ABSTRACT FRAMEWORK FOR THE THEORY OF STATISTICAL SOLUTIONS

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RESUMO

Resumo da Tese apresentada ao PGPIM/UFRJ como parte dos requisitos necessários para a obtenção do grau de Doutor em Matemática (D.Sc.)

UMA FORMULAÇÃO ABSTRATA PARA A TEORIA DE SOLUÇÕES ESTATÍSTICAS

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Soluções estatísticas têm sido usadas principalmente a fim de se entender mais profundamente algumas propriedades de fluxos turbulentos de modo rigoroso. Este tipo de solução é usado como uma alternativa à falta de um resultado sobre boa colocação associada a soluções individuais das equações de Navier-Stokes, um modelo amplamente aceito para fluxos turbulentos. O objetivo deste trabalho é estender a teoria presente de soluções estatísticas a um patamar abstrato que possa então ser aplicado a inúmeros outros problemas de evolução para os quais a boa colocação também não foi estabelecida. Com este propósito, constrói-se uma estrutura abstrata tendo como base um espaço topológico de Hausdorff que representa o espaço de fase de um sistema, e com o conjunto de trajetórias correspondente pertencendo ao espaço de caminhos contínuos neste espaço de fase. Então, após estabelecer as definições de soluções estatísticas neste sentido geral, dois pontos principais são considerados. Em primeiro lugar, prova-se a existência de tais soluções estatísticas em relação a certos problemas de valor inicial. Em segundo lugar, mostra-se a convergência de soluções estatísticas associadas a problemas aproximados dependentes de um parâmetro. Algumas aplicações a modelos específicos são também fornecidas em cada caso como ilustração desta teoria abstrata.

Palavras-chave: soluções estatísticas, soluções estatísticas de trajetórias, equações de Navier-Stokes.

ABSTRACT

Abstract of the Thesis presented to PGPIM/UFRJ as part of the requirements for the degree of Doctor of Science (D.Sc.)

ABSTRACT FRAMEWORK FOR THE THEORY OF STATISTICAL SOLUTIONS

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Statistical solutions have been mainly used for understanding some properties of turbulent flows in a deep and rigorous way. This type of solution is used as an alternative for the lack of a well-posedness result concerned with individual solutions of the Navier-Stokes equations, a widely accepted model for turbulent flows. The aim of this work is to extend the current theory of statistical solutions to an abstract level that allows it to be applied to a wide range of evolution problems which are also not known to be well-posed. For that purpose, an abstract framework is constructed with a general Hausdorff topological space as the phase space of the system, and with the corresponding set of trajectories belonging to the space of continuous paths in that phase space. Then, after establishing the definitions of statistical solutions in this general sense, two main points are addressed. First, the existence of such statistical solutions in regard to some initial-value problems, and secondly, the convergence of statistical solutions associated to approximated problems depending on a parameter. Some model examples are also provided in each case, illustrating the applicability of this abstract theory.

Keywords: statistical solutions, trajectory statistical solutions, Navier-Stokes equations.

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Chapter 1

Introduction

Since its initial steps, the theory of statistical solutions has gone through a significant development, becoming a subject that encompasses a number of concepts from several different areas of Mathematics and with a growing number of applications [3, 6, 11, 15, 17, 20, 24, 40]. However, the current results are mainly given for the Navier-Stokes equations and some modified versions of it. The idea in our work is to extend this theory to a more abstract level, so that similar results could be obtained for other equations which, just like the Navier-Stokes equations, do not have an established result of global well-posedness.

1.1 Historical Background

The concept of statistical solutions has emerged in the context of fluid dynamics in order to provide a rigorous mathematical definition for the notion of ensemble average, commonly used in the study of turbulent flows. In such flows, the relevant physical quantities (e.g., velocity, kinetic energy, and pressure) present a wild variation in space and time, characterizing a highly irregular and unpredictable behavior. Nevertheless, those quantities display a regular behavior when considered with respect to some average. In an attempt to investigate general properties of turbulent flows, one is then naturally led to deal with averages of the desired quantities. Several types of averages are usually considered, such as locally in space, locally in time, and with respect to an ensemble of experiments. Statistical solutions are directly related to this latter notion of average, known as ensemble average.

The mathematical treatment of the equations governing the motion of fluids, namely the Navier-Stokes equations, was started by J. Leray in the 1930's [31, 32, 33], when the concept of weak solutions to these equations was first introduced and studied. These individual weak solutions allow us to obtain rigorous results concerned with time and space averages. However, if one wants to consider ensemble averages, then it is necessary to consider collections of individual solutions and to obtain probability distributions associated with these collections. This new concept requires a statistical approach that has been first developed from Engineering and Physical points of view by R. Reynolds [41], G. Taylor [45], T. von Karman and L. Howarth [44], among others, in the late 1800's and early 1900's.

A few decades later, E. Hopf published a more theoretical work [25] over these statistical ideas, using modern methods of functional analysis. However, a more complete and rigorous theory was only developed in the 1970's, when C. Foias, based on a measure theoretical point of view firstly proposed by G. Prodi, introduced the notion of statistical solutions in phase space [16]. This type of solution consists of a family of measures parametrized by the time variable representing the evolution of probability distributions of a viscous incompressible fluid. Later on, Vishik and Fursikov [52] defined a different notion of statistical solutions, based on a single measure defined on the space of trajectories. More recently, Foias, Rosa and Temam [21, 22] (see also [19]) introduced a slightly modified definition of this latter solution, inspired by the definition given in [52], and which was denoted as Vishik-Fursikov measure, for a measure defined in the space of trajectories. Projecting this measure to the phase space at each time, they obtained a particular type of statistical solution, which is termed a Vishik-Fursikov statistical solution. What is interesting about this new definition is that every Vishik-Fursikov statistical solution is a statistical solution in the sense of Foias-Prodi. Also, besides being more favorable to analysis, the former seems to possess additional properties.

The notion of statistical solutions is mainly used for evolution problems which are not known to be well-posed. In this case, one cannot guarantee that the associated evolution equation generates a well-defined semigroup, i.e. a family of operators $\{S(t)\cdot\}_t$ with t varying in a time interval $I \subset \mathbb{R}$ such that, for each $t \in I$ and each element u_0 in the phase space X, $u = S(t)u_0$ is the unique solution of the given equation satisfying $u(t_0) = u_0$. When such a family of operators is well-defined, one can define the evolution of probability distributions starting from an initial measure μ_0 on X as

$$\mu_t(\cdot) = S(t)\mu_0(\cdot) = \mu_0(S(t)^{-1}\cdot), \quad t \in I.$$

However, if no well-posedness result is available and, consequently, the family $\{S(t)\cdot\}_{t\in I}$ is not necessarily well-defined, one defines the evolution of probability distributions through statistical solutions. Usually, if the evolution problem is well-posed, then both definitions coincide.

1.2 Main Goals

Our purpose in this work is to construct an abstract framework for the theory of statistical solutions focusing on the definitions of Vishik-Fursikov measure and Vishik-Fursikov statistical solution given in [21, 22]. We aimed at extracting the key ideas given in these works and adapt them to a framework as general as possible, so that it would be suitable to a wide range of applications.

This framework is built over a general Hausdorff topological space X and the associated space of continuous paths $\mathcal{C}_{\text{loc}}(I, X)$, over a given time interval $I \subset \mathbb{R}$ (i.e. the space of continuous functions defined on a real interval I and assuming values in X), and endowed with the compact-open topology. A key object in this theory is a subset $\mathcal{U}$ of $\mathcal{C}_{\text{loc}}(I, X)$, which has no special meaning in this abstract level, but which, in the applications, is taken to be the set of (individual) solutions of a given evolutionary system, for which X is a phase space.

Our first general definition of statistical solution is given with respect to this set $\mathcal{U}$ and called a *$\mathcal{U}$-trajectory statistical solution*. This type of solution represents a generalization of the notion of Vishik-Fursikov measure given in [21, 22] and is defined as a tight Borel probability measure on the space of continuous paths in X which is carried by $\mathcal{U}$. The second general definition incorporates the idea of a Vishik-Fursikov statistical solution from [21, 22] and is called a *statistical solution in phase space*. This is defined as a family of Borel probability measures parametrized by a time variable and satisfying, among other conditions, a Liouville-type equation. We also define the notion of a *projected statistical solution*, which is a family of Borel probability measures given by the projections on phase space at each time of a trajectory statistical solution.

The main results of this work may be divided into two parts. In the first one (Chapter 3), our main concern is to prove the existence of statistical solutions in these general senses for certain initial-value problems (Problems 3.1.1 and 3.1.2). In this context, although there is no equation at the abstract level, we consider an interval I closed and bounded at the left and which is interpreted as a time interval. In order to obtain an existence theorem for these abstract initial-value problems, a series of restrictions must be imposed on the set $\mathcal{U}$ that mimic some essential properties of the space of solutions occurring in the applications (Definition 3.2.1). Under those hypotheses, we first prove the existence of a $\mathcal{U}$-trajectory statistical solution such that its projection at the initial time is equal to a given initial tight Borel probability measure on X (Theorem 3.2.1). Next, we prove the existence of a statistical solution in phase space, say $\{\rho_t\}_{t\in I}$, such that, if t_0 represents the initial time (i.e. the left end point of the interval I), then ρ_{t_0} is equal to a given initial tight Borel probability measure on X (Theorem 3.3.2). For this second result, we must assume the existence of an evolution equation of the form

$$u_t = F(t, u(t)), \quad t \in I,$$

and some additional hypotheses must be considered in regard to the function F. In order to illustrate these results, we apply them to the Navier-Stokes equations, a reaction-diffusion equation, a nonlinear wave equation and the Bénard problem.

The second part (Chapter 4) is concerned with the convergence of statistical solutions for models depending on a parameter. Usually, one has an evolution problem which is not known to be well-posed and consider a family of approximated problems with respect to a parameter, each one being well-posed. The idea consists in showing that the statistical solutions associated to these regularized problems, which are defined via their well-defined solution operators, converge to a statistical solution of the limit problem. A general result of this type is obtained by considering a new set of hypotheses, which must be satisfied by the set $\mathcal{U}$ and also by the family of solution operators associated to the approximated problems. Assuming these hypotheses, we prove that, given an initial tight Borel probability measure on X, the sequence of induced measures from this initial measure by each solution operator (see Section 2.2) converges, modulo a subsequence, to a $\mathcal{U}$-trajectory statistical solution (Theorem 4.1.1). Moreover, considering the projections of these measures on X at each time t, we obtain an analogous result for the convergence of statistical solutions in phase space (Theorem 4.1.2). As an application of these results, we consider the Navier-Stokes-α model and the MHD-α model.

We observe that this second part can also be seen as another approach to the first one, i.e. to prove the existence of a statistical solution satisfying a given initial data. The main difference lies in the fact that in the first part the statistical solution is obtained as the limit of a net of measures in trajectory space that is derived from a certain net of approximating measures of the initial measure in phase space, given by the Krein-Milman Theorem. On the other hand, in the second part the statistical solution is obtained as the limit of a sequence of statistical solutions associated to the approximated problems.

As we can see, however, in both cases the statistical solution for the initial value problem is obtained through the limit of an approximating family of measures. Thus, one of the tools we need in these proofs is a compactness result for measures. A result of this kind which is suitable for our abstract context was developed by Topsoe [48, 49, 50] in his works on a generalization of Prohorov's Theorem (see [38]). One of his results states that the uniform tightness of a family of tight Borel probability measures defined on a general Hausdorff space implies that the family is compact with respect to a certain topology which is stronger than the classical weak-star topology for measures. This stronger topology is based on semi-continuity, rather than continuity (see Section 2.3), and is not strictly necessary, but it is a more general and stronger result and it simplifies our presentation.

Some basic definitions and results which are needed throughout this work, including the ones concerned with this stronger topology, are present in Chapter 2. In Appendices A and B we have included further details of some results which are mentioned in the text. More specifically, in Appendix A we explore the relation between the classical weak-star topology and the stronger topology defined by Topsoe in the space of measures. In Appendix B we present the proof of a strengthened energy inequality satisfied by the Navier-Stokes equations for a forcing term which is more general than what it was previously considered in other works.

Chapter 2

Basic Tools

In this chapter we introduce the basic concepts underlying our results. Most definitions and statements are given with respect to a topological space X, which is assumed at first to be only a Hausdorff space, although in some situations further properties may be required. In Section 2.1, we give the notation for some function spaces and, in particular, our basic function space, consisting of all the continuous paths in X defined on a given interval $I \subset \mathbb{R}$. In Section 2.2 we recall some well-known facts and definitions from measure theory, setting the notation that is used later. In Section 2.3 we endow the space of finite measures on X with a suitable topology, named the *weak-star semi-continuity topology*, which allow us in particular to make sense of the convergence of measures. When restricted to the subspace of tight finite measures on X, we guarantee that this topology is Hausdorff (Theorem 2.3.1) and also has a useful compactness property (Theorem 2.3.2), which is essential for proving the main results concerned with the existence and convergence of trajectory statistical solutions in Chapters 3 and 4, respectively.

The remaining subsections are needed for the results concerned with statistical solutions in phase space. For that purpose, we must deal directly with evolution equations and, consequently, with weak derivatives, which are recalled in Section 2.4. In Section 2.5, we introduce a special class of functions which play the role of test functions in the context of statistical solutions, the so-called *cylindrical test functions*. Finally, in Section 2.6 we prove a measurability result for *Nemytskii operators*, a type of function which appears naturally in the theory of partial differential equations.

2.1 Function spaces

When working with measures on topological spaces, it is natural that the topological structure is of fundamental importance. In this regard, we recall a few topologies that will play an important role in this work. Besides the fundamental notion of a Hausdorff topological space, in which two distinct points can be separated by disjoint open sets, we say that a topological space is **completely regular** when every nonempty closed set and every singleton disjoint from it can be separated by a continuous function. A completely regular space in which every singleton is closed is sometimes called a **Tychonoff** space. A topological space is called **Polish** when it is separable and completely metrizable.

When X is a Banach space, we denote its dual by X' and the duality product is denoted by $\langle \cdot, \cdot \rangle_{X',X}$. The norm in X is denoted by $\| \cdot \|_X$, while $\| \cdot \|_{X'}$ denotes the usual duality norm. When X is endowed with its weak topology, we denote it by X_w. Similarly, we consider X' endowed with the weak-star topology, in which case we denote it by X'_{w^*}. We also consider the bidual X'' of X, with duality product $\langle \cdot, \cdot \rangle_{X'',X'}$. For the canonical injection $X \hookrightarrow X''$, we simply write $\langle x, f \rangle_{X'',X'} = \langle f, x \rangle_{X',X}$, for all $x \in X$, $f \in X'$.

Let X and Y be Hausdorff spaces. Denote by $\mathcal{C}(Y, X)$ the space of continuous functions on Y with values in X. The **compact-open topology** in $\mathcal{C}(Y, X)$ is the topology generated by the subbase consisting of sets of the form

$$S(K, U) = \{u \in \mathcal{C}(Y, X) \,|\, u(K) \subset U\},$$

where K is a compact subset of Y and U is an open subset of X. When endowed with the compact-open topology, the space $\mathcal{C}(Y, X)$ is denoted by $\mathcal{C}_{\text{loc}}(Y, X)$ and is a Hausdorff space.

The subscript "*loc*" in $\mathcal{C}_{\text{loc}}(Y, X)$ refers to the fact that this topology considers compact sets in Y. When X is a uniform space, the compact-open topology in $\mathcal{C}_{\text{loc}}(Y, X)$ coincides with the **topology of uniform convergence on compact subsets** [27, Theorem 7.11]. This holds, in particular, when X is a topological vector space, which is the case in the applications that we present in Section 3.4.

Now suppose that Y is an interval $I \subset \mathbb{R}$. In this case, we denote $\mathcal{X} = \mathcal{C}_{\text{loc}}(I, X)$. The space $\mathcal{X}$ is called the **space of continuous paths** in X.

For any $t \in I$, let $\Pi_t : \mathcal{X} \to X$ be the "projection" map at time t defined by

$$\Pi_t u = u(t), \quad \forall u \in \mathcal{X}. \tag{2.1}$$

It is readily verified that Π_t is continuous with respect to the compact-open topology. Indeed, given $u \in \mathcal{X}$, consider a neighborhood U of $u(t)$ in X. Then $S(\{t\}, U)$ is a neighborhood of u in $\mathcal{X}$ such that $\Pi_t S(\{t\}, U) \subset U$.

In addition, we consider the space of bounded and continuous real-valued functions on Y, denoted by $\mathcal{C}_b(Y)$. When Y is a subset of $\mathbb{R}^m$, $m \in \mathbb{N}$, we also consider the space of compactly supported and infinitely differentiable real-valued functions on Y, which is denoted by $\mathcal{C}_c^\infty(Y)$.

2.2 Elements of measure theory

Let $\mathfrak{B}_X$ denote the σ-algebra of Borel sets in X. We denote by $\mathcal{M}(X)$ the set of finite and non-negative Borel measures on X, i.e., the set of non-negative measures μ defined on $\mathfrak{B}_X$ such that $\mu(X) < \infty$. The subset of $\mathcal{M}(X)$ consisting of Borel probability measures is denoted by $\mathcal{P}(X)$. The space $\mathcal{M}(X)$ can be identified with a subset of the dual space $\mathcal{C}_b(X)'$ of the space $\mathcal{C}_b(X)$ of bounded and continuous real-valued functions on X.

A **carrier** of a measure is any measurable subset of full measure, i.e., such that its complement has null measure. If C is a carrier for a measure μ, we say that μ is **carried** by C. If the carrier is a single point $x \in X$, the probability measure is a **Dirac measure** and is denoted δ_x. A probability measure that can be written as a convex combination of Dirac measures is called a **discrete measure**.

Given a family of sets $\mathfrak{K} \subset \mathfrak{B}_X$, we say that a Borel measure μ on X is **inner regular with respect to the family** $\mathfrak{K}$ if

$$\mu(A) = \sup\{\mu(K) \,|\, K \in \mathfrak{K} \text{ and } K \subset A\}, \qquad \forall A \in \mathfrak{B}_X. \tag{2.2}$$

We say that a Borel measure μ is **outer regular with respect to the family** $\mathfrak{K}$ if

$$\mu(A) = \inf\{\mu(V) \,|\, V \in \mathfrak{K} \text{ and } A \subset V\}, \qquad \forall A \in \mathfrak{B}_X. \tag{2.3}$$

A **tight** measure is a non-negative Borel measure which is inner regular with respect to the family of compact subsets of X (such a measure is also called a Radon measure, see [4]). If a Borel measure μ on X is both tight and outer regular with respect to the family of open sets of X, then we say that μ is a **regular** measure. When X is a Polish space, i.e. separable and completely metrizable, every finite Borel measure is regular [1, Theorem 12.7]. In case X is just a metrizable space, every finite Borel measure is inner regular with respect to the family of closed subsets of X and outer regular with respect to the family of open sets of X [1, Theorem 12.5] (such a measure is called **normal** in [1]).

For a compact and metrizable space X, it follows in particular from the result in [1, Theorem 12.5] that every finite Borel measure is tight. The metrizability is indeed a necessary condition, since it is possible to construct a finite Borel measure defined on a certain non-metrizable compact Hausdorff space which is not tight (see [1, Example 12.9]).

Furthermore, a net $\{\mu_\alpha\}_\alpha$ of measures in $\mathcal{M}(X)$ is said to be **uniformly tight** if for every $\varepsilon > 0$ we can find a compact set $K \subset X$ such that

$$\mu_\alpha(X \backslash K) < \varepsilon, \quad \forall \alpha.$$

The set of measures $\mu \in \mathcal{M}(X)$ which are tight will be denoted by $\mathcal{M}(X, \text{tight})$. The subset of $\mathcal{M}(X, \text{tight})$ consisting of probability measures is denoted by $\mathcal{P}(X, \text{tight})$.

Now consider a Hausdorff space Y and let $F : X \to Y$ be a Borel measurable function. Then for every measure μ on $\mathfrak{B}_X$ we can define a measure $F\mu$ on $\mathfrak{B}_Y$ as

$$F\mu(E) = \mu(F^{-1}(E)), \quad \forall E \in \mathfrak{B}_Y,$$

which is called the **induced measure from μ by F on $\mathfrak{B}_Y$**, also known as **push-forward of μ by F**. When μ is a tight measure and F is a continuous function, the induced measure $F\mu$ is also tight.

In regard to the concept of induced measures, we also mention the well-known result that if $\varphi : Y \to \mathbb{R}$ is a $F\mu$-integrable function then $\varphi \circ F$ is μ-integrable and

$$\int_X \varphi \circ F \mathrm{d}\mu = \int_Y \varphi \mathrm{d}F\mu \tag{2.4}$$

(see [1, Theorem 13.46]).

For the sake of notation, if $\mu \in \mathcal{M}(X)$ and f is a μ-integrable function, we write

$$\mu(f) = \int_X f \mathrm{d}\mu.$$

In the case of real numbers, we are also interested in the Lebesgue measure, which we denote by λ, and in the Lebesgue subsets of intervals $I \subset \mathbb{R}$. We denote the σ-algebra of those sets by $\mathfrak{L}_I$.

When we have two topological spaces X and Y, with Y continuously imbedded into X, meaning that there exists a continuous injective map $j : Y \to X$, we are interested in knowing whether the Borel subsets of Y are taken into Borel subsets of X by the injection j. In the particular case that X and Y are Polish spaces, then Theorem 6.8.6 of [4] guarantees that in fact $j(B)$ is Borel in X for any Borel B in Y (the statement of this theorem actually requires X to be more generally a Souslin space, which is defined as a continuous image of a Polish space).

In a related topic, when X is a Banach space, we are also interested in comparing the Borel sets obtained from the strong and the weak topologies. In general, since the strong topology is finer than the weak topology, every Borel set for the weak topology is also a Borel set for the strong topology. Conversely, if X is a separable Banach space, then every Borel set for the strong topology is also a Borel set for the weak topology, so that in this case both Borel σ-algebras coincide. This latter fact is easily proved by observing that, since X is separable, every open set $A \subset X$ can be written as a countable union of open balls in X, say

$$A = \bigcup_{n \in \mathbb{N}} B(x_n; r_n),$$

where $x_n \in X$, $r_n > 0$ and $B(x_n; r_n)$ denotes the open ball centered at x_n with radius r_n in X, i.e.

$$B(x_n; r_n) = \{x \in X \mid \|x - x_n\| < r_n\},$$

with $\|\cdot\|$ being the norm in X. But we may also write

$$B(x_n; r_n) = \bigcup_{k \in \mathbb{N}} \left\{x \in X \mid \|x - x_n\| \leq r_n - \frac{1}{k}\right\} = \bigcup_{k \in \mathbb{N}} C_{n,k}.$$

And since each $C_{n,k}$ is a convex and closed set in the strong topology, it is also closed with respect to the weak topology [5, Theorem III.7]. Thus, $A = \bigcup_n \bigcup_k C_{n,k}$ is a Borel set for the weak topology.

2.3 Topologies for measure spaces and related results

Given a function $f : X \to \mathbb{R}$, consider the mapping $J_f : \mathcal{M}(X) \to \mathbb{R}$ defined as

$$J_f(\mu) = \mu(f) = \int_X f \mathrm{d}\mu, \quad \forall \mu \in \mathcal{M}(X). \tag{2.5}$$

In [49], Topsoe considered a topology in $\mathcal{M}(X)$ obtained as the smallest one for which the mappings J_f are upper semi-continuous, for every bounded and upper semi-continuous real-valued function f on X. Topsoe calls this topology the "weak topology", but in order to avoid any confusion we call it here the **weak-star semi-continuity topology** on $\mathcal{M}(X)$. A sub-basis for this topology is given by the family of sets

$$\{J_f^{-1}((-\infty, b)) \mid \quad b \in \mathbb{R},\ f : X \to \mathbb{R} \text{ is bounded and upper semi-continuous}\}.$$

If a net $\{\mu_\alpha\}_\alpha$ converges to μ with respect to this topology, we denote $\mu_\alpha \overset{wsc}{\rightharpoonup} \mu$.

A more common topology used in $\mathcal{M}(X)$ is the **weak-star topology**, which is the smallest topology for which the mapppings J_f are continuous, for every bounded and continuous real-valued function f on X, i.e., $f \in \mathcal{C}_b(X)$. In this case, a sub-basis is given by the following family:

$$\{J_f^{-1}((a, b)) \mid \quad a, b \in \mathbb{R},\ f : X \to \mathbb{R} \text{ is bounded and continuous}\}.$$

If a net $\{\mu_\alpha\}_\alpha$ converges to μ with respect to this topology, we denote $\mu_\alpha \overset{w^*}{\rightharpoonup} \mu$.

Lemma 2.3.1 below provides some useful characterizations for the weak-star semi-continuity topology (see [49, Theorem 8.1]). According to this result, the weak-star topology is in general weaker than the weak-star semi-continuity topology. Moreover, if X is a completely regular Hausdorff topological space, then these two topologies coincide when restricted to the space $\mathcal{M}(X, \text{tight})$. Also, another characterization implies that the weak-star semi-continuity topology can be equivalently defined as the smallest topology for which the mappings J_f are lower semi-continuous, for every bounded and lower semi-continuous real-valued function f on X. In Appendix A, this simple fact and also the relation between the two topologies are proved in detail by using the sub-basis of each topology.

Lemma 2.3.1. *Let X be a Hausdorff space. For a net $\{\mu_\alpha\}_\alpha$ in $\mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$, consider the following statements:*

(1) $\mu_\alpha \overset{wsc}{\rightharpoonup} \mu$;

(2) $\limsup \mu_\alpha(f) \leq \mu(f)$, *for all bounded and upper semicontinuous function* $f : X \to \mathbb{R}$;

(3) $\liminf \mu_\alpha(f) \geq \mu(f)$, *for all bounded and lower semicontinuous function* $f : X \to \mathbb{R}$;

(4) $\lim_\alpha \mu_\alpha(X) = \mu(X)$ *and* $\limsup \mu_\alpha(F) \leq \mu(F)$, *for all closed set* $F \subset X$;

(5) $\lim_\alpha \mu_\alpha(X) = \mu(X)$ *and* $\liminf \mu_\alpha(G) \geq \mu(G)$, *for all open set* $G \subset X$;

(6) $\lim_\alpha \mu_\alpha(f) = \mu(f)$, *for all bounded and continuous function* $f : X \to \mathbb{R}$.

Then the first five statements are equivalent and each of them implies the last one.

Furthermore, if X is a completely regular space and $\mu \in \mathcal{M}(X, \text{tight})$, then all six statements are equivalent.

Although our framework is based on a general Hausdorff space, the proofs rely on reducing some structures to compact subsets, hence completely regular spaces. Then, since the measures in our proofs are also usually tight, both topologies coincide in this reduced setting, so that we could have very well considered only the weak-star topology. However, we prefer to use the weak-star semi-continuity topology since it is a more natural topology for arbitrary Hausdorff spaces which simplifies our presentation and yields a compactness result in a stronger topology.

When dealing with convergent nets in a given space, a natural question arises as to whether the limits are unique. This requires the given space to be Hausdorff, a condition that we would like to be satisfied by a suitable space of measures. The delicate issue is to determine the minimal hypotheses for that.

If X is a metrizable topological space, then $\mathcal{M}(X)$ turns out to be a Hausdorff space with respect to the weak-star topology (see [1, Section 15.1]), and hence also with respect to the weak-star semi-continuity topology. However, requiring X to be metrizable is too restrictive for our purposes. In an attempt to establish a more general setting for the space X, we were led to work within the space of tight measures $\mathcal{M}(X, \text{tight})$, which

Topsoe proved to be Hausdorff with respect to the weak-star semi-continuity topology for any Hausdorff space X. This key result was a motivation for Topsoe to advance his work on the subject (see [49, Preface]). A proof is given in [49, Theorem 11.2] by showing that the limits of convergent nets are unique. Here we chose to prove it by showing directly that distinct measures in $\mathcal{M}(X)$ can be separated by open sets.

Theorem 2.3.1. *Let X be a Hausdorff space. Then, $\mathcal{M}(X, tight)$ is a Hausdorff space with respect to the weak-star semi-continuity topology.*

The proof is an easy consequence of the following lemma:

Lemma 2.3.2. *Let μ_1, μ_2 be two distinct measures in $\mathcal{M}(X, tight)$ such that $\mu_1(X) = \mu_2(X)$. Then there exists a Borel set $A \subset X$ satisfying $\mu_1(\overline{A}) < \mu_2(\mathring{A})$, where $\overline{A}$ and $\mathring{A}$ denote the closure and interior of A, respectively.*

Proof. Let us suppose by contradiction that

$$\mu_1(\overline{A}) \geq \mu_2(\mathring{A}), \quad \forall A \in \mathfrak{B}_X. \tag{2.6}$$

Consider $E \in \mathfrak{B}_X$ and let K_1, K_2 be arbitrary compact sets in X satisfying $K_1 \subset X \backslash E$ and $K_2 \subset E$. Then, since X is a Hausdorff space, there exist disjoint open sets B_1, B_2 in X such that $K_1 \subset B_1$ and $K_2 \subset B_2$.

In particular, using that B_2 is an open set, it follows from (2.6) that

$$\mu_2(B_2) = \mu_2(B_2^\circ) \leq \mu_1(\overline{B_2}). \tag{2.7}$$

But clearly $\overline{B_2} \subset X \backslash K_1$. Then,

$$\mu_1(\overline{B_2}) \leq \mu_1(X \backslash K_1) = \mu_1(X) - \mu_1(K_1) \tag{2.8}$$

From (2.7) and (2.8), we obtain that

$$\mu_2(K_2) \leq \mu_2(B_2) \leq \mu_1(X) - \mu_1(K_1).$$

Now since K_1 and K_2 were chosen arbitrarily, taking the supremum over all compact sets K_1, K_2 with $K_1 \subset X \backslash E$ and $K_2 \subset E$, it follows that

$$\sup\{\mu_2(K_2) \,|\, K_2 \text{ compact}, K_2 \subset E\} \leq \mu_1(X) - \sup\{\mu_1(K_1) \,|\, K_1 \text{ compact}, K_1 \subset X \backslash E\}.$$

But since μ_1 and μ_2 are tight, then

$$\mu_2(E) \leq \mu_1(X) - \mu_1(X \backslash E) = \mu_1(E).$$

Thus,

$$\mu_2(E) \leq \mu_1(E), \quad \forall E \in \mathfrak{B}_X. \tag{2.9}$$

This implies in particular that

$$\mu_2(X) - \mu_2(E) = \mu_2(X \backslash E) \leq \mu_1(X \backslash E) = \mu_1(X) - \mu_1(E).$$

Using the hypothesis that $\mu_1(X) = \mu_2(X)$, we then obtain

$$\mu_2(E) \geq \mu_1(E), \quad \forall E \in \mathfrak{B}_X. \tag{2.10}$$

Now (2.9) and (2.10) yield $\mu_1 = \mu_2$, which is a contradiction. $\square$

Now let us give the proof of Theorem 2.3.1.

Proof of Theorem 2.3.1. Consider two distinct measures μ_1, μ_2 in $\mathcal{M}(X, \text{tight})$.

Suppose at first that $\mu_1(X) \neq \mu_2(X)$. Let us assume, without loss of generality, that $\mu_1(X) < \mu_2(X)$. Then there exists $\varepsilon > 0$ such that

$$\mu_1(X) < \mu_1(X) + \varepsilon < \mu_2(X) - \varepsilon < \mu_2(X). \tag{2.11}$$

Denote $a = \mu_1(X) + \varepsilon$ and $b = \mu_2(X) - \varepsilon$. Let f be the constant function $f \equiv 1$ on X and let J_f be the corresponding mapping as defined in (2.5). Then it follows from (2.11) that $\mu_1 \in J_f^{-1}((-\infty, a))$ and $\mu_2 \in J_f^{-1}((b, +\infty))$. Since f is in particular a bounded and continuous real-valued function on X, it follows that $J_f^{-1}((-\infty, a))$ and $J_f^{-1}((b, +\infty))$ are open sets in X (see Proposition A.1). These sets are also clearly disjoint. Then μ_1 and μ_2 can be separated by disjoint open sets in X.

Now suppose that $\mu_1(X) = \mu_2(X)$. Then, from Lemma 2.3.2, there exists $A \in \mathfrak{B}_X$ such that $\mu_1(\overline{A}) < \mu_2(\mathring{A})$. The argument now follows analogously to the previous case. Consider $\varepsilon > 0$ satisfying

$$\mu_1(\overline{A}) < \mu_1(\overline{A}) + \varepsilon < \mu_2(\mathring{A}) - \varepsilon < \mu_2(\mathring{A})$$

and denote $a' = \mu_1(\overline{A}) + \varepsilon$ and $b' = \mu_2(\mathring{A}) - \varepsilon$. Since the characteristic function of $\overline{A}$, $\chi_{\overline{A}}$, is bounded and upper semi-continuous, and the characteristic function of $\mathring{A}$, $\chi_{\mathring{A}}$, is bounded and lower semicontinuous, then $J_{\chi_{\overline{A}}}^{-1}((-\infty, a'))$ and $J_{\chi_{\mathring{A}}}^{-1}((b', +\infty))$ are clearly disjoint open sets in X containing μ_1 and μ_2, respectively. Thus, μ_1 and μ_2 can also be separated by disjoint open sets of X in this case.

This proves that $\mathcal{M}(X, \text{tight})$ is a Hausdorff space. $\square$

Moreover, if X is assumed to be a completely regular space, then by Lemma 2.3.1 the weak-star semi-continuity and weak-star topologies are the same in $\mathcal{M}(X, \text{tight})$. Thus, it follows from Theorem 2.3.1 that $\mathcal{M}(X, \text{tight})$ is also Hausdorff with respect to the weak-star topology. We have just proved the following corollary:

Corollary 2.3.1. *Let X be a completely regular space. Then $\mathcal{M}(X, \text{tight})$ is a Hausdorff space with respect to the weak-star topology.*

Remark 2.3.1. If we relaxed the hypothesis in Corollary 2.3.1 by assuming X to be a regular space instead of a completely regular space, then this result would no longer be valid. Indeed, it is possible to construct examples of regular spaces which are not completely regular and containing two distinct points a, b for which every continuous real-valued function satisfies $f(a) = f(b)$ (see [37, 39]). The corresponding space $\mathcal{M}(X, t)$ is then not Hausdorff with respect to the weak-star topology, for it suffices to consider the Dirac measures δ_a and δ_b concentrated on a and b, respectively, and to note that $\delta_a(f) = \delta_b(f)$, for every $f \in \mathcal{C}_b(X)$.

Corollary 2.3.1 implies a useful characterization for the equality between two measures in $\mathcal{M}(X, \text{tight})$ when X is a completely regular space. This is presented in the following result.

Corollary 2.3.2. *Let X be a completely regular space and consider $\mu_1, \mu_2 \in \mathcal{M}(X, \text{tight})$. Then the following statements are equivalent:*

(i) $\mu_1 = \mu_2$*;*

(ii) $\mu_1(\varphi) = \mu_2(\varphi)$*, for every* $f \in \mathcal{C}_b(X)$*.*

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. Let us then prove that (ii) $\Rightarrow$ (i). Suppose on the contrary that $\mu_1 \neq \mu_2$. Since X is completely regular, by Corollary 2.3.1, $\mathcal{M}(X, \text{tight})$ is a Hausdorff space. Then, in particular, there exists a function $\varphi \in \mathcal{C}_b(X)$ and an interval $(a, b) \subset \mathbb{R}$ such that $\mu_1 \in J_\varphi^{-1}((a, b))$, but $\mu_2 \notin J_\varphi^{-1}((a, b))$. However, by hypothesis, $\mu_1(\varphi) = \mu_2(\varphi)$. This implies that $\mu_2 \in J_\varphi^{-1}((a, b))$, which is a contradiction. □

We next state a result of compactness on the space of tight measures $\mathcal{M}(X, \text{tight})$ that is going to be essential for our main result. For a proof of this fact, see [49, Theorem 9.1].

Theorem 2.3.2. *Let X be a Hausdorff topological space and let $\{\mu_\alpha\}_\alpha$ be a net in $\mathcal{M}(X, \text{tight})$ such that $\limsup \mu_\alpha(X) < \infty$. If $\{\mu_\alpha\}_\alpha$ is uniformly tight, then it is compact with respect to the weak-star semi-continuity topology in $\mathcal{M}(X, \text{tight})$.*

The previous theorem allows us to obtain a convergent subnet of a given net in $\mathcal{M}(X, \text{tight})$, provided it satisfies the required conditions. Also, by using Theorem 2.3.1, we can guarantee that the limit of this convergent subnet is unique.

Evidently, all the results shown above are also valid in the space of tight probability measures. In the next section, these results are applied in that space, since it is the natural one in the context of statistical solutions. We consider both the spaces of tight probability measures defined over the Hausdorff space X and over the space of continuous paths $\mathcal{X}$.

2.4 Weak derivatives

Given a Banach space Z and an interval $I \subset \mathbb{R}$, we consider the space $L^1_{loc}(I, Z)$ of locally integrable functions defined on I and with values in Z. Given a function $u \in L^1_{loc}(I, Z)$, we say that u has a derivative in the weak sense when there exists $v \in L^1_{loc}(I, Z)$ such that

$$\int_I u(t)\varphi_t(t)dt = -\int_I v(t)\varphi(t)dt$$

in Z, for all $\varphi \in \mathcal{C}_c^\infty(I)$, where φ_t is the derivative of φ. When such v exists, it is unique in $L^1_{loc}(I, Z)$. In this case v is called the **weak derivative** of u and is denoted u_t. We denote the space of functions with such weak derivatives by

$$W^{1,1}_{loc}(I, Z) = \{u \in L^1_{loc}(I, Z) : u_t \in L^1_{loc}(I, Z)\}.$$

For $w \in Z'$, it follows in particular that

$$\int_I \varphi_t(t)\langle w, u(t)\rangle_{Z',Z}dt = -\int_I \varphi(t)\langle w, u_t(t)\rangle_{Z',Z}dt,$$

for all test functions $\varphi \in \mathcal{C}_c^\infty(I)$. Choosing sequences of test functions converging to the characteristic function of subintervals $[t', t] \subset I$, we find, using the Lebesgue Differentiation Theorem, that

$$\langle w, u(t)\rangle_{Z',Z} - \langle w, u(t')\rangle_{Z',Z} = \int_{t'}^t \langle w, u_t(s)\rangle_{Z',Z}ds, \quad \text{for almost all } t', t \in I. \tag{2.12}$$

In particular, u is almost everywhere equal to a continuous function from I into Z (see [46, Lemma 3.1.1]).

2.5 Cylindrical test functions

Consider a Banach space Y and let $v_1, \ldots, v_k \in Y$, where $k \in \mathbb{N}$. Let ϕ be a continuously differentiable real-valued function on $\mathbb{R}^k$ with compact support. For each $u \in Y'$, define $\Phi(u) \in \mathbb{R}$ by

$$\Phi(u) = \phi(\langle u, v_1\rangle_{Y',Y}, \ldots, \langle u, v_k\rangle_{Y',Y}).$$

The function Φ is clearly continuous from Y' to $\mathbb{R}$ and in fact it is Fréchet differentiable in Y', with Fréchet derivative

$$\Phi'(u) = \sum_{j=1}^{k} \partial_j \phi(\langle u, v_1\rangle_{Y',Y}, \ldots, \langle u, v_k\rangle_{Y',Y}) v_j, \tag{2.13}$$

where $\partial_j \phi$ denotes the derivative of ϕ with respect to its j-th coordinate.

Functions of this form are called **cylindrical test functions** in Y' and play an important role as test functions in the definition of statistical solution in phase space. In that context, we will also consider a Hausdorff topological space X which is assumed to be continuously imbedded in Y'_{w*}, where Y'_{w*} denotes the space Y' endowed with its weak-star topology. Notice that since $u \mapsto \langle u, v\rangle_{Y',Y}$ is continuous in the weak-star topology for any $v \in Y$, the function Φ is also continuous from Y'_{w*} into $\mathbb{R}$. Then, since X is continuously imbedded in Y'_{w*}, we may consider Φ restricted to X, and which is continuous as a function from X into $\mathbb{R}$.

The set of cylindrical test functions is a relatively large set, as can be seen from the Stone-Weierstrass Theorem (see e.g. [13, Theorem IV.6.16]). In fact, consider the closed ball $B_r = \{u \in Y';\ \|u\|_{Y'} \leq r\}$, $r > 0$, which is a compact set in Y'_{w*}, and denote by $\mathcal{S}_r$ the collection of the functions which are the restriction to B_r of the cylindrical test functions. Clearly, $\mathcal{S}_r \subset \mathcal{C}(B_r)$ and, if $\Psi_1, \Psi_2 \in \mathcal{S}_r$, then their sum $\Psi_1 + \Psi_2$ and their product $\Psi_1\Psi_2$ also belong to $\mathcal{S}_r$. By choosing $v \in Y$ with $\|v\|_Y \leq 1$ and $\phi : \mathbb{R} \to \mathbb{R}$ as a continuously differentiable function which is equal to 1 on the interval $[-r, r]$ and is compactly supported, we see that $\Phi(u) = \phi(\langle u, v\rangle_{Y',Y})$ is a cylindrical test function which is equal to 1 on the ball B_r, showing that $\mathcal{S}_r$ contains the unit element. Moreover, if u_1, u_2 are distinct points in B_r, then there exists $v \in Y$ such that $\langle u_1, v\rangle_{Y',Y} \neq \langle u_2, v\rangle_{Y',Y}$. By choosing a continuously differentiable function $\phi : \mathbb{R} \to \mathbb{R}$ which is compactly supported and assumes different values at the points $\langle u_1, v\rangle_{Y',Y}$ and $\langle u_2, v\rangle_{Y',Y}$, we see that $\Phi(u) = \phi(\langle u, v\rangle_{Y',Y})$ is a cylindrical test function which assumes different values at u_1 and u_2, proving that $\mathcal{S}_r$ separates the points in B_r. Therefore, the Stone-Weierstrass Theorem yields that $\mathcal{S}_r$ is dense in $\mathcal{C}(B_r)$, for the uniform topology. This result also holds with B_r replaced by any compact subset of Y'_{w*}. Similarly, since X is continuously imbedded into Y'_{w*}, it can be showed that for any compact subset K of X, the collection of the functions which are the restrictions to K of cylindrical test functions is dense in $\mathcal{C}(K)$.

2.6 The Nemytskii operator

The statistical solutions in a phase space X are directly related to an evolution equation of the form

$$u_t(t) = F(t, u(t)), \tag{2.14}$$

where the unknown u belongs to the space $\mathcal{X} = \mathcal{C}_{\mathrm{loc}}(I, X)$ and F is a given function defined on $I \times Y$, for a time-interval I in $\mathbb{R}$ and a subset Y of X. In this case, it is natural to extend F to an operator G defined on $I \times \mathcal{V}$, where $\mathcal{V}$ is a subset of $\mathcal{X}$. This extended operator G is known as a **Nemytskii operator** in the context of partial differential equations. In the following lemma we prove that, in our general context, if F is measurable then G is also measurable.

Lemma 2.6.1. *Let Y be a Banach space and X be a Hausdorff space such that Y is a subset of X and every Borel subset of Y is a Borel subset of X. Let I be an interval in $\mathbb{R}$ and $\mathcal{V}$ be a subset of $\mathcal{X} = \mathcal{C}_{loc}(I, X)$, endowed with the topology inherited from $\mathcal{X}$. Suppose that $F : I \times Y \to Y'$ is a $(\mathfrak{L}_I \otimes \mathfrak{B}_Y, \mathfrak{B}_{Y'})$-measurable function. Then, the function $G : I \times \mathcal{V} \to Y'$, defined by*

$$G(t, u) = \begin{cases} F(t, u(t)), & \text{if } u(t) \in Y, \\ 0, & \text{if } u(t) \in X \setminus Y, \end{cases} \tag{2.15}$$

is $(\mathfrak{L}_I \otimes \mathfrak{B}_{\mathcal{V}}, \mathfrak{B}_{Y'})$-measurable.

Proof. Consider the functions

$$\begin{aligned} \tilde{F} : I \times X &\to Y' \\ (t, u) &\mapsto \tilde{F}(t, u) = \begin{cases} F(t, u), & \text{if } u \in Y, \\ 0, & \text{if } u \in X \setminus Y, \end{cases} \end{aligned}$$

$$\begin{aligned} \Pi_I : I \times \mathcal{V} &\to I \\ (t, u) &\mapsto t \end{aligned}$$

and

$$\begin{aligned} U : I \times \mathcal{V} &\to X \\ (t, u) &\mapsto u(t). \end{aligned} \tag{2.16}$$

Then Π_I is clearly a $(\mathfrak{L}_I \otimes \mathfrak{B}_{\mathcal{V}}, \mathfrak{L}_I)$-measurable function. Moreover, it is not difficult to see that U is a continuous function and then, in particular, $(\mathfrak{B}_{I \times \mathcal{V}}, \mathfrak{B}_X)$-measurable. But since I is a second countable space it follows that $\mathfrak{B}_{I \times \mathcal{V}} = \mathfrak{B}_I \otimes \mathfrak{B}_{\mathcal{V}}$ [4, Lemma 6.4.2]. Thus, U is also a $(\mathfrak{L}_I \otimes \mathfrak{B}_{\mathcal{V}}, \mathfrak{B}_X)$-measurable function. From this we obtain that the function $(\Pi_I, U) : I \times \mathcal{V} \to I \times X$, defined by $(\Pi_I, U)(t, u) = (t, u(t))$, is $(\mathfrak{L}_I \otimes \mathfrak{B}_{\mathcal{V}}, \mathfrak{L}_I \otimes \mathfrak{B}_X)$-measurable.

Furthermore, we have that $\tilde{F}$ is $(\mathfrak{L}_I \otimes \mathfrak{B}_X, \mathfrak{B}_{Y'})$-measurable. Indeed, let $E \in \mathfrak{B}_{Y'}$ and note that

$$\tilde{F}^{-1}(E) = \begin{cases} F^{-1}(E) \cup (I \times (X \setminus Y)), & \text{if } 0 \in E, \\ F^{-1}(E), & \text{if } 0 \notin E. \end{cases}$$

Since F is $(\mathfrak{L}_I \otimes \mathcal{B}_Y, \mathfrak{B}_{Y'})$-measurable then $F^{-1}(E) \in \mathfrak{L}_I \otimes \mathfrak{B}_Y$. Then, together with the hypothesis that $\mathfrak{B}_Y \subset \mathfrak{B}_X$, this implies that $F^{-1}(E) \in \mathfrak{L}_I \otimes \mathfrak{B}_X$. Moreover, since $X \setminus Y \in \mathfrak{B}_X$, it follows that $\tilde{F}^{-1}(E) \in \mathfrak{L}_I \otimes \mathcal{B}_X$.

Now, since $G = \tilde{F} \circ (\Pi_I, U)$ we conclude that G is $(\mathcal{L}_I \otimes \mathfrak{B}_{\mathcal{V}}, \mathfrak{B}_{Y'})$-measurable. $\square$

Remark 2.6.1. The condition in Lemma 2.6.1 that every Borel subset of Y is a Borel subset of X is not a very restrictive one for the applications. For instance, in view of the results mentioned at the end of Section 2.2, this condition holds when X and Y are separable Banach spaces, with either the weak or the strong topology. This includes, for instance, $Y = W^{1,p}(\Omega)$ and X as the space $L^p(\Omega)$ with either the weak or the strong topology, where $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $1 \leq p < \infty$.

Remark 2.6.2. The subset $\mathcal{V}$ in the statement of Lemma 2.6.1 seems artificial, but it is done in this way in view of the applications, in particular for proving Theorem 3.3.1, where some properties will only be valid for a subset of the whole space $\mathcal{X}$. Moreover, we could actually have proved the result for G defined on $I \times \mathcal{X}$ and then concluded that the restriction of G to $I \times \mathcal{V}$ would also be measurable, provided $\mathcal{V}$ is a Borel subset of $\mathcal{X}$. However, the approach that we follow, working already with a subset $\mathcal{V}$, is more direct and does not require $\mathcal{V}$ to be Borel, although this will be a necessary condition in other places.

Chapter 3

Existence of Statistical Solutions

In this chapter we aim at proving the existence of statistical solutions for some initial-value problems in an abstract setting. First, in Section 3.1, we give our general definitions of statistical solutions in trajectory space and in phase space. In Sections 3.2 and 3.3 we prove the main results on the existence of these general types of statistical solutions with respect to a given initial data. These results require a certain set of hypotheses to be satisfied by a subset $\mathcal{U}$ of the trajectory space $\mathcal{X}$, which we call **hypothesis (H)**. In Section 3.4 we give applications of this general framework for the Navier-Stokes equations, a reaction-diffusion equation, a nonlinear wave equation and the Bénard problem.

3.1 Types of Statistical Solutions

We first define statistical solutions in the space of continuous paths $\mathcal{X}$. They are named **trajectory statistical solutions**, owing to the fact that they are measures carried by a measurable subset of a certain set $\mathcal{U}$ in $\mathcal{X}$ which, in applications, would consist in the set of trajectories, i.e. the set of solutions, in an appropriate sense, of a given evolution equation. At this abstract point, however, there is no evolution equation, and the problem is simply to find a measure carried by a given subset of $\mathcal{X}$. As such, this is a trivial problem, as showed in Remark 3.1.4. The interesting and difficult problem is the corresponding Initial Value Problem 3.1.1. Nevertheless, we start with the following definition.

Definition 3.1.1. *Let X be a Hausdorff topological space and let $I \subset \mathbb{R}$ be an arbitrary interval. Consider $\mathcal{X} = \mathcal{C}_{loc}(I, X)$ and let $\mathcal{U}$ be a subset of $\mathcal{X}$. We say that a Borel probability measure ρ on $\mathcal{X}$ is a **$\mathcal{U}$-trajectory statistical solution over** I (or simply a **trajectory statistical solution**) if*

(i) ρ is tight;

(ii) ρ *is carried by a Borel subset of* $\mathcal{X}$ *included in* $\mathcal{U}$*, i.e. there exists* $\mathcal{V} \in \mathfrak{B}_{\mathcal{X}}$ *such that* $\mathcal{V} \subset \mathcal{U}$ *and* $\rho(\mathcal{X} \setminus \mathcal{V}) = 0$*.*

Remark 3.1.1. Our abstract definition of a trajectory statistical solution was inspired by the concept of a Vishik-Fursikov measure given in [22], which in turn was inspired by the definition of space-time statistical solution defined by Vishik and Fursikov [53] (see also [55]). Such measures are defined within the context of the Navier-Stokes equations and have the property of being carried by their set of weak solutions, called Leray-Hopf weak solutions. In [22, Propositions 2.9 and 2.12] it is proved that the set of Leray-Hopf weak solutions is a Borel set in the corresponding space of continuous paths. However, since we do not know whether this is always the case in every application, we prefer not to assume that $\mathcal{U}$ is Borel, and assume instead that there exists a Borel subset of $\mathcal{U}$ that carries the measure.

Remark 3.1.2. From the Definition 3.1.1, however, we see that $\mathcal{U} \setminus \mathcal{V}$ belongs to the Borel null set $\mathcal{X} \setminus \mathcal{V}$, hence we can certainly say that $\mathcal{U}$ is measurable with respect to the Lebesgue completion of ρ, which we denote by $\bar{\rho}$, and so that $\bar{\rho}$ is carried by $\mathcal{U}$.

Now we define statistical solutions in phase space. Unlike the definition of a trajectory statistical solution, which is given by a single measure defined on $\mathcal{X}$, this second type of statistical solution consists in a family of measures defined on the Hausdorff space X and parametrized by an index t varying in an interval $I \subset \mathbb{R}$. The terminology is again derived from the applications, in which X would stand for the phase space of a certain evolution equation and t the time variable. While t varies in I, the family of measures describes the evolution of statistical information of the system. We write the evolution equation in a general form

$$u_t = F(t, u).$$

We require that $F : I \times Y \to Y'$, where Y is a Banach space such that $Y \subset X \subset Y'$, and Y' is the dual space of Y. For the evolution equation to make sense, it is necessary that $u(t)$ belongs to Y for almost every t and that u_t belongs to the dual space Y'. With this in mind, we recall the space $W^{1,1}_{loc}(I, Y')$ defined in Section 2.4 (with $Z = Y'$) and introduce the space

$$\mathcal{Z} = \{u \in \mathcal{C}_{\text{loc}}(I, X) \cap W^{1,1}_{\text{loc}}(I, Y') : u(t) \in Y \text{ for almost all } t \in I\}. \tag{3.1}$$

It is thus required that $\mathcal{U} \subset \mathcal{Z}$ for the evolution equation to make sense. We then have the following definition of statistical solution in phase space:

Definition 3.1.2. *Let* X *be a Hausdorff space and* Y *be a Banach space such that* $Y \subset X \subset Y'$*, where* Y' *denotes the dual space of* Y*. Let* $I \subset \mathbb{R}$ *be an arbitrary interval.*

Assume $\mathcal{U} \subset \mathcal{Z}$ and suppose $F : I \times Y \to Y'$ is such that

$$u_t(t) = F(t, u(t)), \qquad a.e.\ t \in I,\ \forall u \in \mathcal{U}. \tag{3.2}$$

We say that a family $\{\rho_t\}_{t\in I}$ of Borel probability measures in X is a ***statistical solution in phase space*** *(or simply a* ***statistical solution****) of the evolution equation (3.2) if the following conditions are satisfied:*

(i) The function

$$t \mapsto \int_X \varphi(u)\mathrm{d}\rho_t(u)$$

is continuous on I, for every $\varphi \in \mathcal{C}_b(X)$.

(ii) For almost every $t \in I$, the measure ρ_t is carried by Y and the function $u \mapsto F(t, u)$ is ρ_t-integrable. Moreover, the map

$$t \mapsto \int_X \|F(t, u)\|_{Y'}\mathrm{d}\rho_t(u)$$

belongs to $L^1_{loc}(I)$.

(iii) For any cylindrical test function Φ, it holds

$$\int_X \Phi(u)\mathrm{d}\rho_t(u) = \int_X \Phi(u)\mathrm{d}\rho_{t'}(u) + \int_{t'}^{t}\int_X \langle F(u), \Phi'(u)\rangle_{Y',Y}\mathrm{d}\rho_s(u)\mathrm{d}s, \tag{3.3}$$

for all $t, t' \in I$.

In Definition 3.1.2, equation (3.3) represents a Liouville-type equation similar to that from statistical mechanics. In order to motivate the definition (3.3), let us suppose that a particular statistical solution $\{\rho_t\}_{t\in I}$ is given in the form of a convex combination of Dirac measures,

$$\rho_t = \frac{1}{N}\sum_{n=1}^{N}\delta_{u_n(t)}, \quad t \in I,$$

with equal probability $1/N$, where $N \in \mathbb{N}$, and each u_n is a smooth solution of the system

$$u_t(t) = F(t, u(t)), \quad t \in I.$$

We then formally have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\int_X \Phi(u)\mathrm{d}\rho_s(u) &= \frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{N}\sum_{n=1}^N \Phi(u_n(s)) = \frac{1}{N}\sum_{n=1}^N \frac{\mathrm{d}}{\mathrm{d}s}\Phi(u_n(s)) \\
&= \frac{1}{N}\sum_{n=1}^N \left\langle \frac{\mathrm{d}}{\mathrm{d}s}u_n(s), \Phi'(u_n(s))\right\rangle_{Y',Y} = \frac{1}{N}\sum_{n=1}^N \langle F(s,u_n(s)), \Phi'(u_n(s))\rangle_{Y',Y} \\
&= \int_X \langle F(s,u), \Phi'(u)\rangle_{Y',Y}\mathrm{d}\rho_s(u). \qquad (3.4)
\end{aligned}$$

Thus, integrating with respect to s on $[t', t]$ yields (3.3).

In Subsection 3.3 (Theorem 3.3.1), we prove that the family of measures obtained as the projections of a trajectory statistical solution at each $t \in I$ on the space X is a statistical solution. This shows that every trajectory statistical solution on $\mathcal{X}$ yields a statistical solution on X. However, the converse is not necessarily true. We then call a statistical solution for which the converse is valid, i.e. which can be written as the projections on X of a trajectory statistical solution, a **projected statistical solution**, as defined below.

Definition 3.1.3. *Let X be a Hausdorff space and Y be a Banach space such that $Y \subset X \subset Y'$. Let $I \subset \mathbb{R}$ be an arbitrary interval. Assume $\mathcal{U} \subset \mathcal{Z}$ and suppose $F : I \times Y \to Y'$ is such that*

$$u_t(t) = F(t, u(t)), \qquad a.e.\ t \in I,\ \forall u \in \mathcal{U}.$$

We say that a family $\{\rho_t\}_{t\in I}$ in $\mathcal{P}(X)$ is a ***statistical solution projected from a $\mathcal{U}$-trajectory statistical solution****, or simply a* ***projected statistical solution****, when $\{\rho_t\}_{t\in I}$ is a statistical solution in the sense of Definition 3.1.2 and there exists a $\mathcal{U}$-trajectory statistical solution ρ such that $\rho_t = \Pi_t\rho$, for every $t \in I$.*

Remark 3.1.3. Assuming that Y is a Borel subset of X, we can prove that given a projected statistical solution $\{\rho_t\}_t$, the first statement in item (ii) of Definition 3.1.2 is also a consequence of the fact that every measure ρ_t is written as $\Pi_t\rho$, for some $\mathcal{U}$-trajectory statistical solution ρ, and the assumption that $\mathcal{U} \subset \mathcal{Z}$. In order to prove this, consider the real-valued function $g : X \to [0, +\infty]$ given by

$$g(u) = \begin{cases} 1, & \text{if } u \in Y, \\ +\infty, & \text{if } u \in X \setminus Y. \end{cases}$$

Since $\mathcal{U} \subset \mathcal{Z}$, it follows that $u(s) \in Y$, and hence $g(u(s)) = 1$, for almost every $s \in I$, for all $u \in \mathcal{U}$. Thus,

$$\int_{\mathcal{U}}\int_{t'}^{t} g(u(s))\,\mathrm{d}s\mathrm{d}\rho(u) = \int_{\mathcal{U}}\int_{t'}^{t} 1\,\mathrm{d}s\mathrm{d}\rho(u) = t - t'. \qquad (3.5)$$

Notice that the map $(s, u) \mapsto g(u(s))$ is the composition of the function g with the evaluation operator $U(t, u) = u(t)$ given in (2.16) (with $\mathcal{V} = \mathcal{X}$). Since Y is a Borel set in X, then g is a measurable function on X. Moreover, since the evaluation operator U is continuous and hence measurable from $I \times \mathcal{X}$ into X, it follows that the composition map $(s, u) \mapsto g(u(s))$ is also measurable on $I \times \mathcal{X}$. Now, we apply Tonelli's Theorem [1, Theorem 11.28] to the left-hand side of (3.5) and obtain that

$$\int_{t'}^{t} \int_{\mathcal{U}} g(u(s)) \mathrm{d}\rho(u) \mathrm{d}s = t - t' < \infty.$$

for all $t', t \in I$. This implies that, for almost every $t \in I$,

$$\int_X g(v)\rho_t(v) = \int_{\mathcal{X}} g(u(t)) \mathrm{d}\rho(u) = \int_{\mathcal{U}} g(u(t)) \mathrm{d}\rho(u) < \infty.$$

Therefore $g(v) < \infty$, for ρ_t-almost every v in X, and almost every $t \in I$. Hence, by the definition of g, it follows that $v \in Y$, for ρ_t-almost every v in X, and almost every $t \in I$, which means, in fact, that $\rho_t(X \setminus Y) = 0$, and hence ρ_t is carried by Y, for almost every $t \in I$.

Remark 3.1.4. Note that whenever $\mathcal{U}$ is a nonempty set, we can always obtain a trajectory statistical solution by considering the Dirac measure δ_u, for any element $u \in \mathcal{U}$ (δ_u is tight and $\{u\}$ is a Borel set in $\mathcal{U}$ satisfying $\delta_u(\{u\}) = 1$). A statistical solution can then also be easily obtained by considering the family of projections $\{\delta_{u(t)}\}_{t \in I}$. However, our main concern is not simply the existence of a measure or a family of measures satisfying the properties described in Definitions 3.1.1 or 3.1.2, respectively. Our aim is to prove the existence of such solutions for an initial value problem.

In the case of trajectory statistical solutions, the initial value problem takes the following form:

Problem 3.1.1 (Initial Value Problem for Trajectory Statistical Solutions)**.** Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left, with left end point t_0, and let X be a Hausdorff topological space. Let $\mathcal{X} = \mathcal{C}_{\text{loc}}(I, X)$ be the space of continuous paths in X endowed with the compact open topology. Let $\mathcal{U}$ be a given subset of $\mathcal{X}$. Given an "initial" tight Borel probability measure μ_0 on X, we look for a $\mathcal{U}$-trajectory statistical solution ρ on $\mathcal{X}$ satisfying $\Pi_{t_0}\rho = \mu_0$, i.e. we look for a measure $\rho \in \mathcal{P}(\mathcal{X})$ satisfying conditions (i) and (ii) of Definition 3.1.1 and such that

$$\rho(\Pi_{t_0}^{-1}(A)) = \mu_0(A), \ \forall A \in \mathfrak{B}_X.$$

The corresponding problem for statistical solutions is stated analogously:

Problem 3.1.2 (Initial Value Problem for Statistical Solutions)**.** Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left, with left end point t_0, and let X be a Hausdorff topological space. Given an "initial" tight Borel probability measure μ_0 on X, we look for a statistical solution $\{\rho_t\}_{t\in I}$ satisfying $\rho_{t_0} = \mu_0$.

Note that in the Initial-Value Problems 3.1.1 and 3.1.2 the interval I is considered as being closed and bounded on the left, with left end point t_0. This point t_0 represents the initial time in an application.

3.2 Existence of Trajectory Statistical Solutions

In order to obtain the existence of trajectory statistical solutions in the sense of Definition 3.1.1 and satisfying a given initial data (Problem 3.1.1), the subset $\mathcal{U} \subset \mathcal{X}$ is assumed to satisfy a set of conditions which we call the hypothesis (H). This is described below:

Definition 3.2.1. *Let X be a Hausdorff topological space. Consider an interval $I \subset \mathbb{R}$ closed and bounded on the left with left end point t_0, and let $\mathcal{X} = \mathcal{C}_{loc}(I, X)$. We say that a subset $\mathcal{U} \subset \mathcal{X}$ satisfies the* ***hypothesis (H)*** *if the following conditions are satisfied*

(H1) $\Pi_{t_0}\mathcal{U} = X$*;*

(H2) There exists a family of sets $\mathfrak{K}'(X) \subset \mathfrak{B}_X$ such that

- *(i) Every $K \in \mathfrak{K}'(X)$ is compact in X;*
- *(ii) Every tight Borel probability measure μ_0 on X is inner regular with respect to the family $\mathfrak{K}'(X)$ in the sense of (2.2);*
- *(iii) For every $K \in \mathfrak{K}'(X)$, $\Pi_{t_0}^{-1}K \cap \mathcal{U}$ is compact in $\mathcal{X}$.*

In many applications, $\mathfrak{K}'(X)$ may be considered as the entire family of compact sets of X. This is the case, for instance, in the application to the reaction-diffusion-type equation in Section 3.4.2. In this situation, hypothesis (H2) of Definition 3.2.1 is replaced by the following simpler condition:

(H2') For every compact subset $K \subset X$, $\Pi_{t_0}^{-1}K \cap \mathcal{U}$ is compact in $\mathcal{X}$.

In the applications to the study of statistical solutions of a certain evolution equation, the Hausdorff space X plays the role of the phase space associated to the equation, and the set $\mathcal{U}$ is the set of solutions in a given sense. These solutions are assumed to be continuous functions defined on a real time-interval I and with values in the phase space X, so that $\mathcal{U}$ is a subset of $\mathcal{X} = \mathcal{C}_{\text{loc}}(I, X)$.

Hypothesis (H1) is simply a global existence condition. It requires that for every initial condition $u_0 \in X$, there is a solution $u \in \mathcal{U}$ satisfying $u(t_0) = u_0$ and existing for the whole time interval I, i.e., the initial value problem associated to the evolution equation admits a global solution.

Hypothesis (H2) is essentially an a priori compactness condition on the set of solutions with initial conditions in certain compact sets in the phase space which are sufficient to approximate every tight Borel probability measure. More precisely, item (i) of hypothesis (H2) says that the family $\mathfrak{K}'(X)$ is contained in the family of compact sets in X. Then item (ii) requires that every tight Borel probability measure on X be also inner regular with respect to this subfamily $\mathfrak{K}'(X)$. For instance, in our applications to the Navier-Stokes equations and the Nonlinear Wave equation (Subsections 3.4.1 and 3.4.3), the space X is given by a separable Banach space endowed with its corresponding weak topology, and the family of sets $\mathfrak{K}'(X)$ is defined as the family of strong compact sets in X. In this case, items (i) and (ii) of (H2) are clearly satisfied. The last hypothesis, item (iii) of (H2), is typically obtained through a priori estimates derived from the evolution equation and some compact embedding theorem. Given a set $K \in \mathfrak{K}'(X)$, these estimates allow us to obtain a convergent subsequence of a given sequence of functions in $\Pi_{t_0}^{-1} K \cap \mathcal{U}$. Furthermore, using the strong compactness of K we can assume that the corresponding subsequence in X of the values of these functions at t_0 converge strongly to the value of the limit function at t_0. This is a key step for proving that the limit function belongs to $\mathcal{U}$. It allow us to pass to the limit as $t \to t_0^+$ in the corresponding energy inequality and to prove that the limit function satisfies this energy inequality as well, which is one of the conditions to be considered a solution. We postpone further details to Section 3.4, in which the application of the hypothesis (H) will be clearer.

Now we prove the existence of a solution for Problem 3.1.1. But let us first outline the main ideas of the proof itself.

Starting with an initial measure μ_0 in $\mathcal{P}(X, \text{tight})$, at a given time t_0, our intention is to show the existence of a measure ρ which is a trajectory statistical solution satisfying the initial condition $\Pi_{t_0}\rho = \mu_0$. As usual, this measure ρ is obtained from the limit of a convergent net of measures.

We first consider the case when the initial measure μ_0 is carried by a set K in the family $\mathfrak{K}'(X)$. Since by hypothesis (i) of (H2), K is a compact set in X, then by using the Krein-Milman Theorem we obtain a net $\{\mu_0^\alpha\}_\alpha$ of discrete measures converging to μ_0 in X. Using hypothesis (H1), we can easily extend each discrete initial measure μ_0^α to a discrete measure ρ_α in $\mathcal{X}$, by applying (H1) to each point in the support of μ_0^α. By construction, each ρ_α is a tight measure carried by $\Pi_{t_0}^{-1} K \cap \mathcal{U}$, which by hypothesis (iii) of (H2) is a compact set. This implies that $\{\rho_\alpha\}_\alpha$ is a uniformly tight net and then Theorem 2.3.2 is applied to obtain a subnet converging to some tight measure ρ, also carried by $\Pi_{t_0}^{-1} K \cap \mathcal{U}$, which is in particular a Borel set in $\mathcal{X}$. Thus ρ is a trajectory statistical solution, in the sense of Definition 3.1.1. The fact that ρ satisfies the initial condition, i.e., $\Pi_{t_0}\rho = \mu_0$, follows easily from the uniqueness of the limits in $\mathcal{M}(X, \text{tight})$, guaranteed by Theorem 2.3.1.

The proof of the case when μ_0 is not carried by any set $K \in \mathfrak{K}'(X)$, can be reduced to the previous case by using the hypothesis that μ_0 is a tight measure and thus, in particular, inner regular with respect to the family $\mathfrak{K}'(X)$ by hypothesis (ii) of (H2). The idea consists in decomposing μ_0 as a sum of Borel probability measures, each being carried by a set in $\mathfrak{K}'(X)$. The previous case can then be applied to each of these measures, yielding a countable family of $\mathcal{U}$-trajectory statistical solutions. Our desired measure is then obtained as an appropriate weighted sum of these particular measures.

There are some technical details that we skipped in the previous discussion and which are concerned with the restriction of the approximating measures to convenient compact subsets. If we assumed that our underlying phase space was completely regular, the proof could be made a bit simpler, as these restrictions would no longer be necessary since in completely regular Hausdorff spaces the weak-star semi-continuity topology coincides with the weak-star topology (see Lemma 2.3.1). But again, looking for a higher degree of generality, we assume only that our phase space X is a Hausdorff space.

Theorem 3.2.1. *Let X be a Hausdorff topological space and let I be a real interval closed and bounded on the left with left end point t_0. If $\mathcal{U} \subset \mathcal{X}$ is a subset satisfying hypothesis (H) then for any tight Borel probability measure μ_0 on X there exists a $\mathcal{U}$-trajectory statistical solution ρ on I such that $\Pi_{t_0}\rho = \mu_0$.*

Proof. Let us first suppose that μ_0 is carried by a set $K \in \mathfrak{K}'(X)$, which is a compact set by hypothesis (i) of (H2). Then, using the Krein-Milman Theorem [42, Theorem 3.23] we obtain a net $\{\mu_0^\alpha\}_\alpha$ of discrete measures in $\mathcal{P}(K)$ such that $\mu_0^\alpha \overset{wsc}{\rightharpoonup} \mu_0|_K$. Since each μ_0^α is a discrete measure, there exist $J_\alpha \in \mathbb{N}$, $\theta_j^\alpha \in \mathbb{R}$ with $0 < \theta_j^\alpha \leq 1$ and $u_{0,j}^\alpha \in K$ such that

$$\mu_0^\alpha = \sum_{j=1}^{J_\alpha} \theta_j^\alpha \delta_{u_{0,j}^\alpha},$$

with $\sum_{j=1}^{J_\alpha} \theta_j^\alpha = 1$, for every α.

From (H1) it follows that for each $u_{0,j}^\alpha$ there exists $u_j^\alpha \in \mathcal{U}$ such that $\Pi_{t_0} u_j^\alpha = u_{0,j}^\alpha$. Consider the measure ρ_α defined on $\mathcal{X}$ by

$$\rho_\alpha = \sum_{j=1}^{J_\alpha} \theta_j^\alpha \delta_{u_j^\alpha}.$$

Note that ρ_α belongs to $\mathcal{P}(\mathcal{X}, \text{tight})$ and is carried by $\Pi_{t_0}^{-1} K \cap \mathcal{U}$, which is a compact set by hypothesis (iii) of (H2). Thus, $\{\rho_\alpha\}_\alpha$ is clearly a uniformly tight net. By Theorem

2.3.2, there is a measure ρ in $\mathcal{P}(\mathcal{X}, \text{tight})$ such that, by passing to a subnet if necessary,

$$\rho_\alpha \overset{wsc}{\rightharpoonup} \rho \text{ in } \mathcal{X}. \tag{3.6}$$

Moreover, using Lemma 2.3.1, we find that ρ is carried by $\Pi_{t_0}^{-1} K \cap \mathcal{U}$.

Consider a bounded and upper semicontinuous function $\varphi : X \to \mathbb{R}$ and let $\Pi_{t_0} : \mathcal{X} \to X$ be the projection operator defined in (2.1). Then $\varphi \circ \Pi_{t_0}$ is a bounded function on $\mathcal{X}$. Moreover, since Π_{t_0} is continuous, $\varphi \circ \Pi_{t_0}$ is also an upper semicontinuous function on $\mathcal{X}$. Now applying a change of variables as in (2.4) and using the convergence $\rho_\alpha \overset{wsc}{\rightharpoonup} \rho$ together with Lemma 2.3.1, we obtain that

$$\limsup_\alpha \int_X \varphi \mathrm{d}\Pi_{t_0}\rho_\alpha = \limsup_\alpha \int_{\mathcal{X}} \varphi \circ \Pi_{t_0} \mathrm{d}\rho_\alpha \leq \int_{\mathcal{X}} \varphi \circ \Pi_{t_0} \mathrm{d}\rho = \int_X \varphi \mathrm{d}\Pi_{t_0}\rho.$$

Then, applying Lemma 2.3.1 once again, we obtain that $\Pi_{t_0}\rho_\alpha \overset{wsc}{\rightharpoonup} \Pi_{t_0}\rho$ in X. Further, taking the restrictions of these measures to the compact K, we also have that $\Pi_{t_0}\rho_\alpha|_K \overset{wsc}{\rightharpoonup} \Pi_{t_0}\rho|_K$. On the other hand, we have by construction that

$$\Pi_{t_0}\rho_\alpha|_K = \mu_0^\alpha \overset{wsc}{\rightharpoonup} \mu_0|_K.$$

Adding this to the fact that $\Pi_{t_0}\rho|_K, \mu_0|_K \in \mathcal{P}(K, \text{tight})$, by Theorem 2.3.1 we obtain that $\Pi_{t_0}\rho|_K = \mu_0|_K$. But since $\Pi_{t_0}\rho$ and μ_0 are carried by K we then get that $\Pi_{t_0}\rho = \mu_0$.

Thus, since $\rho \in \mathcal{P}(\mathcal{X}, \text{tight})$ and ρ is carried by the compact and hence Borel set $\Pi_{t_0}^{-1} K \cap \mathcal{U} \subset \mathcal{U}$, we have just proved the existence of a trajectory statistical solution satisfying the initial condition in the case when μ_0 is carried by a set $K \in \mathfrak{K}'(X)$.

Now let us prove the case when μ_0 is not carried by any set $K \in \mathfrak{K}'(X)$. In this case, since μ_0 is a tight Borel probability measure on X, by hypothesis (ii) of (H2) we have that μ_0 is also inner regular with respect to the family $\mathfrak{K}'(X)$. Thus, there exists a sequence $\{K_n\}_n$ of sets in $\mathfrak{K}'(X)$ such that

$$\mu_0(K_{n+1}) > \mu_0(K_n) > 0\,, \ \forall n \in \mathbb{N},$$

and

$$\mu_0(X \backslash K_n) < \frac{1}{n}, \ \forall n \in \mathbb{N}. \tag{3.7}$$

Moreover, we may assume that $K_n \subset K_{n+1}$, for all $n \in \mathbb{N}$.

Let $D_1 = K_1$ and $D_n = K_n \backslash K_{n-1}$, for every $n \geq 2$. Note that

$$\mu_0\left(X \backslash \bigcup_j D_j\right) = \mu_0\left(X \backslash \bigcup_j K_j\right) \leq \mu_0(X \backslash K_n) < \frac{1}{n},$$

for all $n \in \mathbb{N}$. Thus, taking the limit as $n \to \infty$ above, we obtain that μ_0 is carried by $\bigcup_j D_j$. Then, for every $A \in \mathfrak{B}_X$, since the sets D_j, $j \in \mathbb{N}$, are pairwise disjoint, we have

$$\mu_0(A) = \mu_0\left(A \cap \left(\bigcup_j D_j\right)\right) = \sum_{j=1}^{\infty} \mu_0(A \cap D_j).$$

So we may decompose μ_0 as

$$\mu_0 = \sum_j \mu_0(D_j)\mu_0^j,$$

where μ_0^j is the Borel probability measure defined as

$$\mu_0^j(A) = \frac{\mu_0(A \cap D_j)}{\mu_0(D_j)}, \ \forall A \in \mathfrak{B}_X.$$

Note that each μ_0^j is well-defined, since $\mu_0(D_1) = \mu_0(K_1) > 0$ and

$$\mu_0(D_j) = \mu_0(K_j) - \mu_0(K_{j-1}) > 0\,, \ \forall j \geq 2.$$

Also, since each μ_0^j is carried by the set $K_j \in \mathfrak{K}'(X)$, using the first part of the proof, for each $j \in \mathbb{N}$ we obtain a tight Borel probability measure ρ_j carried by $\Pi_{t_0}^{-1}K_j \cap \mathcal{U}$ and such that $\Pi_{t_0}\rho_j = \mu_0^j$.

Let ρ be the Borel probability measure defined by

$$\rho = \sum_j \mu_0(D_j)\rho_j.$$

Observe that

$$\rho\left(\bigcup_l \Pi_{t_0}^{-1}K_l \cap \mathcal{U}\right) = \sum_j \mu_0(D_j)\rho_j(\Pi_{t_0}^{-1}K_j \cap \mathcal{U}) = \sum_j \mu_0(D_j) = 1,$$

where the first and second equalities follow from the fact that ρ_j is carried by $\Pi_{t_0}^{-1}K_j \cap \mathcal{U}$. Thus, ρ is carried by $\bigcup_j \Pi_{t_0}^{-1}K_j \cap \mathcal{U}$, which is a Borel set in $\mathcal{X}$ and is contained in $\mathcal{U}$. The fact that $\Pi_{t_0}\rho = \mu_0$ is also easily verified.

It only remains to show that ρ is a tight measure. In order to prove so, consider a Borel set $\mathcal{A} \in \mathfrak{B}_{\mathcal{X}}$ and $\varepsilon > 0$. Let $n \in \mathbb{N}$ be such that $1/n < \varepsilon/2$. Then, since ρ_j is a tight measure, for each $1 \leq j \leq n$ there exists a compact set $\mathcal{K}_j^n \subset \mathcal{A}$ such that

$$\rho_j(\mathcal{A} \backslash \mathcal{K}_j^n) < \frac{\varepsilon}{2n}.$$

Let $\mathcal{K}^n = \bigcup_{1\leq j\leq n}\mathcal{K}_j^n$. Note that

$$\begin{aligned}
\rho(\mathcal{A}\backslash\mathcal{K}^n) &= \sum_{j=1}^{\infty}\mu_0(D_j)\rho_j(\mathcal{A}\backslash\mathcal{K}^n) \\
&\leq \sum_{j=1}^{n}\rho_j(\mathcal{A}\backslash\mathcal{K}^n) + \sum_{j=n+1}^{\infty}\mu_0(D_j) \\
&< \frac{\varepsilon}{2} + \mu_0(X\backslash K_n).
\end{aligned}$$

Thus, according to (4.9) and the choice of n, it follows that $\rho(\mathcal{A}\backslash\mathcal{K}^n) < \varepsilon$. Since $\mathcal{K}^n$ is a compact set in $\mathcal{X}$, this proves that ρ is tight. □

Remark 3.2.1. Notice that given an initial tight Borel probability measure μ_0 on X, if μ_0 is carried by a set $K \in \mathfrak{K}'(X)$, then the trajectory statistical solution ρ with $\Pi_{t_0}\rho = \mu_0$ obtained in the proof of Theorem 3.2.1 is carried by the Borel set $\Pi_{t_0}^{-1}K \cap \mathcal{U}$. On the other hand, if μ_0 is not carried by any set $K \in \mathfrak{K}'(X)$, then given any sequence of sets K_n in $\mathfrak{K}'(X)$, $n \in \mathbb{N}$, such that $\mu_0(X \setminus K_n) \to 0$, as $n \to \infty$, a trajectory statistical solution ρ with $\Pi_{t_0}\rho = \mu_0$ can be constructed such that it is carried by the Borel set $\mathcal{U}\cap(\bigcup_n \Pi_{t_0}^{-1}K_n)$.

3.3 Existence of Statistical Solutions in Phase Space

As in the previous sections, consider a Hausdorff space X and let I be an arbitrary interval in $\mathbb{R}$. In addition, let Y be a Banach space and denote by Y' its corresponding dual space. We assume that Y is a subset of X and that $X \hookrightarrow Y'_{w*}$ with continuous injection, where Y'_{w*} represents the space Y' endowed with the weak-star topology.

Consider the space $\mathcal{Z}$ defined in (3.1). Given $u \in \mathcal{Z}$ and $w \in Y''$ the bidual of Y, we see from (2.12) that

$$\langle w, u(t)\rangle_{Y'',Y'} - \langle w, u(t')\rangle_{Y'',Y'} = \int_{t'}^{t}\langle w, u_s(s)\rangle_{Y'',Y'}ds, \quad \text{for almost all } t', t \in I.$$

In particular, for $w \in Y \hookrightarrow Y''$, we have that

$$\langle u(t), w\rangle_{Y',Y} = \langle u(t'), w\rangle_{Y',Y} + \int_{t'}^{t}\langle u_s(s), w\rangle_{Y',Y}ds, \quad \text{for almost all } t', t \in I. \tag{3.8}$$

Since $X \hookrightarrow Y'_{w*}$ with continuous injection, we also have $u \in \mathcal{C}_{loc}(I, Y'_{w*})$, so that in fact the equation above holds everywhere:

$$\langle u(t), w\rangle_{Y',Y} = \langle u(t'), w\rangle_{Y',Y} + \int_{t'}^{t}\langle u_s(s), w\rangle_{Y',Y}ds, \ \forall t', t \in I, \ \forall w \in Y. \tag{3.9}$$

Moreover, since $u \in W^{1,1}_{loc}(I, Y')$, then in particular the mapping $t \mapsto \langle u_t(t), w\rangle_{Y',Y}$ belongs to $L^1(t',t;\mathbb{R})$. Therefore, (3.9) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t),w\rangle_{Y',Y} = \langle u_t(t),w\rangle_{Y',Y}, \tag{3.10}$$

for almost every $t \in I$ and that, for each $w \in Y$, the mapping $t \mapsto \langle u(t),w\rangle_{Y',Y}$ is absolutely continuous on I (see [23, Theorem 3.35]).

This latter fact, valid for every function $u \in \mathcal{Z}$, is used in the following theorem, in which we show that the family of measures obtained as the projections of a trajectory statistical solution at each $t \in I$ on X is a statistical solution, in the sense of Definition 3.1.2.

Theorem 3.3.1. *Let X be a Hausdorff space and let Y be a Banach space satisfying*

$$Y \subset X \hookrightarrow Y'_{w*},$$

where the injection $X \hookrightarrow Y'_{w*}$ is continuous, and also $\mathfrak{B}_Y \subset \mathfrak{B}_X$. Consider an interval $I \subset \mathbb{R}$ and a subset $\mathcal{U} \subset \mathcal{X}$. Let ρ be a $\mathcal{U}$-trajectory statistical solution and let $\mathcal{V}$ be a Borel subset of $\mathcal{X}$ such that $\mathcal{V} \subset \mathcal{U}$ and $\rho(\mathcal{V}) = 1$. Suppose that $\mathcal{U} \subset \mathcal{Z}$ and that for every $u \in \mathcal{U}$,

$$u_t(t) = F(t,u(t)), \qquad a.e.\ t \in I. \tag{3.11}$$

Assume that $F : I \times Y \to Y'$ is an $(\mathfrak{L}_I \otimes \mathfrak{B}_Y, \mathfrak{B}_{Y'})$-measurable function such that

$$t \mapsto \int_{\mathcal{V}} \|F(t,u(t))\|_{Y'} \mathrm{d}\rho(u) \in L^1_{loc}(I), \tag{3.12}$$

Then,

$$\int_{\mathcal{V}} \Phi(u(t))d\rho(u) = \int_{\mathcal{V}} \Phi(u(t'))d\rho(u) + \int_{t'}^{t}\int_{\mathcal{V}} \langle F(s,u(s)), \Phi'(u(s))\rangle_{Y',Y} d\rho(u)ds, \tag{3.13}$$

for all $t, t' \in I$ and for all cylindrical test function Φ. Moreover, the function

$$t \mapsto \int_{\mathcal{V}} \varphi(u(t))\mathrm{d}\rho(u) \tag{3.14}$$

is continuous on I for every $\varphi \in \mathcal{C}_b(X)$. In particular, the family of projections $\{\rho_t\}_{t\in I}$, where $\rho_t = \Pi_t\rho$, is a statistical solution in phase space.

Proof. First, note that since $\mathcal{U} \subset \mathcal{Z}$ and $Y \in \mathfrak{B}_X$, then it follows as in Remark 3.1.3 that ρ is carried by $\Pi_t^{-1}Y$, for almost every $t \in I$. Thus, the integrals in (3.12) and (3.13) with respect to ρ in $\mathcal{V}$, with integrands containing the mapping $u \in \mathcal{V} \mapsto F(t,u(t))$, are well-defined almost everywhere in I.

Now consider a cylindrical test function $\Phi : Y' \to \mathbb{R}$ given by

$$\Phi(u) = \phi(\langle u, v_1\rangle_{Y',Y}, \ldots, \langle u, v_k\rangle_{Y',Y}), \quad \forall u \in Y',$$

where ϕ is a continuously-differentiable real-valued function on $\mathbb{R}^k$ with compact support, $k \in \mathbb{N}$, and $v_1, \ldots, v_k \in Y$.

Since $\mathcal{V} \subset \mathcal{Z}$, then for every $u \in \mathcal{V}$ the function $t \mapsto \langle u(t), v_j\rangle_{Y',Y}$ is absolutely continuous on I and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t), v_j\rangle_{Y',Y} = \langle F(t, u(t)), v_j\rangle_{Y',Y}, \quad \forall j = 1, \ldots, k.$$

Thus,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(u(t)) &= \sum_{j=1}^{k} \partial_j\phi(\langle u(t), v_1\rangle_{Y',Y}, \ldots, \langle u(t), v_k\rangle_{Y',Y})\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t), v_j\rangle_{Y',Y} \\ &= \sum_{j=1}^{k} \partial_j\phi(\langle u(t), v_1\rangle_{Y',Y}, \ldots, \langle u(t), v_k\rangle_{Y',Y})\langle F(t, u(t)), v_j\rangle_{Y',Y} \\ &= \langle F(t, u(t)), \Phi'(u(t))\rangle_{Y',Y}, \end{aligned} \tag{3.15}$$

where Φ' is the Fréchet derivative of Φ in Y', given in (2.13).

Let us show that, for every $u \in \mathcal{U}$, the mapping $t \mapsto \Phi(u(t))$ is absolutely continuous on I. Since each $\partial_j\phi$ is bounded in $\mathbb{R}^k$, there exists $M > 0$ such that $\|\nabla\phi(\mathbf{x})\| \leq M$, for every $\mathbf{x} \in \mathbb{R}^k$, where $\|\cdot\|$ denotes the norm in $\mathbb{R}^k$. Then, given any finite sequence of pairwise disjoint subintervals $\{(t_j, s_j)\}_{j=1}^N$ in I, from the Mean Value Theorem we obtain

$$\begin{aligned}&\sum_{j=1}^{N} |\phi(\langle u(s_j), v_1\rangle_{Y',Y}, \ldots, \langle u(s_j), v_k\rangle_{Y',Y}) - \phi(\langle u(t_j), v_1\rangle_{Y',Y}, \ldots, \langle u(t_j), v_k\rangle_{Y',Y})| \\ &\quad \leq M\sum_{j=1}^{N} \|(\langle u(s_j), v_1\rangle_{Y',Y}, \ldots, \langle u(s_j), v_k\rangle_{Y',Y}) - (\langle u(t_j), v_1\rangle_{Y',Y}, \ldots, \langle u(t_j), v_k\rangle_{Y',Y})\| \end{aligned} \tag{3.16}$$

Thus, the absolute continuity of the mapping $t \mapsto \Phi(u(t))$ follows by using that each mapping $t \mapsto \langle u(t), v_j\rangle_{Y',Y}$ is absolutely continuous, for $j = 1, \ldots, k$.

Therefore, from (3.15) we obtain that

$$\Phi(u(t)) = \Phi(u(t')) + \int_{t'}^{t} \langle F(s, u(s)), \Phi'(u(s))\rangle_{Y',Y} ds, \tag{3.17}$$

for every $t, t' \in I$ and every $u \in \mathcal{V}$.

Consider the function

$$\begin{aligned} H: I \times \mathcal{V} &\rightarrow \mathbb{R} \\ (t,u) &\mapsto \Phi(u(t)). \end{aligned}$$

Denote by ι the continuous injection of X into Y'_{w*} and let $U : I \times \mathcal{V} \to X$ be the function defined in (2.16). Then H can be written as the composition $\Phi \circ \iota \circ U$. And since Φ, ι and U are continuous functions, then H is also continuous. Analogously, we obtain that the mapping $(t,u) \mapsto \Phi'(u(t))$ is continuous on $I \times \mathcal{V}$.

Using the function $G : I \times \mathcal{V} \to Y'$ defined in (2.15) we may write

$$\langle G(t,u), \Phi'(u(t))\rangle_{Y',Y} = \begin{cases} \langle F(t,u(t)), \Phi'(u(t))\rangle_{Y',Y}, & \text{if } u(t) \in Y, \\ 0, & \text{if } u(t) \in X \setminus Y \end{cases}$$

Since F is $(\mathfrak{L}_I \otimes \mathfrak{B}_Y, \mathfrak{B}_{Y'})$-measurable by hypothesis, from Lemma 2.6.1 we have that G is $(\mathfrak{L}_I \otimes \mathfrak{B}_{\mathcal{V}}, \mathfrak{B}_{Y'})$-measurable. Therefore, using also the continuity of the function $p : Y' \times Y \to \mathbb{R}$ given by

$$p(u,v) = \langle u, v\rangle_{Y',Y},$$

it follows that the mapping $(t,u) \mapsto \langle G(t,u), \Phi'(u(t))\rangle_{Y',Y}$ is $(\mathfrak{L}_I \otimes \mathfrak{B}_{\mathcal{V}})$-measurable. Thus, from (3.17) we obtain

$$\int_{\mathcal{V}} \Phi(u(t))d\rho(u) = \int_{\mathcal{V}} \Phi(u(t'))d\rho(u) + \int_{\mathcal{V}} \int_{t'}^{t} \langle G(s,u), \Phi'(u(s))\rangle_{Y',Y} ds d\rho(u), \tag{3.18}$$

for every $t, t' \in I$.

Now using that Φ' is bounded in Y and hypothesis (3.12), we may apply Tonelli's Theorem to the second term on the right hand side of (3.18) and obtain that

$$\begin{aligned} \int_{\mathcal{V}} \Phi(u(t))d\rho(u) &= \int_{\mathcal{V}} \Phi(u(t'))d\rho(u) + \int_{t'}^{t} \int_{\mathcal{V}} \langle G(s,u), \Phi'(u(s))\rangle_{Y',Y} d\rho(u)ds \\ &= \int_{\mathcal{V}} \Phi(u(t'))d\rho(u) + \int_{t'}^{t} \int_{\mathcal{V}} \langle F(s,u(s)), \Phi'(u(s))\rangle_{Y',Y} d\rho(u)ds, \end{aligned}$$

for all $t, t' \in I$. This proves the mean equality (3.13).

Now consider a function $\varphi \in \mathcal{C}_b(X)$ and let us prove that the function defined in (3.49) is continuous on I. Given $\tilde{t} \in I$, consider a sequence $\{t_n\}_n$ in I such that $t_n \to \tilde{t}$. Then, since every $u \in \mathcal{V}$ is continuous from I into X, it follows that

$$\varphi(u(t_n)) \to \varphi(u(\tilde{t})), \quad \forall u \in \mathcal{V}.$$

Since φ is in particular a bounded function on X, from the Lebesgue Dominated Convergence Theorem we obtain that

$$\int_{\mathcal{V}} \varphi(u(t_n))\mathrm{d}\rho(u) \to \int_{\mathcal{V}} \varphi(u(\tilde{t}))\mathrm{d}\rho(u).$$

Thus, the function defined in (3.49) is continuous on I. This proves that the family of measures $\{\rho_t\}_{t\in I}$ defined by

$$\rho_t(A) = \Pi_t\rho(A) = \rho(\Pi_t^{-1}A), \quad \forall A \in \mathfrak{B}(X),$$

satisfies condition (i) of Definition 3.1.2.

Also, by (2.4), we have

$$\int_X \|F(t,u)\|_{Y'}\mathrm{d}\rho_t(u) = \int_{\mathcal{V}} \|F(t,u(t))\|_{Y'}\mathrm{d}\rho(u),$$

for almost every $t \in I$. Then from hypothesis (3.12) it follows that $\{\rho_t\}_{t\in I}$ verifies condition (ii) of Definition 3.1.2.

Analogously, from (3.18) we obtain that $\{\rho_t\}_{t\in I}$ satisfies condition (iii) of Definition 3.1.2. Thus, $\{\rho_t\}_{t\in I}$ is a statistical solution. □

The next result provides a solution for Problem 3.1.2. Given an initial measure μ_0 on X, we use Theorem 3.2.1 to obtain a trajectory statistical solution, which is then projected at each time t to yield a family of measures on X. Thanks to Theorem 3.3.1, we then obtain that this family of projections is a statistical solution in phase space.

Theorem 3.3.2. *Let X be a Hausdorff space and let Y be a Banach space satisfying*

$$Y \subset X \hookrightarrow Y'_{w*},$$

where the injection $X \hookrightarrow Y'_{w*}$ is continuous, and also $\mathfrak{B}_Y \subset \mathfrak{B}_X$. Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0 and consider a subset $\mathcal{U} \subset \mathcal{X}$ satisfying hypothesis (H). Suppose that $\mathcal{U} \subset \mathcal{Z}$ and that for every $u \in \mathcal{U}$,

$$u_t(t) = F(t, u(t)), \qquad a.e.\ t \in I. \tag{3.19}$$

Assume that $F : I \times Y \to Y'$ is an $(\mathfrak{L}_I \otimes \mathfrak{B}_Y, \mathfrak{B}_{Y'})$-measurable function and that there exists an $(\mathfrak{L}_I \otimes \mathfrak{B}_X)$-measurable function $\gamma : I \times X \to \mathbb{R}$ such that

$$\int_{t_0}^t \|F(s,u(s))\|_{Y'}\mathrm{d}s \le \gamma(t, u(t_0)), \quad \forall t \in I,\ \forall u \in \mathcal{U}. \tag{3.20}$$

Let μ_0 be a tight Borel probability measure on X such that

$$\int_X \gamma(t, u_0)\, d\mu_0(u_0) < \infty, \tag{3.21}$$

for almost every $t \in I$. Then, there exists a projected statistical solution $\{\rho_t\}_{t\in I}$, associated with a $\mathcal{U}$-trajectory statistical solution, such that $\rho_{t_0} = \mu_0$.

Proof. Since $\mathcal{U}$ is a subset of $\mathcal{X}$ satisfying hypothesis (H) and μ_0 is a tight Borel probability measure on X, from Theorem 3.2.1 it follows that there exists a $\mathcal{U}$-trajectory statistical solution ρ on I such that $\Pi_{t_0}\rho = \mu_0$.

Since ρ is a $\mathcal{U}$-trajectory statistical solution, there exists a subset $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V} \in \mathfrak{B}_{\mathcal{X}}$ and $\rho(\mathcal{V}) = 1$. Then, taking the integral in (4.11) with respect to ρ, applying Tonelli's Theorem and using a change of variables as in (2.4) on the right-hand side, we find that

$$\begin{aligned}\int_{t_0}^t \int_{\mathcal{V}} \|F(s,u(s))\|_{Y'} \mathrm{d}\rho(u)\mathrm{d}s &\leq \int_{\mathcal{V}} \gamma(t, u(t_0))\mathrm{d}\rho(u) = \int_X \gamma(t,u_0)\mathrm{d}\Pi_{t_0}\rho(u_0) \\ &= \int_X \gamma(t,u_0)\mathrm{d}\mu_0(u_0).\end{aligned}$$

Thus from (4.12) we obtain that

$$t \mapsto \int_{\mathcal{V}} \|F(t,u(t))\|_{Y'}\mathrm{d}\rho(u) \in L^1_{loc}(I) \tag{3.22}$$

We then have that F is a $(\mathfrak{L}_I \otimes \mathfrak{B}_Y, \mathfrak{B}_{Y'})$-measurable function satisfying hypotheses (3.11) and (3.12) of Theorem 3.3.1. Applying this result, we conclude that the family $\{\rho_t\}_{t\in I}$, with $\rho_t = \Pi_t\rho$, is a statistical solution satisfying $\rho_{t_0} = \Pi_{t_0}\rho = \mu_0$. $\square$

In the following result, we obtain a mean energy inequality for trajectory statistical solutions.

Proposition 3.3.1. *Consider $\mathcal{U} \subset \mathcal{Z}$. Let ρ be a $\mathcal{U}$-trajectory statistical solution and let $\mathcal{V}$ be a Borel subset of $\mathcal{X}$ such that $\mathcal{V} \subset \mathcal{U}$ and $\rho(\mathcal{V}) = 1$. Suppose that there exist functions $\alpha : I \times X \to \mathbb{R}$ and $\beta : I \times Y \to \mathbb{R}$ satisfying the following conditions*

(i) $(t,u) \mapsto \alpha(t,u(t))$ belongs to $L^1(J \times \mathcal{V}, \lambda \times \rho)$, for every compact subset $J \subset I$;

(ii) $(t,u) \mapsto \beta(t,u(t))$ belongs to $L^1(J \times \mathcal{V}, \lambda \times \rho)$, for every compact subset $J \subset I$;

(iii) For ρ-almost every $u \in \mathcal{V}$ it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t,u(t)) + \beta(t,u(t)) \leq 0, \tag{3.23}$$

in the sense of distributions on I, i.e.,

$$-\int_I \varphi'(s)\alpha(s,u(s))ds + \int_I \varphi(s)\beta(s,u(s))ds \leq 0, \tag{3.24}$$

for all non-negative test functions $\varphi \in \mathcal{C}_c^\infty(I,\mathbb{R})$.

Then,

$$-\int_I\int_{\mathcal{V}} \varphi'(s)\alpha(s,u(s))d\rho ds + \int_I\int_{\mathcal{V}} \varphi(s)\beta(s,u(s))d\rho ds \leq 0, \tag{3.25}$$

for all non-negative test functions $\varphi \in \mathcal{C}_c^\infty(I,\mathbb{R})$.

Proof. The proof follows by integrating (3.24) with respect to ρ on $\mathcal{V}$ and then applying Fubini's Theorem by using hypotheses (i) and (ii). □

The motivation for considering the passage from the inequality (3.23), valid for individual weak solutions, to the mean inequality (3.25) comes from the Navier-Stokes equations (see Section 3.4.1) and other similar equations from fluid flows. It also appears in different types of equations, such as the nonlinear wave equation considered in Section 3.4.3. In some situations, however, such as in the case of the Reaction-Diffusion equation considered in Section 3.4.2, the individual weak solutions satisfies in fact an energy-type equality, and of course this equality is similarly passed on to the statistical solutions. In the case of equality, the test functions are allowed to assume negative values. We state this result as follows, omitting the proof since it follows along the same lines as that of Proposition 3.3.1.

Proposition 3.3.2. *Under the hypothesis of Proposition 3.3.1, if the equality holds in (3.23), then the equality holds in (3.25), for any test function $\varphi \in \mathcal{C}_c^\infty(I,\mathbb{R})$, as well.*

3.4 Applications

3.4.1 Navier-Stokes Equations

The Navier-Stokes equations are a commonly used model in the study of Newtonian turbulent fluids. In their three-dimensional and incompressible form, these equations are written as

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \tag{3.26}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{3.27}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity field, p is the kinematic pressure, $\mathbf{f}$ represents a given body force applied to the fluid and ν is the parameter of kinematic viscosity. We consider

$\mathbf{u}$, p and $\mathbf{f}$ as functions of a space variable $\mathbf{x}$ and a time variable t, with $\mathbf{x}$ varying in a set $\Omega \subset \mathbb{R}^3$ and t varying in an interval $I \subset \mathbb{R}$.

For a physical formulation of the equations we refer the reader to the books by Landau and Lifshitz [30] and Batchelor [2]. For a more mathematical approach, see Ladyzhenskaya [29], Temam [46, 47] and Constantin and Foias [10].

Our main concern in this section is to apply the abstract theory developed in the previous sections to prove the existence of statistical solutions to the Navier-Stokes equations (3.26)-(3.27). For more specific discussions on the notion of statistical solutions to the Navier-Stokes equations, we refer the reader to [16, 19, 54, 55] and also to the more recent paper [22]. The existence result that we obtain in this section has been already proved in [22], a work that has been in fact our inspiration. Nevertheless, our abstract framework, besides applying to a wide range of equations, led to a much simpler proof than that presented in [22]. Moreover, we have also extended the existence result to the Navier-Stokes equations with a more general class of external forces.

For the sake of simplicity, we assume periodic boundary conditions. In this case we consider a periodic domain given by $\Omega = \Pi_{i=1}^{3}(0, L_i)$, where $L_i > 0$, for $i = 1, 2, 3$. This means we are assuming the flow is periodic with period L_i in each spatial direction x_i. We also consider the averages of the flow and of the forcing term to be zero, i.e.,

$$\int_\Omega \mathbf{u}(\mathbf{x}, t)\mathrm{d}\mathbf{x} = 0\,, \quad \int_\Omega \mathbf{f}(\mathbf{x}, t)\mathrm{d}\mathbf{x} = 0.$$

Let $C^\infty_{per}(\Omega, \mathbb{R}^3)$ denote the space of infinitely differentiable and Ω-periodic functions $\mathbf{u}$. We then define the set of periodic test functions with vanishing average and divergence free as

$$\mathcal{V} := \left\{ \mathbf{u} \in \mathcal{C}^\infty_{per}(\Omega, \mathbb{R}^3) \,|\, \nabla \cdot \mathbf{u} = 0 \text{ and } \int_\Omega \mathbf{u}(\mathbf{x})\mathrm{d}\mathbf{x} = 0 \right\}. \tag{3.28}$$

Let H be the closure of $\mathcal{V}$ in $L^2(\Omega, \mathbb{R}^3)$ and let V be the closure of $\mathcal{V}$ in $H^1(\Omega, \mathbb{R}^3)$. The inner product and norm in H are defined, respectively, by

$$(\mathbf{u}, \mathbf{v})_H = \int_\Omega \mathbf{u} \cdot \mathbf{v}\mathrm{d}\mathbf{x} \quad \text{and} \quad |\mathbf{u}|_H = \sqrt{(\mathbf{u}, \mathbf{u})_H},$$

where $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{3} u_i v_i$. In the space V, these are defined as

$$(\!(\mathbf{u}, \mathbf{v})\!)_V = (\nabla \mathbf{u}, \nabla \mathbf{v})_H = \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v}\mathrm{d}x \quad \text{and} \quad \|\mathbf{u}\|_V = \sqrt{(\!(\mathbf{u}, \mathbf{u})\!)_V},$$

where it is understood that $\nabla \mathbf{u} = (\partial u_i / \partial x_j)_{i,j=1}^{3}$ and that $\nabla \mathbf{u} \cdot \nabla \mathbf{v}$ is the componentwise product between $\nabla \mathbf{u}$ and $\nabla \mathbf{v}$. We also consider the space H endowed with its weak topology and denote it by H_w.

Clearly, V is a subset of H. Thus, by identifying H with its dual space H', we obtain the following continuous inclusions

$$V \subset H \equiv H' \subset V'.$$

Thus, since the injection $H \hookrightarrow V'$ is a continuous linear mapping, we also have

$$H_w \hookrightarrow V'_{w*}$$

with continuous injection, where V'_{w*} denotes the space V' endowed with the weak-star topology.

Let A be the **Stokes operator**, defined as $A = -\mathbb{P}\Delta$, where $\mathbb{P} : L^2(\Omega, \mathbb{R}^3) \to H$ is the Leray-Helmholtz projection, i.e., the orthogonal projector in $L^2(\Omega, \mathbb{R}^3)$ onto the subspace of divergence-free vector fields. We denote by $D(A)$ the domain of A, which is defined as the set of functions $\mathbf{u} \in V$ such that $A\mathbf{u} \in H$. In the periodic case with zero average, we have

$$A\mathbf{u} = -\Delta\mathbf{u}\,, \ \forall \mathbf{u} \in D(A) = V \cap H^2(\Omega, \mathbb{R}^3)$$

and A is a positive self-adjoint linear operator with compact inverse, so that it has a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ of positive eigenvalues counted according to their multiplicity, in increasing order, associated with an orthonormal basis $\{\mathbf{w}_i\}_{i\in\mathbb{N}}$ in H. Furthermore, the Poincaré inequality holds, i.e., for all $\mathbf{u} \in V$,

$$\lambda_1 |\mathbf{u}|_H^2 \leq \|\mathbf{u}\|_V^2, \tag{3.29}$$

where $\lambda_1 > 0$ is the first eigenvalue of the Stokes operator.

Also, we denote by $P_k : H \to V$ the Galerkin projector onto the space spanned by the eigenfunctions associated with the first k eigenvalues of the Stokes operator, i.e.

$$P_k\mathbf{u} = \sum_{i=1}^{k} (\mathbf{u}, \mathbf{w}_i)_H \mathbf{w}_i, \quad \forall \mathbf{u} \in H.$$

We observe that given two eigenfunctions $\mathbf{w}_j, \mathbf{w}_k$ of A, we have

$$(\!(\mathbf{w}_j, \mathbf{w}_k)\!)_V = (A\mathbf{w}_j, \mathbf{w}_k)_H = \lambda_j (\mathbf{w}_j, \mathbf{w}_k)_H = \begin{cases} \lambda_j, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

Then for every $\mathbf{u} \in H$, we obtain

$$\|P_k\mathbf{u}\|_V^2 = (\!(P_k\mathbf{u}, P_k\mathbf{u})\!)_V = \sum_{j=1}^{k} |(\mathbf{u}, \mathbf{w}_j)_H|^2 (\!(\mathbf{w}_j, \mathbf{w}_j)\!)_V = \sum_{j=1}^{k} |(\mathbf{u}, \mathbf{w}_j)_H|^2 \lambda_j,$$

from which it follows that P_k is continuous from H to V.

The natural space for the solutions of the Navier-Stokes equations is the space $\mathcal{C}_{loc}(I, H_w)$ of continuous functions from an interval $I \subset \mathbb{R}$ to H_w, endowed with the compact-open topology. This function space can also be seen as the space of weakly continuous functions from I to H.

We recall that the Navier-Stokes equations can be written in the following functional form

$$\mathbf{u}_t + \nu A\mathbf{u} + B(\mathbf{u}, \mathbf{u}) = \mathbf{f} \quad \text{in } V', \tag{3.30}$$

where A is the Stokes operator and $B : V \times V \to V'$ is the bilinear operator defined as

$$B(\mathbf{u}, \mathbf{v}) = \mathbb{P}[(\mathbf{u} \cdot \nabla)\mathbf{v}], \quad \forall \mathbf{u}, \mathbf{v} \in V.$$

In this setting, A is considered as an operator defined on V with values in V', and we have

$$\|A\mathbf{u}\|_{V'} = \sup_{\|\mathbf{v}\|=1} \langle A\mathbf{u}, \mathbf{v}\rangle_{V',V} = \sup_{\|\mathbf{v}\|=1} ((\mathbf{u}, \mathbf{v}))_V = \|\mathbf{u}\|_V,$$

which implies in particular that $A : V \to V'$ is continuous. Moreover, B satisfies the following inequality (see [10, 19])

$$\|B(\mathbf{u}, \mathbf{v})\|_{V'} \leq c|\mathbf{u}|_H^{1/4}\|\mathbf{u}\|_V^{3/4}|\mathbf{v}|_H^{1/4}\|\mathbf{v}\|_V^{3/4}, \quad \forall \mathbf{u}, \mathbf{v} \in V, \tag{3.31}$$

where c is a universal constant. By using this inequality, it is not difficult to see that $B : V \times V \to V'$ is also a continuous operator.

The notion of solution that is considered here is the well-known Leray-Hopf weak solution, which is defined below.

Definition 3.4.1. *Let I be an interval in $\mathbb{R}$ and $\mathbf{f} \in L^2_{loc}(I, V')$. We say that $\mathbf{u}$ is a* ***Leray-Hopf weak solution of the Navier-Stokes equations*** *(3.26)-(3.27)* ***on*** *I if*

(i) $\mathbf{u} \in L^\infty_{loc}(I, H) \cap L^2_{loc}(I, V) \cap \mathcal{C}_{loc}(I, H_w)$;

(ii) $\partial_t \mathbf{u} \in L^{4/3}_{loc}(I, V')$;

(iii) $\mathbf{u}$ *satisfies the weak formulation of the Navier-Stokes equations, i.e.,*

$$\mathbf{u}_t + \nu A\mathbf{u} + B(\mathbf{u}, \mathbf{u}) = \mathbf{f}, \tag{3.32}$$

in V', in the sense of distributions on I;

(iv) $\mathbf{u}$ *satisfies the energy inequality in the sense that for almost all $t' \in I$ and for all $t \in I$ with $t > t'$,*

$$\frac{1}{2}|\mathbf{u}(t)|_H^2 + \nu \int_{t'}^{t} \|\mathbf{u}(s)\|_V^2 \mathrm{d}s \leq \frac{1}{2}|\mathbf{u}(t')|_H^2 + \int_{t'}^{t} \langle \mathbf{f}(s), \mathbf{u}(s)\rangle_{V',V} \mathrm{d}s; \tag{3.33}$$

(v) *If I is closed and bounded on the left, with left end point t_0, then the solution is strongly continuous in H at t_0 from the right, i.e., $\mathbf{u}(t) \to \mathbf{u}(t_0)$ in H as $t \to t_0^+$.*

The set of allowed times t' in (3.33) can be characterized as the points of strong continuity from the right of $\mathbf{u}$ in H. In particular, condition (v) implies that $t' = t_0$ is allowed in that case.

The Leray-Hopf weak solutions of the Navier-Stokes equations also satisfy a strenghtened form of the energy inequality (3.33) of Definition 3.4.1. The proof of this strenghtened energy inequality has been given in [19] for external forces $\mathbf{f}$ in $L^2_{\text{loc}}(I, H)$. However, it turns out that this inequality is also valid if $\mathbf{f}$ belongs to the larger space $L^2_{\text{loc}}(I, V')$, as we state below:

Proposition 3.4.1. *Let $T > 0$ and $\mathbf{f} \in L^2(0, T; V')$. Consider a nonnegative, nondecreasing and continuously-differentiable real-valued function $\psi : [0, \infty) \to \mathbb{R}$ with bounded derivative. If $\mathbf{u}$ is a Leray-Hopf weak solution of the Navier-Stokes equations on $[0, T]$, then*

$$\frac{d}{dt}(\psi(|\mathbf{u}(t)|_H^2)) \leq 2\psi'(|\mathbf{u}(t)|_H^2)[\langle \mathbf{f}(t), \mathbf{u}(t)\rangle_{V',V} - \nu\|\mathbf{u}(t)\|^2]$$

in the sense of distributions on $[0, T]$.

The idea of the proof is to first obtain such inequality for the mollified functions and then to pass to the limit with respect to the mollifier parameter. For the complete proof, see Appendix B.

Given $R > 0$, we denote by $B_H(R)$ the closed ball centered at the origin and with radius R in H. The corresponding closed ball endowed with the weak topology is denoted by $B_H(R)_w$. We then define the following sets of Leray-Hopf weak solutions:

$$\mathcal{U}_I = \{\mathbf{u} \in \mathcal{C}_{loc}(I, H_w) : \mathbf{u} \text{ is a Leray-Hopf weak solution on } I\}, \tag{3.34}$$

$$\mathcal{U}_I(R) = \{\mathbf{u} \in \mathcal{C}_{loc}(I, B_H(R)_w) : \mathbf{u} \text{ is a Leray-Hopf weak solution on } I\}. \tag{3.35}$$

$$\mathcal{U}_I^\sharp = \{\mathbf{u} \in \mathcal{C}_{loc}(I, H_w) : \mathbf{u} \text{ is a Leray-Hopf weak solution on } \mathring{I}\}, \tag{3.36}$$

$$\mathcal{U}_I^\sharp(R) = \{\mathbf{u} \in \mathcal{C}_{loc}(I, B_H(R)_w) : \mathbf{u} \text{ is a Leray-Hopf weak solution on } \mathring{I}\}, \tag{3.37}$$

where $\mathring{I}$ denotes the interior of the interval I.

The following proposition provides some estimates satisfied by every $\mathbf{u} \in \mathcal{U}_I$, for a given interval $I \subset \mathbb{R}$.

Proposition 3.4.2. *Let I be an interval in $\mathbb{R}$ and $\mathbf{f} \in L^2_{loc}(I, V')$. If $\mathbf{u} \in \mathcal{U}_I$ then, for $t' \in I$ allowed in (3.33) and for all $t \in I$ with $t > t'$, the following inequalities hold*

$$|\mathbf{u}(t)|_H^2 \leq |\mathbf{u}(t')|_H^2 + \frac{1}{\nu}\|\mathbf{f}\|^2_{L^2(t',t;V')}, \tag{3.38}$$

$$\int_{t'}^{t} \|\mathbf{u}(s)\|_V^2 \mathrm{d}s \leq \frac{1}{\nu}|\mathbf{u}(t')|_H^2 + \frac{1}{\nu^2}\|\mathbf{f}\|_{L^2(t',t;V')}^2, \tag{3.39}$$

$$\left(\int_{t'}^{t} \|\partial_t \mathbf{u}(s)\|_{V'}^{4/3} \mathrm{d}s\right)^{3/4} \leq \frac{c}{\nu^{3/4}}|\mathbf{u}(t')|_H^2 + \frac{\nu^{5/4}}{\lambda_1^{1/2}} D, \tag{3.40}$$

where c is a universal constant and $D = D(\nu\lambda_1 t', \nu\lambda_1 t)$ is a nondimensional function which depends on the variables t', t and also on the parameters ν, λ_1 and Ω through the nondimensional quantities $\nu\lambda_1|t - t'|$ and $\frac{\lambda_1^{1/4}}{\nu^{3/2}}\|\mathbf{f}\|_{L^2(t,t';V')}$.

Applying the Cauchy-Schwarz, Poincaré and Young inequalities to the second term on the right-hand side of (3.33), we obtain that

$$|\mathbf{u}(t)|_H^2 + \nu \int_{t'}^{t} \|\mathbf{u}(s)\|_V^2 \mathrm{d}s \leq |\mathbf{u}(t')|_H^2 + \frac{1}{\nu}\int_{t'}^{t} \|\mathbf{f}(s)\|_{V'}^2 \mathrm{d}s \tag{3.41}$$

Estimates (3.38) and (3.39) then follow by discarding in each case the appropriate nonnegative term on the left-hand side of (3.41). The last estimate (3.40) is obtained from the functional equation (3.32) by using the following inequality for the bilinear term, which is derived from (3.31) with $\mathbf{v} = \mathbf{u}$:

$$\|B(\mathbf{u}, \mathbf{u})\|_{V'} \leq c|\mathbf{u}|_H^{1/2}\|\mathbf{u}\|_V^{3/2},$$

where c is the same universal constant from above.

The *a priori* estimates (3.38)-(3.40) allow us to prove that $\mathcal{U}_I^\sharp(R)$ is a compact and metrizable space, in the same way as it was done in [22, Proposition 2.2]. Furthermore, one can show that $\mathcal{U}_I^\sharp(R)$ is the closure of the space $\mathcal{U}_I(R)$ with respect to the topology of $\mathcal{C}(I, H_w)$.

The existence of a Leray-Hopf weak solution on a given interval $I \subset \mathbb{R}$ is obtained by using the estimates from Proposition 3.4.2. This proof is a classical result and can be found in many well-known texts [10, 29, 36, 47]. We state it below for completeness.

Theorem 3.4.1. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0 and let $f \in L^2_{loc}(I; V')$. Then, given $\mathbf{u}_0 \in H$, there exists at least one weak solution $\mathbf{u} \in \mathcal{U}_I$ of (3.26)-(3.27) in the sense of Definition 3.4.1 satisfying $\Pi_{t_0}\mathbf{u} = \mathbf{u}_0$.*

From now on, we assume that $I \subset \mathbb{R}$ is an interval closed and bounded on the left, with left end point t_0. Under this assumption, the energy inequality (3.33) is valid for $t' = t_0$.

Consider a compact subinterval $J \subset I$. Then given $\mathbf{u} \in \mathcal{U}_I$ such that $\mathbf{u}(t_0) \in B_H(R)$ for some $R \geq 0$, from (3.38) with $t' = t_0$ it follows that there exists $\tilde{R} \geq R$ such that $\mathbf{u}(t) \in B_H(\tilde{R})$, for every $t \in J$. Thus, the restriction of $\mathbf{u}$ to J belongs to $\mathcal{U}_J(\tilde{R})$.

In order to prove the existence of a trajectory statistical solution for the Navier-Stokes equations satisfying a given initial data, we shall apply Theorem 3.2.1 by considering X

as the space H_w and the general set $\mathcal{U}$ as the set of weak solutions $\mathcal{U}_I$. We now show that $\mathcal{U}_I$ satisfies hypothesis (H).

First, note that hypothesis (H1) is a direct consequence of Theorem 3.4.1. Also, defining $\mathfrak{K}'(H_w)$ as the family of (strong) compact sets in H, it follows that hypotheses (i) and (ii) of (H2) are also satisfied with respect to this family (see proof of Theorem 3.4.2).

The following proposition proves that the remaining hypothesis, (iii) of (H2), is also satisfied.

Proposition 3.4.3. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0 and let K be a set in $\mathfrak{K}'(H_w)$. Then $\Pi_{t_0}^{-1}K \cap \mathcal{U}_I$ is a compact set in $\mathcal{X} = \mathcal{C}_{loc}(I, H_w)$.*

Proof. Let $\mathbf{u} \in \Pi_{t_0}^{-1}K \cap \mathcal{U}_I$ and let $R \geq 0$ be large enough so that $K \subset B_H(R)$. Consider a sequence $\{J_n\}_n$ of compact subintervals of I such that $I = \bigcup_n J_n$. Since $\Pi_{t_0}\mathbf{u} \in K \subset B_H(R)$ and $\mathbf{u} \in \mathcal{U}_I$, from the estimate (3.38) with $t' = t_0$ it follows that there exists a sequence $\{R_n\}_n$ of positive real numbers such that $\Pi_{J_n}\mathbf{u} \in \mathcal{U}_{J_n}(R_n)$, for every n. Thus,

$$\Pi_{t_0}^{-1}K \cap \mathcal{U}_I \subset \bigcap_n \Pi_{J_n}^{-1}\mathcal{U}_{J_n}(R_n). \tag{3.42}$$

Now since each $\mathcal{U}_{J_n}(R_n)$ is a metrizable space, (3.42) implies that $\Pi_{t_0}^{-1}K \cap \mathcal{U}_I$ is also metrizable. Therefore, it suffices to show that $\Pi_{t_0}^{-1}K \cap \mathcal{U}_I$ is sequentially compact.

Let $\{\mathbf{u}_k\}_k$ be a sequence in $\Pi_{t_0}^{-1}K \cap \mathcal{U}_I$. As in the classical proof of existence of weak solutions (Theorem 3.4.1), using the a priori estimates (3.38)-(3.40) on each compact interval J_n and applying a diagonalization method, we obtain a subsequence $\{\mathbf{u}_{k'}\}_{k'}$ and a function $\mathbf{u}$ such that

$$\mathbf{u}_{k'} \to \mathbf{u} \quad \text{in } \mathcal{C}_{loc}(I, H_w) \tag{3.43}$$

as $k' \to \infty$. Moreover, this limit function $\mathbf{u}$ is a weak solution on the interior of I, i.e. $\mathbf{u} \in \mathcal{U}_I^\sharp$ (the condition of strong continuity at t_0, item (v) of Definition 3.4.1, is not guaranteed at this point). From (3.43) we obtain in particular that

$$\mathbf{u}_{k'}(t_0) \to \mathbf{u}(t_0) \quad \text{in } H_w. \tag{3.44}$$

On the other hand, since K is a compact set in H, there exists a further subsequence, which we still denote by $\{\mathbf{u}_{k'}\}_{k'}$, and an element $\mathbf{u}_0 \in K$ such that

$$\mathbf{u}_{k'}(t_0) \to \mathbf{u}_0 \quad \text{in } H. \tag{3.45}$$

From (3.44) and (3.45) it follows that $\mathbf{u}(t_0) = \mathbf{u}_0$, which implies that $\mathbf{u} \in \Pi_{t_0}^{-1}K$. Moreover, we obtain that $\{\mathbf{u}_{k'}(t_0)\}_{k'}$ also converges to $\mathbf{u}(t_0)$ in the strong topology of H. This allow us to prove that $\mathbf{u}$ verifies in addition the last condition of Definition 3.4.1.

Indeed, since each $\mathbf{u}_{k'}$ belongs to $\mathcal{U}_I$, they satisfy in particular the energy inequality (3.33) at $t' = t_0$. Considering the lim inf as $k' \to \infty$ in this inequality we obtain by using the strong convergence of $\{\mathbf{u}_{k'}(t_0)\}_{k'}$ to $\mathbf{u}(t_0)$ in H that

$$\frac{1}{2}|\mathbf{u}(t)|_H^2 + \nu \int_{t_0}^{t} \|\mathbf{u}(s)\|_V^2 \mathrm{d}s \leq \frac{1}{2}|\mathbf{u}(t_0)|_H^2 + \int_{t_0}^{t} \langle \mathbf{f}(s), \mathbf{u}(s)\rangle_{V',V} \mathrm{d}s$$

Then, by taking the lim sup as $t \to t_0^+$ above, we obtain

$$\limsup_{t\to t_0^+} |\mathbf{u}(t)|_H^2 \leq |\mathbf{u}(t_0)|_H^2. \tag{3.46}$$

And since $\mathbf{u} \in \mathcal{C}_{\mathrm{loc}}(I, H_w)$, then

$$|\mathbf{u}(t_0)|_H^2 \leq \liminf_{t\to t_0^+} |\mathbf{u}(t)|_H^2. \tag{3.47}$$

Now (3.46) and (3.47) imply that $\mathbf{u}(t)$ converges in norm to $\mathbf{u}(t_0)$ as $t \to t_0^+$. But since $\mathbf{u}(t)$ also converges weakly to $\mathbf{u}(t_0)$ as $t \to t_0^+$, then

$$\lim_{t\to t_0^+} \mathbf{u}(t) = \mathbf{u}(t_0) \quad \text{in } H.$$

□

Now we are able to prove the existence of a solution for the corresponding Initial-Value Problem 3.1.1 associated to the Navier-Stokes equations.

Theorem 3.4.2. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0 and let $\mathcal{U}_I$ be the set of Leray-Hopf weak solutions of the Navier-Stokes equations on I. Then, given a Borel probability measure μ_0 on H, there exists a $\mathcal{U}_I$-trajectory statistical solution ρ on $\mathcal{C}_{loc}(I, H_w)$ satisfying the initial condition $\Pi_{t_0}\rho = \mu_0$.*

Proof. From Theorem 3.4.1 it follows that the set $\mathcal{U}_I$ satisfies hypothesis (H1) of Definition 3.2.1. Also, if $\mathfrak{K}'(H_w)$ denotes the family of (strong) compact sets in H, then it clearly satisfies hypothesis (i) of (H2). Since H is a Polish space, it follows that any Borel probability measure on H is tight in the sense of being inner regular with respect to the family of compact subsets of H [1, Theorem 12.7]. In particular, since $\mu_0 \in \mathcal{P}(H)$, then μ_0 is inner regular with respect to the family $\mathfrak{K}'(H_w)$. Thus $\mathfrak{K}'(H_w)$ satisfies hypothesis (ii) of (H2). From Proposition 3.4.3, we also obtain that $\mathcal{U}_I$ satisfies hypothesis (iii) of (H2). We then conclude that $\mathcal{U}_I$ verifies hypothesis (H). Moreover, since H is a separable Banach space, the Borel sets in H and H_w coincide. This implies that μ_0 is also a Borel probability measure on H_w. It is also clearly tight on H_w, since every strong compact set is weak compact. Therefore, by Theorem 3.2.1 there exists a $\mathcal{U}_I$ trajectory statistical solution ρ with $\Pi_{t_0}\rho = \mu_0$.

□

Finally, using Theorem 3.4.2 and the strengthened energy inequality from Proposition 3.4.1, we obtain a solution for the corresponding Initial-Value Problem 3.1.2 associated to the Navier-Stokes equations, i.e. the existence of a statistical solution in the sense of Definition 3.1.2 satisfying a given initial data.

Theorem 3.4.3. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0 and let $\mathcal{U}_I$ be the set of Leray-Hopf weak solutions of the Navier-Stokes equations on I. Consider a Borel probability measure μ_0 on H satisfying*

$$\int_H |\mathbf{u}|_H^2 \mathrm{d}\mu_0(\mathbf{u}) < \infty. \tag{3.48}$$

Then there exists a projected statistical solution $\{\rho_t\}_{t\in I}$ of the Navier-Stokes equations (3.26)-(3.27)*, associated with a $\mathcal{U}_I$-trajectory statistical solution, such that*

(i) The initial condition $\rho_{t_0} = \mu_0$ holds;

(ii) The function

$$t \mapsto \int_H \varphi(\mathbf{u}) \mathrm{d}\rho_t(\mathbf{u}) \tag{3.49}$$

is continuous on I, for every bounded and weakly-continuous real-valued function φ on H, and is measurable on I, for every bounded and continuous real-valued function φ on H.

(iii) For any cylindrical test function Φ, it follows that

$$\begin{aligned}\int_H \Phi(\mathbf{u}) \mathrm{d}\rho_t(\mathbf{u}) = \int_H \Phi(\mathbf{u}) \mathrm{d}\rho_{t'}(\mathbf{u}) \\ + \int_{t'}^{t} \int_H \langle \mathbf{f}(s) - \nu A\mathbf{u} - B(\mathbf{u},\mathbf{u}), \Phi'(\mathbf{u})\rangle_{V',V} \mathrm{d}\rho_s(\mathbf{u}) \mathrm{d}s,\end{aligned} \tag{3.50}$$

for all $t, t' \in I$.

(iv) The mean strengthened energy inequality

$$\frac{d}{dt} \int_H (\psi(|\mathbf{u}|_H^2)) \mathrm{d}\rho_t(\mathbf{u}) \leq 2 \int_H \psi'(|\mathbf{u}|_H^2)[\langle \mathbf{f}(t), \mathbf{u}\rangle_{V',V} - \nu \|\mathbf{u}\|_V^2] \mathrm{d}\rho_t(\mathbf{u}) \tag{3.51}$$

is satisfied in the distribution sense on I, for every nonnegative, nondecreasing and continuously-differentiable real-valued function ψ with bounded derivative.

(v) At the initial time, the limit

$$\lim_{t\to t_0^+} \int_H \psi(|\mathbf{u}|_H^2) \mathrm{d}\rho_t(\mathbf{u}) = \int_H \psi(|\mathbf{u}|_H^2) \mathrm{d}\mu_0(\mathbf{u}) \tag{3.52}$$

holds for every function ψ as in (iv).

Proof. We have seen in the proof of Theorem 3.4.2 that the set of Leray-Hopf weak solutions $\mathcal{U}_I$ satisfies the hypothesis (H).

Now let $\mathbf{F} : I \times V \to V'$ be the function defined by

$$\mathbf{F}(t, \mathbf{u}) = \mathbf{f}(t) - \nu A\mathbf{u} - B(\mathbf{u}, \mathbf{u}). \tag{3.53}$$

As previously mentioned, the linear operator $A : V \to V'$ and the bilinear operator $B : V \times V \to V'$ are continuous. This implies that the mapping $\mathbf{u} \mapsto -\nu A\mathbf{u} - B(\mathbf{u}, \mathbf{u})$ is also continuous from V into V'. In particular, the mapping $(t, \mathbf{u}) \mapsto -\nu A\mathbf{u} - B(\mathbf{u}, \mathbf{u})$ is $(\mathfrak{L}_I \otimes \mathfrak{B}_V, \mathfrak{B}_{V'})$-measurable. Further, since $\mathbf{f} \in L^2_{\text{loc}}(I, V')$, we then obtain that $\mathbf{F}$ is a $(\mathfrak{L}_I \otimes \mathfrak{B}_V, \mathfrak{B}_{V'})$-measurable function.

From the functional equation (3.30), it follows that

$$\mathbf{u}_t(t) = \mathbf{F}(t, \mathbf{u}(t)), \quad \forall \mathbf{u} \in \mathcal{U}_I, \quad \forall t \in I.$$

The a priori estimates from Proposition 3.4.2 allow us to obtain a function $\gamma : I \times H_w \to \mathbb{R}$ satisfying

$$\int_{t_0}^{t} \|\mathbf{F}(s, \mathbf{u}(s))\|_{V'} \mathrm{d}s \leq \gamma(t, \mathbf{u}(t_0)), \quad \forall t \in I, \quad \forall \mathbf{u} \in \mathcal{U}_I.$$

And using (3.48) it is not difficult to show that the function

$$t \mapsto \int_{H_w} \gamma(t, \mathbf{u}_0) \mathrm{d}\mu_0(\mathbf{u}_0)$$

belongs to $L^1_{\text{loc}}(I)$.

Also, as mentioned before, we know that $V \subset H_w \hookrightarrow V'_{w*}$ and that the injection of H_w into V'_{w*} is continuous. Moreover, if $B_V(\mathbf{u}_0, r)$ denotes the closed ball in V centered at $\mathbf{u}_0 \in V$ with radius $r > 0$, then

$$\begin{aligned} B_V(\mathbf{u}_0, r) &= \{\mathbf{u} \in V \,|\, \|\mathbf{u} - \mathbf{u}_0\| \leq r\} \\ &= \bigcap_{k \in \mathbb{N}} \{\mathbf{u} \in V \,|\, \|P_k(\mathbf{u} - \mathbf{u}_0)\| \leq r\} \\ &= \bigcap_{k \in \mathbb{N}} \{\mathbf{u} \in H \,|\, \|P_k(\mathbf{u} - \mathbf{u}_0)\| \leq r\}, \end{aligned}$$

and since $P_k : H \to V$ is continuous then every set inside the intersection of the last equality is closed in H. Thus, every closed ball in V is a Borel set in H. This implies that $\mathfrak{B}_V \subset \mathfrak{B}_H$. Moreover, since H is a separable Banach space, then $\mathfrak{B}_H = \mathfrak{B}_{H_w}$. Hence, $\mathfrak{B}_V \subset \mathfrak{B}_{H_w}$.

Then, applying Theorem 3.3.2 with $X = H_w$, $Y = V$, $\mathcal{U} = \mathcal{U}_I$, $\mathbf{F}$ and γ as above, we obtain the existence of a projected statistical solution $\{\rho_t\}_{t \in I}$ associated with a $\mathcal{U}_I$-trajectory statistical solution ρ and such that $\rho_{t_0} = \mu_0$. This means that $\{\rho_t\}_{t \in I}$ satisfies (i), (iii), and the first part of (ii), concerning bounded and weakly-continuous functions φ on H.

Let us prove the second part of property (ii), concerning strongly continuous functions. Consider then a bounded and continuous real-valued function φ on H. Let P_m, $m \in \mathbb{N}$, be the Galerkin projectors. Then, for every $m \in \mathbb{N}$, the function $\varphi \circ P_m$ is bounded and continuous on H_w. Let $\mathcal{V} \subset \mathcal{U}$ be a Borel subset such that $\rho(\mathcal{V}) = 1$. From the first part of (ii), it follows that the function

$$t \mapsto \int_{\mathcal{V}} \varphi(P_m \mathbf{u}(t)) \mathrm{d}\rho(\mathbf{u})$$

is continuous on I, for every $m \in \mathbb{N}$. Then, since the function (3.49) is the pointwise (in t) limit of these functions as $m \to \infty$, it follows that (3.49) is measurable on I. This proves the second part of (ii).

For the proof of (iv), consider the functions $\alpha : I \times H_w \to \mathbb{R}$ and $\beta : I \times V \to \mathbb{R}$ defined respectively by

$$\alpha(t, \mathbf{u}(t)) = \psi(|\mathbf{u}(t)|_H^2) \tag{3.54}$$

and

$$\beta(t, \mathbf{u}(t)) = -2\psi'(|\mathbf{u}(t)|_H^2)[\langle f(t), \mathbf{u}(t)\rangle_{V',V} - \nu\|\mathbf{u}(t)\|_V^2], \tag{3.55}$$

for every $\mathbf{u} \in \mathcal{U}_I$ and $t \in I$. Using the estimates (3.38) and (3.39) with $t' = t_0$, which is allowed for functions in $\mathcal{U}_I$, and using (3.48), we obtain that $\alpha \in L^\infty(J \times \mathcal{V}, \lambda \times \rho)$ and $\beta \in L^1(J \times \mathcal{V}, \lambda \times \rho)$, for every compact subset $J \subset I$, where λ denotes the Lebesgue measure on I. From Proposition 3.4.1, the functions α and β satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t, \mathbf{u}(t)) + \beta(t, \mathbf{u}(t)) \leq 0, \quad \forall t \in I, \quad \forall \mathbf{u} \in \mathcal{U}_I,$$

in the sense of distributions in I. Property (iv) then follows by applying Proposition 3.3.1 with $X = H_w$, $Y = V$, $\mathcal{U} = \mathcal{U}_I$ and with the functions α and β defined in (3.54)-(3.55).

It only remains to prove property (v). Note that for every function ψ as in (3.51), we may write

$$\psi(|\mathbf{u}(t)|_H^2) \leq \psi(0) + \psi'(\xi)|\mathbf{u}(t)|_H^2,$$

for some $0 \leq \xi \leq |\mathbf{u}(t)|^2$. Then, using the boundedness of ψ' and the a priori estimate (3.38) with $t' = t_0$, we find that

$$\psi(|\mathbf{u}(t)|_H^2) \leq C_0 + C_1|\mathbf{u}(t_0)|_H^2,$$

for suitable constants $C_0, C_1 > 0$. Hence, from (3.48), it follows that $\psi(|\mathbf{u}(t)|^2)$ is bounded by a ρ-integrable function in $\mathcal{V}$ which does not depend on t. Then, since every $\mathbf{u} \in \mathcal{U}_I$ is strongly continuous at t_0 and ψ is continuous, we have that

$$\psi(|\mathbf{u}(t)|_H^2) \to \psi(|\mathbf{u}(t_0)|_H^2),$$

ρ-almost everywhere, as $t \to t_0^+$. Therefore, (3.52) follows from the Lebesgue Dominated Convergence Theorem. □

3.4.2 Reaction-Diffusion Equation

In this subsection, we shall analyze the following reaction-diffusion-type equation

$$\frac{\partial u}{\partial t}(\mathbf{x}, t) = a\Delta u(\mathbf{x}, t) - f(u(\mathbf{x}, t), t) + g(\mathbf{x}, t), \quad \mathbf{x} \in \Omega \subset \mathbb{R}^n, \quad t \in I, \tag{3.56}$$

subject to the boundary condition

$$u(\mathbf{x}, t)|_{\mathbf{x}\in\partial\Omega} = 0, \quad \forall t \in I, \tag{3.57}$$

where u is the unknown variable, a is a positive constant, f is the interaction function and g is the external force. Moreover, $I \subset \mathbb{R}$ is an arbitrary interval and $\Omega \subset \mathbb{R}^n$ is a bounded and open subset which is assumed to be smooth.

We follow the same framework and notations from [9, Section XV.3], but in order to simplify the presentation we consider only a scalar equation instead of a system of equations.

Consider the spaces $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$ with respective inner products $(\cdot, \cdot)_H$ and $(\!(\cdot, \cdot)\!)_V$, given by

$$(u, v)_H = \int_\Omega u(\mathbf{x})v(\mathbf{x})\mathrm{d}\mathbf{x}\,, \ \forall u, v \in H,$$

and

$$(\!(u, v)\!)_V = \int_\Omega \nabla v(\mathbf{x}) \cdot \nabla u(\mathbf{x})\mathrm{d}\mathbf{x}\,, \ \forall u, v \in V.$$

The corresponding norms in H and V are denoted by $|\cdot|_H$ and $\|\cdot\|_V$, respectively. Also, consider $V' = H^{-1}(\Omega)$. Then, identifying H with its dual space H', we have

$$V \subset H \equiv H' \subset V',$$

with continuous inclusions and, in particular, $H \hookrightarrow V'_{w*}$ with continuous injection.

We assume that $g \in L^2_{loc}(I, V')$ and that f is a function in $\mathcal{C}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ satisfying the following estimates, for every $v \in \mathbb{R}$ and $s \in \mathbb{R}$:

$$\eta|v|^p - C_1 \le f(v, s)v, \tag{3.58}$$

$$|f(v,s)|^{\frac{p}{p-1}} \leq C_2(|v|^p + 1), \tag{3.59}$$

where $\eta > 0$, $p \geq 2$ and C_1, $C_2 \in \mathbb{R}$ are constants. In [9], the function g is also assumed to be translation bounded in the space $L^2_{loc}(I, V')$. However, we do not consider this hypothesis since it is not needed for our purposes.

As in [9], it follows by using condition (3.59) that if $r \geq \max\{1, n(1/2 - 1/p)\}$ and $u \in L^p_{loc}(I, L^p(\Omega)) \cap L^2_{loc}(I, V)$, then $\partial_t u \in L^q_{loc}(I, H^{-r}(\Omega))$, for $1/p + 1/q = 1$. This implies that the evolution equation (3.56) can be considered in the distribution sense on I, with values in $H^{-r}(\Omega)$.

We then have the following definition of a weak solution for problem (3.56)-(3.57).

Definition 3.4.2. *A weak solution of* (3.56)-(3.57) *is a function $u = u(\mathbf{x}, t)$ on $\Omega \times I$ such that $u \in L^p_{loc}(I, L^p(\Omega)) \cap L^2_{loc}(I, V)$ and u satisfies* (3.56) *in the distribution sense on I, with values in $H^{-r}(\Omega)$.*

Given $R \geq 0$, let $B_H(R)$ be the closed ball centered at the origin and of radius R in H. Consider the following sets of weak solutions:

$$\mathcal{U}_I = \{u \in \mathcal{C}_{\text{loc}}(I, H) \,|\, u \text{ is a weak solution of (3.56)-(3.57) on } I\}, \tag{3.60}$$

$$\mathcal{U}_I(R) = \{u \in \mathcal{C}_{\text{loc}}(I, B_H(R)) \,|\, u \text{ is a weak solution of (3.56)-(3.57) on } I\}, \tag{3.61}$$

The proof of existence of individual weak solutions for the corresponding initial-value problem of (3.56)-(3.57) can be found in [9]. We state it below for completeness.

Theorem 3.4.4. *Consider an interval $I \subset \mathbb{R}$ bounded and closed on the left with left end point t_0. Let $g \in L^2_{loc}(I, V')$ and let $f \in \mathcal{C}(\mathbb{R} \times \mathbb{R})$ be a function satisfying conditions* (3.58) *and* (3.59)*. Then, given $u_0 \in H$, there exists a weak solution u of problem* (3.56)-(3.57) *such that $u \in L^p_{loc}(I, L^p(\Omega)) \cap L^2_{loc}(I, V) \cap L^\infty_{loc}(I, H)$ and $u(t_0) = u_0$.*

The following proposition presents some additional properties satisfied by every weak solution of (3.56)-(3.57) in the sense of Definition 3.4.2. The proof is given in [9, Proposition XV.3.1].

Proposition 3.4.4. *Let $u \in L^p_{loc}(I, L^p(\Omega)) \cap L^2_{loc}(I, V)$ be a weak solution of* (3.56)-(3.57)*. Then*

(i) $u \in \mathcal{C}_{loc}(I, H)$;

(ii) the function $|u(s)|^2_H$ is absolutely continuous on every compact subinterval $J \subset I$ and satisfies the following energy equality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|^2_H + a\|u(t)\|_V + (f(u(t), t), u(t))_H = \langle g(t), u(t)\rangle_{V',V}, \tag{3.62}$$

for almost every $t \in I$, where $\langle \cdot, \cdot \rangle_{V',V}$ denotes the duality product in V.

Now we prove that the set of weak solutions $\mathcal{U}_I$ satisfies the hypothesis (H). We first observe that Theorem 3.4.4 implies that $\Pi_{t_0}\mathcal{U}_I = H$. This means that $\mathcal{U}_I$ satisfies hypothesis (H1) of Definition 3.2.1, with $X = H$.

For the remaining hypothesis, we can show in this case that the stronger condition (H2') holds. This is proved in the following proposition.

Proposition 3.4.5. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0 and let K be a compact subset of H. Then $\Pi_{t_0}^{-1}K \cap \mathcal{U}_I$ is compact in $\mathcal{C}_{loc}(I,H)$.*

Proof. Since $\mathcal{X} = \mathcal{C}_{loc}(I,H)$ is a metrizable space, it suffices to show that $\Pi_{t_0}^{-1}K \cap \mathcal{U}_I$ is sequentially compact. Consider then a sequence $\{u_j\}_j$ in $\Pi_{t_0}^{-1}K \cap \mathcal{U}_I$. Since K is compact there exists $u_0 \in H$ such that, by taking a subsequence if necessary, $u_j(t_0) \to u_0$ in H. This implies in particular that the sequence $\{u_j(t_0)\}$ is bounded in H.

Using condition (3.58) on the energy equality (3.62) for each u_j and integrating from t_0 to t, we obtain that

$$|u_j(t)|_H^2 - |u_j(t_0)|_H^2 + a\int_{t_0}^t \|u_j(s)\|_V^2 \mathrm{d}s + 2\eta\int_{t_0}^t \|u_j(s)\|_p^p \mathrm{d}s \leq \frac{1}{a}\int_{t_0}^t \|g(s)\|_{V'}^2 \mathrm{d}s + 2|t-t_0|C, \tag{3.63}$$

where $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$ and $C = C_1\int_\Omega \mathrm{d}\mathbf{x}$. Consider a sequence $\{J_n\}_n$ of compact subintervals of I such that $I = \bigcup_n J_n$. Then, from the estimate (3.63) and the boundedness of the sequence $\{u_j(t_0)\}$ in H, it follows that, for each n, $\{u_j\}_j$ is a bounded sequence in $L^2(J_n,V) \cap L^p(J_n,L^p(\Omega)) \cap L^\infty(J_n,H)$. Now, using the same arguments as in [9, Theorem XV.3.1] and a diagonalization process, we can obtain a weak solution u of problem (3.56)-(3.57) on I such that, modulo a subsequence, $u_j \to u$ in $\mathcal{C}_{\mathrm{loc}}(I,H)$. In particular, it follows that $u_j(t) \to u(t)$ in H, for every $t \in I$. Thus, $u(t_0) = u_0 \in K$ and we conclude that $u \in \Pi_{t_0}^{-1}K \cap \mathcal{U}_I$, as required. ☐

The existence of a trajectory statistical solution with respect to a given initial data now follows by a simple application of Theorem 3.2.1 for $X = H$ and $\mathcal{U}_I$ as the set of weak solutions of (3.56)-(3.57) over a given interval $I \subset \mathbb{R}$ closed and bounded on the left. Recall that since H is a Polish space then every Borel probability measure on H is tight.

Theorem 3.4.5. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0 and let $\mathcal{U}_I$ be the set of weak solutions of problem (3.56)-(3.57) on I. If μ_0 is a Borel probability measure on H then there exists a $\mathcal{U}_I$-trajectory statistical solution ρ on $\mathcal{C}_{loc}(I,H)$ such that $\Pi_{t_0}\rho = \mu_0$.*

Now we obtain the existence of statistical solutions with respect to a given initial measure.

Theorem 3.4.6. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0 and let μ_0 be a Borel probability measure on H satisfying*

$$\int_H |u|_H^2 \mathrm{d}\mu_0(u) < \infty. \tag{3.64}$$

Then there exists a projected statistical solution $\{\rho_t\}_{t\in I}$ of (3.56)-(3.57), associated with a $\mathcal{U}_I$-trajectory statistical solution, such that

(i) *The initial condition $\rho_{t_0} = \mu_0$ holds;*

(ii) *The function*

$$t \mapsto \int_H \varphi(u)\mathrm{d}\rho_t(u) \tag{3.65}$$

is continuous on I for every bounded and continuous real-valued function φ on H.

(iii) *For any cylindrical test function Φ, it follows that*

$$\int_H \Phi(u)\mathrm{d}\rho_t(u) = \int_H \Phi(u)\mathrm{d}\rho_{t'}(u) + \int_{t'}^t \int_H \langle a\Delta u - f(u,s) + g(s), \Phi'(u)\rangle_{V',V}\, \mathrm{d}\rho_s(u)\mathrm{d}s, \tag{3.66}$$

for all $t, t' \in I$.

(iv) *For every non-negative, nondecreasing and continuously-differentiable real-valued function ψ with bounded derivative, the function*

$$t \mapsto \int_H \psi(|u|_H^2)\mathrm{d}\rho_t(u)$$

is absolutely continuous on I. Moreover, the following mean strengthened energy equality holds in the distribution sense on I:

$$\frac{d}{dt}\int_H \psi(|u|_H^2)\mathrm{d}\rho_t(u) = 2\int_H \psi'(|u|_H^2)[\langle g(t), u\rangle_{V',V} - a\|u\|_V^2 - (f(u,t),u)_H]\mathrm{d}\rho_t(u).$$

Proof. Apply Theorem 3.3.2 and Proposition 3.3.2 for $X = H$, $Y = V$, $\mathcal{U} = \mathcal{U}_I$ and the functions $F : I \times V \to V'$, $\alpha : I \times H \to \mathbb{R}$ and $\beta : I \times V \to \mathbb{R}$ defined as

$$F(t,u) = a\Delta u(t) - f(u(t),t) + g(t),$$

$$\alpha(t,u) = \psi(|u|_H^2)$$

and

$$\beta(t,u) = -2\psi'(|u|_H^2)[\langle g(t), u\rangle_{V',V} - a\|u\|_V^2 - (f(u,t),u)_H].$$

$\square$

3.4.3 Nonlinear Wave Equation

In this subsection, we apply the abstract framework to prove the existence of statistical solutions of a nonlinear hyperbolic-type equation which appears within the theory of Relativistic Quantum Mechanics. We follow the ideas contained in [36, Chap. 1, Sec.1].

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary, denoted $\partial\Omega$, and let $I \subset \mathbb{R}$ be an arbitrary interval. Consider the following equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + |u|^r u = f, \tag{3.67}$$

where $u = u(\mathbf{x}, t)$ is the unknown variable, r is a positive constant and $f = f(\mathbf{x}, t)$ is a given function, with $\mathbf{x} \in \Omega$ and $t \in I$.

We endow equation (3.67) with the following boundary condition:

$$u(\mathbf{x}, t)|_{\mathbf{x} \in \partial\Omega} = 0, \quad \forall t \in I. \tag{3.68}$$

In order to obtain a functional setting for problem (3.67)-(3.68), we introduce the space

$$\tilde{V} = H_0^1(\Omega) \cap L^p(\Omega),$$

where $p = r + 2$. The space $\tilde{V}$ turns into a Banach space when endowed with the norm $\|\cdot\|_{\tilde{V}}$, defined by

$$\|v\|_{\tilde{V}} = \|v\|_{H_0^1} + \|v\|_{L^p}, \quad \forall v \in \tilde{V},$$

where $\|\cdot\|_{H_0^1}$ and $\|\cdot\|_{L^p}$ denote the usual norms in the spaces $H_0^1(\Omega)$ and $L^p(\Omega)$, respectively.

The dual space of $\tilde{V}$ is the space $\tilde{V}' = H^{-1}(\Omega) + L^{p'}(\Omega)$, where $1/p + 1/p' = 1$. The duality product between $\tilde{V}$ and $\tilde{V}'$ is denoted by $\langle \cdot, \cdot \rangle_{\tilde{V}', \tilde{V}}$.

Also, we consider the space $L^2(\Omega)$ endowed with its usual norm and inner product, which are denoted respectively by $\|\cdot\|_{L^2}$ and $((\cdot, \cdot))_{L^2}$. And we assume that f is a function in $L^2_{\text{loc}}(I, L^2(\Omega))$.

We may also rewrite equation (3.67) under the following equivalent form:

$$\begin{cases} \dfrac{\partial u}{\partial t} - v = 0 \\[2ex] \dfrac{\partial v}{\partial t} - \Delta u + |u|^r u = f \end{cases} \tag{3.69}$$

Let us denote the nonlinear term of the second equation in (3.69) through the function $b : \tilde{V} \to \tilde{V}'$ given by

$$b(u) = |u|^r u, \quad \forall u \in \tilde{V}.$$

Furthermore, considering $U = (u, v)$ and the linear operator A defined by

$$AU = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix} U = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ -\Delta u \end{pmatrix},$$

then system (3.69) turns into

$$\frac{\mathrm{d}U}{\mathrm{d}t} + AU + N(U) = G, \tag{3.70}$$

where $N(U)$ and G are the vectors

$$N(U) = \begin{pmatrix} 0 \\ b(u) \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

Furthermore, from (3.68) we immediately obtain the following boundary condition for (3.70):

$$U(\mathbf{x}, t)|_{\mathbf{x} \in \partial\Omega} = 0, \quad \forall t \in I. \tag{3.71}$$

Now we define $V = \tilde{V} \times L^2(\Omega)$, in which the following norm is defined

$$\|U\|_V = \|u\|_{\tilde{V}} + \|v\|_{L^2}, \quad \forall U = (u, v) \in V.$$

When endowed with its corresponding weak topology, the space V is denoted by V_w.

We characterize the dual of V as the space $V' = L^2 \times \tilde{V}'$, with the duality product between $h = (f, g) \in V'$ and $U = (u, v) \in V$ as

$$\langle h, U \rangle_{V', V} = (\!(f, v)\!)_{L^2} + \langle g, u \rangle_{\tilde{V}', \tilde{V}}.$$

With this representation, the usual norm for an element $h = (f, g)$ in the dual space V' can also be written in the form

$$\|h\|_{V'} = \sqrt{\|f\|_{L^2}^2 + \|g\|_{\tilde{V}'}^2}.$$

We now give the definition of a weak solution of problem (3.70)-(3.71).

Definition 3.4.3. *Let $I \subset \mathbb{R}$ be an interval and let $f \in L^2_{loc}(I; L^2(\Omega))$. We say that $U = U(t) = (u(t), v(t))$ is a* ***weak solution of problem*** (3.70)-(3.71) ***on*** *I if the following conditions are satisfied:*

(i) $U \in L^\infty_{loc}(I; V)$;

(ii) $U \in \mathcal{C}_{loc}(I, V_w)$;

(iii) U satisfies

$$\frac{\mathrm{d}U}{\mathrm{d}t} + AU + N(U) = G \quad \text{in } V', \tag{3.72}$$

in the sense of distributions on I.

(iv) For almost every $\tilde{t} \in I$, U satisfies the following energy inequality

$$E(U(t)) \leq E(U(\tilde{t})) + \int_{\tilde{t}}^{t} \langle \check{G}(s), U(s)\rangle_{V',V} \mathrm{d}s, \tag{3.73}$$

for every $t \in I$ with $t > \tilde{t}$, where $\check{G} = (f, 0)$ and

$$E(U) = E(u,v) = \frac{1}{2}\|u\|^2_{H^1_0} + \frac{1}{p}\|u\|^p_{L^p} + \frac{1}{2}\|v\|^2_{L^2}. \tag{3.74}$$

The set of times $\tilde{t}$ for which (3.73) is valid can be characterized as the points of strong continuity from the right of U, and they form a set of total measure in I.

(v) If I is closed and bounded on the left, with left end point t_0, then U is strongly continuous at t_0 from the right, i.e. $U(t) \to U(t_0)$ in V as $t \to t_0^+$.

For any $R > 0$, let $B_V(R)$ denote the closed ball of radius R in V. We then define the following sets of weak solutions of problem (3.70)-(3.71):

$$\mathcal{U}_I = \{U \in \mathcal{C}_{loc}(I, V_w) : U \text{ is a weak solution of (3.70)-(3.71) on } I\}, \tag{3.75}$$

$$\mathcal{U}_I(R) = \{U \in \mathcal{C}(I, B_V(R)_w) : U \text{ is a weak solution of (3.70)-(3.71) on } I\}, \tag{3.76}$$

Next we state an existence theorem of individual weak solutions for the initial value problem associated to the system (3.70)-(3.71). The proof is given in [36, Theorem 1.1, Chap. 1, Sec. 1]. Although the regularity conditions (ii), (iv) and (v) of Definition 3.4.3 are not explicitly written in the statement of the theorem in this reference, they are obtained along the lines of the proof.

Theorem 3.4.7. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0 and let $f \in L^2_{loc}(I; L^2(\Omega))$. Then, given $U_0 \in V$, there exists at least one weak solution $U \in \mathcal{U}_I$ of (3.70)-(3.71) in the sense of Definition 3.4.3 satisfying $\Pi_{t_0} U = U_0$.*

Consider now $I \subset \mathbb{R}$ an interval closed and bounded on the left with left end point t_0. In this case, item (v) of Definition 3.4.3 implies that the energy inequality (3.73) is valid for $\tilde{t} = t_0$.

Let $U = (u, v) \in \mathcal{U}_I$ such that $U(t_0) \in B_V(R)$, for some $R > 0$. From the energy inequality (3.73) with $\tilde{t} = t_0$, it follows that

$$E(U(t)) \leq R + \frac{1}{2}\int_{t_0}^{t} \|f(s)\|^2_{L^2}\mathrm{d}s + \frac{1}{2}\int_{t_0}^{t} \|v(s)\|^2_{L^2}\mathrm{d}s, \tag{3.77}$$

for every $t \in I$, which also yields

$$\frac{1}{2}\|v(t)\|_{L^2}^2 \leq R + \frac{1}{2}\int_{t_0}^{t} \|f(s)\|_{L^2}^2 \mathrm{d}s + \frac{1}{2}\int_{t_0}^{t} \|v(s)\|_{L^2}^2 \mathrm{d}s. \tag{3.78}$$

Then, given a compact subinterval $J \subset I$, by applying Grönwall's inequality in (3.78) we obtain that $\|v(\cdot)\|_{L^2}$ is uniformly bounded on J. From the estimate (3.77), it then follows that there exists $\tilde{R} \geq R$ such that $U(t) \in B_V(\tilde{R})$, for every $t \in J$. Thus, the restriction of U to J belongs to $\mathcal{U}_J(\tilde{R})$.

We shall now prove that the set of weak solutions $\mathcal{U}_I$ satisfies the hypothesis (H).

Theorem 3.4.7 shows, in an equivalent form, that $\Pi_{t_0}\mathcal{U}_I = V_w$. Thus, the set $\mathcal{U}_I$ satisfies hypothesis (H1) with $X = V_w$.

Now define

$$\mathfrak{K}'(V_w) = \{K \subset V_w \,|\, K \text{ is a (strongly) compact set in } V\}.$$

By analogous arguments used in Subsection 3.4.1, we obtain that every Borel probability measure μ_0 on V is tight with respect to the family $\mathfrak{K}'(V_w)$. Then, considering $X = V_w$, it follows that the family $\mathfrak{K}'(V_w)$ satisfies hypotheses (i) and (ii) of (H2).

The next proposition shows that $\mathcal{U}_I$ also satisfies hypothesis (iii) of (H2).

Proposition 3.4.6. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0 and let K be a set in $\mathfrak{K}'(V_w)$. Then $\Pi_{t_0}^{-1}K \cap \mathcal{U}_I$ is a compact set in $\mathcal{X} = \mathcal{C}_{loc}(I, V_w)$.*

Proof. Let $R > 0$ be such that $K \subset B_V(R)$ and let $\{J_n\}_n$ be a sequence of compact subsets of I such that

$$I = \bigcup_n J_n.$$

Then, from the energy inequality (3.73) with $\tilde{t} = t_0$, one obtains that for each n there exists a positive real number $R_n \geq R$ such that

$$\Pi_{J_n}U \in \mathcal{U}_{J_n}(R_n), \quad \forall U \in \Pi_{t_0}^{-1}K \cap \mathcal{U}_I,$$

for every n, which implies that

$$\Pi_{t_0}^{-1}K \cap \mathcal{U}_I \subset \bigcap_n \Pi_{J_n}^{-1}\mathcal{U}_{J_n}(R_n).$$

Since $\mathcal{U}_{J_n}(R_n)$ is a subset of $\mathcal{C}_{loc}(I, B_V(R_n)_w)$, which is a metrizable space, it follows that $\Pi_{t_0}^{-1}K \cap \mathcal{U}_I$ is also metrizable. Thus, it is enough to prove that $\Pi_{t_0}^{-1}K \cap \mathcal{U}_I$ is a sequentially compact space.

Consider then a sequence $\{U_k\}_k$ in $\Pi_{t_0}^{-1}K \cap \mathcal{U}_I$. Since $U_k(t_0) \in K$ and K is a compact set in V, there exists $U_0 \in V$ and a subsequence $\{k_j\}_j$ such that

$$U_{k_j}(t_0) \to U_0 \quad \text{in } V. \tag{3.79}$$

Following classical arguments used for the existence of weak solutions (see [36, Chap. 1, Sec.1]), we obtain a priori estimates that allow us to pass to the limit on each compact set J_n. Then, using a diagonalization process, we obtain a further subsequence (which we still denote by $\{U_{k_j}\}_j$) and a function U defined on the interval I such that $\{U_{k_j}\}_j$ converges to U in $\mathcal{C}_{loc}(I, V_w)$ and U is a weak solution on the interior of I. Thanks to (3.79) we have at the initial time that $U(t_0) = U_0 \in K$, so that $U \in \Pi_{t_0}^{-1} K$. Then, as in the case of the Navier-Stokes equations (see the proof of Proposition 3.4.3), using the energy inequality and the fact that the convergence (3.79) at the initial time is in the strong topology, we obtain that U is strongly continuous at the initial time t_0, so that $U \in \mathcal{U}_I$. Therefore, $U \in \Pi_{t_0}^{-1} K \cap \mathcal{U}_I$, proving that $\Pi_{t_0}^{-1} K \cap \mathcal{U}_I$ is compact. $\square$

Thus, applying Theorem 3.2.1 for $X = V_w$ and $\mathcal{U}$ as the set of weak solutions $\mathcal{U}_I$, we obtain the following result on the existence of a trajectory statistical solution for the nonlinear wave equation with respect to a given initial data.

Theorem 3.4.8. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0 and let $\mathcal{U}_I$ be the set of weak solutions of problem (3.70)-(3.71) on I. If μ_0 is a Borel probability measure on V then there exists a $\mathcal{U}_I$-trajectory statistical solution ρ on $\mathcal{C}_{loc}(I, V_w)$ such that $\Pi_{t_0}\rho = \mu_0$.*

In the next result we prove the existence of a statistical solution of problem (3.70)-(3.71) in the sense of Definition 3.1.2 with respect to a given initial data.

Theorem 3.4.9. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0 and let $\mathcal{U}_I$ be the set of weak solutions of problem (3.70)-(3.71) on I. Consider a Borel probability measure μ_0 on V satisfying*

$$\int_V E(U) \mathrm{d}\mu_0(U) < \infty, \tag{3.80}$$

with E as defined in (3.74). Then there exists a projected statistical solution $\{\rho_t\}_{t\in I}$ of (3.67), associated with a $\mathcal{U}_I$-trajectory statistical solution, such that

(i) The initial condition $\rho_{t_0} = \mu_0$ holds;

(ii) The function

$$t \mapsto \int_V \varphi(U) \mathrm{d}\rho_t(U) \tag{3.81}$$

is continuous on I, for every bounded and weakly-continuous real-valued function φ on V, and is measurable on I, for every bounded and continuous real-valued function φ on V.

(iii) For any cylindrical test function Φ, it follows that

$$\int_V \Phi(U)\mathrm{d}\rho_t(U) = \int_V \Phi(U)\mathrm{d}\rho_{t'}(U) + \int_{t'}^{t}\int_V \langle G - AU - N(U), \Phi'(U)\rangle_{V',V}\mathrm{d}\rho_s(U)\mathrm{d}s, \quad (3.82)$$

for all $t, t' \in I$.

(iv) The mean strengthened energy inequality

$$\frac{d}{dt}\int_V \psi(E(U))\mathrm{d}\rho_t(U) \leq \int_V \psi'(E(U))\langle \check{G}(t), U(t)\rangle_{V',V}\mathrm{d}\rho_t(U) \quad (3.83)$$

is satisfied in the distribution sense on I, for every nonnegative, nondecreasing and continuously-differentiable real-valued function ψ with bounded derivative, where $\check{G} = (f, 0)$.

(v) At the initial time, the limit

$$\lim_{t\to t_0^+}\int_V \psi(E(U))\mathrm{d}\rho_t(U) = \int_V \psi(E(U))\mathrm{d}\mu_0(U) \quad (3.84)$$

holds for every function ψ as in (iv).

Proof. The proof follows by arguments similar to the ones used in Theorem 3.4.3, considering $X = V_w$, $Y = V$ and the functions $F : I \times V \to V'$, $\alpha : I \times V_w \to \mathbb{R}$ and $\beta : I \times V \to \mathbb{R}$ defined respectively as

$$F(t, U) = G(t) - AU(t) - N(U(t)),$$

$$\alpha(t, U) = \psi(E(U(t))),$$

and

$$\beta(t, U) = -\psi'(E(U(t)))\langle \check{G}(t), U(t)\rangle_{V',V},$$

for every (t, U) in the corresponding domains. □

3.4.4 Bénard Problem

In this section we consider a model for a phenomenon of convection in fluids, namely the Bénard problem, which consists of the Navier-Stokes equations coupled with an equation for the temperature via the Boussinesq approximation [2, 34].

We shall analyze the three-dimensional case for a homogeneous and incompressible fluid in the region $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, 0 < x_3 < h\}$. At the lower surface $x_3 = 0$, the fluid is heated at a constant temperature T_0, while at the upper surface $x_3 = h$, the fluid is at a temperature $T_1 < T_0$, also constant. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the canonical orthonormal basis in $\mathbb{R}^3$. Then, through the Boussinesq approximation one obtains the following equations describing the evolution of the velocity field $\mathbf{u}$, the pressure p and the temperature T:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = g\alpha(T - T_1)\mathbf{e}_3, \tag{3.85}$$

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)T - \kappa \Delta T = 0, \tag{3.86}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{3.87}$$

where g is the acceleration of gravity, α is the volume-expansion coefficient of the fluid, ν is the kinematic viscosity and κ is the coefficient of thermometric conductivity.

We also consider zero velocity field at the boundaries $x_3 = 0$ and $x_3 = h$ and periodic boundary conditions in the directions x_1 and x_2, so that the boundary conditions for problem (3.85)-(3.87) are given as

$$\mathbf{u} = 0 \quad \text{at} \quad x_3 = 0 \quad \text{and} \quad x_3 = h, \tag{3.88}$$

$$T = T_0 \quad \text{at} \quad x_3 = 0, \quad T = T_1 \quad \text{at} \quad x_3 = h, \tag{3.89}$$

$$p, \mathbf{u}, T \quad \text{are periodic in the } x_1 \text{ and } x_2 \text{ directions,} \tag{3.90}$$

where the last condition means that

$$\psi(x_1 + L_1, x_2, x_3) = \psi(x_1, x_2, x_3), \quad \forall (x_1, x_2, x_3) \in \mathbb{R}^2 \times (0, h)$$

$$\psi(x_1, x_2 + L_2, x_3) = \psi(x_1, x_2, x_3), \quad \forall (x_1, x_2, x_3) \in \mathbb{R}^2 \times (0, h)$$

for some positive real numbers L_1 and L_2, and ψ being any of the functions in condition (3.90).

In order to simplify the analysis of the problem, we define a background temperature $T_{b,\varepsilon}$, given by

$$T_{b,\varepsilon}(x_1, x_2, x_3) = \begin{cases} 0, & \text{for } 0 \le x_3 < h - \varepsilon, \\ \dfrac{(T_1 - T_0)}{\varepsilon}(x_3 - h + \varepsilon), & \text{for } h - \varepsilon \le x_3 \le h, \end{cases}$$

where ε is a positive real number which is chosen appropriately later. Then, we introduce a change of variables for the temperature by considering $\theta = T - T_0 - T_{b,\varepsilon}$, in terms of which, problem (3.85)-(3.87) is rewritten as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = g\alpha\theta\mathbf{e}_3 + g\alpha(T_{b,\varepsilon} + T_0 - T_1)\mathbf{e}_3, \tag{3.91}$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla)\theta - \kappa \Delta \theta = -(\mathbf{u} \cdot \nabla) T_{b,\varepsilon} + \kappa \Delta T_{b,\varepsilon}, \tag{3.92}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{3.93}$$

with the following boundary conditions

$$\mathbf{u} = 0 \quad \text{at} \quad x_3 = 0 \quad \text{and} \quad x_3 = h, \tag{3.94}$$

$$\theta = 0 \quad \text{at} \quad x_3 = 0 \quad \text{and} \quad x_3 = h, \tag{3.95}$$

$$p, \mathbf{u}, \theta \quad \text{are periodic in the } x_1 \text{ and } x_2 \text{ directions.} \tag{3.96}$$

Note that the boundary conditions for the temperature in the x_3 direction are now zero, justifying the introduction of this new variable. In fact, there are many possible choices for this background temperature. The one we use here has been chosen so as to yield uniform in time *a priori* estimates.

Let us now introduce the function spaces which are necessary in the following analysis. Consider the domain $\Omega = (0, L_1) \times (0, L_2) \times (0, h)$ and define

$$\mathcal{V}_1 = \{\mathbf{v} = \mathbf{w}|_\Omega \, : \, \mathbf{w} \in (\mathcal{C}_c^\infty(\mathbb{R}^2 \times (0,h)))^3, \, \nabla \cdot \mathbf{w} = 0,$$

$$\mathbf{w} \text{ is } L_1\text{-periodic in the } x_1 \text{ direction and } L_2\text{-periodic in the } x_2 \text{ direction}\},$$

and

$$\mathcal{V}_2 = \{\theta = \phi|_\Omega \, : \, \phi \in \mathcal{C}_c^\infty(\mathbb{R}^2 \times (0,h)), \, \phi \text{ is } L_1\text{-periodic in the } x_1 \text{ direction and}$$

$$L_2\text{-periodic in the } x_2 \text{ direction}\},$$

where $\mathcal{C}_c^\infty(\mathbb{R}^2 \times (0,h))$ denotes the set of infinitely differentiable and compactly supported functions in $\mathbb{R}^2 \times (0,h)$.

Then, let V_1 be the closure of $\mathcal{V}_1$ with respect to the $(H_0^1(\Omega))^3$ norm and H_1 be the closure of $\mathcal{V}$ with respect to the $(L^2(\Omega))^3$ norm. Also, let V_2 be the closure of $\mathcal{V}_2$ with respect to the $H_0^1(\Omega)$ norm and H_2 be the closure of $\mathcal{V}_2$ with respect to the $L^2(\Omega)$ norm. We then define the Hilbert spaces $V = V_1 \times V_2$ and $H = H_1 \times H_2$.

The inner product and norm in V_1 are defined as

$$(\!(\mathbf{v}, \tilde{\mathbf{v}})\!)_1 = \sum_{i,j=1}^{3} \int_\Omega \nabla v_i \cdot \nabla \tilde{v}_j \mathrm{d}x, \quad \forall \mathbf{v}, \tilde{\mathbf{v}} \in V_1,$$

$$\|\mathbf{v}\|_1 = (\!(\mathbf{v}, \mathbf{v})\!)_1^{1/2}, \quad \forall \mathbf{v} \in V_1.$$

Similarly for V_2,

$$((\theta,\tilde{\theta}))_2 = \int_\Omega \nabla\theta\cdot\nabla\tilde{\theta}\mathrm{d}x, \quad \forall\theta,\tilde{\theta}\in V_2,$$

$$\|\theta\|_2 = ((\theta,\theta))_2^{1/2}, \quad \forall\theta\in V_2.$$

Then, we define the following inner product and norm in the product space $V = V_1\times V_2$:

$$((z,\tilde{z}))_V = ((\mathbf{v},\tilde{\mathbf{v}}))_1 + \gamma((\theta,\tilde{\theta}))_2, \quad \forall z=(\mathbf{v},\theta),\ \tilde{z}=(\tilde{\mathbf{v}},\tilde{\theta})\in V,$$

$$\|z\|_V = ((z,z))_V^{1/2}, \quad \forall z\in V,$$

where γ is a positive parameter making the above definition dimensionally correct. Like ε, the parameter γ is chosen appropriately later.

Similarly, the inner products and norms of the spaces H_1 and H_2 are the usual ones from $(L^2(\Omega))^3$ and $L^2(\Omega)$, and are denoted respectively by $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$, with norms $|\cdot|_1$ and $|\cdot|_2$. The inner product and norm in the space H are then defined accordingly:

$$(z,\tilde{z})_H = (\mathbf{v},\tilde{\mathbf{v}})_1 + \gamma(\theta,\tilde{\theta})_2, \quad \forall z=(\mathbf{v},\theta),\ \tilde{z}=(\tilde{\mathbf{v}},\tilde{\theta})\in H,$$

$$|z|_H = (z,z)_H^{1/2}, \quad \forall z\in H.$$

We identify H_1 and H_2 with their respective duals and consider the dual spaces V_1' and V_2' of V_1 and V_2, respectively, so that $V_1\subset H_1 = H_1'\subset V_1'$ and $V_2\subset H_2 = H_2'\subset V_2'$, with continuous and dense injections.

In the product space, we characterize the dual of V as the space $V' = V_1'\times V_2'$, with the duality product between $h=(\mathbf{f},g)\in V'$ and $z=(\mathbf{u},\theta)\in V$ given by

$$\langle h,z\rangle = \langle \mathbf{f},\mathbf{u}\rangle_1 + \gamma\langle g,\theta\rangle_2,$$

where $\langle\cdot,\cdot\rangle_i$ denotes the duality product in V_i, $i=1,2$. With this representation, the usual norm for an element $h=(\mathbf{f},g)$ in the dual space $V' = V_1'\times V_2'$ can also be written in the form

$$\|h\|_{V'} = \sqrt{\|\mathbf{f}\|_{V_1'}^2 + \gamma\|g\|_{V_2'}^2}, \tag{3.97}$$

where $\|\cdot\|_{V_i'}$ denotes the usual norm of the dual space V_i', $i=1,2$.

Similarly, H is identified with its dual $H' = H_1'\times H_2' = H_1\times H_2$ with a norm analogous to (3.97), and we have the continuous and dense injections $V\subset H = H'\subset V$.

We rewrite the system (3.91)-(3.93) in the following functional form

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \nu A_1\mathbf{u} + B_1(\mathbf{u},\mathbf{u}) = g\alpha\theta\mathbf{e}_3 + g\alpha(T_{b,\varepsilon}+T_0-T_1)\mathbf{e}_3, \quad \text{in } V_1', \tag{3.98}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} + \kappa A_2\theta + B_2(\mathbf{u},\theta) = -(\mathbf{u}\cdot\nabla)T_{b,\varepsilon} + \kappa\Delta T_{b,\varepsilon}, \quad \text{in } V_2', \tag{3.99}$$

where $B_1(\cdot,\cdot) : V_1 \times V_1 \to V_1'$ and $B_2(\cdot,\cdot) : V_1 \times V_2 \to V_2'$ are the bilinear operators defined by duality as

$$\langle B_1(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_1 = ((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})_1 \quad \forall \mathbf{u},\ \mathbf{v},\ \mathbf{w} \in V_1,$$

$$\langle B_2(\mathbf{v},\theta),\tilde{\theta}\rangle_2 = ((\mathbf{v}\cdot\nabla)\theta,\tilde{\theta})_2 \quad \forall \mathbf{v} \in V_1,\ \forall\theta,\ \tilde{\theta} \in V_2.$$

Furthermore, $A_1 : V_1 \to V_1'$ and $A_2 : V_2 \to V_2'$ are the linear operators defined by duality according to

$$\langle A_1\mathbf{v},\tilde{\mathbf{v}}\rangle_1 = ((\mathbf{v},\tilde{\mathbf{v}}))_1, \quad \forall \mathbf{v},\ \tilde{\mathbf{v}} \in V_1,$$

$$\langle A_2\theta,\tilde{\theta}\rangle_2 = ((\theta,\tilde{\theta}))_2, \quad \forall\theta,\ \tilde{\theta} \in V_2.$$

Both these operators can be seen as positive and self-adjoint closed operators with compact inverse when restricted to their domain $D(A_j)$ in H_j, $j = 1, 2$, given by

$$D(A_1) = \{\mathbf{v} \in V_1;\ A_1\mathbf{v} \in H_1\},$$

$$D(A_2) = \{\theta \in V_2;\ A_2\theta \in H_2\}.$$

We let λ_1 and λ_2 denote the smallest eigenvalues of each of these operators and set $\lambda_0 = \min\{\lambda_1, \lambda_2\}$.

We can also write equations (3.98)-(3.99) in the compact form

$$\frac{\mathrm{d}z}{\mathrm{d}t} + Az + B(z,z) + Rz = 0 \quad \text{in } V',$$

where

$$z = (\mathbf{u}, \theta),$$

$$Az = (\nu A_1\mathbf{u}, \kappa A_2\theta),$$

$$Rz = (-g\alpha\theta\mathbf{e}_3 - g\alpha(T_{b,\varepsilon} + T_0 - T_1)\mathbf{e}_3, (\mathbf{u}\cdot\nabla)T_{b,\varepsilon} - \kappa\Delta T_{b,\varepsilon})$$

and

$$B(z,\tilde{z}) = (B_1(\mathbf{u},\tilde{\mathbf{u}}), B_2(\mathbf{u},\tilde{\theta})), \quad \forall z = (\mathbf{u},\theta),\ \tilde{z} = (\tilde{\mathbf{u}},\tilde{\theta}).$$

The following definition provides a notion of weak solution to the problem (3.91)-(3.93). We denote by H_w the space H endowed with the weak topology.

Definition 3.4.4. *Let $I \subset \mathbb{R}$ be an interval. We say that $z = (\mathbf{u},\theta)$ is a* ***Leray-Hopf weak solution of the Bénard problem*** (3.91)-(3.96) ***on*** *I if*

(i) $z \in L^2_{loc}(I;V) \cap L^\infty_{loc}(I;H) \cap \mathcal{C}_{loc}(I,H_w)$;

(ii) $\partial_t z \in L^{4/3}_{loc}(I;V')$;

(iii) z satisfies

$$\frac{\mathrm{d}z}{\mathrm{d}t} + Az + B(z,z) + Rz = 0 \quad \text{in } V'$$

in the sense of distributions on I.

(iv) For almost every $t' \in I$, $z = (\mathbf{u}, \theta)$ satisfies the following energy inequalities

$$\frac{1}{2}|\mathbf{u}(t)|_1^2 + \nu \int_{t'}^{t} \|\mathbf{u}(s)\|_1^2 \mathrm{d}s \le \frac{1}{2}|\mathbf{u}(t')|_1^2 + \int_{t'}^{t} (g\alpha\theta(s)\mathbf{e}_3, \mathbf{u}(s))_1 \mathrm{d}s \tag{3.100}$$

$$\begin{aligned}\frac{1}{2}|\theta(t)|_2^2 + \kappa \int_{t'}^{t} \|\theta(s)\|_2^2 \mathrm{d}s \quad &\le \quad \frac{1}{2}|\theta(t')|_2^2 + \int_{t'}^{t} \langle (\mathbf{u}\cdot\nabla)T_{b,\varepsilon}, \theta\rangle_2 \mathrm{d}s \\ &\quad + \kappa \int_{t'}^{t} \langle \Delta T_{b,\varepsilon}, \theta\rangle_2 \mathrm{d}s,\end{aligned} \tag{3.101}$$

for every $t \in I$ with $t > t'$. The set of times t' for which (3.100) and (3.101) are valid can be characterized as the points of strong continuity from the right of $\mathbf{u}$ and θ, and they form a set of total measure in I.

(v) If I is closed and bounded on the left, with left end point t_0, then z is strongly continuous at t_0 from the right, i.e. $z(t) \to z(t_0)$ in H as $t \to t_0^+$.

For any $R \ge 0$, let $B_H(R)$ be the closed ball with radius R in H and denote by $B_H(R)_w$ the closed ball endowed with the weak topology. Based on Definition 3.4.4, we consider the following trajectory spaces associated to the Bénard problem:

$$\mathcal{U}_I = \{z \in \mathcal{C}_{\mathrm{loc}}(I, H_w) \,|\, z \text{ is a weak solution of problem (3.91)-(3.96) on } I\}, \tag{3.102}$$

$$\mathcal{U}_I(R) = \{z \in \mathcal{C}_{\mathrm{loc}}(I, B_H(R)_w) \,|\, z \text{ is a weak solution of problem (3.91)-(3.96) on } I\}, \tag{3.103}$$

where $\mathring{I}$ denotes the interior of the interval I.

By choosing γ sufficiently large and ε sufficiently small, one obtains suitable estimates for the weak solutions in $\mathcal{U}_I$.

Proposition 3.4.7. *Let $I \subset \mathbb{R}$ be an interval and let $z \in \mathcal{U}_I$. Suppose γ satisfies*

$$\gamma > \frac{4(g\alpha)^2}{\nu\kappa\lambda_0^2} \tag{3.104}$$

and ε satisfies

$$0 < \varepsilon^2 < \frac{\nu}{\gamma(T_0 - T_1)^2}\left(\frac{\kappa}{4} - \frac{(g\alpha)^2}{\gamma\nu\lambda_0^2}\right). \tag{3.105}$$

Then, for every $t' \in I$ for which (3.100) and (3.101) are valid, and for every $t \in I$ with $t > t'$, the following estimates hold

$$|z(t)|_H^2 \le |z(t')|_H^2 \,\mathrm{e}^{-\eta\lambda_0(t-t')} + 2\frac{\kappa\gamma}{\eta\lambda_0}\frac{L_1L_2}{\varepsilon}(T_1 - T_0)^2(1 - \mathrm{e}^{-\eta\lambda_0(t-t')}), \tag{3.106}$$

$$\int_{t'}^{t} \|z(s)\|_V^2 \mathrm{d}s \leq \frac{1}{\eta}|z(t')|_H^2 + 2\frac{\kappa\gamma}{\eta}\frac{L_1 L_2}{\varepsilon}(T_1 - T_0)^2(t - t'), \tag{3.107}$$

$$\left(\int_{t'}^{t} \|\partial_t z(s)\|_{V'}^{4/3} \mathrm{d}s\right)^{3/4} \leq \frac{C}{(\nu\kappa)^{3/8}}|z(t')|_H^2 + C\frac{(\nu\kappa)^{1/8}}{\lambda_0^{3/2}}(t - t') + C\frac{(\nu\kappa)^{5/8}}{\lambda_0^{1/2}}, \tag{3.108}$$

where $\eta = \min\{\nu, \kappa\}$ *and* C *is a nondimensional constant which depends on the parameters* ν, κ, λ_0, g, α, T_0, T_1, γ, ε, L_1 *and* L_2 *through nondimensional combinations of them.*

The *a priori* estimates (3.106)-(3.108) allow us to prove the existence of weak solutions of the initial-value problem associated to system (3.91)-(3.93) in the sense of Definition 3.4.4. The proof follows in a way similar to the classical result of existence of Leray-Hopf weak solutions of the Navier-Stokes equations [10, 29, 46]. The choice of the background flow and of the parameters γ and ε were based on the formulation given in [7] for the Bénard problem in two dimensions (see also [26] for the three-dimensional case in which the boundary conditions are, however, fully homogeneous, with the flow driven instead by a forcing distributed within the domain). We then obtain the following result.

Theorem 3.4.10. *Let* $z_0 \in H$. *Then there exists at least one weak solution of problem* (3.91)-(3.96) *in the sense of Definition 3.4.4 satisfying* $z(t_0) = z_0$.

Let us now consider $I \subset \mathbb{R}$ as an interval closed and bounded on the left with left end point t_0. Define

$$R_0 = \left(2\frac{\kappa\gamma}{\eta\lambda_0}\frac{L_1 L_2}{\varepsilon}\right)^{1/2}(T_1 - T_0).$$

If $R \geq R_0$ and $z \in \mathcal{U}_I$ is such that $z(t_0) \in B_H(R)$, it follows from (3.106) with $t' = t_0$ that $z(t) \in B_H(R)$, for every $t \geq t_0$. Thus, $z \in \mathcal{U}_I(R)$.

In order to show that the abstract framework we constructed is valid for the Bénard problem, consider X as H_w and the abstract set of trajectories $\mathcal{U}$ as the set of weak solutions $\mathcal{U}_I$ defined in (3.102).

First, observe that hypothesis (H1) of Definition 3.2.1 is readily verified by the set $\mathcal{U}_I$ thanks to Theorem 3.4.10.

Also, let $\mathfrak{K}'(H_w)$ be the family of (strongly) compact sets in H. Then, hypotheses (i) and (ii) of (H2) are clearly satisfied by this family. The following proposition asserts that $\mathfrak{K}'(H_w)$ satisfies in addition hypothesis (iii) of (H2). The proof follows by arguments analogous to the ones in Theorem 3.4.3, using the energy inequalities (3.100)-(3.101) and the a priori estimates (3.106)-(3.108), provided the parameters γ and ε satisfy conditions (3.104) and (3.105), respectively.

Proposition 3.4.8. *Let* $I \subset \mathbb{R}$ *be an interval closed and bounded on the left with left end point* t_0 *and let* K *be a set in* $\mathfrak{K}'(H_w)$. *Then* $\Pi_{t_0}^{-1}K \cap \mathcal{U}_I$ *is a compact set in* $\mathcal{X} = \mathcal{C}_{loc}(I, H_w)$.

Thus, the set of weak solutions $\mathcal{U}_I$ satisfies hypothesis (H). Applying Theorem 3.2.1

for $X = H_w$ and $\mathcal{U}$ as $\mathcal{U}_I$, we then obtain the existence of a trajectory statistical solution for problem (3.91)-(3.96) with respect to a given initial measure.

Theorem 3.4.11. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0 and let $\mathcal{U}_I$ be the set of weak solutions of problem (3.91)-(3.96) on I. Then, given a Borel probability measure μ_0 on H, there exists a $\mathcal{U}_I$-trajectory statistical solution ρ on $\mathcal{C}_{loc}(I, H_w)$ satisfying the initial condition $\Pi_{t_0}\rho = \mu_0$.*

The following theorem shows the existence of a statistical solution for problem (3.91)-(3.96) with respect to a given initial data. This corresponds to a solution of Problem 3.1.2 in this case.

Theorem 3.4.12. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0 and let $\mathcal{U}_I$ be the set of weak solutions of problem (3.91)-(3.96) on I. Consider a Borel probability measure μ_0 on H satisfying*

$$\int_H |z|_H^2 \mathrm{d}\mu_0(z) < \infty. \tag{3.109}$$

Then there exists a projected statistical solution $\{\rho_t\}_{t\in I}$ of problem (3.91)-(3.96), associated with a $\mathcal{U}_I$-trajectory statistical solution, such that

(i) The initial condition $\rho_{t_0} = \mu_0$ holds;

(ii) The function

$$t \mapsto \int_H \varphi(z)\mathrm{d}\rho_t(z) \tag{3.110}$$

is continuous on I, for every bounded and weakly-continuous real-valued function φ on H, and is measurable on I, for every bounded and continuous real-valued function φ on H.

(iii) For any cylindrical test function Φ, it follows that

$$\int_H \Phi(z)\mathrm{d}\rho_t(z) = \int_H \Phi(z)\mathrm{d}\rho_{t'}(z) + \int_{t'}^{t}\int_H \langle -Az - B(z,z) - Rz, \Phi'(z)\rangle_{V',V}\mathrm{d}\rho_s(z)\mathrm{d}s, \tag{3.111}$$

for all $t, t' \in I$.

(iv) The mean strengthened energy inequality

$$\frac{d}{dt}\int_H (\psi(|z|_H^2))\mathrm{d}\rho_t(z) \leq \int_H \psi'(|z|_H^2)\left[2\frac{\kappa\gamma}{\eta\lambda_0}\frac{L_1L_2}{\varepsilon}(T_1 - T_0)^2 - \eta\|z\|_V^2\right]\mathrm{d}\rho_t(z) \tag{3.112}$$

is satisfied in the distribution sense on I, for every nonnegative, nondecreasing and continuously-differentiable real-valued function ψ with bounded derivative.

(v) At the initial time, the limit

$$\lim_{t\to t_0^+}\int_H \psi(|z|_H^2)\mathrm{d}\rho_t(z) = \int_H \psi(|z|_H^2)\mathrm{d}\mu_0(z) \tag{3.113}$$

holds for every function ψ as in (iv).

Proof. The proof follows by applying Theorem 3.3.2 and Proposition 3.3.1 for $X = H_w$, $Y = V$, $\mathcal{U} = \mathcal{U}_I$ and the functions

$$F(t,z) = -Az - B(z,z) - Rz, \quad \forall (t,z) \in I \times V,$$

$$\alpha(t,z) = \psi(|z(t)|_H^2), \quad \forall (t,z) \in I \times H_w,$$

and

$$\beta(t,z) = \psi'(|z|_H^2)\left[2\frac{\kappa\gamma}{\eta\lambda_0}\frac{L_1L_2}{\varepsilon}(T_1 - T_0)^2 - \eta\|z\|_V^2\right], \quad \forall (t,z) \in I \times V.$$

□

Chapter 4

Convergence of Statistical Solutions

A useful tool for dealing with evolution equations which do not have an established result of well-posedness is to consider regularized well-posed approximations of the equation with respect to a parameter. One then analyzes whether the solutions of the approximated problems converge to a solution of the limit problem. In this chapter, we want to address this question with respect to statistical solutions, in an abstract sense.

In order to obtain the convergence of the statistical solutions of the approximated problems, we must use some well-known results on the continuity properties of measures and integrals. More specifically, we use a result of this type for measures which states that the measure of the elements of a monotone decreasing convergent sequence of sets converge to the measure of the limit set. In regard to Lebesgue integrals, we use the classical Dominated Convergence Theorem. We point out that both results are stated with respect to sequences: a sequence of sets in the first result and a sequence of functions in the second one.

Although the first result also holds, more generally, for a net of sets (this follows by using [28, Proposition 10, Chapter 1]), the Dominated Convergence Theorem is not valid for nets in general. To see this, consider for instance the family of all finite subsets of $[0, 1]$, denoted by D and ordered by inclusion. For each $\alpha \in D$, let χ_α be the characteristic function of the set α. It is clear that $\{\chi_\alpha\}_{\alpha \in D}$ converges pointwise to $\chi_{[0,1]}$. On the other hand, $\int_0^1 \chi_\alpha(t)\mathrm{d}t = 0$, for all $\alpha \in D$, and $\int_0^1 \chi_{[0,1]}(t)\mathrm{d}t = 1$. Therefore, $\int_0^1 \chi_\alpha(t)\mathrm{d}t \nrightarrow \int_0^1 \chi_{[0,1]}(t)\mathrm{d}t$.

For that reason, in the results of this chapter we assume that the parameters associated to the approximated problems vary in a countable set.

In Section 4.1, we prove the main result on the convergence of approximated statistical solutions to a statistical solution of the limit equation. We first prove this result for trajectory statistical solutions and then obtain as an easy consequence the convergence of statistical solutions in phase space. For these results to hold, we need a new set of hypotheses, which is called *hypothesis* $(\tilde{H})$, following a similar terminology from Chapter

3. This new hypothesis ($\tilde{\mathrm{H}}$) must be satisfied by a pair formed by a subset of the trajectory space, which represents the set of individual solutions of the limit equation, and by the sequence of solution operators associated to the regularized equations.

Next, in Section 4.2 we give an application of these abstract results of convergence for two specific models: the Navier-Stokes-α (NS-α) equations and the Magnetohydrodynamics-α (MHD-α) equations. These equations consist in regularized approximations of the Navier-Stokes equations and the Magnetohydrodynamics equations, respectively. For the NS-α model, the convergence of the corresponding approximated statistical solutions to a statistical solution of the Navier-Stokes equations was already proved in [6]. Our idea here is to show this result as an application of the abstract framework we constructed.

Once this abstract framework is available, it can also be used to obtain the convergence of other regularized approximations, such as the Galerkin approximations, the Leray-α model, and so on.

4.1 Convergence in trajectory and phase spaces

In this section we consider X as a Hausdorff space and $I \subset \mathbb{R}$ an interval closed and bounded on the left with left end point t_0. Also, let $\mathcal{X} = \mathcal{C}_{\text{loc}}(I, X)$ be the space of continuous paths on I with values in X, endowed with the compact-open topology (see Section 2.1).

Given a family of sets $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}$, its *topological lim sup* is defined as

$$\limsup_n \mathcal{A}_n = \bigcap_n \overline{\bigcup_{j\geq n} \mathcal{A}_j},$$

where the overline stands for the closure in $\mathcal{X}$.

We define below the set of hypotheses which are needed in the proof of the convergence results.

Definition 4.1.1. *Let X be a Hausdorff space. Consider an interval $I \subset \mathbb{R}$ closed and bounded on the left with left end point t_0, and let $\mathcal{X} = \mathcal{C}_{loc}(I, X)$. Given a family of functions $\{S_n\}_{n\in\mathbb{N}}$, with $S_n : X \to \mathcal{X}$, and a subset $\mathcal{U} \subset \mathcal{X}$, we say that the pair $(\{S_n\}_{n\in\mathbb{N}}, \mathcal{U})$ satisfies the hypothesis ($\tilde{H}$) if the following conditions hold*

($\tilde{H}$1) For each $n \in \mathbb{N}$, $S_n : X \to \mathcal{X}$ is a continuous function;

($\tilde{H}$2) $\Pi_{t_0} S_n(u_0) \to u_0$ as $n \to \infty$, for every $u_0 \in X$;

($\tilde{H}$3) There exists a family of sets $\mathfrak{K}'(X) \subset \mathfrak{B}_X$ such that

(i) *Every $K \in \mathfrak{K}'(X)$ is compact in X;*

(ii) *Every tight Borel probability measure μ_0 on X is inner regular with respect to the family $\mathfrak{K}'(X)$ in the sense of (2.2);*

(iii) *For every $K \in \mathfrak{K}'(X)$ there exists a compact set $\mathcal{K} \subset \mathcal{X}$ such that*

$$S_n(K) \subset \mathcal{K}, \quad \forall n \in \mathbb{N};$$

(iv) *For every $K \in \mathfrak{K}'(X)$,*

$$\limsup_n S_n(K) \subset \mathcal{U}.$$

In order to gain a better insight on how the hypothesis ($\tilde{\text{H}}$) would fit into a specific problem, let us think in terms of some applications. Given a family of regularized approximations of a certain evolution equation, the space X is to be considered as the phase space associated to the equation and $\mathcal{U}$ as the set of solutions of the limit problem, in a given sense. These solutions are assumed to be continuous functions on the interval $I \subset \mathbb{R}$ with values in X, i.e. $\mathcal{U} \subset \mathcal{X}$, where $\mathcal{X} = \mathcal{C}_{\text{loc}}(I, X)$. If the approximated equations are well-posed, for each $n \in \mathbb{N}$ we usually define the operator $S_n : X \to \mathcal{X}$ as the mapping which takes an initial condition u_0 in X to the unique solution u^n of the corresponding approximated equation with parameter n such that $u^n(t_0) = u_0$. Note that we are also assuming that the solutions of the approximated problems lie in the space $\mathcal{X}$. Usually, these approximated solutions belong to a more regular space, which is contained in $\mathcal{X}$.

From the well-posedness, it follows that each S_n is well-defined and continuous, so that hypothesis ($\tilde{\text{H}}$1) is verified. Hypothesis ($\tilde{\text{H}}$2) follows immediately by the definition of S_n, since for every $u_0 \in X$ and $n \in \mathbb{N}$, $\Pi_{t_0} S_n(u_0) = u_0$. In some cases, such as in Galerkin approximations, we may need to consider a sequence of approximated initial conditions as well, but nevertheless, we still have the convergence $\Pi_{t_0} S_n(u_0) \to u_0$ as $n \to \infty$.

In case X is given by a separable Banach space endowed with its corresponding weak topology, we may define $\mathfrak{K}'(X)$ as the family of strong compact sets in X. Then item (i) of hypothesis ($\tilde{\text{H}}$3) is clearly satisfied, and item (ii) is a consequence of the fact that every Borel probability measure on a separable Banach space (or, more generally, on a Polish space) is tight [1, Theorem 12.7], i.e., inner regular with respect to the family $\mathfrak{K}'(X)$. Items (iii) and (iv) of ($\tilde{\text{H}}$3) are typically a consequence of the a priori estimates satisfied by the solutions of the regularized equations. For item (iv), a result on the convergence of the individual approximated solutions to an individual solution of the limit problem is needed. This means that for a given initial condition u_0, the sequence of solutions of the approximated equations with initial condition u_0, i.e. $\{S_n u_0\}_n$, converge in the trajectory space to a solution u of the limit equation with initial condition u_0.

The verification of this set of hypotheses within a specific problem will be more clear with the applications to the Navier-Stokes-α model and the MHD-α model that we present in Section 4.2.

Now that we have established the necessary set of hypotheses in the context of convergence of statistical solutions, let us prove the main results of this section. We start with the convergence of trajectory statistical solutions. But before presenting the proof, we shall make a brief outline with its general ideas.

Consider an initial tight Borel probability measure μ_0 on X. Given a family of continuous operators $S_n : X \to \mathcal{X}$ and a subset $\mathcal{U} \subset \mathcal{X}$ such that $(\{S_n\}_{n\in\mathbb{N}}, \mathcal{U})$ satisfies hypothesis $(\tilde{\mathrm{H}})$, we want to prove that the sequence of induced measures from μ_0 by each operator S_n, i.e. $\{S_n\mu_0\}_{n\in\mathbb{N}}$, converges, possibly modulo a subsequence, to a $\mathcal{U}$-trajectory statistical solution ρ satisfying the initial condition $\Pi_{t_0}\rho = \mu_0$. As in Theorem 3.2.1, we divide the proof into two complementary cases: first, when μ_0 is carried by a set $K \in \mathfrak{K}'(X)$; and secondly, when μ_0 is not carried by any set in this family.

In the first case, we use hypothesis (iii) of $(\tilde{\mathrm{H}}3)$ in order to show that the sequence $\{S_n\mu_0\}_{n\in\mathbb{N}}$ is uniformly tight. Also, from hypothesis $(\tilde{\mathrm{H}}1)$ it follows that each measure $S_n\mu_0$ is tight. Thus, applying Theorem 2.3.2, we obtain a subsequence of $\{S_n\mu_0\}_n$ converging weak star to a tight Borel probability measure ρ on $\mathcal{X}$. This limit measure ρ is the candidate for a $\mathcal{U}$-trajectory statistical solution satisfying the initial condition. In order to show that $\Pi_{t_0}\rho = \mu_0$, we first prove that $\Pi_{t_0}\rho(\varphi) = \mu_0(\varphi)$, for every function $\varphi \in \mathcal{C}_b(X)$. This is obtained via the Dominated Convergence Theorem, by using the convergence from hypothesis $(\tilde{\mathrm{H}}2)$. Then, using that $\Pi_{t_0}\rho$ and μ_0 are both carried by compact subsets of X (therefore completely regular spaces with respect to the induced topology), from Corollary 2.3.2 we obtain that $\Pi_{t_0}\rho = \mu_0$. Finally, we use hypotheses (iii) and (iv) of $(\tilde{\mathrm{H}}3)$ in order to show that ρ is carried by a Borel subset of $\mathcal{U}$, so that ρ is a $\mathcal{U}$-trajectory statistical solution satisfying the given initial condition.

The second case is proved analogously to the corresponding case in the proof of Theorem 3.2.1, by using items (i) and (ii) of hypothesis $(\tilde{\mathrm{H}}3)$.

Theorem 4.1.1. *Let X be a Hausdorff space and let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0. Consider a pair $(\{S_n\}_{n\in\mathbb{N}}, \mathcal{U})$ satisfying hypothesis $(\tilde{\mathrm{H}})$ and let μ_0 be a tight Borel probability measure on X. Then, the sequence $\{\rho_n\}_{n\in\mathbb{N}}$, with $\rho_n = S_n\mu_0$, for all $n \in \mathbb{N}$, has a subsequence converging in the weak star semicontinuity topology to a Borel probability measure ρ on $\mathcal{X}$ which is a $\mathcal{U}$-trajectory statistical solution satisfying the initial condition $\Pi_{t_0}\rho = \mu_0$.*

Proof. Let us suppose at first that μ_0 is carried by a set $K \in \mathfrak{K}'(X)$. From hypothesis (iii) of $(\tilde{\mathrm{H}}3)$, there exists a compact set $\mathcal{K} \subset \mathcal{X}$ such that $S_n(K) \subset \mathcal{K}$, for every $n \in \mathbb{N}$. We then have

$$\rho_n(\mathcal{X} \setminus \mathcal{K}) \leq \rho_n(\mathcal{X} \setminus S_n(K)) = \mu_0(S_n^{-1}(\mathcal{X} \setminus S_n(K)) \leq \mu_0(X \setminus K) = 0, \quad \forall n \in \mathbb{N}.$$

Thus, $\{\rho_n\}_{n\in\mathbb{N}}$ is uniformly tight. Also, since μ_0 is tight and, by hypothesis $(\tilde{\mathrm{H}}1)$, S_n

is continuous, then $\rho_n = S_n\mu_0$ is tight, for every $n \in \mathbb{N}$. Therefore, from Theorem 2.3.2, there is a subsequence of $\{\rho_n\}_n$, which we still denote by $\{\rho_n\}_n$, and a tight Borel probability measure ρ on $\mathcal{X}$ such that $\rho_n \overset{wsc}{\rightharpoonup} \rho$. This implies in particular that $\rho_n \overset{w^*}{\rightharpoonup} \rho$.

We shall now prove that this limit measure ρ is a $\mathcal{U}$-trajectory statistical solution and that $\Pi_{t_0}\rho = \mu_0$.

For the second assertion, let us first prove that

$$\int_X \varphi(u_0)\mathrm{d}\Pi_{t_0}\rho(u_0) = \int_X \varphi(u_0)\mathrm{d}\mu_0(u_0), \quad \forall \varphi \in \mathcal{C}_b(X).$$

Applying a change of variables in the integral of the left-hand side, we obtain

$$\int_X \varphi(u_0)\mathrm{d}\Pi_{t_0}\rho(u_0) = \int_{\mathcal{X}} \varphi(\Pi_{t_0}u)\mathrm{d}\rho(u). \tag{4.1}$$

Since $\rho_n \overset{w^*}{\rightharpoonup} \rho$ and $\varphi \circ \Pi_{t_0} \in \mathcal{C}_b(\mathcal{X})$, then

$$\int_{\mathcal{X}} \varphi(\Pi_{t_0}u)\mathrm{d}\rho(u) = \lim_n \int_{\mathcal{X}} \varphi(\Pi_{t_0}u)\mathrm{d}\rho_n(u). \tag{4.2}$$

Now using that $\rho_n = S_n\mu_0$ and changing the variables, we have

$$\int_{\mathcal{X}} \varphi(\Pi_{t_0}u)\mathrm{d}\rho_n(u) = \int_X \varphi(\Pi_{t_0}S_n(u_0))\mathrm{d}\mu_0(u_0). \tag{4.3}$$

Therefore, (4.1), (4.2) and (4.3) yield

$$\int_X \varphi(u_0)\mathrm{d}\Pi_{t_0}\rho(u_0) = \lim_n \int_X \varphi(\Pi_{t_0}S_n(u_0))\mathrm{d}\mu_0(u_0). \tag{4.4}$$

For each $n \in \mathbb{N}$, consider $F_n : X \to \mathbb{R}$ given by

$$F_n(u_0) = \varphi(\Pi_{t_0}S_n(u_0)), \quad \forall u_0 \in X.$$

Since $\varphi \in \mathcal{C}_b(X)$, $\Pi_{t_0} \in \mathcal{C}_{\mathrm{loc}}(\mathcal{X}, X)$ and $S_n \in \mathcal{C}_{\mathrm{loc}}(X, \mathcal{X})$, then $F_n \in \mathcal{C}_b(X)$. Also, from hypothesis ($\tilde{\mathrm{H}}2$) it follows that F_n converges pointwise to φ. Furthermore,

$$|F_n(u)| \leq \|\varphi\|_{L^\infty(X)}, \quad \forall u \in X.$$

Therefore, by the Dominated Convergence Theorem, it follows that

$$\lim_n \int_X \varphi(\Pi_{t_0}S_n(u_0))\mathrm{d}\mu_0(u_0) = \int_X \varphi(u_0)\mathrm{d}\mu_0(u_0).$$

Thus, the last identity combined with (4.4) yields

$$\int_X \varphi(u_0) \mathrm{d}\Pi_{t_0}\rho(u_0) = \int_X \varphi(u_0) \mathrm{d}\mu_0(u_0), \quad \forall \varphi \in \mathcal{C}_b(X). \tag{4.5}$$

Now since each ρ_n is carried by $\mathcal{K}$ and $\rho_n \overset{wsc}{\rightharpoonup} \rho$, then it follows from Lemma 2.3.1 that ρ is also carried by $\mathcal{K}$. Thus, $\Pi_{t_0}\rho$ is carried by $\Pi_{t_0}\mathcal{K}$. Moreover, since μ_0 is carried by K, from (4.5) it follows in particular that

$$\int_{\Pi_{t_0}\mathcal{K}\cup K} \varphi(u_0) \mathrm{d}\Pi_{t_0}\rho(u_0) = \int_{\Pi_{t_0}\mathcal{K}\cup K} \varphi(u_0) \mathrm{d}\mu_0(u_0), \quad \forall \varphi \in \mathcal{C}(\Pi_{t_0}\mathcal{K} \cup K). \tag{4.6}$$

Using that $\Pi_{t_0}\mathcal{K} \cup K$ is a compact space and hence completely regular, and that μ_0 and $\Pi_{t_0}\rho$ are tight measures, from (4.6) and Corollary 2.3.2 it follows that $\Pi_{t_0}\rho$ and μ_0 coincide on $\Pi_{t_0}\mathcal{K} \cup K$. However, since these measures are actually carried by $\Pi_{t_0}\mathcal{K} \cup K$, then $\Pi_{t_0}\rho = \mu_0$ on X.

In order to prove that ρ is in addition a $\mathcal{U}$-trajectory statistical solution, we must find a Borel set $\mathcal{V} \in \mathfrak{B}_\mathcal{X}$ such that $\mathcal{V} \subset \mathcal{U}$ and $\rho(\mathcal{V}) = 1$.

First, we claim that

$$\rho\left(\limsup_{n\in\mathbb{N}} S_n(K)\right) = 1. \tag{4.7}$$

Indeed, recall that

$$\limsup_n S_n(K) = \bigcap_k \overline{\bigcup_{j\geq k} S_j(K)}.$$

Since $\rho_n \overset{wsc}{\rightharpoonup} \rho$ in $\mathcal{P}(\mathcal{X})$ and $\overline{\bigcup_{j\geq k} S_j(K)}$ is closed, we obtain from Lemma 2.3.1 that

$$\rho\left(\overline{\bigcup_{j\geq k} S_j(K)}\right) \geq \limsup_{n\in\mathbb{N}} \rho_n\left(\overline{\bigcup_{j\geq k} S_j(K)}\right), \tag{4.8}$$

for every $k \in \mathbb{N}$.

Now, we observe that for all $n \in \mathbb{N}$ such that $k \leq n$,

$$\rho_n\left(\overline{\bigcup_{j\geq k} S_j(K)}\right) \geq \rho_n\left(S_n(K)\right),$$

and since $\rho_n(S_n(K)) = 1$, then

$$\rho_n\left(\overline{\bigcup_{j\geq k} S_j(K_i)}\right) = 1, \quad \forall n \in \mathbb{N} \text{ with } k \leq n.$$

Thus, from (4.8) we obtain that

$$\rho\left(\overline{\bigcup_{j\geq k} S_j(K)}\right) = 1.$$

As before, from hypothesis (iii) of ($\tilde{\text{H}}3$), we know that there exists a compact set $\mathcal{K} \subset \mathcal{X}$ such that $S_n(K) \subset \mathcal{K}$, for every $n \in \mathbb{N}$. Now for each $k \in \mathbb{N}$, consider the set $E_k = \overline{\bigcup_{j\geq k} S_j(K)}$. Then $E_k \subset \mathcal{K}$, for every k, and in particular each E_k is a compact set. Since $\{E_k\}_{k\in\mathbb{N}}$ is also a non-decreasing sequence, it follows that (see [1, Theorem 10.8])

$$\rho\left(\bigcap_k \overline{\bigcup_{j\geq k} S_j(K)}\right) = \lim_k \rho\left(\overline{\bigcup_{j\geq k} S_j(K)}\right) = 1,$$

which proves (4.7). Thus, ρ is carried by the set $\mathcal{A} = \limsup_n S_n(K) \subset \mathcal{X}$, which is clearly a Borel subset of $\mathcal{X}$ since K is compact in X and $S_n : X \to \mathcal{X}$ is continuous, for every $n \in \mathbb{N}$. Moreover, from hypothesis (iv) of ($\tilde{\text{H}}3$), we have that $\mathcal{A} \subset \mathcal{U}$. Therefore, ρ is a $\mathcal{U}$-trajectory statistical solution. This proves the particular case when μ_0 is carried by a set $K \in \mathfrak{K}'(X)$.

Now let us assume that μ_0 is not carried by any set $K \in \mathfrak{K}'(X)$. In this case, the proof follows by arguments similar to the ones used in the proof of Theorem 3.2.1. The idea consists in using the fact that μ_0 is a tight measure with respect to the family $\mathfrak{K}'(X)$ in order to obtain an increasing sequence of sets $\{K_n\}_n$ in $\mathfrak{K}'(X)$ such that $\mu_0(K_{n+1}) > \mu_0(K_n) > 0$, for all $n \in \mathbb{N}$, and

$$\mu_0(X \setminus K_n) < \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{4.9}$$

We then define $D_1 = K_1$ and $D_n = K_n \setminus K_{n-1}$, for every $n \geq 2$, and decompose μ_0 as

$$\mu_0 = \sum_j \mu_0(D_j)\mu_0^j.$$

Since each μ_0^j is carried by the set $K_j \in \mathfrak{K}'(X)$, using the first part of the proof, we obtain the existence of a tight Borel probability measure ρ_j carried by a Borel set $\mathcal{A}_j \subset \mathcal{U}$ and such that $\Pi_{t_0}\rho_j = \mu_0^j$. Then, we prove that the Borel probability measure ρ defined by

$$\rho = \sum_j \mu_0(D_j)\rho_j$$

is tight and is carried by the Borel set $\bigcup_l \mathcal{A}_l \subset \mathcal{U}$. Moreover, ρ satisfies the initial condition $\Pi_{t_0}\rho = \mu_0$. This proves the second case. $\square$

Remark 4.1.1. In the proof of Theorem 4.1.1, we have seen that given an initial tight

Borel probability measure μ_0 on X, in case μ_0 is carried by a set $K \in \mathfrak{K}'(X)$, the $\mathcal{U}$-trajectory statistical solution ρ satisfying $\Pi_{t_0}\rho = \mu_0$ which was obtained is carried by the Borel set $\limsup_n S_n(K)$, a subset of $\mathcal{U}$. On the other hand, when μ_0 is not carried by any set $K \in \mathfrak{K}'(X)$, then we obtain a $\mathcal{U}$-trajectory statistical solution ρ with $\Pi_{t_0}\rho = \mu_0$ being carried by $\bigcup_j \limsup_n S_n(K_j)$, where $\{K_j\}_j$ is a sequence of sets in $\mathfrak{K}'(X)$ such that $\mu_0(X \setminus K_j) \to 0$, as $j \to \infty$.

We now show the result on the convergence of statistical solutions in phase space. First, let us define in which sense these solutions converge. We say that a family of statistical solutions in phase space $\{\{\rho_t^n\}_{t\in I}\}_{n\in\mathbb{N}}$ converge to a statistical solution $\{\rho_t\}_{t\in I}$ if, for each $t \in I$, $\{\rho_t^n\}_{n\in\mathbb{N}}$ converges in the weak star topology to ρ_t as $n \to \infty$, i.e.

$$\lim_{n\to\infty} \int_X \varphi(u_0)\mathrm{d}\rho_t^n(u_0) = \int_X \varphi(u_0)\mathrm{d}\rho_t(u_0), \quad \forall t \in I \text{ and } \forall \varphi \in \mathcal{C}_b(X).$$

The proof follows easily by using the previous result, Theorem 4.1.1. The idea consists in considering the projections of the measures $S_n\mu_0$ on X at each $t \in I$, i.e. $\Pi_t S_n \mu_0$. We observe that, in the context of an application, for every $n \in \mathbb{N}$, each operator $\Pi_t S_n : X \to X$ is the same as $U_n(t, t_0) : X \to X$, where $\{U_n(t, s)\}_{t,s\in I, t\geq s}$ is the evolution process associated to the regularized equation with parameter n [8, Chapter 1].

In the statement below, the space $\mathcal{Z}$ is the one defined in (3.1), Section 3.1.

Theorem 4.1.2. *Let X be a Hausdorff space and let Y be a Banach space satisfying*

$$Y \subset X \hookrightarrow Y'_{w*},$$

where the injection $X \hookrightarrow Y'_{w*}$ is continuous, and also $\mathfrak{B}_Y \subset \mathfrak{B}_X$. Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0. Consider a pair $(\{S_n\}_{n\in\mathbb{N}}, \mathcal{U})$ satisfying hypothesis $(\tilde{H})$. Suppose that $\mathcal{U} \subset \mathcal{Z}$ and that for every $u \in \mathcal{U}$,

$$u_t(t) = F(t, u(t)), \qquad a.e.\ t \in I. \tag{4.10}$$

Assume that $F : I \times Y \to Y'$ is an $(\mathfrak{L}_I \otimes \mathfrak{B}_Y, \mathfrak{B}_{Y'})$-measurable function and that there exists an $(\mathfrak{L}_I \otimes \mathfrak{B}_X)$-measurable function $\gamma : I \times X \to \mathbb{R}$ such that

$$\int_{t_0}^t \|F(s, u(s))\|_{Y'}\mathrm{d}s \leq \gamma(t, u(t_0)), \quad \forall t \in I,\ \forall u \in \mathcal{U}. \tag{4.11}$$

Let μ_0 be a tight Borel probability measure on X such that

$$\int_X \gamma(t, u_0)\, d\mu_0(u_0) < \infty, \tag{4.12}$$

for almost every $t \in I$. Then, there exists a subsequence of $\{\{\Pi_t S_n \mu_0\}_{t\in I}\}_{n\in\mathbb{N}}$ converging *to a family of tight Borel probability measures $\{\rho_t\}_{t\in I}$ on X, which is a projected statistical solution, associated with a $\mathcal{U}$-trajectory statistical solution ρ, and satisfying $\rho_{t_0} = \mu_0$.*

Proof. From the proof of Theorem 4.1.1, there exists a $\mathcal{U}$-trajectory statistical solution ρ on $\mathcal{X}$ such that $\Pi_{t_0}\rho = \mu_0$ and, modulo a subsequence, $S_n\mu_0 \overset{w^*}{\rightharpoonup} \rho$. Then, since for every $t \in I$, $\Pi_t : \mathcal{X} \to X$ is a continuous function, it follows that $\Pi_t\rho$ is a tight measure and

$$\Pi_t S_n \mu_0 \overset{w^*}{\rightharpoonup} \Pi_t\rho, \quad \forall t \in I.$$

Now, by using the same arguments from the proof of Theorem 3.3.2, it follows by applying Theorem 3.3.1 that the family of Borel probability measures $\{\rho_t\}_{t\in I} = \{\Pi_t\rho\}_{t\in I}$ is a projected statistical solution satisfying $\rho_{t_0} = \Pi_{t_0}\rho = \mu_0$. ☐

4.2 Applications

4.2.1 Navier-Stokes-α Equations

In this section we apply the results of Section 4.1 in order to prove the convergence of statistical solutions of the Navier-Stokes-α equations given with respect to their well-defined solution operator, to a statistical solution of the Navier-Stokes equations in the sense of Definition 3.1.2.

Consider a periodic domain $\Omega = \prod_{i=1}^{3}(0, L_i)$ in $\mathbb{R}^3$, where $L_i > 0$, for $i = 1, 2, 3$, and let $I \subset \mathbb{R}$ be an interval. The three-dimensional Navier-Stokes-α equations in $\Omega \times I$ are given by

$$\mathbf{v}_t - \nu\Delta\mathbf{v} + \mathbf{u} \times (\nabla \times \mathbf{v}) + \nabla p = \mathbf{f}, \tag{4.13}$$

$$\mathbf{v} = \mathbf{u} - \alpha^2\Delta\mathbf{u}, \tag{4.14}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{4.15}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is the unknown (filtered) velocity field, $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ is an auxiliary variable, $p = p(\mathbf{x}, t)$ is the pressure, $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ is the external force, and $\alpha > 0$ is a constant.

We consider the same functional setting for the Navier-Stokes-α model introduced by Vishik, Titi and Chepyzhov in [51].

Let H and V be the same spaces defined for the Navier-Stokes equations in Section 3.4.1, i.e. H is the closure of the set of test functions $\mathcal{V}$ given in (3.28) with respect to the $L^2(\Omega, \mathbb{R}^3)$ norm and V is the closure of $\mathcal{V}$ with respect to the $H^1(\Omega, \mathbb{R}^3)$ norm. We also consider the same notation for the inner products and norms in V and H introduced in Section 3.4.1.

For each interval $I \subset \mathbb{R}$, consider the following functional space

$$\mathcal{F}_I = \{\mathbf{z} : \mathbf{z}(\cdot) \in L^2_{loc}(I;V) \cap L^\infty_{loc}(I;H), \partial_t \mathbf{z}(\cdot) \in L^2_{loc}(I;D(A)')\}. \tag{4.16}$$

The space $\mathcal{F}_I$ is the natural space for the solutions of the Navier-Stokes-α model. We endow this space with a weak-type topology, which we call $\tau_{\mathcal{F}}$ *topology*. This can be defined in terms of nets as follows: a net of functions $\{\mathbf{z}_\gamma\}_\gamma \subset \mathcal{F}_I$ converges to a function $\mathbf{z} \in \mathcal{F}_I$ with respect to the topology $\tau_{\mathcal{F}}$ if for each compact interval $J \subset I$,

$$\mathbf{z}_\gamma \overset{*}{\rightharpoonup} \mathbf{z} \text{ in } L^\infty(J;H), \quad \mathbf{z}_\gamma \rightharpoonup \mathbf{z} \text{ in } L^2(J;V), \quad \text{and} \quad \partial_t \mathbf{z}_\gamma \rightharpoonup \partial_t \mathbf{z} \text{ in } L^2(J;D(A)').$$

Consider also the following Banach space

$$\mathcal{F}_I^b = \{\mathbf{z} : \mathbf{z}(\cdot) \in L^2_b(I;V) \cap L^\infty(I;H), \partial_t \mathbf{z}(\cdot) \in L^2_b(I;D(A)')\}, \tag{4.17}$$

with norm given by

$$\|\mathbf{z}\|_{\mathcal{F}_b} = \|\mathbf{z}\|_{L^2_b(I,V)} + \|\mathbf{z}\|_{L^\infty(I,H)} + \|\partial_t \mathbf{z}\|_{L^2_b(I,D(A)')},$$

where

$$\|\mathbf{z}\|_{L^2_b(I,V)} = \sup_{\{t \in I : t+1 \in I\}} \int_t^{t+1} \|\mathbf{z}(s)\|^2 ds,$$

and

$$\|\partial_t \mathbf{z}\|_{L^2_b(I,D(A)')} = \sup_{\{t \in I : t+1 \in I\}} \int_t^{t+1} \|\partial_t \mathbf{z}(s)\|^2_{D(A)'} ds.$$

Defining $\mathbf{w} = (1+\alpha^2 A)^{1/2}\mathbf{u}$, we may rewrite equation (4.13) in the following functional form

$$\mathbf{w}_t + \nu A\mathbf{w} + (1+\alpha^2 A)^{-1/2}\tilde{B}((1+\alpha^2 A)^{-1/2}\mathbf{w}, (1+\alpha^2 A)^{1/2}\mathbf{w}) = (1+\alpha^2 A)^{-1/2}\mathbf{f}, \quad \text{in } D(A)', \tag{4.18}$$

where $\mathbf{v}$ is recovered by the relation $\mathbf{v} = (1+\alpha^2 A)^{1/2}\mathbf{w}$. We recall that A represents the Stokes operator, and $\tilde{B} : V \times V \to V'$ is given by $\tilde{B}(\mathbf{u},\mathbf{v}) = \mathbb{P}[\mathbf{u} \times (\nabla \times \mathbf{v})]$, where $\mathbb{P} : L^(\Omega)^3 \to H$ is the Leray-Helmholtz projection.

We now provide a notion of solution for the equation (4.18).

Definition 4.2.1. *Let $I \subset \mathbb{R}$ be an interval and consider $\mathbf{f} \in L^2_{loc}(I;H)$. We say that a function $\mathbf{w}$ is a Leray-Hopf weak solution of (4.18) on I if:*

(i) $\mathbf{w} \in L^\infty_{loc}(I;H) \cap L^2_{loc}(I;V)$;

(ii) $\partial_t \mathbf{w} \in L^2_{loc}(I;D(A)')$;

(iii) $\mathbf{w} \in \mathcal{C}_{loc}(I;H)$;

(iv) $\mathbf{w}$ *satisfies* (4.18) *in* $D(A)'$, *in the sense of distributions on* I;

(v) *For all* $t', t \in I$ *with* $t > t'$, $\mathbf{w}$ *satisfies the following energy equality*

$$\frac{1}{2}|\mathbf{w}(t)|^2 + \nu \int_{t'}^{t} \|\mathbf{w}(s)\|^2 ds = \frac{1}{2}|\mathbf{w}(t')|^2 + \int_{t'}^{t} ((1+\alpha^2 A)^{-1/2}\mathbf{f}(s), \mathbf{w}(s)) ds. \quad (4.19)$$

Given an interval $I \subset \mathbb{R}$ and $R > 0$, we define the following sets of solutions of (4.18) in the sense of Definition 4.2.1:

$$\mathcal{U}_I^\alpha = \{\mathbf{u} \in \mathcal{C}_{loc}(I; H) : \mathbf{u} \text{ is a weak solution of (4.18) on } I\}, \quad (4.20)$$

$$\mathcal{U}_I^\alpha(R) = \{\mathbf{u} \in \mathcal{C}_{loc}(I; B_H(R)) : \mathbf{u} \text{ is a weak solution of (4.18) on } I\}, \quad (4.21)$$

where $B_H(R)$ denotes the closed ball in H centered at the origin with radius R.

Suppose now that $\mathbf{f} \in L^\infty(I, H)$. The next proposition yields some a priori estimates satisfied by every solution $\mathbf{w} \in \mathcal{U}_I^\alpha$ in this case. These estimates were adapted from [51, Corollary 3.2].

Proposition 4.2.1. *Let* $I \subset \mathbb{R}$ *be an interval and* $\mathbf{f} \in L^2_{loc}(I; H)$. *If* $\mathbf{w} \in \mathcal{U}_I^\alpha$ *then, for all* $t', t \in I$ *with* $t > t'$, *the following inequalities hold*

$$|\mathbf{w}(t)|^2 \leq |\mathbf{w}(t')|^2 \, \mathrm{e}^{-\nu\lambda_1(t-t')} + \frac{1}{\lambda_1^2 \nu^2} \|\mathbf{f}\|^2_{L^\infty([t',t],H)} (1 - \mathrm{e}^{-\nu\lambda_1(t-t')}), \quad (4.22)$$

$$\left(\int_{t'}^{t} \|\mathbf{w}(s)\|^2 \mathrm{d}s \right)^{1/2} \leq \frac{1}{\nu^{1/2}} |\mathbf{w}(t')| + \lambda_1^{1/4} \nu M (t-t')^{1/2}, \quad (4.23)$$

$$\left(\int_{t'}^{t} \|\partial_t \mathbf{w}(s)\|^2_{D(A)'} \mathrm{d}s \right)^{1/2} \leq \frac{c}{\lambda_1^{1/4} \nu^{1/2}} |\mathbf{w}(t')|^2 + \frac{\nu^{3/2}}{\lambda_1^{3/4}} M + \nu^{5/2} \lambda_1^{1/4} M (t-t'), \quad (4.24)$$

where c *is a universal constant and* M *is a non-dimensional constant which depends only on non-dimensional combinations of the parameters* ν, λ_1 *and* $\|\mathbf{f}\|_{L^\infty([t',t],H)}$.

Based on the *a priori* estimates for the Navier-Stokes-α model, we now define an auxiliary functional space which plays an essential role in the subsequent results, due to its compactness property. Given an interval $I \subset \mathbb{R}$, let $\{J_n\}_{n \in \mathbb{N}}$ be a sequence of compact subintervals of I such that $J_n \subset J_{n+1}$, for all $n \in \mathbb{N}$, and $I = \bigcup_{n=1}^{\infty} J_n$. Then, given $R \geq 0$, we define

$$\mathcal{Y}_I(R) = \bigcap_{n=1}^{\infty} \Pi_{J_n}^{-1} \mathcal{Y}_{J_n}(R), \quad (4.25)$$

where, for any compact subinterval $J \subset I$,

$$\mathcal{Y}_J(R) = \Big\{ \mathbf{w} \in \mathcal{C}_{loc}(J; H_w) : |\mathbf{w}(t)| \leq R, \|\mathbf{w}\|_{L^2([s,t],V)} \leq \frac{1}{\nu^{1/2}} R + \lambda_1^{1/4} \nu M |t-s|^{1/2}, \text{ and}$$

$$\|\partial_t \mathbf{w}\|_{L^2([s,t],D(A)')} \leq \frac{c}{\lambda_1^{1/4}\nu^{1/2}} R^2 + \frac{\nu^{3/2}}{\lambda_1^{3/4}} M + \nu^{5/2}\lambda_1^{1/4} M|t-s|, \forall s,t \in J \Big\}, \quad (4.26)$$

where c and M are the same constants from Proposition 4.2.1. We consider $\mathcal{Y}_I(R)$ as a topological space endowed with the induced topology from $\mathcal{C}_{\text{loc}}(I, H_w)$.

Denote

$$R_0 = \frac{1}{\nu\lambda_1}\|\mathbf{f}\|_{L^\infty(I,H)},$$

and let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0. Note that, for any $R \geq R_0$, if $|\mathbf{w}(t_0)| \leq R$, then every $\mathbf{w} \in \mathcal{U}_I^\alpha$ satisfy the estimates in (4.26), thanks to the a priori estimates (4.22)-(4.24). We then have $\mathcal{U}_I^\alpha(R) \subset \mathcal{Y}_I(R)$, for every $R \geq R_0$.

The space $\mathcal{Y}_I(R)$ has also some useful properties, which we state below. The proof can be found in [6].

Lemma 4.2.1. *Let $I \subset \mathbb{R}$ be an arbitrary interval in $\mathbb{R}$ and let $R \geq 0$. Then, $\mathcal{Y}_I(R)$ is a compact and metrizable space with respect to the induced topology from $\mathcal{C}_{loc}(I, H_w)$.*

From now on, unless otherwise stated, we assume that $I \subset \mathbb{R}$ is an interval closed and bounded on the left with left end point t_0.

It was proved in [18] that the Navier-Stokes-α equations are well-posed. Then, for each $\alpha > 0$, we may define a solution operator S_α given by

$$\begin{aligned} S_\alpha : H_w &\rightarrow \mathcal{C}_{\text{loc}}(I, H_w) \\ \mathbf{w}_0 &\mapsto S_\alpha(\mathbf{w}_0) = \mathbf{w}, \end{aligned} \quad (4.27)$$

where $\mathbf{w}$ is the unique weak solution of (4.18) in the sense of Definition 4.2.1 satisfying $\mathbf{w}(t_0) = \mathbf{w}_0$.

Let $\mathcal{U}_I$ be the set of Leray-Hopf weak solutions of the Navier-Stokes equations on I, as defined in (3.34). From now on, we assume that the parameter α varies in a countable set, say $\{\alpha_n\}_{n\in\mathbb{N}}$ with $\alpha_n \to 0$ as $n \to \infty$. We shall prove that the pair $(\{S_{\alpha_n}\}_{n\in\mathbb{N}}, \mathcal{U}_I)$ satisfies the hypothesis (H) given in Definition 4.1.1. For that purpose, consider $X = H_w$, $\mathcal{X} = \mathcal{C}_{\text{loc}}(I, H_w)$ and

$$\mathfrak{K}'(H_w) = \{K \subset H_w \,|\, K \text{ is a compact set in } H\}.$$

Hypothesis ($\tilde{\text{H}}1$) follows from the well-posedness of the Navier-Stokes-α_n equations and hypothesis ($\tilde{\text{H}}2$) follows directly from the definition of S_{α_n}. In hypothesis ($\tilde{\text{H}}3$), items

(i) and (ii) follow from the definition of the set $\mathfrak{K}'(H_w)$ and from the fact that H is a Polish space. In the following propositions, we show that the remaining items, (iii) and (iv) of ($\tilde{\mathrm{H}}3$), also hold.

Proposition 4.2.2. *Let $K \in \mathfrak{K}'(H_w)$. Then, there exists a compact set $\mathcal{K} \subset \mathcal{C}_{loc}(I, H_w)$ such that $S_{\alpha_n}(K) \subset \mathcal{K}$, for all $n \in \mathbb{N}$.*

Proof. Let $R \geq R_0$ be such that $K \subset B_H(R)_w$. Thus, $S_{\alpha_n}(K) \subset S_{\alpha_n}(B_H(R)_w)$, for all $n \in \mathbb{N}$. Moreover, we have $S_{\alpha_n}(B_H(R)_w) \subset \mathcal{U}_I^{\alpha_n}(R)$. Indeed, given $\mathbf{w} \in S_{\alpha_n}(B_H(R)_w)$, it follows that $\mathbf{w} \in \mathcal{U}_I^{\alpha_n}$ and $\mathbf{w}(t_0) \in B_H(R)_w$. Then, from the estimate (4.22) with $t' = t_0$, we obtain that $|\mathbf{w}(t)| \leq R$, for all $t \in I$, so that $\mathbf{w} \in \mathcal{U}_I^{\alpha_n}(R)$. Finally, using that $\mathcal{U}_I^{\alpha_n}(R) \subset \mathcal{Y}_I(R)$ and that $\mathcal{Y}_I(R) \subset \mathcal{C}_{loc}(I, H_w)$ is compact (Lemma 4.2.1), we conclude the proof. □

In [51, Theorem 3.1], Vishik, Titi and Chepyzhov proved the convergence of solutions of the Navier-Stokes-α_n model to solutions of the Navier-Stokes equations. This result was proved in the case where $\mathbf{f} \in H$ and $I = [0, \infty)$, which was of interest to them. However, it is not difficult to see that the proof can be adapted to the case when I is an arbitrary interval and $\mathbf{f} \in L^\infty(I, H)$. We state this result below:

Theorem 4.2.1. *Let $I \subset \mathbb{R}$ be an arbitrary interval and $\mathbf{f} \in L^\infty(I, H)$. Let $\{\mathbf{w}_n\}$ be a bounded sequence in $\mathcal{F}_I^b$ such that each $\mathbf{w}_n$ is a solution of the Navier-Stokes-α_n model on I, with $\alpha_n \to 0$ as $n \to \infty$, and $\mathbf{w}_n \to \mathbf{w}$ in $\tau_{\mathcal{F}}$ as $n \to \infty$, for some $\mathbf{w} \in \mathcal{F}_I^b$. Then $\mathbf{w}$ is a Leray-Hopf weak solution of the three-dimensional Navier-Stokes equations on $\mathring{I}$.*

Using the previous result, we are able to show that hypothesis (iv) of ($\tilde{\mathrm{H}}3$) is also satisfied.

Proposition 4.2.3. *Consider a set $K \in \mathfrak{K}'(H_w)$. Then, there exists a compact set $\tilde{K}$ in H_w such that*

$$\limsup_n S_{\alpha_n}(K) \subset \Pi_{t_0}^{-1}\tilde{K} \cap \mathcal{U}_I.$$

Proof. As in the proof of Proposition 4.2.2, we may consider $R \geq R_0$ such that $S_{\alpha_n}(K) \subset \mathcal{Y}_I(R)$, for all $n \in \mathbb{N}$. Therefore, since $\mathcal{Y}_I(R)$ is a metrizable space (Lemma 4.2.1), given $\mathbf{w} \in \limsup_n S_{\alpha_n}(K)$ there exists a sequence $\{\mathbf{w}_k\}_k$ such that $\mathbf{w}_k \in S_{\alpha_k}(K)$ and $\mathbf{w}_k \to \mathbf{w}$ in $\mathcal{Y}_I(R)$. This implies in particular that $\mathbf{w}_k(t_0) \to \mathbf{w}(t_0)$ in the topology of H_w. Thus, since K is compact in H and, consequently, in H_w, and since $\{\mathbf{w}_k(t_0)\}_k \subset K$, we obtain that $\mathbf{w}(t_0) \in K$, so that $\mathbf{w} \in \Pi_{t_0}^{-1}K$. Furthermore, from the *a priori* estimates (4.22)-(4.24) it follows that $\{\mathbf{w}_k\}_k$ is bounded in $\mathcal{F}_I^b$. Then, there exists a subsequence, which we still denote by $\{\mathbf{w}_k\}_k$, such that $\mathbf{w}_k \to \mathbf{v}$ in the $\tau_{\mathcal{F}}$-topology, for some $\mathbf{v} \in \mathcal{F}_I^b$. Now from Theorem 4.2.1, $\mathbf{v}$ is a weak solution of the Navier-Stokes equations on $\mathring{I}$. Further, since $\mathbf{w}_k \to \mathbf{v}$ in the $\tau_{\mathcal{F}}$-topology then we also have that $\mathbf{w}_k \to \mathbf{v}$ in the topology of $\mathcal{Y}_I(R)$. Thus,

$\mathbf{w} = \mathbf{v}$, so that $\mathbf{w} \in \mathcal{U}_I^\sharp(R)$, where $\mathcal{U}_I^\sharp(R)$ denotes the set of Leray-Hopf weak solutions of the Navier-Stokes equations on $\mathring{I}$ with values in $B_H(R)_w$, as defined in (3.37). Note that, if $\tilde{K} = B_H(R)_w$, a compact set in H_w, then it is clear that $\mathcal{U}_I(R) \subset \Pi_{t_0}^{-1}\tilde{K} \cap \mathcal{U}_I$. Thus, taking the closure in the topology of $\mathcal{C}_{\text{loc}}(I, H_w)$, we obtain that $\mathcal{U}_I^\sharp(R) \subset \overline{\Pi_{t_0}^{-1}\tilde{K} \cap \mathcal{U}_I}$. But since $\Pi_{t_0}^{-1}\tilde{K} \cap \mathcal{U}_I$ is closed in $\mathcal{C}_{\text{loc}}(I, H_w)$ (Proposition 3.4.3), then $\overline{\Pi_{t_0}^{-1}\tilde{K} \cap \mathcal{U}_I} = \Pi_{t_0}^{-1}\tilde{K} \cap \mathcal{U}_I$. We then conclude that $\mathbf{w} \in \Pi_{t_0}^{-1}\tilde{K} \cap \mathcal{U}_I$, which proves the result. □

Using the previous results, we now prove that given an initial measure μ_0, the sequence formed by the family of induced measures from μ_0 by each solution operator S_{α_n} has a subsequence which converges to a trajectory statistical solution of the Navier-Stokes equations.

Theorem 4.2.2. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0. Let $\mathcal{U}_I$ be the set of Leray-Hopf weak solutions of the Navier-Stokes equations on I and, for each $n \in \mathbb{N}$, let S_{α_n} be the solution operator associated to the Navier-Stokes-α_n equations defined in (4.27). If μ_0 is a Borel probability measure on H, then $\{S_{\alpha_n}\mu_0\}_{n\in\mathbb{N}}$ has a subsequence converging in the weak star topology to a $\mathcal{U}_I$-trajectory statistical solution ρ on $\mathcal{C}_{loc}(I, H_w)$ such that $\Pi_{t_0}\rho = \mu_0$.*

Proof. Let us verify that the pair $(\{S_{\alpha_n}\}_{n\in\mathbb{N}}, \mathcal{U}_I)$ satisfies the hypothesis ($\tilde{\text{H}}$) of Definition 4.1.1. Since the Navier-Stokes-α_n equations are well-posed, then each $S_{\alpha_n} : X \to \mathcal{X}$ is a continuous function, so that hypothesis ($\tilde{\text{H}}$1) is satisfied. By the definition of S_{α_n}, hypothesis ($\tilde{\text{H}}$2) also clearly holds. Hypothesis (i) of ($\tilde{\text{H}}$3) is immediate from the definition of the set $\mathfrak{K}'(H_w)$. Furthermore, since H is a Polish space, then any Borel probability measure on H is tight, i.e. inner regular with respect to the family of compact subsets of H [1, Theorem 12.7]. This implies that every Borel probability measure μ_0 on H is inner regular with respect to the family $\mathfrak{K}'(H_w)$. Thus, $\mathfrak{K}'(H_w)$ satisfies hypothesis (ii) of ($\tilde{\text{H}}$3). From Propositions 4.2.2 and 4.2.3, we also have that hypothesis (iii) and (iv) of ($\tilde{\text{H}}$3) are satisfied. Therefore, the pair $(\{S_{\alpha_n}\}_{n\in\mathbb{N}}, \mathcal{U}_I)$ satisfies hypothesis (H). Thus, by Theorem 4.1.1, there exists a subsequence of $\{S_{\alpha_n}\mu_0\}_{n\in\mathbb{N}}$ converging weak star to a $\mathcal{U}_I$-trajectory statistical solution ρ on $\mathcal{C}_{\text{loc}}(I, H_w)$ with $\Pi_{t_0}\rho = \mu_0$. □

Finally, in the next result we show that the statistical solutions of the Navier-Stokes-α_n equations, defined via the solution operator S_{α_n}, converge, modulo a subsequence, to a statistical solution of the Navier-Stokes equations in the sense of Definition 3.1.2.

Theorem 4.2.3. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0 and let $\mathcal{U}_I$ be the set of Leray-Hopf weak solutions of the Navier-Stokes equations on I. For each $n \in \mathbb{N}$, let S_{α_n} be the solution operator associated to the Navier-Stokes-α_n equations, defined in (4.27). Consider a Borel probability measure μ_0 on H satisfying*

$$\int_H |\mathbf{u}|_H^2 \mathrm{d}\mu_0(\mathbf{u}) < \infty. \tag{4.28}$$

Then, there exists a subsequence of $\{\{\Pi_t S_{\alpha_n}\mu_0\}_{t\in I}\}_{n\in\mathbb{N}}$ *converging to a statistical solution* $\{\rho_t\}_{t\in I}$ *of the Navier-Stokes equations such that* $\rho_{t_0} = \mu_0$.

Proof. From the proof of Theorem 4.2.2, we know that the pair $(\{S_{\alpha_n}\}_{n\in\mathbb{N}}, \mathcal{U}_I)$ satisfies the hypothesis ($\tilde{\mathrm{H}}$). Furthermore, using (4.28) and the same arguments from the proof of Proposition 3.4.3, it follows that the function $\mathbf{F}$ given in (3.53) satisfies the hypotheses of Theorem 4.1.2. The statement then follows by applying this theorem. $\square$

4.2.2 Magnetohydrodynamics-α Equations

In this section we analyze a regularized approximation of the magnetohydrodynamic equations (MHD equations), namely, the MHD-α model. By applying the abstract results of Section 4.1 in this case, we are able to show the convergence of statistical solutions of the MHD-α model to a statistical solution of the MHD equations.

As in the previous section, consider a periodic domain $\Omega = \Pi_{i=1}^{3}(0, L_i)$, where $L_i > 0$, for $i = 1, 2, 3$, and let $I \subset \mathbb{R}$ be an interval. The three-dimensional magnetohydrodynamic equations for a homogeneous incompressible resistive viscous fluid in the region Ω are given by

$$\frac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - (\mathbf{B}\cdot\nabla)\mathbf{B} + \nabla p + \frac{1}{2}\nabla|\mathbf{B}|^2 = 0, \tag{4.29}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \eta\Delta\mathbf{B} + (\mathbf{v}\cdot\nabla)\mathbf{B} - (\mathbf{B}\cdot\nabla)\mathbf{v} = 0, \tag{4.30}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{4.31}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{4.32}$$

where $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ is the velocity field, $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$ is the magnetic field and $p = p(\mathbf{x}, t)$ is the pressure, with $(\mathbf{x}, t) \in \Omega \times I$. Moreover, the constants $\nu > 0$ and $\eta > 0$ are the kinematic viscosity and the magnetic diffusivity, respectively.

The MHD equations result from a coupling of the Maxwell's equations for the magnetic field $\mathbf{B}$ with the Navier-Stokes equations for the velocity field $\mathbf{v}$ and pressure p. For further details concerned with the MHD equations, we refer to [14] and [43].

Let H and V be the same spaces defined in Section 3.4.1. Consider $\mathbb{H} = H \times H$ and $\mathbb{V} = V \times V$. Also, denote by $\mathbb{H}_w$ the space $\mathbb{H}$ endowed with the weak topology.

The inner product and norm in the product space $\mathbb{H}$ are defined by

$$(\Phi, \tilde{\Phi})_{\mathbb{H}} = (\mathbf{v}, \tilde{\mathbf{v}}) + (\mathbf{B}, \tilde{\mathbf{B}}), \quad \forall \Phi = (\mathbf{v}, \mathbf{B}),\ \tilde{\Phi} = (\tilde{\mathbf{v}}, \tilde{\mathbf{B}}) \in \mathbb{H},$$

and

$$|\Phi|_{\mathbb{H}} = (\Phi, \Phi)_{\mathbb{H}}^{1/2}, \quad \forall \Phi \in \mathbb{H}.$$

In the space $\mathbb{V}$, these are defined as

$$(\!(\Phi, \tilde{\Phi})\!)_{\mathbb{V}} = (\!(\mathbf{v}, \tilde{\mathbf{v}})\!) + (\!(\mathbf{B}, \tilde{\mathbf{B}})\!), \quad \forall \Phi = (\mathbf{v}, \mathbf{B}),\ \tilde{\Phi} = (\tilde{\mathbf{v}}, \tilde{\mathbf{B}}) \in \mathbb{V},$$

and

$$\|\Phi\|_{\mathbb{V}} = (\!(\Phi, \Phi)\!)_{\mathbb{V}}^{1/2}, \quad \forall \Phi \in \mathbb{V}.$$

Let $\mathbb{H}'$ and $\mathbb{V}'$ be the dual spaces of $\mathbb{H}$ and $\mathbb{V}$, respectively. We identify $\mathbb{H}$ with its dual space $\mathbb{H}'$, so that $\mathbb{V} \subset \mathbb{H} \equiv \mathbb{H}' \subset \mathbb{V}'$, with continuous and dense injections. We then rewrite equations (4.29)-(4.32) in the following functional form

$$\mathbf{v}_t + \nu A\mathbf{v} + B(\mathbf{v},\mathbf{v}) - B(\mathbf{B},\mathbf{B}) = 0 \quad \text{in } V', \tag{4.33}$$

$$\mathbf{B}_t + \eta A\mathbf{B} + B(\mathbf{v},\mathbf{B}) - B(\mathbf{B},\mathbf{v}) = 0 \quad \text{in } V', \tag{4.34}$$

where A is the Stokes operator and $B(\mathbf{u},\mathbf{w}) = \mathbb{P}[(\mathbf{u}\cdot\nabla)\mathbf{w}]$, for every $\mathbf{u},\mathbf{w} \in V$, with $\mathbb{P}$ being the Leray-Helmholtz projection.

We can also write equations (4.33)-(4.34) in the compact form

$$\Phi_t + \mathbb{A}\Phi + \mathbb{B}(\Phi,\Phi) = 0, \tag{4.35}$$

where

$$\Phi = (\mathbf{v},\mathbf{B}),$$

$$\mathbb{A}\Phi = (\nu A\mathbf{v}, \eta A\mathbf{B}),$$

and

$$\mathbb{B}(\Phi,\tilde{\Phi}) = (B(\mathbf{v},\tilde{\mathbf{v}}) - B(\mathbf{B},\tilde{\mathbf{B}}), B(\mathbf{v},\tilde{\mathbf{B}}) - B(\mathbf{B},\tilde{\mathbf{v}})), \quad \forall \Phi = (\mathbf{v},\mathbf{B}),\ \tilde{\Phi} = (\tilde{\mathbf{v}},\tilde{\mathbf{B}}).$$

The following definition provides a notion of solution for the MHD equations (4.29)-(4.32).

Definition 4.2.2. *Given an interval $I \subset \mathbb{R}$, we say that $\Phi = (\mathbf{v},\mathbf{B})$ is a Leray-Hopf weak solution of the MHD equations (4.29)-(4.32) on I if*

(i) $\Phi \in L^\infty_{loc}(I,\mathbb{H}) \cap L^2_{loc}(I,\mathbb{V}) \cap \mathcal{C}_{loc}(I,\mathbb{H}_w)$;

(ii) $\partial_t\Phi \in L^{4/3}_{loc}(I,\mathbb{V}')$;

(iii) $\Phi = (\mathbf{v},\mathbf{B})$ *satisfies*

$$\mathbf{v}_t + \nu A\mathbf{v} + B(\mathbf{v},\mathbf{v}) - B(\mathbf{B},\mathbf{B}) = 0 \quad \textit{in } V' \tag{4.36}$$

and

$$\mathbf{B}_t + \eta A\mathbf{B} + B(\mathbf{v},\mathbf{B}) - B(\mathbf{B},\mathbf{v}) = 0 \quad \textit{in } V', \tag{4.37}$$

in the sense of distributions on I;

(iv) For almost every $t' \in I$ and for every $t > t'$, $\Phi = (\mathbf{v},\mathbf{B})$ satisfies the following energy inequality

$$|\mathbf{v}(t)|^2 + |\mathbf{B}(t)|^2 + 2\int_{t'}^{t}(\nu\|\mathbf{v}(s)\|^2 + \eta\|\mathbf{B}(s)\|^2)\mathrm{d}s \le |\mathbf{v}(t')|^2 + |\mathbf{B}(t')|^2; \tag{4.38}$$

(v) *If I is closed and bounded on the left, with left end point t_0, then Φ is strongly continuous in $\mathbb{H}$ at t_0 from the right, i.e. $\Phi(t) \to \Phi(t_0)$ in $\mathbb{H}$ as $t \to t_0^+$.*

A proof of the existence of a Leray-Hopf weak solution of the MHD equations satisfying a given initial condition $\Phi_0 = (\mathbf{v}_0, \mathbf{B}_0) \in \mathbb{H}$ is classical and can be found in [14].

We also consider the following sets of weak solutions of the MHD equations, in the sense of Definition 4.2.2, for an interval $I \subset \mathbb{R}$ and a real number $R > 0$:

$$\mathcal{U}_I = \{\Phi \in \mathcal{C}_{\rm loc}(I, \mathbb{H}_w) : \Phi \text{ is a Leray-Hopf weak solution of (4.29)-(4.32) on } I\}, \quad (4.39)$$

$$\mathcal{U}_I(R) = \{\Phi \in \mathcal{C}_{\rm loc}(I, B_{\mathbb{H}}(R)_w) : \Phi \text{ is a Leray-Hopf weak solution of (4.29)-(4.32) on } I\}. \quad (4.40)$$

$$\mathcal{U}_I^\sharp = \{\Phi \in \mathcal{C}_{\rm loc}(I, \mathbb{H}_w) : \Phi \text{ is a Leray-Hopf weak solution of (4.29)-(4.32) on } \mathring{I}\}, \quad (4.41)$$

$$\mathcal{U}_I^\sharp(R) = \{\Phi \in \mathcal{C}_{\rm loc}(I, B_{\mathbb{H}}(R)_w) : \Phi \text{ is a Leray-Hopf weak solution of (4.29)-(4.32) on } \mathring{I}\}, \quad (4.42)$$

where $\mathring{I}$ denotes the interior of the interval I and $B_{\mathbb{H}}(R)_w$ is the closed ball in $\mathbb{H}$ centered at the origin with radius R and endowed with the weak topology.

Given a parameter $\alpha > 0$, the MHD-α model in the periodic domain Ω is defined as

$$\frac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{u}\cdot\nabla)\mathbf{v} + \sum_{j=1}^{3} \mathbf{v}_j \nabla \mathbf{u}_j - (\mathbf{B}\cdot\nabla)\mathbf{B} + \frac{1}{2}\nabla|\mathbf{B}|^2 + \nabla p = 0, \quad (4.43)$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{B} - (\mathbf{B}\cdot\nabla)\mathbf{u} - \eta\Delta\mathbf{B} = 0, \quad (4.44)$$

$$\mathbf{v} = (1 - \alpha^2\Delta)\mathbf{u}, \quad (4.45)$$

$$\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{B} = \nabla\cdot\mathbf{v} = 0, \quad (4.46)$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is the "filtered" velocity field, $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$ is the magnetic field and $p = p(\mathbf{x}, t)$ is the "filtered" pressure.

We now define a notion of solution for the MHD-α model:

Definition 4.2.3. *Given an interval $I \subset \mathbb{R}$, we say that $(\mathbf{u}, \mathbf{B})$ is a Leray-Hopf weak solution of the MHD-α equations (4.43)-(4.46) on I if*

(i) $\mathbf{u} \in \mathcal{C}_{loc}(I, V) \cap L^2_{loc}(I, D(A))$ *and* $\mathbf{B} \in \mathcal{C}_{loc}(I, H) \cap L^2_{loc}(I, V)$;

(ii) $\partial_t \mathbf{u} \in L^2_{loc}(I, H)$ *and* $\partial_t \mathbf{B} \in L^2_{loc}(I, V')$;

(iii) $(\mathbf{u}, \mathbf{B})$ *satisfies*

$$\partial_t((1+\alpha^2 A)\mathbf{u}) + \nu A((1+\alpha^2 A)\mathbf{u}) + \tilde{B}(\mathbf{u}, (1+\alpha^2 A)\mathbf{u}) - B(\mathbf{B}, \mathbf{B}) = 0 \quad \text{in } D(A)', \quad (4.47)$$

and

$$\partial_t \mathbf{B} + \eta A\mathbf{B} + B(\mathbf{u},\mathbf{B}) - B(\mathbf{B},\mathbf{u}) = 0 \quad \text{in } V', \tag{4.48}$$

in the sense of distributions on I;

(iv) For every $t, t' \in I$, $(\mathbf{u},\mathbf{B})$ satisfies the following energy equality

$$|\mathbf{u}(t)|^2 + \alpha^2\|\mathbf{u}(t)\|^2 + |\mathbf{B}(t)|^2 + 2\int_{t'}^{t} (\nu(\|\mathbf{u}(s)\|^2 + \alpha^2|A\mathbf{u}(s)|^2) + \nu\|\mathbf{B}(s)\|^2)\mathrm{d}s = |\mathbf{u}(t')|^2 + \alpha^2\|\mathbf{u}(t')\|^2 + |\mathbf{B}(t')|^2, \tag{4.49}$$

In [35], it was proved that the MHD-α equations are globally well-posed in $V \times H$. More precisely, given $\mathbf{u}_0 \in V$ and $\mathbf{B}_0 \in H$, there exists a unique solution $(\mathbf{u},\mathbf{B})$ of the MHD-α equations in the sense of Definition 4.2.3 satisfying $(\mathbf{u}(t_0),\mathbf{B}(t_0)) = (\mathbf{u}_0,\mathbf{B}_0)$ and which depends continuously on the initial data.

Given an interval $I \subset \mathbb{R}$ and $R > 0$, we consider the following sets of weak solutions of the MHD-α equations in the sense of Definition 4.2.3:

$$\mathcal{U}_I^\alpha = \{(\mathbf{u},\mathbf{B}) \in \mathcal{C}_{\mathrm{loc}}(I,\mathbb{H}_w) : (\mathbf{u},\mathbf{B}) \text{ is a weak solution of (4.43)-(4.46) on } I\}, \tag{4.50}$$

$$\mathcal{U}_I^\alpha(R) = \{(\mathbf{u},\mathbf{B}) \in \mathcal{C}_{\mathrm{loc}}(I,B_{\mathbb{H}}(R)_w) : (\mathbf{u},\mathbf{B}) \text{ is a weak solution of (4.43)-(4.46) on } I\}. \tag{4.51}$$

One may also consider the corresponding sets of weak solutions of the MHD-α equations on the interior of the interval I, $\mathring{I}$. However, these are not needed in the subsequent results.

In the next proposition we present some *a priori* estimates which are satisfied by every weak solution $(\mathbf{u},\mathbf{B}) \in \mathcal{U}_I^\alpha$. For more details on how to obtain these estimates, see [35].

Proposition 4.2.4. *Let $I \subset \mathbb{R}$ be an interval. Then, given $(\mathbf{u},\mathbf{B}) \in \mathcal{U}_I^\alpha$, for all $t', t \in I$ with $t > t'$, the following inequalities hold*

$$|\mathbf{u}(t)|^2 + \alpha^2\|\mathbf{u}(t)\|^2 + |\mathbf{B}(t)|^2 \leq |\mathbf{u}(t')|^2 + \alpha^2\|\mathbf{u}(t')\|^2 + |\mathbf{B}(t')|^2 \tag{4.52}$$

$$\left(\int_{t'}^{t} (\|\mathbf{u}(s)\|^2 + \alpha^2|A\mathbf{u}(s)|^2 + \|\mathbf{B}(s)\|^2)\mathrm{d}s\right)^{1/2} \leq \frac{1}{(2\kappa)^{1/2}}(|\mathbf{u}(t')|^2 + \alpha^2\|\mathbf{u}(t')\|^2 + |\mathbf{B}(t')|^2), \tag{4.53}$$

$$\left(\int_{t'}^{t} (\|\mathbf{u}(s)\|^2_{D(A)'} + \|\mathbf{B}(s)\|^2_{D(A)'})\mathrm{d}s\right)^{1/2} \leq \frac{C}{(\lambda_1\nu\eta)^{1/4}}(|\mathbf{u}(t')|^2 + \alpha^2\|\mathbf{u}(t')\|^2 + |\mathbf{B}(t')|^2) + (\nu\eta)^{5/4}\lambda_1^{1/4}C|t-t'|, \tag{4.54}$$

where $\kappa = \min\{1,\nu,\eta\}$, λ_1 is the first eigenvalue of the Stokes operator and C is a nondi-

mensional constant which depends on the parameters ν, η and λ_1 through nondimensional combinations of them.

Now, given an interval $I \subset \mathbb{R}$, consider a sequence $\{J_n\}_n$ of compact subintervals of I such that $J_n \subset J_{n+1}$, for all $n \in \mathbb{N}$, and $I = \bigcup_{n=1}^{\infty} J_n$. Then, for $R \geq 0$, we define

$$\mathcal{Y}_I(R) = \bigcap_{n=1}^{\infty} \Pi_{J_n}^{-1} \mathcal{Y}_{J_n}(R),$$

where, for any compact subinterval $J \subset I$,

$$\begin{aligned}\mathcal{Y}_J(R) = \Big\{ \Phi = (\mathbf{w}, \mathbf{B}) \in \mathcal{C}(J, \mathbb{H}_w) : |\Phi(t)|_{\mathbb{H}} \leq R, \ \forall t \in J; \quad \|\Phi\|_{L^2(J,\mathbb{V})} \leq (2\kappa)^{-1/2} R; \\ \text{and there exists } \alpha \geq 0 \text{ such that } \|(\partial_t (1+\alpha^2 A)^{-1/2}\mathbf{w}, \partial_t \mathbf{B})\|_{L^2(J, D(A)' \times D(A)')} \leq \\ \frac{C}{(\lambda_1 \nu \eta)^{1/4}} R^2 + (\nu\eta)^{5/4} \lambda_1^{1/4} C |t-s|, \forall s,t \in J \Big\}, \quad (4.55)\end{aligned}$$

where κ and C are the same constants from Proposition 4.2.4.

As in the case of the NS-α model (Lemma 4.2.1), one obtains analogously that $\mathcal{Y}_I(R)$ is a compact and metrizable space with respect to the induced topology from $\mathcal{C}_{\text{loc}}(I, \mathbb{H}_w)$.

Notice that the space $\mathcal{Y}_I(R)$ is directly connected with the *a priori* estimates for the MHD-α model. Indeed, if $I \subset \mathbb{R}$ is an interval closed and bounded on the left with left end point t_0, then given $R \geq 0$ and $(\mathbf{u}, \mathbf{B}) \in \mathcal{U}_I^{\alpha}$ with $|((1+\alpha^2 A)^{1/2}\mathbf{u}(t_0), \mathbf{B}(t_0))|_{\mathbb{H}} \leq R$, from the estimates (4.52)-(4.54) it follows that $\Phi = ((1+\alpha^2 A)^{1/2}\mathbf{u}, \mathbf{B}) \in \mathcal{Y}_I(R)$.

From now on, consider $I \subset \mathbb{R}$ as an interval closed and bounded on the left with left end point t_0. We shall apply the abstract framework of Section 4.1 by considering $X = \mathbb{H}_w$, $\mathcal{X} = \mathcal{C}_{\text{loc}}(I, \mathbb{H}_w)$ and

$$\mathfrak{K}'(\mathbb{H}_w) = \{ K \subset \mathbb{H}_w \,|\, K \text{ is a compact set in } \mathbb{H} \}.$$

Since, for each $\alpha > 0$, the MHD-α equations are well-posed, we may consider a solution operator S_α given by

$$\begin{aligned} S_\alpha : \mathbb{H}_w &\rightarrow \mathcal{C}_{\text{loc}}(I, \mathbb{H}_w) \\ \Phi_0 &\mapsto S_\alpha(\Phi_0) = \Phi, \end{aligned} \quad (4.56)$$

with $\Phi_0 = (\mathbf{w}_0, \mathbf{B}_0)$ and $\Phi = (\mathbf{w}, \mathbf{B})$, where $\mathbf{w} = (1+\alpha^2 A)^{1/2}\mathbf{u}$ and $(\mathbf{u}, \mathbf{B})$ is the unique weak solution of (4.29)-(4.32) with initial data $((1+\alpha^2 A)^{-1/2}\mathbf{w}_0, \mathbf{B}_0)$ in the sense of Definition 4.2.3.

We assume from now on that the parameter α varies in a countable set, say $\{\alpha_n\}_{n \in \mathbb{N}}$, with $\alpha_n \to 0$ as $n \to \infty$. Then, we show that the family of operators $\{S_{\alpha_n}\}_n$ satisfies hypothesis $(\tilde{\mathrm{H}})$.

Hypothesis ($\tilde{\mathrm{H}}1$), ($\tilde{\mathrm{H}}2$) and items (i) and (ii) of ($\tilde{\mathrm{H}}3$) follow immediately from the definitions. In the following proposition we show that $\{S_{\alpha_n}\}_n$ also satisfies item (iii) of ($\tilde{\mathrm{H}}3$).

Proposition 4.2.5. *Let $K \in \mathfrak{K}'(\mathbb{H}_w)$. Then, there exists a compact set $\mathcal{K} \subset \mathcal{C}_{loc}(I, \mathbb{H}_w)$ such that $S_{\alpha_n}(K) \subset \mathcal{K}$, for all $n \in \mathbb{N}$.*

Proof. Since K is a compact set in $\mathbb{H}$, there exists $R > 0$ such that $K \subset B_{\mathbb{H}}(R)$, where $B_{\mathbb{H}}(R)$ denotes the closed ball in $\mathbb{H}$ of radius R and centered at the origin. Thus, $S_{\alpha_n}(K) \subset S_{\alpha_n}(B_{\mathbb{H}}(R))$. On the other hand, from the *a priori* estimates (4.52)-(4.54) and the definition of S_{α_n}, it follows that $S_{\alpha_n}(B_{\mathbb{H}}(R)) \subset \mathcal{Y}_I(R)$, for every $n \in \mathbb{N}$. Therefore, $S_{\alpha_n}(K) \subset \mathcal{Y}_I(R)$, for every $n \in \mathbb{N}$, and since $\mathcal{Y}_I(R)$ is a compact set in $\mathcal{C}_{\mathrm{loc}}(I, \mathbb{H}_w)$, we conclude the proof. ◻

In order to show the remaining hypothesis, item (iv) of ($\tilde{\mathrm{H}}3$), we need the following result on the convergence of individual solutions of the MHD-α equations to an individual solution of the MHD equations. The statement below has been adapted from [35, Theorem 4.1], in which a sequence of solutions of the MHD-α equations was considered with all having the same initial data. Here we consider a sequence of solutions of these equations with the corresponding initial data also varying in a sequence, which is assumed to be uniformly bounded in a given sense. With this condition, the uniform boundedness obtained from the *a priori* estimates still hold, and the proof follows along the same lines as in [35, Theorem 4.1].

Theorem 4.2.4. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0. Consider $\{\mathbf{u}_0^n\}_n \subset V$ and $\{\mathbf{B}_0^n\}_n \subset H$. Suppose that there exists $R > 0$ such that*

$$|\mathbf{u}_0^n|^2 + {\alpha_n}^2 \|\mathbf{u}_0^n\|^2 + |\mathbf{B}_0^n|^2 \leq R, \quad \forall n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, let $(\mathbf{u}_{\alpha_n}, \mathbf{B}_{\alpha_n})$ be a weak solution of the MHD-α_n equations on I satisfying $(\mathbf{u}_{\alpha_n}(t_0), \mathbf{B}_{\alpha_n}(t_0)) = (\mathbf{u}_0^n, \mathbf{B}_0^n)$ and let $\mathbf{v}_{\alpha_n} = (1 + \alpha_n^2 A)\mathbf{u}_{\alpha_n}$. Then, there are subsequences $\{\mathbf{u}_{\alpha_j}\}_j$, $\{\mathbf{v}_{\alpha_j}\}_j$ and $\{\mathbf{B}_{\alpha_j}\}_j$ and a weak solution $(\mathbf{v}, \mathbf{B})$ of the MHD equations (4.29)-(4.32) on $\mathring{I}$ such that

(i) For every compact subinterval $J \subset I$, $\mathbf{u}_{\alpha_j} \to \mathbf{v}$ and $\mathbf{B}_{\alpha_j} \to \mathbf{B}$ weakly in $L^2(J, V)$ and strongly in $L^2(J, H)$;

(ii) For every compact subinterval $J \subset I$, $\mathbf{v}_{\alpha_j} \to \mathbf{v}$ weakly in $L^2(J, H)$ and strongly in $L^2(J, V')$;

(iii) $\mathbf{u}_{\alpha_j}(t) \to \mathbf{v}(t)$ and $\mathbf{B}_{\alpha_j}(t) \to \mathbf{B}(t)$ in $\mathcal{C}_{loc}(I, H_w)$.

Lemma 4.2.2. *Let* $\mathbf{f} \in \mathcal{C}_{loc}(I, H_w)$ *and consider a sequence* $\{\mathbf{f}_n\}_n \subset \mathcal{C}_{loc}(I, V)$ *such that* $\mathbf{f}_n \to \mathbf{f}$ *in* $\mathcal{C}_{loc}(I, H_w)$ *as* $n \to \infty$ *and suppose that there exists* $M > 0$ *such that*

$$(|\mathbf{f}_n(t)|^2 + \alpha_n^2\|\mathbf{f}_n(t)\|^2)^{1/2} \le M, \quad \forall t \in I, \forall n \in \mathbb{N}. \tag{4.57}$$

Then,

$$(1 + \alpha_n^2 A)^{1/2}\mathbf{f}_n \to \mathbf{f} \quad \text{in } \mathcal{C}_{loc}(I, H_w) \text{ as } n \to \infty. \tag{4.58}$$

Proof. Consider $\mathbf{w} \in H$ and $J \subset I$ a compact subinterval. It suffices to prove that

$$\sup_{t\in J} |((1 + \alpha_n^2 A)^{1/2}\mathbf{f}_n - \mathbf{f}, \mathbf{w})| \to 0 \text{ as } n \to \infty. \tag{4.59}$$

Consider $\varepsilon > 0$. Since $\mathbf{f} \in \mathcal{C}_{\text{loc}}(I, H_w)$ and $\{\mathbf{f}_n\}_n$ satisfies (4.57), there exists $K > 0$ such that

$$(|\mathbf{f}_n(t)|^2 + \alpha_n^2\|\mathbf{f}_n(t)\|^2)^{1/2} + |\mathbf{f}(t)| \le K, \quad \forall t \in J, \forall n \in \mathbb{N}. \tag{4.60}$$

Further, since V is dense in H, there exists $\mathbf{v} \in V$ such that

$$|\mathbf{w} - \mathbf{v}| < \frac{\varepsilon}{2K}. \tag{4.61}$$

Using that $(1 + \alpha_n^2 A)^{1/2}$ is a self-adjoint operator, we may write

$$((1 + \alpha_n^2 A)^{1/2}\mathbf{f}_n - \mathbf{f}, \mathbf{v}) = (\mathbf{f}_n, (1 + \alpha_n^2 A)^{1/2}\mathbf{v} - \mathbf{v}) + (\mathbf{f}_n - \mathbf{f}, \mathbf{v}). \tag{4.62}$$

Now since $\alpha_n \to 0$, it follows that $(1 + \alpha_n^2 A)^{1/2}\mathbf{v} \to \mathbf{v}$ in H. Also, by hypothesis, $\mathbf{f}_n \to \mathbf{f}$ in $\mathcal{C}_{\text{loc}}(I, H_w)$. Therefore, there exists $N \in \mathbb{N}$ such that

$$\sup_{t\in J} |(\mathbf{f}_n(t) - \mathbf{f}(t), \mathbf{v})| < \frac{\varepsilon}{4}, \quad \forall n \ge N, \tag{4.63}$$

and

$$|(1 + \alpha_n^2 A)^{1/2}\mathbf{v} - \mathbf{v}| < \frac{\varepsilon}{4K}, \quad \forall n \ge N. \tag{4.64}$$

Then, from (4.60) and (4.62)-(4.64), we obtain that

$$\begin{aligned}
&\sup_{t\in J} |((1 + \alpha_n^2 A)^{1/2}\mathbf{f}_n(t) - \mathbf{f}(t), \mathbf{v})| \le \\
&\qquad \le \sup_{t\in J} |(\mathbf{f}_n(t), (1 + \alpha_n^2 A)^{1/2}\mathbf{v} - \mathbf{v})| + \sup_{t\in J} |(\mathbf{f}_n(t) - \mathbf{f}(t), \mathbf{v})| \\
&\qquad \le |(1 + \alpha_n A)^{1/2}\mathbf{v} - \mathbf{v})| \sup_{t\in J} |\mathbf{f}_n(t)| + \sup_{t\in J} |(\mathbf{f}_n(t) - \mathbf{f}(t), \mathbf{v})| < \frac{\varepsilon}{2},
\end{aligned} \tag{4.65}$$

for every $n \geq N$. Finally, from (4.60), (4.61) and (4.65), it follows that

$$
\begin{aligned}
\sup_{t\in J} |((1+\alpha_n^2 A)^{1/2}\mathbf{f}_n(t) - \mathbf{f}(t), \mathbf{w})| \leq \\
\leq \sup_{t\in J} |((1+\alpha_n^2 A)^{1/2}\mathbf{f}_n(t) - \mathbf{f}(t), \mathbf{w}-\mathbf{v})| + \sup_{t\in J} |((1+\alpha_n^2 A)^{1/2}\mathbf{f}_n(t) - \mathbf{f}(t), \mathbf{v})| \\
< |\mathbf{w}-\mathbf{v}| \sup_{t\in J} |(1+\alpha_n^2 A)^{1/2}\mathbf{f}_n(t) - \mathbf{f}(t)| + \frac{\varepsilon}{2} \\
< \frac{\varepsilon}{2K} \sup_{t\in J} [(|\mathbf{f}_n(t)|^2 + \alpha_n^2 \|\mathbf{f}_n(t)\|^2)^{1/2} + |\mathbf{f}(t)|] + \frac{\varepsilon}{2} < \varepsilon, \quad (4.66)
\end{aligned}
$$

for every $n \geq N$, proving (4.59). □

From Theorem 4.2.4 and Lemma 4.2.2, we obtain immediately the following result.

Theorem 4.2.5. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0. Consider $\{\mathbf{w}_0^n\}_n \subset H$ and $\{\mathbf{B}_0^n\}_n \subset H$. Suppose that there exists $R > 0$ such that*

$$
|\mathbf{w}_0^n|^2 + |\mathbf{B}_0^n|^2 \leq R, \quad \forall n \in \mathbb{N}. \tag{4.67}
$$

For each $n \in \mathbb{N}$, consider $\mathbf{u}_0^n = (1+\alpha_n^2 A)^{-1/2}\mathbf{w}_0^n$ and let $(\mathbf{u}_{\alpha_n}, \mathbf{B}_{\alpha_n})$ be a weak solution of the MHD-α_n equations (4.43)-(4.46) on I with initial data $(\mathbf{u}_0^n, \mathbf{B}_0^n)$. Define $\mathbf{w}_{\alpha_n} = (1+\alpha_n^2 A)^{1/2}\mathbf{u}_{\alpha_n}$. Then, there exist a subsequence $(\mathbf{w}_{\alpha_j}, \mathbf{B}_{\alpha_j})_j$ and a weak solution $(\mathbf{v}, \mathbf{B})$ of the MHD equations (4.29)-(4.32) on I such that $\mathbf{w}_{\alpha_j} \to \mathbf{v}$ and $\mathbf{B}_{\alpha_j} \to \mathbf{B}$ in $\mathcal{C}_{loc}(I, H_w)$.

Proof. Since each $\mathbf{w}_0^n$ belongs to H, then, for every $n \in \mathbb{N}$, $\mathbf{u}_0^n = (1+\alpha_n^2 A)^{-1/2}\mathbf{w}_0^n$ is in V. Moreover, from (4.67) it follows that

$$
|\mathbf{u}_0^n|^2 + \alpha_n^2 \|\mathbf{u}_0^n\|^2 + |\mathbf{B}_0^n|^2 = |\mathbf{w}_0^n|^2 + |\mathbf{B}_0^n|^2 \leq R, \quad \forall n \in \mathbb{N}.
$$

Then, by Theorem 4.2.4 there exists a subsequence $\{(\mathbf{u}_{\alpha_j}, \mathbf{B}_{\alpha_j})\}_j$ and a Leray-Hopf weak solution $(\mathbf{v}, \mathbf{B})$ os the MHD equations such that $\mathbf{u}_{\alpha_j} \to \mathbf{v}$ and $\mathbf{B}_{\alpha_j} \to \mathbf{B}$ in $\mathcal{C}_{\text{loc}}(I, H_w)$. Furthermore, from the estimate (4.52), it follows that

$$
|\mathbf{u}_{\alpha_j}(t)|^2 + \alpha_j^2 \|\mathbf{u}_{\alpha_j}(t)\|^2 + |\mathbf{B}_{\alpha_j}(t)|^2 \leq |\mathbf{u}_0|^2 + \alpha_j^2 \|\mathbf{u}_0\|^2 + |\mathbf{B}_0|^2, \quad \forall j.
$$

Thus, from Lemma 4.2.2, we also obtain that $\mathbf{w}_{\alpha_j} \to \mathbf{v}$ in $\mathcal{C}_{\text{loc}}(I, H_w)$. □

We now show that item (iv) of hypothesis ($\tilde{\text{H}}3$) is also satisfied by the family $\{S_{\alpha_n}\}_n$. Recall that, in the present case, $\mathcal{U}_I$ denotes the set of Leray-Hopf weak solutions of the MHD equations on I.

Proposition 4.2.6. *Consider a set $K \in \mathfrak{K}'(\mathbb{H}_w)$. Then, there exists a compact set $\tilde{K}$ in $\mathbb{H}_w$ such that*

$$
\limsup_n S_{\alpha_n}(K) \subset \Pi_{t_0}^{-1}\tilde{K} \cap \mathcal{U}_I.
$$

Proof. Consider $R > 0$ such that $K \subset B_{\mathbb{H}}(R)$. Thus, it follows that $S_{\alpha_n}(K) \subset \mathcal{Y}_I(R)$, for every $n \in \mathbb{N}$. Therefore, since $\mathcal{Y}_I(R)$ is metrizable, given $\Phi \in \limsup_n S_{\alpha_n}(K)$, there exists a sequence $\{\Phi_k\}_k$ in $\mathcal{Y}_I(R)$ such that each Φ_k belongs to $S_{\alpha_k}(K)$, for some $\alpha_k > 0$, and $\Phi_k \to \Phi$ in $\mathcal{Y}_I(R)$. This implies, in particular, that $\Pi_{t_0}\Phi_k = \Phi_k(t_0) \to \Phi(t_0) = \Pi_{t_0}\Phi$ in $\mathbb{H}_w$. Thus, since $\{\Phi_k(t_0)\}_k \subset K$ and K is compact in $\mathbb{H}$, hence compact in $\mathbb{H}_w$, we obtain that $\Phi \in \Pi_{t_0}^{-1}(K)$.

Furthermore, since $\{\Phi_k(t_0)\}_k$ is uniformly bounded in $\mathbb{H}_w$, by Theorem 4.2.5 there exists a subsequence of $\{\Phi_k\}_k$, which we still denote by $\{\Phi_k\}_k$ and a weak solution $\tilde{\Phi}$ of the MHD equations on $\mathring{I}$ such that $\Phi_k \to \tilde{\Phi}$ in $\mathcal{C}_{\text{loc}}(I, \mathbb{H}_w)$. Thus, it follows that $\tilde{\Phi} = \Phi$ and, in particular, $\Phi \in \mathcal{U}_I^\sharp(R)$. On the other hand, considering $\tilde{K} = B_{\mathbb{H}}(R)_w$, we have that $\mathcal{U}_I^\sharp(R) \subset \overline{\Pi_{t_0}^{-1}\tilde{K} \cap \mathcal{U}_I}$, where the overline denotes the closure in $\mathcal{C}_{\text{loc}}(I, \mathbb{H}_w)$. Moreover, as in Proposition 3.4.3, we can prove analogously that $\Pi_{t_0}^{-1}\tilde{K} \cap \mathcal{U}_I$ is closed in $\mathcal{C}_{\text{loc}}(I, \mathbb{H}_w)$, so that $\overline{\Pi_{t_0}^{-1}\tilde{K} \cap \mathcal{U}_I} = \Pi_{t_0}^{-1}\tilde{K} \cap \mathcal{U}_I$. We then conclude that $\Phi \in \Pi_{t_0}^{-1}\tilde{K} \cap \mathcal{U}_I$, which concludes the proof. □

Now, from the previous results and using similar arguments as in Theorem 4.2.2, we obtain that, given an initial Borel probability measure μ_0 on $\mathbb{H}$, the family of measures $\{S_{\alpha_n}\mu_0\}_n$ converges, modulo a subsequence to a trajectory statistical solution of the MHD equations.

Theorem 4.2.6. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0. Let $\mathcal{U}_I$ be the set of Leray-Hopf weak solutions of the MHD equations on I and, for each $n \in \mathbb{N}$, let S_{α_n} be the solution operator associated to the MHD-α_n equations defined in (4.56). If μ_0 is a Borel probability measure on $\mathbb{H}$, then $\{S_{\alpha_n}\mu_0\}_{n\in\mathbb{N}}$ has a subsequence converging in the weak star topology to a $\mathcal{U}_I$-trajectory statistical solution ρ on $\mathcal{C}_{loc}(I, \mathbb{H}_w)$ such that $\Pi_{t_0}\rho = \mu_0$.*

Finally, we obtain the convergence of statistical solutions of the MHD-α_n equations to a statistical solution of the MHD equations, satisfying a given initial data. The proof follows by using the energy inequality (4.38) and similar arguments to the ones used in Theorem 4.2.3, applied to the pair $(\{S_{\alpha_n}\}_{n\in\mathbb{N}}, \mathcal{U}_I)$ associated to the MHD-α_n and MHD equations, and the function $\mathbf{F} : I \times \mathbb{V} \to \mathbb{V}'$ given by

$$\mathbf{F}(t, \Phi) = -\mathbb{A}\Phi - \mathbb{B}(\Phi, \Phi).$$

Theorem 4.2.7. *Let $I \subset \mathbb{R}$ be an interval closed and bounded on the left with left end point t_0 and let $\mathcal{U}_I$ be the set of Leray-Hopf weak solutions of the MHD equations on I. For each $n \in \mathbb{N}$, let S_{α_n} be the solution operator associated to the MHD-α_n equations, defined in (4.56). Consider a Borel probability measure μ_0 on $\mathbb{H}$ satisfying*

$$\int_{\mathbb{H}} |\Phi|_{\mathbb{H}}^2 \mathrm{d}\mu_0(\Phi) < \infty. \tag{4.68}$$

Then, there exists a subsequence of $\{\{\Pi_t S_{\alpha_n}\mu_0\}_{t\in I}\}_{n\in\mathbb{N}}$ *converging to a statistical solution* $\{\rho_t\}_{t\in I}$ *of the MHD equations such that* $\rho_{t_0} = \mu_0$.

Appendix A

Comparing topologies in the space of measures

Let X be a Hausdorff space and let $\mathcal{M}(X)$ be the space of finite Borel measures on X. We shall analyze the relation between the two topologies on $\mathcal{M}(X)$ defined in Section 2.3, namely the *weak-star topology* and the *weak-star semi-continuity topology*.

Let us fix a notation and denote the weak-star topology by τ_c and the weak-star semi-continuity topology by τ_{sc}.

As we can see, in the definition of the weak-star semi-continuity topology, the condition of continuity is relaxed to upper semi-continuity, for both the function f and the corresponding mapping J_f, defined as

$$J_f(\mu) = \mu(f) = \int_X f \mathrm{d}\mu, \quad \forall \mu \in \mathcal{M}(X).$$

This subtle difference makes the weak-star semi-continuity topology stronger than the weak-star topology, i.e. $\tau_c \subset \tau_{\text{sc}}$. In order to prove this, we first note that an equivalent way of defining the weak-star semi-continuity topology is to replace the condition of upper semi-continuity by lower semi-continuity. In other words, if we denote by τ'_{sc} the smallest topology for which the mappings J_f are lower semi-continuous, for every bounded and lower semi-continuous real-valued function f on X, then $\tau'_{\text{sc}} = \tau_{\text{sc}}$. We prove this in the following proposition.

Proposition A.1. *In the space $\mathcal{M}(X)$, the topologies τ_{sc} and τ'_{sc} are equivalent.*

Proof. Note that τ'_{sc} is the topology generated by the sub-basis

$$\{J_f^{-1}((a, +\infty)) \mid \quad a \in \mathbb{R},\ f : X \to \mathbb{R} \text{ is bounded and lower semi-continuous}\}.$$

Thus, in order to first obtain that $\tau'_{\text{sc}} \subset \tau_{\text{sc}}$, it suffices to prove that every element from this sub-basis belongs to τ_{sc}.

Given a bounded and lower semi-continuous function $f : X \to \mathbb{R}$, consider the function $g = -f$. Then g is clearly a bounded and upper semi-continuous real-valued function on X. Moreover, for every $a \in \mathbb{R}$, we have

$$J_f^{-1}((a, +\infty)) = J_g^{-1}((-\infty, -a)).$$

And since the set on the right-hand side belongs to τ_{sc}, it follows that $\tau'_{\text{sc}} \subset \tau_{\text{sc}}$.

The inverse inclusion is proved analogously. $\square$

Now the inclusion $\tau_c \subset \tau_{\text{sc}}$ is easily obtained:

Proposition A.2. *In the space $\mathcal{M}(X)$, the topology τ_{sc} is stronger than the topology τ_c.*

Proof. Consider $a, b \in \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a bounded and continuous function. Note that

$$J_f^{-1}((a, b)) = J_f^{-1}((a, +\infty)) \cap J_f^{-1}((-\infty, b)).$$

Now the first set on the right-hand side belongs to τ_{sc} since f is in particular bounded and lower semi-continuous on X, and the second set on the right-hand side belongs to τ_{sc} since f is also in particular bounded and upper semi-continuous on X. Therefore, $J_f^{-1}((a, b)) \in \tau_{\text{sc}}$, proving that every set from the sub-basis of τ_c belongs to τ_{sc}. Thus, $\tau_c \subset \tau_{\text{sc}}$. $\square$

Appendix B

Strengthened energy inequality for the Navier-Stokes equations

In the same setting introduced in Section 3.4.1, we consider the Navier-Stokes equations with periodic boundary conditions and prove that the Leray-Hopf weak solutions satisfy a strengthened form of the usual energy inequality. This result has been proved in [12] for a forcing term $f \in L^2_{\text{loc}}(I, H)$ and for a function $\psi : [0, \infty) \to \mathbb{R}$ which is nonnegative, nondecreasing, absolutely continuous and with bounded derivative. Here we assume that ψ is continuously differentiable, but consider, more generally, a term f in $L^2_{\text{loc}}(I, V')$. In order to simplify we consider $I = [0, T]$, for some $T > 0$.

Theorem B.1. *Let $T > 0$ and $f \in L^2(0, T; V')$. Consider a nonnegative, nondecreasing and continuously-differentiable real-valued function $\psi : [0, \infty) \to \mathbb{R}$ with bounded derivative. If $\mathbf{u}$ is a weak solution of the Navier-Stokes equations on $[0, T]$, then*

$$\frac{d}{dt}(\psi(|\mathbf{u}(t)|^2)) \leq 2\psi'(|\mathbf{u}(t)|^2)[\langle f(t), \mathbf{u}(t)\rangle_{V',V} - \nu\|\mathbf{u}(t)\|^2]$$

in the sense of distributions on $[0, T]$.

Proof. Define

$$\xi(\cdot) = |\mathbf{u}(\cdot)|^2 \quad \text{and} \quad g(\cdot) = 2(\langle f(\cdot), \mathbf{u}(\cdot)\rangle_{V',V} - \nu\|\mathbf{u}(\cdot)\|^2).$$

Since $\mathbf{u}$ is a Leray-Hopf weak solution of the NSE, it follows that $\xi \in L^\infty(0, T)$ and $g \in L^1(0, T)$. We then want to prove that the inequality

$$\frac{d}{dt}(\psi \circ \xi(t)) \leq (\psi \circ \xi(t))g(t)$$

holds in the sense of distributions on $[0, T]$.

Consider a non-negative mollifier $\rho \in \mathcal{C}_c^\infty(\mathbb{R})$ and, for any $\varepsilon > 0$, define $\rho_\varepsilon(\cdot) = \varepsilon^{-1}\rho(\cdot/\varepsilon)$. Let $\tilde{\xi}$ and $\tilde{g}$ be the extensions of ξ and g by zero on $\mathbb{R}$ outside $[0,T]$, respectively. Also, let $\overline{\psi} : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\overline{\psi}(t) = \begin{cases} \psi(t) & \text{, if } t \geq 0 \\ \psi(0) & \text{, if } t < 0 \end{cases}.$$

Note that ψ is continuous on $\mathbb{R}$.

For any $\varepsilon > 0$, define $\xi_\varepsilon := \rho_\varepsilon * \tilde{\xi}$ and $g_\varepsilon := \rho_\varepsilon * \tilde{g}$. And for any $\delta > 0$, let $\psi_\delta := \rho_\delta * \overline{\psi}$.

Observe that

$$\xi_\varepsilon'(t) = \frac{d}{dt}\int_{\mathbb{R}} \frac{1}{\varepsilon}\rho\left(\frac{t-s}{\varepsilon}\right)\tilde{\xi}(s)\mathrm{d}s = \int_{\mathbb{R}} \frac{1}{\varepsilon^2}\rho'\left(\frac{t-s}{\varepsilon}\right)\tilde{\xi}(s)\mathrm{d}s = -\int_{\mathbb{R}} \varphi_{t,\varepsilon}'(s)\tilde{\xi}(s)\mathrm{d}s,$$

where

$$\varphi_{t,\varepsilon}(s) = \frac{1}{\varepsilon}\rho\left(\frac{t-s}{\varepsilon}\right), \forall s \in \mathbb{R}.$$

Given $t \in (0,T)$, we can obtain a sufficiently small $\varepsilon > 0$ such that $\operatorname{supp}\varphi_{t,\varepsilon} \subset (0,T)$. Indeed, if $\operatorname{supp}\rho \subset [-R,R]$, $R > 0$, taking ε such that

$$0 < \varepsilon < \min\left\{\frac{T-t}{R}, \frac{t}{R}\right\}, \tag{B.1}$$

it is easy to see that $\operatorname{supp}\varphi_{t,\varepsilon} \subset [t-\varepsilon R, t+\varepsilon R] \subset (0,T)$.

Fix $t \in (0,T)$ and ε satisfying (B.1). Since $\mathbf{u}$ is a weak solution, it satisfies the energy inequality $\xi' \leq g$ in the sense of distributions on $[0,T]$. Using this and the fact that $\operatorname{supp}\varphi_{t,\varepsilon} \subset (0,T)$, we have

$$\begin{aligned} \xi_\varepsilon'(t) &= -\int_{\mathbb{R}} \varphi_{t,\varepsilon}'(s)\tilde{\xi}(s)\mathrm{d}s = -\int_0^T \varphi_{t,\varepsilon}'(s)\xi(s)\mathrm{d}s \\ &\leq \int_0^T \varphi_{t,\varepsilon}(s)g(s)\mathrm{d}s = \int_{\mathbb{R}} \varphi_{t,\varepsilon}(s)\tilde{g}(s)\mathrm{d}s = g_\varepsilon(t). \end{aligned}$$

Now since ξ_ε and ψ_δ are smooth functions, we have

$$\frac{d}{dt}(\psi_\delta \circ \xi_\varepsilon(t)) = (\psi_\delta' \circ \xi_\varepsilon(t))\xi_\varepsilon'(t),$$

in the classical sense on $\mathbb{R}$.

Then, given a non-negative test function $\varphi \in \mathcal{C}_c^\infty(0,T)$, let $\eta > 0$ be such that $\operatorname{supp}\varphi \subset [\eta, T-\eta]$. Hence, for every $\delta > 0$ and ε satisfying

$$0 < \varepsilon < \min\left\{\frac{T-\eta}{R}, \frac{\eta}{R}\right\},$$

we obtain that

$$\int_0^T (\psi_\delta \circ \xi_\varepsilon)'(t)\varphi(t)\mathrm{d}t = \int_0^T (\psi_\delta' \circ \xi_\varepsilon)(t)\xi_\varepsilon'(t)\varphi(t)\mathrm{d}t \leq \int_0^T (\psi_\delta' \circ \xi_\varepsilon)(t)g_\varepsilon(t)\varphi(t)\mathrm{d}t.$$

Thus,

$$-\int_0^T (\psi_\delta \circ \xi_\varepsilon)(t)\varphi'(t)\mathrm{d}t \leq \int_0^T (\psi_\delta' \circ \xi_\varepsilon)(t)g_\varepsilon(t)\varphi(t)\mathrm{d}t. \tag{B.2}$$

We now must take the vanishing limits of ε and δ in order to obtain our desired inequality. Let us consider the term on the left-hand side of (B.2) first. Note that, for every $t \geq 0$,

$$\begin{aligned} \psi_\delta'(t) = \rho_\delta * \overline{\psi}'(t) &= \int_{\mathbb{R}} \overline{\psi}'(t-s)\frac{1}{\delta}\rho\left(\frac{s}{\delta}\right)\mathrm{d}s \\ &\leq \sup_{\tau\in(0,t+\delta R)} \psi'(\tau)\int_{\mathbb{R}} \frac{1}{\delta}\rho\left(\frac{s}{\delta}\right)\mathrm{d}s = \sup_{\tau\in(0,t+\delta R)} \psi'(\tau), \end{aligned} \tag{B.3}$$

where the last equality follows from the fact that ρ is a mollifier. Using a similar argument, we can show that

$$0 \leq \sup_{t\in\mathbb{R}} \xi_\varepsilon(t) \leq \operatorname{ess\,sup}_{(0,T)}\xi =: M. \tag{B.4}$$

Then, from (B.3), (B.4) and applying the Mean Value Theorem to the function ψ_δ, we obtain that

$$\begin{aligned} \left|\int_0^T (\psi_\delta \circ \xi_\varepsilon)(t)\varphi'(t)\mathrm{d}t \right.&- \left.\int_0^T (\psi_\delta \circ \xi)(t)\varphi'(t)\mathrm{d}t\right| \leq \\ &\leq \left(\sup_{\tau\in(0,T)} |\varphi'(\tau)|\right)\left(\sup_{\tau\in(0,M)} \psi_\delta'(\tau)\right)\int_0^T |\xi_\varepsilon(t)-\xi(t)|\mathrm{d}t \\ &\leq \left(\sup_{\tau\in(0,T)} |\varphi'(\tau)|\right)\left(\sup_{\tau\in(0,M+\delta R)} \psi'(\tau)\right)\int_0^T |\xi_\varepsilon(t)-\xi(t)|\mathrm{d}t. \end{aligned}$$

But since ξ_ε converges to $\tilde{\xi}$ in $L^1(\mathbb{R})$ as ε goes to 0, then $\xi_\varepsilon|_{[0,T]}$ converges to ξ in $L^1(0,T)$. From the inequality above, it then follows that

$$\int_0^T (\psi_\delta \circ \xi_\varepsilon)(t)\varphi'(t)\mathrm{d}t \to \int_0^T (\psi_\delta \circ \xi)(t)\varphi'(t)\mathrm{d}t, \text{ as } \varepsilon \to 0. \tag{B.5}$$

Now let us analyze the term on the right-hand side of (B.2). Observe that

$$\begin{aligned}
&\left|\int_0^T (\psi_\delta' \circ \xi_\varepsilon)(t) g_\varepsilon(t)\varphi(t)\mathrm{d}t - \int_0^T (\psi_\delta' \circ \xi)(t) g(t)\varphi(t)\mathrm{d}t\right| \leq \\
\leq\; & \left|\int_0^T (\psi_\delta' \circ \xi_\varepsilon)(t)(g_\varepsilon(t) - g(t))\varphi(t)\mathrm{d}t\right| + \left|\int_0^T [(\psi_\delta' \circ \xi_\varepsilon)(t) - (\psi_\delta' \circ \xi)(t)] g(t)\varphi(t)\mathrm{d}t\right| \\
\leq\; & (\sup_{\tau\in(0,M+\delta R)} \psi'(\tau))(\sup_{\tau\in(0,T)} |\varphi(\tau)|) \int_0^T |g_\varepsilon(t) - g(t)|\mathrm{d}t \\
& + \left|\int_0^T [(\psi_\delta' \circ \xi_\varepsilon)(t) - (\psi_\delta' \circ \xi)(t)] g(t)\varphi(t)\mathrm{d}t\right|.
\end{aligned}$$

Then since $g_\varepsilon|_{[0,T]} \to g$ in $L^1(0,T)$, the first term on the right-hand side of the last inequality above vanishes as $\varepsilon \to 0$. For the second term, note that since $\xi_\varepsilon|_{[0,T]} \to \xi$ in $L^1(0,T)$, there is a subsequence $\{\xi_{\varepsilon'}\}_{\varepsilon'}$ such that

$$\xi_{\varepsilon'}(t) \to \xi(t) \text{ q.t.p. in } [0,T], \text{ as } \varepsilon' \to 0.$$

And since ψ_δ' is continuous in $\mathbb{R}$,

$$(\psi_\delta' \circ \xi_{\varepsilon'}(t) \to (\psi_\delta' \circ \xi)(t) \text{ q.t.p. in [0,T]}.$$

Also,

$$|[(\psi_\delta' \circ \xi_\varepsilon)(t) - (\psi_\delta' \circ \xi)(t)] g(t)\varphi(t)| \leq 2(\sup_{\tau\in(0,M+\delta R)} \psi'(\tau))(\sup_{\tau\in[0,T]} |\varphi(\tau)|)|g(t)| \in L^1(0,T).$$

Then, by the Dominated Convergence Theorem, taking a subsequence if necessary we have that

$$\left|\int_0^T [(\psi_\delta' \circ \xi_\varepsilon)(t) - (\psi_\delta' \circ \xi)(t)] g(t)\varphi(t)\mathrm{d}t\right| \to 0, \text{ as } \varepsilon \to 0.$$

Thus,

$$\int_0^T (\psi_\delta' \circ \xi_\varepsilon)(t) g_\varepsilon(t)\varphi(t)\mathrm{d}t \to \int_0^T (\psi_\delta' \circ \xi)(t) g(t)\varphi(t)\mathrm{d}t, \text{ as } \varepsilon \to 0. \tag{B.6}$$

From (B.2), (B.5) and (B.6), we have

$$-\int_0^T (\psi_\delta \circ \xi)(t)\varphi'(t)\mathrm{d}t \leq \int_0^T (\psi_\delta' \circ \xi)(t) g(t)\varphi(t)\mathrm{d}t. \tag{B.7}$$

Again, we must analyze the vanishing limit of δ in each individual term of (B.7).

Let us assume that $0 < \delta < 1$. For the term on the left-hand side, note that for every $L > 0$ and $r \in [0,L]$, we have

$$|\psi_\delta(r)| = \left|\int_{\mathbb{R}} \frac{1}{\delta}\rho\left(\frac{s}{\delta}\right)\tilde{\psi}(r-s)\mathrm{d}s\right| \leq \sup_{\tau\in[L-R,L+R]} |\tilde{\psi}(\tau)| \int_{\mathbb{R}} \frac{1}{\delta}\rho\left(\frac{s}{\delta}\right)\mathrm{d}s = \sup_{\tau\in[0,L+R]} |\psi(\tau)|.$$

Then,

$$\sup_{t\in[0,T]} |(\psi_\delta \circ \xi)(t)| \leq \sup_{\tau\in[0,M+R]} |\psi(\tau)|, \tag{B.8}$$

where $M := \text{ess sup}_{(0,T)}\xi$, as before.

Furthermore, since ψ is continuous on $[0,\infty)$, from the properties of mollifiers we know that

$$\psi_\delta(r) \to \psi(r),\ \forall r \in [0,\infty).$$

And since ξ is nonnegative, we have

$$\psi_\delta \circ \xi \to \psi \circ \xi \text{ q.t.p. in } [0,T]. \tag{B.9}$$

From (B.8), (B.9) and the Dominated Convergence Theorem, we then obtain that

$$\int_0^T (\psi_\delta \circ \xi)(t)\varphi'(t)\mathrm{d}t \to \int_0^T (\psi \circ \xi)(t)\varphi'(t)\mathrm{d}t.$$

An analogous argument applies to the term on the right-hand side of (B.7). Indeed, it suffices to note that ψ' is continuous on $[0,\infty)$ and that from (B.3) and from the fact that $0 < \delta < 1$,

$$\sup_{t\in(0,T)} \psi'_\delta \circ \xi(t) \leq \sup_{\tau\in(0,M+R)} \psi'(\tau).$$

Taking the limit as $\delta \to 0$ in (B.7), we finally obtain that

$$-\int_0^T (\psi \circ \xi)(t)\varphi'(t)\mathrm{d}t \leq \int_0^T (\psi' \circ \xi)(t)g(t)\varphi(t)\mathrm{d}t,$$

for every nonnegative function $\varphi \in \mathcal{C}_c^\infty(0,T)$. $\square$

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