(x) \in T_x W^{ss}(\sigma)$. We conclude that $X(x) \in E^s_x$ for all $x \in W^{ss}(\sigma)$ and now Claim (1) applies. □

□

The Lemma 4.2.1 implies the following two useful properties.

Lemma 4.2.3. *Let X be a transitive sectional Anosov flow C^1 in M. If $O \subset M(X)$ is a periodic orbit of X, then O is a hyperbolic saddle-type periodic orbit. In addition, if $p \in O$ then the set*

$$\{q \in W^{uu}_X(p) : M(X) = \omega_X(q)\}$$

is dense in $W^{uu}_X(p)$.

Proof. By Lemma 4.2.1, we have that O is hyperbolic and saddle-type. Let W be an open set in $W^{uu}_X(p)$. This set W exists since the point p belongs to the periodic orbit O which is hyperbolic. Define

$$B = \bigcup_{0\le t\le 1} X_t(W).$$

This set has dimension at least two, and so,

$$B' = \bigcup_{x\in B} W^{ss}_X(x)$$

contains an open set V with $B\cap V \neq \emptyset$.

Since $M(X)$ is the maximal invariant of X, $B\subset W^u_X(p)$ and $p\in O$, we obtain $B\cap V\subset M(X)$. Let $q\in M(X)$ such that $M(X)=\omega_X(q)$. Then, the forward orbit of q intersects V and so it intersects B' too. It follows from the definition of B' that the positive orbit of q is asymptotic to the forward orbit of some $q'\in B$. In particular, $M(X)=\omega_X(q)=\omega_X(q')$. This proves that $\{q\in W^{uu}_X(p) : M(X)=\omega_X(q)\}$ is dense in $M(X)$ as desired. □

The Theorem 4.2.2 implies the following two useful properties.

Proposition 4.2.4. *Let X be a transitive sectional Anosov flow C^1 of M. Let σ be a singularity of X in $M(X)$ (so σ is Lorenz-like by Theorem 4.2.2). Then, there is a singular-cross section Σ^t, Σ^b of σ in M such that*

$$(M(Y))\cap(\partial^h\Sigma^t\cup\partial^h\Sigma^b)=\emptyset,$$

for every C^r vector field Y close to X.

Proof. The equality in Theorem 4.2.2 implies that the negative orbit of every point in $W^{ss}_X(\sigma)\setminus\{\sigma\}$ leaves $M(X)$. Hence, we can arrange a singular-cross section Σ^t, Σ^b nearby σ, such that

$$M(X)\cap(\partial^h\Sigma^t\cup\partial^h\Sigma^b)=\emptyset.$$

Since $M(X)$ is the maximal invariant of X and the boundary of Σ^t, Σ^b is compact, we can find $T > 0$ such that

$$X_T(M) \cap (\partial^h \Sigma^t \cup \partial^h \Sigma^b) = \emptyset.$$

Hence

$$Y_T(M) \cap (\partial^h \Sigma^t \cup \partial^h \Sigma^b) = \emptyset,$$

for all C^r vector field close to X. The result follows since $\cap_{t>0} Y_t(M) \subset Y_T(U)$. □

4.2.1.1 Refinement of singular cross-sections and induced foliation

Recall, X denotes a sectional Anosov flow of a compact n-manifold M, $n \geq 3$, $X \in \mathcal{X}^\infty(\mathcal{M})$. Let $M(X)$ be the maximal invariant set of X.

In the same way of [8], we obtain an induced foliation $\mathcal{F}$ on Σ by projecting $\mathcal{F}^{ss}$ onto Σ, where $\mathcal{F}^{ss}$ denotes the invariant continuous contracting foliation on a neighborhood of $M(X)$ [21].

Let σ be a Lorenz-like singularity of a C^1 vector field X in $\mathcal{X}^1(M)$, and Σ^t, Σ^b be a singular cross-section of σ. Thus, for σ we recall that,

$$\begin{aligned} &dim(W^{ss}_X(\sigma)) = s, \text{ then} \\ &dim(W^{s}_X(\sigma)) = s+1 \text{ and } dim(W^{u}_X(\sigma)) = n-s-1, \\ &dim(\Sigma^*) = s + (n-s-1) = n-1. \end{aligned} \tag{4.2}$$

For the refinement, since $\Sigma^* = B^u[0,1] \times B^{ss}[0,1]$, we will set up a family of singular cross-sections as follows: given $0 < \Delta \leq 1$ small, we define $\Sigma^{*,\Delta} = B^u[0,\Delta] \times B^{ss}[0,1]$, such that

$$l^* \subset \Sigma^{*,\Delta} \subset \Sigma^*,$$

$(i.e., \quad l^* = \{0\} \times B^{ss}[0,1]) \subset (\Sigma^{*,\Delta} = B^u[0,\Delta] \times B^{ss}[0,1]) \subset (\Sigma^* = B^u[0,1] \times B^{ss}[0,1])$,

where we fix a coordinate system (x^*, y^*) in Σ^* $(* = t, b)$. We will assume that $\Sigma^* = \Sigma^{*,1}$.

We will use this notation throughout this lemma and the Theorem A proof.

Lemma 4.2.5. *Let X be a transitive sectional Anosov flow C^1 of M. Let σ be a singularity of X in $M(X)$. Let Y^n be a sequence of vector fields converging to X in the C^1 topology. Let O_n be a periodic orbit of Y^n such that the sequence $\{O_n : n \in I\!N\}$ accumulates on σ. If $0 < \Delta \leq 1$ and Σ^t, Σ^b is a singular cross-section of σ, then there is n such that either*

$$O_n \cap int(\Sigma^{t,\Delta}) \neq \emptyset \quad or \quad O_n \cap int(\Sigma^{b,\Delta}) \neq \emptyset.$$

Proof. Since O_n accumulates on $\sigma \in M(X)$ and $M(X)$ is maximal invariant, we have that $O_n \subset M(X)$ for all n large (recall $Y^n \to X$ as $n \to \infty$). Let us fix a fundamental domain D_ϵ of the vector field's flow X_t restricted to the local stable manifold $W^s_{loc}(\sigma)$ ([44]) for $\epsilon > 0$ as follows:

$$D_\epsilon = S_\epsilon \cup S_{-\epsilon} \cup C_\epsilon, \text{ where:}$$

$$\begin{aligned}
S_\epsilon &= \{x \in \mathbb{R}^{s+1}| \quad \Sigma_{i=1}^s x_i^2 + (x_{s+1} - \epsilon)^2 = 1, \quad \wedge \quad x_{s+1} \geq \epsilon\},\\
S_{-\epsilon} &= \{x \in \mathbb{R}^{s+1}| \quad \Sigma_{i=1}^s x_i^2 + (x_{s+1} - \epsilon)^2 = 1, \quad \wedge \quad x_{s+1} \leq -\epsilon\},\\
C_\epsilon &= \{x \in \mathbb{R}^{s+1}| \quad \Sigma_{i=1}^s x_i^2 = 1, \quad \wedge \quad x_{s+1} \in [-\epsilon, \epsilon]\}.
\end{aligned}$$

As $W^s_{loc}(\sigma)$ is $(s+1)$-dimensional and D_ϵ is homeomorphic to the sphere (s)-dimensional, by construction D_ϵ intersects $W^{ss}_X(\sigma)$ in $C_\epsilon|_{x_{s+1}=0}$, that is a sphere $(s-1)$-dimensional. Note that the orbits of all point in $C_\epsilon|_{x_{s+1}=0}$ together with σ yield $W^{ss}_X(\sigma)$. In particular, $C_\epsilon|_{x_{s+1}=0} \notin M(X)$ by Theorem 4.2.2. Also note that for all ϵ, D_ϵ is a fundamental domain.

Let $\tilde{D}_\epsilon$ be a cross section of X such that $W^s_{loc}(\sigma) \cap \tilde{D}_\epsilon = D_\epsilon$. It follows that $\tilde{D}_\epsilon$ is a $(n-1)$-cylinder, and so we can consider a system coordinated (x, s) with $x \in D_\epsilon$ and $s \in I^u$. Thus, by using this system coordinate we can construct a family of singular cross-sections $\Sigma^t_\delta, \Sigma^b_\delta$ (for all $\delta \in [-\epsilon, \epsilon]$) by setting

$$\Sigma^t_\delta = \{(x, s) \in \tilde{D}_\epsilon : x \in S_\delta, s \in I^u\},$$

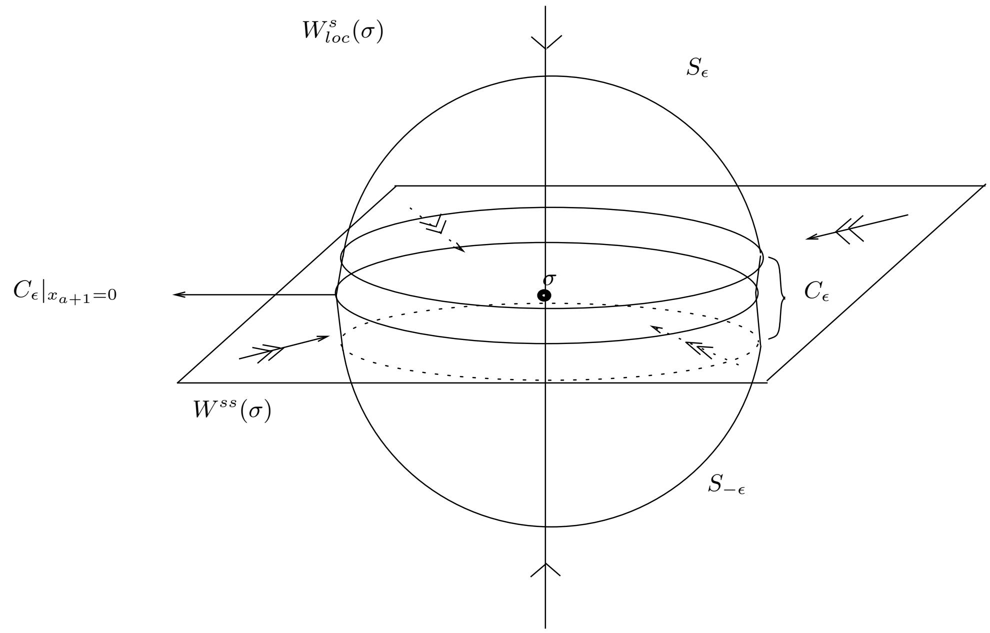


Figure 4.1: The fundamental domain.

$$\Sigma^b_\delta = \{(x, s) \in \tilde{D}_\epsilon : x \in S_{-\delta}, s \in I^u\}.$$

Due to the smooth variation of $W^{ss}_Y(\sigma(Y))$ with respect to Y close to X we can assume that $\sigma(Y) = \sigma$ and that $W^{ss}_{loc,Y}(\sigma(Y) = W^{ss}_{loc}(\sigma)$ for every Y close to X. By choosing D_ϵ so close to σ we can further assume that $\tilde{D}_\epsilon$ is a cross-section of Y, for every Y close to X. We claim that there is $\delta > 0$ such that the conclusion of the lemma holds for $\Sigma^t = \Sigma^t_\delta$ and $\Sigma^b = \Sigma^b_\delta$. Indeed, we first note that under the cylindrical coordinate system (x, s), one has $\Sigma^{*,\Delta} = \Sigma^*_\Delta$, for all $0 < \Delta \leq \delta$ $(* = t, b)$. Otherwise, if the conclusion of the claim fails, it implies that O_n intersects $\tilde{D}_\epsilon \setminus (\Sigma^t_\Delta \cup \Sigma^b_\Delta)$ for all $\Delta > 0$ small. Further, we would find a sequence of periodic points such that $p_n \in O_n$ (for all n large) and $p_n = (x_n, s_n)$, with $x_n \in C_\Delta$ and $s_n \to 0$ as $n \to \infty$. As Δ is arbitrary and $s_n \to 0$ we conclude that p_n converges to a point in $C_\Delta|_{x_{s+1}=0}$, by passing to a subsequence if necessary. As $s_n \to 0$, it implies that the intersection tends to the (s)-dimensional sphere D_ϵ.

As $O_n \subset M(Y^n)$, $Y^n \to X$ and $M(Y^n)$ is ϵ-C^1-close to $M(X)$ for all n ($n \in \mathbb{N}$), by using the above arguments, there exists a point $z \in (C_\epsilon|_{x_{s+1}=0})$ such that $z \in M(X)$. This contradicts Theorem 4.2.2 and the proof follows. □

4.3 Proof of Theorem A

We prove the theorem by contradiction. Let X be a transitive sectional Anosov flow C^1 of M. Then, we suppose that there exists a sequence $X^n \xrightarrow{C^1} X$ such that every X^n exhibits a non-singular attractor $A^n \in M(X^n)$ arbitrarily close to $M(X)$ and since A^n also is arbitrarily close to $M(X)$, we can assume that A^n belongs to $M(X)$ for all n. It follows from the definition of attractor that each A^n is compact, invariant and nonempty. As A^n is non-singular by hypothesis, then the Lemma 4.2.1 and the Lemma 4.2.3 imply the following:

$$\begin{array}{l} A^n \text{ is a hyperbolic attractor of type saddle of } X^n \text{ for all } n, \\ \text{and since } A^n \text{ is non-singular for all } n \text{, obviously } A^n \text{ is not} \\ \text{a singularity of } X^n \text{ for all } n. \end{array} \tag{4.3}$$

Recall, we denote by $Sing(X)$ the set of singularities of X and $Cl(A)$ the closure of A, $A \subset M$. Moreover, given $\delta > 0$ and $A \subset M$, we define $B_\delta(A) = \{x \in M : d(x, A) < \delta\}$ where $d(\cdot, \cdot)$ is the metric in M.

In the same way of [28], let us consider the following lemma that similarly provides useful features for the higher dimension case. The lemma gives one description about the behavior of the attractors.

Lemma 4.3.1. *The sequence of attractors A^n accumulates on $Sing(X)$, i.e.*

$$Sing(X) \bigcap Cl\left(\bigcup_{n \in \mathbb{N}} A^n\right) \neq \emptyset.$$

Proof. We prove the lemma by contradiction. Then, we suppose that there is $\delta > 0$, such that

$$B_\delta(Sing(X)) \bigcap \left(\bigcup_{n \in I\!N} A^n \right) = \emptyset. \tag{4.4}$$

In the same way as in [28], we define

$$H = \bigcap_{t \in I\!R} X_t \left(M \setminus B_{\delta/2}(Sing(X)) \right).$$

It should be noted that H is a invariant set and $Sing(X) \cap H = \emptyset$. Additionally, H is a nonempty compact set [28]. By using the Lemma 4.2.1 we conclude that H is a hyperbolic set. So, we denote by $E^s \oplus E^X \oplus E^u$ the corresponding hyperbolic splitting (see Definition 2.2.1).

By the stability of hyperbolic sets we can fix a neighborhood W of H and $\epsilon > 0$ such that if Y is a vector field C^r close to X and H_Y is a compact invariant set of Y in W then :

$$\begin{array}{l} H_Y \text{ is hyperbolic and its hyperbolic splitting } E^{s,Y} \oplus E^Y \oplus E^{u,Y}. \\ dim(E^u) = dim(E^{u,Y}),\ dim(E^s) = dim(E^{s,Y}). \\ \text{The manifolds } W^{uu}_Y(x, \epsilon),\ x \in H_Y, \text{ have uniform size } \epsilon. \end{array} \tag{4.5}$$

As $X^n \to X$, we have that:

$$\begin{array}{l} \bigcap_{t \in I\!R} X^n_t(M \setminus B_{\delta/2}(Sing(X)) \subset W, \text{ for all } n \text{ large.} \\ A^n \subset M \setminus B_{\delta/2}(Sing(X)) \text{ for all } n, \text{ and } A^n \subset W \text{ for all } n \text{ large.} \\ \text{If } x^n \in A^n \text{ so that } x^n \text{ converges to some } x \in M, \text{ then } x \in H. \\ \text{If } w \in W^{uu}_{X^n}(x^n, \epsilon), \text{ the tangent vectors of } W^{uu}_{X^n}(x^n, \epsilon) \\ \text{in this point are in } E^{u,X^n}_w. \\ \text{As } x^n \to x, W^{uu}_{X^n}(x^n, \epsilon) \to W^{uu}_X(x, \epsilon) \text{ in the sense of } C^1 \text{ submanifolds [45].} \\ \text{And } \angle(E^{u,X^n}, E^u) \longrightarrow 0, \text{ if } n \to \infty \text{[28].} \end{array} \tag{4.6}$$

Thus, we fix an open set $U \subset W^{uu}_X(x,\epsilon)$ containing the point x.
By (4.3), it follows that the periodic orbits of X^n in A^n are dense in A^n [Anosov closing lemma]. Particularly, we can assume that each x^n is a periodic point of A^n. As $M(X) \cap Sing(X) \neq \emptyset$ and $M(X)$ is a transitive set, it follows from the Lemma 4.2.3 that there exists $q \in U$, $0 < \delta_1 < \delta_2 < \frac{\delta}{2}$ and $T > 0$ such that $X_T(q) \in B_{\delta_1}(Sing(X))$.
By using [44, Tubular Flow Box Theorem], there is an open set V_q containing q such that $X_T(V_q) \subset B_{\delta_1}(Sing(X))$ and, as $X^n \to X$ it follows that

$$X^n_T(V_q) \subset B_{\delta_2}(Sing(X)), \tag{4.7}$$

for all n large (see Figure 4.2).

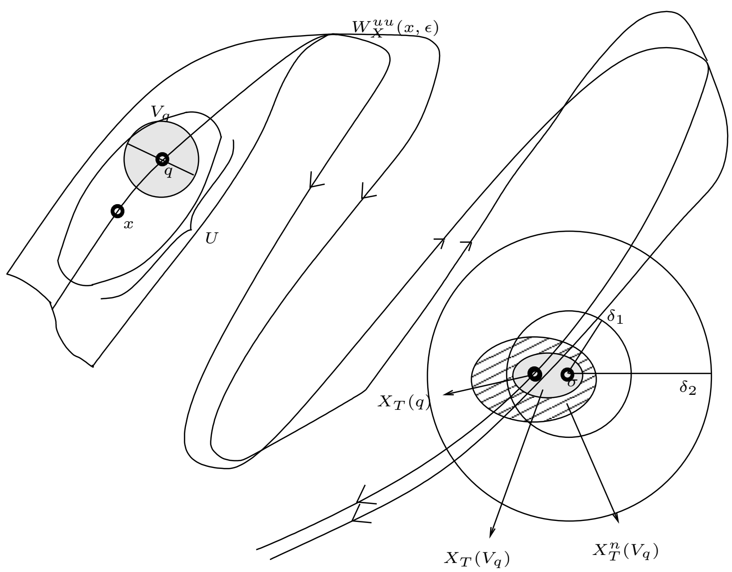


Figure 4.2: Tubular Flow Box Theorem for $X_T(V_q)$.

In addition, $W^{uu}_{X^n}(x^n,\epsilon) \cap V_q \neq \emptyset$ for n large enough, since $W^{uu}_{X^n}(x^n,\epsilon) \to W^{uu}_X(x,\epsilon)$ and $q \in U \subset W^{uu}_X(x,\epsilon)$. Applying (4.7) to X^n for n large we have

$$X^n_T(W^{uu}_{X^n}(x^n,\epsilon)) \cap B_{\delta_2}(Sing(X)) \neq \emptyset.$$

In particular $W^{uu}_{X^n}(x^n,\epsilon) \subset W^u_{X^n}(x^n)$, then the invariance of $W^u_{X^n}(x^n)$ implies

$$W^u_{X^n}(x^n) \cap B_{\delta/2}(Sing(X)) \neq \emptyset.$$

Observe that $W^u_{X^n}(x^n) \subset A^n$ since $x^n \in A^n$ and A^n is an attractor. We conclude that

$$A^n \cap B_\delta(Sing(X)) \neq \emptyset.$$

This contradicts (4.4) and the proof follows. □

Proof of Theorem A: By using the Lemma 4.3.1 there exists $\sigma \in M(X)$ such that

$$\sigma \in Sing(X) \bigcap Cl\left(\bigcup_{n \in I\!N} A^n\right).$$

By Theorem 4.2.2 we have that σ is Lorenz-like and satisfies

$$M(X) \cap W^{ss}_X(\sigma) = \{\sigma\}.$$

By Proposition 4.2.4, we can choose Σ^t, Σ^b, singular-cross section for σ and $M(X)$ such that

$$M(X) \cap \left(\partial^h \Sigma^t \cup \partial^h \Sigma^b\right) = \emptyset.$$

As $X^n \to X$ we have that Σ^t, Σ^b is singular-cross section of X^n too, thus we can assume that $\sigma(X^n) = \sigma$ and $l^t \cup l^b \subset W^s_{X^n}(\sigma)$ for all n. [Implicit function theorem].

We have that the splitting $E^s \oplus E^c$ persists by small perturbations of X [21]. The dominance condition [Definition 2.2.3-(2)] together with [17, Proposition 2.2] imply that for $* = t, b$ one has

$$T_x\Sigma^* \cap \left(E^s_x \oplus E^X_x\right) = T_x l^*,$$

for all $x \in l^*$.

Denote by $\angle(E, F)$ the angle between two linear subspaces. The last equality implies that there is $\rho > 0$ such that

$$\angle(T_x\Sigma^* \cap E^c_x, T_x l^*) > \rho,$$

for all $x \in l^*$ $(* = t, b)$. In this way, since $E^{c,n} \to E^c$ as $n \to \infty$ we have for n large enough that

$$\angle(T_x\Sigma^* \cap E^{c,n}_x, T_x l^*) > \frac{\rho}{2}, \tag{4.8}$$

for all $x \in l^*$ $(* = t, b)$.

As in the previous subsection 4.2.1.1 we fix a coordinate system $(x, y) = (x^*, y^*)$ in Σ^* such that

$$\Sigma^* = B^u[0,1] \times B^{ss}[0,1], \qquad l^* = \{0\} \times B^{ss}[0,1]$$

with respect to (x, y). Also, given $\Delta > 0$ we define $\Sigma^{*,\Delta} = B^u[0,\Delta] \times B^{ss}[0,1]$.

Hereafter, $\Pi^* : \Sigma^* \to B^u[0,1]$ will be the projection such that $\Pi^*(x, y) = x$. We will denote the line field in Σ^{*,Δ_0} by F^n, where

$$F^n_x = T_x\Sigma^* \cap E^{c,n}_x, \quad x \in \Sigma^{*,\Delta_0}.$$

Remark 4.3.2. *The continuity of $E^{c,n}$ and (4.8) imply that there is $\Delta_0 > 0$ such that for every n large the line F^n is* transverse to Π^*. *By this we mean that $F^n(z)$ is* not tangent to the curves $(\Pi^*)^{-1}(c)$, for every $c \in B^u[0, \Delta_0]$.

Recall that A^n is a hyperbolic attractor of type saddle of X^n for all n (see (4.3)) and the periodic orbits of X^n in A^n are dense in A^n ([45]). As $\sigma \in Cl\,(\cup_{n\in\mathbb{N}} A^n)$, we can find a sequence of periodic orbits $(O_n)_{n\in\mathbb{N}}$, such that $O_n \in A^n$ and accumulating on σ. It follows from the Lemma 4.2.5 applied to $Y^n = X^n$ that there exists $n_0 \in \mathbb{N}$ such that either

$$O_{n_0} \cap int(\Sigma^{t,\Delta_0}) \neq \emptyset \text{ or } O_{n_0} \cap int(\Sigma^{b,\Delta_0}) \neq \emptyset.$$

As $O_{n_0} \subset A_{n_0}$ we conclude that either

$$A^{n_0} \cap int(\Sigma^{t,\Delta_0}) \neq \emptyset \text{ or } A^{n_0} \cap int(\Sigma^{b,\Delta_0}) \neq \emptyset.$$

We shall assume that $A^{n_0} \cap int(\Sigma^{t,\Delta_0}) \neq \emptyset$ (analogous proof for the case $* = b$). Note that $\partial^h\Sigma^{t,\Delta_0} \subset \partial^h\Sigma^t$ by definition. Then, by using Proposition 4.2.4 one has

$$A \cap \partial^h\Sigma^{t,\Delta_0} = \emptyset.$$

As A^{n_0} and Σ^{t,Δ_0} are compact non-empty sets, it follows that $A^{n_0} \cap \Sigma^{t,\Delta_0}$ is a compact nonempty subset of Σ^{t,Δ_0}, and thus there exists $p \in \Sigma^{t,\Delta_0} \cap A^{n_0}$ such that

$$dist(\Pi^t(\Sigma^{t,\Delta_0} \cap A^{n_0}), 0) = dist(\Pi^t(p), 0),$$

where $dist$ denotes the distance in $B^u[0, \Delta_0]$. Note that $dist(\Pi^t(p), 0)$ is the minimum distance of $\Pi^t(\Sigma^{t,\Delta_0} \cap A^{n_0})$ to 0 in $B^u[0, \Delta_0]$.

As $p \in A^{n_0}$, we have that $W^u_{X^{n_0}}(p)$ is a well defined submanifold, since that A^{n_0} is hyperbolic set (4.3), and $dim(E^c) = dim(E^{c,n_0})$(4.5).

By domination [Definition 2.2.3-(2)], $T_z(W^u_{X^{n_0}}(p)) = E^{c,n_0}_z$ for every $z \in W^u_{X^{n_0}}(p)$ and hence, $dim(W^u_{X^{n_0}}(p)) = (n - s - 1)$ (4.2). Next, we can ensure that

$$T_z(W^u_{X^{n_0}}(p)) \cap T_z\Sigma^{t,\Delta_0} = E^{c,n_0}_z \cap T_z\Sigma^{t,\Delta_0} = F^{n_0}_z$$

for every $z \in W^u_{X^{n_0}}(p) \cap \Sigma^{t,\Delta_0}$.
First, note that the last equality shows that $W^u_{X^{n_0}}(p) \cap \Sigma^{t,\Delta_0}$ is transversal, and therefore there exists some compact submanifold inside of $W^u_{X^{n_0}}(p) \cap \Sigma^{t,\Delta_0}$. We denote this compact submanifold by K^{n_0}. Thus, by construction $p \in K^{n_0}$[See (4.3)] and K^{n_0} is tangent to F^{n_0}, since $K^{n_0} \subset W^u_{X^{n_0}}(p) \cap \Sigma^{t,\Delta_0}$.

Remark 4.3.3. *Since $dim(E^{c,n_0}) = dim(W^u_{X^{n_0}}(p)) = (n - s - 1)$, by construction we have that $dim(B^u[0, \Delta_0]) = (n - s - 1)$.*
We have that $W^u_{X^{n_0}}(p) \cap \Sigma^{t,\Delta_0}$ is a submanifold of M, since $W^u_{X^{n_0}}(p) \cap \Sigma^{t,\Delta_0}$ is transversal and nonempty and $W^u_{X^{n_0}}(p)$, Σ^{t,Δ_0} are submanifolds of M. Note that $dim(W^u_{X^{n_0}}(p)) + dim(\Sigma^{t,\Delta_0}) \geq n$.

Since F^{n_0} is *transverse* to Π^t, one has that K^{n_0} is *transverse* to Π^t (i.e. K^{n_0} is transverse to the curves $(\Pi^t)^{-1}(c)$, for every $c \in B^u[0, \Delta_0]$). Let us denote the image of K^{n_1} by the projection Π^t in $B^u[0, \Delta_0]$ by $K^{n_1}_1$, i.e., $\Pi^t(K^{n_1}) = K^{n_1}_1$. Note that $K^{n_1}_1 \subset B^u[0, \Delta_0]$ and $\Pi^t(p) \in int(K^{n_1}_1)$.

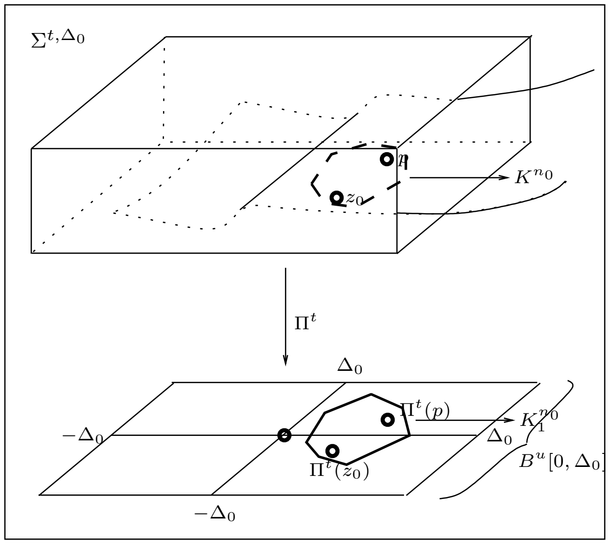


Figure 4.3: The projection $\Pi^t(K^{n_0}) = K_1^{n_0}$.

As $dim(K_1^{n_0}) = dim(B^u[0, \Delta_0])$ (by Remark 4.3.3), there exists $z_0 \in K^{n_0}$ such that

$$dist(\Pi^t(z_0), 0) < dist(\Pi^t(p), 0).$$

It follows from the property of attractor that $W^{uu}_{X^{n_0}}(p, \epsilon) \subset W^u_{X^{n_0}}(p) \subset A_{n_0}$. Thus, $K^{n_0} \subset \Sigma^{t,\Delta_0} \cap A^{n_0}$ and $p \in A^{n_0}$.

As $A^{n_0} \cap \partial^h \Sigma^{t,\Delta_0} = \emptyset$ (by Proposition 4.2.4) and $dim(K_1^{n_0}) = dim(B^u[0, \Delta_0])$ (by Remark 4.3.3), we conclude that

$$dist(\Pi^t(\Sigma^{t,\Delta_0} \cap A^{n_0}), 0) = 0.$$

Given that A^{n_0} is closed, this last equality implies

$$A^{n_0} \cap l^t \neq \emptyset.$$

Since $l^t \subset W^s_{X^{n_0}}(\sigma)$ and A^{n_0} is closed invariant set for X^{n_0} we conclude that $\sigma \in A^{n_0}$. We have proved that A^{n_0} contains a singularity of X^{n_0}. But A^{n_0} is a hyperbolic attractor of X^{n_0} by the Property (4.3), and this leads to that $A^{n_0} = \{\sigma\}$. Finally, by using the Property (4.3) we obtain a contradiction and the proof follows. □

Chapter 5

Finiteness and existence of attractors and repellers on sectional hyperbolic sets

5.1 Introduction

The dynamical systems describe different properties about the evolution of initial states, asymptotic behavior and relationships between system's elements. However, most of these systems' behavior might be very complex, therefore, finding the link between them becomes a difficult task.

It is well known that many of these properties come from physics phenomena. In the sixties some definitions appeared that tried to explain these behaviors and properties, such as *attractors* and *repellers*. These concepts are well-known and play a fundamental role in the dynamical systems theory. They have received some mathematical interpretations, such as *turbulence* that appears in the classical paper [49] which, simultaneously, provides existence of attractors for particular vector fields. Also, strange attractor [42], wild strange attractor [51] or non trivial attractor, among others. Since a repeller is an attractor for the reverse flow, it is clear that this result provides existence for repellers too. Thereby, stressing the importance of attractors, we highlight the classical construction of the geometric Lorenz models [1], [20] or the multidimensional Lorenz attractors [12]. They provide a wide range of results at research of dynamical systems theory, particularly hyperbolic and sectional hyperbolic theories on three dimensional manifolds. **The study of sectional hyperbolic attractors for higher dimensional flows is, however, mostly open**.

Thus, the aim of this chapter, on sectional hyperbolic sets of vector fields on compact higher dimensional manifolds, is to research two very important related problems, namely, how many attractors and repellers can arise from small perturbations and, also, the possible appearance of repellers from small perturbations. In this way we focus our interest on the particular result given by [28] (finiteness) and some examples containing repellers (existence). For this reason, the present chapter is divided into two important sections.

5.1.1 About finiteness and Bonatti's conjecture

Particularly, [28] asserts that, for every sectional hyperbolic transitive attracting set Λ of a vector field X on a compact 3-manifold, there are neighborhoods $\mathcal{U}$ of X and U of Λ such that the number of attractors in U of a vector field in $\mathcal{U}$ is less than one plus the number of equilibria of X.

This result was extended later in [4] by allowing Λ to be an attracting set contained in the nonwandering set (rather than transitive). An extension of [28] to higher dimensions was recently obtained in [25] (previous chapter). This results provide information about which conditions one has attractors's finiteness. We can also mention the recent paper [13] studying the similar problem but for one-dimensional maps with discontinuities.

Indeed, we remove both the transitivity and nonwandering hypotheses in order to obtain robust finiteness of attractors and repellers. Here, we obtain an upper bound for the number of attractors and repellers that can appear from small perturbations of vector field (this improves [25], [28]).

This in turn, provides an important corollary which is directly motivated and related to one of the Bonatti's conjectures [11]. Being more precise, one of the Palis's conjectures asserts that generic diffeomorphisms far from homoclinic tangencies have only finitely many sinks and sources [43]. For the surfaces case, this conjecture was proved for C^1 diffeomorphisms by Pujals and Sambarino [46]. Subsequently, Bonatti gives a slightly stronger conjecture, namely, generic diffeomorphisms that are far from homoclinic tangencies have only finitely many attractors and repellers [11].

Recently, Crovisier in [16] proves that a sectional hyperbolic three dimensional flows is open and dense in the open set of flows which are far from homoclinic tangencies.

Then, by combining this result with our main theorem of finiteness, we obtain the Bonatti's conjecture mentioned above, but for three dimensional flows.

5.1.2 About repeller's existence

The following section arises after observing some examples of 3-dimensional sectional hyperbolic sets containing repellers. Among these examples, we name particularly those containing repellers in different sceneries. Firstly, we find the well-known example of a non transitive sectional hyperbolic set (without singularities), containing a non trivial repeller (e.g. *Anomalous Anosov flow* [18]). Secondly, [29] exhibits an example of a transitive sectional hyperbolic set containing singularities, but without non trivial repellers. Finally, [14] shows that there exists non transitive sectional hyperbolic sets, containing both singularities and non trivial hyperbolic repellers. This in turns leads our next step, that it would be to discuss about existence of repellers. Note that repeller's occurrence in the previous examples is given when the set is non transitive. Also, it is not difficult to prove that every transitive sectional hyperbolic set has no proper repellers. Therefore, these observations motivate the following question:

1. Under what conditions are there hyperbolic repellers close to transitive sectional hyperbolic sets?

There are works related with our main question, since a repeller contains the stable manifold of all their points [30], [5].
On the other hand, note that the transitivity of a set implies that it is contained in the nonwandering set, then in order to improve our result it is natural to ask:

1. Are there sectional hyperbolic sets with singularities contained in the nonwandering set containing hyperbolic repellers?

2. Are there hyperbolic repellers close to sectional hyperbolic sets with singularities contained in the nonwandering set?

This questions have been our motivation and we give a negative answer to these ones. In fact, we prove a robustly non-existence of repellers on a connected sectional hyperbolic set which both has singularities and consists of nonwandering points.

5.1.3 Main theorems

Let us state our results in a more precise way.

Theorem B. *For every sectional hyperbolic set* Λ *of a vector field* X *on a compact manifold there are neighborhoods* $\mathcal{U}$ *of* X, U *of* Λ *and* $n_0 \in \mathbb{N}$ *such that*

$$\#\{L \subset U : L \text{ is an attractor or repeller of } Y \in \mathcal{U}\} \leq n_0.$$

Theorem C. *Let X be a vector field with singularities of a compact n-manifold M, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \subset M$ be a connected sectional hyperbolic set of X. If $\Lambda \subset \Omega(X)$, then there are neighborhoods $\mathcal{U} \subset \mathcal{X}^1(M)$ of X and $U \subset M$ of Λ such that if $Y \in \mathcal{U}$, Y has no repeller in U.*

To finish we state the following corollaries of our results. Recall that a *sectional Anosov flow* is a vector field whose maximal invariant set is sectional hyperbolic [27].

Corollary 5.1.1. *For every sectional Anosov flow of a compact manifold there are a neighborhood $\mathcal{U}$ and $n_0 \in \mathbb{N}$ such that*

$$\#\{L \text{ is an attractor or repeller of } Y \in \mathcal{U}\} \leq n_0.$$

The following result provides the Bonatti's conjecture proof for the three dimensional flows.

Corollary 5.1.2. *Generic three dimensional flows that are far from homoclinic tangencies have only finitely many attractors and repellers.*

5.2 Preliminaries

In this section, we recall some results on sectional hyperbolic sets, and we obtain some useful results for the main theorems.

Let Λ be a sectional hyperbolic set of a C^1 vector field X of M. Recall, we denote by $Sing(X)$ the set of singularities of the vector field X and by $Cl(A)$ the closure of A, $A \subset M$.

As the previous chapter, let us use the Lemma 4.2.1 and the Theorem 4.2.2. Although our scenario does not have the transitivity hypothesis, the Theorem 4.2.2 has the same conclusion on the intersection of the singularity's strong stable manifold with the set. But, it is losing the property of all its singularities be Lorenz-like, i.e., one has the following theorem.

Theorem 5.2.1. *Let Λ be a sectional hyperbolic set of a C^1 vector field X of M. If $\sigma \in Sing(X) \cap \Lambda$, then $\Lambda \cap W^{ss}_X(\sigma) = \{\sigma\}$.*

Next we explain briefly how to obtain sectional hyperbolic sets nearby Λ from vector fields close to X. Fix a neighborhood U with compact closure of Λ as in Lemma 4.2.1. Define

$$\Lambda_X = \cap_{t\in\mathbb{R}} X_t(Cl(U)).$$

Note that Λ_X is sectional hyperbolic and $\Lambda \subset \Lambda_X$. Likewise, if Y is a C^1 vector field close to X, we define the continuation

$$\Lambda_Y = \cap_{t\in\mathbb{R}} Y_t(Cl(U)).$$

Definition 5.2.2. *Let A and B be compact sets of M, and let $d(\cdot,\cdot)$ be a metric on M. Define the* Hausdorff distance *between A and B by*

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right\}.$$

We define $K(M) = \{A \subset M \mid A \text{ is compact}\}$.

Remark 5.2.3. *We have that d_H is a metric on $K(M)$ and the metric space $(K(M), d_H)$ is compact by* Blaschke's selection theorem.

In the previous chapter a version of the following proposition appears (Proposition 4.2.4), by using that the set is isolated and transitive. Here, we prove the same conclusion without these hypotheses and with different tools.

Proposition 5.2.4. *Let Λ be a sectional hyperbolic set of a C^1 vector field X of M. Let σ be a Lorenz-like singularity of X in Λ. Let Y^n be a sequence of vector fields converging to X in the C^1 topology. Then, there are a neighborhood $U \subset M$ of Λ, a singular-cross section Σ^t, Σ^b of σ in M and $N \in \mathbb{N}$ enough large such that for every $n \geq N$ one has*

$$(\Lambda_{Y^n}) \cap (\partial^h \Sigma^t \cup \partial^h \Sigma^b) = \emptyset.$$

Proof. We fix the neighborhood U of Λ as in Lemma 4.2.1.

By using the Hausdorff's metric and by Remark (5.2.3) we have that there exists a subsequence of sectional hyperbolic sets in $(\Lambda_{Y^n})_{n\in\mathbb{N}}$ that converges to a compact invariant set in $Cl(U)$. Without loss of generality, we say that the sequence itself converges to a compact invariant set in $Cl(U)$, i.e.,

$$\Lambda_{Y^n} \xrightarrow{h} \tilde{\Lambda},$$

where $\tilde{\Lambda}$ is a compact invariant set and "$\cdot \xrightarrow{h} \cdot$" denotes the convergence on the Hausdorff distance.

Since $\Lambda_{Y^n} \xrightarrow{h} \tilde{\Lambda}$, given $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$ one has

$$d_H(\Lambda_{Y^n}, \tilde{\Lambda}) < \frac{\epsilon}{4}. \tag{5.1}$$

Let us prove that $\tilde{\Lambda} = \Lambda_X$. Firstly, for $m \in \mathbb{N}$, every vector field Y^n and X we define the following sets

$$\Lambda^m_{Y^n} = \cap_{|t|\leq m} Y^n_t(CL(U))) \quad \text{and} \quad \Lambda^m_X = \cap_{|t|\leq m} X_t(CL(U))). \tag{5.2}$$

show

By construction we have that $\Lambda^m_{Y^n} \xrightarrow{h} \Lambda_{Y^n}$ and $\Lambda^m_X \xrightarrow{h} \Lambda_X$ if $m \to \infty$.

Secondly, it follows from the (5.2) that there exists $M_1 = M_1(n) \in \mathbb{N}$ and $M_2 \in \mathbb{N}$ such that if $m \geq M_1$ and $k \geq M_2$ then

$$\begin{aligned} &d_H(\Lambda_{Y^n}, \Lambda^m_{Y^n}) < \tfrac{\epsilon}{4} \quad \text{and} \\ &d_H(\Lambda_X, \Lambda^k_X) < \tfrac{\epsilon}{4}. \end{aligned} \tag{5.3}$$

Since $Y^n \overset{C^1}{\to} X$, given $m \in \mathbb{N}$, there exists $N_2(m) \in \mathbb{N}$ such that if $n \geq N_2(m)$ one has

$$\begin{aligned} &\Longrightarrow \quad d_H(\cap_{|t|\leq m} Y^n_t(CL(U)), \cap_{|t|\leq m} X_t(CL(U))) < \tfrac{\epsilon}{4} \\ &\Longrightarrow \quad d_H(\Lambda^m_{Y^n}, \Lambda^m_X) < \tfrac{\epsilon}{4}. \end{aligned} \tag{5.4}$$

Let $M = \max\{M_1(N_1), M_2\} \in \mathbb{N}$ and $N = \max\{N_1, N_2(M(N_1))\}$. Thus, by using (5.1),(5.3) and (5.4) we have

$$\begin{aligned} d_H(\tilde{\Lambda}, \Lambda_X) &\leq d_H(\tilde{\Lambda}, \Lambda_{Y^N}) + d_H(\Lambda_{Y^N}, \Lambda_X) \\ &\leq d_H(\tilde{\Lambda}, \Lambda_{Y^N}) + d_H(\Lambda_{Y^N}, \Lambda^M_{Y^N}) + d_H(\Lambda^M_{Y^N}, \Lambda^M_X) \\ &\quad + d_H(\Lambda^M_X, \Lambda_X) \\ &< \epsilon, \end{aligned}$$

and it follows that $\tilde{\Lambda} = \Lambda_X$.

Since Λ_X is sectional hyperbolic, by using the Theorem 5.2.1 for Λ_X, the equality implies that the negative orbit of every point in $W^{ss}_X(\sigma) \setminus \{\sigma\}$ does not intersect $Cl(U)$. Hence, we can arrange a singular-cross section Σ^t, Σ^b nearby σ such that

$$\Lambda_X \cap (\partial^h \Sigma^t \cup \partial^h \Sigma^b) = \emptyset.$$

Since Λ_X is maximal invariant of $Cl(U)$ and the boundary of Σ^t, Σ^b is compact we can find $T > 0$ such that

$$X_T(Cl(U)) \cap (\partial^h \Sigma^t \cup \partial^h \Sigma^b) = \emptyset.$$

Thus, there exists $N \in \mathbb{N}$ such that for every $n \geq N$ one has

$$Y^n_T(Cl(U)) \cap (\partial^h \Sigma^t \cup \partial^h \Sigma^b) = \emptyset.$$

The result follows since $\Lambda_{Y^n} \subset Y^n_T(Cl(U))$. □

Remark 5.2.5. *We can observe that if there is $x \in \Lambda \setminus W^{ss}(\sigma)$ such that $\sigma \in \omega_X(x) \subset \Lambda$, then σ is Lorenz-like and satisfies the Theorem 5.2.1 [8, Theorem 4, Proposition 1], [26, Theorem 2.5, Proposition 2.6]. It shows that the outstanding dynamic of the system is found around of the Lorenz-like singularities. We shall see this in more detail in the next chapter.*

Corollary 5.2.6. *Let X be a C^1 vector field of a compact n-manifold M, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \subset M$ be a sectional hyperbolic set of X. Let σ be a singularity of X in Λ. Then, there is a neighborhood V of $W^{ss}(\sigma) \setminus \{\sigma\}$ such that*

$$\Lambda \cap V = \emptyset.$$

Proof. The equality in Theorem 5.2.1 implies that the negative orbit of every point in $W^{ss}_X(\sigma)\setminus\{\sigma\}$ does not intersect Λ. Given $x \in (W^{ss}(\sigma)\setminus\{\sigma\})$, we denote by l_x the distance of x to Λ, i.e., $l_x = d(x, \Lambda)$. Then, we define the neighborhood V as follows

$$V = \bigcup_{x \in W^{ss}(\sigma)\setminus\{\sigma\}} B(x, \frac{l_x}{2}).$$

Thus, by construction V satisfies that $\Lambda \cap V = \emptyset$ and the proof follows. □

5.3 Finiteness

We start by recalling some useful definitions to prove lemmas and propositions that provide very important properties on sectional hyperbolic sets, that in our case support the main theorems' proofs.

Henceforth, for $\delta > 0$ we define $B_\delta(A) = \{x \in M : d(x, A) < \delta\}$, where $d(\cdot, \cdot)$ is the metric in M.

Lemma 5.3.1. *Let X be a C^1 vector field of a compact n-manifold M, $X \in \mathcal{X}^1(M)$. Let $\Lambda \in M$ be a hyperbolic set of X. Then, there are a neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X, a neighborhood $U \subset M$ of Λ and $n_0 \in \mathbb{N}$ such that*

$$\#\{L \subset U : L \text{ is homoclinic class of } Y \in \mathcal{U}\} \leq n_0,$$

for every vector field $Y \in \mathcal{U}$.

Proof. We prove the Lemma by contradiction. So, we suppose that there exists a sequence of vector fields $(X^n)_{n\in\mathbb{N}} \subset \mathcal{U}$, $X^n \overset{C^1}{\to} X$ and such that

$$\#\{L \subset U : L \text{ is homoclinic class of } X^n\} \geq n.$$

It is well known [22] that the periodic orbits are dense in $L \subset \Lambda^n = \Lambda_{X^n}$, for all $n \in \mathbb{N}$. Recall that the homoclinic classes are pairwise disjoint on hyperbolic sets.

Let us consider $\eta > 0$ such that $0 < \eta < \frac{\epsilon}{2}$, where $\epsilon > 0$ is given by (2.1). Let $\cup_{x\in Cl(U)} B(x, \frac{\eta}{2})$ be the collection of open balls of radius $\frac{\eta}{2}$ covering $Cl(U)$.
Since $Cl(U)$ is a compact neighborhood of Λ, this one admits a finite sub-coverage, i.e., there exists some sub-collection consisting only of finitely many open balls of radius $\frac{\eta}{2}$ which also covers $Cl(U)$. We denote this finite number by n_0.

It follows from definition of homoclinic class and (2.1) that given points p_1 and p_2 of L, if $d(p_1, p_2) < \eta$ then

$$W^{ss}_X(p_1, \epsilon) \cap W^{uu}_X(p_2, \epsilon) \neq \emptyset. \tag{5.5}$$

In particular, if we choose $N \in \mathbb{N}$ such that $N > n_0$, we have that the sequence's element X^N exhibits more than N homoclinic classes in $Cl(U)$. Since $Cl(U)$ is covered by n_0 balls and $N > n_0$, it states that X^N exhibits at least two homoclinic classes contained at same $\frac{\eta}{2}$-ball. We denote by L^N_1 and L^N_2 these ones homoclinic classes.

Since L^N_1 and L^N_2 are homoclinic classes, there are periodic points p^1 and p^2 of L^N_1 and L^N_2 respectively satisfying (5.5), and it states that p^1 and p^2 belongs to the same homoclinic class. Therefore, it shows that $L^N_1 = L^N_2$.

Then, we obtain that X^N exhibits finite homoclinic classes in U. This is a contradiction and the proof follows. □

Lemma 5.3.2. *Let X be a C^1 vector field of a compact n-manifold M, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \in M$ be a sectional hyperbolic set of X. Let Y^n be a sequence of vector fields converging to X in the C^1 topology. Then, there is a neighborhood $U \subset M$ of Λ, such that if R^n is a repeller of Y^n, $R^n \subset \Lambda_{Y^n}$ for each $n \in \mathbb{N}$, then the sequence $(R^n)_{n\in\mathbb{N}}$ of repellers does not accumulate on the singularities of X, i.e.,*

$$Sing(X) \bigcap (\cap_{N>0} Cl(\cup_{m\geq N} R^n)) = \emptyset.$$

Proof. Fix the neighborhood U of Λ as in Lemma 4.2.1.
Assume by contradiction that

$$Sing(X) \bigcap (\cap_{N>0} Cl(\cup_{m\geq N} R^n)) \neq \emptyset.$$

Then, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$, with $x_{n_k} \in R^{n_k} \subset \Lambda_{Y^{n_k}}$ for all $k \in \mathbb{N}$ and such that $x_{n_k} \longrightarrow \sigma$, $\sigma \in Sing(X)$. Without loss of generality, we can suppose that the sequence itself converges to σ.
Let $\epsilon > 0$ be given by (2.1). As $x_n \longrightarrow \sigma$,

$$W^{ss}_{Y^n}(x^n, \epsilon) \longrightarrow W^{ss}_X(\sigma, \epsilon) \tag{5.6}$$

in the sense of C^1 manifolds [45]. Then, for $N \in \mathbb{N}$ enough large, and by using the Corollary 5.2.6 we have that

$$W^{ss}_{Y^N}(x^N, \epsilon) \cap V \neq \emptyset. \tag{5.7}$$

By using the Hausdorff's metric and by Remark (5.2.3) we have that the repellers sequence converges to a compact invariant set in Λ_X, i.e.,

$$R^n \xrightarrow{h} R, \tag{5.8}$$

where R is a compact invariant set and "$\cdot \xrightarrow{h} \cdot$" denotes the convergence on the Hausdorff distance. Thus $R \subset \Lambda_X$.
Since that $W^{ss}(x^n, \epsilon)$ is included in R^n, we have that the limit $W^{ss}(\sigma, \epsilon)$ is included in R ((5.6) and (5.8)).

By using (5.7) and (5.8), we obtain that $W^{ss}(\sigma, \epsilon) \subset R \subset \Lambda_X$ and therefore $\Lambda_X \cap V \neq \emptyset$. This is a contradiction.

□

Proposition 5.3.3. *Let X be a C^1 vector field of a compact n-manifold M, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \subset M$ be a sectional hyperbolic set of X. Then, there are neighborhoods $\mathcal{U}$ of X, U of Λ and $n_0 \in \mathbb{N}$ such that*

$$\#\{A \subset U : A \text{ is an attractor of } Y \in \mathcal{U}\} \leq n_0.$$

Proof. First, we fix the neighborhood U of Λ as in Lemma 4.2.1. We prove the proposition by contradiction. So, we suppose that for $n \in \mathbb{N}$, one has that for all neighborhood $\mathcal{U}$ of X there exists $Y \in \mathcal{U}$ such that

$$\#\{A \subset U : A \text{ is an attractor of } Y \in \mathcal{U}\} \geq n.$$

Then, we consider a sequence of vector fields $(X^n)_{n\in\mathbb{N}}$, such that $X^n \overset{C^1}{\to} X$, each one vector field X^n exhibiting an attractor A^n in U. So, we have a sequence of attractors $(A^n)_{n\in\mathbb{N}}$ in U. By compactness we can suppose that the attractors are non-singular, since the singularities are isolated.

We assert that the sequence $(A^n)_{n\in\mathbb{N}}$ of attractors accumulates on the singularities of X, otherwise if $Sing(X) \bigcap (\cap_{N>0} Cl(\cup_{m\geq N} A^n)) = \emptyset$, then it could exist $\delta > 0$, such that $B_\delta(Sing(X)) \bigcap (\cup_{n\in I\!N} A^n) = \emptyset$.

Thus, we define

$$H = \cap_{t\in I\!R} X_t \left(U \setminus B_{\delta/2}(Sing(X))\right). \tag{5.9}$$

By definition H is a invariant set and $Sing(X) \cap H = \emptyset$. Additionally, H is a compact set as Λ is too, and therefore H is a nonempty compact set ([28, Lemma 3.2], [25, Lemma 4.1]). It follows from the Lemma 4.2.1 that H is a hyperbolic set and by using the Lemma 5.3.1 there is $n_0 \in \mathbb{N}$ such that the sequence of attractors is bounded by n_0, that is a contradiction.

Then, the sequence $(A^n)_{n\in\mathbb{N}}$ of attractors accumulates on the singularities of X, i.e., $Sing(X)\bigcap(\cap_{N>0}Cl(\cup_{m\geq N}A^n)) \neq \emptyset$. Thus, there exists $\sigma \in U$ such that $\sigma \in Sing(X)\bigcap(\cap_{N>0}Cl(\cup_{m\geq N}A^n))$.

Recall that we use in the previous chapter the transitivity hypothesis exactly for obtain attractors accumulation on the singularities. Thus, in the same way of the Theorem A, we obtain that the last equality implies that there is $n_1 \in \mathbb{N}$ such that

$$A^{n_1} \cap l^t \neq \emptyset.$$

Since $l^t \subset W^s_{X^{n_1}}(\sigma)$ and A^{n_1} is a closed invariant set for X^{n_1}, we obtain that $\sigma \in A^{n_1}$. By hypothesis A^n is non-singular for all $n \in \mathbb{N}$, so this leads to a contradiction and the proof follows.

□

5.3.1 Existence

The following lemma will be useful for the existence's proof related with repellers on sectional hyperbolic sets.

Lemma 5.3.4. *Let X be a C^1 vector field of a compact n-manifold M, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \in M$ be a connected sectional hyperbolic set of X with singularities. If $\Lambda \subset \Omega(X)$, then Λ has no repellers.*

Proof. We prove the lemma by contradiction. Since Λ has singularities, it cannot be a repeller. Then, we assume that Λ at least contains a repeller that will be denote by R. It follows from definition of repeller that there exists U isolant block negatively invariant of R, such that $R = \cap_{t\leq 0}X_t(U)$, where $U \subset \Lambda$.

Since Λ is a connected set and $R \subset \Lambda$, we can pick $p \in \Lambda$ such that $p \in int(U) \setminus R$. So, we assert that there exists time $\tau < 0$ such that $p \notin X_\tau(U)$ and therefore $X_{-\tau}(p) \notin U$. As R is compact, there is an open neighborhood $V \subset int(U) \setminus R$ of p such that $X_{-\tau}(V)$

is not contained in U. By construction $p \in V \subset \Omega(X)$, thus we can choose time $T > 0$ with $T > -\tau > 0$, such that $X_T(V) \cap V \neq \emptyset$. So, there exists $q \in X_T(V) \cap V$ and one has

$$\begin{array}{llll} \Rightarrow & q \in V & \text{and} & q \in X_T(V) \\ \Rightarrow & q \in V & \text{and} & X_{-T}(q) \in V \\ \Rightarrow & X_{-\tau}(q) \in X_{-\tau}(V) & \text{and} & X_{-\tau}(X_{-T}(q)) \in X_{-\tau}(V) \\ \Rightarrow & X_{-\tau-T}(q) \in X_{-\tau}(V) \not\subseteq U. & & \end{array}$$

This shows that in particular $X_{-T-\tau}(q) \notin U$ with $-T-\tau < 0$. However, $q \in U$ and given that $X_t(q) \in U$ for all $t \leq 0$, this is a contradiction. □

5.4 Proof of the main theorems

5.4.1 Proof of Theorem B

Proof. We prove the theorem by contradiction. Let X be a C^1 vector field of a compact n-manifold M, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \in M$ be a sectional hyperbolic set of X. Then, we suppose that there is a sequence of vector fields $(X^n)_{n\in\mathbb{N}} \subset \mathcal{X}^1(M)$, $X^n \overset{C^1}{\to} X$ such that every vector field X^n exhibits n attractors or repellers, with $n > n_0$. It follows from the Proposition 5.3.3 that there are neighborhoods $\mathcal{U} \subset \mathcal{X}^1(M)$ of X and $U \subset M$ of Λ such that the attractors in U are finite for all vector field Y in $\mathcal{U}$. Thus, we are left to prove only for the repeller case. We denote by R^n a repeller of X^n in Λ_{X^n}. Since Λ_{X^n} is arbitrarily close to Λ_X and $R^n \in \Lambda_{X^n}$, R^n is also arbitrarily close to Λ_X. Therefore, we can assume that R^n belongs to Λ_X for all n.

Let $(R^n)_{n\in\mathbb{N}}$ be the sequence of repellers contained in Λ_X. By using the Lemma 5.3.2 we have that

$$Sing(X) \bigcap (\cap_{N>0} Cl(\cup_{m\geq N} R^n)) = \emptyset.$$

Then, we can find $\delta > 0$, such that $B_\delta(Sing(X)) \bigcap (\cup_{n\in\mathbb{N}} R^n) = \emptyset$.

As in (5.9) we define $H = \bigcap_{t\in I\!R} X_t\left(U \setminus B_{\delta/2}(Sing(X))\right)$. It follows from the Lemma 4.2.1 that H is a hyperbolic set and beside the Lemma 5.3.1 we have that there are neighborhoods $\mathcal{U} \subset \mathcal{X}^1(M)$ of X, $U \subset M$ of H and $n_1 \in \mathbb{N}$ such that

$$\#\{R \subset U : R \text{ is a repeller of } Y \in \mathcal{U}\} \leq n_1 \leq n_0.$$

Note that the last inequality holds for every vector field $Y \in \mathcal{U}$, but this leads to a contradiction, since by hypothesis we have that

$$\#\{R \subset H : R \text{ is a repeller of } Y \in \mathcal{U}\} \geq n > n_0.$$

□

5.4.1.1 Proof of the corollaries

Proof of Corollary 5.1.1

Proof. Since X is a sectional Anosov flow, then its maximal invariant $M(X)$ is a sectional hyperbolic set for X. By using the Theorem B for $M(X)$ the proof follows.

□

Proof of Corollary 5.1.2

Proof. From [16], we have that a sectional hyperbolic three dimensional flows is open and dense in the open set of flows which are far from homoclinic tangencies. By using Theorem B we obtain finiteness of attrators and repellers and the proof follows.

□

5.4.2 Proof of Theorem C

Proof. Before starting the proof of Theorem C, recall the Definition 5.2.2 and Remark 5.2.3. We prove the theorem by contradiction. It follows from the Lemma 5.3.4 that Λ has no repellers.

Then we suppose that for all neighborhood U of Λ, there exists a vector field C^1 close to X exhibiting a repeller in U. Thus, we begin by considering a sequence of the vector fields $(X^n)_{n\in\mathbb{N}} \subset \mathcal{X}^1(M)$, with $X^n \overset{C^1}{\to} X$ and each one is exhibiting a repeller R^n in U.

So, using the Lemma 5.3.2 we have that $Sing(X) \bigcap (\cap_{N>0} Cl(\cup_{m\geq N} R^n)) = \emptyset$. Then, we can find $\delta > 0$ such that $B_\delta(Sing(X)) \bigcap (\cup_{n\in\mathbb{N}} R^n) = \emptyset$.

We define $H = \bigcap_{t\in I\!R} X_t \left(U \setminus B_{\frac{\delta}{2}}(Sing(X))\right)$ and hence we can assume that H is a hyperbolic set [25]. Beside definition of H and from the Lemma 5.3.2, we can suppose that the sequence $(R^n)_{n\in\mathbb{N}}$ is contained in U.

By using the Hausdorff's metric and by Remark (5.2.3) we have that the repellers sequence converges to a compact invariant set in H, i.e.,

$$R^n \overset{h}{\to} R, \tag{5.10}$$

where R is a compact invariant set. Therefore $R \subset H \subset \Lambda$.

As R^n is a hyperbolic set, we can choose periodic point $p^n \in R^n$ for every $n \in \mathbb{N}$ and it follows that the sequence $(p^n)_{n\in\mathbb{N}}$ converges to a point $p \in R$. From the hyperbolicity H and as $X^n \longrightarrow X$, we have that $W^{ss}_{X^n}(p^n, \epsilon) \longrightarrow W^{ss}_X(p, \epsilon)$ in the sense of C^1 submanifolds [45], where $\epsilon > 0$ is given by (2.1).

It is well known that the repeller sets contain the stable manifold of all its points. So, $W^{ss}_{X^n}(p^n, \epsilon) \subset R^n$ for all $n \in \mathbb{N}$ and by (5.10), we have that $W^{ss}_X(p, \epsilon) \subset W^s_X(O_p) \subset R$, since R is a compact invariant set.

From the above we obtain $Cl(W^s_X(O_p)) \subset R$ and in particular $Cl(W^s_X(O_p))$ is a hyperbolic set contained in $R \subset \Lambda \subset \Omega(X)$, which can be used to construct a hyperbolic repeller inside R. Specifically, the α-limit set $\alpha(W^s_X(O_p))$ would be such a set and note that $\alpha(W^s_X(O_p)) \subset Cl(W^s_X(O_p)) \subset R$. Finally, there is a hyperbolic repeller contained in $\Lambda \subset \Omega(X)$.

This is a contradiction by Lemma 5.3.4 and the result follows.

□

Chapter 6

Existence of periodic orbits for sectional Anosov flows

6.1 Introduction

A well known problem in dynamics is to investigate the existence of periodic orbits for flows on compact manifolds. This problem has a satisfactory solution under certain circunstancies. In fact, every Anosov flow of a compact manifold has not only one but infinitely many periodic orbits instead.

In this Chapter we shall investigate this problem not for Anosov but for the sectional Anosov flows introduced in [31]. It is known for instance that every sectional Anosov flow of a compact 3-manifold has a periodic orbit, this was proved in [8]. In the transitive case (i.e. with a dense orbit in the maximal invariant set) it is known that the maximal invariant set consists of a homoclinic class and, therefore, the flow has infinitely many periodic orbits [2]. Another relevant result by Reis [47] proves the existence of infinitely many periodic orbits under certain conditions. Our goal here is to extend [8] to the higher dimensional setting. More precisely, we shall prove that every sectional Anosov flow (or, equivalently, every sectional hyperbolic attracting set of a flow) on a compact manifold has a periodic orbit.

Let us state our result in a precise way.

Theorem D. *Every sectional Anosov flow on a compact manifold has a periodic orbit.*

An equivalent version of this result is as follows.

Given $\Lambda \in M$ compact, recall that Λ is an *attracting* set if $\Lambda = \cap_{t>0} X_t(U)$ for some compact neighborhood U of it, where this neighborhood is often called *isolating block*. It is well known that the isolating block U can be chosen to be positively invariant, i.e., $X_t(U) \subset U$ for all $t > 0$. We call a sectional hyperbolic set with the above property as a sectional hyperbolic attracting.

Thus, the Theorem D is equivalent to the result below.

Theorem E. *Every sectional hyperbolic attracting set of a C^1 vector field on a compact manifold has a periodic orbit.*

Theorem D will be proved extending the arguments in [8] to the higher dimensional setting. Indeed, in Section 6.2 we provide useful definitions for Lorenz-like singularity, singular-cross sections and triangular maps for the higher dimensional case (n-triangular maps for short) in the sectional hyperbolic context. Also, we extend some definitions, lemmas and propositions necessaries for the next sections. In Section 6.3 we give sufficient conditions for a hyperbolic n-triangular map to have a periodic point. In Section 6.4 we prove that hyperbolic n-triangular maps satisfying these hypotheses have a periodic point and we prove the Theorem D.

6.2 Triangular maps in higher dimensions

In this section, we recall some definitions of Chapter 3 behind to Lorenz-like singularities (Definition 3.1.1) and singular cross-sections (Definition 3.2.1) in the sectional hyperbolic context. Also, we set certain maps defined on a finite disjoint union of these singular cross-sections. As in [8], these ones are discontinuous maps still preserving the continuous foliation (but not necessarily constant). Thus, on compact manifolds of dimension $n \geq 3$, particularly to group of these maps we shall call them n-triangular maps.

6.2.1 Preliminaries and useful results

Considering the definitions of Chapter 3, we show the following lemmas presenting an elementary but very useful dichotomy for the singularities of a sectional hyperbolic sets. Also, it will be necessary recall the Hyperbolic lemma 4.2.1 and the Theorem 4.2.2 in Chapter 4.

The following results appear in [8], but these versions are a modification for the sectional hyperbolic scenario in the higher dimensional case.

Theorem 6.2.1. *Let X be a sectional Anosov flow of M. Let σ be a singularity of $M(X)$. If there is $x \in M(X) \setminus W^{ss}(\sigma)$ such that $\sigma \in \omega_X(x)$, then σ is Lorenz-like and satisfies*

$$M(X) \cap W^{ss}_X(\sigma) = \{\sigma\}.$$

Proof. The equality follows from Theorem 4.2.2. We assume that $M(X)$ is connected for, otherwise, we consider the connected components. Suppose that $\sigma \in M(X) \cap Sing(X)$ satisfies $\sigma \in \omega_X(x)$ for some $x \in M(X) \backslash W^{ss}(\sigma)$. Let us prove that σ is Lorenz-like. Since $M(X)$ is maximal invariant of X we have $\omega_X(x) \in M(X)$ and so $\sigma \in M(X)$. Assume by contradiction that σ is not Lorenz-like. Then, by (3.1) we have that $dim(W^{ss}(\sigma)) =$

$dim(W^s(\sigma))$ and so $W^{ss}(\sigma) = W^s(\sigma)$. Since $x \notin W^{ss}(\sigma)$, one has $\omega_X(x) \cap (W^s(\sigma) \setminus \{\sigma\}) \neq \emptyset$. But recall that $\omega_X(x) \in M(X)$ and beside with $W^{ss}(\sigma) = W^s(\sigma)$, we obtain that

$$M(X) \cap (W^{ss}_X(\sigma) \setminus \{\sigma\}) \neq \emptyset,$$

contradicting the equality in Theorem 4.2.2. This proves the result.

□

Proposition 6.2.2. *Let X be a sectional Anosov flow of M. If $M(X)$ has no Lorenz-like singularities, then $M(X)$ has a periodic orbit.*

Proof. Let x be a point in $M(X) \setminus Sing(X)$. We claim that $\omega_X(x)$ has no singularities. Indeed, suppose by contradiction that $\omega_X(x)$ has a singularity σ. By hypothesis $M(X)$ has no Lorenz-like singularities and so σ is not Lorenz-like too. Hence $W^{ss}(\sigma) = W^s(\sigma)$ by (3.1), and by using the Theorem 4.2.2 one has

$$M(X) \cap (W^s_X(\sigma) \setminus \{\sigma\}) = \emptyset.$$

It follows in particular that $(W^s_X(\sigma) \setminus \{\sigma\})$ does not belong to $M(X)$. Since $x \in (M(X) \setminus Sing(X))$ we conclude that $x \notin W^s_X(\sigma) = W^{ss}_X(\sigma)$. It follows from Theorem 6.2.1 that σ is Lorenz-like. This is a contradiction and the claim follows.

Now we conclude the proof of the proposition. Clearly $\omega_X(x) \subset M(X)$ since $M(X)$ is compact. The claim and Lemma 4.2.1 imply that $\omega_X(x)$ is a hyperbolic set. It follows from the Shadowing Lemma for flows [22] that there is a periodic orbit of X_t close to $\omega_X(x)$. Since $M(X)$ is the maximal invariant, we have that such a periodic orbit is contained in $M(X)$. Then we obtain the result. □

6.2.2 n-Triangular maps

We begin by reminding the three-dimensional case [8], where the authors choose the cross-sections as copies of $[0, 1] \times [0, 1]$ and they define maps on a finite disjoint union of these one copies called *triangular maps*. This concept is frequently used by maps on $[0, 1] \times [0, 1]$ preserving the constant vertical foliation. Also, they assume two hypotheses imposing certain amount of differentiability close to the points whose iterates fall eventually in the interior of Σ.

For the higher dimensional case, we will define a certain maps on a finite disjoint union of singular cross-sections Σ. Here, we will modify the triangular map's definition and we will impose some suitable properties in order to define a *triangular hyperbolic map*. These maps could be discontinuous and they will preserve still the continuous foliation (but not necessarily constant). Thus, on compact manifolds of dimension $n \geq 3$, particularly to group of these maps we shall call them *n-triangular maps*.

By using the definitions about singular cross-sections, we will provide the following definitions.

Definition 6.2.3. *Let Σ be a disjoint union of finite singular cross-sections Σ_i, $i = 1, \ldots, k$. . We denote by l_{0_i} to the singular leaf of the singular cross-section Σ_i. Here L_0 stands the union of singular leaves, i.e.,*

$$L_0 = \bigcup_{i=1}^{n} l_{0_i}.$$

Recall that $\partial^v \Sigma_i$ is the union of the boundary submanifolds which are parallel to l_{0_i}. In the same way we set by $\partial^v \Sigma$ as

$$\partial^v \Sigma = \bigcup_{i=1}^{n} \partial^v \Sigma_i.$$

Hereafter, given a map F we will denote its domain by $Dom(F)$.

Definition 6.2.4. *Let $F : Dom(F) \subset \Sigma \to \Sigma$ be a map and x a point in $Dom(F)$. We say that x is a* periodic point *of F if there is $n \geq 1$ such that $F^j(x) \in Dom(F)$, for all $0 \leq j \leq n-1$ and $F^n(x) = x$.*

Definition 6.2.5. *We say that a submanifold c of Σ is a* k-surface *if it is the image of a C^1 injective map $c : Dom(c) \subset \mathbb{R}^k \to \Sigma$, with $Dom(c)$ being I^k and $k \leq n-1$. For simplicity, hereafter c stands the image of this one map. A k-surface c is* vertical *if it is the graph of a C^1 map $g : I^{n-k-1} \to I^k$, i.e., $c = \left\{(g(y), y) : y \in I^k\right\} \subset \Sigma$.*

Definition 6.2.6. *A continuous foliation $\mathcal{F}_i$ on a component Σ_i of Σ is called* vertical *if its leaves are vertical s-surfaces and $\partial^v\Sigma \subset \mathcal{F}_i$, where $s = dim(B^{ss}[0,1])$. A* vertical foliation *$\mathcal{F}$ of Σ is a foliation which restricted to each component Σ_i of Σ is a vertical foliation.*

It follows from the definition above that the leaves L of a vertical foliation $\mathcal{F}$ are vertical s-surfaces, hence differentiable ones. In particular, the tangent space T_xL is well defined for all $x \in L$.

Remark 6.2.7. *Note that, given a singular cross-section Σ equipped with a vertical foliation $\mathcal{F}$, one has that $dim(\mathcal{F}) = dim(B^{ss}[0,1]) = s$, and each leaf L of $\mathcal{F}$ has the same dimension of $W^{ss}(\sigma)$, being σ the Lorenz-like singularity associated to Σ.*

Definition 6.2.8. *Let $F : Dom(F) \subset \Sigma \to \Sigma$ be a map and $\mathcal{F}$ be a vertical foliation on Σ. We say that F* preserves *$\mathcal{F}$ if for every leaf L of $\mathcal{F}$ contained in $Dom(F)$ there is a leaf $f(L)$ of $\mathcal{F}$ such that $F(L) \subset f(L)$ and the restricted map $F/_L : L \to f(L)$ is continuous.*

If $\mathcal{F}$ is a vertical foliation on Σ a subset $B \subset \Sigma$ is a *saturated set* for $\mathcal{F}$ if it is an union of leaves of $\mathcal{F}$. We shall write $\mathcal{F}$-saturated for short.

Definition 6.2.9 (**n-Triangular map**). *A map $F : Dom(F) \subset \Sigma \to \Sigma$ is called* n-triangular *if it preserves a vertical foliation $\mathcal{F}$ on Σ such that $Dom(F)$ is $\mathcal{F}$-saturated and $dim(\Sigma) = n-1$, with $n \geq 3$. Note that a 3-triangular map is the classical triangular map.*

6.2.3 Hyperbolic n-triangular maps

In the same way of [8], in order to find periodic points for n-triangular maps, we also introduce some kind of hyperbolicity for these maps. The hyperbolicity will be defined through cone fields in Σ and we denote by $T\Sigma$ the tangent bundle of Σ. Given $x \in \Sigma$, $\alpha > 0$ and a linear subspace $V_x \subset T_x\Sigma$, we denote by $C_\alpha(x, V_x) \equiv C_\alpha(x)$ the cone around V_x in $T_x\Sigma$ with inclination α, namely

$$C_\alpha(x) = \{v_x \in T_x\Sigma : \angle(v_x, V_x) \leq \alpha\}.$$

Here, $\angle(v_x, V_x)$ denotes the angle between a vector v_x and the subspace V_x. A *cone field* in Σ is a continuous map $C_\alpha : x \in \Sigma \to C_\alpha(x) \subset T_x\Sigma$, where $C_\alpha(x)$ is a cone with constant inclination α on $T_x\Sigma$. A cone field C_α is called *transversal* to a vertical foliation $\mathcal{F}$ on Σ if T_xL is not contained in $C_\alpha(x)$ for all $x \in L$ and all $L \in \mathcal{F}$.

Now we can define hyperbolic n-triangular map.

Definition 6.2.10 (**Hyperbolic n-triangular map**). *Let $F : Dom(F) \subset \Sigma \to \Sigma$ be a n-triangular map with associated vertical foliation $\mathcal{F}$. Given $\lambda > 0$ we say that F is* λ-hyperbolic *if there is a cone field C_α in Σ such that:*

1. *C_α is transversal to $\mathcal{F}$.*
2. *If $x \in Dom(F)$ and F is differentiable at x, then*

$$DF(x)(C_\alpha(x)) \subset Int(C_{\alpha/2}(F(x)))$$

and

$$\| DF(x) \cdot v_x \| \geq \lambda \cdot \| v_x \|,$$

for all $v_x \in C_\alpha(x)$.

6.3 Periodic points for hyperbolic n-triangular maps

In the three dimensional case [8], the authors impose certain conditions or properties so-called hypotheses (H1) and (H2) on triangular maps. With this conditions the periodic point arose on the triangular map. Therefore, the general tools for searching our definitions is trying and reproducing these hypotheses, in the higher dimensional setting. Here, the topology turns out to play a significant role in this extension, imposing certain restrictions on the manifolds, kind of foliations and singular cross-sections one may have. In this section we give sufficient conditions for a hyperbolic n-triangular map to have a periodic point.

6.3.1 Hypotheses (A1)-(A2)

They impose some regularity around those leaves whose iterates *eventually fall into* $\Sigma \setminus (\partial^v \Sigma)$. To state them we will need the following definition. If $\mathcal{F}$ is foliation we use the notation $L \in \mathcal{F}$ to mean that L is a leaf of $\mathcal{F}$.

Definition 6.3.1. *Let* $F : Dom(F) \subset \Sigma \to \Sigma$ *be a triangular map such that* $\partial^v \Sigma \subset Dom(F)$. *For all* $L \in \mathcal{F}$ *contained in* $Dom(F)$ *we define the (possibly* ∞*) number* $n(L)$ *as follows:*

1. *If* $F(L) \subset \Sigma \setminus (\partial^v \Sigma)$ *we define* $n(L) = 0$.
2. *If* $F(L) \subset \partial^v \Sigma$ *we define*

$$n(L) = \sup\{n \geq 1 : F^i(L) \subset Dom(F) \text{ and}$$

$$F^{i+1}(L) \subset \partial^v \Sigma, \forall 0 \leq i \leq n-1\}.$$

Essentially $n(L)+1$ gives the first non-negative iterate of L falling into $\Sigma \setminus (\partial^v\Sigma)$.

Remark 6.3.2. *On the other hand, note that if $n(L_*) \geq 1$ for $L_* \in \mathcal{F}$ contained in $Dom(F)$, for every neighborhood S of L_*, $F(S) \cap \partial^v\Sigma \neq \emptyset$. We denote*

$$V_{L_*}(S) \equiv V_{L_*} = F(S) \cap \partial^v\Sigma.$$

Therefore $V_{L_*} \neq \emptyset$ and if $L_* \notin \partial^v\Sigma$, V_{L_*} splits $F(S)$ in two connected components S_1', S_2'. It shows that there exists three connected components S_0, S_1, S_2 of S such that (See 6.2)

$$S = S_0 \cup S_1 \cup S_2 \text{ and}$$
$$F(S_0) = V_{L_*},\ F(S_1) = S_1',\ \text{and } F(S_2) = S_2'.$$

Given $L \in \mathcal{F}$ contained in $Dom(F)$, the number $n(L)$ and the neighborhood V_{L_*} play fundamental role in the following definition.

Definition 6.3.3 (Hypotheses (A1)-(A2)). *Let $F\colon Dom(F) \subset \Sigma \to \Sigma$ be a n-triangular map such that $\partial^v\Sigma \subset Dom(F)$. We say that F satisfies:*

(A1) *If $L \in \mathcal{F}$ satisfies $L \subset Dom(F)$ and $n(L) = 0$, then there is a $\mathcal{F}$-saturated neighborhood S of L in Σ such that the restricted map $F/_S$ is C^1. (See Figure 6.1).*

(A2) *If $L_* \in \mathcal{F}$ satisfies $L_* \subset Dom(F)$, $1 \leq n(L_*) < \infty$ and*

$$F^{n(L_*)}(L_*) \subset Dom(F),$$

then there is a connected neighborhood $S \subset Dom(F)$ of L_* such that $S = S_0 \cup S_1 \cup S_2$ (see Remark (6.3.2)) and the connected components S_1, S_2 (possibly equal if $L_* \subset \partial^v\Sigma$) beside V_{L_*} satisfying the properties below:

1. *Both $F(S_1)$ and $F(S_2)$ are contained in $\Sigma \setminus (\partial^v\Sigma)$.*

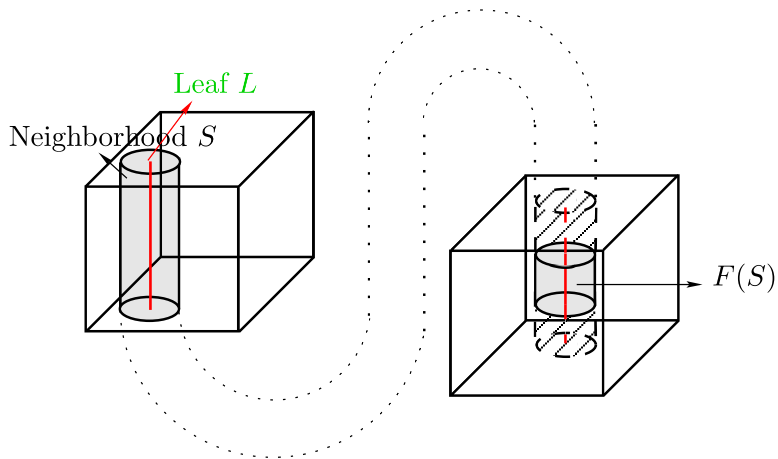


Figure 6.1: Hypothesis A_1

2. *$F^i(V_{L_*}) \subset \partial^v\Sigma$ for all $0 \leq i \leq n(L_*)$, i.e., as V_{L_*} is $\mathcal{F}$-saturated, for any $L' \in V_{L_*}$ one has that $n(L') = n(L_*) - 1$ (or $n(L') = n(L_*)$ for the case $L_* \in \partial^v\Sigma$)*

3. *For each $j \in \{0, 1, 2\}$, there is $0 \leq n^j(L_*) \leq n(L_*) + 1$ such that if $y_l \in S_j$ is a sequence converging to $y \in L_*$, then $F(y_l)$ is a sequence converging to $F^{n^j(L_*)}(y)$. If $n^j(L_*) = 1$, then F is C^1 in $S_j \cup V_{L_*}$. Note that $n^0(L_*) = n(L_*)$ by 2.*

4. *If $L_* \subset \Sigma \setminus (\partial^v\Sigma)$ (and so $S_1 \neq S_2$), then either $n^1(L_*) = 1$ and $n^2(L_*) > 1$ or $n^1(L_*) > 1$ and $n^2(L_*) = 1$. (See Figure 6.2).*

6.3.2 Existence on hyperbolic n-triangular maps

The following theorem will deal with the existence of periodic points for hyperbolic n-triangular maps satisfying (A1) and (A2). Since the conditions (A1) and (A2) generalize the conditions in [8], also we have that the three dimensional Lorenz attractor return map is an example for us. Recall the Lorenz attractor return map has a periodic point and it is a λ-hyperbolic n-triangular map satisfying (A1) and (A2) with λ large and $Dom(F) = \Sigma \setminus L_0$. Indeed, the main motivation is show the following theorem for

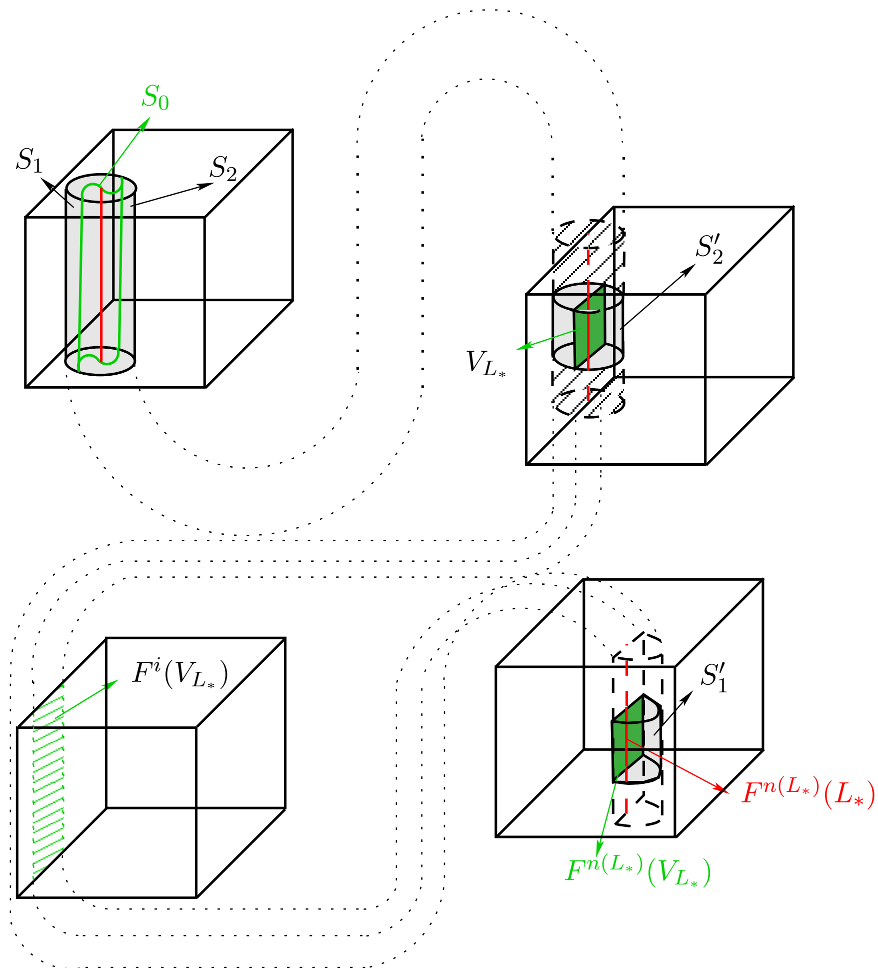


Figure 6.2: Hypothesis A_2

higher dimensional case. More precisely,

Theorem 6.3.4. *Let F be a λ-hyperbolic n-triangular map satisfying (A1) and (A2) with $\lambda > 2$ and $Dom(F) = \Sigma \setminus L_0$. Then, F has a periodic point.*

Since the proof of Theorem 6.3.4 is technical, we include some preliminaries for its proof and we will prove in the next section.

6.4 Existence of the periodic point

In this section we shall prove the Theorem 6.3.4. The proof follow the same way of [8]. We will extend and modify some results for the higher dimensional case. In Subsection 6.4.1 we present preliminary lemmas for the proof of the Theorem 6.3.4. In Subsection 6.5 we prove the theorem.

6.4.1 Preliminary lemmas

Hereafter we fix Σ as in Subsection 3.2. Let k be the number of components of Σ. We shall denote by SL the leaf space of a vertical foliation $\mathcal{F}$ on Σ. It turns out that SL is a disjoint union of k-copies $B_1^u[0,1], \cdots, B_k^u[0,1]$ of $B^u[0,1]$. We denote by $\mathcal{F}_B$ the union of all leaves of $\mathcal{F}$ intersecting B. If $B = \{x\}$, then $\mathcal{F}_x$ is the leaf of $\mathcal{F}$ containing x. If $S, B \subset \Sigma$ we say that S *cover* B whenever $B \subset \mathcal{F}_S$.

The lemma below quotes some elementary properties of $n(L)$ in Definition 6.3.1.

Lemma 6.4.1. *Let $F\colon Dom(F) \subset \Sigma \to \Sigma$ be a n-triangular map with associated vertical foliation $\mathcal{F}$. If $L \in \mathcal{F}$ and $L \subset Dom(F)$, then:*

1. *If F* has no periodic points *and $\partial^v\Sigma \subset Dom(F)$, then*
$$n(L) \leq 2k.$$
2. *$n(L) = 0$ if and only if $F(L) \subset \Sigma \setminus (\partial^v\Sigma)$.*
3. *$F^i(L) \subset \partial^v\Sigma$ for all $1 \leq i \leq n(L)$.*
4. *If $F^{n(L)}(L) \subset Dom(F)$, then $F^{n(L)+1}(L) \subset \Sigma \setminus (\partial^v\Sigma)$.*

If $F : Dom(F) \subset \Sigma \to \Sigma$ is a n-triangular map with associated foliation $\mathcal{F}$, then we also have an associated u-dimensional map

$$f : Dom(f) \subset SL \to SL.$$

This map allows us to obtain certain geometric properties for the singular cross-section whole. We use this map in the definition below.

Definition 6.4.2. *Let $F : Dom(F) \subset \Sigma \to \Sigma$ a triangular map with associated foliation $\mathcal{F}$ and $f : Dom(f) \subset SL \to SL$ its associated u-dimensional map. Then we define the following limit sets:*

$$\mathcal{V} = \{f(B) : B \in \mathcal{F}, B \subset Dom(F) \text{ and } B \subset \partial^v\Sigma\}.$$

$$\mathcal{L} = \bigcup \left\{ K_i : i \in \{1, ..., k\}, \lim_{L \to L_{0i}} f(L) \text{ exists and such that } K_i = \lim_{L \to L_{0i}} f(L) \right\}.$$

The lemma below is a direct consequence of **(A2)**.

Lemma 6.4.3. *Let $F\colon Dom(F) \subset \Sigma \to \Sigma$ a n-triangular map satisfying* **(A2)** *and $\mathcal{F}$ be its associated foliation. Let L_* be a leaf of $\mathcal{F}$, $L_* \subset Dom(F)$, $1 \leq n(L_*) < \infty$ and $F^{n(L_*)}(L_*) \subset Dom(F)$. If there is a sequence $(L_k)_{k\in\mathbb{N}}$ such that $L_k \to L_*$, then:*

(1) If $\#\{L : L \in (V_{L_*} \cap (F(L_k))_{k\in\mathbb{N}})\} = \infty$, then there exists a subsequence $(L_{k_l})_{l\in\mathbb{N}}$ such that $Lim_{l\to\infty} F(L_{k_l}) = F(L_*)$.

(2) If $\#\{L : L \in (S_1' \cap (F(L_k))_{k\in\mathbb{N}})\} = \infty$, then there exists a subsequence $(L_{k_l})_{l\in\mathbb{N}}$ such that $Lim_{l\to\infty} F(L_{k_l}) = F^{n_1(L_*)}(L_*)$.

(3) If $\#\{L : L \in (S_2' \cap (F(L_k))_{k\in\mathbb{N}})\} = \infty$, then there exists a subsequence $(L_{k_l})_{l\in\mathbb{N}}$ such that $Lim_{l\to\infty} F(L_{k_l}) = F^{n_2(L_*)}(L_*)$.

In each case the corresponding limits belong to

$$\partial^v\Sigma \cup \mathcal{V}.$$

Proof. By hypotheses F satisfies the property $(A2)$, so there is a connected neighborhood $S = S_0 \cup S_1 \cup S_2$ of L_*. If we have (1), as $L_k \to L_*$, one has that

$\#\{L : L \in (S_0 \cap (L_k)_{k\in\mathbb{N}})\} = \infty$. Then, there exists a subsequence $(L_{k_l})_{l\in\mathbb{N}} \subset S_0$ such that $L_{k_l} \to L_*$. Thus, $Lim_{l\to\infty}F(L_{k_l}) = F(L_*)$.

If we have (2) beside the (A2) property, in the same way one has that there exists a subsequence $(L_{k_l})_{l\in\mathbb{N}} \subset S_1$ such that $L_{k_l} \to L_*$ and $Lim_{l\to\infty}F(L_{k_l}) = F^{n_1(L_*)}(L_*)$. Analogously for (3). □

Given a map $F\colon Dom(F) \subset \Sigma \to \Sigma$ we define its *discontinuity set* $D(F)$ by

$$D(F) = \{x \in Dom(F) : F \text{ is discontinuous in } x\}. \tag{6.1}$$

In the sequel we derive useful properties of $Dom(F)$ and $D(F)$.

Lemma 6.4.4. *Let F be a n-triangular map satisfying* **(A1)**, *$F : Dom(F) \subset \Sigma \to \Sigma$ and $\mathcal{F}$ be its associated foliation. If $L \in \mathcal{F}$ and $L \subset D(F)$, then $F(L) \subset \partial^v\Sigma$.*

Proof. Suppose by contradiction that $L \subset D(F)$ and $F(L) \subset \Sigma\backslash(\partial^v\Sigma)$. These properties are equivalent to $n(L) = 0$ by Lemma 6.4.1-(2). Then, by using **(A1)**, there is a neighborhood of L in Σ restricted to which F is C^1. In particular, F would be continuous in L which is absurd. □

Lemma 6.4.5. *Let F be a n-triangular map satisfying* **(A1)-(A2)**, *$F : Dom(F) \subset \Sigma \to \Sigma$ and $\mathcal{F}$ be its associated foliation. If F has no periodic points and $\partial^v\Sigma \subset Dom(F)$, then $Dom(F) \setminus D(F)$ is $\mathcal{F}$-saturated, open in $Dom(F)$ and $F/_{(Dom(F)\setminus D(F))}$ is C^1.*

Proof. In order to prove the lemma, it suffices to show that $\forall x \in Dom(F) \setminus D(F)$ there is a neighborhood S of $\mathcal{F}_x$ in Σ such that $F/_S$ is C^1. To find S we proceed as follows. Fix $x \in Dom(F) \setminus D(F)$. As $Dom(F)$ is $\mathcal{F}$-saturated, one has $\mathcal{F}_x \subset Dom(F)$ and so $n(\mathcal{F}_x)$ is well defined. By using the Lemma 6.4.1-(1), one has

$$n(\mathcal{F}_x) < \infty.$$

If $n(\mathcal{F}_x) = 0$, then the neighborhood S of $L = \mathcal{F}_x$ in **(A1)** works.

By simplicity, if $n(\mathcal{F}_x) \geq 1$ let us denote $L_* = \mathcal{F}_x$. Clearly $1 \leq n(L_*) < \infty$ and Definition 6.3.1 of $n(L_*)$ implies $f^{n(L_*)}(L_*) \subset \partial^v\Sigma$. By hypothesis $\partial^v\Sigma \subset Dom(F)$ and then

$$F^{n(L_*)}(L_*) \subset Dom(F).$$

So, we can choose S as the neighborhood of L_* in **(A2)**. Let us prove that this neighborhood works.

First we claim that $L_* \subset \partial^v\Sigma$. Indeed, if $L_* \subset \Sigma \setminus (\partial^v\Sigma)$, then S has three different connected components S_0, S_1, S_2. By **(A2)**-(4) we can assume $n^1(L_*) > 1$ where $n_1(L)$ comes from **(A2)**-(3). Choose sequence $x_i^1 \in S_1 \to x$ then $F(x_i^1) \to F^{n_1(L_*)}(x)$ by **(H2)**-(3). As F is continuous in x we also have $F(x_i^1) \to F(x)$ and then $F^{n^1(L_*)}(x) = F(x)$ because limits are unique. Thus, $F^{n^1(L_*)-1}(x) = x$ because F is injective and so x is a periodic point of F since $n^1(L_*) - 1 \geq 1$. This contradicts the non-existence of periodic points for F. The claim is proved.

The claim implies that S has a two components, i.e, S_0 and $S_1 = S_2$. By using **(A2)**, for the component S_0 one has $n^0(L_*) = 0$ and $n^1(L_*) = n^2(L_*) = 1$ since F is continuous in $x \in L_*$. Then, $F/_S$ is C^1 by the last part of **(A2)**-(3). This finishes the proof. □

Lemma 6.4.6. *Let $F : Dom(F) \subset \Sigma \to \Sigma$ be a triangular map satisfying* **(A1)-(A2)**. *If F has no periodic points and $Dom(F) = \Sigma \setminus L_0$, then $Dom(F) \setminus D(F)$ is open in Σ.*

Proof. $Dom(F)$ is open in Σ because $Dom(F) = \Sigma \setminus L_0$ and L_0 is closed in Σ. $Dom(F) \setminus D(F)$ is open in $Dom(F)$ by Lemma 6.4.5 because F has no periodic points and $\partial^v\Sigma \subset \Sigma \setminus L_0 = Dom(F)$. Thus $Dom(F) \setminus D(F)$ is open in Σ. □

6.4.2 Leaf class

Now, we need to introduce some definitions for the next result. We say that L is related with L' ($L \sim L'$) if we have that $n(L), n(L') \geq 1$ and $L, L' \subset S_0(L) \cap S_0(L')$.

Definition 6.4.7. *Let L be a leaf of $\mathcal{F}$ and $L \in Dom(F)$. If $n(L) \geq 1$, we define the* leaf class *associated to the leaf L as*

$$\langle L \rangle = \{L' \in Dom(F) \,|L' \sim L\}.$$

If $n(L) = 0$, we say that $\langle L \rangle = \{L\}$.

Remark 6.4.8. *If X is a vector field of codimension 1, i.e., $dim(E^c) = 2$, one has that $\langle L \rangle = \{L\}$ and so $dim(\langle L \rangle) = dim(\{L\}) = s$*

Lemma 6.4.9. *Let L be a leaf in $D(F)$. Then $\langle L \rangle$ is a $(s+1)$-submanifold (or s-submanifold if $\langle L \rangle = \{L\}$) of Σ, and whose boundary belong to $\partial^v\Sigma$.*

Proof. By using the Lemma 6.4.5, $D(F)$ is closed in $\Sigma \setminus L_0$. So, given $L \in Dom(F)$ by (A2) there is $S(L) = S(L)_0 \cup S(L)_1 \cup S(L)_2$. Then, $Cl(S(L)_0) \subset D(F)$, and as $D(F)$ is $\mathcal{F}$-saturated, one has that $Cl(S(L)_0) \setminus S(L)_0 = \partial S(L)_0$ are leaves. Thus, for each $L' \in \partial S(L)_0 \subset D(F)$, by using the Lemma 6.4.4, we obtain that $F(L') \subset \partial^v\Sigma$. Then, again by (A2) there exists $S(L') = S(L')_0 \cup S(L')_1 \cup S(L')_2$. In the same way, we proceeds analogously for $S(L')$. Since Σ has finite diameter, we conclude that $\langle L \rangle$ has boundary in $\partial^v\Sigma$ (See Figure 6.3). □

We have that if $L \in D(F)$, then $F(L) \in \partial^v\Sigma$ (Lemma 6.4.4), and this motivate the following definition.

Definition 6.4.10. *We define the* discontinuous class of leaves *by the set*

$$\langle D(F) \rangle = \{\langle L \rangle \,|L \in D(F)\}.$$

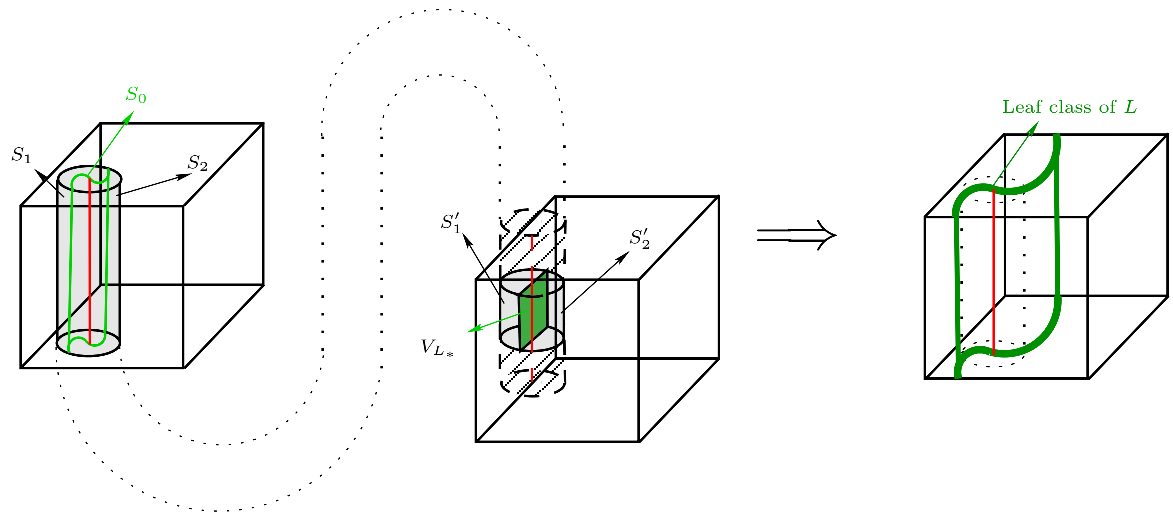


Figure 6.3: Leaf class of L

Definition 6.4.11. *A subset B of Σ is $\mathcal{F}$-*discrete *if it corresponds to a set of leaves whose only points of accumulations are the leaves in L_0.*

Lemma 6.4.12. *If F has no periodic points, then $\langle D(F)\rangle$ is discrete.*

Proof. By contradiction, we suppose that $\langle D(F)\rangle$ is not $\mathcal{F}$-discrete. Then, there is an open neighborhood U of L_0 in Σ such that $\langle D(F)\rangle \setminus \mathcal{A}$ contains infinitely many classes $\langle L_n\rangle$, where

$$\mathcal{A} = \{\langle L\rangle \in \langle D(F)\rangle \,|\langle L\rangle \cap U \neq \emptyset\}\,.$$

By using Lemma 6.4.5, $D(F)$ is closed in $Dom(F) = \Sigma \setminus L_0$, and so $\langle D(F)\rangle$ is closed in $Dom(F) = \Sigma \setminus L_0$ too. Since $D(F) \setminus U$ is closed in $Dom(F) \setminus U$, one has that $\langle D(F)\rangle \setminus \mathcal{A}$ is closed. As U is an open neighborhood of L_0 and $Dom(F) = \Sigma \setminus L_0$ we obtain that $Dom(F) \setminus U$ is compact in Σ. Henceforth $\langle D(F)\rangle \setminus \mathcal{A}$ is compact. So, without loss of generality, we can assume that $\langle L_n\rangle$ converges to a class $\langle L_*\rangle$ of $\langle D(F)\rangle \setminus \mathcal{A}$. By construction $\langle L_*\rangle \subset Dom(F)$. Since $\langle L_n\rangle \subset D(F)$ we have $F(\langle L_*\rangle) \subset \partial^v\Sigma$ by Lemma 6.4.4. It follows that $n(W_*) \geq 1$, for all $W_* \in \langle L_*\rangle$. We also have $n(W_*) \leq 2k < \infty$ by Lemma 6.4.1-(2) since F has no periodic points and $\partial^v\Sigma \subset \Sigma \setminus L_0 = Dom(F)$, for all

$W_* \in \langle L_* \rangle$. By Definition 6.3.1 we have $f^{n(L_*)}(L_*) \subset \partial^v \Sigma \subset Dom(F)$.

By definition $\langle L_n \rangle \cap \langle L_* \rangle = \emptyset$ for all $n \in \mathbb{N}$. Now, by using the property (A2), for each L_n and for L_* we can choose the following neighborhood associated to $\langle L_n \rangle$ and $\langle L_* \rangle$ as follows:

$$CS_n = \bigcup_{W \in \langle L_n \rangle} S(W) \quad and \quad CS_* = \bigcup_{W_* \in \langle L* \rangle} S(W_*).$$

Since $\langle L_n \rangle$ is compact, one has that

$$CS_n = \bigcup_{i=1}^{k} S(W_i) \quad and \quad CS_* = \bigcup_{i=1}^{k} S(W_{*,i}).$$

As $\langle L_n \rangle \to \langle L_* \rangle$ and $\langle L_n \rangle \cap \langle L_* \rangle = \emptyset$ we can assume $\langle L_n \rangle \subset CS_* \setminus \langle L_* \rangle$ for all n. As $\langle L_n \rangle \cap \langle L_* \rangle = \emptyset$ for all n we can further assume that $\langle L_n \rangle \subset CS_{1,*}$ where $CS_{1,*}$ is one of the (possibly equal) connected components of $CS_* \setminus \langle L_* \rangle$, i.e.,

$$CS_{1,*} = \bigcup_{i=1}^{k} S_1(W_{*,i}).$$

As $F(S_1(L_n)) \subset \Sigma \backslash (\partial^v \Sigma)$ for all $n \in \mathbb{N}$ by **(A2)**-(1) we conclude that $F(\langle L_n \rangle) \subset \Sigma \backslash (\partial^v \Sigma)$ for all n. However, $F(\langle L_n \rangle) \subset \partial^v \Sigma$ by Lemma 6.4.4 since $L_n \subset D(F)$ a contradiction. This proves the lemma.

□

6.4.3 Bands in higher dimensions

We need to extend some definitions for next lemmas and propositions. Let $\tilde{H}(L, L')$ be a cylinder such that $L, L' \in \partial H$, whose diameter is l and where l represent the distance of L to L', i.e., $l = dist(L, L')$. A *vertical band* in Σ between two vertical s-surfaces L, L' in the same component Σ is nothing but a cylinder H formed by the connected component that contain both L and L'. Let us denote by $H(L, L')$ and $H[L, L']$ the open and vertical band respectively.

Given a u-surface c, we say that c is *tangent* to C_α if $Dc(t) \in C_\alpha(c(t))$ for all $t \in Dom(c) \subset \mathbb{R}^u$. A C_α*-spine* of a vertical band $H(L, L')$(or $H[L, L']$) is a u-surface $c \subset H(L, L')$ tangent to C_α, such that $\partial c \subset \partial H(L, L')$ and $int(c) \subset H(L, L')$.

Lemma 6.4.13. *Let $c \subset Dom(F) \setminus D(F)$ be and open u-surface transversal to $\mathcal{F}$. If there is $n \geq 1$ and open C^1 u-surface c^* whose clausure $Cl(c^*) \subset c$ and such that $F^i(c^*) \subset (Dom(F) \setminus D(F)$ for all $0 \leq i \leq n-1$ and $F^n(c^*)$ covers c, then F has a periodic point.*

Proof. We prove the lemma by contradiction. Then, we suppose that F has no periodic point. So, by using the Lemma 6.4.5, $Dom(F) \setminus D(F)$ is saturated and $F\big|_{Dom(F)\setminus D(F)}$ is C^1. Then, c and c^*, projects (via $\mathcal{F}$) into two u-balls in SL still denoted by c and c^* respectively. The assumptions imply that $f^i(c)$ is defined for all $0 \leq i \leq n-1$ and

$$Cl(c^*) \subset c \subset f^n(c^*).$$

Then, f^n has a periodic point L_{**}. As $F^n(L_{**}) \subset f(L_{**}) = L_{**}$ and $F^n|_{L_{**}}$ is continuous, then the *Brower fixed point Theorem* implies that F^n has a fixed point. This fixed point represents a periodic point of F. □

Lemma 6.4.14. *F carries a u-surface $c \subset Dom(F) \setminus D(F)$ tangent to C_α (with volume $V(c)$) into a u-surface tangent to C_α (with volume $\geq \lambda \cdot V(c)$).*

Proof. See [8]. □

Lemma 6.4.15. *Suppose that F has no periodic points. Let L, L' be different leaves in $D(F)$ such that the open vertical band $H(L, L') \subset Dom(F) \setminus D(F)$. If c is a C_α-spine of $H(L, L')$, then $F(Int(c))$ covers a vertical band $H(W, W')$ with*

$$W, W' \subset \partial^v \Sigma \cup \mathcal{V}.$$

Proof. By using Lemma 6.4.5 we have that $F/_{H(L,L')}$ is C^1 because $H(L,L') \subset Dom(F) \setminus D(F)$. And Lemma 6.4.4 implies

$$F(L), F(L') \subset \partial^v \Sigma \tag{6.2}$$

because $L, L' \subset D(F)$. Since $L, L' \subset Dom(F)$, one has that $n(L), n(L')$ are defined. By (6.2) we have $n(L), n(L') \geq 1$. Then, $1 \leq n(L), n(L') < \infty$ by Lemma 6.4.1-(1) since F has no periodic points and $\partial^v \Sigma \subset \Sigma \setminus L_0 = Dom(F)$. By the same reason

$$F^{n(L)}(L), F^{n(L')}(L') \subset Dom(F).$$

If there exists a sequence converging to L or L', by using the Lemma 6.4.3 exist the limit $F(L_n)_{n\in\mathbb{N}}$ and this limit belong to $\partial^v\Sigma \cup \mathcal{V}$. Let W and W' be these limits respectively. Now, let c be a C_α-spine of $H(L,L')$. To fix ideas we assume $\partial c \subset \partial H(L,L')$, and this implies that there are $p, q \in \partial c$ such that $p \in L$ and $q \in L'$. As $Int(c) \subset H(L,L') \subset Dom(F) \setminus D(F)$ we have that $F(Int(c))$ is defined. As $F/_{H(L,L')}$ is C^1 we have that $F(Int(c))$ is a u-surface whose boundary containing points that belong in $\partial^v\Sigma \cup \mathcal{V}$.

Clearly $W \neq W'$ because F preserves $\mathcal{F}$. Then, $\mathcal{F}_{F(Int(c))} = H(W, W')$ is an open vertical band, where $W, W' \subset \partial^v\Sigma \cup \mathcal{V}$. □

Definition 6.4.16. *Let p be a point of M. We define the* radius of injectivity $inj(p)$ *at a point p as the largest radius for which the exponential map at p is a injective map, i.e.,*

$$inj(p) = Sup\{r > 0\,|exp_p : B(0,r) \longrightarrow M \quad is \quad injective\}.$$

Also, we say that the radius of injectivity of the manifold M is the infimum of the radius at all points, i.e,.

$$inj(M) = Inf\,\{inj(p)\,|p \in M\}.$$

The proof is based on the following result of [8], that it will be modified for the higher dimensional case.

Lemma 6.4.17. *Suppose that F has no periodic points. For every open u-surface $c \subset Dom(F) \setminus D(F)$ tangent to C_α there are an open u-surface $c^* \subset c$ and $n'(c) > 0$ such that $F^j(c^*) \subset Dom(F) \setminus D(F)$ for all $0 \leq j \leq n'(c) - 1$ and $F^{n'(c)}(c^*)$ covers a vertical band $H(W, W')$ with*

$$W, W' \subset \partial^v \Sigma \cup \mathcal{V} \cup \mathcal{L}.$$

Proof. Let $c \subset Dom(F) \setminus D(F)$ be an u-surface tangent to C_α. The following claim will be proved by modifying arguments used in [8],[20] and a similar argument used by [12] beside the radius of injectivity definition.

Claim 6.4.3:

There are an open u-surface $c^{**} \subset c$ and $n''(c) > 0$ such that $F^j(c^{**}) \subset Dom(F) \setminus D(F)$ for all $0 \leq j \leq n''(c) - 1$ and $F^{n''(c)}(c^{**})$ covers an open vertical band

$$H(L, L') \subset Dom(F) \setminus \langle D(F) \rangle,$$

where L, L' are different leaves in $D(F) \cup L_0$.

Proof. For every open u-surface $c' \subset Dom(F) \setminus D(F)$ tangent to C_α we define

$$N(c') = \sup \left\{ n \geq 1 : F^j(c') \subset Dom(F) \setminus \langle D(F) \rangle, \forall 0 \leq j \leq n - 1 \right\}.$$

Note that $1 \leq N(c') < \infty$ because $\lambda > 1$ and Σ has finite diameter. In addition, $F^{N(c')}(c')$ is a u-surface tangent to C_α with

$$F^{N(c')}(c') \cap (D(F) \cup L_0) \neq \emptyset$$

because $Dom(F) = \Sigma \setminus L_0$.

Define the number β by

$$\beta = (1/2) \cdot \lambda.$$

Then, $\beta > 1$ since $\lambda > 2$. Define $c_1 = c$ and $N_1 = N(c_1)$.

Since $F^{N_1}(c_1)$ is a open u-surface, if $F^{N_1}(c_1)$ intersects $\langle D(F)\rangle \cup L_0$ in a unique leaf class $\langle L_1\rangle$, then $F^{N_1}(c_1) \setminus \langle L_1\rangle$ has two connected components. In this case we define

- $c_2^* =$ the connected component of $F^{N_1}(c_1) \setminus \langle L_1\rangle$ that contain a ball whose radius of injectivity is greater or equal to any ball of the complement.
- $c_2 = F^{-N_1}(c_2^*)$.

The following properties hold,

1) $c_2 \subset c_1$ and then c_2 is an open u-surface tangent to C_α.

2) $F^j(c_2) \subset Dom(F) \setminus \langle D(F)\rangle$, for all $0 \leq j \leq N_1$.

3) $V(F^{N_1}(c_2)) \geq \beta \cdot V(c_1)$.

In fact, the first property follows because $F^{N_1}/_{\mathcal{F}_{c_2}}$ is injective and C^1. The second one follows from the definition of $N_1 = N(c_1)$ and from the fact that $c_2^* = F^{N_1}(c_2)$ does not intersect any leaf in $\langle D(F)\rangle \cup L_0$. The third one follows from Lemma 6.4.14 because

$$V(F^{N_1}(c_2)) = V(c_2^*) \geq (1/2) \cdot V(F^{N_1}(c_1)) \geq$$

$$\geq (1/2) \cdot \lambda^{N_1} V(c_1) \geq (1/2) \cdot \lambda V(c_1) = \beta \cdot V(c_1)$$

since $\lambda > 2$ and $N_1 \geq 1$.

Next we define $N_2 = N(c_2)$. The second property implies $N_2 > N_1$. As before, if $F^{N_2}(c_2)$ intersects $\langle D(F)\rangle \cup L_0$ in a unique leaf class $\langle L_2\rangle$, then $F^{N_2}(c_2) \setminus \langle L_2\rangle$ has two connected components. In such a case we define analogously c_3^* and also $c_3 = F^{-N_2}(c_3^*)$.
As before

$$V(F^{N_3}(c_3)) = V(c_3^*) \geq (1/2) \cdot V(F^{N_2}(c_2)) \geq (1/2) \cdot \lambda^{N_2 - N_1} V(F^{N_1}(c_2)) \geq \beta^2 V(c_1)$$

because of the third property. So,

1) $c_3 \subset c_2$ and c_3 is an open u-surface tangent to C_α.

2) $F^j(c_3) \subset Dom(F) \setminus \langle D(F) \rangle$ for all $0 \leq j \leq N_2$.

3) $V(F^{N_2}(c_3)) \geq \beta^2 \cdot V(c_1)$.

In this way we get a sequence $N_1 < N_2 < N_3 < \cdots < N_l < \cdots$ of positive integers and a sequence $c_1, c_2, c_3, \cdots c_l, \cdots$ of open u-surfaces (in c) such that the following properties hold $\forall l \geq 1$

1) $c_{l+1} \subset c_l$ and c_{l+1} is an open u-surface tangent to C_α.

2) $F^j(c_{l+1}) \subset Dom(F) \setminus \langle D(F) \rangle$ for all $0 \leq j \leq N_l$.

3) $V(F^{N_l}(c_{l+1})) \geq \beta^l \cdot V(c_1)$.

The sequence c_l must stop by Property (3) since Σ has finite diameter. So, *there is a first integer l_0 such that $F^{N(c_{l_0})}(c_{l_0})$ intersects $\langle D(F) \rangle \cup L_0$ in two different leaves class $\langle L \rangle, \langle L' \rangle$*. Note that these classes must be contained in the same component of Σ since $F^{N(c_{l_0})}(c_{l_0})$ is connected. Hence, we can suposse that the vertical band $H(L, L')$ bounded by L, L' is well defined. We can assume that $H(L, L') \subset Dom(F) \setminus \langle D(F) \rangle$ because $\langle D(F) \rangle$ is $\mathcal{F}$-discrete by Lemma 6.4.12. Choosing $c^{**} = c_{l_0}$ and $n''(c) = N_{l_0}$ we get the result. □

Now we finish the proof of Lemma 6.4.17. Let $c^{**}, n''(c)$ and $L, L' \subset D(F) \cup L_0$ be as in Claim 6.4.3. We have three possibilities: $L, L' \subset D(F)$; $L \subset L_0$ and $L' \subset D(F)$; $L \subset D(F)$ and $L' \subset L_0$. We only consider the two first cases since the later is similar to the second one.

First we assume that $L, L' \subset D(F)$. As $F^{n''(c)}(c^{**})$ is tangent to C_α, and covers $H(L, L')$, we can assume that $F^{n''(c)}(c^{**})$ itself is a C_α-spine of $H(L, L')$. Then, applying Lemma 6.4.15 to this spine, one gets that $F^{n''(c)+1}(c^{**})$ covers a vertical band $H(W, W')$ with

$$W, W' \subset \partial^v \Sigma \cup \mathcal{V}$$

In this case the choices $c^* = c^{**}$ and $n'(c) = n''(c) + 1$ satisfy the conclusion of Lemma 6.4.17.

Finally, we assume that $L \subset L_0$ and $L' \subset D(F)$. As $L \subset L_0$ we have $L = L_{0i}$ for some $i = 1, \cdots, k$.

On the one hand, $H(L_{0i}, L') = H(L, L') \subset Dom(F) \setminus \langle D(F) \rangle$ and then $\langle D(F) \rangle \cap H(L_{0i}, L') = \emptyset$. So, Lemma 6.4.12 implies that exists the limit K_i. Consequently,

$$K_i \in \mathcal{L}.$$

Additionally, $F(L') \subset \partial^v \Sigma$ by Lemma 6.4.4 since $L' \subset D(F)$. It follows that $1 \leq n(L')$ and also $n(L') \leq 2k$ by Lemma 6.4.1-(1) since F has no periodic points and $\partial^v \Sigma \subset \Sigma \setminus L_0 = Dom(F)$. Since $F^{n(L')}(L') \subset \partial^v \Sigma$ by the definition of $n(L')$ we obtain

$$F^{n(L')}(L') \subset Dom(F).$$

Then, Lemma 6.4.3 applied to L' implies that $lim_{L \to L'} f(L) = f^*(L')$ exists and satisfies

$$f * (L') \subset (\partial^v \Sigma) \cup \mathcal{L}.$$

But $F(H(L_{0i}, L'))$ (and so $F(F^{n''(c)}(c^{**})))$ covers $H(K_i, f * (L'))$ since $H(L_{0i}, L') \subset Dom(F) \setminus \langle D(F) \rangle$. Setting $W = K_i$ and $W' = f * (L')$ we get

$$W, W' \subset (\partial^v \Sigma) \cup \mathcal{V} \cup \mathcal{L}.$$

(Recall the definition of $\mathcal{V}$ in Definition 6.4.2) Then, $F(F^{n''(c)}(c^{**}))$ covers (W, W') as in the statement. Choosing $c^* = c^{**}$ and $n'(c) = n''(c) + 1$ we obtain the result. □

6.5 Proof of the Theorem 6.3.4

Finally, we prove Theorem 6.3.4. Let F be a λ-hyperbolic n-triangular map satisfying **(A1)-(A2)** with $\lambda > 2$ and $Dom(F) = \Sigma \setminus L_0$. We assume by contradiction that the following property holds:

(P) F has no periodic points.

Since $\partial^v\Sigma \subset \Sigma \setminus L_0$ and $\Sigma \setminus L_0 = Dom(F)$ we also have

$$\partial^v\Sigma \subset Dom(F).$$

Then, the results in the previous subsections apply. In particular, we have that $Dom(F) \setminus D(F)$ is open in Σ (by Lemma 6.4.6) and that $\langle D(F)\rangle$ is $\mathcal{F}$-discrete (by Lemma 6.4.12-(1)). All together imply that $Dom(F) \setminus D(F)$ is open-dense in Σ.

Now, let $\mathcal{B}$ be a family of open vertical bands of the form $H(W, W')$ with

$$W, W' \subset \partial^v\Sigma \cup \mathcal{V} \cup \mathcal{L}.$$

Given $\langle L\rangle$ and $\langle L'\rangle$, note that by compactness, the leaf classes are covers by a finite number of vertical bands. Thus, we have that $\mathcal{B} = \{B_1, \cdots, B_m\}$ is a finite set. In $\mathcal{B}$ we define the relation $B \leq B'$ if and only if there are an open u-surfaces $c \subset B$ tangent to C_α with closure $\mathrm{CL}(c) \subset Dom(F) \setminus \langle D(F)\rangle$, an open u-surface $c^* \subset c$ and $n > 0$ such that

$$F^j(c^*) \subset Dom(F) \setminus \langle D(F)\rangle\,, \qquad \forall 0 \leq j \leq n-1,$$

and $F^n(c^*)$ covers B'.

As $Dom(F) \setminus D(F)$ is open-dense in Σ, and the bands in $\mathcal{B}$ are open, we can use Lemma 6.4.17 to prove that for every $B \in \mathcal{B}$ there is $B' \in \mathcal{B}$ such that $B \leq B'$. Then, we can construct a chain

$$B_{j_1} \leq B_{j_1} \leq B_{j_2} \leq \cdots,$$

with $j_i \in \{1, \cdots, m\}$ $(\forall i)$ and $j_1 = 1$. As $\mathcal{B}$ is finite it would exist a closed sub-chain

$$B_{j_i} \leq B_{j_{i+1}} \leq \cdots \leq B_{j_{i+s}} \leq B_{j_i}.$$

Hence there a positive integer n such that $F^n(B_{j_i})$ covers B_{j_i}. Applying Lemma 6.4.13 to suitable u-surfaces $c^* \subset Cl(c^*) \subset c \subset B_{j_i}$ we obtain that F *has a periodic point.* This contradicts **(P)** and the proof follows.

6.6 Proof of the Main Theorem

Proof of Theorem D

Proof. Let X be a sectional Anosov flow on a compact n-manifold M. We prove the Theorem by contradiction, i.e, to prove that $M(X)$ has a periodic orbit we assume that this is not so. Fix $\lambda > 2$. As there is a singular cross-section Σ close to $M(X)$ such that if F is the return map of the refinement $\Sigma(\delta)$ of Σ (see Subsection 4.2.1.1), then there is $\delta > 0$ such that F is a λ-hyperbolic n-triangular map with $Dom(F) = \Sigma \setminus L_0$. We also have that F satisfies (A1) and (A2) in Subsection 6.3.1 since Σ is close to Λ. Then, F has a periodic point by Theorem 6.3.4 since $\lambda > 2$. This periodic point belongs to a periodic orbit of X_t which in turns belongs to $M(X)$ since it is maximal invariant. Consequently X_t has a periodic orbit in $M(X)$, a contradiction. This contradiction proves the result. □

Chapter 7

Existence of venice masks with two singularities

7.1 Introduction

As stated in Chapter 4, we talk about the relationship between the hyperbolic and sectional hyperbolic theory. Recall, the sectional hyperbolic sets and sectional Anosov flows were introduced in [31] and [27] respectively as a generalization of the hyperbolic sets and Anosov flows. They contain important examples such as the saddle-type hyperbolic attracting sets, the geometric and multidimensional Lorenz attractors [1], [12], [20].

One motivation is to look properties that are preserved or which are not in the new scenario. Particularly, we can mention two important properties related to hyperbolic sets which are not satisfied by all sectional hyperbolic sets. The first is the spectral decomposition theorem [52]. It says that an attracting hyperbolic set $\Lambda = Cl(Per(X))$ is a finite disjoint union of homoclinic classes, where $Per(X)$ is the set of periodic points of X. The second says that an Anosov flow on a closed manifold is transitive if and only if it has dense periodic orbits.

The results above are false for sectional Anosov flows, i.e., sets whose maximal invariant is a sectional-hyperbolic set [35]. Specifically, it is proved that there exists a sectional Anosov flow such that it is supported on a compact 3-manifold, it has dense periodic orbits, is the union non disjoint of two homoclinic classes but is not transitive. So, a sectional Anosov flow is a *Venice mask* if it has dense periodic orbits which is not transitive.

Naturally, it arise examples of three dimensional venice masks. In the same direction, [36] improves results in [35] about venice masks containing a unique singularity [35] and it shows a three dimensional example containing three singularities. After, [10] exhibits and constructs a three dimensional example with an unique singularity. The only known examples of venice masks have one or three singularities, and they are characterized by having two properties: are the union non disjoint of two homoclinic classes and the intersection of its homoclinic classes is the closure of the unstable manifold of a singularity.

Thus, in order to provide new examples and to extend works related with this theory, the above observations motivate the following questions:

1. Are there venice masks with two singularities?

2. Are there venice masks whose intersection of its homoclinic classes is different to the closure of the unstable manifold of a singularity?

Since the aim is find new examples with different features, these in turn induce questions about which type of manifold supports such examples. We give a affirmative answer to them on particular three compact manifolds. In fact, we show two examples of Venice masks with two singularities. Each one is union of two different homoclinic classes. However for the first, the intersection of homoclinic classes is just a hyperbolic periodic orbit. Whereas for the second, the intersection of homoclinic classes is the closure of the unstable manifold of two singularities.

Let us state our results in a more precise way.

Theorem F. *There exist a Venice mask X with two singularities supported on a 3-manifold M, such that:*

- $M(X)$ *is the union of two homoclinic classes* $\mathcal{H}_X^1, \mathcal{H}_X^2$.
- $\mathcal{H}_X^1 \cap \mathcal{H}_X^2 = O$, *where* O *is a hyperbolic periodic orbit.*

Theorem G. *There exist a Venice mask Y with two singularities supported on a 3-manifold N, such that:*

- $N(Y)$ *is the union of two homoclinic classes* $\mathcal{H}_Y^1, \mathcal{H}_Y^2$.
- $\mathcal{H}_Y^1 \cap \mathcal{H}_Y^2 = Cl(W^u(\sigma_1) \cup W^u(\sigma_2))$, *where* σ_1, σ_2 *are the singularities of* Y.

In section 7.3.2, we shall be described briefly this construction by using one-dimensional and two-dimensional maps. In section 7.4.1, from modifications on the previous maps in Section 7.3.2 and by considering a plug, we shall prove the Theorem F. In the same way, in Section 7.4.2, by using the venice mask with a unique singularity, the Theorem G will be obtained by gluing a particular plug preserving the original flow.

7.2 Preliminaries

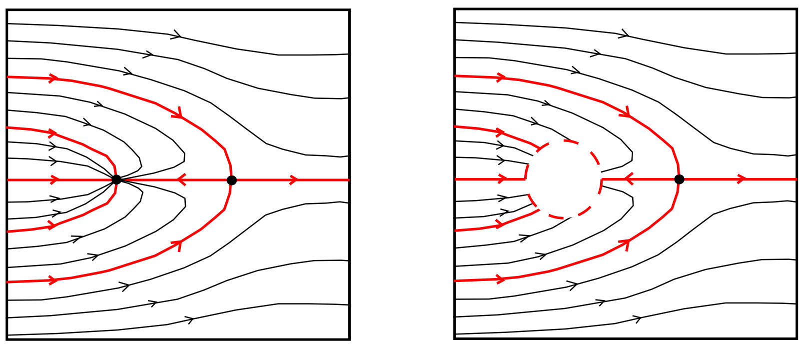
7.2.1 Original plugs

In order to obtain the three-dimensional vector field of our example, we begin by considering the well known *Plykin attractor* and the *Cherry flow* (See [48], [44]).

We give a sketch of the flow construction. It will be constructed through three steps, firstly by modifying the Cherry flow. In fact, we consider a vector field in the square whose flow is described in Figure 7.1 a). Note that this vector field has two equilibria: a saddle σ and a sink p. For σ one has that its eigenvalues $\{\lambda_s, \lambda_u\}$ of σ satisfy the relation

$$\lambda_s < 0 < -\lambda_s < \lambda_u.$$

We have depicted a small disk D centered at the attracting equilibrium p Figure 7.1 b). Note that the flow is pointing inward the edge of the disk. This finishes the first step for the construction.

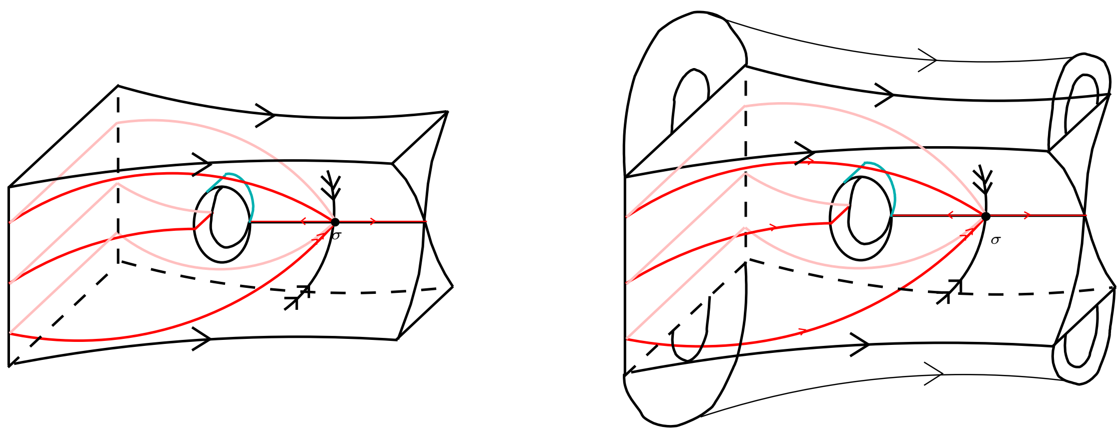


$a)$ $b)$

Figure 7.1: Cherry flow.

For the second step we multiply the above vector field by a strong contraction λ_{ss} in order to obtain the vector field described in Figure 7.2 a). We can choose λ_{ss} such that $-\lambda_{ss}$ be large, so the resulting vector field will have a Lorenz-like singularity and this new eigenvalue will be associated with the strong manifold of the singularity. This yields a Cherry flow box and finishes the second step for the construction.

From Plykin attractor follows that the construction must have at least two holes inasmuch as we will use certain return map. Then, the final step is to glue two handles that provides two holes and the three dimensional vector field above in order to obtain the vector field whose flow is given in Figure 7.2 b). Hereafter the resulting vector field will be called of Plug 7.2.



a) *b*)

Figure 7.2: Cherry flow box and Plug 7.2.

The hole indicated in this Figure 7.2 is nothing but the disk D times a compact interval. Again, note that the flow is pointing inward the edge of the hole by construction. For this reason, we take a solid 3-ball and we define a flow on this one. Indeed, the flow has no singularities, it acts as in Figure 7.3 and will be used for to glue the hole's bound with this one. Hereafter the resulting vector field will be called of Plug 7.3.

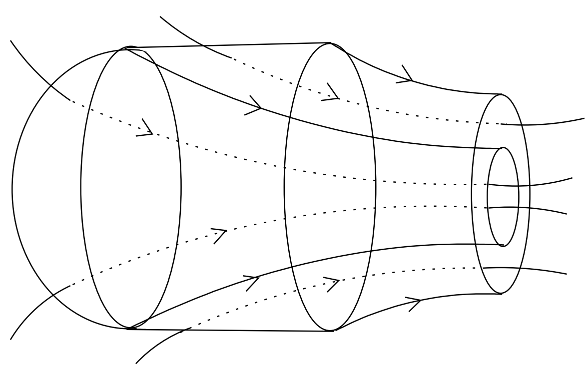


Figure 7.3: Plug 7.3.

7.3 Modified maps

We begin by considering the construction made in [10] like model in order of obtain the vector fields X and Y of the main theorems. Recall that the original model provides tools for a three dimensional example with a unique singularity. The main aim is modify the original maps, in order to make a suspension of the modify maps via the new plugs. For this purpose, we will do such modifications followed by its original maps.

7.3.1 One-dimensional map

Thus, in the same way of [10], we consider the branched 1-manifold $\mathcal{B}$ consisting of a compact interval and a circle with branch point b. We cut $\mathcal{B}$ along b to obtain a compact interval which we assume to be $[0, 1]$ for simplicity. In $[0, 1]$ we consider three points $0 < d_1 < d_* < d_2 < 1$, where d_* is depicted also in the Figure 7.4. These will be the discontinuity points of f as a map of $[0, 1]$. The set $\mathcal{B} \setminus \{d_*\}$ will be the domain of f. We define $f : \mathcal{B} \setminus \{d_*\} \to \mathcal{B}$ in a way that its graph in $[0, 1]$ is the one in Figure 7.4.

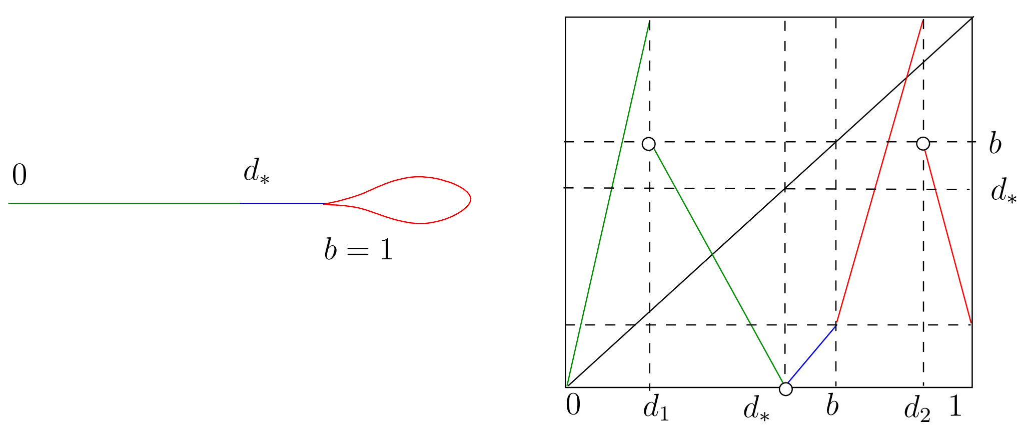


Figure 7.4: The quotient space and one-dimensional map.

By construction one has that f satisfies the following hypotheses:

(H1): $Dom(f) = [0,1] \setminus \{d_*\}$.

(H2): $f(0) = 0$; $f(d_1) = f(d_2) = 1$; $f(1) = f(b) \in (0, d_1)$.

(H3): $f(d_1+) = f(d_2+) = b$; $f(d_1-) = f(d_2-) = 1$; $f(d_*+) = f(d_*.) = 0$.

(H4): $f([0, d1]) = [0, 1]$; $f((d_1, d_*)) = (0, b)$; $f((d_*, d_2]) = (0, 1]$; $f((d_2, 1]) = [f(b), b)$.

(H5): f is expanding, i.e., f is C^1 in $Dom(f)$ and there is $\lambda > 1$ such that $|f'(x)| \geq \lambda$, for each $x \in Dom(f)$.

7.3.2 Modified one-dimensional map

We realize a modification of the above map f . Denote $d_* = d^+$ and let $f^+ : \mathcal{B}^+ \setminus \{d^+\} \to \mathcal{B}^+$ be in a way that its graph in $[0, 1]$ is the one in Figure 7.5.
Here, there exist $\varepsilon > 0$ small such that $\int_0^{d_1} \sqrt{[(f)'(x)]^2 + 1}dx < \int_0^{d_1} \sqrt{[(f^+)'(x)]^2 + 1}dx < \int_0^{d_1} \sqrt{[(f)'(x)]^2 + 1}dx + \varepsilon$ and $\int_{d_*}^{b} \sqrt{[(f)'(x)]^2 + 1}dx < \int_{d_*}^{b} \sqrt{[(f^+)'(x)]^2 + 1}dx < \int_{d_*}^{b} \sqrt{[(f)'(x)]^2 + 1}dx + \varepsilon$. Moreover f^+ satisfies **(H1)-(H5)**. We define $f^-(x) = f(-x)$ and denote $-d^+ = d^-$. $f^- : \mathcal{B}^- \setminus \{d^-\} \to \mathcal{B}^-$.

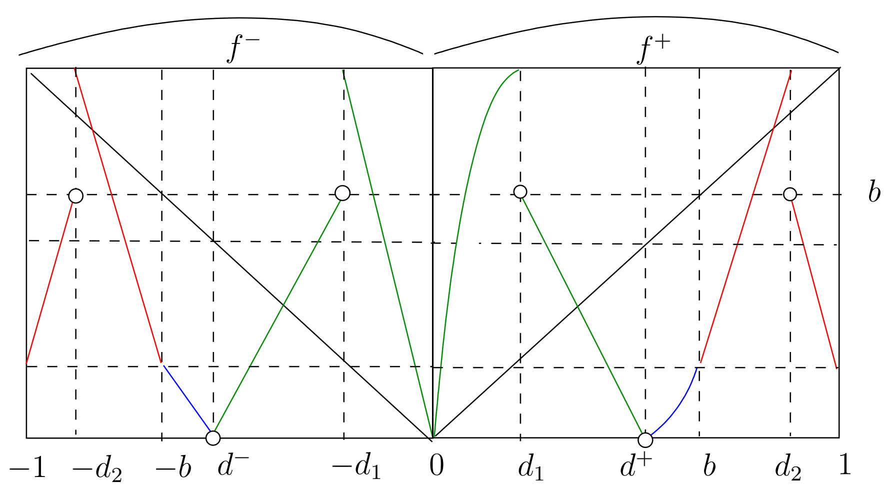


Figure 7.5: Modified one-dimensional map.

These following results examining the properties of f and appears in [10]. This in turns through a structure closely related to [10] and by construction we obtain the same properties for the f^+ map.

Definition 7.3.1. *We say that f is* locally eventually onto (leo for short) *if given any open interval $I \subset [0,1]$ there is $m \geq 0$ such that $f^m(I) = [0,1]$.*

Theorem 7.3.2. *f^+ is* leo.

Corollary 7.3.3. *The periodic points of f^+ are dense in $\mathcal{B}$. If $x \in \mathcal{B}$, then*

$$\mathcal{B} = Cl\left(\bigcup_{n\geq 0}(f^+)^{-n}(x)\right).$$

7.3.3 Two-dimensional map

Consider the twice punctured planar region R depicted in Figure 7.6. It is formed by: two half-annuli A, F, and four rectangles B, C, D, E. There is a middle vertical line denoted by l. Note that l defines a plane reflexion throughout denoted by θ. We assume $\theta(D) = C$, $\theta(E) = B$ and $\theta(F) = A$. In particular, $\theta(R) = R$ and $\theta(d^+) = d^-$, where the vertical segments d^-, d^+ correspond to the right-hand and left-hand boundary curves of B and D respectively. We define $H^- = A \cup B \cup C$ and $H^+ = D \cup E \cup F$.

Let $\mathcal{F}$ denote the foliation of R formed by line segments depicted in Figure 7.6. Note that the middle leaf l is a leaf of $\mathcal{F}$. In addition, the segments forming $\mathcal{F}$ are vertical in the rectangular components B, C, D, E and radial in the annuli components A, F.

We use the standard notation $\mathcal{F}_x$ to denote the leaf of $\mathcal{F}$ containing $x \in R$. Identifying points in the same leaf of $\mathcal{F}$ we get the quotient space K. Note that $0 \in K$ represents the leaf l in the quotient space. The manifold K is obtained by gluing two copies $\mathcal{B}^+$,

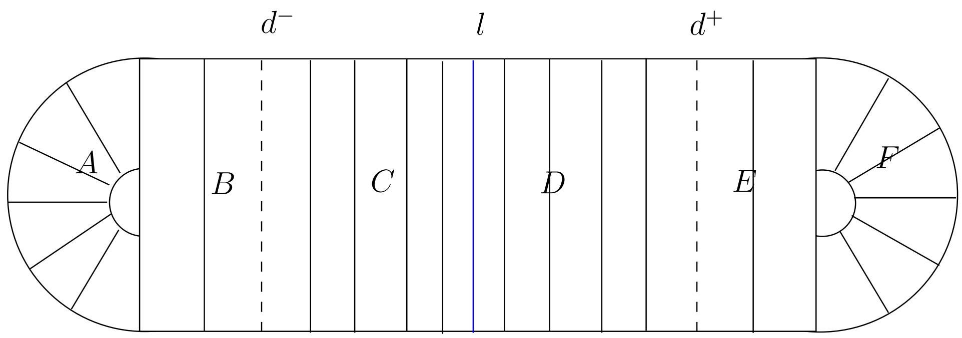


Figure 7.6: Region R.

$\mathcal{B}^-$ of the branched manifold $\mathcal{B}$ along the boundary point 0.

By using the set $R \setminus \{d^-, d^+\}$ as a domain in order to obtain a suitable map, we can define the C^∞ map $G : R \setminus \{d^-, d^+\} \to Int(R)$ in a way that its image is as indicated in Figure 7.7. We require the following hypotheses:

(G1): G and θ commute, i.e., $G \circ \theta = \theta \circ G$.

(G2): G preserves and contracts the foliation $\mathcal{F}$.

(G3): Let $g : K \setminus \{d^-, d^+\} \to K$ be the map induced by G in the leaf space K.

Then, the map f^+ defined by $f^+ = g|_{\mathcal{B}^+}$ satisfies the hypotheses **(H1)-(H5)**, with $f = f^+$, $\mathcal{B} = \mathcal{B}^+$ and $d_* = d^+$.

Properties of G

- By **(G1)**, H^+ and H^- are invariant under G.
- Since G contracts $\mathcal{F}$ (**(G2)**) we have that $W^s(x, G)$ is union of leaves of $\mathcal{F}$. It follows from **(G2)**, **(G3)** and the expansiveness in **(H5)** that all periodic points of G are hyperbolic saddles.
- By **(G1)** we have that $G(l) \subset l$ and so G has a fixed point P in l. Clearly one has $\pi(P) = 0$.

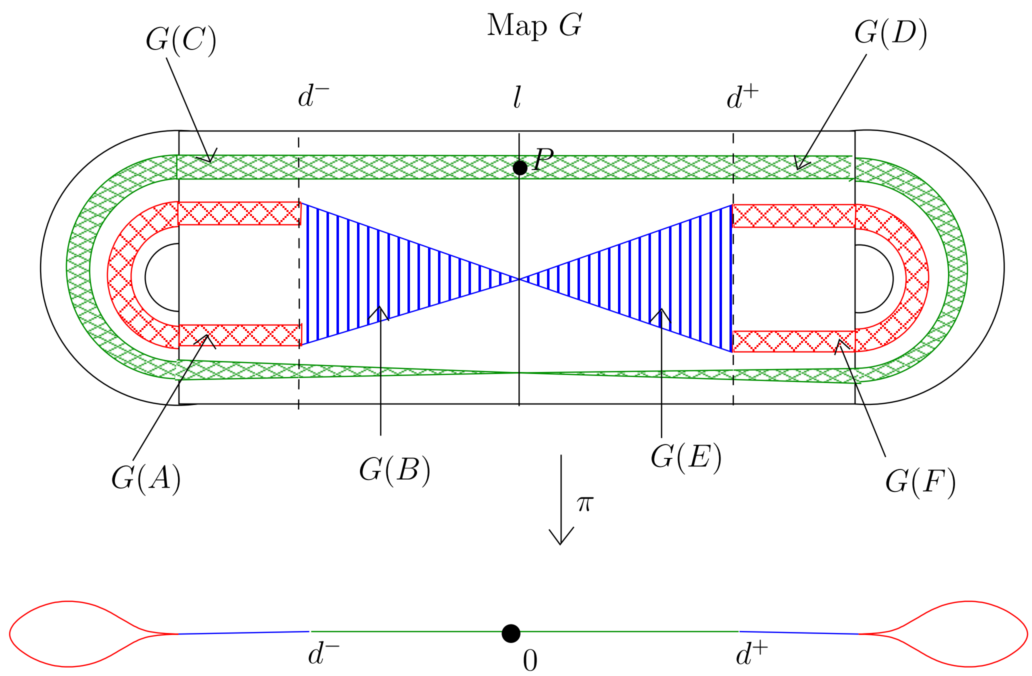


Figure 7.7: Two-dimensional map and quotient space

Define

$$A_G^- = Cl\left(\bigcap_{n\geq 1} G^n(H^-)\right), \qquad A_G^+ = Cl\left(\bigcap_{n\geq 1} G^n(H^+)\right).$$

Theorem 7.3.4. *A_G^- and A_G^+ are homoclinic classes and $P \in A_G^+ \cap A_G^-$.*

7.3.4 Modified two-dimensional map

Consider the twice punctured planar region R in 7.6, and define the C^∞ map $H : R \setminus \{d^-, d^+\} \to Int(R)$ in a way that its image is as indicated in Figure 7.8. We require the following hypotheses:

(L1): H^-, H^+ are invariant under H. $H(H^- \setminus \{d^-\}) \subset H^-$ and $H(H^- \setminus \{d^+\}) \subset H^+$.

(L2): H preserves and contracts the foliation $\mathcal{F}$.

(L3): Let $h : K \setminus \{d^-, d^+\} \to K$ be the map induced by H in the leaf space K.

Then, the map $f^{+(-)}$ defined by $f^{+(-)} = h|_{\mathcal{B}^{+(-)}}$ satisfies the hypotheses **(H1)-(H5)**, $\mathcal{B} = \mathcal{B}^{+(-)}$ and $d_* = d^{+(-)}$.

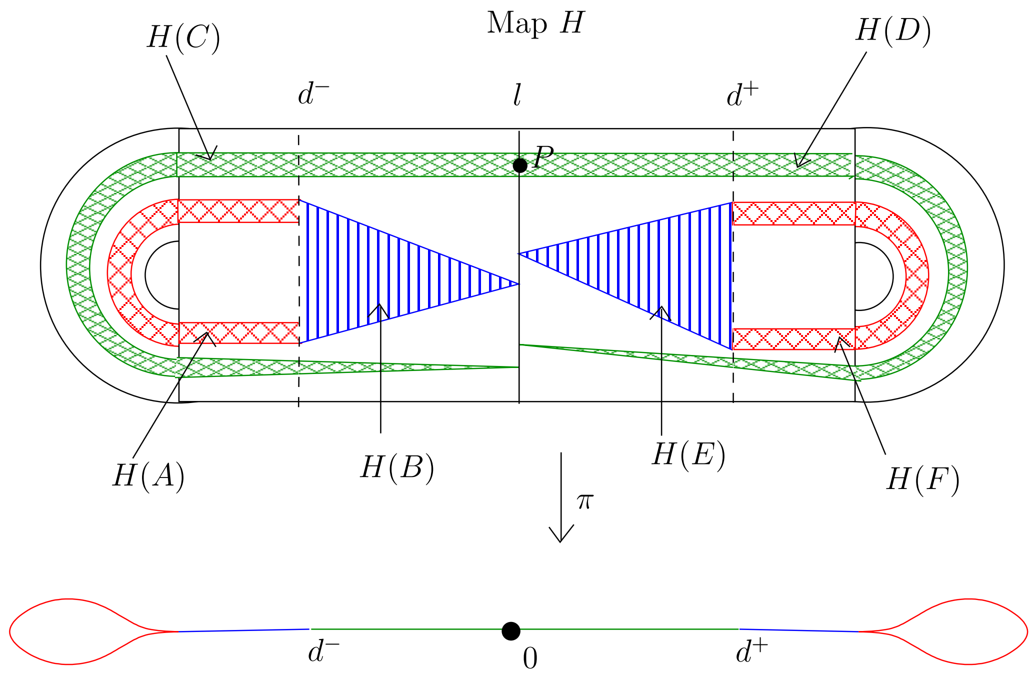


Figure 7.8: The quotient space and modified two-dimensional map.

We observe that **(L1)** implies $H(l) \subset l$ and by contraction, H has a fixed point $P \in l$. Again, for

$$A_H^- = Cl\left(\bigcap_{n\geq 1} H^n(H^-)\right), \qquad A_H^+ = Cl\left(\bigcap_{n\geq 1} H^n(H^+)\right)$$

we have that A_H^+ and A_H^- are homoclinic classes and $\{P\} = A_H^+ \cap A_H^-$.

7.3.5 Venice mask with one singularity

Recall, by considering the original maps (Subsection 7.3.1, 7.3.3), and by using the plugs 7.2, 7.3, in [10] was construct the venice mask example with one singularity. Here, we provides a graphic idea in order to compare it with the new examples.

The Figure 7.9 *a*) shows the flow, whereas the Figure 7.9 *b*) shows the ambient manifold that supports this one. The ambient manifold is a solid bi-torus excluding two tori neighborhoods V_1, V_2 associated to two repelling periodic orbits O_1, O_2 respectively.

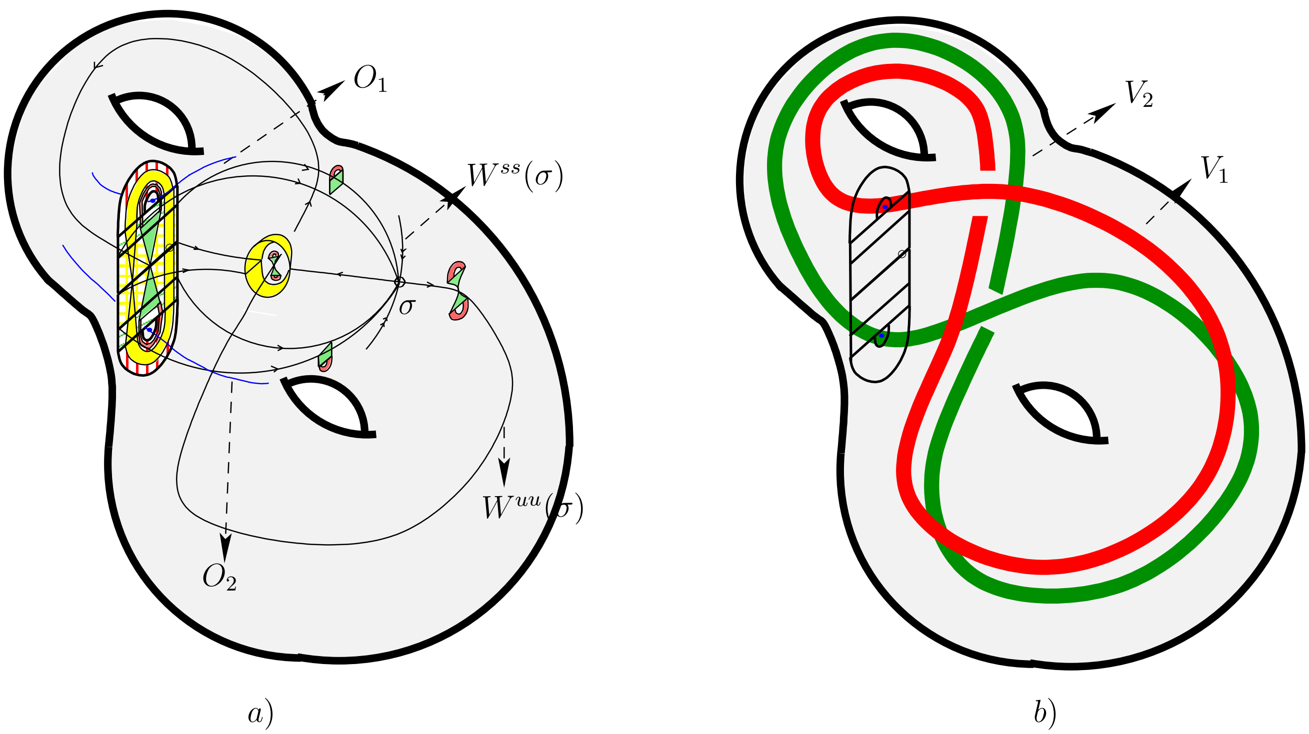


Figure 7.9: Venice mask with one singularity

7.4 Venice mask's examples with two-singularities

7.4.1 Vector field X and Example 1.

In this section, we construct a vector field X which will satisfy the properties in the Theorem F by using the subsection 7.3.2 and 7.3.4.
We begin by considering a vector field as the Cherry flow described in Figure 7.1, with the same conditions of subsection 7.3.2.

We called this flow of A and we proceed to perturb the flow A following the ideas of the well known DA-Attractor introduced by Smale (see [48]). Let U be a neighborhood (relatively small) of σ. We can obtain a flow φ^t such that $supp(\varphi^t - id) \subset U$ (Figure 7.10 a)). Also, the derivate of the flow at σ with respect to canonical basis in $T_\sigma Q$ is

$$D\varphi^t_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix}.$$

We deform such a flow in order to obtain a one-parameter family of flows $B^t = \varphi^t \circ A$. Let $\tau > 0$ such that $e^\tau \lambda_s > 1$, so σ is a source for B^τ. Moreover, the new map has three fixed points on $W^s_X(\sigma)$, σ a source and σ_1, σ_2 saddles. Moreover, there exists a neighborhood V of σ (not containing σ_1 and σ_2) contained in U such that $B^\tau_s(V) \supset V$ for all $s > 0$. (Figure 7.10 b))

Thus, we obtain a vector field as the square Q whose flow A is described in Figure 7.10.

Now, we remove two small disks D_1, $D_2 = V$ centered at the attracting equilibrium p and at the repelling equilibrium σ respectively. (Figure 7.10 c))

In the next step, we multiply the above vector field by a strong contraction λ_{ss} in order to obtain the similar vector field described in Figure 7.2 b). We choose λ_{ss} such that σ_1 and σ_2 are Lorenz-like. Let I_1, I_2 be compact intervals, with the same direction of the

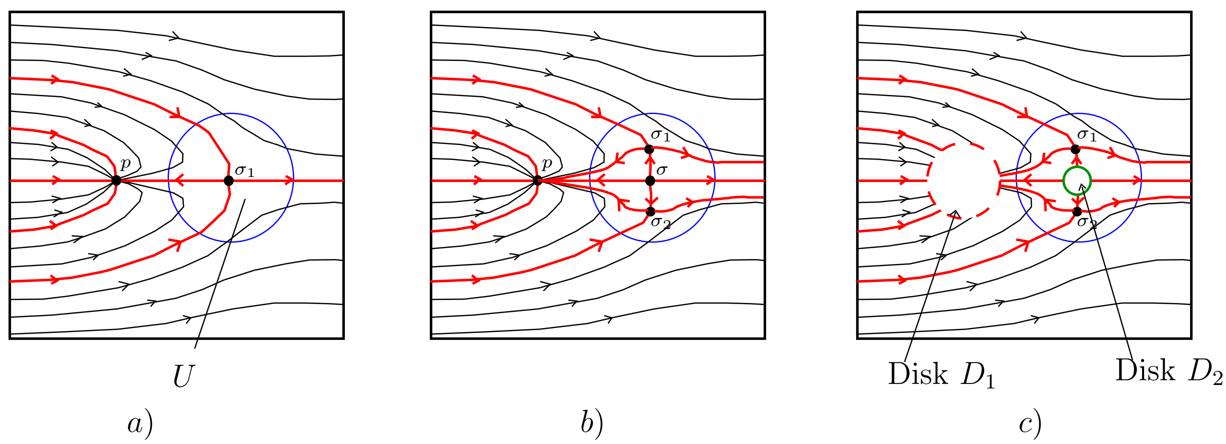


Figure 7.10: Perturbed Cherry flow

strong contraction.

Now, we consider an interval $I_0 = I_1 \times \{p_0\}$, where p_0 is the point of intersection between $W^u_X(\sigma)$ and the disk D_1. We realize a modification in the flow such that a branch of $W^u_X(\sigma_1)$ intersects a connected component of $I_0 \setminus \{p_0\}$ and a branch of $W^u_X(\sigma_2)$ intersects the other connected component of $I_0 \setminus \{p_0\}$ (See 7.11).

The final step is to glue two handles on the 3-dimensional vector field above in order to obtain the vector field whose flow is given in Figure 7.11 *a*). The resulting vector field is what we shall call Plug X.

In the same way as in Figure 7.2, in this case, by multiplying the above vector field by a strong contraction generate two holes and it is nothing but the disks D_1 times a compact interval I_1, and D_2 times a compact interval I_2. Also, let us to use the Plug 7.3 and apply on the hole associated to D_1 and note that the interval I_2 is chosen such that $D_2 \times I_2$ produces the third hole on the ambient manifold. It generates a solid tritorus (see Figure 7.11 *b*)).

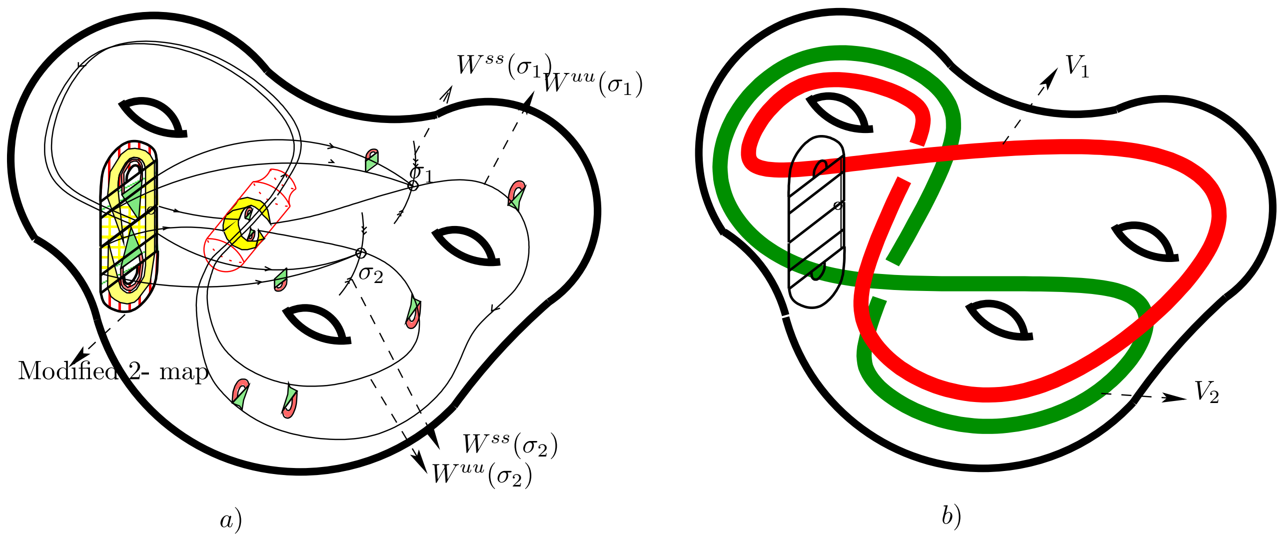


Figure 7.11: Plug X and its associated manifold.

Then, we construct a vector field X on a solid tritorus ST_1 in a way that $X_t(ST_1) \subset Int(ST_1)$ for all $t > 0$ and X is transverse to the boundary of the solid tritorus. The flow is obtained gluing plugs X and 7.3 as indicated in Figure 7.11 a).

We require the following hypotheses:

(X1): There are two repelling periodic orbits O_1, O_2 in $Int(ST_1)$ crossing the holes of R.

(X2): There are two solid tori neighborhoods $V_1, V_2 \subset Int(ST_1)$ of O_1, O_2 with boundaries transverse to X_t such that if $M = ST_1 \setminus (V_1 \cup V_2)$, then M is a compact neighborhood with smooth boundary transverse to X_t and $X_t(M) \subset M$ for $t > 0$. As M is a solid tritorus with two solid tori removed, we have that M is connected as indicated in Figure 7.11 b).

(X3): $R \subset M$ and the return map H induced by X in R satisfies the properties **(L1)-(L3)** in Section 7.3.4. Moreover,

$$\{q \in M : X_t(q) \notin R, \forall t \in \mathbb{R}\} = \{\sigma_1, \sigma_2\}.$$

Now, define

$$A^+ = Cl\left(\bigcup_{t\in\mathbb{R}} X_t(A_H^+)\right) \qquad \text{and} \qquad A^- = Cl\left(\bigcup_{t\in\mathbb{R}} X_t(A_H^-)\right).$$

Proposition 7.4.1. *$W_X^u(\sigma_1) \subset A^+$ and $W_X^u(\sigma_2) \subset A^-$.*

Proof. If $x \in H^+$ is a periodic point of H, then $G^n(x) \in R$ for all $n \leq 0$ and so $x \in A_H^+ = Cl(\bigcap_{n\geq 1} H^n(H^+))$. Therefore $x \in A^+$ (for $A_{H^+} \subset A^+$) and by invariance of A^+, the full orbit of x is contained in A^+.

Second, the periodic points of f^+ in **(L3)** are dense in $\mathcal{B}$ by Corollary 2.2. Then, the periodic points of H accumulate on d^+ in both connected components of $H^+ \setminus d^+$. Since d^+ is contained in $W_X^s(\sigma_1)$, the full X_t-orbit of the periodic points of H accumulating d^+ also accumulate on $W_X^u(\sigma_1)$. Then $W_X^u(\sigma_1) \subset A^+$ because A^+ is closed. Analogously, we have $W_X^u(\sigma_2) \subset A^-$. □

Define $A_H = A_H^+ \cup A_H^-$ and

$$A = Cl\left(\bigcup_{t\in\mathbb{R}} X_t(A_H)\right),$$

Lemma 7.4.2. *A^+ and A^- are homoclinic classes of X and $A = A^+ \cup A^-$.*

Proof. See [10]. □

Proposition 7.4.3. *X is a sectional Anosov flow.*

Proof. In the same way of [10], we define $M(X) = A$ and by definition A is a sectional-hyperbolic set. Indeed, how $A = A_1 \cup A_2$ is union of homoclinic classes then A has dense periodic orbits (Birkhoff-Smale Theorem). Moreover, of the hypotheses **(L2)** and **(L3)** follows that every periodic orbit of X contained in A has a hyperbolic splitting $T_O M = E^s_O \oplus E^X_O \oplus E^u_O$. Here, E^s_O is due to **(L2)**, E^u_O by **(L3)** and E^X_O is the one-dimensional subbundle over O induced by X. Let $Per(A)$ be the union of the periodic orbits of X contained in A. Define the splitting

$$T_{Per(A)}M = F^s_{Per(A)} \oplus F^c_{Per(A)},$$

where $F^s_x = E^s_x$ and $F^c_x = E^X_x \oplus E^u_x$ for $x \in Per(A)$. As every periodic orbit in M of every vector field C^1 close to X is hyperbolic of saddle type, we can use the arguments in [38] to prove that the splitting $T_{Per(A)}M = F^s_{Per(A)} \oplus F^c_{Per(A)}$ over $Per(A)$ extends to a sectional-hyperbolic splitting $T_A M = F^s_A \oplus F^c_A$ over the whole $A = Cl(Per(A))$.

We conclude that X is a sectional Anosov flow on M.

□

Proof of Theorem F.

By using the Lemma 7.4.2 and the Proposition 7.4.3 we have that X is a sectional Anosov flow and $M(X)$ is the union of two homoclinic classes H^1_X, H^2_X, where $H^1_X = A^+$ and $H^2_X = A^-$. Since $\{P\} = A^+_H \cap A^-_H$, it implies that $H^1_X \cap H^2_X = O$, with O the orbit associated to P. In particular X is a Venice mask, and by construction it has two singularities.

7.4.2 Vector field Y and Example 2.

In this section, we construct a vector field Y which will satisfy the properties in the Theorem G by using the results from [10].

Firstly, in order to obtain the vector field Y, we begin by considering the venice mask with one singularity. Unlike the previous section, in this case we will not perturb the flow. Moreover, we will change the flow by preserving the plugs 7.2, 7.3 and we will remove a connected component of the flow and its ambient manifold, as depicted in Figure 7.12.

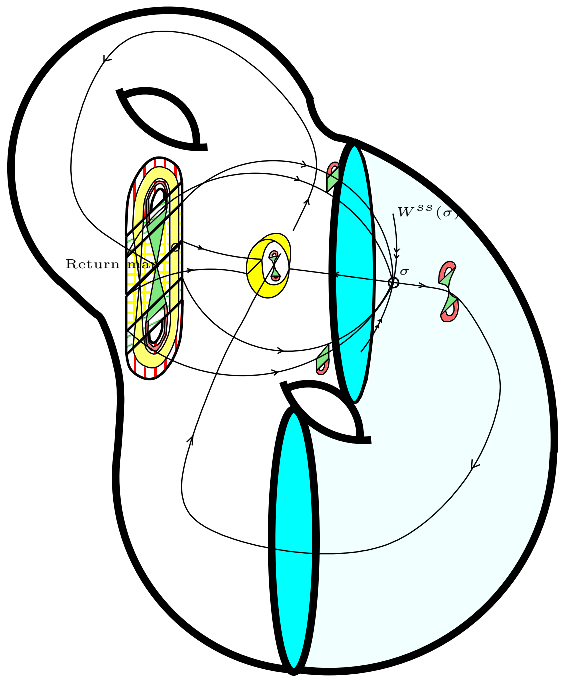


Figure 7.12: Connected componet to remove.

The main aim of remove a connected component will be glue a new plug with different features, properties and that provides other singularity. This process is done in simple steps. (see Figure 7.13). Indeed, the important steps are Figure 7.13 c), d) and since we want a plug by containing a singularity, we will see that the this one has a hole, which is produced by the singularity.

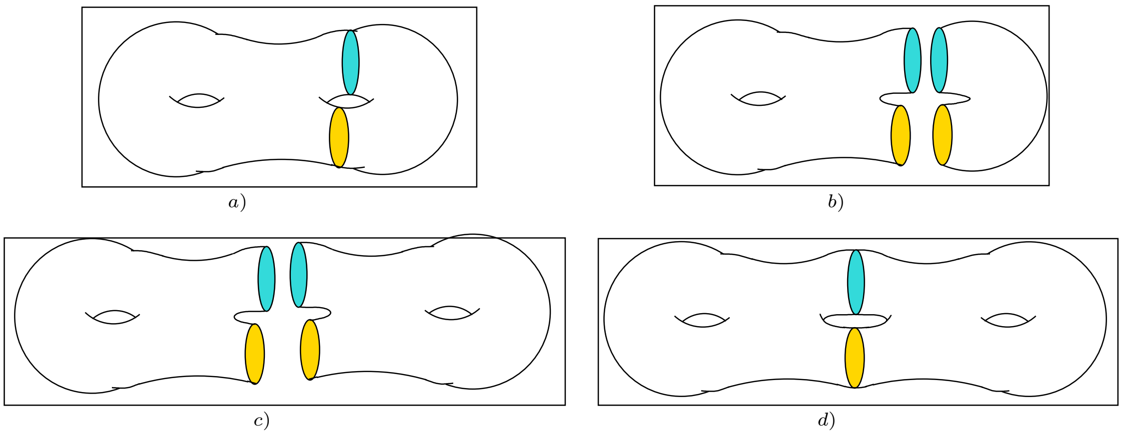


Figure 7.13: Steps by gluing the new plug.

7.4.3 Flow through of the faces

We begin by considering the plug 7.2 described in Figure 7.2 with the same conditions of subsections 7.3.1, 7.3.3.

For this purpose we need to observe with detail the flow behavior through of the faces removed. Indeed, we consider the vector field in the square whose flow is described in Figure 7.2.

Thus, it will be constructed the new plug through two steps. Firstly, we will be depicted a circle that represents the face 1 on the Cherry flow and let us to observe the flow behavior. It should be noted that this vector field exhibits two leaves which belong to the region R and converge to the singularity, i.e., the region R exhibits two singular leaves. Note that these leaves are crossing outward to the face 1. In addition, note that there are trajectories crossing inward to the face 1 too, such as the branch unstable manifold of the singularity. This shows that extensive analysis is necessary for understand the flow behavior to the face 1.

We can observe that the top and bottom region of the singular leaves saturated by the flow are crossing through the face 1, i.e., the flow is pointing outward of the face 1.

By studying the complement of these regions, we have that the behavior of the leaves is depicted as Figure 7.14. Here, this region exhibits two tangent leaves, whereas the other leaves intersect the region twice, i.e., the other leaves cross and return.

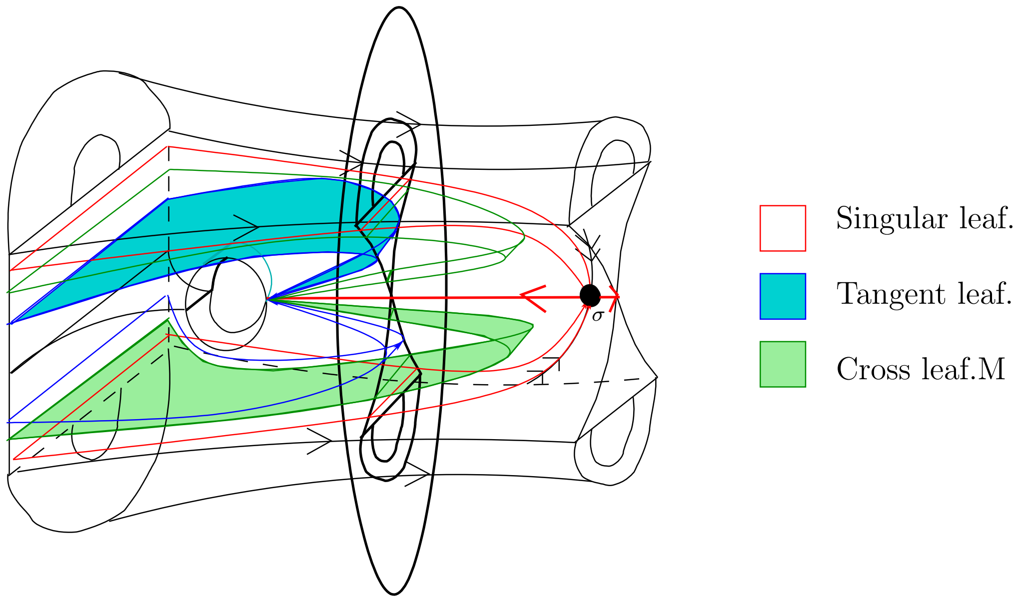


Figure 7.14: Flow through of the face 1.

Also, we must research the flow behavior inside to the face 1, but in the complement of Cherry box flow. However, we can to observe that the behavior flow is extended to the whole circle. This finishes the first step.

Now, we must to observe the flow behavior on the face 2. But in this case, is easy to verify that all trajectories are crossing inward to the face 2. Thus, the flow through of the two faces is depicted in the following figure
Then, we construct a plug Y containing a singularity σ_2. Consequently, the dynamical system can be transferred by means of plug Y surgery from one bitorus onto another

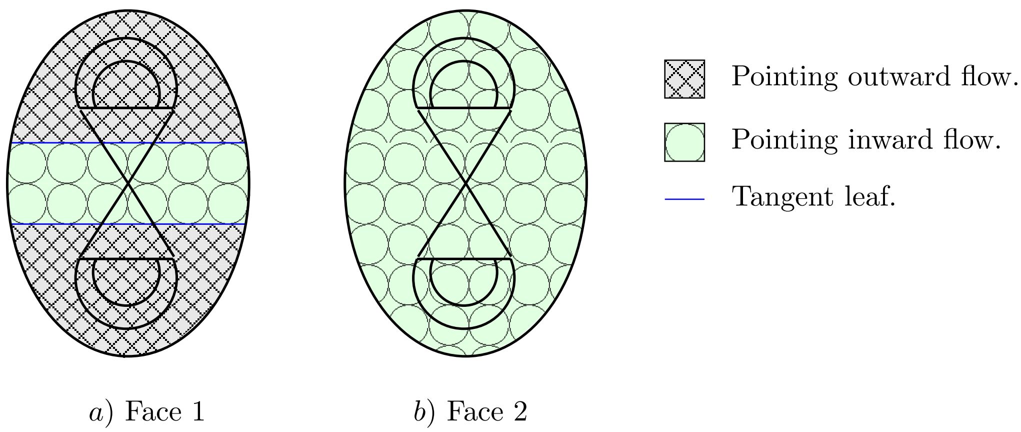


Figure 7.15: Direction of flow through the faces.

manifold exporting some of its properties. This singularity generates a hole and this in turns generates a solid tritorus ST_2 in a way that $Y_t(ST_2) \subset Int(ST_2)$ for all $t > 0$ and Y is transverse to the boundary tritorus. The flow is obtained gluing the plugs 7.2, 7.3 with the plug Y as indicated in Figure 7.16. Indeed, the hole is generated by the unstable manifold of the singularity σ_2.

In the same way from the previous subsection, we require some hypotheses for the ambient manifold (after of gluing).

($\hat{X}$1): There are two repelling periodic orbits O_1, O_2 in $Int(ST_2)$ crossing the holes of R.

($\hat{X}$2): There are two solid tori neighborhoods $V_1, V_2 \subset Int(ST_2)$ of O_1, O_2 with boundaries transverse to Y_t such that if $N = ST_2 \setminus (V_1 \cup V_2)$, then N is a compact neighborhood with smooth boundary transverse to Y_t and $Y_t(N) \subset N$ for $t > 0$. As N is a solid tritorus with two solid tori removed, we have that N is connected.

($\hat{X}$3): $R \subset N$ and the return map H induced by Y in R satisfies the properties **(H1)-(H3)** in Section 7.3.4. Moreover,

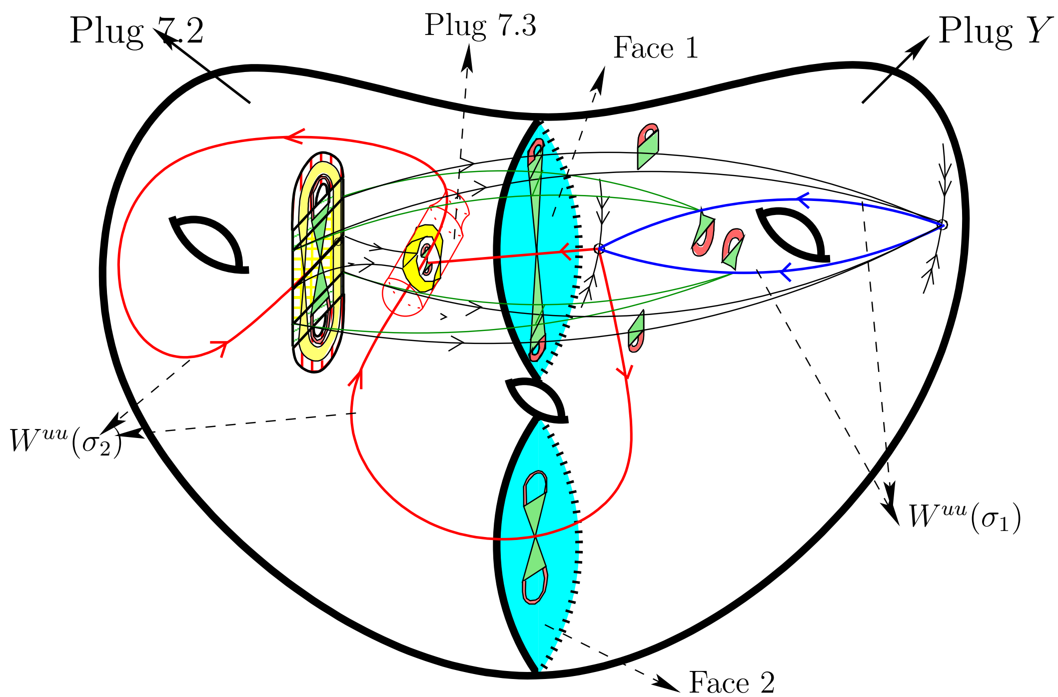


Figure 7.16: Plug Y

$$\{q \in N : Y_t(q) \notin R, \forall t \in \mathbb{R}\} = Cl(W_Y^{uu}(\sigma_2)).$$

Now, define

$$\hat{A}^{+}_{=}Cl\left(\bigcup_{t\in\mathbb{R}} Y_t(\hat{A}_H^+)\right) \qquad \text{and} \qquad \hat{A}^- = Cl\left(\bigcup_{t\in\mathbb{R}} Y_t(\hat{A}_H^-)\right).$$

By using the Propositions 7.4.1, 7.4.3 and Lemma 7.4.2 we can obtain that the intersection of homoclinic classes is the closure of the unstable manifold of two singularities. In particular one has:

Proof of Theorem G.

By using the Lemma 7.4.2 and the Proposition 7.4.3 we have that Y is a sectional Anosov flow and $N(Y)$ is the union of two homoclinic classes H_Y^1, H_Y^2, where $H_Y^1 = \hat{A}^+$ and $H_Y^2 = \hat{A}^-$. It implies that $H_Y^1 \cap H_Y^2 = Cl(W_Y^u(\sigma_1) \cup W_Y^u(\sigma_2))$. In particular Y is a Venice mask, and by construction it has two singularities.

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