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Abstract

We show that there is a residual set of the space of homeomorphisms of a compact smooth boundaryless manifold which its elements have total upper metric mean dimension. The notion of metric mean dimension for a dynamical system was introduced by Lindenstrauss and Weiss in 2000 and this refines the topological entropy for dynamical systems with infinite topological entropy.

Key words: dynamical systems, topological entropy, metric mean dimension, pseudo-horseshoe

Contents

1 Introduction

In order to study how physical events evolve along a timescale, mathematicians developed an area called dynamical systems. The goal is to study qualitative behavior of the iterates of a continuous function $f : X \to X$, where X is a topological space. We define the pair (X, f) as a dynamical system. Several tools were developed in the last century with the aim of studying this qualitative behavior, and one of them is called topological entropy.

The topological entropy counts the number of different orbits, the iterates of a function f at a given $x \in X$, in exponential scales, up to arbitrarily small errors. It is a useful tool to measure how chaotic is a dynamical system. When a dynamical system have infinite topological entropy and we still want to study its orbits, we recur to the metric mean dimension. Precisely, given f an element of the space of homeomorphisms $\text{Homeo}(X, d)$ of a metric space (X, d) we can calculate its metric mean dimension.

The goal of this text is to prove the following theorem already proved by M. Carvalho, F. B. Rodrigues and P. Varandas in [2]:

Theorem. Let X be a compact smooth boundaryless manifold of dimension m strictly greater than one endowed with a metric d. There exists a residual subset $\mathfrak{R} \subset \text{Homeo}(X, d)$ such that

$$\overline{\text{mdim}_M}(X, d, g) = m, \text{ for all } g \in \mathfrak{R}.$$

The definition of a metric here is the distance between points in the manifold and $\overline{\text{mdim}_M}(X, d, g)$ is the notation of the upper metric mean dimension of a continuous function $g : X \to X$, where (X, d) is a metric space.

The metric mean dimension is the metric version of the mean dimension, a concept proposed by Gromov in [4] which may be viewed as a dynamical analogue of the topological dimension and is invariant under topological conjugacy. A crucial result given by Velozo and Velozo in [10] says that the upper metric mean dimension of a dynamical system on a manifold is upper bounded by its dimension.

In Chapter 2 and Appendix A there is an introduction to topological dynamical systems and topological entropy. To understand it, the student only needs some background in topology and metric spaces. Again, in Chapter 2 is the definition of mean dimension, upper metric mean dimension and lower metric mean dimension. In order to obtain a new invariant to distinguish maps with intinite topological entropy, Lindenstrauss and Weiss introduced in [6] the notion of metric mean dimension of a continuous function f of a metric space (X, d).

In Chapter 3 we talk about absorbing disks and how to use it to define periodic attracting points for an homeomorphism. In resume, we prove that the set of homeomorphisms that have a periodic attracting point is dense in $\text{Homeo}(X, d)$ and use this fact to prove that there is a residual subset $\mathcal{H}$ of $\text{Homeo}(X, d)$ such that every $f \in \mathcal{H}$ have absorbing disks of arbitrarily small diameter. Absorbing disks are necessary because we need neighborhoods of the periodic point that are strictly contained in some chart of the manifold.

A discussion about pseudo-horseshoes is done in Chapter 4. Pseudo-horseshoes are just a generalization of the usual concept of a horseshoe, the difference is that it isn't necessarily invariant. Given $f \in \mathcal{H}$ there is $g \in \text{Homeo}(X, d)$ close enough to f such that it have a sufficiently large separating set induced by the pseudo-horseshoes. The idea is to compose f with small perturbations along the orbit of a periodic attracting point. We also observe that the set of all g that satisfy this property is residual in $\text{Homeo}(X, d)$.

Finally, in Chapter 5, we use pseudo-horseshoes to construct such $g \in \text{Homeo}(X, d)$ and show the existence of the residual set $\mathfrak{R}$. In this chapter is also proved the main result, using these large enough separating sets for $g \in \mathfrak{R}$ to prove that the upper metric mean dimension of g can be arbitrarily positive number and therefore it is the dimension of the manifold.

2 Metric Mean Dimension

2.1 Topological entropy

Let (X, d) be a compact metric space and $f : X \to X$ a continuous function. We say that the pair (X, f) is a dynamical system. Given $n \in \mathbb{N}$, define $d_n : X \times X \to \mathbb{R}$ as

$$d_n(x, y) = \max\{d(x, y), d\Big(f(x), f(y)\Big), ..., d\Big(f^{n-1}(x), f^{n-1}(y)\Big)\}.$$

With this we can define the *dynamical balls* as $B_n(x, \varepsilon) = \{y \in X; d_n(x, y) < \varepsilon\}$ and observe that d_n still a metric on X and generates the same topology as d.

Definition 2.1. We say that $E \subset X$ is a (n, ε)-cover set if $X = \bigcup\limits_{x \in E} B_n(x, \varepsilon)$. And denote by $N(n, \varepsilon)$ the minimum cardinality of a (n, ε)-cover set for X.

By the compactness of X, $N(n, \varepsilon)$ is a positive finite real number, so it makes sense to consider the following

$$h(f, \varepsilon) = \limsup_{n \to \infty} \frac{\log N(n, \varepsilon)}{n}.$$

Definition 2.2. Define the *topological entropy* of the dynamical system (X, f) as

$$h(f) = \lim_{\varepsilon \to 0} h(f, \varepsilon).$$

Proposition 2.1. Given (X, d_X) and (Y, d_Y) compact metric spaces, $f : X \to X$, $g : Y \to Y$ continuous functions and $\pi : X \to Y$ continuous surjective such that $\pi \circ f = g \circ \pi$, that is, f and g are semiconjugate, so $h(f) \geq h(g)$.

Proof. Since π is continuous defined in a compact metric space, then it is uniform continuous. That is, for every $\varepsilon > 0$, there is $\delta > 0$ such that if $d_X(x_1, x_2) < \delta$, so $d_Y(y_1, y_2) < \varepsilon$, where $y_i = \pi(x_i)$.

Consider E_X the smallest (n, δ)-cover set for X, that is, $N_X(n, \delta) = \#E_X$, and is such that $X \subset \bigcup\limits_{x \in E_X} B_n(x, \delta)$. Note that

$$Y = \pi(X) \subset \bigcup_{x \in E_X} \pi\Big(B_n(x, \delta)\Big) \subset \bigcup_{x \in E_X} B_n(\pi(x), \varepsilon).$$

This means that if E_Y is the smallest (n, ε)-cover for Y, then $\#E_Y \leq \#E_X$. Therefore $N_X(n, \delta) \geq N_Y(n, \varepsilon)$. Thus $h(f, \delta) \geq h(g, \varepsilon)$, and finally, $h(f) \geq h(g)$. □

Remark 2.1. Observe that if π is an homeomorphism, then f and g are topologically conjugate and $h(f) = h(g)$.

Definition 2.3. Call $A \subset X$ a (n, ε)-separated set if to any distinct points $x, y \in A$, we have $d_n(x, y) \geq \varepsilon$. Denote by $S(n, \varepsilon)$ the maximal cardinality of a (n, ε)-separated set for X.

Proposition 2.2. For all $\varepsilon > 0$ and $n \in \mathbb{N}$, $N(n, \varepsilon) \leq S(n, \varepsilon) \leq N(n, \varepsilon/2)$.

Proof. Given $\varepsilon > 0$ and $n \in \mathbb{N}$, let $A \subset X$ be a (n, ε)-separated set of maximum cardinality $S(n, \varepsilon)$.

Claim 1: $S(n, \varepsilon) < \infty$.

Suppose $\{x_1, x_2, ...\}$ an enumeration of A. If $\mathcal{A} := \bigcup_{i=1}^{\infty} B_n(x_i, \varepsilon)$ is not a open cover of X, consider a neighborhood W_x of $x \in \mathcal{A}^c$ such that $W_x \cap A = \varnothing$. Observe that A cannot have accumulation points. Then $\mathcal{A} \cup (\bigcup_{x \in \mathcal{A}^c} W_x)$ is a open cover of X. By compactness of X, there is finite elements $\{x_{k_1}, ..., x_{k_\ell}\}$ of A such that $\bigcup_{i=1}^{\ell} B_n(x_{k_i}, \varepsilon)$ covers A, contradiction.

Claim 2: $X \subset \bigcup_{x \in A} B_n(x, \varepsilon)$.

Suppose that there is $y \in X \setminus \bigcup_{x \in A} B_n(x, \varepsilon)$. Then for all $x \in A$, $d_n(x, y) \geq \varepsilon$, since y isn't inside any dynamical ball centered in elements of A. Therefore $A \cup \{y\}$ is a (n, ε)-separated set bigger than A, contradiction with the maximum cardinality of A.

With this claim we have that A is also a (n, ε)-cover set and thus $N(n, \varepsilon) \leq S(n, \varepsilon)$.

To see the other inequality, suppose E a $(n, \varepsilon/2)$-cover set with minimum cardinality $N(n, \varepsilon/2)$. So we have that $X \subset \bigcup_{x \in E} B_n(x, \varepsilon/2)$. For $x \in E$ and y, z distinct points in $B_n(x, \varepsilon/2)$, $d_n(y, z) \leq d_n(y, x) + d_n(x, z) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$. Therefore $B_n(x, \varepsilon/2)$ cannot have more than one point of A. Thus $S(n, \varepsilon) \leq N(n, \varepsilon/2)$. □

Easily we get that

$$\frac{\log N(n, \varepsilon)}{n} \leq \frac{\log S(n, \varepsilon)}{n} \leq \frac{\log N(n, \varepsilon/2)}{n}.$$

Taking the lim sup on above inequality

$$h(f, \varepsilon) \leq \limsup_{n \to \infty} \frac{\log S(n, \varepsilon)}{n} \leq h(f, \varepsilon/2).$$

Now taking the limit with $\varepsilon \to 0$ we have that

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log N(n, \varepsilon)}{n} = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log S(n, \varepsilon)}{n}.$$

In order to clarify further notations, define

$$\text{Sep}(f, \varepsilon) = \limsup_{n \to \infty} \frac{\log S(n, \varepsilon)}{n}. \tag{2.1}$$

That is, (n, ε)-separated sets are another way to calculate topological entropy.

On a compact metric space, a Lipschitz function has finite topological entropy. If the dynamics is just continuous, the topological entropy may be infinite. Furthermore, Yano proved in [11] that, on compact smooth manifolds with dimension greater than one, the set of homeomorphisms having infinite topological entropy is residual. So the topological entropy is no longer an effective label to classify them and we need a more refined definition to distinguish maps with infinite entropy and that is the metric mean dimension.

2.2 Upper and lower metric mean dimension

In this section we will define the mean dimension and metric mean dimension for any continuous map $f : X \to X$, when (X, d) a compact metric space. The notion of mean dimension for a topological dynamical system (X, f), which will be denoted by $\text{mdim}(X, f)$, was introduced by M. Gromov in [4]. It is another invariant under topological conjugacy.

Let $\mathcal{A} = \{A_\alpha, \alpha \in I\}$ be an open cover of X. If $\mathcal{B} = \{B_\beta, \beta \in J\}$ are just open subsets of X, not necessarily an open cover, define $\mathcal{A} \vee \mathcal{B} = \{A_\alpha \cap B_\beta, \alpha \in I, \beta \in J\}$. For $f^{-n}(\mathcal{A}) := \{f^{-n}(A_\alpha), A_\alpha \in \mathcal{A}\}$, define

$$\mathcal{A}_0^{n-1} = \mathcal{A} \vee f^{-1}(\mathcal{A}) \vee f^{-2}(\mathcal{A}) \vee ... \vee f^{-n+1}(\mathcal{A}).$$

Set

$$\text{ord}(\mathcal{A}) = \sup_{x \in X} \sum_{A_\alpha \in \mathcal{A}} 1_{A_\alpha}(x) - 1 \quad \text{and} \quad \mathcal{D}(\mathcal{A}) = \min_{\mathcal{C} \prec \mathcal{A}} \text{ord}(\mathcal{C})$$

where 1_{A_α} is the indicator function and $\mathcal{C} \prec \mathcal{A}$ means that $\mathcal{C}$ is a partition of X finer than $\mathcal{A}$.

Definition 2.4. The *mean dimension* of $f : X \to X$ is defined as

$$\text{mdim}(X, f) = \sup_{\mathcal{A}} \lim_{n \to \infty} \frac{\mathcal{D}(\mathcal{A}_0^{n-1})}{n}.$$

We have that $\text{mdim}(X, f) = 0$ if $\dim(X) < \infty$, where $\dim(X)$ is the topological dimension of X (see [6]). Still in [6], Lindenstrauss and Weiss introduced the notions of upper metric mean dimension and lower mean dimension for any continuous map f on X. This notion depends on the metric d on X, consequently it is not invariant under topological conjugacy, and it is zero for any map with finite topological entropy.

Definition 2.5. The *lower metric mean dimension* and the *upper metric mean dimension* of (X, d, f) are defined by

$$\underline{\text{mdim}_M}(X, d, f) = \liminf_{\varepsilon \to 0} \frac{\text{Sep}(f, \varepsilon)}{|\log \varepsilon|} \quad \text{and} \quad \overline{\text{mdim}_M}(X, d, f) = \limsup_{\varepsilon \to 0} \frac{\text{Sep}(f, \varepsilon)}{|\log \varepsilon|},$$

respectively.

If both limits coincides then the *metric mean dimension* of (X, d, f) is denoted simply by $\text{mdim}_M(X, d, f)$.

Note that if $h(f)$ is finite, that is, if exists some $M \in \mathbb{R}$ such that $h(f) < M$, then, for ε sufficiently small, $\text{Sep}(f, \varepsilon) < M$. Therefore

$$\overline{\text{mdim}_M}(X, d, f) = \limsup_{\varepsilon \to 0} \frac{\text{Sep}(f, \varepsilon)}{|\log \varepsilon|} \leq \limsup_{\varepsilon \to 0} \frac{M}{|\log \varepsilon|} = 0.$$

So, in order to study homeomorphisms with infinite topological entropy, the metric mean dimension is an appropriate tool.

Definition 2.6. Set $\mathcal{M} := \{\rho; \rho \text{ is a metric on } X \text{ equivalent to } d\}$ and take

$$\underline{\text{mdim}_M}(X, f) = \inf_{\rho \in \mathcal{M}} \underline{\text{mdim}_M}(X, \rho, f) \quad \text{and} \quad \overline{\text{mdim}_M}(X, f) = \inf_{\rho \in \mathcal{M}} \overline{\text{mdim}_M}(X, \rho, f)$$

Then $\underline{\text{mdim}}_M(X, f)$ and $\overline{\text{mdim}_M}(X, f)$ are topological invariant. The inequalities

$$\text{mdim}(X, f) \leq \underline{\text{mdim}}_M(X, d, f) \leq \overline{\text{mdim}_M}(X, d, f)$$

always hold and the proof for the first inequality can be seen in [6]. Historically, the idea of metric mean dimension have been useful to distinguish dynamical systems acting on infinite-dimensional spaces. A crucial result about metric mean dimension is proved in [10] and says that

$$\overline{\text{mdim}_M}(X, d, f) \leq m, \tag{2.2}$$

where m is the dimension of X, and we will use this result to prove the main theorem.

3 Absorbing disks

Let X be a compact manifold of dimension m endowed with a metric d, for simplicity we will refer to them as (X, d). Denote by $\text{Homeo}(X, d)$ the set of homeomorphisms of (X, d). If $B_1(0)$ is the closed unit ball in $\mathbb{R}^m$, then call $B \subset X$ a *disk* if it is homeomorphic to $B_1(0)$.

Definition 3.1. A closed subset $B \subset X$ is called *n-absorbing* for an homeomorphism f of X if $f^n(B)$ is contained in the interior of B, and B is said to be *absorbing* if it is n-absorbing for some $n \in \mathbb{N}$.

Note that if B is a n-absorbing disk, then, by the Brouwer fixed point theorem, B contains a periodic point p by f with period n.

Definition 3.2. Call a point $p \in X$ *periodic attracting point* for f if there is a n-absorbing disk B satisfying:

1. $\text{diam}(f^i(B)) < \text{diam}(B)$ for every $1 \leq i \leq n-1$;
2. $\bigcap_{j \geq 0} f^{jn}(B) = \{p\}$.

Observe that, since f is a bijection, the last equality implies that $f^n(p) = p$. Indeed, applying the pre-image of f^n we have that

$$f^{-n}\Big(\bigcap_{j \geq 0} f^{jn}(B)\Big) = \{f^{-n}(p)\}.$$

Therefore

$$f^{-n}\Big(\bigcap_{j \geq 0} f^{jn}(B)\Big) = \bigcap_{j \geq 0} f^{(j-1)n}(B) = f^{-n}(B) \cap \bigcap_{j \geq 0} f^{jn}(B) = f^{-n}(B) \cap \{p\} = \{p\}.$$

Since B is a n-absorbing disk, we have that $f^n(B) \subset B$, therefore $f^{(j+1)n}(B) \subset f^{jn}(B)$, for $j \geq 0$. We can conclude that

$$\bigcap_{j \geq 0}^{k} f^{jn}(B) = f^{kn}(B), \text{ thus } \bigcap_{j \geq 0} f^{jn}(B) = \lim_{j \to \infty} f^{jn}(B). \tag{3.1}$$

Remark 3.1. Given a periodic attracting point, it is possible to choose a n-absorbing disk $W \subset B$ satisfying $f^i(W) \cap W = \varnothing$ for every $1 \leq i \leq n-1$. Indeed, let $\varepsilon > 0$ such that $B_\varepsilon(p) \cap B_\varepsilon(f^i(p)) = \varnothing$, for all $1 \leq i \leq n-1$. By the continuity of each f^i, there is some sufficiently small $\delta > 0$ such that

$$B_\delta(p) \subset B_\varepsilon(p) \quad \text{and} \quad f^i(B_\delta(p)) \subset B_\varepsilon(f^i(p)),$$

and by what we have discussed in (3.1) there is some $J \geq 0$ such that $f^{Jn}(B) \subset B_\delta(p)$. Since $f^n(B) \subset \text{int}(B)$, then

$$f^n(f^{Jn}(B)) = f^{Jn+n}(B) = f^{Jn}(f^n(B)) \subset \text{int}\Big(f^{Jn}(B)\Big).$$

Then $f^{Jn}(B)$ is also a n-absorbing disk for f.

Let F be an homeomorphism of X. We have that in [5], Hurley talks about $CR(F)$, that is, the set of chain recurrent points of F. A point $q \in X$ is a element of $CR(F)$ if for each $\varepsilon > 0$ there is a finite sequence $x_0, x_1, ..., x_n$ with $n \geq 1$, $x_0 = x_n = q$ and $d(F(x_i), x_{i+1}) < \varepsilon$ for $0 \leq i \leq n-1$. Particularly, a periodic point for F is in $CR(F)$. We will need this next proposition because we want to prove density of functions with periodic points.

Proposition 3.1. Let $F \in \text{Homeo}(X, d)$, then there is $q \in CR(F)$.

Proof. By the Birkhoff Recurrence Theorem, as you can see in Appendix A, there exists some point $q \in X$ that is recurrent to F, that is, there exists a sequence $(n_k)_{k\in\mathbb{N}}$ such that $F^{n_k}(q) \to q$ as $k \to \infty$.

Let $\varepsilon > 0$ and $\ell \in \mathbb{N}$ the smallest positive integer such that $F^{\ell}(q) \in B_{\varepsilon}(q)$. Since F is continuous and X is compact, there is some $\rho > 0$ such that if $d(x_1, x_2) < \rho$, then $d(F(x_1), F(x_2)) < \varepsilon/2$.

For $0 < \delta < \min\{\rho, \varepsilon/2\}$ and $1 \leq i \leq \ell - 2$, choose $z_i \in B_{\delta}\big(F^i(q)\big)$, then $F(z_i) \in B_{\varepsilon/2}(F^{i+1}(q))$. Observe that $d(F(z_i), z_{i+1}) < \varepsilon$. Finally, we have $z_{\ell-1} = F^{\ell-1}(q)$. Setting q as z_0 and z_ℓ we are done. □

The next two lemmas are necessary to prove Proposition 3.2. The following lemma is similar to a theorem proved in [8], but here we only need an homeomorphism in X, there they made a diffeomorphism. Because of this, a different proof is given here, which we need to give credits to [1] for most of the ideas in this proof. We will make the notation cleaner and set $|.|$ as the euclidean norm in $\mathbb{R}^m$.

Lemma 3.1. Let X be a manifold of dimension greater or equal than 2 with distance d coming from a Riemann metric. Suppose that a finite collection $\{(p_i, q_i) \in X \times X; i = 1, ..., n\}$ of pairs of points of X is specified, together with a small positive constant $\varepsilon > 0$ such that:

1. For each i, $d(p_i, q_i) < \varepsilon$;
2. If $i \neq j$, then $p_i \neq p_j$ and $q_i \neq q_j$.

Then there exists $\psi \in \text{Homeo}(X, d)$ such that:

(a) $d(\psi(x), x) < 2\varepsilon$ and $d(\psi^{-1}(x), x) < 2\varepsilon$ for every $x \in X$;

(b) $\psi(p_i) = q_i$ for $i = 1, ..., n$.

Proof. Firstly we will show a simpler version of the lemma on $\mathbb{R}$. Fix $\varepsilon > 0$. Assume that we have only a point p and that $p = 0$. Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a tent function such that

$$\sigma(x) = 0 \text{ for any } x \in (-\infty, -\varepsilon] \cup [\varepsilon, \infty) \text{ and } \sigma(0) = 1.$$

That is,

$$\sigma(x) = \begin{cases} 0, & \text{if } x \in (-\infty, -\varepsilon] \cup [\varepsilon, \infty); \\ \frac{x}{\varepsilon} + 1, & \text{if } x \in [-\varepsilon, 0]; \\ \frac{-x}{\varepsilon} + 1, & \text{if } x \in [0, \varepsilon]. \end{cases}$$

Define, for any $q \in (-\varepsilon, \varepsilon)$,

$$\varphi_q(x) = x + \sigma(x)q \text{ for any } x \in \mathbb{R}.$$

Thus, φ_q is an homeomorphism of $\mathbb{R}$ and $\varphi_q(x) = x$ if $x \in (-\infty, \varepsilon] \cup [\varepsilon, \infty)$. Observe that $\varphi_q([-\varepsilon, \varepsilon]) = [-\varepsilon, \varepsilon]$ and $|x - \varphi_q(x)| = \sigma(x)|q| < \varepsilon$.

Step 2: We adapt this proof to $\mathbb{R}^m$. We can still assume that a point p is the origin of $\mathbb{R}^m$. Given a point q in the euclidean space such that $|q| < \varepsilon$, we may rotate the coordinate axes so that the point q sits on the first axis (this rotation is obtained from a multiplication by a matrix which preserve the distances), so that $q = (q_1, 0, ..., 0) \in \mathbb{R}^m$ and $|q_1| < \varepsilon$. Take $\varphi_q : \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$\varphi_q(x_1, x_2, ..., x_m) = (x_1 + \sigma(x_1)q_1, x_2, ..., x_m).$$

Therefore φ_q is an homeomorphism of $\mathbb{R}^m$ and $\varphi_q(x) = x$ if $|x_1| \geq \varepsilon$, for $x = (x_1, ..., x_m) \in \mathbb{R}^m$. Observe that $\varphi_q(0) = q$, $\varphi_q(B_\varepsilon(0)) = B_\varepsilon(0)$ and, again, $|x - \varphi_q(x)| = \sigma(x_1)|q_1| < \varepsilon$.

Step 3: Finally, we will prove this lemma on X. Let $\{(p_i, q_i) \in X \times X; i = 1, ..., n\}$ such that if $i \neq j$, then $p_i \neq p_j$ and $q_i \neq q_j$. The idea is that, for all i, $p_i \neq q_j$, for all j. Fix $\varepsilon > 0$ such that, for each i, $d(p_i, q_i) < \varepsilon$. Consider $\phi_i : B_\varepsilon(0) \subset \mathbb{R}^m \to B_\varepsilon(p_i) \subset X$ a chart such that $\phi_i(0) = p_i$. We have two cases, if $\{p_j, q_j; j \neq i\} \cap B_\varepsilon(p_i) \neq \varnothing$, then we need to be careful to not perturbate these points. If we don't have any of these other points in $B_\varepsilon(p_i)$, we can take directly the homeomorphism $\varphi_{\phi_i^{-1}(q_i)} : \mathbb{R}^m \to \mathbb{R}^m$ obtained above. Observe that $\varphi_{\phi_i^{-1}(q_i)}(0) = \phi_i^{-1}(q_i)$, $\varphi_{\phi_i^{-1}(q_i)}(B_\varepsilon(0)) = B_\varepsilon(0)$ and $\varphi_{\phi_i^{-1}(q_i)}$ is the identity for $|x| \geq \varepsilon$. Set $\psi_{q_i} : X \to X$ defined by

$$\psi_{q_i}(x) = \begin{cases} x & \text{if } x \notin B_\varepsilon(p_i), \\ \phi_i \circ \varphi_{\phi_i^{-1}(q_i)} \circ \phi_i^{-1}(x) & \text{if } x \in B_\varepsilon(p_i). \end{cases}$$

Thus ψ_{q_i} is an homeomorphism of X such that $\psi_{q_i}(p_i) = q_i$ and $d(\psi_{q_i}(x), x) < 2\varepsilon$. Also note that $d(\psi_{q_i}^{-1}(x), x) < 2\varepsilon$.

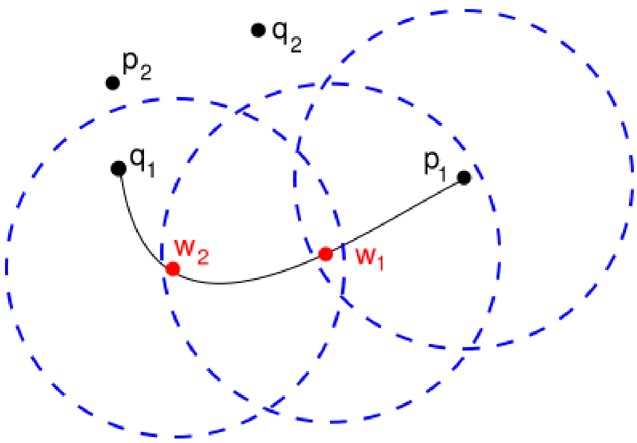

Figure 1 – Here we are seeing the points in the manifold X as points in $\mathbb{R}^m$.

If there exists some p_j or q_j, $j \neq i$, such that $p_j \in B_\varepsilon(p_i)$ or $q_j \in B_\varepsilon(p_i)$, then we need to take auxiliary homeomorphisms. Firstly, take γ a path in $B_\varepsilon(0)$ such that $\gamma(0) = 0$,

$\gamma(1) = \phi_i^{-1}(q_i)$ and $d(z,\gamma) \geq \delta$, for some $\delta > 0$ and for all $z \in \phi_i^{-1}\Big(\{p_j, q_j; j \neq i\}\Big)$, then exists $w_0, w_1, ..., w_t \in \gamma$ with $d(w_i, w_{i+1}) < \delta$, $w_0 = 0$ and $w_t = \phi_i^{-1}(q_i)$. Now, define

$$\widetilde{\varphi}_{w_{i+1}}(x) = \varphi_{w_{i+1}-w_i}(x - w_i) + w_i$$

that is clearly an homeomorphism such that $\widetilde{\varphi}_{w_{i+1}}(w_i) = w_{i+1}$ and $\widetilde{\varphi}_{w_{i+1}}(x) = x$ if $|x - w_i| \geq \delta$, for all $i \in \{0, ..., t-1\}$.

Note that here we used the homeomorphism φ_q previously define for δ in the place of ε. Set $\psi_{q_i} : X \to X$ defined by

$$\psi_{q_i}(x) = \begin{cases} x & \text{if } x \notin B_\varepsilon(p_i), \\ \phi_i \circ \widetilde{\varphi}_{w_t} \circ ... \circ \widetilde{\varphi}_{w_1} \circ \phi_i^{-1}(x) & \text{if } x \in B_\varepsilon(p_i). \end{cases}$$

Therefore ψ_{q_i} is an homeomorphism of X such that $\psi_{q_i}(p_i) = q_i$, $d(\psi_{q_i}(x), x) < 2\varepsilon$ and $d(\psi_{q_i}^{-1}(x), x) < 2\varepsilon$. Lastly, define $\psi : X \to X$ by

$$\psi := \psi_{q_n} \circ \psi_{q_{n-1}} \circ ... \circ \psi_{q_1},$$

then ψ is clearly an homeomorphism of the manifold X such that $\psi(p_i) = q_i$, for $1 \leq i \leq n$, $d(\psi(x), x) < 2\varepsilon$ and $d(\psi^{-1}(x), x) < 2\varepsilon$ for every $x \in X$. □

The next lemma is proved in Hurley's paper [5].

Lemma 3.2. Suppose that $w : (0, \infty) \to [0, \infty)$ is continuous, nonincreasing, $w(t) \to \infty$ as t decreases to 0, and that $w(t) = 0$ for all $t \geq \delta$, $\delta > 0$. Define $\alpha(y) = e^{-w(|y|)}y$ for $y \neq 0$ and $\alpha(0) = 0$. Then α is an homeomorphism on $\mathbb{R}^m$, and $\alpha(y) = y$ whenever $|y| \geq \delta$.

Proof. We only need to prove continuity at 0. If $(y_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^m$ such that $y_n \to 0$, then is easy to see that $e^{-w(|y_n|)} \to 0$. Thus, $|\alpha(y_n)| = e^{-w(|y_n|)}|y_n| \to 0$.

To see surjectivity, observe that, for $v \in \mathbb{R}^m, |v| = 1$ and $R := \{tv; t \in (0, \infty)\}$, $\alpha(R) \subset R$. Let $w \in R$. If $|w| \geq \delta$, then $\alpha(w) = w$. For $|w| < \delta$, we will study the function $h(t) = e^{-w(t)}t$, $t \in (0, \infty)$. If $\varepsilon > 0$ is a arbitrary small number, then $h(\varepsilon) = e^{-w(\varepsilon)}\varepsilon < \varepsilon$. And for a sequence $t_n \to \delta$, $h(t_n) \to \delta$. By the intermediate value theorem, exists $t_w \in (0, \delta)$ such that $h(t_w) = |w|$. Then $\alpha(t_w v) = h(t_w)v = |w|v = w$.

To verify that it is injective, suppose that exists y, z in some R such that $\alpha(y) = \alpha(z)$ and $y = \gamma z$ for some positive constant γ, which we may assume is bigger than 1. This means that $|y| > |z|$, so that $-w(|y|) \geq -w(|z|)$. Therefore we have the following contradiction,

$$|\alpha(y)| = e^{-w(|y|)}|y| = e^{-w(|y|)}\gamma|z| > e^{-w(|z|)}|z| = |\alpha(z)|.$$

For the homeomorphism part, we will prove that α is a closed function. Let $F \subset \mathbb{R}^m$ be a closed set, $B_\delta := \{x \in \mathbb{R}^m; |x| \leq \delta\}$ and $C_\delta := \{x \in \mathbb{R}^m; |x| \geq \delta\}$. Then we have that

$$\alpha(F) = \alpha\Big((F \cap B_\delta) \cup (F \cap C_\delta)\Big) = \alpha(F \cap B_\delta) \cup \alpha(F \cap C_\delta) = \alpha(F \cap B_\delta) \cup (F \cap C_\delta).$$

Since $F \cap B_\delta$ is a compact set, then $\alpha(F \cap B_\delta)$ is also compact, and $F \cap C_\delta$ is clearly closed. Thus $\alpha(F)$ is a closed set. □

3.1 Density of functions with periodic attracting points

We know that Homeo(X, d) is a complete metric space if endowed with the metric

$$D(f,g) = \max_{x \in X}\{d(f(x), g(x)), d(f^{-1}(x), g^{-1}(x))\}$$

and we will use this metric from now on.

Proposition 3.2. If $q \in CR(F)$ and $\varepsilon > 0$, then there is $f \in$ Homeo(X, d) with $D(F, f) < \varepsilon$ and such that q is a periodic attracting point for f.

Proof. Let F be an homeomorphism of X, $q \in CR(F)$ and $\varepsilon > 0$. We will show first that there is $h \in$ Homeo(X, d) with $D(F, h) < \varepsilon/2$ which has q as a periodic point.

Since F^{-1} is a continuous function and X is compact, then for $\varepsilon/2$ exists some $\rho > 0$ such that if $d(x_1, x_2) < \rho$, then $d(F^{-1}(x_1), F^{-1}(x_2)) < \varepsilon/2$, for all $x_1, x_2 \in X$.

Let $\delta > 0$ such that $\delta < \min\{\rho/2, \varepsilon/4\}$ and select a δ-chain $q_0, q_1, ..., q_n$ with $q_0 = q_n = q$. If two of the $q_i's$ with $0 \leq i < n$ are equal, then we can delete on of them and all the $q_j's$ between them to get a shorter δ-chain from q to q, so there is no loss in generality in assuming that $q_i \neq q_j$ for $0 \leq i < j < n$. For $0 \leq i < n$, let $p_{i+1} = f(q_i)$, so all the $p_i's$ are distinct and $d(q_i, p_i) < \delta$ for $1 \leq i \leq n$.

By lemma 3.1, there is an homeomorphism ψ of X such that $\psi(p_i) = q_i$ for each i, $d(x, \psi(x)) < 2\delta$ and $d(x, \psi^{-1}(x)) < 2\delta$, for every $x \in X$. Setting $h = \psi \circ F$, we have that h is an homeomorphism of X such that

$$d(F(x), \psi(F(x))) = d(F(x), h(x)) < 2\delta < \varepsilon/2,$$

and

$$d(F^{-1}(x), h^{-1}(x)) = d(F^{-1}(x), F^{-1}(\psi^{-1}(x))) < \varepsilon/2,$$

because $d(x, \psi^{-1}(x)) < 2\delta < \rho$. Therefore, h is our desired homeomorphism that have q as a periodic point of period n.

Now we will show that there is an homeomorphism f such that $D(h, f) < \varepsilon/2$ which has q as an attracting periodic point of period n.

Consider $\theta \in (0, \delta)$ such that $B_\theta(h^i(q)) \cap B_\theta(h^j(q)) = \varnothing$, $0 < i < j < n$. Since each h^i is a continuous function, for $0 < i < n$, exists some $\kappa > 0$ such that if $x \in B_\kappa(q)$, then $h^i(x) \in B_{\theta/2}(h^i(q))$ and $B_\kappa(q) \cap B_\theta(h^i(q)) = \varnothing$. For $\lambda := \min\{\theta, \kappa\}$, denote $V_i := B_\lambda(h^i(q))$, $0 \leq i < n$, and consider $W \subset V_0$ be a neighborhood of q such that W is a disk and $h^n(W) \subset B_{\lambda/2}(q)$, observe that $h^i(W) \subset V_i$. Let $\phi_i : B_1(0) \subset \mathbb{R}^m \to V_i \subset X$ be charts such that $\phi_i(0) = h^i(q)$, $0 \leq i < n$. Let $\mu \in (0, \lambda)$ such that $B_\mu(q) \subset W$ is a strictly inclusion, that is, $2\mu <$ diam(W).

For $0 < i < n$, consider $\delta_i \in (0, 1)$ the smallest positive number such that $\phi_i^{-1}(h^i(W)) \subset B_{\delta_i}(0)$ and $\varepsilon_i \in (0, \delta_i)$ such that

$$B_{\varepsilon_i}(0) \subset \phi_i^{-1}\Big(B_\mu(h^i(q)) \cap h^i(W)\Big).$$

Now define the following homeomorphism $\tau_i : B_1(0) \to B_1(0)$ as

$$\tau_i(\delta_i, \varepsilon_i, y) = \begin{cases} \frac{\varepsilon_i}{\delta_i} y, & \text{if } |y| \leq \delta_i; \\ \left[\frac{(1-\varepsilon_i)|y|}{1-\delta_i} + \frac{(\varepsilon_i - \delta_i)}{1-\delta_i}\right] \frac{y}{|y|}, & \text{if } \delta_i < |y| \leq 1. \end{cases}$$

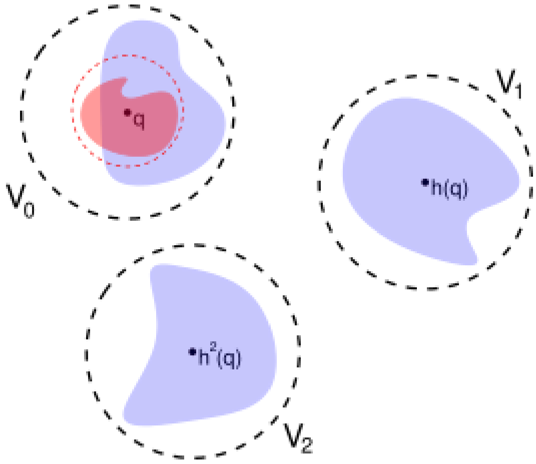

Figure 2 – In blue, there is W and its iterates by h. In red, we can see $h^n(W)$ inside the ball $B_{\lambda/2}(q)$, for $n = 3$.

Observe that τ_i is such that $\tau_i\Big(B_{\delta_i}(0)\Big) = B_{\varepsilon_i}(0)$, then $\tau_i\Big(\phi_i^{-1}(h^i(W))\Big) \subset B_{\varepsilon_i}(0)$ which implies that

$$\phi_i \circ \tau_i \circ \phi_i^{-1}(h^i(W)) \subset \phi_i(B_{\varepsilon_i}(0)) \subset B_\mu(h^i(q)).$$

Define $\gamma_i : V_i \to V_i$ as $\gamma_i = \phi_i \circ \tau_i \circ \phi_i^{-1}$ and $W_1 = \gamma_1(h(W))$. Since $W_1 \subset B_\mu(h(q))$, we have that $\operatorname{diam}(W_1) < \operatorname{diam}(W)$. Note that $W_1 \subset h(W)$ because $\phi_1(B_{\varepsilon_1}(0)) \subset h(W)$, then $h(W_1) \subset h^2(W)$. For $2 \leq i < n$ we will define W_i as $W_i = \gamma_i(h(W_{i-1}))$. We have that $\operatorname{diam}(W_i) < \operatorname{diam}(W)$ because

$$W_i = \gamma_i(h(W_{i-1})) \subset \gamma_i(h^n(W)) \subset B_\mu(h^i(q)) \cap h^i(W).$$

Observe that $W_{n-1} \subset h^{n-1}(W)$, therefore $h(W_{n-1}) \subset h^n(W)$.

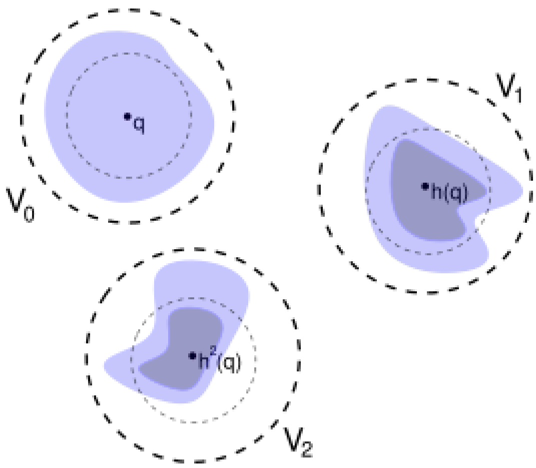

Figure 3 – Here W is drew differently to illustrate better what we are doing: the ball inside W is $B_\mu(q)$ and the neighborhoods in dark blue are W_1 and W_2. See that $\operatorname{diam}(W_i) < \operatorname{diam}(W)$, for $i \in \{1, 2\}$, if q is 3-periodic.

Define $\sigma : \phi_0^{-1}(W) \subset B_1(0) \to B_1(0)$ as $\sigma = \phi_0^{-1} \circ h^n \circ \phi_0$. Let $\eta = \sup_{y \in \phi_0^{-1}(W)} |\sigma(y)|$. Note that $\eta \leq 1$. For $r > 0$ we will define the following auxiliary function a)

$$M_1(r) = \begin{cases} \min\{|y|; y \in \phi_0^{-1}(W) \text{ and } |\sigma(y)| \geq r\}, & r \in (0,\eta) \\ c_1, & r \in [\eta, \infty), \end{cases}$$

where $c_1 = \lim_{s\to\eta^-} M_1(s)$. We have that c_1 is well defined because, for $r \in (0,\eta)$, M_1 is limited by 1 and nondecreasing. In fact, for $r_1 > r_2$, note that

$$\{y; y \in \phi_0^{-1}(W) \text{ and } |\sigma(y)| \geq r_1\} \subset \{y; y \in \phi_0^{-1}(W) \text{ and } |\sigma(y)| \geq r_2\},$$

then $M_1(r_1) \geq M_1(r_2)$. Therefore M_1 is a continuous function. Note that

$$M_1(\frac{|\sigma(z)|}{k}) \leq |z|,$$

for all $z \in \phi_0^{-1}(W)$ and any $k \in \mathbb{N}$, because $z \in \{y \in \phi_0^{-1}(W); |\sigma(y)| \geq |\sigma(z)| \geq |\sigma(z)|/k\}$. We also have that $c_1 > 0$, because if $c_1 = 0$, then $M_1(r) = 0$ for every $r \in (0,\eta)$. This means that $|\sigma(0)| > 0$, contradiction.

Note that $b_1 := \lim_{r\to 0} M_1(r)$ is such that $b_1 = 0$. If $b_1 > 0$, note that, for $r \in (0,\infty)$, $M_1(r) \geq b_1$, then for $|y| < b_1$, $|\sigma(y)| < r$, for every $r > 0$. This means that exists $y \neq 0$ such that $\sigma(y) = 0$, absurd.

The intuition is that if σ is pushing points away from the origin, then $M_1(r)$ should be smaller than r.

b) $M_2(r) = \psi(r)M_1(r)$, where ψ is a continuous positive function satisfying $\psi(r) = 1$ if $r \leq \eta/2$, ψ is constant on $[\eta,\infty)$ and is chosen so that $M_2(\eta) = k$, that is, $\psi(\eta) = k/c_1$, k will be later a positive integer big enough, so here k will be any positive integer. Precisely, we have that

$$\psi(r) = \begin{cases} 1, & \text{if } r \leq \eta/2; \\ \Big(-2 + \frac{2k}{c_1}\Big)\frac{x}{\eta} + 2 - \frac{k}{c_1}, & \text{if } \eta/2 \leq r \leq \eta; \\ k/c_1, & \text{if } \eta \leq r. \end{cases}$$

c) $M_3(r) = r/M_2(r)$ for $r \leq \eta$ and $M_3(r) = \eta/k$ for $r \geq \eta$; observe that M_3 is also continuous.

d) $M(r) = \max\{M_3(s); s \geq r\}$, so that $M(r) \geq M_3(r)$ and M is nonincreasing.

Define $w(r) = \ln\Big(k \cdot M(r)/\eta\Big)$; it is clear that w is continuous, nonincreasing, and equal to 0 for $r \geq \eta$.

Case 1: Suppose that w satisfies all the hypotheses of the lemma 3.2, that is, $w \to \infty$ as $r \to 0$. Let $\alpha : \mathbb{R}^m \to \mathbb{R}^m$ be given by the lemma, that is, $\alpha(y) = e^{-w(|y|)}y$. Consider $\tau_0 : B_1(0) \to B_1(0)$ as $\tau_0 = \tau_0(\delta_0, \varepsilon_0, y)$, where $\delta_0 := \eta$ and $\varepsilon_0 := \eta/2$. Define $\gamma_0 : V_0 \to V_0$ as $\gamma_0 = \phi_0 \circ \alpha \circ \tau_0 \circ \phi_0^{-1}$, and $\beta : X \to X$ as

$$\beta(x) = \begin{cases} x, & \text{if } x \notin \bigcup_{i=0}^{n-1} V_i; \\ \gamma_i(x), & \text{if } x \in V_i. \end{cases}$$

Note that γ_i is the identity on the frontier of V_i, remember we defined $\gamma_i = \phi_i \circ \tau_i \circ \phi_i^{-1}$ for $1 \leq i < n$. Thus, by construction, $\beta \in \text{Homeo}(X, d)$. We also have that $d(x, \beta(x)) < 2\delta$ and $d(x, \beta^{-1}(x)) < 2\delta$ because $\beta(V_i) = V_i$ and $\text{diam}(V_i) < 2\delta$.

In this case our desired function f will be defined as $f = \beta \circ h$. We have that $f^i(W) = W_i$, for $0 < i < n$, then $\text{diam}(f^i(W)) < \text{diam}(W)$. Observe that, for $y \in \phi_0^{-1}(W)$,

$$|(\phi_0^{-1} \circ \gamma_0 \circ h^n)(\phi_0(y))| = |\alpha\Big(\frac{\sigma(y)}{2}\Big)| = e^{-w(|\sigma(y)|/2)}\frac{|\sigma(y)|}{2} = \frac{\eta|\sigma(y)|}{2k \cdot M(|\sigma(y)|/2)}$$

$$\leq \frac{|\sigma(y)|}{2k \cdot M_3(|\sigma(y)|/2)} = \frac{2 \cdot |\sigma(y)| \cdot M_2(|\sigma(y)|/2)}{2k|\sigma(y)|} = \frac{M_1(|\sigma(y)|/2)}{k} \leq \frac{|y|}{k} \leq \frac{1}{k}.$$

Therefore we have that for $k \geq 2$ a positive integer such that $B_{k^{-1}}(0) \subset \phi_0^{-1}(W)$,

$$\phi_0^{-1}(f^n(W)) = \phi_0^{-1}(\gamma_0(h(W_{n-1}))) \subset \phi_0^{-1}(\gamma_0(h^n(W))) \subset B_{k^{-1}}(0).$$

Inductively, we get that, for $\varphi := \gamma_0 \circ h^n$,

$$f^{jn}(W) \subset \varphi^j(W) \subset \phi_0(B_{k^{-j}}(0)).$$

Then we finally get the following property,

$$\bigcap_{j \geq 0} f^{jn}(W) = \{q\}.$$

Thus, q is an periodic attracting point for f with W as a n-absorbing disk. Lastly we will prove that $D(h, f) < \varepsilon/2$. Note that

$$d(h(x), f(x)) = d\Big(h(x), \beta(h(x))\Big) < 2\delta < \varepsilon/2.$$

And

$$d(h^{-1}(x), f^{-1}(x)) = d\Big(h^{-1}(x), h^{-1}(\beta^{-1}(x))\Big) < \varepsilon/2,$$

because $d(x, \beta^{-1}(x)) < 2\delta < \rho$.

Case 2: We now have that $w \to \xi$ or $w \to -\infty$ as $r \to 0$. Suppose that

$$\lim_{r\to 0} \ln\Big(\frac{k \cdot M(r)}{\eta}\Big) = -\infty, \text{ then } \lim_{r\to 0} M(r) = 0.$$

For every $\nu > 0$, exists some $s_1 > 0$ such that if $r \in (0, s_1)$, then $m(r) \in (0, \nu)$. Since $M(r) \geq M_3(r)$, then $M_3(r) \in (0, \nu)$. Then, for $0 < r < \min\{s_1, \eta/2\}$,

$$M_3(r) = \frac{r}{M_1(r)} \in (0, \nu), \text{ and therefore } r < \nu \cdot M_1(r).$$

Now supposing that

$$\lim_{r\to 0} \ln\Big(\frac{k \cdot M(r)}{\eta}\Big) = \xi, \text{ then } \lim_{r\to 0} M(r) = \frac{\eta \cdot e^{\xi}}{k}.$$

Since M is nonincreasing, for every $\nu > 0$, exists $s_2 > 0$ such that if $r \in (0, s_2)$, then $M(r) \in (c-\nu, c)$, with $c = \eta \cdot e^{\xi}/k$. Applying the same arguments we have that $r < c \cdot M_1(r)$, for $0 < r < \min\{s_2, \eta/2\}$. Therefore, let $s \in (0, \eta/2)$ be such that for $0 < r < s$, then $r < c \cdot M_1(r)$.

Let $t \geq 2$ be a positive integer such that $B_{t^{-1}}(0) \subset \phi_0^{-1}(W)$ and $l \in \mathbb{N}$ such that $t \cdot c \leq l$, $t \leq l$ and $\eta < s \cdot l$. Then

$$\frac{|\sigma(y)|}{l^2} < \frac{c}{l} \cdot M_1\Big(\frac{|\sigma(y)|}{l}\Big) \leq \frac{c}{l}|y| \leq \frac{|y|}{t} \leq \frac{1}{t}.$$

Consider $\tau_0 : B_1(0) \to B_1(0)$ as $\tau_0 = \tau_0(\delta_0, \varepsilon_0, y)$, where $\delta_0 := \eta$ and $\varepsilon_0 := \eta/l^2$ and define $\gamma_0 : V_0 \to V_0$ as $\gamma_0 = \phi_0 \circ \tau_0 \circ \phi_0^{-1}$, and $\beta : X \to X$ as

$$\beta(x) = \begin{cases} x, & \text{if } x \notin \bigcup_{i=0}^{n-1} V_i; \\ \gamma_i(x), & \text{if } x \in V_i. \end{cases}$$

Note that γ_i is the identity on the frontier of V_i, remember we defined $\gamma_i = \phi_i \circ \tau_i \circ \phi_i^{-1}$ for $0 \leq i < n$. Thus, by construction, $\beta \in \text{Homeo}(X, d)$. We also have that $d(x, \beta(x)) < 2\delta$ and $d(x, \beta^{-1}(x)) < 2\delta$ because $\beta(V_i) = V_i$ and $\text{diam}(V_i) < 2\delta$.

As we did on the other case, we define f as $f = \beta \circ h$. Note that $f^i(W) = W_i$, for $0 < i < n$, then $\text{diam}(f^i(W)) < \text{diam}(W)$, and clearly $D(h, f) < \varepsilon/2$.

By construction, $\tau_0\Big(B_\eta(0)\Big) \subset B_{t^{-1}}(0)$. Then

$$\phi_0^{-1}(f^n(W)) = \phi_0^{-1}(\gamma_0(h(W_{n-1}))) \subset \phi_0^{-1}(\gamma_0(h^n(W))) \subset B_{t^{-1}}(0).$$

Inductively, we get that, for $\varphi := \gamma_0 \circ h^n$,

$$f^{jn}(W) \subset \varphi^j(W) \subset \phi_0(B_{t^{-j}}(0)), \text{ therefore } \bigcap_{j \geq 0} f^{jn}(W) = \{q\}.$$

Thus W is the n-absorbing disk for the periodic attracting point q of f. □

Remark 3.2. Given W a n-absorbing disk for a periodic attracting point q, we know, by Remark 3.1, that there exists $J \in \mathbb{N}$ such that $f^{Jn}(W)$ is a n-absorbing disk for q and $\text{diam}\Big(f^{Jn}(W)\Big) < \varepsilon$, given $\varepsilon > 0$. We can apply the last theorem and get a new function g sufficiently close to f so that $\text{diam}(g^i\Big(f^{Jn}(W)\Big)) < \text{diam}\Big(f^{Jn}(W)\Big)$. We observe this because we are interested in arbitrarily small n-absorbing disks that defines a periodic attracting point q.

3.2 Residual set

For this section we will need some background on Baire spaces (see Appendix B). When a set is open in $\text{Homeo}(X, d)$ we say that it is C^0-open, and the same when the set is dense. We will use this type of notation when dealing with homeomorphisms of a compact space.

We say that having f a periodic attracting point is a C^0*quasi-robust property*. Precisely, there is some $\delta > 0$ such that for every $g \in \text{Homeo}(X, d)$ where $D(f, g) < \delta$ the following conditions hold:

1. if B is a n-absorbing disk for $f \in \text{Homeo}(X, d)$, then B is n-absorbing for g.

   Indeed, since $f^n(B) \subsetneq \text{int}(B)$, exists some $\rho > 0$ such that $d(f^n(B), \partial B) > \rho$. By lemma B.1, there exists some $\delta_1 > 0$ such that if $D(f, g) < \delta_1$, then $D(f^n, g^n) < \rho/2$. Thus $g^n(B) \subsetneq \text{int}(B)$.

2. if B is a n-absorbing disk for $f \in \text{Homeo}(X, d)$, then for every $1 \leq i < n$ the disk $f^i(B)$ is n-absorbing for g.

   We know that $f^i(B)$ still a n-absorbing disk for f, then there is some ρ_i such that $d\Big(f^n(f^i(B)), \partial f^i(B)\Big) > \rho_i$. For $0 < \eta < \min\{\rho_1, ..., \rho_{n-1}\}$, there is some $\delta_2 > 0$ such that if $D(f, g) < \delta_2$, then $D(f^n, g^n) < \eta/2$. Therefore, $g^n(f^i(B)) \subsetneq \text{int}\Big(f^i(B)\Big)$, for every $1 \leq i < n$.

3. for every $\varepsilon > 0$ we may find some $J \geq 0$ such that $f^{Jn}(B)$ has diameter smaller than ε and is a n-absorbing disk for g.

   By what we have discussed in relation (3.1), there is some $J \geq 0$ such that $f^{Jn}(B) \subset B_\varepsilon(p)$, and $f^{Jn}(B)$ still a n-absorbing disk for f. Therefore there is some $\rho' > 0$ such that $d(f^{(J+1)n}(B), \partial f^{Jn}(B)) > \rho'$. We know that there is some $\delta_3 > 0$ such that if $D(f, g) < \delta_3$, then $D(f^n, g^n) < \rho'/2$. Thus $g^n\Big(f^{Jn}(B)\Big) \subsetneq \text{int}\Big(f^{Jn}(B)\Big)$.

   Now we only have to set $\delta < \min\{\delta_1, \delta_2, \delta_3\}$ and we are done.

We also proved that the set of functions that have a n-absorbing disk of diameter ε is C^0-open and dense, and we are ready to prove the following lemma.

Lemma 3.3. Residual homeomorphisms have absorbing disks of arbitrarily small diameter.

Proof. Since $\text{Homeo}(X, d)$ is a complete metric space with respect to the metric D, then it is a Baire space. Now just take the intersection of the sets

$$\mathcal{H} := \bigcap_{n\in\mathbb{N}} \mathcal{H}_n = \{f \in \text{Homeo}(X, d); f \text{ has an absorbing disk with diameter at most } 1/n\}$$

and we finish the proof. □

4 Pseudo-horseshoes

For this chapter and from now on we will consider (X, d) a compact smooth boundaryless manifold with dimension greater than one. Take in $\mathbb{R}^m$ the maximum norm, that is,

$$\|(x_1, ..., x_m)\| = \max_{1 \leq i \leq m} |x_i|.$$

For $r > 0$ and $x \in \mathbb{R}^m$, consider $B_r^m(x) := \{y \in \mathbb{R}^m; \|x - y\| \leq r\}$. Also define, for $1 \leq j \leq m$, $\pi_j : \mathbb{R}^m \to \mathbb{R}^j$ the projection on the first j coordinates, that is, if $x = (x_1, ..., x_m) \in \mathbb{R}^m$, then $\pi_j(x) = (x_1, ..., x_j) \in \mathbb{R}^j$.

Definition 4.1. Let $\delta > 0$, $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$ in $\mathbb{R}^m$. For a fixed positive integer $N \geq 2$, consider a partition $\mathcal{P}$ of the interval $B_\delta^1(0) = (-\delta, \delta)$ such that $\mathcal{P} = \{t_0, t_1, ..., t_N\}$, with $0 \notin \mathcal{P}$, $t_0 = -\delta$ and $t_N = \delta$. Take $U \subset \mathbb{R}^m$ an open set containing $B_{2\delta}^m(x)$. We say that a homeomorphism $\tau : U \to \mathbb{R}^m$ has a *pseudo-horseshoe of type N at scale δ connecting x to y* if the following conditions hold:

1. $\tau(x) = y$;
2. $\tau\Big(B_\delta^m(x)\Big) \subset \operatorname{int}\Big(B_\delta^{m-1}(\pi_{m-1}(y))\Big) \times (y_m - 2\delta, y_m + 2\delta)$;
3. For i an even integer,
$$\tau\left(B_\delta^{m-1}(\pi_{m-1}(x)) \times \{x_m + t_i\}\right) \subset \operatorname{int}(B_\delta^{m-1}(\pi_{m-1}(y))) \times (y_m - 2\delta, y_m - \delta);$$
4. For j an odd integer,
$$\tau\left(B_\delta^{m-1}(\pi_{m-1}(x)) \times \{x_m + t_j\}\right) \subset \operatorname{int}(B_\delta^{m-1}(\pi_{m-1}(y))) \times (y_m + \delta, y_m + 2\delta);$$
5. For each $i \in \{0, ..., N-1\}$, the intersection
$$V_i = B_\delta^m(y) \cap \tau\left(B_\delta^{m-1}(x) \times [x_m + t_i, x_m + t_{i+1}]\right)$$
is connected and satisfies:
   a) $V_i \cap (B_\delta^{m-1}(y) \times \{-\delta\}) \neq \varnothing$;
   b) $V_i \cap (B_\delta^{m-1}(y) \times \{\delta\}) \neq \varnothing$;
   c) each connected component of $V_i \cup \partial B_\delta^m(y)$ is simply connected.

Observe that, when $x = y$, τ has a compact invariant subset similar to a horseshoe, which is semiconjugate to a subshift of a finite type, so we define generally as pseudo-horseshoe. We call each V_i by *vertical strip* of τ, and we denote that collection of vertical strips by $\mathcal{V}_\tau$. Similarly we define the *horizontal strip* of τ by $H_i = \tau^{-1}(V_i)$.

This definition is both topological and geometrical, because we consider homeomorphisms and assume that a certain scale is preserved and identify a preferable vertical direction.

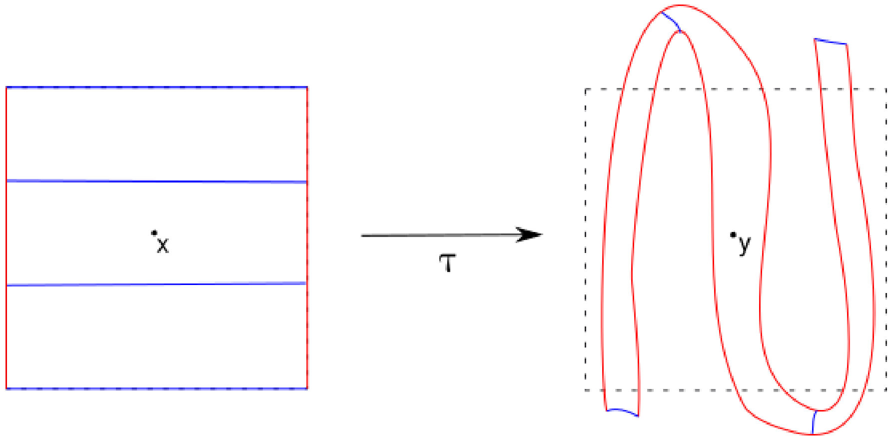

Figure 4 – An example of a pseudo-horseshoe of type 3 at scale δ connecting x to y.

Definition 4.2. Let $\varepsilon > 0$ and $\tau : U \to \mathbb{R}^m$ a homeomorphism with a pseudo-horseshoe of type N at scale δ connecting x to y. The pseudo-horseshoe associated to τ is said to be *ε-separating* if we may choose the collection $\mathcal{V}_\tau$ so that $\inf\{\|a-b\|; a \in V_i, b \in V_j\} > \varepsilon$ for every $i \neq j$, that is, the Hausdorff distance between distinct vertical strips is bigger than ε.

Following proposition will show us how to construct such homeomorphisms having pseudo-horseshoes. It is similar to what is proved in [11] and most of the ideas used here are made there. For $N \in \mathbb{N}$ such that $N \geq 2$, we define the set $\mathcal{S}_N$ by the following: we say that $g \in \mathcal{S}_N$ if X have charts $\varphi : U \subset \mathbb{R}^m \to X$, $\psi : V \subset \mathbb{R}^m \to X$ such that $\varphi \circ g \circ \psi^{-1} : U \to \mathbb{R}^m$ has a pseudo-horseshoe of type N at scale δ connecting 0 to itself.

Proposition 4.1. $\mathcal{S}_N$ is a open and dense set of $\mathrm{Homeo}(X, d)$.

Proof. We will prove first the density of such homeomorphisms. Let $\varepsilon > 0$ and f be a homeomorphism of X. By the uniform continuity of f, exists some $\kappa > 0$ such that if $d(x_1, x_2) < \kappa$, then $d(f(x_1), f(x_2)) < \kappa$ and $d(f^{-1}(x_1), f^{-1}(x_2)) < \varepsilon$, for all $x_1, x_2 \in X$. Consider $\lambda > 0$ such that $\lambda < \min\{\kappa, \varepsilon\}$.

For $y \in X$, consider the following charts $\varphi : U \subset \mathbb{R}^m \to X$ and $\psi : V \subset \mathbb{R}^m \to X$ such that $\varphi(0) = y$, $\psi(0) = f(y)$ and $\mathrm{diam}\big(\psi(V)\big) < \lambda/2$. We can also suppose that $\mathrm{int}\Big(B^m_{2\delta}(0)\Big)$ is contained in both U and V. By continuity of f we can say that $f \circ \varphi(U) \subset \psi(V)$, then $\psi^{-1} \circ f \circ \varphi(U) \subset V$. For N a positive integer greater or equal than two and a partition $\mathcal{P}$ of the interval $B^1_\delta(0) = (-\delta, \delta)$ such that $\mathcal{P} = \{t_0, t_1, ..., t_N\}$, with $0 \notin \mathcal{P}$, $t_0 = -\delta$ and $t_N = \delta$, it is easy to suppose that there exists a homeomorphism ρ of $\mathbb{R}^m$ satisfying:

1. $\rho(0) = 0$;
2. ρ is the identity in ∂V;
3. $\rho\Big(\psi^{-1} \circ f \circ \varphi(\{(0, ..., 0) \times B^1_\delta(0)\})\Big) \subset \mathrm{int}(B^{m-1}_\delta(0)) \times (-2\delta, 2\delta)$;
4. For i an even integer,

$$\rho\left(\psi^{-1} \circ f \circ \varphi\,(0, ..., 0, t_i)\right) \in \mathrm{int}(B^{m-1}_\delta(0)) \times (-2\delta, -\delta);$$

5. For j an odd integer,

$$\rho\left(\psi^{-1}\circ f\circ\varphi\,(0,...,0,t_j)\right)\in \operatorname{int}(B_\delta^{m-1}(0))\times(\delta,2\delta).$$

Take $r\in(0,\delta)$ sufficiently small such that

(a) $\rho\Big(\psi^{-1}\circ f\circ\varphi(B_r^{m-1}(0)\times B_\delta^1(0))\Big)\subset \operatorname{int}(B_\delta^{m-1}(0)\times(-2\delta,2\delta))$

(b) For i an even integer,

$$\rho\left(\psi^{-1}\circ f\circ\varphi\left(B_r^{m-1}(0)\times\{t_i\}\right)\right)\subset \operatorname{int}(B_\delta^{m-1}(0))\times(-2\delta,-\delta);$$

(c) For j and odd integer,

$$\rho\left(\psi^{-1}\circ f\circ\varphi\left(B_r^{m-1}(0)\times\{t_j\}\right)\right)\subset \operatorname{int}(B_\delta^{m-1}(0))\times(\delta,2\delta).$$

We can easily construct a homeomorphism σ of $B_{2\delta}^m(0)$ satisfying supp $\sigma\subset \operatorname{int}(B_{2\delta}^m(0))$ and $\sigma\Big(B_\delta^{m-1}(0)\times\{x\}\Big)=B_r^{m-1}(0)\times\{x\}$ for every $x\in B_\delta^1(0)$. We will now define a homeomorphism g of X by

$$g(x)=\begin{cases}\psi\circ\rho\circ\psi^{-1}\circ f\circ\varphi\circ\sigma\circ\varphi^{-1}\circ f^{-1}(x), & \text{if } x\in\psi(V);\\ x, & \text{if } x\notin\psi(V).\end{cases}$$

From what we constructed, we can also suppose that the item (5) of Definition 4.1 holds for ρ, so this homeomorphism have a pseudo-horseshoe of type N at scale δ connecting 0 to itself. Now we just need to compose it with f. Since $\operatorname{diam}\Big(\psi(V)\Big)<\lambda/2$, then $d(g(x),x)<\lambda$ and $d(g^{-1}(x),x)<\lambda$. By one side we have $d(g(f(x)),f(x))<\varepsilon$ and by the other side $d(f^{-1}(g^{-1}(x)),f^{-1}(x))<\varepsilon$, because $\lambda<\kappa$. This proves that $D(g\circ f,f)<\varepsilon$.

To prove the open property we just need to choose some small $\theta_f>0$ such that a perturbation of ρ by θ_f doesn't destroy all the properties of the pseudo-horseshoe, and we can calculate that number by taking the minimum distance between some key sets of the pseudo-horseshoe. □

Definition 4.3. Let $f\in \operatorname{Homeo}(X,d)$ and constants $0<\alpha<1$, $\delta>0$, $\varepsilon>0$ sufficiently small and $n\in\mathbb{N}$. Then f has a $(\delta,\varepsilon,n,\alpha)$-*pseudo-horseshoe* if there is a pairwise disjoint family of open subsets U_i, $i\in\{0,1,...,n-1\}$, of X so that

$$f(U_i)\cap U_{(i+1)\bmod n}\neq\varnothing,$$

for all i, and a collection ϕ_i of homeomorphisms

$$\phi_i: B_1^m(0)\subset\mathbb{R}^m\to U_i\subset X$$

satisfying, for every $0\le i\le n-1$, the following statements:

1. $(f\circ\phi_i)\Big(B_{2\delta}^m(0)\Big)\subset U_{(i+1)\bmod n}$;
2. The homeomorphism

$$\psi_i: B_1^m(0)\to\mathbb{R}^m \text{ defined as } \psi_i=\phi_{(i+1)\bmod n}^{-1}\circ f\circ\phi_i$$

has a pseudo-horseshoe of type $\lfloor(1/\varepsilon)^{\alpha\cdot m}\rfloor$ at scale δ connecting $x=0$ to itself and such that:

a) there are families $\{V_{i,j}\}_j$ and $\{H_{i,j}\}_j$ of vertical and horizontal strips, respectively, with $j \in \{1, 2, ..., \lfloor (1/\varepsilon)^{\alpha \cdot m} \rfloor\}$, such that $H_{i,j} = \psi_i^{-1}(V_{i,j})$;

b) for every $j_1 \neq j_2 \in \{1, 2, ..., \lfloor (1/\varepsilon)^{\alpha \cdot m} \rfloor\}$ we have

$$\min\left\{ \inf\{\|a - b\|\,; a \in V_{i,j_1}, b \in V_{i,j_2}\}, \inf\{\|z - w\|\,; z \in H_{i,j_1}, w \in H_{i,j_2}\}\right\} > \varepsilon.$$

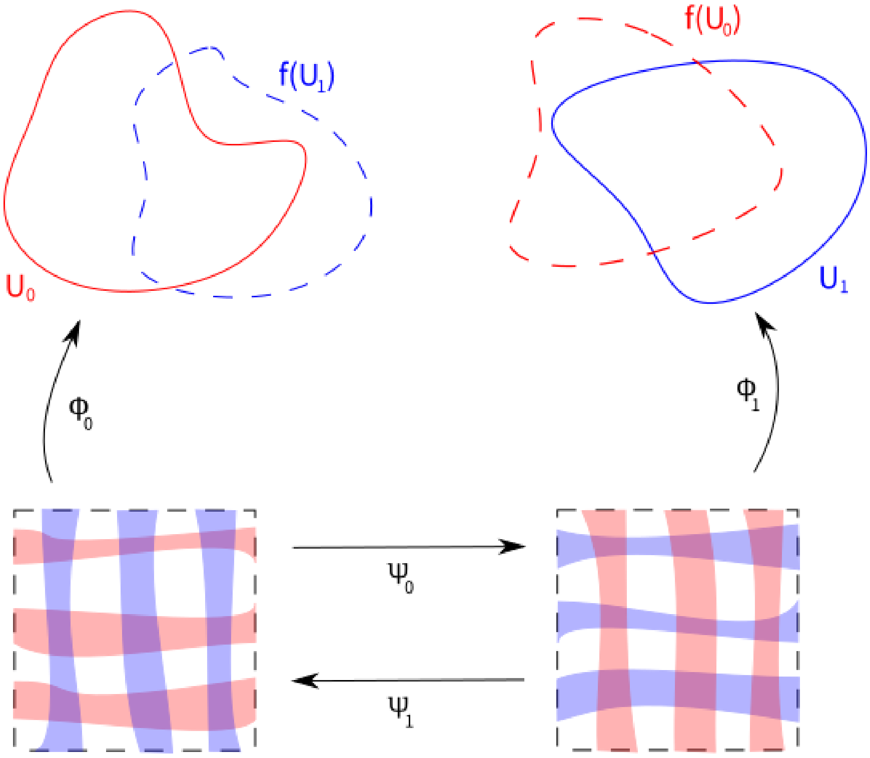

Figure 5 – An example of a $(\delta, \varepsilon, n, \alpha)$-pseudo-horseshoe for $n = 2$.

Regarding the parameters that identify the pseudo-horseshoe, we note that ε is the scale at which a large number of finite orbits is separated, which is inversely proportional to ε and involves α; and α is conditioned by the space in the manifold needed to build the convenient amount of ε-separated points.

Remark 4.1. Observe that if we want to create $\lfloor (1/\varepsilon)^{\alpha \cdot m} \rfloor$ points that are ε-separated inside a ball with diameter 2δ we need that $(2\lfloor (1/\varepsilon)^{\alpha \cdot m} \rfloor - 1)^m \cdot \varepsilon^m < (2\delta)^m$. Thus,

$$\sqrt[m]{\left(\frac{1}{\varepsilon}\right)^{\alpha \cdot m}} \cdot \varepsilon < 2\delta \text{ or, equivalently, } 0 < \varepsilon < \sqrt[1-\alpha]{2\delta}. \tag{4.1}$$

That is, α is in the interval $(0, 1)$.

Definition 4.4. We say that $f \in \mathrm{Homeo}(X, d)$ has a *coherent* $(\delta, \varepsilon, n, \alpha)$*-pseudo-horseshoe* if the pseudo-horseshoe satisfies the following extra condition:

3. For every $0 \leq i \leq n - 1$ and every $j_1 \neq j_2 \in \{1, 2, ..., \lfloor (1/\varepsilon)^{\alpha \cdot m} \rfloor\}$, the horizontal strip $H_{(i+1) \bmod n, j_1}$ crosses the vertical strip V_{i,j_2}.

We mean by crossing that there exists a foliation of each horizontal strip $H_{i,j} \subset B^m_\delta(0)$ by a family $\mathcal{C}_{i,j}$ of continuous curves $c : [0,1] \to H_{i,j}$ such that $\psi_i(c(0)) \in B^{m-1}_\delta(0) \times \{-\delta\}$ and $\psi_i(c(1)) \in B^{m-1}_\delta \times \{\delta\}$. On the manifold, the intermediate value theorem ensures that $\widehat{H}_{i,j} := \phi_i(H_{i,j}) \subset U_i$ crosses every vertical strip $\widehat{V}_{i,j} := f(\widehat{H}_{i,j})$ as well.

About coherent $(\delta, \varepsilon, n, \alpha)$-pseudo-horseshoes there are two main features. Firstly, if f has a $(\delta, \varepsilon, n, \alpha)$-pseudo-horseshoe, then it persist by C^0 perturbations of f. Secondly, if f has a coherent $(\delta, \varepsilon, n, \alpha)$-pseudo-horseshoe, then the composition $\psi_{n-1} \circ ... \circ \psi_0$ contains $\lfloor (1/\varepsilon)^{\alpha \cdot m} \rfloor^n$ horizontal strips which are mapped onto vertical strips and are ε-separated by f. In particular, any homeomorphism f that has a coherent $(\delta, \varepsilon, n, \alpha)$-pseudo-horseshoe also has a (n, ε)-separated set with $\lfloor (1/\varepsilon)^{\alpha \cdot m} \rfloor^n$ elements.

To estimate the metric mean dimension using local charts taking values in Euclidean coordinates, the separation scale has to be preserved by charts. For this reason, we assume that the charts ϕ_i, $0 \le i \le n-1$, are bi-Lipschitz, and thus we require the compact manifold to be smooth.

4.1 Separating sets

In this section we will use the same notation as in Chapter 2 to refer to separated sets cardinality, that is, remember that $S(n, \varepsilon)$ is the maximal cardinality of a (n, ε)-separated set.

Proposition 4.2. Assume that X is a smooth compact manifold of dimension m. If $f \in \text{Homeo}(X, d)$ has a coherent $(\delta, \varepsilon, n, \alpha)$-pseudo-horseshoe, then

$$S(\ell, C^{-1}\varepsilon) \ge \left(\left\lfloor \left(\frac{1}{\varepsilon} \right)^{\alpha \cdot m} \right\rfloor \right)^{\ell}, \tag{4.2}$$

for all $\ell \in \mathbb{N}$ such that $\ell \ge n$, where $C > 1$ is the upper bound of the bi-Lipschitz constants of all charts.

Proof. Call $N = \lfloor (1/\varepsilon)^{\alpha \cdot m} \rfloor$. By hypothesis, there are charts ϕ_i, $0 \le i \le n-1$, such that the maps $\psi_i = \phi^{-1}_{(i+1) \bmod n} \circ f \circ \phi_i$ has a pseudo-horseshoe of type N at scale δ whose its horizontal strips $\{H_{i,j}\}, 1 \le j \le N$, in the domain $B^m_\delta(0)$ of ψ_i are ε-separated and the same holds for the vertical strips $\{V_{i,j}\}, 1 \le j \le N$, in the image of ψ_i.

Define the horizontal and vertical strips, respectively, on the manifold X by

$$\widehat{H}_{i,j} = \phi_i(H_{i,j}) \text{ and } \widehat{V}_{i,j} = f(\widehat{H}_{i,j}) = (f \circ \phi_i)(H_{i,j}).$$

Note that, by construction,

$$\phi^{-1}_{(i+1) \bmod n}(\widehat{V}_{i,j}) = (\phi^{-1}_{(i+1) \bmod n} \circ f \circ \phi_i)(H_{i,j}) = \psi_i(H_{i,j}) = V_{i,j}$$

is a vertical strip in the image of the pseudo-horseshoe ψ_i.

Now consider the following non-empty compact subsets of X:

$$\begin{array}{rcl}
j \in \{1, ..., N\} & \mapsto & \widehat{K}_{0,j} := \widehat{H}_{0,j} \\
j_1, j_2 \in \{1, ..., N\} & \mapsto & \widehat{K}_{1,j_1,j_2} := f^{-1}(\widehat{V}_{0,j_1} \cap \widehat{H}_{1,j_2}) = f^{-1}(f(\widehat{K}_{0,j_1}) \cap \widehat{H}_{1,j_2}) \\
\vdots & & \vdots \\
j_1, j_2, ..., j_n \in \{1, ..., N\} & \mapsto & \widehat{K}_{n-1,j_1,j_2,...,j_n} := f^{-(n-1)}(f^{n-1}(\widehat{K}_{n-2,j_1,j_2,...,j_{n-1}}) \cap \widehat{H}_{n-1,j_n}).
\end{array}$$

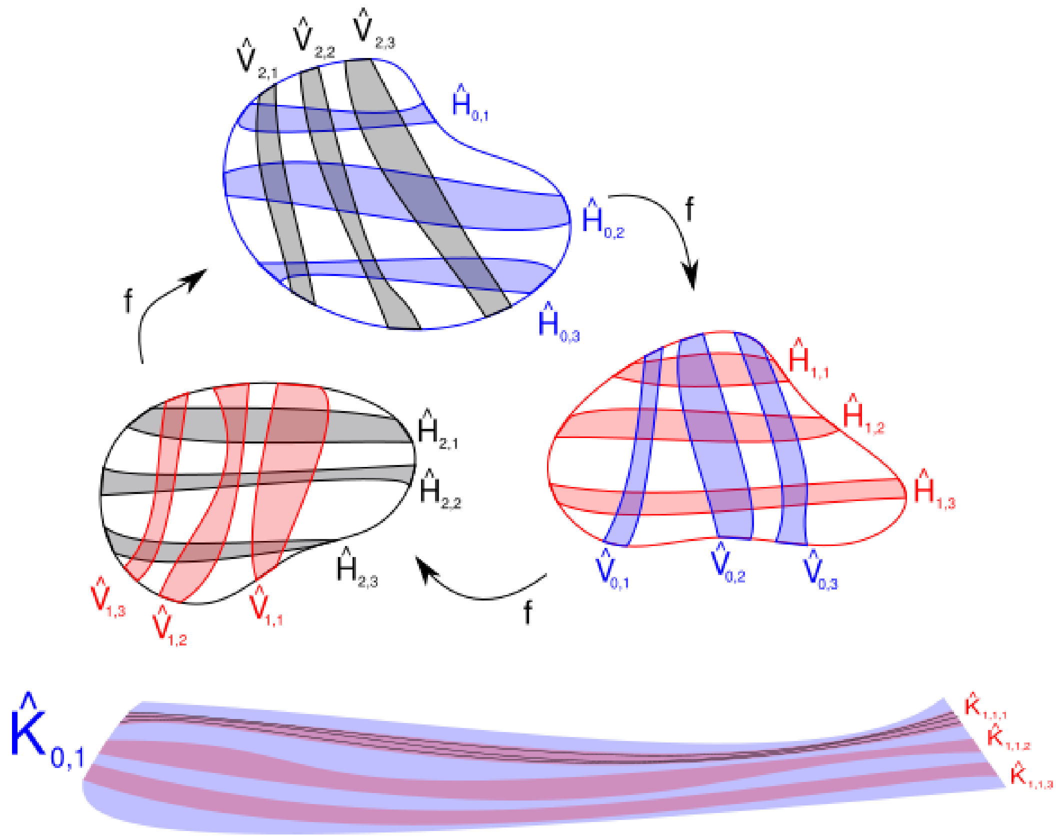

Figure 6 – An example of coherent pseudo-horseshoes generating the compact subsets $\widehat{K}_{n-1,j_1,j_2,\ldots,j_n}$ for $n = 3$. The process is very similar to Cantor sets.

Since X is a smooth manifold, by hypothesis, we may assume that all the maps $\{\phi_i^{\pm 1}; 0 \le i \le n-1\}$ are Lipschitz with Lipschitz constant bounded by a uniform constant $C > 1$. Particularly, by item 2(b) in Definition 4.3, there are at least N points which are $(C^{-1}\varepsilon)$-separated by f in $\widehat{K}_{0,j}$. In fact, just choose N points z_j in $\widehat{K}_{0,j}$, $1 \le j \le N$, then $\left\|\phi_1^{-1}(f(z_{j_1})) - \phi_1^{-1}(f(z_{j_2}))\right\| > C^{-1}\varepsilon$.

Claim. With the previous notation, for $(j_1, j_2) \neq (J_1, J_2)$, $x \in \widehat{K}_{1,j_1,j_2}$ and $y \in \widehat{K}_{1,J_1,J_2}$, then we have that x and y are $(2, C^{-1}\varepsilon)$-separated.

Indeed, by hypothesis, ϕ_1^{-1} is C-Lipschitz and suppose that $j_1 \neq J_1$, then

$$d_2(x, y) \ge d(f(x), f(y)) \ge \mathrm{dist}(\widehat{V}_{0,j_1}, \widehat{V}_{0,J_1}) \ge C^{-1}\mathrm{dist}(V_{0,j_1}, V_{0,J_1}) > C^{-1}\varepsilon$$

where

$$\mathrm{dist}(A, B) := \begin{cases} \inf\{\|a - b\|; a \in A, b \in B\} & \text{if } A, B \in \mathbb{R}^m, \\ \inf\{d(a, b); a \in A, b \in B\} & \text{if } A, B \subset X. \end{cases}$$

On the other way, if $j_1 = J_1$ and $j_2 \neq J_2$, then $f(x), f(y) \in \widehat{V}_{0,j_1}$, but they lie in different horizontal strips, that is, $f(x) \in \widehat{H}_{1,j_2}$ and $f(y) \in \widehat{H}_{1,J_2}$. So

$$d_2(x, y) \ge d(f(x), f(y)) \ge \mathrm{dist}(\widehat{H}_{1,j_2}, \widehat{H}_{1,J_2}) \ge C^{-1}\mathrm{dist}(H_{1,j_2}, H_{1,J_2}) > C^{-1}\varepsilon.$$

And we have proved the claim.

Remind that we have associated to $(j_1, j_2, ..., j_n) \in \{1, 2, ..., N\}^n$ the non-empty compact set

$$\widehat{K}_{n-1,j_1,j_2,...,j_n} = f^{-(n-1)}(f^{n-1}(\widehat{K}_{n-2,j_1,j_2,...,j_{n-1}}) \cap \widehat{H}_{n-1,j_n})$$

and observe that, whenever $(j_1, j_2, ..., j_n) \neq (J_1, J_2, ..., J_n)$, then there is $1 \leq i \leq n$ such that $j_i \neq J_i$. For $x \in \widehat{K}_{n-1,j_1,j_2,...,j_n}$ and $y \in \widehat{K}_{n-1,J_1,J_2,...,J_n}$, if $i = 1$, then $f(x) \in \widehat{V}_{0,j_1}$ and $f(y) \in \widehat{V}_{0,J_1}$ and we are done. For $2 \leq i \leq n$, then $f^{i-1}(x) \in \widehat{H}_{i-1,j_i}$ and $f^{i-1}(y) \in \widehat{H}_{i-1,J_i}$. Therefore

$$d_n(x, y) \geq d(f^{i-1}(x), f^{i-1}(y)) \geq \text{dist}(\widehat{H}_{i-1,j_i}, \widehat{H}_{i-1,J_i}) \geq C^{-1}\text{dist}(H_{i-1,j_i}, H_{i-1,J_i}) > C^{-1}\varepsilon.$$

This proves that

$$S(n, C^{-1}\varepsilon) \geq N^n,$$

that is, there are N^n points which are $(n, C^{-1}\varepsilon)$-separated by f in the union of the sets $\widehat{K}_{n-1,j_1,j_2,...,j_n}$.

Suppose $\ell > n$, $\ell \in \mathbb{N}$, and define, for $j_1, j_2, ..., j_n, ..., j_\ell \in \{1, ..., N\}$,

$$\widehat{K}_{\ell-1,j_1,...,j_\ell} := f^{-(\ell-1)}(f^{\ell-1}(\widehat{K}_{\ell-2,j_1,...,j_{\ell-1}}) \cap \widehat{H}_{\ell-1 \bmod n,j_\ell}).$$

By the same previous argument we can say that if $(j_1, ..., j_\ell) \neq (J_1, ..., J_\ell)$, $x \in \widehat{K}_{\ell-1,j_1,...,j_\ell}$ and $y \in \widehat{K}_{\ell-1,J_1,...,J_\ell}$, then there is some $1 \leq i \leq \ell$ such that $j_i \neq J_i$ and therefore $f^{i-1}(x) \in \widehat{H}_{i-1,j_i}$ and $f^{i-1}(y) \in \widehat{H}_{i-1,J_i}$. Now is clearly that $d_\ell(x, y) > C^{-1}\varepsilon$ and we are done since we can choose N^ℓ different points that are $(\ell, C^{-1}\varepsilon)$-separated by f, each point in $\widehat{K}_{\ell-1,j_1,...,j_\ell}$. □

Now we just need some simple algebraic manipulation on the equation 4.2 to show the following result.

Corollary 4.1. Under the assumptions of Proposition 4.2 one has

$$\text{Sep}(f, \varepsilon) = \limsup_{n\to+\infty} \frac{\log S(n, C^{-1}\varepsilon)}{n} \geq \alpha \cdot m \cdot |\log \varepsilon|, \tag{4.3}$$

where the Sep notation is from (2.1).

5 Residual homeomorphisms have full metric mean dimension

5.1 A C^0-pertubation lemma along orbits

Now we are interested in construct coherent pseudo-horseshoes inside absorbing disks with small diameter, because we can control the size of the iterates of an absorbing disk. The argument depends on a finite number of C^0-perturbations of the initial dynamics on disjoint supports.

Given that X is a smooth compact boundaryless manifold, we can get a finite atlas $\mathcal{A}$ whose charts are bi-Lipschitz. If $r := r(\mathcal{A}) > 0$ denotes the Lebesgue covering number of the charts, by Remark 3.2, we can choose an absorbing disk that defines the periodic attracting point with diameter smaller than r and, thus, every iterates of the absorbing disk is inside some neighborhood defined by some chart.

Lemma 5.1. Given $\eta > 0$ and $f \in \mathcal{H}$, there exist $n \in \mathbb{N}$ and $\delta > 0$ such that, for every $\alpha \in (0, 1)$ and $\varepsilon > 0$ sufficiently small, we may find $g \in \text{Homeo}(X, d)$ satisfying:

1. g has a coherent $(\delta, \varepsilon, n, \alpha)$-pseudo-horseshoe;
2. $D(f, g) < \eta$.

Proof. Let $f \in \mathcal{H}$ and $\eta > 0$. We remember, from Lemma 3.3, that given $\eta > 0$, then each $f \in \mathcal{H}$ has a n-absorbing disk B with diameter smaller than η, for some $n \in \mathbb{N}$, and there exists some $\tilde{\eta} > 0$ such that for every $\tilde{f} \in \text{Homeo}(X, d)$ where $D(f, \tilde{f}) < \tilde{\eta}$ the disk B still n-absorbing for $\tilde{f}$. Consider now a neighborhood $\mathcal{W}_f$ in $\text{Homeo}(X, d)$ such that $\tilde{f} \in \mathcal{W}_f$ if $D(f, \tilde{f}) < \beta$, where $\beta := \min\{\eta, \tilde{\eta}\}$.

By Brouwer's fixed point theorem, f have a periodic point p of period n in B. Let ϕ_i be a bi-Lipschitz chart, for every $0 \le i \le n-1$, from $B^m_\kappa(0) \subset \mathbb{R}^m$ onto some open neighborhood U_i of $f^i(p)$ contained in the disk $f^i(B)$ and such that $\phi_i(0) = f^i(p)$, for some small $\kappa > 0$. We can obtain these charts by restrictions of charts in the atlas $\mathcal{A}$ and composing with translations, which do not affect the value of C, the upper bound of the bi-Lipschitz constants of all charts.

By the uniform continuity of f, exists some $\theta > 0$ such that if $d(x_1, x_2) < \theta$, then $d(f(x_1), f(x_2)) < \beta$ and $d(f^{-1}(x_1), f^{-1}(x_2)) < \beta$, for all $x_1, x_2 \in X$. Let $\lambda > 0$ such that $\lambda < \min\{\theta, \beta/2\}$, then for every perturbation $h \in \text{Homeo}(X, d)$ of the identity whose support has diameter smaller than λ, $d(h(f(x)), f(x)) < \beta$ and $d(f^{-1}(h^{-1}(x)), f^{-1}(x)) < \beta$, because $d(h^{-1}(x), x) < \theta$, for all $x \in X$. This proves that $h \circ f \in \mathcal{W}_f$ and therefore $D(h \circ f, f) < \eta$. We also want that the ball $B_\lambda(f^i(p))$ is strictly contained in U_i and that $B_\lambda(f^i(p)) \cap B_\lambda(f^j(p)) = \varnothing$, for every $0 \le i < j \le n-1$, so we can reduce λ if necessary.

We will suppose that $n \ge 2$. The other case, $n = 1$, is just an adaptation of the arguments using only the same chart ϕ_0.

Step 1: Consider $\delta > 0$ such that $\phi_i(B^m_{2\delta}(0)) \subset B_\lambda(f^i(p))$ and $f(\phi_i(B^m_{2\delta}(0))) \subset B_\lambda(f^{i+1}(p))$, for every $0 \le i \le n-1$. Fix $N = \lfloor (1/\varepsilon)^{\alpha \cdot m} \rfloor$, and remember from the inequality in (4.1) that ε depends on δ and α. We may assume that the map

$$\phi_1^{-1} \circ f \circ \phi_0 : B^m_{2\delta}(0) \to W_1$$

where $W_i := \phi_i^{-1}(B_\lambda(f^i(p)))$, is well defined, fixes the origin and is a homeomorphism onto its image. Note that $B^m_{2\delta}(0)$ is also in W_i. By the same argument that we used in Proposition 4.1 we can construct a homeomorphism $\rho_1 : W_1 \to W_1$ that fix ∂W_1 and $\rho_1 \circ \phi_1^{-1} \circ f \circ \phi_0$ has a pseudo-horseshoe of type N at scale δ connecting 0 to itself. So, there exists a family $\mathcal{V}_1 = \{V_{1,i}\}, 0 \leq i \leq N-1$, of connected disjoint vertical strips such that

$$V_{1,i} = (\rho_1 \circ \phi_1^{-1} \circ f \circ \phi_0)(H_{1,i}) \subset B^m_\delta(0)$$

for some connected subset

$$H_{1,i} \subset B^{m-1}_r(0) \times [t_i, t_{i+1}],$$

where $\mathcal{P} = \{t_0, t_1, ..., t_N\}$ is a partition of $(-\delta, \delta)$. We also want to ensure that the vertical strips $V_{1,i}$ and horizontal strips $H_{1,i}$ are a distance ε apart, and to obtain this we can possibly make a extra C^0-perturbation in $B^m_\delta(0)$, so we will assume that ρ_1 do this job. Let $h_1 \in \text{Homeo}(X, d)$ be a homeomorphism such that

$$h_1(x) := \begin{cases} \phi_1 \circ \rho_1 \circ \phi_1^{-1}(x), & \text{if } x \in B_\lambda(f(p)); \\ x, & \text{if } x \notin B_\lambda(f(p)). \end{cases}$$

By construction, the diameter of the support of h_1 is smaller than λ, so $f_1 := h_1 \circ f$ belongs to $\mathcal{W}_f$ and $D(f_1, f) < \eta$. Moreover, in $B^m_{2\delta}(0)$ one has

$$\phi_1^{-1} \circ f_1 \circ \phi_0 = \phi_1^{-1} \circ h_1 \circ f \circ \phi_0 = \rho_1 \circ \phi_1^{-1} \circ f \circ \phi_0,$$

that is, $f_1 \in \mathcal{S}_N$. Observe that $f_1(p) = h_1(f(p)) = f(p)$.

Step 2: Proceeding as in Step 1, for $0 \leq i \leq n-1$, we can construct homeomorphisms $\rho_i : W_i \to W_i$ that fix ∂W_i and

$$\rho_i \circ \phi_i^{-1} \circ f \circ \phi_{(i-1) \bmod n} : B^m_{2\delta}(0) \to W_i$$

has a pseudo-horseshoe of type N at scale δ connecting 0 to itself. Thus we can define

$$h_i(x) := \begin{cases} \phi_i \circ \rho_i \circ \phi_i^{-1}(x), & \text{if } x \in B_\lambda(f^i(p)); \\ x, & \text{if } x \notin B_\lambda(f^i(p)). \end{cases}$$

Note that the support of all perturbations are disjoint, because λ is chosen to be sufficiently small, so we can define

$$g := h_0 \circ h_1 \circ ... \circ h_{n-1} \circ f.$$

So $g \in \mathcal{W}_f$ and thus $D(f, g) < \eta$. Note that $g^i(p) = f^i(p)$, that is, p is a n-periodic point for g and B still a n-absorbing disk.

All the perturbations h_i can be made so that the pseudo-horseshoe induced by them are ε-separated and such that g has a coherent $(\delta, \varepsilon, n, \alpha)$-pseudo-horseshoe. So we are done. □

Remark 5.1. Observe that the set of all g such that g has a coherent $(\delta, \varepsilon, n, \alpha)$-pseudo-horseshoe is a C^0-open set because we can find some sufficiently small perturbations of the auxiliary homeomorphisms h_i such that they doesn't destroy the structure we want.

5.2 Main theorem

Given $f \in \mathcal{H}$ (see Lemma 3.3), there is an open set $\mathcal{O}_f \subset \text{Homeo}(X, d)$ such that every $g \in \mathcal{O}_f$ have a coherent $(\delta, \varepsilon, n, \alpha)$-pseudo-horseshoe, for some $\delta > 0$, $n \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\varepsilon > 0$ sufficiently small. From Corollary 4.1, we have that

$$\text{Sep}(g, \varepsilon) \geq \alpha \cdot m \cdot |\log \varepsilon|.$$

For $(\alpha_k)_{k\in\mathbb{N}}$ a strictly increasing sequence in $(0, 1)$ converging to 1, fix a strictly decreasing sequence $(\varepsilon_k)_{k\in\mathbb{N}}$, respecting the inequality at (4.1), which converges to zero. Now consider the set $\mathcal{O}_f^k \subset \text{Homeo}(X, d)$ such that for every $g \in \mathcal{O}_f^k$, g have a coherent $(\delta, \varepsilon_k, n, \alpha_k)$-pseudo-horseshoe. Note that $\mathcal{O}_f^k$ is an open and dense set, because $\mathcal{H}$ is dense and by Remark 5.1. Therefore, consider the residual set

$$\mathfrak{R} = \bigcap_{k\in\mathbb{N}} \mathcal{O}_f^k.$$

If $g \in \mathfrak{R}$, we have that $\text{Sep}(g, \varepsilon_k) \geq \alpha_k \cdot m \cdot |\log \varepsilon_k|$. Thus,

$$\overline{\text{mdim}_M}(X, d, g) = \limsup_{\varepsilon\to 0} \frac{\text{Sep}(g, \varepsilon)}{|\log \varepsilon|} \geq \lim_{k\to\infty} \frac{\text{Sep}(g, \varepsilon_k)}{|\log \varepsilon_k|} \geq m \lim_{k\to\infty} \alpha_k = m.$$

Remember that, by Remark 2.2, we have an upper bound for the upper metric mean dimension. So we have the following theorem:

Theorem 5.1. *Let X be a compact smooth boundaryless manifold of dimension m strictly greater than one endowed with a metric d. There exists a residual subset $\mathfrak{R} \subset \text{Homeo}(X, d)$ such that*

$$\overline{\text{mdim}_M}(X, d, g) = m, \text{ for all } g \in \mathfrak{R}.$$

5.3 An example

Consider the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, we will give an example of homeomorphism that have upper full metric mean dimension on $\mathbb{T}^2$.

Suppose a sequence $(\delta_k)_{k\in\mathbb{N}}$ such that

$$\sum_{k=1}^{\infty} \delta_k^2 = \frac{1}{2\pi}. \tag{5.1}$$

Take $(x_k)_{k\in\mathbb{N}}$ a sequence in $\mathbb{T}^2$ such that $B_{\delta_k}(x_k) \cap B_{\delta_i}(x_i) \neq \varnothing$, for all $i < k$ in $\mathbb{N}$. This is possible because of (5.1).

For $(\alpha_k)_{k\in\mathbb{N}}$ a strictly increasing sequence in $(0, 1)$ converging to 1, fix a strictly decreasing sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ which converges to zero. For every $k \in \mathbb{N}$, consider the homeomorphism $h_k : \mathbb{T}^2 \to \mathbb{T}^2$ such that outside the ball $B_{\delta_k}(x_k)$ is the identity, fixing its boundary, and inside this ball h_k have a coherent $(\delta_k/2, \varepsilon_k, 1, \alpha_k)$-pseudo-horseshoe. Since $n = 1$, we have the usual invariant horseshoe. Define the following homeomorphism $g : \mathbb{T}^2 \to \mathbb{T}^2$

$$g = \lim_{k\to\infty} h_k \circ h_{k-1} \circ \ldots \circ h_1.$$

We have that $\text{Sep}(g, \varepsilon_k) \geq \alpha_k \cdot 2 \cdot |\log \varepsilon_k|$. Therefore, doing the same argument as previously,

$$\overline{\text{mdim}_M}(\mathbb{T}^2, d, g) = 2.$$

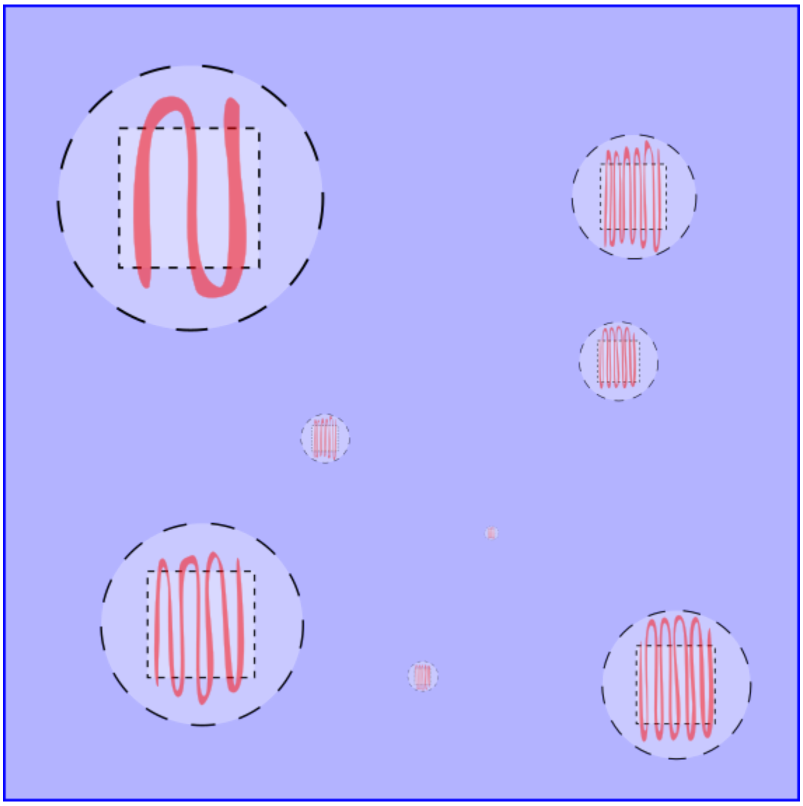

Figure 7 – To illustrate how g is acting in $\mathbb{T}^2$.

5.4 Future works

There are a few questions that can lead to new results and they are:

(a) Can we show that the Baire residual subset with full metric mean dimension $\mathfrak{R}$ contains small curves?

(b) Is it true that the set of homeomorphisms with metric mean dimension equal to some positive β is dense?

(c) If we induce the dynamical system to the hyperspace, the space of the closed sets of a manifold, is it true that residual homeomorphisms have infinite metric mean dimension?

A Appendix: Birkhoff Recurrence Theorem

Let X be a Hausdorff topological space and $f : X \to X$ a continuous function. Given $x \in X$, your *positive orbit* is $O_f^+(x) = \{x, f(x), f(f(x)), ...\}$.

Definition A.1. A set $A \subset X$ is called *invariant*, with relation to f, if $f(A) \subset A$, and is *completely invariant* if $f(A) = A$.

Proposition A.1. $\overline{O_f^+(x)}$ is an invariant set.

Proof. Take $z \in f(\overline{O_f^+(x)})$, then there is $y \in \overline{O_f^+(x)}$ such that $f(y) = z$. We know that there is a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to y$, with $x_n \neq y$ and $f(x_n) \neq z$. Since f is continuous and X is Hausdorff, thus $f(x_n) \to z$. Note that each x_n is in $O_f^+(x)$, then $f(x_n) \in O_f^+(x)$. Therefore $z \in \overline{O_f^+(x)}$. □

Definition A.2. A set $Y \subset X$ is called *minimal* if Y is closed, non-empty, invariant and is the smallest set with these properties, that is, a strict subset of Y cannot be minimal. We call $f : X \to X$ a *minimal function* if X is a minimal set, with relation to f.

Observe that if $Y \subset X$ is minimal, then $f(Y) \subset Y$ and for $g := f|_Y$, g is a minimal function.

Proposition A.2. Let $f : X \to X$ a continuous function and $Z \subset X$ a compact and invariant set, then there is $Y \subset Z$ minimal set.

Proof. Suppose $\mathcal{E} := \{Y \subset Z; Y \text{ is a non-empty, closed and invariant set}\}$. Note that $\mathcal{E} \neq \varnothing$ because $Z \in \mathcal{E}$ and every compact in a Hausdorff space is closed. Let $\{Y_\alpha\}$, $\alpha \in I$, be a chain in $\mathcal{E}$ and $Y := \bigcap_\alpha Y_\alpha$.

We have that Y is compact because is closed, intersection of closed sets, and is a subset of a compact, and is non-empty because is an intersection of compact sets. We also have that Y is invariant because

$$f(Y) = f\left(\bigcap_\alpha Y_\alpha\right) \subset \bigcap_\alpha f(Y_\alpha) \subset \bigcap_\alpha Y_\alpha = Y.$$

Therefore $Y \in \mathcal{E}$ and is a lower bound. Finally, by Zorn's lemma, we can conclude that $\mathcal{E}$ has a minimal element. □

Definition A.3. A point $x \in X$ is *recurrent* if for every neighborhood U of x there is $n \geq 1$ such that $f^n(x) \in U$.

Lemma A.1. A point $x \in X$ is recurrent if, and only if, $x \in \overline{O_f^+(x)}$.

Lemma A.2. A set $Y \subset X$ is minimal if, and only if, for all $y \in Y$, $\overline{O_f^+(y)} = Y$.

Proof. Since $Y \subset X$ is minimal, Y is closed and invariant. Then, for all $y \in Y$, $\overline{O_f^+(y)} \subset Y$. We also have that $\overline{O_f^+(y)}$ is non-empty, closed and invariant. By the minimality of Y, $\overline{O_f^+(y)} = Y$.

On the other way, suppose that, for all $y \in Y$, $\overline{O_f^+(y)} = Y$. Consider $B \subset Y$ such that B is non-empty, closed and invariant, then for all $y \in B$, $\overline{O_f^+(y)} \subset B$. Since $\overline{O_f^+(y)} = Y$, $B = Y$ and Y is minimal. $\square$

Theorem A.1 (Birkhoff Recurrence Theorem)**.** Given X a compact topological space and $f : X \to X$ a continuous function, then there is a recurrent point in X.

Proof. Clearly X is a compact and invariant set to f, then there is $Y \subset X$ minimal set. For all $y \in Y$, $\overline{O_f^+(y)} = Y$, then y is recurrent. $\square$

The material for this appendix and for topological entropy in Chapter 2 were taken from class notes during my undergraduate course (see [3]).

B Appendix: Baire Spaces

Let (X, d) be a compact metric space. Consider $\text{Homeo}(X, d)$ the space of homeomorphisms $f : X \to X$ and the metric D defined as

$$D(f, g) := \max_{x \in X} \{d(f(x), g(x)), d(f^{-1}(x), g^{-1}(x))\}$$

for every $f, g \in \text{Homeo}(X, d)$. In this appendix we will refer to the metric space of homeomorphisms as (H, D). The material for this chapter in the appendix were taken from [7] and [9].

Proposition B.1. The function $\sigma_k : \text{H} \to \text{H}$ defined as $\sigma_k(f) = f^k$ is continuous with respect to the metric D, for every $k \in \mathbb{N}$.

Proof. Note that σ_1 is the identity and clearly is continuous. We will prove first that σ_2 is continuous.

Fix $\varepsilon > 0$. Let $\rho > 0$ such that if $d(x_1, x_2) < \rho$, then $d\Big(f(x_1), f(x_2)\Big) < \varepsilon/2$ and $d\Big(f^{-1}(x_1), f^{-1}(x_2)\Big) < \varepsilon/2$, for every $x_1, x_2 \in X$. Therefore, for $\delta < \min\{\rho, \varepsilon/2\}$ and g an homeomorphism such that $D(f, g) < \delta$,

$$d\Big(f^2(x), g^2(x)\Big) \leq d\Big(f^2(x), f(g(x))\Big) + d\Big(f(g(x)), g^2(x)\Big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ and}$$

$$d\Big(f^{-2}(x), g^{-2}(x)\Big) \leq d\Big(f^{-2}(x), f^{-1}(g^{-1}(x))\Big) + d\Big(f^{-1}(g^{-1}(x)), g^{-2}(x)\Big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for every $x \in X$. Thus $D(f^2, g^2) < \varepsilon$. Note that we also have $D(f^{-2}, g^{-2}) < \varepsilon$. Now we will prove that if σ_k is continuous, then σ_{k+1} is also continuous.

Let $\varepsilon > 0$, and $\eta > 0$ such that if $D(f, g) < \eta$, then $D(f^k, g^k) < \rho$. Observe that $d\Big(f^{k+1}(x), f(g^k(x))\Big) < \varepsilon/2$. Then, for $\delta < \min\{\eta, \varepsilon/2\}$ and $g \in \text{Homeo}(X, d)$ such that $D(f, g) < \delta$,

$$d\Big(f^{k+1}(x), g^{k+1}(x)\Big) \leq d\Big(f^{k+1}(x), f(g^k(x))\Big) + d\Big(f(g^k(x)), g^{k+1}(x)\Big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ and}$$

$$d\Big(f^{-(k+1)}(x), g^{-(k+1)}(x)\Big) \leq d\Big(f^{-(k+1)}(x), f^{-1}(g^{-k}(x))\Big) + d\Big(f^{-1}(g^{-k}(x)), g^{-(k+1)}(x)\Big) < \varepsilon.$$

Therefore $D(f^{k+1}, g^{k+1}) < \varepsilon$. □

Definition B.1. Let (X, d) be a metric space. A sequence (x_n) of points of X issaid to be a *Cauchy sequence* in (X, d) if it has the property that given $\varepsilon > 0$, there is an integer N such that

$$d(x_n, x_m) < \varepsilon \;\text{ whenever }\; n, m \geq N.$$

The matric space (X, d) is said to be *complete* if every Cauchy sequence in X converges.

Theorem B.1. (H, D) is a complete metric space.

Recall that if A is a subset of a space X, the *interior* of A is defined as the union of all open sets of X that are contained in A. To say that A has *empty interior* is to say then that A contains no open set of X other than the empty set. Equivalently, A has empty interior if every point of A is a limit point of the complement of A, that is, if the complement of A is dense in X.

Definition B.2. A space X is said to be a *Baire space* if the following condition holds: Given any countable collection $\{A_n\}$ of closed sets of X each of which has empty interior in X, their union $\bigcup A_n$ also has empty interior in X.

A subset A of a space X is said to be of the *first category* in X if it is contained in the union of a countable collection of closed sets of X having empty interiors in X; otherwise, it is said to be of the *second category* in X. In a Baire space, the complement of any set of first category is called a *residual set*.

Theorem B.2 (Baire category theorem)**.** If X is a compact Hausdorff space or a complete metric space, then X is a Baire space.

Therefore, (H, D) is a Baire space. Note that if U_α, $\alpha \in I$, is an open and dense set in H, then $\bigcup_{\alpha\in I} U_\alpha^c$ is of first category. Thus

$$\left(\bigcup_{\alpha\in I} U_\alpha^c\right)^c = \bigcap_{\alpha\in I} U_\alpha \quad \text{is a residual set, for } I \text{ an enumerable set.}$$

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