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Cofinality Spectrum Problems and Applications in Model Theory and Set Theory

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Dissertação de mestrado apresentada ao Programa de Pós-graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Mestre em Matemática

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To my grandfather, José dos Anjos (in memoriam).

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Resumo

A pergunta sobre a relação entre $\mathfrak{p}$ e $\mathfrak{t}$, caracteristicas cardinais do contínuo, se manteve aberta por décadas, mais precisamente se $\mathfrak{p} = \mathfrak{t}$. Num trabalho recente de Malliaris e Shelah, foi desenvolvida uma nova ferramenta, designadamente *problema de espectro de cofinalidade*, que nos permite responder de maneira afirmativa a questão de se $\mathfrak{p} = \mathfrak{t}$. A utilidade dessa nova ferramenta não para por ai, ela também nos permite refinar as fronteiras na área de teoria de classificação, mostrando que a properiedade SOP_2 é uma condição suficiente para uma teoria completa estar na classe máxima da order de Keisler. O fato dessa ferramenta ter aplicações em teoria de modelos e teoria de conjuntos, e nos permitir transportar uma questão de uma área para a outra, nos indica ela como um objeto promissor no arsenal de ambas as áreas. Nesta dissertação apresentamos o contexto que dá origem a essa ferramenta e suas principais aplicações até o momento.

Palavras-chave: Teoria de Modelos, Teoria de Conjuntos

Abstract

A decades-long question in set theory is if the cardinal characteristics $\mathfrak{p}$ and $\mathfrak{t}$ are equal. In a recent work of Malliaris and Shelah, they developed a new tool, namely *cofinality spectrum problem*, that can answer the problem of $\mathfrak{p} = \mathfrak{t}$. The usefulness of this new tool does not stop there, as it can also refine the known boundaries in classification theory, by showing that the property SOP_2 is a sufficient condition for maximality in Keisler's order. The fact that the same tool has applications in model theory and set theory, and its ability to transport a problem from one field to another, shows it as a promising tool in the arsenal of both fields. We present the context in which this new tool arises and its main known applications.

Keywords: Model Theory, Set Theory

Contents

1 Introduction

The notions of set theory and model theory necessary to understand the constructions present in this dissertation can be found in Chapter 2, so the reader unfamiliar with certain concepts can use that chapter to get the basic knowledge needed to proceed.

1.1 Cardinal characteristics of the continuum

With Cantor's proof that $\aleph_0 < 2^{\aleph_0} = \mathfrak{c}$, a natural following question was the continuum hypothesis, that is, if $\mathfrak{c} = \aleph_1$, see [2]. With the works of Cohen we know that, relative to the consistency of ZFC, there are models of ZFC where the continuum hypothesis is false, so we can ask ourselves how we can construct cardinalities in between. That is one motivating question behind cardinal characteristics of the continuum and will be the focus of this introduction. Let κ be a cardinality defined as the least size of a set with a given property P. If we can show that $\aleph_0 < \kappa \leqslant \mathfrak{c}$ and, relative to the consistency of ZFC and perhaps some large cardinal assumption, there exists a model of ZFC where $\kappa < \mathfrak{c}$, then κ is called a *cardinal characteristic of the continuum*.

A common way of attacking this problem is to look at subsets of $[\mathbb{N}]^{\aleph_0}$ (the set of infinite subsets of $\mathbb{N}$) or ${}^{\mathbb{N}}\mathbb{N}$ (the set of functions from $\mathbb{N}$ to $\mathbb{N}$), with a given property P that is a candidate to generate sizes less than $\mathfrak{c}$. With this idea we choose properties that are false for any countable subset but are true for some uncountable subset. The properties of interest are, in a lot of cases, inspired by other areas of mathematics, as seen for exemple in [1, page 417], where **null**, the family of subsets of $\mathbb{R}$ with Lebesgue measure zero, and **meager**, the family of subsets of $\mathbb{R}$ that are meager in the usual topology, are used as basis to construct interesting subsets.

In the compilation work of [31] we see that the majority of the study in cardinal characteristics of the continuum have been focused, historically, on six classic cardinals, denoted by $\mathfrak{p}, \mathfrak{t}, \mathfrak{a}, \mathfrak{b}, \mathfrak{d}$ and $\mathfrak{s}$. Mapping all the inequalities between these cardinals was central to the area. At the time of [32], only two inequalities were still open, between $\mathfrak{a}$ and $\mathfrak{d}$, and between $\mathfrak{p}$ and $\mathfrak{t}$. With Shelah's 2004 result [29], where he constructed a model satisfying $\mathfrak{d} < \mathfrak{a}$, relative to the consistency of ZFC, the only question left was if $\mathfrak{p} = \mathfrak{t}$ is demonstrable in ZFC or there is a model where $\mathfrak{p} < \mathfrak{t}$ (it was already known that $\mathfrak{p} \leqslant \mathfrak{t}$). It is worth mentioning that the opinion of the experts in the area was, in general, that probably there was a model where $\mathfrak{p} < \mathfrak{t}$, see [9].

Let's define $\mathfrak{p}$ and $\mathfrak{t}$, to better illustrate what we are working with. Define $A \subseteq^* B$ as saying that the set $A \setminus B$ is finite, that is, A is a subset of B except for a finite number of elements. Given sets A and $\mathcal{F}$, we say that A is a *pseudo-intersection* of $\mathcal{F}$ if $(\forall B \in \mathcal{F})(A \subseteq^* B)$. A set X is said to have the *strong finite intersection property* (sfip) if the intersection of any finite number of elements of X is infinite. We say that $\mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0}$ is a *tower* if it is linearly ordered by $\supseteq^*$ and has no infinite pseudo-intersection.

Definition A. The cardinal $\mathfrak{p}$ is defined as the smallest size of a family $\mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0}$ such that $\mathcal{F}$ has the strong finite intersection property but has no infinite pseudo-intersection, while the cardinal $\mathfrak{t}$ is defined as the smallest size of a tower.

The cardinal $\mathfrak{p}$ is also the cardinality of the least number of nowhere dense sets needed to cover a compact topological space, see [23], again connecting these cardinals with other areas.

Since both are defined by families of subsets of $\mathbb{N}$ we have that $\mathfrak{p} \leqslant \mathfrak{c}$ and $\mathfrak{t} \leqslant \mathfrak{c}$. It is clear that $\mathfrak{p} \leqslant \mathfrak{t}$ because a tower has the sfip. If there exists a set with the sfip but no infinite pseudo-intersection then $\aleph_1 \leqslant \mathfrak{p}$, because let $\mathcal{F} = \{F_n : n \in \omega\} \subseteq [\mathbb{N}]^{\aleph_0}$ be a countable family with the sfip, and define recursively $a_n = \min(\cap_{k \leqslant n} F_n \setminus \{a_k : k \in n\})$, $a_0 = \min(F_0)$. The set $\{a_n : n \in \omega\}$ is an infinite pseudo-intersection for $\mathcal{F}$, because for each F_n we have, by construction, that $\{a_k : k \geqslant n\} \subseteq F_n$. In classic works, see [31], it has been shown that indeed there exists a set with the sfip with no infinite pseudo-intersection and a tower, so they are not vacuous definitions. All the above properties together guarantee that $\aleph_1 \leqslant \mathfrak{p} \leqslant \mathfrak{t} \leqslant \mathfrak{c}$.

In the recent work of Malliaris and Shelah [20] they were able to answer the decades-long question of whether $\mathfrak{p} = \mathfrak{t}$, with the use of a new technology, that of cofinality spectrum problems.

Theorem B ([20], Theorem 14.1)**.** $\mathfrak{p} = \mathfrak{t}$.

In Section 4.2 we will show how this new technology is used to attack the problem of $\mathfrak{p} = \mathfrak{t}$.

1.2 Classifying theories

A major problem of model theory today is the problem of classifying complete theories in a useful way, where we want to group theories with a certain property in a way that represents their level of complexity. A major advance in this direction was Shelah's classification theory, compiled in [27]. Shelah's work introduced the notion of *stable* and *unstable* theories, defined as follows.

A complete theory T is λ-*stable* if for every $\mathcal{M} \models T$ and any $A \subseteq M$ with $|A| = \lambda$, the set $S(A)$, of complete types (see Section 2.3) over A, is of size no more than λ, otherwise T is λ-*unstable*. The theory T is *unstable* if it is λ-unstable for all λ, otherwise T is *stable*.

So stability is a combinatorial property of the models of the theory. Moreover, stability has a lot of interesting characterizations, with the one most relevant to this work being that T is stable if and only if it doesn't contain a formula with the *order property*, defined as follows. A formula $\varphi(x; y)$ has the *order property* if there exist sequences $\langle a_i : i < \omega \rangle, \langle b_j : j < \omega \rangle$ in a structure $\mathcal{M}$ such that $i \leqslant j$ if and only if $\mathcal{M} \models \varphi(a_i; b_j)$ (A formula with the order property is also called an *unstable formula*).

For example, the theory of a non-abelian free group is stable [26], while Peano arithmetic is unstable.

The theories with the order property, that is, the unstable theories, can be further characterized as the complete theories with at least one of two important properties, the *independence property (IP)* and the *strict order property (SOP)*.

Definition A. Let T be a complete theory, and $\varphi(x; y)$ be a formula of T. The formula $\varphi(x; y)$ has the *independence property (IP)* if in some model $\mathcal{M} \models T$ there exists a sequence $\langle a_i : i < \omega \rangle$ such that for all $\sigma, \tau \in [\omega]^{<\aleph_0}$ (the set of finite subsets of ω) with $\sigma \cap \tau = \emptyset$, and all i

$$\exists x((i \in \sigma \to \mathcal{M} \models \varphi(x; a_i)) \wedge (j \in \tau \to \mathcal{M} \models \neg\varphi(x; a_j))).$$

We say that T has IP if there is a formula of T that has IP.

Definition B. Let T be a complete theory, and $\varphi(x; y)$ be a formula of T. The formula $\varphi(x; y)$ has the *strict order property (SOP)* if in some model $\mathcal{M} \models T$ there exists a sequence $\langle a_i : i < \omega \rangle$ such that for all i, j

$$i < j \leftrightarrow \mathcal{M} \models \exists x(\neg\varphi(x; a_i) \wedge \varphi(x; a_j)).$$

We say that T has SOP if there is a formula of T that has SOP.

For example any theory of the random graph has the independence property but does not have the strict order property, on the other hand a linear order like $\mathrm{Th}((\mathbb{Q}, <))$ has the strict order property.

So we have

$$\text{unstable} = OP = SOP \cup IP.$$

The further classification of stable complete theories has seen a fruitful development over the years, see [27], while the knowledge about unstable complete theories had not seen nearly as much advances.

With an approach independent of Shelah's work, Keisler developed another way of classifying complete theories using saturation of regular ultrapowers, that of Keisler's order.

Definition C (Keisler's Order [11]). Let T_1, T_2 be complete, countable first-order theories. We say that $T_1 \trianglelefteq T_2$ if for all infinite λ, all $M_1 \models T_1, M_2 \models T_2$, and any regular ultrafilter D over λ, if M_2^λ/D is λ^+-saturated then M_1^λ/D must be λ^+-saturated.

Keisler's order is actually a preorder, but its equivalence classes are the interesting objects. When we talk about Keisler's order we will be always referring to the order between its equivalence classes.

Keisler proved in the same paper the following result.

Theorem D. [11] *Keisler's order admits a minimum class and a maximum class.*

The following informal diagram depicts the structure of Keisler's order known at the time.

$$\mathbf{T}_{\min} \trianglelefteq \cdots ??? \cdots \trianglelefteq \mathbf{T}_{\max}.$$

A few years after Keisler's development, Shelah, while enriching the known classification of Keisler's order classes, showed that this order also witnesses the change in complexity between stable and unstable theories.

Theorem E ([27], Chapter VI). *Keisler's order admits classes $\mathbf{T}_{\min}, \mathbf{T}_2$ and $\mathbf{T}_{\max}$ that relate as follows*

1. $\mathbf{T}_{\min} \triangleleft \mathbf{T}_2$;
2. $\mathbf{T}_2 \triangleleft \mathbf{T}_{\max}$;
3. $\mathbf{T}_{\min} \triangleleft \mathbf{T}_{\max}$.

In more details:

- *the class $\mathbf{T}_{\min}$ consists of the complete theories without the finite cover property;*
- *the class $\mathbf{T}_2$ consists of the stable complete theories with the finite cover property;*
- *for the class $\mathbf{T}_{\max}$ we don't have a model theoretic characterization to this day, but it was known that it contains all linear orders.*

In addition, $\mathbf{T}_{\min} \cup \mathbf{T}_2$ is precisely the set of the stable complete theories.

We can update our diagram with this new information.

$$\mathbf{T}_{\min} \triangleleft \mathbf{T}_2 \triangleleft \cdots ??? \cdots \trianglelefteq \mathbf{T}_{\max}.$$

Later Shelah proved in [28] that a weaker property, SOP_3, is also sufficient for a theory to be in the maximum class of Keisler's order.

This result pointed to Keisler's order as a promising way of attacking the problem of classifying unstable theories.

In this context, the work of Malliaris and Shelah in [20] showed that an even weaker property, SOP$_2$, is sufficient for a complete theory to be in the maximum class. The details are expanded in section 4.1.

Malliaris and Shelah continue to this day the project of classifying the structure of Keisler's order, with fruitful results such as the existence of a minimum unstable class $\mathbf{T}_3$ right after $\mathbf{T}_2$ [17, Section 5] and that there are an infinite number of classes [21].

1.3 Why trees and cuts?

In chapter 5 we will study, given a regular ultrafilter D over λ, what happens to the cuts in $(\mathbb{N},<)^\lambda/D$, and conclude that as long as D has λ^+-treetops, a property about the trees living in the ultrapowers using D, then $(\mathbb{N},<)^\lambda/D$ does not have any cut with size less than or equal to λ.

A natural question is why are we searching for a relation between cuts and trees. The motivation behind it is our objective of section 4.1, that is to show that any theory with SOP_2 is in the maximum class of Keisler's order.

First let us look at the property SOP_2.

Definition A. The theory T has SOP_2 if there is a formula $\psi(x;\bar{y})$ such that in some (equivalently any) model $\mathcal{M} \models T$ there exists $\{\bar{a}_\eta : \eta \in {}^{<\kappa}\mu\}$, called an SOP_2-tree for ψ, such that:

1. for $\eta, \rho \in {}^{<\kappa}\mu$ incompatible, that is $\neg(\eta \trianglelefteq \rho) \wedge \neg(\rho \trianglelefteq \eta)$, we have that $\{\psi(x;\bar{a}_\eta), \psi(x;\bar{a}_\rho)\}$ is inconsistent.

2. for $\eta \in {}^{\kappa}\mu$, $\{\psi(x;\bar{a}_{\eta\restriction i}) : i < \kappa\}$ is consistent, making it a type.

So, in $\mathcal{M}$, we have the tree ${}^{<\kappa}\mu$ and for each node $\eta \in {}^{<\kappa}\mu$ we associate a value $\bar{a}_\eta$. Elements of different branches are mutually inconsistent parameters for ψ, and the set of all elements of a branch of length κ is a set of mutually consistent parameters for ψ.

By the definition of SOP_2 we can see why it is natural to look at the trees inside our theory. And, not surprisingly, we will prove a theorem about the relation between the tree property stated before and SOP_2, namely that for a complete theory T with SOP_2 and a regular ultrafilter D, the ultrafilter D has λ^+-treetops if and only if every SOP_2-type is realized in any ultrapower of models of T by D, where SOP_2-types are special types constructed using a formula with SOP_2.

Given that the set of all SOP_2-types is just a subset of the set of all types we have that saturation implies realization of all SOP_2-types, and by the theorem mentioned above, it implies that D has λ^+-treetops.

So

$$T \text{ is } \lambda^+\text{-saturated by } D \Rightarrow D \text{ realizes all } SOP_2\text{-types} \Rightarrow D \text{ has } \lambda^+\text{-treetops.}$$

The definition of "T is λ^+-saturated by D" will be given in Section 4.1, but think of the intuitive idea that every ultrapower of a model of T by D is λ^+-saturated. Now, to understand where the cuts come from we need to look at Keisler's set theoretic characterization of the maximum class.

Theorem B (Keisler's characterization of the maximum class)**.** *There is a maximum class in Keisler's order, which consists precisely of those complete theories T such that for any regular ultrafilter D over λ, we have*

$$T \textit{ is } \lambda^+\textit{-saturated by } D \Leftrightarrow D \textit{ is } \lambda^+\textit{-good.}$$

The definition of λ^+-good ultrafilter will be given in Section 4.1. Nonetheless, the important property proved in there is that D is λ^+-good if and only if there are no cuts in $(\mathbb{N}, <)^I/D$ of size less than or equal to λ (this last property will be denoted by $\mathcal{C}(D) = \emptyset$).

Since our objective is to show that any complete theory T with SOP_2 is in the maximum class, we need to show that for any regular ultrafilter D over λ we have that

$$T \text{ is } \lambda^+\text{-saturated by } D \Leftrightarrow D \text{ is } \lambda^+\text{-good.}$$

By the definition of λ^+-good ultrafilters it will be easy to prove that λ^+-goodness implies λ^+-saturation. So the difficult part will be to prove that λ^+-saturation implies λ^+-goodness.

On the one hand we have that λ^+-saturation implies λ^+-treetops and on the other we have that D is λ^+-good if and only if $\mathcal{C}(D) = \emptyset$. So the only bridge left is to show that λ^+-treetops implies that $\mathcal{C}(D) = \emptyset$. This is why the objective of chapter 5 is to show exactly that.

Figure 1 – Chain of consequences from saturation to goodness

$$
\begin{array}{c}
T \text{ is } \lambda^+\text{-saturated by } D \\
\Downarrow \\
D \text{ realizes all } SOP_2\text{-types} \\
\Downarrow \\
D \text{ has } \lambda^+\text{-treetops} \\
\Downarrow \scriptstyle\text{Chap.5} \\
\mathcal{C}(D) = \emptyset \\
\Downarrow \\
D \text{ is } \lambda^+\text{-good}
\end{array}
$$

2 Preliminaries

2.1 Theories and Models

First, we need to define some essential objects of model theory, languages, which enable us to construct formulas, and structures, which give semantic meaning to those formulas.

Definition 2.1.1 ([22], Definition 1.1.1)**.** A *language* $\mathcal{L}$ is given by specifying the following data:

1. a set of function symbols $\mathcal{F}$ and positive integers n_f for each $f \in \mathcal{F}$;
2. a set of relation symbols $\mathcal{R}$ and positive integers n_R for each $R \in \mathcal{R}$;
3. a set of constant symbols $\mathcal{C}$.

The numbers n_f and n_R tell us that f is a function of n_f variables and R is an n_R-ary relation.

Any or all the sets $\mathcal{F}, \mathcal{R}$, and $\mathcal{C}$ may be empty. For example the language of ZFC has only one binary relation symbol and no function or constant symbols, namely $\mathcal{L}_{\text{ZFC}} = \{\in\}$.

With a language $\mathcal{L}$, we can define the notion of a structure using that language.

Definition 2.1.2 ([22], Definition 1.1.2)**.** An $\mathcal{L}$-*structure* $\mathcal{M}$ is given by the following data:

1. a nonempty set M called the *universe, domain,* or *underlying set* of $\mathcal{M}$;
2. a function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $f \in \mathcal{F}$;
3. a set $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$;
4. an element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$.

We refer to $f^{\mathcal{M}}, R^{\mathcal{M}}$, and $c^{\mathcal{M}}$ as the *interpretations* of the symbols f, R, and c. We often write the structure as $\mathcal{M} = (M, f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}} : f \in \mathcal{F}, R \in \mathcal{R}, c \in \mathcal{C})$. We will use the notation $A, B, M, N, \ldots$ to refer to the underlying sets of the structures $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \ldots$.

With this in mind we use the language $\mathcal{L}$ to create formulas describing, in first-order, properties of $\mathcal{L}$-structures. Formulas will be strings of symbols built using the symbols of $\mathcal{L}$, variable symbols $v_1, v_2, \ldots$, the equality symbol $=$, the Boolean connectives $\wedge, \vee$, and

$\neg$, which we read as "and", "or", and "not", the quantifiers $\exists$ and $\forall$, which we read as "there exists" and "for all", and parentheses (,).

Definition 2.1.3 ([22], Definition 1.1.4)**.** The set of $\mathcal{L}$*-terms* is the smallest set $\mathcal{T}$ such that

1. $c \in \mathcal{T}$ for each constant symbol $c \in \mathcal{C}$,
2. each variable symbol $v_i \in \mathcal{T}$ for $i = 1, 2, \ldots$, and
3. if $t_1, \ldots, t_{n_f} \in \mathcal{T}$ and $f \in \mathcal{F}$, then $f(t_1, \ldots, t_{n_f}) \in \mathcal{T}$.

Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and that t is a term built using variables from $\bar{v} = (v_{i_1}, \ldots, v_{i_m})$. We want to interpret t as a function $t^{\mathcal{M}} : M^m \to M$. For s a subterm of t and $\bar{a} = (a_{i_1}, \ldots, a_{i_m}) \in M$, we inductively define $s^{\mathcal{M}}(\bar{a})$ as follows:

1. if s is a constant symbol c, then $s^{\mathcal{M}}(\bar{a}) = c^{\mathcal{M}}$;
2. if s is the variable v_{i_j}, then $s^{\mathcal{M}}(\bar{a}) = a_{i_j}$;
3. if s is the term $f(t_1, \ldots, t_{n_f})$, where f is a function symbol of $\mathcal{L}$ and $t_1, \ldots, t_{n_f}$ are terms, then $s^{\mathcal{M}}(\bar{a}) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{n_f}^{\mathcal{M}}(\bar{a}))$.

The function $t^{\mathcal{M}}$ is defined by $\bar{a} \mapsto t^{\mathcal{M}}(\bar{a})$.

We are now ready to define $\mathcal{L}$-formulas.

Definition 2.1.4 ([22], Definition 1.1.5)**.** We say that φ is an *atomic* $\mathcal{L}$*-formula* if φ is either

1. $t_1 = t_2$, where t_1 and t_2 are terms, or
2. $R(t_1, \ldots, t_{n_R})$, where $R \in \mathcal{R}$ and $t_1, \ldots, t_{n_R}$ are terms.

The set of $\mathcal{L}$*-formulas* is the smallest set $\mathcal{W}$ containing the atomic formulas such that

1. if φ is in $\mathcal{W}$, then $\neg\varphi$ is in $\mathcal{W}$,
2. if φ and ψ are in $\mathcal{W}$, then $(\varphi \wedge \psi)$ and $(\varphi \vee \psi)$ are in $\mathcal{W}$, and
3. if φ is in $\mathcal{W}$, then $\exists v_i \varphi$ and $\forall v_i \varphi$ are in $\mathcal{W}$.

Throughout this dissertation it is assumed that we are always working in first-order logic, that is, we can use quantifiers in the construction of formulas; however, any quantifier is bounded to only one universe. For example, let $\mathcal{L} = \{+, \cdot, 0, 1\}$ be the language of arithmetic, and $\mathcal{M}$ the $\mathcal{L}$-structure $(\mathbb{N}, +, \cdot, 0, 1)$. We cannot have a formula of the form

$\exists X(X = (0,1,1+1))$ translate to "there exists a sequence of natural numbers X such that $X = (0,1,2)$", because the quantifier $\exists$ ranges only in $\mathbb{N}$ and not on the set of sequences of $\mathbb{N}$.

We say that a variable v *occurs freely* in a formula φ if it is not inside the scope of a $\exists v$ or $\forall v$ quantifier; otherwise, we say that it is *bound*. For example, using the language $\mathcal{L} = \{0\}$, the variable v occurs freely in the formula $v = 0$ while it is bound in the formula $\exists v(v = 0)$. We call a formula a *sentence* if it has no free variables.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure. We will define a complete notion of truth inside of $\mathcal{M}$. By that I mean that each $\mathcal{L}$-sentence will either true or false inside of $\mathcal{M}$. On the other hand, if φ is a formula with free variables $v_1, \dots, v_n$, we will think of φ as expressing a property of elements of M^n. We often write $\varphi(v_1, \dots, v_n)$ to make explicit the free variables in φ. We must define what it means for $\varphi(v_1, \dots, v_n)$ to hold of $(a_1, \dots, a_n) \in M^n$.

Definition 2.1.5. Let φ be a formula with free variables from $\bar{v} = (v_{i_1}, \dots, v_{i_m})$, and let $\bar{a} = (a_{i_1}, \dots, a_{i_m}) \in M^m$. We inductively define $\mathcal{M} \models \varphi(\bar{a})$ as follows:

1. if φ is $t_1 = t_2$, then $\mathcal{M} \models \varphi(\bar{a})$ if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$;
2. if φ is $R(t_1, \dots, t_{n_R})$, then $\mathcal{M} \models \varphi(\bar{a})$ if $(t_1^{\mathcal{M}}(\bar{a}), \dots, t_{n_R}^{\mathcal{M}}(\bar{a}) \in R^{\mathcal{M}}$;
3. if φ is $\neg\psi$, then $\mathcal{M} \models \varphi(\bar{a})$ if $\mathcal{M} \not\models \psi(\bar{a})$;
4. if φ is $(\psi \wedge \theta)$, then $\mathcal{M} \models \varphi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ and $\mathcal{M} \models \theta(\bar{a})$;
5. if φ is $(\psi \vee \theta)$, then $\mathcal{M} \models \varphi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ or $\mathcal{M} \models \theta(\bar{a})$;
6. if φ is $\exists v_j(\psi(\bar{v}, v_j))$, then $\mathcal{M} \models \varphi(\bar{a})$ if there is $b \in M$ such that $\mathcal{M} \models \psi(\bar{a}, b)$;
7. if φ is $\forall v_j(\psi(\bar{v}, v_j))$, then $\mathcal{M} \models \varphi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a}, b)$ for all $b \in M$.

If $\mathcal{M} \models \varphi(\bar{a})$ we say that $\mathcal{M}$ *satisfies* $\varphi(\bar{a})$ or $\varphi(\bar{a})$ is *true* in $\mathcal{M}$.

We will use throughout the dissertation some common useful abbreviations. For example $\varphi \to \psi$ as an abbreviation for $\neg\varphi \vee \psi$, and $\bigwedge_{i=1}^n \varphi_i$ as an abbreviation for $\varphi_1 \wedge \dots \wedge \varphi_n$.

Let $\mathcal{L}$ be a language. An *$\mathcal{L}$-theory* T is simply a set of $\mathcal{L}$-sentences. We say that an $\mathcal{L}$-structure $\mathcal{M}$ is a *model* of T and write $\mathcal{M} \models T$ if $\mathcal{M} \models \varphi$ for all sentences $\varphi \in T$.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. The language $\mathcal{L}_A$ is defined as the language $\mathcal{L}$ with the addition of a constant symbol for each element of A. The set $\mathrm{Th}_A(\mathcal{M})$ is composed of all $\mathcal{L}_A$-sentences φ such that $\mathcal{M} \models \varphi$. We usually write $\mathrm{Th}(\mathcal{M})$ as the abbreviation for $\mathrm{Th}_\emptyset(\mathcal{M})$. The set $\mathrm{Th}(\mathcal{M})$ is a straightforward example of a theory, because $\mathcal{M}$ is by definition a model of $\mathrm{Th}(\mathcal{M})$.

Definition 2.1.6. Let T be an $\mathcal{L}$-theory. We say that T is *satisfiable* if T has a model. We say that T is *finitely satisfiable* if every finite subset of T is satisfiable.

Let T be an $\mathcal{L}$-theory and φ and $\mathcal{L}$-sentence. We say that φ is *true* in T if $\mathcal{M} \models \varphi$ for all models $\mathcal{M}$ of T. If φ is true in T we write $T \models \varphi$. We now have a semantic lens to view theories and sentences, given that to check if $T \models \varphi$ we need to look at how the models of T interpret the sentence φ.

Another important concept is that of a proof of a sentence φ from a theory T. A *proof* of φ from T is a finite sequence of $\mathcal{L}$-formulas $\psi_1, \dots, \psi_m$ such that $\psi_m = \varphi$ and $\psi_i \in T$ or ψ_i follows from $\psi_1, \dots, \psi_{i-1}$ by a simple logical rule for each i. An example of a simple logical rule is modus ponens, where if we have $\varphi \in T$ and $(\varphi \to \psi) \in T$, then it follows that $\psi \in T$. We write $T \vdash \varphi$ if there is a proof of φ from T.

Definition 2.1.7. Let T be an $\mathcal{L}$-theory. We say that T is *inconsistent* if $T \vdash (\varphi \wedge \neg\varphi)$ for some sentence φ, otherwise we say that T is *consistent*.

We now have a syntactic lens to view theories and sentences, given that to check if $T \vdash \varphi$ we need only to look at formulas through a mechanical process of simple logical rules, devoid of any interpretation for the symbols of the language.

The following theorem states that, when working in first-order logic, the syntactic notion and the semantic notion always agree with each other.

Theorem 2.1.8 (Gödel's Completeness Theorem, [22], Theorem 2.1.2)**.** *Let T be an $\mathcal{L}$-theory and φ an $\mathcal{L}$-sentence. Then $T \models \varphi$ if and only if $T \vdash \varphi$.*

A direct consequence of the completeness theorem is the equivalence between consistency and satisfiability.

Corollary 2.1.9 ([22], Corollary 2.1.3)**.** *Let T be an $\mathcal{L}$-theory. Then T is consistent if and only if T is satisfiable.*

Now we have a direct consequence of this corollary that is important enough to be stated as a theorem.

Theorem 2.1.10 (Compactness Theorem, [22], Theorem 2.1.4)**.** *Let T be an $\mathcal{L}$-theory. Then T is satisfiable if and only if T is finitely satisfiable.*

This theorem is a cornerstone of model theory as it enables us to assert the existence of a model for a theory T even when we cannot construct such a model, as long as we construct a model for each finite subset of T.

Definition 2.1.11 ([22], Definition 2.2.1)**.** An $\mathcal{L}$-theory T is called *complete* if for any $\mathcal{L}$-sentence φ either $T \models \varphi$ or $T \models \neg\varphi$.

Looking back at the theory $T := \mathrm{Th}(\mathcal{M})$ that we defined before, it is a complete theory, because for every $\mathcal{L}$-sentence φ, either $\mathcal{M} \models \varphi$ or $\mathcal{M} \models \neg\varphi$, so either $\varphi \in T$ or $\neg\varphi \in T$.

Now we define the usual morphisms of model theory and how they relate with each other.

Definition 2.1.12 ([22], Definition 1.1.3)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures with universes M and N, respectively. An $\mathcal{L}$*-embedding* $\eta : \mathcal{M} \to \mathcal{N}$ is a one-to-one map $\eta : M \to N$ that preserves the interpretation of all the symbols of $\mathcal{L}$. More precisely:

1. $\eta(f^{\mathcal{M}}(m_1, \ldots, m_l)) = f^{\mathcal{N}}(\eta(m_1), \ldots, \eta(m_l))$ for all f function symbols of $\mathcal{L}$ and $m_1, \ldots, m_l \in M$.
2. $(m_1, \ldots, m_l) \in R^{\mathcal{M}}$ if and only if $(\eta(m_1), \ldots, \eta(m_l)) \in R^{\mathcal{N}}$ for all R relation symbols of $\mathcal{L}$ and $m_1, \ldots, m_l \in M$.
3. $\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for all c constant symbols of $\mathcal{L}$.

A bijective $\mathcal{L}$-embedding is called an $\mathcal{L}$*-isomorphism*. If $M \subseteq N$ and the inclusion map is an $\mathcal{L}$-embedding, we say either that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ or that $\mathcal{N}$ is an *extension* of $\mathcal{M}$.

Definition 2.1.13 ([22], Definition 2.3.1)**.** If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures, then an $\mathcal{L}$-embedding $\eta : \mathcal{M} \to \mathcal{N}$ is called an *elementary embedding* if

$$\mathcal{M} \models \varphi(m_1, \ldots, m_l) \Longleftrightarrow \mathcal{N} \models \varphi(\eta(m_1), \ldots, \eta(m_l))$$

for all $\mathcal{L}$-formulas $\varphi(v_1, \ldots, v_l)$ and all $m_1, \ldots, m_l \in M$. If $\mathcal{M}$ is a substructure of $\mathcal{N}$, we say that it is an *elementary substructure*, and write $\mathcal{M} \prec \mathcal{N}$ if the inclusion map is elementary. We also say that $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$.

Definition 2.1.14 ([22], Definition 1.1.9)**.** We say that two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent* and write $\mathcal{M} \equiv \mathcal{N}$ if

$$\mathcal{M} \models \varphi \Longleftrightarrow \mathcal{N} \models \varphi$$

for all $\mathcal{L}$-sentences φ.

Proposition 2.1.15 ([22], Theorem 1.1.10)**.** *Suppose that $\eta : \mathcal{M} \to \mathcal{N}$ is an isomorphism. Then $\mathcal{M} \equiv \mathcal{N}$.*

Remark. On the other hand being elementarily equivalent does not imply being isomorphic. For example, let D be a non-principal ultrafilter over ω, then by Łoś's Theorem (stated in Section 2.2) we have that $\mathbb{N} \equiv \mathbb{N}^{\omega}/D$, but they are not isomorphic ($\mathbb{N}^{\omega}/D$ has cardinality $2^{\aleph_0}$).

Definition 2.1.16 ([22], Exercise 1.4.15)**.** Let $\mathcal{L}$ and $\mathcal{L}'$ be languages such that $\mathcal{L} \subseteq \mathcal{L}'$. If $\mathcal{M}'$ is an $\mathcal{L}'$-structure, then by ignoring the interpretation of the symbols in $\mathcal{L}' \setminus \mathcal{L}$ we get an $\mathcal{L}$-structure $\mathcal{M}$. We call $\mathcal{M}$ a *reduct* of $\mathcal{M}'$ and $\mathcal{M}'$ an *expansion* of $\mathcal{M}$.

2.2 Ultraproducts

In model theory we want to be able to construct new structures using known structures as a base, and the ultraproduct construction is one such way. Ultraproducts will be central for the definition of cofinality spectrum problems and Keisler's Order, mainly for their property of preserving truth of sentences, a consequence of the main theorem of this section, Łoś's Theorem.

To define ultraproducts we first need to define and explore the properties of filters.

Definition 2.2.1. Let I be a nonempty set. A set $D \subseteq \mathcal{P}(I)$ is said to be a *filter over* I if:

1. $I \in D$;
2. if $X, Y \in D$, then $X \cap Y \in D$;
3. if $X \in D$ and $X \subseteq Z \subseteq I$, then $Z \in D$.

Definition 2.2.2. Let E be a subset of $\mathcal{P}(I)$. By the *filter generated by* E we mean the intersection of all filters over I which include E, that is,

$$D = \bigcap \{F \subseteq \mathcal{P}(I) : E \subseteq F \text{ and } F \text{ is a filter over } I\}.$$

If a filter D is not $\mathcal{P}(I)$ then D is called a *proper filter*. By the third property of filters we can see that being a proper filter is the same as not having $\emptyset$ in the filter, and by the second property of filters we have that $\emptyset \notin D$ if and only if every finite intersection of elements of D is nonempty, so this prompts a definition.

Definition 2.2.3. A set X is said to have the *finite intersection property* if the intersection of any finite number of elements of X is nonempty.

So by the discussion above we have that D is a proper filter if and only if D is a filter with the finite intersection property.

We can also strengthen this property asking that the intersection be not only nonempty but also infinite. This strengthening will be necessary to define the cardinal characteristic $\mathfrak{p}$.

Definition 2.2.4. A set X is said to have the *strong finite intersection property* if the intersection of any finite number of elements of X is infinite.

We can add a completeness property when talking about filters, asking that, for a set $X \in \mathcal{P}(I)$, we have that X is in D or its complement, I/X, is in D. A filter that is proper and complete is called an *ultrafilter*.

Definition 2.2.5. Let D be a filter over I. D is said to be an *ultrafilter over* I if for all $X \in \mathcal{P}(I)$,

$$X \in D \longleftrightarrow I \setminus X \notin D.$$

Proposition 2.2.6 ([3], Proposition 4.1.2)**.** *D is an ultrafilter over I if and only if D is a maximal proper filter over I.*

As long as the axiom of choice is true we can always extend sets with the finite intersection property into an ultrafilter, using Zorn's Lemma.

Theorem 2.2.7 ([3], Proposition 4.1.3)**.** *If $E \subset \mathcal{P}(I)$ and E has the finite intersection property, then there exists an ultrafilter D over I such that $E \subseteq D$.*

Corollary 2.2.8 ([3], Corollary 4.1.4)**.** *Any proper filter over I can be extended to an ultrafilter over I.*

Suppose I is a nonempty set, D is a proper filter over I and for each $i \in I$, A_i is a nonempty set. Let

$$C = \prod_{i \in I} A_i$$

be the Cartesian product of these sets. Given the ordered structure of the Cartesian product, each ordered I-tuple in C can be seen as a function f with domain I such that for each $i \in I$, $f(i) \in A_i$. For two functions $f, g \in C$, we say that f and g are D-equivalent, denoted $f \equiv_D g$, if and only if

$$\{i \in I : f(i) = g(i)\} \in D.$$

Proposition 2.2.9 ([3], Proposition 4.1.5)**.** *The relation $\equiv_D$ is an equivalence relation over the set C.*

When using the equivalence relation $\equiv_D$, the filter D is, in a sense, telling us which subsets of I are "large", given that they tell us how many indexes have to agree for two elements to be considered equal to the equivalence relation. The subsets not in D are the "null" sets in this context, that is, the sets where even if the elements disagree in them, it does not matter for the equivalence relation.

Let f/D be the equivalence class of f, that is,

$$f/D = \{g \in C : f \equiv_D g\}.$$

Definition 2.2.10. The *reduced product of A_i modulo D*, denoted $\prod_{i\in I} A_i/D$, is the set of all equivalence classes of $\equiv_D$. That is,

$$\prod_{i\in I} A_i/D = \{f/D : f \in \prod_{i\in I} A_i\}.$$

In the special case where D is an ultrafilter over I, the reduced product $\prod_{i\in I} A_i/D$ is called an *ultraproduct*. In the case when all the sets A_i are the same, say $A_i = A$, the reduced product may be written A^I/D, and it is called the *reduced power of A modulo D*. Again, in the special case where D is an ultrafilter over I, the reduced power A^I/D is called an *ultrapower*.

Now we can define the same concepts for structures.

Definition 2.2.11 ([3], Definition 4.1.6)**.** Let $\mathcal{L}$ be a language, I be a nonempty set, D be a proper filter over I, and for each $i \in I$ we let $\mathcal{A}_i$ be an $\mathcal{L}$-structure with universe A_i. The *reduced product* $\prod_{i\in I} \mathcal{A}_i/D$ is the $\mathcal{L}$-structure described as follows:

1. The universe of $\prod_{i\in I} \mathcal{A}_i/D$ is $\prod_{i\in I} A_i/D$.

2. Let R be a n-ary relation symbol of $\mathcal{L}$. The interpretation of R in $\prod_{i\in I} \mathcal{A}_i/D$ is the relation S such that

$$S(f^1/D, \dots, f^n/D) \longleftrightarrow \{i \in I : R^{\mathcal{A}_i}(f^1(i), \dots, f^n(i))\} \in D.$$

3. Let F be a n-ary function symbol of $\mathcal{L}$. The interpretation of F in $\prod_{i\in I} \mathcal{A}_i/D$ is the function G such that

$$G(f^1/D, \dots, f^n/D) = \langle F^{\mathcal{A}_i}(f^1(i), \dots, f^n(i)) : i \in I\rangle/D.$$

4. Let c be a constant symbol of $\mathcal{L}$. The interpretation of c in $\prod_{i\in I} \mathcal{A}_i/D$ is the constant $b \in \prod_{i\in I} A_i/D$ such that

$$b = \langle c^{\mathcal{A}_i} : i \in I\rangle/D.$$

This definition is well-defined because both S and G do not depend on the representatives $f^1, \dots, f^n$ chosen for $f^1/D, \dots, f^n/D$.

As before, in the special case where D is an ultrafilter over I, the reduced product $\prod_{i\in I} \mathcal{A}_i/D$ is called an *ultraproduct*. In the case when all the models $\mathcal{A}_i$ are the same, say $\mathcal{A}_i = \mathcal{A}$, the reduced product may be written $\mathcal{A}^I/D$, and it is called the *reduced power of $\mathcal{A}$ modulo D*. Again, in the special case where D is an ultrafilter over I, the reduced power $\mathcal{A}^I/D$ is called an *ultrapower*.

Theorem 2.2.12 (Expansion Theorem, [3], Theorem 4.1.8)**.** *Let $\mathcal{L}$ and $\mathcal{L}'$ be languages such that $\mathcal{L} \subseteq \mathcal{L}'$. Let I be an nonempty set and for each $i \in I$ let $\mathcal{A}_i$ be a model for $\mathcal{L}$ and $\mathcal{B}_i$ an expansion of $\mathcal{A}_i$ to $\mathcal{L}'$. Let D be a filter over I. Then the reduced product $\prod_{i\in I} \mathcal{B}_i/D$ is an expansion of the reduced product $\prod_{i\in I} \mathcal{A}_i/D$ to $\mathcal{L}'$.*

This theorem says that the following diagram commutes:

$$\begin{tikzcd}
\prod_{i\in I} \mathcal{A}_i/D \arrow[r] & \prod_{i\in I} \mathcal{B}_i/D \\
\mathcal{A}_i \arrow[u] \arrow[r] & \mathcal{B}_i \arrow[u]
\end{tikzcd}$$

The following lemma gives us a powerful way of checking the value of terms when applied to elements of the ultraproduct.

Lemma 2.2.13. *Let D be an ultrafilter over I, and for each $i \in I$ let $\mathcal{A}_i$ be a model for $\mathcal{L}$. Then, for any term $t(x_1, \ldots, x_n)$ of $\mathcal{L}$ and elements $f^1/D, \ldots, f^n/D \in \prod_{i\in I} A_i/D$, we have*

$$t^{\prod_{i\in I} \mathcal{A}_i/D}(f^1/D, \ldots, f^n/D) = \langle t^{\mathcal{A}_i}(f^1(i), \ldots, f^n(i)) : i \in I\rangle/D.$$

We can use this lemma to show that the truth value of any sentence in $\prod_{i\in I} \mathcal{A}_i/D$ is given by how many of the index structures $\mathcal{A}_i$ agrees with the sentence, giving us a fundamental result of ultraproducts.

Theorem 2.2.14 (Łoś's Theorem)**.** *Let D be an ultrafilter over I, and for each $i \in I$ let $\mathcal{A}_i$ be a model for $\mathcal{L}$. Then, for any formula $\varphi(x_1, \ldots, x_n)$ of $\mathcal{L}$ and elements $f^1/D, \ldots, f^n/D \in \prod_{i\in I} A_i/D$, we have*

$$\prod_{i\in I} \mathcal{A}_i/D \models \varphi(f^1/D, \ldots, f^n/D) \iff \{i \in I : \mathcal{A}_i \models \varphi(f^1(i), \ldots, f^n(i))\} \in D.$$

Corollary 2.2.15 ([3], Corollary 4.1.10)**.** *Let $\mathcal{A}^I/D$ be an ultrapower of $\mathcal{A}$. Then $\mathcal{A} \equiv \mathcal{A}^I/D$.*

2.3 Types

Types are a tool that enable us to, among other properties, construct desired models and classify theories. In this section we will define and state important properties of types. At the end we will present an important example of the use of types in the context of linear orders. It will be a useful example as we will always work with linear orders throughout the dissertation.

Definition 2.3.1 ([22], Definition 4.1.1)**.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure, A be a subset of the universe M and p be a set of $\mathcal{L}_A$-formulas φ with n free variables $v_1, \ldots, v_n$. We call p an *n-type over A* if $p \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable. We say that p is a *complete n-type* if for all $\mathcal{L}_A$-formulas φ with n free variables we have $\varphi \in p$ or $\neg\varphi \in p$. We let $S_n^{\mathcal{M}}(A)$ be the set of all complete n-types over A, and $S^{\mathcal{M}}(A)$ be the set of all complete types over A, that is, $S^{\mathcal{M}}(A) = \bigcup_{n \geqslant 1} S_n^{\mathcal{M}}(A)$.

By the compactness theorem, we could replace "$p \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable" by "every finite subset of $p \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable".

Definition 2.3.2 ([22], Definition 4.1.2)**.** If p is an n-type over A, we say that $\bar{a} \in M^n$ *realizes* p if $\mathcal{M} \models \varphi(\bar{a})$ for all $\varphi \in p$. If p is not realized in $\mathcal{M}$ we say that $\mathcal{M}$ *omits* p.

We can redefine p being a type as "all finite subsets of p are realized in $\mathcal{M}$".

We can also label some special types by how they are constructed.

Definition 2.3.3. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, A be a subset of the universe M, p be an n-type over A, and $\psi(v_1, \ldots, v_n; u_1, \ldots, u_m)$ be a formula with $n + m$ free variables. We say that p is a *ψ-type* if there exists a set of parameters $B \subseteq A^m$ such that $p = \{\psi(\bar{v}; \bar{a}_i) : \bar{a}_i \in B\}$.

Next we state some results that enable us to view types as representatives for elements that may not exist in $\mathcal{M}$ but that do exist in some elementary extension of $\mathcal{M}$.

Theorem 2.3.4 ([22], Proposition 4.1.3)**.** *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$, and p an n-type over A. There is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ such that p is realized in $\mathcal{N}$.*

Define $\mathrm{tp}^{\mathcal{M}}(\bar{m}/A) := \{\varphi(v_1, \ldots, v_n) \in \mathcal{L}_A : \mathcal{M} \models \varphi(m_1, \ldots, m_n)\}$. We say that $\mathrm{tp}^{\mathcal{M}}(\bar{m}/A)$ is the complete type over A generated by $\bar{m}$.

Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$ and $A \subseteq M$. Then $\mathrm{Th}_A(\mathcal{M}) = \mathrm{Th}_A(\mathcal{N})$. Thus $S_n^{\mathcal{M}}(A) = S_n^{\mathcal{N}}(A)$. With this, Theorem 2.3.4 yields a characterization of complete types.

Corollary 2.3.5 ([22], Corollary 4.1.4)**.** *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$ and p an n-type over A. $p \in S_n^{\mathcal{M}}(A)$ if and only if there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\bar{b} \in N^n$ such that $p = tp^{\mathcal{N}}(\bar{b}/A)$.*

So every complete type of a structure is generated by a single tuple of elements in a sufficiently large elementary extension of that structure.

Theorem 2.3.6 ([22], Proposition 4.1.5)**.** *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. Let $\bar{a}, \bar{b} \in M^n$ such that $tp^{\mathcal{M}}(\bar{a}/A) = tp^{\mathcal{M}}(\bar{b}/A)$. Then, there exists an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and an automorphism σ of $\mathcal{N}$ fixing all elements of A such that $\sigma(\bar{a}) = \bar{b}$.*

Inspired by the topology on the space of complete types with basic open sets $[\varphi] = \{p : \varphi \in p\}$ we have the following definition.

Definition 2.3.7 ([22], Definition 4.2.1)**.** Let T be an $\mathcal{L}$-theory, $\varphi(v_1, \cdots, v_n)$ an $\mathcal{L}$-formula such that $T \cup \{\varphi(\bar{v})\}$ is satisfiable, and p an n-type. We say that φ *isolates* p if

$$T \models \forall \bar{v}(\varphi(\bar{v}) \to \psi(\bar{v}))$$

for all $\psi \in p$.

Note that if p is a complete type and $\varphi(\bar{v})$ isolates p, then

$$(T \models \varphi(\bar{v}) \to \psi(\bar{v})) \iff \psi(\bar{v}) \in p$$

for all $\mathcal{L}$-formulas $\psi(\bar{v})$.

So p is non isolated if and only if for every $\mathcal{L}$-formula $\varphi(\bar{v})$ which is consistent with T, there exists $\psi(\bar{v}) \in p$ such that $\varphi(\bar{v}) \wedge \neg\psi(\bar{v})$ is consistent with T.

Proposition 2.3.8 ([22], Proposition 4.2.2)**.** *If $\varphi(\bar{v})$ isolates p, then p is realized in any model of $T \cup \{\exists \bar{v}\varphi(\bar{v})\}$. In particular, if T is complete, then every isolated type is realized.*

Let T be a complete $\mathcal{L}$-theory and p an n-type. This proposition says that if T has a model that omits p then p is non isolated. The converse of this property is also true for any consistent theory in a countable language. Even though it will not be useful for this dissertation, it is stated for compilation purposes.

Theorem 2.3.9 (Omitting types theorem)**.** *Let $\mathcal{L}$ be a countable language, T an $\mathcal{L}$-theory, and p a non isolated n-type. Then there is a countable $\mathcal{M} \models T$ omitting p.*

Now we define the property of *saturation*. It is an essential property that will enable us to make strong characterizations in the future.

Definition 2.3.10 ([22], Definition 4.3.1)**.** Let κ be an infinite cardinal and T a theory. We say that $\mathcal{M} \models T$ is *κ-saturated* if, for all $A \subseteq M$, if $|A| < \kappa$ and $p \in S_n^{\mathcal{M}}(A)$, then p is realized in $\mathcal{M}$.

We say that $\mathcal{M}$ is *saturated* if it is $|M|$-saturated.

Proposition 2.3.11 ([22], Proposition 4.3.2)**.** *Let $\kappa \geqslant \aleph_0$. The following are equivalent:*

1. *$\mathcal{M}$ is κ-saturated;*
2. *if $A \subseteq M$ with $|A| < \kappa$ and p is a (possibly incomplete) n-type over A, then p is realized in $\mathcal{M}$;*
3. *if $A \subseteq M$ with $|A| < \kappa$ and $p \in S_1^{\mathcal{M}}(A)$, then p is realized in $\mathcal{M}$.*

So we need only look at complete 1-types to determine if a model $\mathcal{M}$ is κ-saturated.

Proposition 2.3.12 ([22], Corollary 4.3.5)**.** *If $\mathcal{M}, \mathcal{N} \models T$ are countable saturated models, then $\mathcal{M} \cong \mathcal{N}$.*

Another important property that relates closely to saturation is *compactness*.

Definition 2.3.13 ([27], Chapter I, Definition 1.2)**.** A structure $\mathcal{M}$ is λ-*compact* if every type in $\mathcal{M}$ of cardinality less than λ is realized in $\mathcal{M}$.

Lemma 2.3.14 ([27], Chapter I, Theorem 1.9)**.** *Let $\mathcal{L}$ be a language and λ be a cardinal such that $\lambda > |\mathcal{L}|$. Then for any $\mathcal{L}$-structure $\mathcal{M}$, we have that $\mathcal{M}$ is λ-compact if and only if it is λ-saturated.*

The usefulness of this lemma is that we will, in general, show that a structure is λ-saturated by proving that it is λ-compact, as long as the hypothesis apply. Given that we will only work with theories in countable languages, we have, for any infinite cardinal λ, that $\lambda^+ > |\mathcal{L}|$. Then, this lemma enables us to work with λ^+-compactness and λ^+-saturation interchangeably. This will be useful in chapter 4.1, as λ^+-saturation will be very important there.

Example 2.3.15 (Dense Linear Orders)**.** Let $\mathcal{L} = \{<\}$. Let $\mathcal{M} = (M, <)$ be a dense linear order and let $A \subseteq M$. Let $p \in S_1^{\mathcal{M}}(A)$. For every a in A, because p is complete, exactly one of the formulas $v = a, v < a$ or $v > a$ is in p.

With this in mind we can, for every $p \in S_1^{\mathcal{M}}(A)$, define a pre-cut [1] on $(A, <)$ by $L_p = \{a \in A : \langle a < v\rangle \in p\}$ and $U_p = \{a \in A : \langle a > v\rangle \in p\}$, where one of two cases is true, or (L_p, U_p) is a cut in A, that is, there is not an element of A in between both sets, or there is exactly one $a \in A$ where a is between both set and the formula $v = a$ is in p. On the other hand, every pre-cut (L, U) with the same properties (being a cut or missing one element) determines a type p with $L_p = L$ and $U_p = U$, by the use of the density property. Since the atomic formulas with one free variable on $\mathcal{L}_A$ are only of the form $(v = a), (v < a)$ or $(v > a)$, this determination is unique and well defined.

Remark 2.3.16. The representation of types by cuts is true for every linear order. Density is only used to ensure characterization. The fact that even in a non dense linear order like $(\mathbb{N}, <)$ we can represent types by cuts will be important in Section 4.1.

With this we can show that $(\mathbb{Q}, <)$ and any other dense linear order, are saturated. By proposition 2.3.11 we need only to search for 1-types. Since $|\mathbb{Q}| = \aleph_0$, suppose $A \subseteq \mathbb{Q}$ is finite, in particular $A = \{a_1, \cdots, a_m\}$ with $a_1 < \cdots < a_m$. As seen above, we can find all complete 1-types over A by describing all possible cuts of A. We have m complete types

[1] The definition of pre-cut and cut used in this example can be found in Chapter 3, Definition 3.0.1

realized in A by $v = a_k$ for each k with $L = \{a \in A : a < a_k\}$ and $U = \{a \in A : a_k < a\}$, and $m+1$ complete types not realized in A with, for each $k \in \{0, \cdots, m\}$, $L = \{a \in A : a < \frac{a_k + a_{k+1}}{2}\}$ and $U = \{a \in A : \frac{a_k + a_{k+1}}{2} < a\}$, defining $a_0 := 0$ and $a_{m+1} := a_m$. (The use of $\frac{a_k + a_{k+1}}{2}$ is arbitrary, any element of $\mathbb{Q}$ defining the same cut can be chosen.) The types realized in A are obviously realized in $\mathbb{Q}$, for the ones not realized in A we have three cases, all using the fact that L always has a largest element l, and U has a least element u.

1. For $L = \emptyset$ and $U = A$, the formula $v < u$ isolates the type;

2. For $L = A$ and $U = \emptyset$, the formula $l < v$ isolates the type;

3. For $L \neq \emptyset$ and $U \neq \emptyset$, the formula $l < v < u$ isolates the type.

Then, by proposition 2.3.8, every complete 1-type over A is realized in $\mathbb{Q}$, since A was arbitrary we conclude that $(\mathbb{Q}, <)$ is saturated.

2.4 Forcing

The method of forcing is a tool, similar to ultraproducts, used to construct new structures from known ones. Forcing will be one of the many tools used in the proof of $\mathfrak{p} = \mathfrak{t}$, presented in section 4.2.

Definition 2.4.1 ([13], Definition III.3.1)**.** $(\mathbb{P}, \leqslant, \mathbb{1})$ is a *forcing poset* if $(\mathbb{P}, \leqslant)$ is a poset with $\mathbb{1}$ as its maximum.

Let $p, q \in \mathbb{P}$. In this context $q < p$ reads q *is stronger than* (or *extends*) p. If an $r \in \mathbb{P}$ with $r \leqslant q$ and $r \leqslant p$ does not exist we say that p and q are *incompatible* ($p \perp q$), otherwise they are *compatible*.

A subset $D \subseteq \mathbb{P}$ is said to be *dense* if for any $p \in \mathbb{P}$ there is a $q \in D$ such that $q \leqslant p$.

Now we introduce a new, more general definition of filters in the context of forcing posets.

Definition 2.4.2 ([13], Definition III.3.10)**.** A subset $D \subseteq \mathbb{P}$ is a *filter on* $\mathbb{P}$ if:

1. $\mathbb{1} \in D$;

2. Any two elements $p, q \in D$ are compatible inside of D, that is, there is an $r \in D$ such that $r \leqslant p$ and $r \leqslant q$;

3. If $p \in D, q \in \mathbb{P}$ and $p \leqslant q$ then $q \in D$. (Upward closed)

Remark. A filter with the definition introduced in the ultraproducts section is simply a filter with the forcing poset definition using $(\mathbb{P}, \leqslant) = (\mathcal{P}(I), \subseteq)$ and $\mathbb{1} = I$.

Definition 2.4.3 ([13], Definition IV.2.2)**.** Let $\mathcal{M}$ be a transitive model and $(\mathbb{P}, \leqslant, \mathbb{1}) \in M$ be a forcing poset. A filter G on $\mathbb{P}$ is said to be $\mathbb{P}$-generic over $\mathcal{M}$ if $G \cap D \neq \emptyset$ for all dense $D \subseteq \mathbb{P}$ such that $D \in M$.

Lemma 2.4.4 ([13], Lemma IV.2.3)**.** *Let $\mathcal{M}$ be a countable transitive model for $ZF - P$ and $(\mathbb{P}, \leqslant, \mathbb{1}) \in M$ be a forcing poset. Then for every $p \in \mathbb{P}$, there exists a filter G on $\mathbb{P}$ such that $p \in G$ and G is $\mathbb{P}$-generic.*

This lemma is essential for the construction of forcing as it guarantees the existence of generic filters. A reader unfamiliar with model theory may find it strange that we assume a seemingly strong hypothesis of countability; nevertheless it is conveniently resolved using the strong result of the Löwenheim-Skolem Theorem, which enables us to go from any infinite model to a countable model.

An element $r \in \mathbb{P}$ is an *atom* of $(\mathbb{P}, \leqslant, \mathbb{1})$ if there are no $p, q \leqslant r$ with $p \perp q$. $(\mathbb{P}, \leqslant, \mathbb{1})$ is *atomless* if it doesn't contain an atom.

Lemma 2.4.5 ([13], Lemma IV.2.4)**.** *If $(\mathbb{P}, \leqslant, \mathbb{1})$ is atomless and the filter G is $\mathbb{P}$-generic over $\mathcal{M}$, then $G \notin M$.*

What we want to do now is use a generic filter G to extend our ZFC model $\mathcal{M}$ to a ZFC model $\mathcal{M}[G]$ where G, and all the elements that it implies, exist.

Assuming that $\mathcal{M}$ is a model of ZFC, then we want $\mathcal{M}[G]$ to be the smallest model of ZFC containing $M \cup \{G\}$ as a subset of it. The intuition behind the construction of $\mathcal{M}[G]$ will be, in a way, similar to that of field extensions. To illustrate let us look at $\mathbb{Q}(\sqrt{2})$.

We can think of $\mathbb{Q}(\sqrt{2})$ as the field closure of $\mathbb{Q} \cup \{\sqrt{2}\}$, that is, the smallest field that contains $\mathbb{Q} \cup \{\sqrt{2}\}$, namely $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. However we can take another approach for the construction of $\mathbb{Q}(\sqrt{2})$, using the set of polynomials $\mathbb{Q}[x]$. Even though $\sqrt{2}$ is not in $\mathbb{Q}$, we can look at its minimal polynomial in $\mathbb{Q}[x]$, namely $x^2 - 2$, as a sort of "name" for $\sqrt{2}$ (For this context we will have a name associated with multiple values, namely all of its roots, but this is a problem that will not exist in the case of forcing), and as we know, the quotient of $\mathbb{Q}[x]$ by the ideal generated by $x^2 - 2$, that is $\mathbb{Q}[x]/(x^2 - 2)$, is isomorphic to $\mathbb{Q}(\sqrt{2})$. So we could use this construction as the definition of $\mathbb{Q}(\sqrt{2})$.

For the construction of $\mathcal{M}[G]$ we will do something similar. There will be a set of special elements $M^{\mathbb{P}} \subset M$, called the set of $\mathbb{P}$-names in M, and a value function outside of $\mathcal{M}$ dependent on G, $\mathrm{val}(x, G)$, which associates with each name its unique value in $\mathcal{M}[G]$.

The universe of $\mathcal{M}[G]$ will be, by definition, the set

$$M[G] = \{\mathrm{val}(x, G) : x \in M^{(\mathbb{P})}\},$$

that is the set of values of all the $\mathbb{P}$-names in M, using G as the parameter. It can be shown that if $\mathcal{M}$ is a model of ZFC, then $\mathcal{M}[G]$ is the smallest model of ZFC containing $M \cup \{G\}$ as a subset of it, as desired. For any element $x \in M[G]$ we will use the notation $\tilde{x}$ to represent a name for it (The value of a name is unique, however there can be lots of names with the same value; nonetheless, we can always select one of them).

Another desired property of this construction is that we want to be able to reason about sentences in $\mathcal{M}[G]$, even inside of $\mathcal{M}$, but the fact that all the names live in M is not sufficient for this; we have to, in a way, be able to reason about the value of a name even inside of $\mathcal{M}$. What will enable us to do this is the forcing relation.

The set $\mathcal{FL}_{\mathbb{P}}^{M}$, called *the* $\mathbb{P}$ *forcing language in* M, is the set of all the formulas using the binary relation $\in$ and $\mathbb{P}$-names in M as constant symbols.

Definition 2.4.6 ([13], Definition IV.2.22)**.** Let $\mathcal{M}$ be a countable transitive model of $ZF - P$, let $(\mathbb{P}, \leqslant, \mathbb{1}) \in M$ be a forcing poset and ψ be a sentence of $\mathcal{FL}_{\mathbb{P}}^{M}$. Then $p \Vdash_{\mathbb{P},\mathcal{M}} \psi$ holds if $\mathcal{M}[G] \models \psi$ for all filters G on $\mathbb{P}$ such that $p \in G$ and G is $\mathbb{P}$-generic over $\mathcal{M}$. We omit the subscripts $\mathbb{P}, \mathcal{M}$ on the $\Vdash$ when they are clear from context. "$p \Vdash \psi$" is read "p *forces* ψ".

This definition seems, at first sight, to be far away from our objective, given that it looks outside of $\mathcal{M}$ and into $\mathcal{M}[G]$, not only for one generic G but for all of them. However it is true that, fixed a formula ψ, not only can we reason if $p \Vdash \psi$ inside of $\mathcal{M}$, using the definability lemma below, but it is also the case that any sentence true in $\mathcal{M}[G]$ is forced by some $p \in G$, by the truth lemma below.

Theorem 2.4.7 (Definability Lemma; [13], Lemma IV.2.25)**.** *Let $\mathcal{M}$ be a countable transitive model for $ZF - P$, let $\varphi(x_1, \ldots, x_n)$ be a formula in $\mathcal{L} = \{\in\}$, with all free variables shown. Then*

$$\begin{gathered}\{(p, \mathbb{P}, \leqslant, \mathbb{1}, \vartheta_1, \ldots, \vartheta_n) : (\mathbb{P}, \leqslant, \mathbb{1}) \textit{ is a forcing poset } \wedge p \in \mathbb{P} \wedge \\ (\mathbb{P}, \leqslant, \mathbb{1}) \in M \wedge \vartheta_1, \ldots, \vartheta_n \in M^{\mathbb{P}} \wedge p \Vdash_{\mathbb{P},\mathcal{M}} \varphi(\vartheta_1, \ldots, \vartheta_n)\}\end{gathered}$$

is definable in $\mathcal{M}$ without parameters.

Theorem 2.4.8 (Truth Lemma; [13], Lemma IV.2.24)**.** *Let $\mathcal{M}$ be a countable transitive model of $ZF - P$, let $(\mathbb{P}, \leqslant, \mathbb{1}) \in M$ be a forcing poset, let ψ be a sentence of $\mathcal{FL}_{\mathbb{P}} \cap M$, and let G be a $\mathbb{P}$-generic filter over $\mathcal{M}$. Then $\mathcal{M}[G] \models \psi$ if and only if there is a $p \in G$ such that $p \Vdash \psi$.*

It is important to note the necessity of a fixed formula ψ for the definability lemma. If we could define the forcing notion uniformly for all ψ, then the truth lemma would imply that the formula $\exists p \in G(p \Vdash \psi(x_1, \ldots, x_n))$ is a definition of truth inside of $\mathcal{M}[G]$. This would contradict Tarski's undefinability theorem.

3 The motivation behind Cofinality Spectrum Problems

Our objective in this chapter is to begin attacking one of the problems presented in the introduction, that is to show that for any regular ultrafilter D that has the treetops property it is also true that $\mathcal{C}(D) = \emptyset$, and seeing how the idea of cofinality spectrum problems comes naturally from some early results.

We'll first formalize the objects presented in the introduction.

Let J be a linear order and $C_1, C_2 \subseteq J$. We say that $C_1 < C_2$ if for all $s_1 \in C_1$ and $s_2 \in C_2$ we have $s_1 <_J s_2$.

Definition 3.0.1. Let J be a linear order, and $C_1, C_2 \subseteq J$.

1. We say that (C_1, C_2) is a *pre-cut* of J if $C_1 <_J C_2$, the set C_1 is downward closed, and the set C_2 is upward closed.

2. We say that (C_1, C_2) is a *cut* of J if it is a pre-cut of J and $J = C_1 \cup C_2$.

3. For a pre-cut (C_1, C_2) of J, we say that the *cofinality* of (C_1, C_2), denoted by $\mathrm{cf}(C_1, C_2)$, is equal to (κ_1, κ_2) when
   - κ_1 is the cofinality of C_1.
   - κ_2 is the coinitiality (or downward cofinality) of C_2.

4. Suppose (C_1, C_2) is a pre-cut of J and $\mathrm{cf}(C_1, C_2) = (\kappa_1, \kappa_2)$.
   - We say that the κ_1-indexed sequence $\bar{\ell} = \langle \ell_\alpha : \alpha < \kappa_1 \rangle$ *witnesses* $\mathrm{cf}(C_1) = \kappa_1$ if $\bar{\ell}$ is $<_J$-increasing and cofinal in C_1.
   - We say that the κ_2-indexed sequence $\bar{u} = \langle u_\beta : \beta < \kappa_2 \rangle$ *witnesses* $\mathrm{dcf}(C_2) = \kappa_2$ if $\bar{u}$ is $<_J$-decreasing and coinitial in C_2.
   - We say that $(\bar{l}, \bar{u})$ *witnesses* $\mathrm{cf}(C_1, C_1) = (\kappa_1, \kappa_2)$ if $\bar{l}$ witnesses $\mathrm{cf}(C_1) = \kappa_1$ and $\bar{u}$ witnesses $\mathrm{dcf}(C_2) = \kappa_2$.

5. We say that J has a (κ_1, κ_2)-cut if there exists a cut in J with cofinality (κ_1, κ_2).

Definition 3.0.2. Let D be an ultrafilter over an index I. Then

$$\mathcal{C}(D) = \{(\kappa_1, \kappa_2) : \kappa_1, \kappa_2 \text{ regular}, \kappa_1 + \kappa_2 \leqslant |I|, (\mathbb{N}, <)^I/D \text{ has a } (\kappa_1, \kappa_2)\text{-cut}\}.$$

Definition 3.0.3. Let $(L, <_L)$ be a linear order and X be a set. We say that a subset $A \subseteq L$ is an *initial segment* of L if there is an element $m \in L$ where $A = \{\ell \in L : \ell <_L m\}$. We indicate by ${}^{<(L,<_L)}X$ the set of all functions from an initial segment of L to X.

Definition 3.0.4 (Trees)**.** Let $(L, <)$ be a discrete linear order with first element and X be a set. For a subset T of ${}^{<L}X$ we say that $(T, \subseteq)$ is a *tree* if it is closed under initial segments, that is, if $t \in T$ and $s \subseteq t$, then $s \in T$. For any $b \in T$, we define the length of b, denoted by $\lg(b)$, as $\min(L \setminus \operatorname{dom}(b))$, and we define the value of b, denoted $\operatorname{val}(b)$, as $b(\max(\operatorname{dom}(b)))$, that is, $b(\lg(b) - 1)$ (where $\lg(b) - 1$ just means the predecessor of $\lg(b)$). We denote an arbitrary tree by $(T, \trianglelefteq)$.

Definition 3.0.5. Let $\mathcal{M}$ be a structure in a language $\mathcal{L}$. We say that $\mathcal{M}$ *interprets a tree* if there exist $\mathcal{L}$-formulas $\varphi_T, \varphi_\trianglelefteq$ that define in $\mathcal{M}$ sets $T^\mathcal{M}, \trianglelefteq^\mathcal{M}$ such that $(T^\mathcal{M}, \trianglelefteq^\mathcal{M})$ represents a tree inside of $\mathcal{M}$, that is, there exist in $\mathcal{M}$ definable sets $L^\mathcal{M}, <^\mathcal{M}, X^\mathcal{M}$ such that $\mathcal{M} \models$ "$(L^\mathcal{M}, <^\mathcal{M})$ is a discrete linear order with first element, the set $T^\mathcal{M}$ is a subset of ${}^{<L^\mathcal{M}}X^\mathcal{M}$ closed by initial segments, and $\trianglelefteq^\mathcal{M}$ is the order by initial segment in $T^\mathcal{M}$".

Note that for a structure to interpret a tree it needs to be a model of a rich enough theory to be able to express and define the concepts of *tree*, *linear order*, etc., as required by the definition above. For example, the structure $\mathcal{M} = (\mathcal{H}(\kappa), \in)$ for some uncountable cardinal κ, will be often used (cf. Theorem 3.0.10).

Definition 3.0.6 (Treetops)**.** Let D be an ultrafilter over I and κ a regular cardinal. We say that D has *κ-treetops* when for any structure $\mathcal{M}$ that interprets a tree $(T^\mathcal{M}, \trianglelefteq^\mathcal{M})$, for any regular cardinal $\gamma < \kappa$ and any $\trianglelefteq^\mathcal{N}$-increasing sequence $\langle a_i : i < \gamma \rangle$ in $(T^\mathcal{N}, \trianglelefteq^\mathcal{N})$, where $\mathcal{N} = \mathcal{M}^I/D$, there is $a^* \in T^\mathcal{N}$ such that $a_i \trianglelefteq^\mathcal{N} a^*$ for all $i < \gamma$.

Definition 3.0.7. We say that a filter D over I is *κ-regular* when there is a collection $\bar{X} = \{X_i : i < \kappa\} \subseteq D$ such that for each $t \in I$,

$$|\{i < \kappa : t \in X_i\}| < \aleph_0.$$

That is, any infinite subset of $\bar{X}$ has empty intersection. Such a collection is called a *κ-regularizing family*. We call D *regular* when it is $|I|$-regular.

To recap, now with the objects formally defined, we want to show that given a regular ultrafilter D over λ with λ^+-treetops we have $\mathcal{C}(D) = \emptyset$, that is, there are no cuts of size less than or equal to λ in $(\mathbb{N}, <)^\lambda/D$.

Simplifying the problem, we can ask first if there are no *symmetric* cuts of size less than or equal to λ. The following lemma will prove that there are indeed no symmetric cuts. The method of the proof is very important as it will be a motivating factor for the construction of cofinality spectrum problems and the same ideas will be used again in later proofs found in chapter 5.

Lemma 3.0.8. *Let D be a regular ultrafilter over λ with λ^+-treetops. Then, for every regular $\kappa < \lambda^+$, the set $\mathcal{C}(D)$ has no (κ, κ)-cuts.*

Before we prove this theorem we need to lay some groundwork about the structures used in the proof, but first lets see what is the idea behind the proof. We want to suppose that there is a regular $\kappa < \lambda^+$ such that there is a symmetric cut (κ, κ) in $\mathcal{C}(D)$, witnessed by a pair of sequences $(\langle a_\alpha : \alpha < \kappa\rangle, \langle b_\beta : \beta < \kappa\rangle)$ in $(\mathbb{N}, <)^\lambda/D$, and show by contradiction that there is a tree in some ultraproduct of D with an increasing sequence of size κ that does not have an upper bound. The way to do it is to construct a tree where the value of its nodes is a pair of elements of $(\mathbb{N}, <)^\lambda/D$ and any time we "go up" in the tree, the value of the nodes gets closer together, that is, for x and y in the tree, if $x \triangleleft y$, then we have $\mathrm{val}(x) = (a, b)$ and $\mathrm{val}(y) = (c, d)$ with $a < c < d < b$. Given this tree, we will show that there is a sequence of size κ of elements of this tree such that the value of the α element of the sequence is the pair (a_α, b_α), that is, any time we "go up" in this sequence the elements of the sequences $\langle a_\alpha : \alpha < \kappa\rangle$ and $\langle b_\alpha : \alpha < \kappa\rangle$ gets closer to the "center" of the cut. If such a sequence exists than by treetops there would be an upper bound c_*, with value $(c_*(0), c_*(1))$, for this sequence inside this tree; however, by the definition of our tree this would imply that $a_\alpha < c_*(0) < c_*(1) < b_\alpha$ for any $\alpha < \kappa$, contradicting the hypothesis that (a, b) is a cut.

To formalize this idea we need to find a structure that enables us to define this particular tree of pairs of elements of $(\mathbb{N}, <)^\lambda/D$ and all the auxiliary tools, like the length function and the value function. Luckily, we have a model of (ZFC - P) using the set of hereditarily countable sets. It will suffice for our construction.

Definition 3.0.9 ([13], Definition I.13.27)**.** Let κ be an infinite cardinal. Define $H(\kappa) = \{x : |\mathrm{trcl}(x)| < \kappa\}$, that is, $H(\kappa)$ is the set of all sets x such that the cardinality of its transitive closure ($|\mathrm{trcl}(x)|$) is smaller than κ. The elements of $H(\kappa)$ are said to be *hereditarily of cardinality* $< \kappa$. In particular the elements of $H(\omega_1)$ are said to be *hereditarily countable.*

Theorem 3.0.10 ([13], Theorem II.2.1)**.** *Let κ be an uncountable regular cardinal. Then $(H(\kappa), \in)$ is a model of* ZFC *without the power set axiom.*

We will use the structure $(H(\omega_1), \in)$ as the universe to build trees. So the structures used for our proof will be as follows:

Let $\mathcal{M} = (\mathbb{N}, <)$ and $\mathcal{M}^+ = (H(\omega_1), \in)$, together with their ultrapowers by D, given by $\mathcal{N} = (\mathbb{N}, <)^\lambda/D$ and $\mathcal{N}^+ = (H(\omega_1), \in)^\lambda/D$. Now, we will prove the basic properties about these structures necessary to execute the idea of the proof explained before.

First let's understand the relations between the four structures at hand. Given $\eta : M \to M^+$ an embedding, define

$$\bar{\eta} : M^\lambda/D \to (M^+)^\lambda/D$$
$$a_D \mapsto (\eta \circ a)_D$$

remembering that if a is a representative of an element a_D of M^λ/D then it is a function from λ to M, so $\eta \circ a$ is a function from λ to M^+. Then $\bar{\eta}$ is an embedding from M^λ/D to $(M^+)^\lambda/D$ (This proposition can be extended to ultraproducts, using appropriate η_t for each $t < \lambda$ when selecting $a(t)$). So in our case using the natural embedding η from $(\mathbb{N}, <)$ to $(H(\omega_1), \in)$ where each natural number goes to its ordinal definition, and $d_{\mathcal{M}}$, $d_{\mathcal{M}^+}$ the natural embeddings to each ultrapower, that is, $d_{\mathcal{M}}(a) = (a, a, \ldots)/D$ and the same for $d_{\mathcal{M}^+}$, we have the following relations:

$$\begin{tikzcd} \mathcal{M}^\lambda/D \arrow[r, "\bar{\eta}"] & (\mathcal{M}^+)^\lambda/D \\ \mathcal{M} \arrow[u, "d_{\mathcal{M}}"] \arrow[r, "\eta"'] & \mathcal{M}^+ \arrow[u, "d_{\mathcal{M}^+}"'] \end{tikzcd}$$

Now, an important property that will be necessary is that if $\eta(\mathbb{N}) = \omega$ is definable in $\mathcal{M}^+$ by φ_ω, then φ_ω defines $\bar{\eta}(\mathbb{N}^\lambda/D)$ in $\mathcal{N}^+$, that is, the set defined by φ_ω in $\mathcal{N}^+$ represents the nonstandard integers. We will prove this property in a more general setting.

Proposition 3.0.11. *Let $\mathcal{M}$ and $\mathcal{M}^+$ be structures with η as an embedding from $\mathcal{M}$ to $\mathcal{M}^+$, let D be an ultrafilter. If $\eta[M]$ is definable in $\mathcal{M}^+$ (without parameters) by $\psi(x)$, then $\psi(x)$ defines $\bar{\eta}[M^\lambda/D]$ in $(\mathcal{M}^+)^\lambda/D$.*

Proof. Given b_D in $(M^+)^\lambda/D$, we know by Łoś's theorem that $(\mathcal{M}^+)^\lambda/D \models \psi(b_D)$ if and only if $\mathcal{M}^+ \models \psi(b[i])$ for D-many is, and $\psi(b[i])$ is true in $\mathcal{M}^+$ precisely when $b[i] \in \eta[M]$, on the other hand $b_D \in \bar{\eta}[M^\lambda/D]$ if and only if there is $a_D \in M^\lambda/D$ such that $b_D = (\eta \circ a)_D$, so by Łoś's theorem $b[i] = (\eta \circ a)[i]$ for D-many is (given that M^λ/D is the universe of the model $\mathcal{M}^\lambda/D$, we have that $a[i]$ is in M for all i.), that is, $b[i] \in \eta[M]$ for D-many is, which concludes that $(\mathcal{M}^+)^\lambda/D \models \psi(b_D)$ if and only if $b_D \in \bar{\eta}[M^\lambda/D]$. □

Remark. In the last proposition, even if we were in an extension of $\mathcal{M}^\lambda/D$, it would not be a problem for the proof, given that the intersection of elements of an ultrafilter is also in the ultrafilter, in this case would be the set of indexes for which $b[i] = (\eta \circ a)[i]$ and the set of indexes for which $a[i] \in M$.

Another property that will be essential for the proof is the pseudo-finite property and how it behaves when going from $\mathcal{M}^+$ to $\mathcal{N}^+$. We say that the pair of structures $(\mathcal{M}, \mathcal{M}^+)$ has the *pseudo-finite property* if $\mathcal{M}$ has a linear order relation $<$, there is an embedding η from $\mathcal{M}$ to $\mathcal{M}^+$ and each nonempty, $\mathcal{M}^+$-definable $<$-bounded subset of $\eta[M]$ has a $<$-least and $<$-greatest element. For our special case we know that each nonempty $<$-bounded subset of $\mathbb{N}$ has a $<$-least and $<$-greatest element, so in particular, using our embedding η, each nonempty $\in$-bounded subset of ω definable in $(H(\omega_1), \in)$ has a $\in$-least and $\in$-greatest element. For each $A \subseteq M^+$ defined by $\varphi(x; c)$ and ω defined by $\psi_\omega(x)$, this property can be expressed with the following sentence

$$(B_\varphi(c) \wedge \neg E_\varphi(c) \wedge S_\varphi(c)) \to (\exists y(m_\varphi(y; c)) \wedge \exists z(M_\varphi(z; c))))$$

where

- $B_\varphi(w) = \langle \exists x(\forall y(\varphi(y; w) \to (y \in x \vee y = x))) \rangle$ (A is bounded.)
- $E_\varphi(w) = \forall x(\neg\varphi(x, w))$ (A is empty.)
- $S_\varphi(w) = \forall x(\varphi(x; w) \to \psi_\omega(x))$ (A is a subset of ω.)
- $M_\varphi(z; w) = \langle \varphi(z; w) \wedge \forall y(\varphi(y; w) \to (y \in z \vee y = z)) \rangle$ (A has a maximum element.)
- $m_\varphi(z; w) = \langle \varphi(z; w) \wedge \forall y(\varphi(y; w) \to (z \in y \vee z = y)) \rangle$ (A has a minimum element.)

So, if $A \subseteq M^+$ is a nonempty, bounded, definable subset of ω then it has a minimum and maximum. What is important to emphasize is that this is not a property of $\mathcal{M}$ or $\mathcal{M}^+$ alone, it states that subsets of $\eta[M]$ with specific properties in $\mathcal{M}^+$ have a minimum and maximum, so it is a property of the pair $(\mathcal{M}, \mathcal{M}^+)$, but when the bigger structure is clear from the context we can say that $\mathcal{M}$ is pseudo-finite.

Now we want to show that this property is also true for the pair $(\mathcal{N}, \mathcal{N}^+)$.

Let $A \subseteq N^+$ be a definable set in $\mathcal{N}^+$ by $\varphi(x; b_D)$. Let b be a representative for b_D. For each $\alpha < \lambda$, let $A_\alpha \subseteq M^+$ be the set defined by $\varphi(x; b(\alpha))$ in $\mathcal{M}^+$. Since A_α is a definable set in $\mathcal{M}^+$ the sentence

$$(B_\varphi(b(\alpha)) \wedge \neg E_\varphi(b(\alpha)) \wedge S_\varphi(b(\alpha))) \to (\exists y(m_\varphi(y; b(\alpha))) \wedge \exists z(M_\varphi(z; b(\alpha)))))$$

is true. Given that this is true for all α, we conclude trivially by Łoś's Theorem that the sentence

$$(B_\varphi(b_D) \wedge \neg E_\varphi(b_D) \wedge S_\varphi(b_D)) \to (\exists y(m_\varphi(y; b_D)) \wedge \exists z(M_\varphi(z; b_D))))$$

is true in $\mathcal{N}^+$. So the pseudo-finite property is carried from $(\mathcal{M}, \mathcal{M}^+)$ to $(\mathcal{N}, \mathcal{N}^+)$.

Now we want to show that the desired tree and it's auxiliary elements, the functions length and value, are definable in $\mathcal{N}^+$, we will do this by first defining then in $\mathcal{M}^+$ and then carrying the definitions to $\mathcal{N}^+$. Let $(T, \trianglelefteq)$ be the tree whose elements are all the finite sequences of pairs of natural numbers (we will work with elements of $\mathbb{N}$ and ω interchangeably depending on the context), partially ordered by initial segment. In $\mathcal{M}^+$ we have the following properties:

1. ω is a $\mathcal{M}^+$-definable set, T is a $\mathcal{M}^+$-definable set and $\trianglelefteq$ is a $\mathcal{M}^+$-definable relation;
   - $\varphi_\omega, \varphi_T, \varphi_\trianglelefteq$
2. The elements of T are exactly all the functions from an initial segment of ω into $\omega \times \omega$;
   - $\varphi_T(x) \leftrightarrow (\varphi_f(x) \wedge \exists z(\forall y(\varphi_\omega(z) \wedge (y \in \mathrm{dom}(a) \leftrightarrow y \in z))) \wedge (\forall y(y \in \mathrm{range}(x) \rightarrow \varphi_{\omega\times\omega}(y))));$
   - where $\varphi_{\omega\times\omega}(y) = \exists z(\exists w(\varphi_\omega(z) \wedge \varphi_\omega(w) \wedge \varphi_{(z,w)}(y)))$
   - and $\varphi_{(z,w)}(y) = \forall x(x \in y \rightarrow (z \in x \wedge \forall k(k \neq z \rightarrow k \notin x)) \vee (z \in x \wedge w \in x \wedge \forall k(k \neq z \wedge k \neq w \rightarrow k \notin x)))$.
3. The following are $\mathcal{M}^+$-definable, uniformly for $a \in T$:
   a) the length function lg;
      - $\varphi_{\mathrm{lg}}(a,b) = \varphi_T(a) \wedge \varphi_\omega(b) \wedge \forall y(y \in \mathrm{dom}(a) \leftrightarrow y \in b)$
   b) the function giving max(dom(a)), that is, lg(a)-1;
      - $\varphi_{\max(\mathrm{dom})}(a,b) = \varphi_T(a) \wedge b \in \mathrm{dom}(a) \wedge \forall y(y \in \mathrm{dom}(a) \rightarrow (y \in b \vee y = b))$
   c) for each $n \leqslant \max(\mathrm{dom}(a))$, the evaluation function $a(n)$;
      - $\varphi_{\mathrm{eval}}(a,n,m) = \varphi_T(a) \wedge n \in \mathrm{lg}(a) \wedge \varphi_{\omega\times\omega}(m) \wedge (n,m) \in a$
   d) for each $n \leqslant \max(\mathrm{dom}(a))$, the projection functions $a(n)(0)$ and $a(n)(1)$.
      - $\varphi_{\mathrm{proj}}(a,n,p,q) = \varphi_T(a) \wedge n \in \mathrm{lg}(a) \wedge (p = 0 \vee p = 1) \wedge (p = 0 \rightarrow \exists x(\varphi_{(q,x)}(a(n)))) \wedge (p = 1 \rightarrow \exists x(\varphi_{(x,q)}(a(n))))$

We then carry T from $\mathcal{M}^+$ to $\mathcal{N}^+$ using its defining formula φ_T. The elements of this new tree $T^{\mathcal{N}^+}$ in $\mathcal{N}^+$ are functions from an initial segment of the nonstandard integers into pairs of nonstandard integers. This is true by using Łoś's Theorem in the sentence

$$\forall x(\varphi_T(x) \rightarrow (\varphi_f(x) \wedge \exists z(\forall y(\varphi_\omega(z) \wedge (y \in \mathrm{dom}(a) \leftrightarrow y \in z))) \wedge \\ (\forall y(y \in \mathrm{range}(x) \rightarrow \varphi_{\omega\times\omega}(y)))))$$

and the fact that, by proposition 3.0.11, φ_ω will define in $\mathcal{N}^+$ the set representing $(\mathbb{N},<)^\lambda/D$. The same will occur with the other objects defined above, for example now the length function lg can have nonstandard value.

Now that we have guaranteed all the needed properties of our structures we can proceed with the idea behind the proof without problems.

Proof of theorem 3.0.8.

We are supposing by contradiction that there is a pair of sequences $(\langle a_\alpha : \alpha < \kappa\rangle, \langle b_\beta : \beta < \kappa\rangle)$ that witnesses a (κ,κ)-cut in $(\mathbb{N},<)^\lambda/D$.

Let $\psi(x)$ be the formula

$$\psi(x) = \forall n \forall m((\varphi_T(x) \wedge \varphi_\omega(n) \wedge \varphi_\omega(m) \wedge (n < m \leqslant \max(\mathrm{dom}(x)))) \\ \rightarrow x(n)(0) < x(m)(0) < x(m)(1) < x(n)(1))$$

The formula ψ defines a subtree of T where $n < m < \max(\mathrm{dom}(x))$ implies $x(n)(0) < x(m)(0) < x(m)(1) < x(n)(1)$. Denote by T_* the subtree of $T^{\mathcal{N}^+}$ defined by ψ.

Now we construct by induction two κ-sequences, one of elements c_α of T_* and one of elements n_α of $\omega^{\mathcal{N}^+}$ where:

1. for all $\beta < \alpha < \kappa$, we have that $\mathcal{N}^+ \models c_\beta \trianglelefteqslant c_\alpha$;
2. for all $\alpha < \kappa$, we have that $n_\alpha = \max(\mathrm{dom}(c_\alpha))$;
3. for all $\alpha < \kappa$, we have that $c_\alpha(n_\alpha)(0) = a_\alpha$ and $c_\alpha(n_\alpha)(1) = b_\alpha$.

For the base case, let $c_0 = \langle (a_0, b_0) \rangle$. When $\alpha = \beta + 1$, let $c_\alpha = c_\beta{}^\frown \langle (a_\alpha, b_\alpha) \rangle$ and $n_\alpha = n_\beta + 1$. When α is a limit ordinal, by the treetops hypothesis there is $c_* \in T_*$ such that $c_\beta \trianglelefteqslant c_*$ for all $\beta < \alpha$. Let $n_* = \max(\mathrm{dom}(c_*))$. By the definition of T_* and the fact that the order $\trianglelefteqslant$ is by initial segment we have that, for $\beta < \alpha$, $c_\beta(n_\beta, 0) = c_*(n_\beta, 0) < c_*(n_*, 0) < c_*(n_*, 1) < c_*(n_\beta, 1) = c_\beta(n_\beta, 1)$, that is, $a_\beta < c_*(n_*, 0) < c_*(n_*, 1) < b_\beta$ for all $\beta < \alpha$, but it may also be the case that $a_\alpha < c_*(n_*, 0) < c_*(n_*, 1) < b_\alpha$, so we need to restrict our element c_* to before a point where that happens. We can then concatenate it with the element $\langle (a_\alpha, b_\alpha) \rangle$ without failing the definition of T_*. For this we use the following set

$$A = \{n \leqslant n_* : c_*(n)(0) < a_\alpha \wedge b_\alpha < c_*(n)(1)\}.$$

The set A is nonempty, because n_β is in it for any $\beta < \alpha$, bounded, trivially by n_*, and definable with c_* as a parameter, so it has a maximum m_*. Necessarily $c_\beta \trianglelefteqslant c_* \restriction_{m_*}$ for each $\beta < \alpha$, because $n_\beta \in A$ for each $\beta < \alpha$, so $n_\beta \leqslant m_*$. Now we can concatenate without problems, so $c_\alpha := (c_* \restriction_{m_*})^\frown \langle (a_\alpha, b_\alpha) \rangle$ and $n_\alpha := m_*$.

Now we have the sequence $\bar{c} = \langle c_\alpha : \alpha < \kappa \rangle$ that is a non-decreasing path in a branch of T_*, where each c_α is an element with (a_α, b_α) in its node. Again by treetops, there exists an element $c_* \in T_*$ with $c_\alpha \trianglelefteqslant c_*$ for all $\alpha < \kappa$. Let $n_* = \max(\mathrm{dom}(c_*))$. Then, for each $\alpha < \kappa$, by definition of T_* we have

$$a_\alpha = c_\alpha(n_\alpha)(0) = c_*(n_\alpha)(0) < c_*(n_*)(0) < c_*(n_*)(1) < c_*(n_\alpha)(1) = c_\alpha(n_\alpha)(1) = b_\alpha.$$

This implies that both $c_*(n_*)(0)$ and $c_*(n_*)(1)$ realize the cut $(\bar{a}, \bar{b})$, contradiction the hypothesis that $(\bar{a}, \bar{b})$ represents a cut.

□

A way to look at cofinality spectrum problems is that they have the necessary conditions for us to use the methods of this last proof. Informally we will have four structures $\mathcal{M}, \mathcal{M}^+, \mathcal{N}, \mathcal{N}^+$ and a set of formulas Δ, where Δ will define what linear orders inside $\mathcal{M}$ and $\mathcal{N}$ we care about (for example in this last lemma we have a trivial case where the linear order we care about is all of $\mathcal{M}$ and $\mathcal{N}$, but it could be the case that $\mathcal{M}$ was a bigger structure with ω inside it and the formula defining ω would be in Δ), and the structures have the following similar picture as before

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\theta"] & \mathcal{N}^+ \\ \mathcal{M} \arrow[u, "\preccurlyeq"] \arrow[r, "\eta"] & \mathcal{M}^+ \arrow[u, "\preccurlyeq"] \end{tikzcd}$$

where η, θ are embeddings, and $\mathcal{M}^+$ and $\mathcal{N}^+$ are rich enough to talk about the trees that we care about (the relevant trees will be the ones constructed using the elements of the linear orders defined by Δ). We will now define these properties formally.

Definition 3.0.12 (Enough Set Theory for Trees (ESTT) [20], Definition 2.3)**.** Let $\mathcal{N}^+$ be a structure and Δ a nonempty set of formulas in the language of $\mathcal{N}^+$. We say that $(\mathcal{N}^+, \Delta)$ has *enough set theory for trees* when the following conditions are true.

1. Δ consists of first-order formulas $\varphi(\bar{x}, \bar{y}; \bar{z})$, with $\ell(\bar{x}) = \ell(\bar{y})$;
2. For each formula $\varphi \in \Delta$ and each parameter $\bar{c} \in {}^{\ell(\bar{z})}N^+$, $\varphi(\bar{x}, \bar{y}; \bar{c})$ defines a discrete linear order with a first and last element on $\{\bar{a} : \mathcal{N}^+ \models \varphi(\bar{a}, \bar{a}; \bar{c})\ \}$;
3. The family of all linear orders defined this way will be denoted $\mathrm{Or}(\Delta, \mathcal{N}^+)$. Specifically, each $\mathbf{a} \in \mathrm{Or}(\Delta, \mathcal{N}^+)$ is a tuple $(X_\mathbf{a}, \leqslant_\mathbf{a}, \varphi_\mathbf{a}, \bar{c}_\mathbf{a}, d_\mathbf{a})$, where:
    a) $X_\mathbf{a}$ denotes the underlying set $\{\bar{a} : \mathcal{N} \models \varphi_\mathbf{a}(\bar{a}, \bar{a}, \bar{c}_\mathbf{a})\}$;
    b) $\bar{x} \leqslant_\mathbf{a} \bar{y}$ abbreviates the formula $\varphi_\mathbf{a}(\bar{x}, \bar{y}, \bar{c}_\mathbf{a})$;
    c) $d_\mathbf{a}$ is a bound for the length of elements in the associated tree; it is often, but not always, $\max(X_\mathbf{a})$. If $d_\mathbf{a}$ is finite we call $\mathbf{a}$ trivial.
4. For each $\mathbf{a} \in \mathrm{Or}(\Delta, \mathcal{N}^+)$, $(X_\mathbf{a}, \leqslant_\mathbf{a})$ is pseudofinite, meaning that any bounded, nonempty, $\mathcal{N}^+$-definable subset has a $\leqslant_\mathbf{a}$-greatest and $\leqslant_\mathbf{a}$-least element;
5. For each pair $\mathbf{a}$ and $\mathbf{b}$ in $\mathrm{Or}(\Delta, \mathcal{N}^+)$, there is $\mathbf{c}$ in $\mathrm{Or}(\Delta, \mathcal{N}^+)$ such that
    a) there exists an $\mathcal{N}^+$-definable bijection $\mathrm{Pr} : X_\mathbf{a} \times X_\mathbf{b} \to X_\mathbf{c}$ such that the coordinate projections are $\mathcal{N}^+$-definable;
    b) if $d_\mathbf{a}$ is not finite in $X_\mathbf{a}$ and $d_\mathbf{b}$ in not finite in $X_\mathbf{b}$, then also $d_\mathbf{c}$ is not finite in $X_\mathbf{c}$.

6. For some nontrivial $\mathbf{a} \in \mathrm{Or}(\Delta, \mathcal{N}^+)$, there is $\mathbf{c} \in \mathrm{Or}(\Delta, \mathcal{N}^+)$ such that $X_{\mathbf{c}} = \mathrm{Pr}(X_{\mathbf{a}} \times X_{\mathbf{a}})$ and the ordering $\leqslant_{\mathbf{c}}$

$$\mathcal{N}^+ \models (\forall x \in X_{\mathbf{a}})(\exists y \in X_{\mathbf{c}})(\forall x_1, x_2 \in X_{\mathbf{a}})(\max(x_1, x_2) \leqslant_{\mathbf{a}} x \Leftrightarrow \mathrm{Pr}(x_1, x_2) \leqslant_{\mathbf{c}} y);$$

7. To the family of distinguished orders, we associate a family of trees, as follows. for each formula $\varphi(\bar{x}, \bar{y}; \bar{z})$ in Δ there are formulas $\psi_0, \psi_1, \psi_2, \psi_3$ of the language of $\mathcal{N}^+$ such that for any $\mathbf{a} \in \mathrm{Or}(\Delta, \mathcal{N}^+)$ with $\varphi_{\mathbf{a}} = \varphi$

   a) $\psi_0(\bar{x}; \bar{c}_{\mathbf{a}})$ defines a set, denoted $T_{\mathbf{a}}$, of functions from $X_{\mathbf{a}}$ to $X_{\mathbf{a}}$, where the domain of each one is an initial segment of $X_{\mathbf{a}}$;

   b) $\psi_1(\bar{x}, \bar{y}; \bar{c}_{\mathbf{a}})$ defines a function $\mathrm{lg}_{\mathbf{a}} : T_{\mathbf{a}} \to X_{\mathbf{a}}$ satisfying

      i. for all $b \in T_{\mathbf{a}}$, we have that $(\mathrm{lg}_{\mathbf{a}}(b) - 1) \leqslant_{\mathbf{a}} d_{\mathbf{a}}$;

      ii. for all $b \in T_{\mathbf{a}}$, we have that $\mathrm{lg}_{\mathbf{a}}(b) = \min(X_{\mathbf{a}} \setminus \mathrm{dom}(b))$.

   c) $\psi_2(\bar{x}, \bar{y}, \bar{c})$ defines a function from $\{(b, a) : b \in T_{\mathbf{a}}, a \in X_{\mathbf{a}}, a <_{\mathbf{a}} \mathrm{lg}_{\mathbf{a}}(b)\}$ into $X_{\mathbf{a}}$ whose value is called $\mathrm{val}_{\mathbf{a}}(b, a)$, and abbreviated $b(a)$.

      i. if $c \in T_{\mathbf{a}}$, $\mathrm{lg}_{\mathbf{a}}(c) \leqslant_{\mathbf{a}} d_{\mathbf{a}}$ and $a \in X_{\mathbf{a}}$, then $c^\frown\langle a \rangle$ exists, that is, there is $c' \in T_{\mathbf{a}}$ such that $\mathrm{lg}_{\mathbf{a}}(c') = \mathrm{lg}_{\mathbf{a}}(c) + 1, c'(\mathrm{lg}_{\mathbf{a}}(c)) = a$ and

      $$(\forall b <_{\mathbf{a}} \mathrm{lg}_{\mathbf{a}}(c))(c(b) = c'(b));$$

      ii. $\psi_0(\bar{x}, \bar{c})$ implies that if $b_1 \neq b_2 \in T_{\mathbf{a}}$ and $\mathrm{lg}_{\mathbf{a}}(b_1) = \mathrm{lg}_{\mathbf{a}}(b_2)$, then for some $n <_{\mathbf{a}} \mathrm{lg}_{\mathbf{a}}(b_1)$ we have that $b_1(n) \neq b_2(n)$.

   d) $\psi_3(\bar{x}, \bar{y}; \bar{c})$ defines the partial order $\trianglelefteq_{\mathbf{a}}$ on $T_{\mathbf{a}}$ given by initial segment, that is, such that $b_1 \trianglelefteq_{\mathbf{a}} b_2$ implies

      i. $\mathrm{lg}_{\mathbf{a}}(b_1) \leqslant_{\mathbf{a}} \mathrm{lg}_{\mathbf{a}}(b_2)$;

      ii. $(\forall a <_{\mathbf{a}} \mathrm{lg}_{\mathbf{a}}(b_1))(b_1(a) = b_2(a))$.

The family of all $T_{\mathbf{a}}$ defined this way will be denoted $\mathrm{Tr}(\Delta, \mathcal{N}^+)$. We refer to elements of this family as trees.

Lets take another informal look at this definition to understand what we get from each hypothesis.

1. The set Δ will consist of the **formulas that define the orders that we are interested**, and all the formulas needed for Δ to have properties (5) and (6);

2. For each formula $\varphi \in \Delta$ and each parameter $\bar{c}$, $\varphi(\bar{x}, \bar{y}; \bar{c})$ defines a **discrete linear order with a first and last element** on a subset of the universe, that is, **every element in the domain has a successor, except the last element, and**

every element has a predecessor, except the first element. An example of such structure is $\mathbb{N} + (\mathbb{R} \times \mathbb{Z}) + \mathbb{N}^*$, where $\mathbb{N}^*$ represents the natural numbers with the reverse order. Actually all possible structures are similar to this, with some linear order L in place of $\mathbb{R}$ to set the number of copies of $\mathbb{Z}$ in between, $\mathbb{N} + (L \times \mathbb{Z}) + \mathbb{N}^*$;

3. $\mathrm{Or}(\Delta, \mathcal{N})$ is the set of all such linear orders.

4. Each $\mathbf{a}$ is pseudo-finite. Given the definition we used for lemma 3.0.8, we are saying that, looking at $(X_{\mathbf{a}}, \leqslant_{\mathbf{a}})$ as a model of linear order, the pair $((X_{\mathbf{a}}, \leqslant_{\mathbf{a}}), \mathcal{N}^+)$ is pseudo-finite, meaning that **any bounded, nonempty, $\mathcal{N}^+$-definable subset has a $\leqslant_{\mathbf{a}}$-greatest and $\leqslant_{\mathbf{a}}$-least element**. Since there is a last element, every nonempty subset is bounded. This pseudo-finite property tells us that, for example looking at $\mathbb{N} + \mathbb{Z} + \mathbb{N}^*$, subsets $\mathbb{N}, \mathbb{Z}, \mathbb{N}^*$, among others cannot be $\mathcal{N}^+$-definable;

5. **This enables us to work with Cartesian products**;

6. For each nontrivial $\mathbf{a}$, thinking of $X_{\mathbf{c}}$ as $X_{\mathbf{a}} \times X_{\mathbf{a}}$, there exists $\mathbf{c}$ such that for every $x \in X_{\mathbf{a}}$ we have that $(x_1, x_2) \leqslant_{\mathbf{c}} (x, x)$ if and only if $x_1 \leqslant_{\mathbf{a}} x$ and $x_2 \leqslant_{\mathbf{a}} x$. Thinking of $X_{\mathbf{c}}$ as just bijective to $X_{\mathbf{a}} \times X_{\mathbf{a}}$, this says that there exists $\mathbf{c}$ such that for each $x \in X_{\mathbf{a}}$ there is $y \in X_{\mathbf{c}}$ such that the same properties apply with y in place of (x, x);

7. **For each order $\mathbf{a} \in \mathrm{Or}(\Delta, \mathcal{N})$ we define an associate tree, $T_{\mathbf{a}}$, with the following properties**:

   a) The elements of $T_{\mathbf{a}}$ are sequences of $X_{\mathbf{a}}$ indexed by $X_{\mathbf{a}}$, that is, partial functions from $X_{\mathbf{a}}$ to $X_{\mathbf{a}}$ where the domain is an initial segment of $X_{\mathbf{a}}$;

   b) There is a length function $\lg_{\mathbf{a}} : T_{\mathbf{a}} \to X_{\mathbf{a}}$ such that for all $b \in T_{\mathbf{a}}$, $\lg_{\mathbf{a}}(b) \leqslant_{\mathbf{a}} d_{\mathbf{a}}$ and $\lg_{\mathbf{a}}(b) = \max(\mathrm{dom}(b))$;

   c) There is a value function $\mathrm{val}_{\mathbf{a}}$ defined in a subset of $T_{\mathbf{a}} \times X_{\mathbf{a}}$ where $a <_{\mathbf{a}} \lg_{\mathbf{a}}(b), (a, b) \in T_{\mathbf{a}} \times X_{\mathbf{a}}$;

      i. if $c \in T_{\mathbf{a}}$, $\lg_{\mathbf{a}}(c) \leqslant_{\mathbf{a}} d_{\mathbf{a}}$ and $a \in X_{\mathbf{a}}$, then $c^\frown\langle a \rangle$ exists, that is, **for every element c of the tree, if it is not too large, that is, if its length is less than or equal to $d_{\mathbf{a}}$, then we can add to the end of the sequence c any element of $X_{\mathbf{a}}$, and the new extended element exists in the tree $T_{\mathbf{a}}$**;

      ii. if $b_1 \neq b_2 \in T_{\mathbf{a}}$ and $\lg_{\mathbf{a}}(b_1) = \lg_{\mathbf{a}}(b_2)$ then for some $n <_{\mathbf{a}} \lg_{\mathbf{a}}(b_1)$, $b_1(n) \neq b_2(n)$.

   d) **There is a partial order $\trianglelefteq_{\mathbf{a}}$ on $T_{\mathbf{a}}$ given by initial segment**, that is, such that $b \trianglelefteq_{\mathbf{a}} c$ implies:

      i. $\lg_{\mathbf{a}}(b) \leqslant_{\mathbf{a}} \lg_{\mathbf{a}}(c)$;

      ii. $(\forall a <_{\mathbf{a}} \lg_{\mathbf{a}}(b))(b(a) = c(a))$.

e) $\mathrm{Tr}(\Delta, \mathcal{N}^+)$ is the set of all such trees.

Remark. The definition of tree used above is not necessarily closed by initial segment, that is, the existence of an element of the tree doesn't imply that a truncation of the sequence is also an element of the tree. But the important property for our applications is the ability to extend the elements, because we generally want a bigger element with the desired properties, which is indeed present. It might be a little strange to call it a tree without being closed by initial segment, but at any given element, if we look only at its extensions, it is then almost closed by initial segment (if we stop at the given element).

This definition of ESTT gives the groundwork necessary to define cofinality spectrum problems, in a similar way that for theorem 3.0.8 we needed to guarantee essentially the properties of ESTT for the structures $\mathcal{M}^+$ and $\mathcal{N}^+$. We define next the tool that is central to this dissertation, the *cofinality spectrum problems.*

Definition 3.0.13 (Cofinalty Spectrum Problems (CSP) [20],Definition 2.5)**.** We say that $(\mathcal{M}, \mathcal{N}, \mathcal{M}^+, \mathcal{N}^+, T, \Delta)$ is a *cofinality spectrum problem* when

1. $\mathcal{M} \preccurlyeq \mathcal{N}$;
2. $T \supseteq \mathrm{Th}(\mathcal{M})$ is a theory in a possibly larger vocabulary;
3. Δ is a set of formulas in the language of $\mathcal{M}$, that is, we are interested in studying the orders of $\mathcal{L}(\mathcal{M}) = \mathcal{L}(\mathcal{N})$ in the presence of the additional structure of $\mathcal{L}(\mathcal{M}^+) = \mathcal{L}(\mathcal{N}^+)$.
4. There are embeddings from $\mathcal{M}, \mathcal{N}$ to $\mathcal{M}^+, \mathcal{N}^+$, respectively, so that $\mathcal{M}^+ \preccurlyeq \mathcal{N}^+ \models T$ and $(\mathcal{N}^+, \Delta)$ has enough set theory for trees.

Now that we have the definition of CSP at hand, we want to talk about cuts and the treetops property inside of CSPs, given that they are a tool to attack the problems presented in the introduction. For this we can define, given a CSP $\mathbf{s}$, a set $\mathcal{C}^{\mathrm{ct}}(\mathbf{s})$, similar to $\mathcal{C}(D)$ of lemma 3.0.8, that gives us the information about the cuts inside the CSP, and a set $\mathcal{C}^{\mathrm{ttp}}(\mathbf{s})$ that gives us the information about the treetops property inside the CSP. Given a CSP $\mathbf{s}$, we denote by $\mathrm{Or}(\mathbf{s})$ the set $\mathrm{Or}(\Delta, \mathcal{N}^+)$.

$$\begin{aligned}
\mathcal{C}^{\mathrm{ct}}(\mathbf{s}) = \{(\kappa_1, \kappa_2) \quad & : \text{There is in } (X_{\mathbf{a}}, \leqslant_{\mathbf{a}}) \text{ a } (\kappa_1, \kappa_2)\text{-cut for some } \mathbf{a} \in \mathrm{Or}(\mathbf{s})\} \\
\mathcal{C}^{\mathrm{ttp}}(\mathbf{s}) = \{\kappa \geqslant \aleph_0 \quad & : \text{There is a strictly increasing sequence of cofinality} \\
& \kappa \text{ with no upper bound in } T_{\mathbf{a}} \text{ for some } \mathbf{a} \in \mathrm{Or}(\mathbf{s})\}
\end{aligned}$$

Let $\mathfrak{t}_{\mathbf{s}}$ be defined as the minimum of the set $\mathcal{C}^{\mathrm{ttp}}(\mathbf{s})$ and $\mathfrak{p}_{\mathbf{s}}$ be defined as the minimum of the set $\{\kappa : (\kappa_1, \kappa_2) \in \mathcal{C}^{\mathrm{ct}}(\mathbf{s}) \text{ and } \kappa = \kappa_1 + \kappa_2\}$.

Definition 3.0.14 (treetops for CSP)**.** Let $\mathbf{s}$ be a cofinality spectrum problem and λ a regular cardinal. When $\lambda \leqslant \mathfrak{t}_{\mathbf{s}}$ we say that $\mathbf{s}$ has λ-treetops.

This definition reflects the same property of treetops for ultrafilters, given that by definition of $\mathfrak{t}_{\mathbf{s}}$, if $\kappa < \lambda \leqslant \mathfrak{t}_{\mathbf{s}}$ and $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$ then any strictly increasing κ-sequence of elements of $T_{\mathbf{a}}$ has an upper bound in $T_{\mathbf{a}}$.

For λ an infinite cardinal, define

$$\mathcal{C}(\mathbf{s}, \lambda) = \{(\kappa_1, \kappa_2) : \kappa_1 + \kappa_2 < \lambda \text{ and } (\kappa_1, \kappa_2) \in \mathcal{C}^{\mathrm{ct}}(\mathrm{s})\}.$$

The central result about cofinality spectrum problems present in the work of Mariallis and Shelah is the following:

Theorem 3.0.15 ([20], Theorem 9.1)**.** *Let $\boldsymbol{s}$ be a cofinality spectrum problem. Then*

$$\mathcal{C}(\boldsymbol{s}, \mathfrak{t}_{\boldsymbol{s}}) = \emptyset,$$

that is, there are no cuts of size less than $\mathfrak{t}_{\boldsymbol{s}}$ in any order $\boldsymbol{a} \in Or(\boldsymbol{s})$, in other words $\mathfrak{t}_{\boldsymbol{s}} \leqslant \mathfrak{p}_{\boldsymbol{s}}$.

In the following chapter we will show how this result can be applied to both problems showed in the introduction, that of maximality of SOP_2 and of $\mathfrak{p} = \mathfrak{t}$.

For the proof of this result, we will construct and prove, in chapter 5, a special case in the context of pseudo-finite ultraproducts instead of in general cofinality spectrum problems; notwithstanding, this special case is sufficient for both applications present in the next chapter.

4 Applications of Cofinality Spectrum Problems

4.1 Keisler's Order

In this section we will introduce the Keisler's order, a way of comparing complexity between complete theories by how easy it is to saturate one theory in comparison with another one using regular ultrafilters. We then will present the objective of weakening the sufficient condition for a theory to be in the maximum class of this order, to theories with the 2-strict order property (SOP_2). We will construct a CSP that relates to the problem at hand and show that we can use the main result of CSP, Theorem 3.0.15, to reach the objective of weakening the sufficient condition.

4.1.1 Basic definitions and properties

Using regular ultrafilters (cf. Definition 3.0.7) we can build Keisler's order $\trianglelefteq$, which is actually a preorder in the class of complete theories, but it becomes a partial order between the $\trianglelefteq$-equivalence classes of complete theories.

Definition 4.1.1 (Keisler's Order)**.** Let T_1 and T_2 be complete, countable first-order theories. We say that $T_1 \trianglelefteq T_2$ if for all infinite λ, all $M_1 \models T_1, M_2 \models T_2$, and all D regular ultrafilter over λ, if $\mathcal{M}_2^\lambda/D$ is λ^+-saturated then $\mathcal{M}_1^\lambda/D$ must be λ^+-saturated.

The use of regular ultrafilters is essential to the definition of Keisler's order, because for this to be a preorder between complete theories we need it to be reflexive and transitive. It is clearly transitive by definition, and the use of regular ultrafilters for the ultrapower will guarantee that it is reflexive given the following theorem and the fact that we are working only with complete theories.

Theorem 4.1.2. *Let $\mathcal{M}$, $\mathcal{N}$ be structures and D a regular ultrafilter over λ. If $\mathcal{M} \equiv \mathcal{N}$ then*

$$\mathcal{M}^\lambda/D \text{ is saturated} \Leftrightarrow \mathcal{N}^\lambda/D \text{ is saturated.}$$

With this theorem at hand we can also talk about, for a regular ultrafilter D and a complete theory T, the theory T *being λ^+-saturated by D*, where it means that for any model $\mathcal{M} \models T$ we have that $\mathcal{M}^\lambda/D$ is λ^+-saturated.

Remark. This enables us to rephrase Keisler's order with this new definition as:

We have that $T_1 \trianglelefteq T_2$ if for all infinite λ, and all D regular ultrafilter over λ, if T_2 is λ^+-saturated by D then T_1 must be λ^+-saturated by D.

Next is a definition of a special type of ultrafilter introduced by Keisler that will enable his characterization of the maximum class of Keisler's order.

For a set X, let $[X]^{<\aleph_0}$ be the set of finite subsets of X.

Definition 4.1.3 (Good Ultrafilters)**.** The filter D over I is said to be μ^+-good if every monotonic function $f : [\mu]^{<\aleph_0} \to D$ has a multiplicative refinement, that is, there exists a $f' : [\mu]^{<\aleph_0} \to D$ where

1. $\forall u \in [\mu]^{<\aleph_0}$ we have that $f'(u) \subseteq f(u)$ (f' is a refinement of f);
2. $\forall u, v \in [\mu]^{<\aleph_0}$ we have that $f'(u) \cap f'(v) = f'(u \cup v)$ (f' is multiplicative).

Remark. The hypothesis of monotonicity can be dropped from the definition of good ultrafilters, because for any function f we can construct a monotonic refinement f' given by $f'(u) = f(u) \cap \bigcap_{v \subseteq u} f'(v)$.

This definition may seem strange at first sight. What multiplicative refinements of monotonic functions has to do with saturation of types? I will try to present a more intuitive path towards this definition to show why it is not as arbitrary as it may seem. The idea behind this approach is thanks to [14].

Firstly, our objective is to find some property of an ultrafilter D over I that guarantees that ultraproducts constructed using D will be λ^+-saturated. As stated in 2.3.14, it is equivalent to show that the ultraproduct is λ^+-compact.

Let $\mathcal{N} = \prod_i \mathcal{M}_i/D$ be an ultraproduct by D. Let $p \in S_1^{\mathcal{N}}$ be a complete 1-type such that $|p| < \lambda^+$. Let $\mu = |p|$, and enumerate p using μ, so $p = \{\varphi_\alpha(x) : \alpha \in \mu\}$. As we know, p being a type is telling us that it is finitely satisfiable, that is, any finite subset of p has a realization in $\mathcal{N}$. This property can be state in a different way in terms of a function from $[\mu]^{<\aleph_0}$ to the filter D.

Let d_0 be defined as the following function

$$\begin{aligned} d_0 : [\mu]^{<\aleph_0} &\to \mathcal{P}(I) \\ u &\mapsto \{i \in I : \mathcal{M}_i \models \exists x \bigwedge_{\alpha \in u} \varphi_\alpha(x)\}. \end{aligned}$$

We call d_0 the *Łoś's map* of p. We can see that the property "p is finitely satisfiable" is equivalent to stating that the codomain of d_0 is actually D.

Suppose that p is finitely satisfiable. We have that $\mathcal{N} \models \exists x \bigwedge_{\alpha \in \theta} \varphi_\alpha(x)$, so by Łoś's Theorem the subset of I in the definition is indeed an element of D. The other direction is also a consequence of Łoś's Theorem.

We can see that d_0 is monotonic, that is, $u \subseteq u'$ implies that $d_0(u) \supseteq d_0(u')$. This is not surprising as it is saying that if a set of formulas $\{\varphi_\alpha(x) : \alpha \in u'\}$ is realized in $\mathcal{N}$ then any subset of it is also realized in $\mathcal{N}$.

What we want to do now is to find a way of stating that p is satisfiable, using only its Łoś's map d_0. Let $\mathcal{A} = \langle A_i \in [\mu]^{<\aleph_0} : i \in I \rangle$ be a sequence of finite subsets of μ indexed by I with the following properties:

1. for all $i \in I$ we have that $i \in d_0(A_i)$;

2. for all $\theta \in \mu$ we have that $\{i \in I : \theta \in A_i\} \in D$.

We will call a sequence $\mathcal{A}$ with such properties an *actualization* of d_0.

We will show that if there exists an actualization of d_0 then p is satisfiable. Property (1) guarantees that we can pick, for each $i \in I$, an $a_i \in M_i$ such that $\mathcal{M}_i \models \bigwedge_{\theta \in A_i} \varphi_\theta(a_i)$. Let $a = \langle a_i : i \in I \rangle$. Now, property (2) guarantees that this element a is a realization of p, because for any formula $\varphi_\theta(x)$ in p we have that $\mathcal{N} \models \varphi_\theta(a)$, by consequence of Łoś's Theorem.

Now, we have a sufficient condition for satisfiability that only involves the Łoś's map. We will move the sufficient condition further by showing that there exists an actualization of d_0 if and only if d_0 has a multiplicative refinement that is locally finite.

Suppose that there exists an actualization $\mathcal{A} = \langle A_i \in [\mu]^{<\aleph_0} : i \in I \rangle$ of d_0. Define the function $d' : [\mu]^{<\aleph_0} \to D$ such that for each $u \in [\mu]^{<\aleph_0}$ we have $d'(u) := \{i \in I : u \subseteq A_i\}$. We have that d' is multiplicative, that is, for any two elements $u, v \in [\mu]^{<\aleph_0}$ it is true that $d'(u) \cap d'(v) = d'(u \cup v)$, by definition of d'. The function d' is also locally finite, that is, for all $i \in I$ we have that $\sup\{|u| : i \in d'(u)\} < \aleph_0$, because we cannot have $u \subseteq A_i$ with $|u| > |A_i|$. Finally, it is also the case that d' is a refinement of d_0, because for $i \in d'(u)$ we have that $u \subseteq A_i$ and $i \in d_0(A_i)$, concluding that $i \in d_0(u)$.

On the other hand, suppose that there exists a locally finite multiplicative refinement d' of d_0. For every $i \in I$ define the set $d'^{-1}i := \{u \in [\mu]^{<\aleph_0} : i \in d'(u)\}$. Given that d' is multiplicative we have that $d'^{-1}i$ is closed under finite union. This together with the fact that d' is locally finite guarantees us that $\bigcup d'^{-1}i$ is finite, concluding that $\bigcup d'^{-1}i \in d'^{-1}i$. For each $i \in I$ let $B_i := \bigcup d'^{-1}i$. The sequence $\langle B_i : i \in I \rangle$ is an actualization of d'. It is easy to show that, given the fact that d' is a refinement of d_0, this sequence is also an actualization of d_0. For the first property of actualization, let $i \in d'(B_i)$. By refinement we have that $i \in d_0(B_i)$. The second property actually does not care for the underlying function d', so it works for d_0 as well. Then the sequence $\langle B_i : i \in I \rangle$ is an actualization of d_0.

By this approach we can see that if every monotonic function from $[\mu]^{<\aleph_0}$ to D has a locally finite multiplicative refinement then every ultraproduct by D is μ^+-saturated. It is clear that the multiplicative refinement comes directly from the definition of λ^+-good ultrafilters; however, how do we guarantee the locally finite part?

It is actually the case that all the ultrafilters that we will work with has a property that implies the existence of locally finite refinements.

Definition 4.1.4 ([27], Chapter VI, Definition 1.3)**.** Let D be an ultrafilter over I, and λ be a cardinal. We say that D is λ-incomplete if there are $X_i \in D$ for $i < \alpha < \lambda$ such that $\bigcap_{i<\alpha} X_i = \emptyset$

Lemma 4.1.5 ([14], Lemma 2.1)**.** *Let D be an $\aleph_1$-incomplete ultrafilter over I. Then every monotonic function f has a locally finite monotonic refinement f'.*

This property of $\aleph_1$-incompleteness comes directly from the regularity property.

Lemma 4.1.6. *Let D be a regular ultrafilter over I. Then D is $\aleph_1$-incomplete.*

Proof. Just pick any subset, of size $\aleph_0$, of a regularizing family of D. By the definition of regularizing family the intersection will be empty. □

So with the discussion about good ultrafilters that we just had, we conclude the following theorem that encapsulates the motivation behind good ultrafilters.

Theorem 4.1.7. *Let T be a complete theory and D a regular ultrafilter. Then*

$$D \textit{ is } \lambda^+\textit{-good} \Rightarrow T \textit{ is } \lambda^+\textit{-saturated by } D.$$

We restate the definition of Łoś's map in the context of ψ-types, together with the definition of a distribution, that will be an important concept for our context of regular ultrafilters.

Definition 4.1.8. Let $\mathcal{N} = \mathcal{M}^I/D$ be a regular ultrapower, $J \subseteq I$, $|J| = \mu$, and $p = \{\psi(x; a_i) : i \in J\}$ a ψ-type in $\mathcal{N}$.

1. The *Łoś's map of* p is the function $d_0 : [J]^{<\aleph_0} \to D$ given by
$$u \in [J]^{<\aleph_0} \mapsto \left\{ s \in I : \mathcal{M} \models \exists x \bigwedge_{j\in u} \psi(x; a_j(s)) \right\}$$
2. A *distribution for* p is a function $d : [J]^{<\aleph_0} \to D$ such that:
    a) the function d refines the Łoś's map of p, that is, $d(u) \subseteq d_0(u)$ for all $u \in J$;
    b) the range of d is a μ-regularizing family for D;
    c) the function d is monotonic, that is, $u \subseteq v$ implies $d(u) \supseteq d(v)$ (informally, d still "perceives" the property "if an element c realizes a type q then it realizes all of its subtypes").

4.1.2 Keisler's characterization of the maximum class

With the definitions of last section at hand we have what is needed to state Keisler's characterization of the maximum class of his order.

Theorem 4.1.9 (Keisler's characterization of the maximum class)**.** *There is a maximum class in Keisler's order, which consists precisely of those complete theories T such that for any regular ultrafilter D over λ, we have*

$$T \text{ is } \lambda^+\text{-saturated by } D \Leftrightarrow D \text{ is } \lambda^+\text{-good.} \tag{$\star$}$$

To prove this characterization we will need to do some groundwork beforehand. But first lets talk about why this characterization must be true. This comes essentially by how Keisler's order is constructed. The property $\star$ is special because it enables us to "carry down" the saturation from T to any other complete theory, much the same way as being in the maximum class does. Let me be more precise by what this means. Let D be a regular ultrafilter over λ. Let T be a complete theory with property $\star$ and T' any other complete theory. if T is λ^+-saturated by D then D is λ^+-good, but we know that if D is λ^+-good and regular then it λ^+-saturates any theory, so T' is λ^+-saturated by D, here we "carried down" the saturation from T to T', and since the choice of regular D was arbitrary we have $T' \trianglelefteq T$. Now lets look at the other side of the characterization. Let T be a complete theory in the maximum class. Since D being λ^+-good and regular always implies λ^+-saturation, the non-trivial part is, given T λ^+-saturated by D, proving that D is λ^+-good. Thinking of the contrapositive, we want to show that if D is not λ^+-good then T is not λ^+-saturated by D. Remember that by the construction of Keisler's order, any saturation is carried from the more complex to the less complex theory, as we just talked about with the property $\star$, so inversely, the non saturation is carried from the less complex to the more complex. If we can show that for a non λ^+-good filter D there exists a theory T' that is not λ^+-saturated then this will be carried to T since T is in the maximum class. The following lemmas will enable us to show that indeed there exists such a theory.

Definition 4.1.10. Let $\mathcal{N} = \mathcal{M}^I/D$ be a regular ultrapower, $J \subseteq I$, and $p = \{\psi(x; a_i) : i \in J\}$ a ψ-type in $\mathcal{N}$. A distribution $d : [I]^{<\aleph_0} \to D$ is said to be *accurate* if for each index $t \in I$ and each finite subset $\{i_1, \ldots, i_n\} \subset A_t := \{j : t \in d(\{j\})\}$, we have that $t \in d(\{i_1, \ldots, i_n\})$ if and only if $\mathcal{M} \models \exists x \bigwedge_{k \leqslant n} \psi(x; a_{i_k}(t))$.

It is easy to see that the Łoś's map d_0 has the accurate property given how it is defined, however it may not be a distribution. An accurate distribution is, in a sense, one that acts similarly to the Łoś's map.

Lemma 4.1.11. *Let T be a complete theory, $\mathcal{M} \models T$, λ an infinite cardinal, D a regular ultrafilter over λ, $\mathcal{N} = \mathcal{M}^\lambda/D$, φ a formula in the language of T, $A \subset N$ with $|A| \leqslant \lambda$, p a φ-type over A. Then an accurate distribution of p always exists.*

Proof. We can construct an accurate distribution using the Łoś's map as a starting point. Write p as $\{\varphi(x; a_i) : i \in \lambda\}$. Let d_0 be the Łoś's map of p. Let $X = \{X_i : i \in \lambda\}$ be a λ-regularizing family for D. Define for the singletons the function

$$d_1(\{i\}) := d_0(\{i\}) \cap X_i.$$

This guarantees that d_1 refines the Łoś's map and the range of d_1 by the set of singletons is a λ-regularizing family (Note that there can't be the case that the regularizing family shrinks in size, because they can't be empty given that they are elements of the ultrafilter, and if there is an infinite number of is such that all $d_0(\{i\}) \cap X_i$ are equal we have that X was not a regularizing family to begin with). Now we need to extend this property to every element in the domain while making the function d_1 monotonic. We gain this by doing an intersection with the original Łoś's map as follows

$$d_1(\{i_0, \ldots, i_n\}) := (\bigcap_{j \leqslant n} d_1(\{i_j\})) \cap d_0(\{i_0, \ldots, i_n\}).$$

The function $d_1 : [\lambda]^{<\aleph_0} \to D$ is an accurate distribution of p by construction. □

Lemma 4.1.12. *Let T be a complete theory, $\mathcal{M} \models T$, λ an infinite cardinal, D a regular ultrafilter over λ, $\mathcal{N} = \mathcal{M}^\lambda/D$, φ a formula in the language of T, $A \subset N$ with $|A| \leqslant \lambda$, $p \in S(A)$ a φ-type. Then the following are equivalent:*

1. *Some distribution of p has a multiplicative refinement;*
2. *Every accurate distribution of p has a multiplicative refinement;*
3. *The type p is realized in $\mathcal{N}$.*

Proof. We shall divide the proof into three implications.

(2) ⇒ (1) Given by Lemma 4.1.11.

(1) ⇒ (3) Let d be the distribution and d' its multiplicative refinement. By the definition of distribution, the set $\{u \in [\lambda]^{<\aleph_0} : t \in d(u)\}$ is finite for all $t \in I$. Because otherwise the set $\{d(u) : t \in d(u)\}$ would be infinite, which contradicts the fact that the range of d is a regularizing family, concluding that any distribution is locally finite. Given that d is already a refinement of the Łoś's map d_0, we have that d' is a locally finite multiplicative refinement of d_0. By the discussion about good ultrafilters introduced in last section, we conclude that there exists an actualization of d_0, which implies that p is realized in $\mathcal{N}$.

(3) ⇒ (2) Let α be a realization of $p = \{\varphi(x; a_i) : i \in \lambda\}$. Let d be any accurate distribution of p. Define a function d' where for each $u \in [\lambda]^{<\aleph_0}$ we have

$$d'(u) := \left\{ t \in \lambda : \mathcal{M} \models \bigwedge_{j \in u} \varphi(\alpha(t); a_j(t)) \right\} \cap d(u).$$

The function d' refines d by construction. Let $u, v \in [\lambda]^{<\aleph_0}$. If $t \in d'(u \cup v)$ then $\mathcal{M} \models \bigwedge_{j \in u \cup v} \varphi(\alpha(t); a_j(t))$ and $t \in d(u \cup v)$. We have directly that $\mathcal{M} \models \bigwedge_{j \in u} \varphi(\alpha(t); a_j(t))$ and $\mathcal{M} \models \bigwedge_{k \in v} \varphi(\alpha(t); a_k(t))$, and by monotonicity of d we also have that $t \in d(u) \cap d(v)$, so $t \in d'(u) \cap d'(v)$, concluding that d' is monotonic. If $t \in d'(u) \cap d'(v)$ then $\mathcal{M} \models \bigwedge_{j \in u} \varphi(\alpha(t); a_j(t))$ and $\mathcal{M} \models \bigwedge_{k \in v} \varphi(\alpha(t); a_k(t))$, and also $t \in d(u) \cap d(v)$. We have directly that $\mathcal{M} \models \bigwedge_{j \in u \cup v} \varphi(\alpha(t); a_j(t))$, since the same element $\alpha(t)$ realized both u and v. Using this together with the fact that d is accurate and $t \in d(u) \cap d(v)$ we conclude that $t \in d(u \cup v)$. This gives us that $t \in d'(u \cup v)$, concluding that d' is multiplicative, and thus a multiplicative refinement of d. $\square$

Proof of 4.1.9. Let T be a complete theory.

($\Leftarrow$) Suppose that T has property $\star$. Let T' be any complete theory, λ any infinite cardinal and D any regular ultrafilter over λ. If D is λ^+-saturated for T, then D is λ^+-good, which implies by theorem 4.1.7 that D is λ^+-saturated for T', so $T' \trianglelefteq T$. Given that the choice of T' was arbitrary, we conclude that T is in the maximum class.

($\Rightarrow$) Suppose now that T is in the maximum class. Let λ be any infinite cardinal and D any regular ultrafilter over λ. We need to show that if D is λ^+-saturated for T the D is λ^+-good, since the other direction is given already by 4.1.7. We prove by contrapositive. Supposing that D is not λ^+-good, we construct a theory T' where T' is not λ^+-saturated by D, and use the hypothesis that T is in the maximum class to conclude that T is also not λ^+-saturated by D. Details can be found in [15, Observation 1.12]. $\square$

Now we retain our attention to some other sufficient conditions for a complete theory to be in the maximum class. One such property, as presented in the introduction, is the strict order property.

Definition 4.1.13. Let T be a complete theory, and $\varphi(x; y)$ be a formula of T. The formula $\varphi(x; y)$ has the *strict order property (SOP)* if in some model $\mathcal{M} \models T$ there exists a sequence $\langle a_i : i < \omega \rangle$ such that for all i, j

$$i < j \leftrightarrow \mathcal{M} \models \exists x(\neg\varphi(x; a_i) \wedge \varphi(x; a_j)).$$

We say that T has SOP if there is a formula of T that has SOP.

Theorem 4.1.14 ([27, Chapter VI, Theorem 4.3])**.** *Any complete theory T with SOP is in the maximum class of Keisler's order.*

It is easy to see that the complete theory $\mathrm{Th}((\mathbb{N}, <))$ has the strict order property given by the formula $\varphi(x, y) = x < y$. We simply choose any increasing sequence $\langle a_i : i < \omega \rangle$

in $\mathbb{N}$ and for $i < j$ choose $x = a_i$ and it will guarantee that $\neg\varphi(x; a_i) \wedge \varphi(x; a_j)$. So the maximum class indeed exists.

The next result relates a filter D being λ^+-good with the set $\mathcal{C}(D)$ (cf. Definition 3.0.2).

Corollary 4.1.15. *Let D be a regular ultrafilter over I, with $|I| = \lambda$. Then $\mathcal{C}(D) = \emptyset$ if and only if D is λ^+-good.*

Proof. First it is important to restate the representation of complete 1-types made in Example 2.3.15 of Section 2.3. There, we demonstrated that given a linear order $\mathcal{M} = (M, <)$ and a subset $A \subseteq M$, the set of complete 1-types over A ($S_1^{\mathcal{M}}(A)$) is represented by cuts and "almost" cuts, that is, if $p \in S_1^{\mathcal{M}}(A)$ then there are L_p and U_p associated with p where (L_p, U_p) is a cut in A or it is a pre-cut missing only one element of A.

Suppose $\mathcal{C}(D) = \emptyset$. We know that $\mathcal{N} = (\mathbb{N}, <)^I/D =: (N, <)$ is a linear order by Łoś's Theorem, so we can use the representation by cuts. For any $A \subseteq N$, if a type $p \in S_1^{\mathcal{N}}(A)$ has a formula of the form $v = a$ then it is clearly realized in N by a, so we will only look at complete types represented by cuts in A, that is, those without an equality. Let $A \subset N$ with $|A| \leqslant \lambda$. Let $p \in S_1^{\mathcal{N}}(A)$ and (L_p, U_p) be its associated cut in A. If L_p is empty just pick an element in N that is below every element of U_p. If U_p is empty do the opposite. If L_p is finite, let m_L be the maximum of L_p and construct the sequence $\langle S^n(m_L) : n \in \omega\rangle$ and pick any sequence $\langle u_\alpha : \alpha \in \theta\rangle$ coinitial in U_p ($\theta \leqslant |U_p| \leqslant |A| \leqslant \lambda$). The pair $(\langle S^n(m_L) : n \in \omega\rangle, \langle u_\alpha : \alpha \in \theta\rangle)$ represents a pre-cut in $\mathcal{N}$, and given that $\mathcal{C}(D) = \emptyset$ there is an element in N realizing it. If U_p is finite do the opposite. If both L_p and U_p are infinite then, given that $\mathcal{C}(D) = \emptyset$, we know that (L_p, U_p) is not a cut so there is an element in N realizing it. With the representation by cuts and the hypothesis that $\mathcal{C}(D) = \emptyset$ we proved that $\mathcal{N} = (\mathbb{N}, <)^I/D$ is λ^+-saturated. We showed just before this corollary that $(\mathbb{N}, <)^I/D$ is in the maximum class of Keisler's order, so $(\mathbb{N}, <)^I/D$ being λ^+-saturated implies that D is λ^+-good.

Now suppose that D is λ^+-good. Again, given that $(\mathbb{N}, <)^I/D$ is in the maximum class, D being λ^+-good implies that $(\mathbb{N}, <)^I/D$ is λ^+-saturated (this is also true directly by the fact that D is regular, given by Theorem 4.1.7). Using the representation by cuts we state that any cut in $\mathcal{C}(D)$ is associate with some complete 1-type, and λ^+-saturation implies that there is an element of N realizing the type, that is, in between the cut, a contradiction, concluding that $\mathcal{C}(D)$ must be empty. $\square$

4.1.3 Constructing a CSP to attack our problem

Our next objective is to use the main theorem about CSP, $\mathcal{C}(\mathbf{s}, \mathbf{t_s}) = \emptyset$, to prove that if D has λ^+-treetops (cf. Definition 3.0.6) then $\mathcal{C}(D) = \emptyset$, whenever D is a regular ultrafilter. The set $\mathcal{C}(D)$ looks at all the cuts in $(\omega, \in)^I/D$ with size less than or equal to λ, however by Łoś's theorem $(\omega, \in)^I/D$ doesn't have a maximum, so it cannot be an element of $\mathrm{Or}(\mathbf{s})$ for any CSP $\mathbf{s}$. We will show next that, because D is a regular ultrafilter, there exists a discrete linear order with first and last element that has the information of all the elements of $\mathcal{C}(D)$, this linear order will be the one used in the CSP.

Lemma 4.1.16. *Let D be a regular ultrafilter over I, $|I| = \lambda$. For any $n < \omega$, write $<_n$ for the order on ω restricted to n, that is to $\{0, 1, \dots, n-1\}$. Then there exists a sequence $\bar{n} = \bar{n}(D) = \langle n_t : t \in I\rangle \in {}^I\omega$ such that for all regular cardinals κ_1, κ_2 with $\kappa_1 + \kappa_2 \leqslant \lambda$, the following are equivalent:*

1. *$(\kappa_1, \kappa_2) \in \mathcal{C}(D)$, that is $(\omega, <)^I/D$ has a (κ_1, κ_2)-cut;*
2. *$\prod_t (n_t, <_{n_t})/D$ has a (κ_1, κ_2)-cut.*

Proof. Let $R = \{X_i : i \in \lambda\}$ be a regularizing family of D. For $t \in I$, let $n_t := |\{i \in \lambda : t \in X_i\}| + 1$, and $\bar{n}(D) := \langle n_t : t \in I\rangle$.

(2) ⇒ (1) This direction is simple, given that we can extend $\prod_t (n_t, <_{n_t})/D$ to $(\omega, <)^I/D$ using the inclusion function from $\prod_t (n_t, <_{n_t})$ to $(\omega, <)^I$, and using this function will preserve cuts. Note that this direction is true for any tuple of natural numbers.

(1) ⇒ (2) Let $(\kappa_1, \kappa_2) \in \mathcal{C}(D)$. By definition of $\mathcal{C}(D)$ we have that $\kappa_1 + \kappa_2 < \lambda$, so there exists an injection $d : \kappa_1 \sqcup \kappa_2 \to R$. For each $t \in I$, there are fewer that n_t $(\alpha, 0)$ and $(\beta, 1)$ in $\kappa_1 \sqcup \kappa_2$, $\alpha \in \kappa_1, \beta \in \kappa_2$, such that $t \in d((\alpha, 0))$ or $t \in d((\beta, 1))$, because we defined n_t using the regularizing family in a way that forces this property. Let $(\langle a_\alpha : \alpha \in \kappa_1\rangle, \langle b_\beta : \beta \in \kappa_2\rangle)$ be a representative for a (κ_1, κ_2)-cut in $(\omega, <)^I/D$. We construct, for each $t \in I$, the set $Y_t := \{a_\alpha(t) : t \in d((\alpha, 0))\} \cup \{b_\beta(t) : t \in d((\beta, 1))\}$ (that is, for a fixed $t \in I$, we go through each $\alpha \in \kappa_1$ and pick $a_\alpha(t)$ if $t \in d((\alpha, 0))$, then we do the same for each $\beta \in \kappa_2$ checking if $t \in d((\beta, 1))$). The set Y_t is a linearly ordered subset of $(\omega, <)$ with fewer than n_t elements (making it well ordered). For each $t \in I$ we can construct an order preserving injection $h_t : (Y_t, <_{Y_t}) \to (n_t, <_{n_t})$ where the range is an interval, by assigning the first element of Y_t to 0, then the second to 1 and so on. Let $h := \prod_t h_t/D$. Then, by Łoś's Theorem, $(\langle h(a_\alpha) : \alpha \in \kappa_1\rangle, \langle h(b_\beta) : \beta \in \kappa_2\rangle)$ represents a (κ_1, κ_2)-cut in $\prod_t (n_t, <_{n_t})/D$. Concluding that when there is a (κ_1, κ_2)-cut in $(\omega, <)^\lambda/D$ we can, using the regularity of D, construct a (κ_1, κ_2)-cut in $\prod_t (n_t, <_{n_t})/D$. □

Another important remark for this lemma is the fact that $\prod_t (n_t, <_{n_t})/D$ is an infinite pseudo-finite discrete linear order with first and last element, which shows that the results that we will prove in Chapter 5 can be used to state the same theorems that we will prove in this section, even without the generality of CSPs.

Definition 4.1.17. Let D be a regular ultrafilter over I and $\mathcal{M} = (\omega, <)$. If $\bar{n} = \langle n_t : t \in I\rangle \in {}^I\omega$ is a sequence satisfying the conclusion of theorem 4.1.16 for D and $(X, <_X) \subseteq \mathcal{M}^I/D$ is given by

$$(X, <_X) := \prod_t (n_t, <_{n_t})/D.$$

then we say $(X, <_X)$ *captures pseudo-finite cuts.*

Theorem 4.1.18. *Let D be a regular ultrafilter over I, $|I| = \lambda$. Let $\mathcal{M} = (\omega, \in)$ and $\mathcal{M}^+ = (H(\omega_1), \in)$, together with their ultrapowers by D, given by $\mathcal{N} = (\omega, \in)^I/D$ and $\mathcal{N}^+ = (H(\omega_1), \in)^I/D$. There exists a set of formulas Δ with $\varphi(x, y; z) = \langle x \leqslant y \in z\rangle \in \Delta$ ($x \leqslant y$ abbreviates $x \in y \vee x = y$), such that*

1. *$\boldsymbol{s} = (\mathcal{M}, \mathcal{N}, \mathcal{M}^+, \mathcal{N}^+, Th(\mathcal{M}^+), \Delta)$ is a cofinality spectrum problem, and*

2. *some nontrivial $\boldsymbol{a} \in Or(\boldsymbol{s})$ captures pseudo-finite cuts.*

We will say that a CSP $\boldsymbol{s}$ given by this theorem is a CSP associated with D.

Proof. We already proved in Chapter 3 that $\mathcal{M}^+$ and $\mathcal{N}^+$ can define trees of elements of $\mathcal{M}$ and $\mathcal{N}$ respectively, and that both the pairs $(\mathcal{M}, \mathcal{M}^+)$ and $(\mathcal{N}, \mathcal{N}^+)$ have the pseudo-finite property.

By lemma 4.1.16 we know that looking for cuts, with size less than or equal to λ, in $(\omega, \in)^I/D$ is the same as looking in $\prod_t (n_t, <_{n_t})/D$, where the latter is a discrete linear order with first and last elements. Given that in the definition of cuts we assume that the sequences representing the cut are strictly increasing and decreasing, respectively, the cuts in $\prod_t (n_t, <_{n_t})/D$ are the same as the ones in $\prod_t (n_t, \leqslant_{n_t})/D$, so we can look at the latter instead. Now it remains to show that $\prod_t (n_t, \leqslant_{n_t})/D$ is $\mathcal{N}^+$-definable.

Let $\varphi(x, y; z) = x \leqslant y \in z$ ($x \leqslant y$ abbreviates $x \in y \vee x = y$). It is clear that in $\mathcal{M}^+$ we can define $(n_t, \leqslant_{n_t})$ using φ for all $t \in I$, simply using n_t as the parameter, that is, $\varphi(x, x; n_t)$ defines the universe and $\varphi(x, y; n_t)$ defines the relation. Let $\bar{n}_D = \langle n_t : t \in I\rangle/D$. Let $a_D \in \mathcal{N}^+$ be such that $\varphi(a_D, a_D; \bar{n}_D)$, by Łoś's Theorem $\varphi(a_D, a_D; \bar{n}_D)$ is true in $\mathcal{N}^+$ if and only if $\varphi(a(t), a(t), n_t)$ is true in $\mathcal{M}^+$ for D-many ts. This implies that $a(t) \in n_t$ for D-many ts, concluding that $a_D \in \prod_t n_t/D$, and by reciprocity $a_D \in \prod_t n_t/D$ implies $\varphi(a_D, a_D; \bar{n}_D)$, so $\varphi(x, x; \bar{n}_D)$ defines $\prod_t n_t/D$ in $\mathcal{N}^+$. Now we show that the formula $\varphi(x, y; \bar{n}_D)$ defines the relation $\prod_t \leqslant_{n_t}/D$ in $\mathcal{N}^+$. Let $a_D, b_D \in \prod_t n_t/D$, let $L = \{t \in I : a(t) \leqslant b(t)\}$ and $G = \{t \in I : b(t) \in a(t)\}$. Since the index models $(n_t, \leqslant_{n_t})$ are linear orders, L and G are complementary in I, so by the ultrafilter property $L \in D$ or $G \in D$, concluding, by Łoś's Theorem, that $a_D \leqslant b_D$ or $b_D \in a_D$, that is, $\varphi(a_D, b_D; \bar{n}_D)$ or $\neg\varphi(a_D, b_D; \bar{n}_D)$. So $\varphi(x, y; \bar{n}_D)$ defines the relation $\prod_t \leqslant_{n_t}/D$ in $\mathcal{N}^+$.

So by using a set $\Delta \supseteq \{\varphi(x, y; z)\}$ for the construction of our CSP we can select the order $\prod_t (n_t, <_{n_t})/D$ inside of the CSP, and use our theorem about cuts. $\square$

Corollary 4.1.19. *Let D be a regular ultrafilter over I, $|I| = \lambda$, and $\boldsymbol{s}$ a CSP associated with D. For κ_1, κ_2 regular with $\kappa_1 + \kappa_2 \leqslant \lambda$ the following are equivalent:*

1. $(\kappa_1, \kappa_2) \in \mathcal{C}(\boldsymbol{s}, \lambda^+)$*;*
2. $(\kappa_1, \kappa_2) \in \mathcal{C}(D)$*.*

Proof.

(1) $\Rightarrow$ (2) Let $(\kappa_1, \kappa_2) \in \mathcal{C}(\mathbf{s}, \lambda^+)$. Let $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$ be given by theorem 4.1.18(2), where it captures pseudo-finite cuts. By 5.2.2, if an element of $\mathrm{Or}(\mathbf{s})$ witness a (κ_1, κ_2)-cut, then all elements witness a (κ_1, κ_2)-cut, so the order $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$ witness a (κ_1, κ_2)-cut, concluding with theorem 4.1.16 that $(\kappa_1, \kappa_2) \in \mathcal{C}(D)$.

(2) $\Rightarrow$ (1) Let $(\kappa_1, \kappa_2) \in \mathcal{C}(D)$. Again by theorems 4.1.18(2) and 4.1.16, the existence of $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$ that captures pseudo-finite cuts concludes that $(\kappa_1, \kappa_2) \in \mathcal{C}(\mathbf{s}, \lambda^+)$. $\square$

Remark 4.1.20. We can actually add the following claim in the equivalences of the theorem above

1. There is a (κ_1, κ_2)-cut in some $\mathcal{N}^+$-definable linearly ordered set.

Implying that, for the case of regular ultrafilters, the cuts in $\mathcal{C}(\mathbf{s}, \lambda^+)$ are the same for all CSPs of the form of theorem 4.1.18 regardless of the choice of Δ, as long as $\langle x \leqslant y < z \rangle \in \Delta$ and Δ has the properties necessary in the definition of CSP. The sketch of the proof is as follows:

Suppose $(\kappa_1, \kappa_2) \in \mathcal{C}(\mathbf{s}, \lambda^+)$. Then clearly the new claim is true, given that any element of $\mathrm{Or}(\mathbf{s})$ is a $\mathcal{N}^+$-definable linearly ordered set.

Now suppose the new claim. Let $(Y, <_Y)$ be a $\mathcal{N}^+$-definable linear order and $(\langle a_\alpha : \alpha \in \kappa_1\rangle, \langle b_\beta : \beta \in \kappa_2\rangle)$ a representation of a (κ_1, κ_2)-cut in $(Y, <_Y)$. Let $A = \{a_\alpha : \alpha \in \kappa_1\} \cup \{b_\beta : \beta \in \kappa_2\}$. We want to show that there exists an order preserving internal function that maps A to $X_{\mathbf{a}}$, for some non-trivial $\mathbf{a} \in \mathrm{Or}(\mathbf{s})$, and its range is an interval. This function will move the cut from $(Y, <_Y)$ to $X_{\mathbf{a}}$. Similar to Lemma 4.1.16, create $d : \kappa_1 \sqcup \kappa_2 \to R$ one-to-one function, use d to define, for each $t \in I$, the set $X_t := \{a_\alpha(t) : t \in d((\alpha, 0))\} \cup \{b_\beta(t) : t \in d((\beta, 1))\}$. By the property of the regularizing family each X_t is finite. Now just create, for each $t \in I$, an auxiliary order preserving function h_t from X_t to $|X_t| \subseteq n_t$ (n_t is the number of elements of the regularizing family that contain t, plus one.), then combine all of then into the (internal) function $h := \prod_t h_t/D$ in $(M^+)^\lambda/D$, by Łoś's theorem it satisfies the properties required.

Corollary 4.1.21. *Let D be a regular ultrafilter over I and $\mathbf{s}$ a CSP associated with D. Let κ be regular with $\kappa \leqslant \lambda$. If D has κ^+-treetops, then $\kappa^+ \leqslant \mathfrak{t}_{\mathbf{s}}$.*

Proof. Assume D has λ^+-treetops. This implies that for any tree definable in $\mathcal{M}^+$, and any increasing sequence of size less than κ^+ in the tree in $\mathcal{N}^+$ induced by the tree in $\mathcal{M}^+$, we have that this sequence has an upper bound. So for any order $\mathbf{b} \in \mathrm{Or}(\mathbf{s})$, and any increasing sequence of size less than κ^+ in the tree $T_{\mathbf{b}}$, that lives in $\mathcal{N}^+$ and is internal, that sequence has an upper bound. Concluding that $\kappa^+ \leqslant \mathfrak{t}_{\mathbf{s}}$. □

Remark 4.1.22. The result of last theorem can be strengthened to an equivalence, using the regularity of D in a similar way to what was done in theorem 4.1.16. The sketch of the proof is as follows:

We prove the contrapositive. Suppose that D does not have κ^+-treetops, that is, there exists a definable tree $(T, \trianglelefteq_T)$ in $\mathcal{M}^+$ such that there is an increasing sequence $\langle c_\alpha : \alpha \in \kappa\rangle$ in $(T^{\mathcal{N}^+}, \trianglelefteq_T{}^{\mathcal{N}^+})$ with no upper bound. With the same technique as before we pick a one-to-one function $d : \kappa \to R$ into the regularizing family and use it to construct finite trees $(T_t, \trianglelefteq_{T_t})$ such that $\prod_t (T_t, \trianglelefteq_{T_t})/D$ is a subtree of $(T^{\mathcal{N}^+}, \trianglelefteq_T{}^{\mathcal{N}^+})$. We also construct (for almost every $t \in I$) one-to-one order preserving functions $f_t : (T_t, \trianglelefteq_{T^t}) \to (T_{\mathbf{a}}^{\mathcal{M}^+}, \trianglelefteq_{\mathbf{a}}{}^{\mathcal{M}^+})$. Let $f = \prod_t f_t/D$. We need to show that $\langle f(c_\alpha) : \alpha < \kappa\rangle$ has no upper bound. Suppose by contradiction that there is an upper bound b_*. Let d' be the map given by

$$\alpha \mapsto \{t \in d(\alpha) : f(c_\alpha)[t] \trianglelefteq b_*[t] \text{ and } f_t \text{ is injective and respects the partial ordering.}\}$$

The set $B_t = \{f(c_\alpha)[t] : \alpha < \kappa \text{ and } t \in d'(\alpha)\}$ is finite and well-ordered by $\trianglelefteq$ (because of the property in d' where $f(c_\alpha)[t] \trianglelefteq b_*[t]$ (the set of predecessors is well-ordered)) so it has a maximum b_t. Let $c_* := \prod_t f_t^{-1}(b_t)/D$. We have that $c_* \in T^{\mathcal{N}^+}$ and is an upper bound for $\langle c_\alpha : \alpha < \kappa\rangle$, a contradiction. We conclude that $\langle f(c_\alpha) : \alpha < \kappa\rangle$ has no upper bound, so $\mathfrak{t}_{\mathbf{s}} < \kappa^+$.

Theorem 4.1.23. *Let D be a regular ultrafilter over I, with $|I| = \lambda \geqslant \aleph_0$. If D has λ^+-treetops, then D is λ^+-good.*

Proof. Let $\mathbf{s}$ be a CSP associated with D, given by theorem 4.1.18. By theorem 3.0.15, $\mathcal{C}(\mathbf{s}, \mathfrak{t}_{\mathbf{s}}) = \emptyset$. By corollary 4.1.21 we have that $\lambda^+ \leqslant \mathfrak{t}_{\mathbf{s}}$, so $\mathcal{C}(\mathbf{s}, \lambda^+) = \emptyset$. With corollary 4.1.19 we get that $\mathcal{C}(D) = \emptyset$. Concluding with theorem 4.1.15 that D is λ^+-good. □

So with this we connected the notion of treetops with that of good ultrafilters. This may seem irrelevant but remember that our objective is to study the maximum class of Keisler's order, where the non-trivial property of Keisler's characterization is that saturation implies goodness. So in the next section we will show that in the case of SOP_2 theories, which we want to show are maximal, saturation implies treetops, then making use of these last results to conclude goodness.

4.1.4 SOP_2 implies maximality in Keisler's Order

Definition 4.1.24. Let T be a complete theory, and $\psi(x;\bar{y})$ be a formula of T. The formula $\psi(x;\bar{y})$ has the *2-strict order property* (SOP_2) if in some model $\mathcal{M} \models T$ there exists $\{\bar{a}_\eta : \eta \in {}^{<\kappa}\mu\}$, called an SOP_2-tree for ψ, such that:

1. for $\eta, \rho \in {}^{<\kappa}\mu$ incompatible, that is $\neg(\eta \trianglelefteq \rho) \quad \& \quad \neg(\rho \trianglelefteq \eta)$, we have that $\{\psi(x;\bar{a}_\eta), \psi(x;\bar{a}_\rho)\}$ is inconsistent.
2. for $\eta \in {}^{\kappa}\mu$, $\{\psi(x;\bar{a}_{\eta\restriction i}) : i < \kappa\}$ is consistent, making it a type.

We say that T has SOP_2 if there is a formula of T that has SOP_2.

So, in $\mathcal{M}$, we have the tree ${}^{<\kappa}\mu$ and for each node $\eta \in {}^{<\kappa}\mu$ we associate a value $\bar{a}_\eta$. Elements of different branches are mutually inconsistent parameters for ψ, and the set of all elements of a branch is a set of mutually consistent parameters for ψ.

Definition 4.1.25. Let T be a theory with SOP_2, D a regular ultrafilter over I and $\mathcal{M}$ a model of T. Within $\mathcal{M}^I/D$, by *SOP_2-κ-type* we mean a type $p(x) = \{\psi(x;a_\ell) : \ell < \kappa\}$ where $\psi(x;y)$ has SOP_2 and D-almost all of projections to the index model come from an SOP_2-tree for ψ. We say that D realizes all SOP_2-types if for all $\mathcal{M} \models T$, all SOP_2-$|I|$-types are realized in $\mathcal{M}^I/D$.

Lemma 4.1.26. *Let T be a theory with SOP_2 and $\psi(x;y)$ be a formula with SOP_2. Let D be a regular ultrafilter over I, with $|I| = \lambda$, and $\mathcal{M} \models T$ with $\mathcal{N} = \mathcal{M}^I/D$. Then the following are equivalent:*

1. *Every SOP_2-type is realized in $\mathcal{N}$;*
2. *Every SOP_2-type $p = \{\psi(x;a_\ell) : \ell < \lambda\}$ in $\mathcal{N}$ has a distribution d such that for D-almost all $s \in I$, for all $i, j < \lambda$,*

$$s \in d(\{i\}) \cap d(\{j\}) \Rightarrow a_i[s] \text{ and } a_j[s] \text{ are comparable}$$

3. *D has λ^+-treetops.*

Proof. We shall divide the proof into four implications, using property (2) as a bridge between properties (1) and (3).

(1) $\Rightarrow$ (2) Let p be an SOP_2-type in $\mathcal{N}$. By (1) there is $\alpha \in N$ a realization of p, and by the regularity of D there is $\{X_i : i < \lambda\}$ a λ-regularizing family for D. Define the function $d : [\lambda]^{<\aleph_0} \to D$ as:

a) $\{i\} \mapsto \{s \in I : \mathcal{M} \models \psi(\alpha[s], a_i[s])\} \cap X_i$;

b) for $|u| > 1$, $u \mapsto \bigcap\{d(\{i\}) : i \in u\}$.

In (a), the first part "$\{s \in I : \mathcal{M} \models \psi(\alpha[s], a_i[s])\}$" guarantees that if $s \in d(\{i\}) \cap d(\{j\})$ then a_i and a_j are comparable, and the second part "$\cap X_i$" guarantees that the image of d is a λ-regularizing family (Note that there can't be the case that the regularizing family shrinks in size, because if there are an infinite number of is such that $\{s \in I : \mathcal{M} \models \psi(\alpha[s], a_i[s])\} \cap X_i$ are equal we have that $\{X_i\}$ was not a λ-regularizing family to begin with, and they can't be empty because they are elements of the ultrafilter). In (b), "$u \mapsto \bigcap\{d(\{i\}) : i \in u\}$" completes the function guaranteeing that d refines the Łoś's map of p and that d is monotonic.

(2) ⇒ (1) Let $p = \{\psi(x; a_\ell) : \ell < \lambda\}$ be an SOP_2-type in $\mathcal{N}$, d a distribution given by (2), and d_0 the Łoś's map of p. Let $A \in D$ be the set of indexes $s \in I$ such that for all $i < \lambda$ we have that $a_i[s]$ is in an SOP_2-tree for ψ and property (2) holds. The set A is in D because it is an intersection of two elements of D. Let $d'(u) := \bigcap_{i \in u} d(\{i\})$, that is, for the unitary elements we have $d'(\{i\}) := d(\{i\})$ and we construct other elements by intersection of the unitary case. We have that $s \in d'(u) \cap d'(v)$ if and only if $s \in \bigcap_{i\in u} d(\{i\}) \cap \bigcap_{i \in v} d(\{i\}) = \bigcap_{i \in u \cup v} d(\{i\}) = d'(u \cup v)$, concluding that d' is multiplicative. The range of d' is a λ-regularizing family, because the range by the unitary elements is already a λ-regularizing family and the addition of all the intersections do not change this fact. We will use these together with property (2) and SOP_2 to show that d' is a refinement of the Łoś's map d_0. Let $s \in d'(u)$. By multiplicativity we have that $s \in \bigcap_{i \in u} d'(\{i\})$. By property (2) we have that all elements $a_i[s], a_j[s]$ are comparable in an SOP_2-tree for ψ in $\mathcal{M}$, with $i, j \in u$. By the SOP_2 property this is the same as stating that $\mathcal{M} \models \exists x \bigwedge_{i \in u} \psi(x, a_i[s])$, so $s \in d_0(u)$. This concludes that d' is a multiplicative refinement of d_0 such that its range is a regularizing family. In other words, d' is a multiplicative distribution for p. Now we use Lemma 4.1.12 to conclude that p has a realization in $\mathcal{N}$.

(3) ⇒ (2) Let $p = \{\psi(x; a_\ell) : \ell < \lambda\}$ be an SOP_2-type. For each $i \in \lambda$, let $(P_i, \trianglelefteq_i)$ be an SOP_2-tree for ψ in $\mathcal{M}$ where $a_\ell[i] \in P_i$ for all $\ell < \lambda$ (if i is a projection where the parameters do not come from an SOP_2-tree just pick $P_i = \emptyset$). Let $(P, \trianglelefteq) := (\prod_{i\in\lambda} P_i/D, \prod_{i \in \lambda} \trianglelefteq_i /D)$ be the induced tree in $\mathcal{N}$. Since p is a SOP_2-type, all of a_is are comparable. Suppose, without loss of generality, that the sequence $\langle a_i : i < \lambda\rangle$ is $\trianglelefteq$-increasing, so by (3) there is an upper bound c. Let $\{X_i : i < \lambda\}$ be a λ-regularizing family for D. Define the function $d : [\lambda]^{<\aleph_0} \to D$ as:

a) $\{i\} \mapsto \{s \in I : a_i[s] \trianglelefteq_i c[s]\} \cap X_i$;

b) for $|u| > 1$, $u \mapsto \bigcap\{d(\{i\}) : i \in u\}$.

Similar to before, in (a), the first part "$\{s \in I : a_i[s] \trianglelefteq_i c[s]\}$" guarantees that if $s \in d(\{i\}) \cap d(\{j\})$ then a_i and a_j are comparable, and the second part "$\cap X_i$"

guarantees that the image of d is a λ-regularizing family. In (b), "$u \mapsto \bigcap\{d(\{i\}) : i \in u\}$" completes the function guaranteeing that d refines the Łoś's map of p and that d is monotonic.

(2) $\Rightarrow$ (3) Let $\mathcal{M}'$ be a structure that interprets a tree $(T^{\mathcal{M}'}, \trianglelefteq_T^{\mathcal{M}'})$, and $\mathcal{N}' = (\mathcal{M}')^I/D$. Let $\langle c_\alpha : \alpha \in \lambda \rangle$ be an increasing sequence in $(T^{\mathcal{N}'}, \trianglelefteq_T^{\mathcal{N}'})$. We construct the type $\{c_\alpha \trianglelefteq_T^{\mathcal{N}'} x : \alpha \in \lambda\}$ in $(T^{\mathcal{N}'}, \trianglelefteq_T^{\mathcal{N}'})$. Realizing this type is the same as finding an upper bound for the sequence. Let d_T be some distribution for this type, which exists by 4.1.11. By the definition of distribution, the set $\{u \in [\lambda]^{<\aleph_0} : t \in d_T(u)\}$ is finite for all $t \in I$. Because otherwise the set $\{d_T(u) : t \in d_T(u)\}$ would be infinite, which contradicts the fact that the range of d_T is a regularizing family.

Let $(P, \trianglelefteq_P)$ be an SOP_2-tree in $\mathcal{M}$ for the formula $\psi(x; y)$ of the hypothesis. With this we may construct an SOP_2-type $p' = \{\psi(x; a_\alpha) : \alpha \in \lambda\}$ associated with the sequence $\langle c_\alpha : \alpha \in \lambda \rangle$ such that:

- for each $t \in I$, let $a_\beta(t) \trianglelefteq_P a_\alpha(t)$ if and only if $c_\beta(t) \trianglelefteq_T^{\mathcal{M}'} c_\alpha(t)$ (For example, if for a certain $t \in I$ we have that the sequence $\langle c_\alpha(t) : \alpha \in \lambda \rangle$ is increasing, then simply choose a branch of the SOP_2-tree and select the elements of this branch as the parameters a_α accordingly, otherwise choose a_α randomly);
- let $a_\alpha \in M^I/D$ such that $a_\alpha = \prod_{t \in I} a_\alpha(t)/D$.

By construction together with Łoś's Theorem we have that p' is indeed an SOP_2-type in $\mathcal{N}$. By hypothesis there is a distribution d' of p' satisfying property (2). For each $t \in I$ define the set $C_t := \{c_\alpha(t) : \alpha \in \lambda, t \in d'(\{\alpha\})\}$ which is finite, as we know that the set $\{\alpha : t \in d'(\{\alpha\})\}$ is finite, and $\trianglelefteq_T^{\mathcal{M}'}$-linearly ordered in $(T^{\mathcal{M}}, \trianglelefteq_T^{\mathcal{M}'})$. Choose $c^* \in (T^{\mathcal{M}}, \trianglelefteq_T^{\mathcal{M}'})^I$ such that $c^*(t)$ is the $\trianglelefteq_T^{\mathcal{M}'}$-maximum element of C_t for each $t \in I$, and conclude that c^*/D is an upper bound for the sequence $\langle c_\alpha : \alpha \in \lambda \rangle$. □

Corollary 4.1.27. *Let T be a complete theory with SOP_2 and D be a regular ultrafilter over I, $|I| = \lambda$. Then we have the following implications:*

$$D \text{ realizes all } SOP_2\text{-types} \Rightarrow D \text{ has } \lambda^+\text{-treetops} \Rightarrow \mathcal{C}(D) = \emptyset \Rightarrow D \text{ is } \lambda^+\text{-good.}$$

Proof. Just a compilation of theorems 4.1.26, 4.1.23 and 4.1.15. □

Theorem 4.1.28. *Let T be a complete theory with SOP_2. If D is a regular ultrafilter over I then*

$$T \text{ is } \lambda^+\text{-saturated by } D \Rightarrow D \text{ is } \lambda^+\text{-good}$$

Proof. If T is λ^+-saturated by D then D saturates all SOP_2-types, so by corollary 4.1.27 we have that D is λ^+-good. □

Theorem 4.1.29 (Maximality of SOP_2)**.** *Every theory with SOP_2 is maximal in Keisler's order.*

Proof. Directly by theorems 4.1.7 and 4.1.28. □

4.2 $\mathfrak{p} = \mathfrak{t}$

In this section we will use the technology of CSPs to solve the question about the inequality between the cardinal characteristics $\mathfrak{p}$ and $\mathfrak{t}$.

Firstly we define some important properties. Define $A \subseteq^* B$ as saying that the set $A \setminus B$ is finite, that is, A is a subset of B except for a finite number of elements. Let $\mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0}$ be a family. We say that a set $A \subseteq \mathbb{N}$ is a *pseudo-intersection* for $\mathcal{F}$ if for all $B \in \mathcal{F}$ we have that $A \subseteq^* B$. The family $\mathcal{F}$ is a *tower* if it is linearly ordered by $\subseteq^*$ and has no infinite pseudo-intersection. Lastly we restate the strong finite intersection property, defined in 2.2.4. A set X is said to have the *strong finite intersection property* (sfip) if the intersection of any finite number of elements of X is infinite.

Definition 4.2.1. The cardinal $\mathfrak{p}$ is defined as the smallest size of a family $\mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0}$ such that $\mathcal{F}$ has the strong finite intersection property but has no infinite pseudo-intersection. The cardinal $\mathfrak{t}$ is defined as the smallest size of a tower.

Another cardinal characteristic that will be useful for this section is the bounding number, defined as $\mathfrak{b} = \min\{|B| : B \subseteq {}^{\omega}\omega \text{ is unbounded in } ({}^{\omega}\omega, \leqslant^*)\}$.

We have directly that $\mathfrak{p} \leqslant \mathfrak{t}$, because a tower has the sfip. As long as there exists a family with the sfip and no infinite pseudo-intersection then $\aleph_1 \leqslant \mathfrak{p}$, because let $\mathcal{F} = \{F_n : n \in \omega\} \subseteq [\mathbb{N}]^{\aleph_0}$ be a countable family with the sfip, and define recursively $a_n = \min(\cap_{k\leqslant n} F_n \setminus \{a_k : k \in n\})$, $a_0 = \min(F_0)$. The set $\{a_n : n \in \omega\}$ is an infinite pseudo-intersection for $\mathcal{F}$, because for each F_n we have, by construction, that $\{a_k : k \geqslant n\} \subseteq F_n$.

For the proof of $\mathfrak{p} = \mathfrak{t}$ we will use

$$\mathbf{Q} = ([\mathbb{N}]^{\aleph_0}, \subseteq^*)$$

as a forcing poset. So $\mathbf{Q}$ only has infinite subsets as elements.

Lemma 4.2.2. *Any generic filter G of $\mathbf{Q}$ is a non-principal ultrafilter on $\mathcal{P}(\mathbb{N})^{\mathcal{V}}$.*

Proof. Select any element $a \in \mathbf{Q}$. Let A be the open set generated by a in $\mathbf{Q}$, that is

$$A = \{x \in \mathbf{Q} : x \leqslant a\} = \{x \in \mathbf{Q} : x \subseteq^* a\}.$$

If a^c is finite, then A is a dense subset of $\mathbf{Q}$. Because for every element $x \in \mathbf{Q}$, the intersection of x and a must be infinite, given that a^c is finite, making $x \cap a$ an element of A.

If a^c is also infinite, let A_c be the open set generated by a^c in $\mathbf{Q}$, now $A \cup A_c$ is a dense subset of $\mathbf{Q}$. Because for every element $x \in \mathbf{Q}$, if the intersection of x and a is finite, then $x \cap a^c$ is infinite, as otherwise x would be finite, making $x \cap a$ or $x \cap a^c$ an element of $A \cup A_c$. This shows that G is an ultrafilter on $\mathcal{P}(\mathbb{N})^{\mathcal{V}}$.

And G cannot be principal as it does not contain any finite sets. □

Definition 4.2.3. Now we define some objects that will be used throughout the section.

1. Let $\mathcal{V}$ be a transitive model of ZFC.
2. Let $\mathcal{M} = (H(\omega_1), \in)$.
3. Let $\mathbf{Q} = ([\mathbb{N}]^{\aleph_0}, \subseteq^*)$ be our forcing poset.
4. Let G be a generic filter of $\mathbf{Q}$ over $\mathcal{V}$. Let $\tilde{G}$ be a $\mathbf{Q}$-name for G.
5. Given $\mathcal{M}, \mathbf{Q}$ and G, define the generic ultrapower $\mathcal{M}^\omega/G$ in $\mathcal{V}[G]$ as the model $\mathcal{N} \in \mathcal{V}[G]$ with universe $\{f/G : f \in (M^\omega)^{\mathcal{V}}\}$ such that
$$\text{“}\mathcal{N} \models f_1/G \in f_2/G\text{” if and only if } \{n \in \omega : f_1(n) \in f_2(n)\} \in G.$$
Given that we defined the truth in $\mathcal{N}$ using the result of Łoś's Theorem, we conclude that the model $\mathcal{V}[G]$ will see internally the object $\mathcal{N}$ as the ultrapower of $\mathcal{M}$ by G, as desired. As we know by the results of forcing, we can talk about these objects inside of $\mathcal{V}$.
6. In $\mathcal{V}$, let $\tilde{\mathcal{N}}$ be the $\mathbf{Q}$-name of $\mathcal{N}$, that is, the model with universe $\{f/\tilde{G} : f \in (M^\omega)^{\mathcal{V}}\}$ such that
$$\Vdash (\text{“}\tilde{\mathcal{N}} \models f_1/\tilde{G} \in f_2/\tilde{G}\text{” if and only if } \{n \in \omega : f_1(n) \in f_2(n)\} \in \tilde{G}).$$

We will now build a cofinality spectrum problem to attack the question of $\mathfrak{p} = \mathfrak{t}$. Let $\mathcal{M} = \mathcal{M}^+ = (H(\omega_1), \in)$ and $\mathcal{N} = \mathcal{N}^+$ as above.

Definition 4.2.4. Working in $\mathcal{V}[G]$, let $\mathcal{M}$ be as defined before. Let Δ_{f} be the set of all first-order formulas $\varphi(x, y; \bar{z})$ in the vocabulary of $\mathcal{M}$, that is, $\{\in\}$, such that if $\bar{c} \in M^{\ell(\bar{z})}$, then $\varphi(x, t; \bar{c})$ defines a linear order on the finite set $A^{\mathcal{M}}_{\varphi,\bar{c}} := \{a \in M : \mathcal{M} \models \varphi(a, a, \bar{c})\}$, denoted by $\leqslant^{\mathcal{M}}_{\varphi,\bar{c}}$. Where $\ell(\bar{z})$ is the arity of $\bar{z}$. We require $\ell(x) = \ell(y)$ but do not require $\ell(x) = 1$.

If $\mathcal{M}, \mathcal{N}$ are as above and $\varphi \in \Delta_{\mathrm{f}}$, then we can use Łoś's Theorem to prove the following:

1. for each $\bar{c} \in N^{\ell \bar{y}}$, we have that $\varphi(x, y; \bar{c})$ is a discrete linear order on the set $A^{\mathcal{N}}_{\varphi,\bar{c}} = \{a \in N : \mathcal{N} \models \varphi(a, a, \bar{c})\}$;

2. each nonempty $\mathcal{N}$-definable subset of $A^{\mathcal{N}}_{\varphi,\bar{c}}$ has a first and last elements;

3. we may in $\mathcal{N}$ identify $(A^{\mathcal{N}}_{\varphi,\bar{c}}, \leqslant^{\mathcal{N}}_{\varphi,\bar{c}})$ with a definable subset of some

$$\langle (X_n, \leqslant_n) : n < \omega \rangle / G,$$

where each X_n is finite and linearly ordered by $\leqslant_n$.

Lemma 4.2.5 ([20], Claim 14.6)**.** *Working in $\mathcal{V}[G]$, the tuple $(\mathcal{M}, \mathcal{M}^+, \mathcal{N}, \mathcal{N}^+, \mathrm{Th}(\mathcal{M}), \Delta_{\mathrm{f}})$ is a cofinality spectrum problem.*

We can see by the choice Δ_{f} that any order $\mathbf{a}$ in this CSP is built from the ultraproduct of finite linear orders. This shows that we can use, for this section, the results of Chapter 5 in the same way as the general results of CSP.

Lemma 4.2.6. *Working in $\mathcal{V}[G]$, let $\boldsymbol{s}$ be the CSP from lemma 4.2.5. Then $\mathfrak{t} \leqslant \mathfrak{t}_{\boldsymbol{s}}$.*

Proof. This proof has a technical construction that is essential to it, however I would like to first present the general idea behind the proof and later show the technical part to fill the gaps.

Let $\theta < \mathfrak{t}$. Suppose there is, in $\mathcal{V}[G]$, a sequence $\langle \tilde{f}_\alpha / \tilde{G} : \alpha < \theta \rangle$ $\trianglelefteq^{\tilde{\mathcal{N}}}$-increasing in $({}^{<\omega}\omega, \trianglelefteq)^{\tilde{\mathcal{N}}}$ that is unbounded. That is,

$$\mathcal{V}[G] \models \text{“}\langle \tilde{f}_\alpha / \tilde{G} : \alpha < \theta \rangle \text{ is } \trianglelefteq^{\tilde{\mathcal{N}}} \text{-increasing and unbounded in } ({}^{<\omega}\omega, \trianglelefteq)^{\tilde{\mathcal{N}}}.$$

By the Truth Lemma (2.4.8), there exists a $B \in G$ such that

$$B \Vdash \text{“}\langle \tilde{f}_\alpha / \tilde{G} : \alpha < \theta \rangle \text{ is } \trianglelefteq^{\tilde{\mathcal{N}}} \text{-increasing and unbounded in } ({}^{<\omega}\omega, \trianglelefteq)^{\tilde{\mathcal{N}}}\text{”}.$$

Given that, we want to find a $C \in \mathbf{Q}$ stronger then B, that forces the existence of an upper bound for $\langle \tilde{f}_\alpha / \tilde{G} : \alpha < \theta \rangle$, because the property of the sequence being unbounded will be carried from B to C, giving us a contradiction. Even if $C \notin G$, Lemma 2.4.4 guarantees us that there exists a generic filter H where $C \in H$. Given that $C \subseteq^* B$ and H is a filter, we have that $B \in H$. By the truth lemma we conclude that in $\mathcal{V}[H]$ it is true that the above sequence is unbounded and has a bound, giving a contradiction. So it cannot be the case that such a set $C \in \mathbf{Q}$ exists, even if it is not in G.

For each $\alpha < \theta$, we define

$$Y_\alpha := \bigcup \{\{n\} \times ({}^{<\omega}\omega)^{[f_\alpha(n)]} : n \in B\},$$

where $({}^{<\omega}\omega)^{[\nu]} = \{\eta \in {}^{<\omega}\omega : \nu \trianglelefteq \eta\}$, and let $Y := \{Y_\alpha : \alpha < \theta\}$. So each Y_α contains, as a disjoint union for each n, all the possible upper bounds for $f_\alpha(n)$ in ${}^{<\omega}\omega$ (We want the union to be disjoint because, for a given n, we want to know the elements that extend $f_\alpha(n)$, which in principle have nothing in common with the elements extending other ns).

Now we want to take advantage of the fact that $\theta < \mathfrak{t}$, and to do that we need to restrict our Y_αs. For each n we construct a finite, nonempty $s_n \subset {}^{<\omega}\omega$ that will work as the pool of available elements for each n, and with that a new Y'_α where, for each n, we pick only the extensions of $f_\alpha(n)$ that are in s_n, and we guarantee that, for all but a finite number of indexes, it is nonempty. Let

$$Y_* := \bigcup\{\{n\} \times s_n : n \in B\}, \qquad Y'_\alpha := \bigcup\{\{n\} \times (s_n)^{[f_\alpha(n)]} : n \in B\}.$$

Suppose that such s_ns exists (We will make the formal construction at the end of the proof), we can then continue our proof using $Y' := \{Y'_\alpha : \alpha < \theta\}$.

Now we show that if $\alpha < \beta$ then $Y'_\alpha \supseteq^* Y'_\beta$, making the set Y' well-ordered by $\supseteq^*$. Firstly, if $\alpha < \beta$ then $B' = \{n \in B : f_\alpha(n) \ntrianglelefteq f_\beta(n)\}$ is finite (equivalent to "$B \subseteq^* \{n : f_\alpha(n) \trianglelefteq f_\beta(n)\}$" and to "$B \setminus \{n : f_\alpha(n) \trianglelefteq f_\beta(n)\}$ is finite"), as otherwise we would have $B' \in \mathbf{Q}$ stronger than B with $B' \Vdash$ "$f_\alpha/\tilde{G} \ntrianglelefteq f_\beta/\tilde{G}$", contradicting B. Knowing that in Y'_α there are only a finite number of elements associated with each n, namely every element in s_n, and there only a finite number of ns in which $f_\alpha(n) \ntrianglelefteq f_\beta(n)$, there can be only a finite number of elements in $Y'_\beta \setminus Y'_\alpha$.

We can use any bijection between Y_* and $\mathbb{N}$ to look at the subsets of Y_* as though they are subsets of $\mathbb{N}$, which will enable us to use the hypothesis of $\theta < \mathfrak{t}$. Given that $(Y', \supseteq^*)$ is a well-ordered, in particular linearly ordered, set equivalent to a family $\mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0}$ with size $\theta < \mathfrak{t}$ we conclude that there exists an infinite pseudo-intersection Z of Y', because otherwise $\mathcal{F}$ would be a tower of size $\theta < \mathfrak{t}$. That is, $Z \subseteq^* Y'_\alpha$ for all $\alpha < \theta$, which implies that $Z \subseteq^* Y_*$. We can easily refine Z to Z' where it is still an infinite pseudo-intersection and also $Z' \subseteq Y_*$.

Let C be the set of indexes for which Z' is nonempty, that is

$$C := \{n \in B : Z' \cap (\{n\} \times s_n) \neq \emptyset\}.$$

The set C is infinite. Because $Z' \subseteq^* Y'_\alpha$ for all $\alpha < \theta$, the set Z' is infinite, and the set s_n is finite for each $n \in B$, so there are an infinite number of $n \in B$ such that Z' and Y'_α share an element of the form (n, σ). We conclude then that $C \in \mathbf{Q}$.

We can use Z' to construct an upper bound as such. For each $n \in C$, choose $\nu_n \in s_n$ such that $(n, \nu_n) \in Z'$, and for each $n \in \mathbb{N} \setminus C$ let $\nu_n = \langle 0 \rangle$. The element $\langle \nu_n : n < \omega \rangle/\tilde{G}$ is forced by C to be an upper bound for the sequence, as show below.

For each $\alpha < \theta$ define C_α as the set of indexes for which $f_\alpha(n) \trianglelefteq \nu_n$, that is

$$C_\alpha := \{n \in C : f_\alpha(n) \trianglelefteq \nu_n\}.$$

For all $\alpha < \theta$ we have that $C \setminus C_\alpha$ is finite, because otherwise we would have an infinite number of elements of Z' (at least one for each $n \in C \setminus C_\alpha$) that aren't in Y_α, contradicting $Z' \subseteq^* Y'_\alpha$. Concluding that $C \subseteq^* C_\alpha$ and, as a corollary, if $C \in G$ then $C_\alpha \in G$ for all $\alpha < \theta$.

This concludes that

$$C \Vdash \text{“}\langle \nu_n : n < \omega \rangle / \tilde{G} \text{ is an upper bound for } \langle \tilde{f}_\alpha / \tilde{G} : \alpha < \theta \rangle \text{ in } {}^{<\omega}\omega^{\tilde{N}}\text{”}.$$

By the discussion at the start of the proof, the existence of such a $C \in \mathbf{Q}$ gives us a contradiction even if $C \notin G$.

Now for the technical part, the construction of s_n. For this we will use the fact that $\mathfrak{t} \leqslant \mathfrak{b}$, the bounding number defined as $\mathfrak{b} = \min\{|B| : B \subseteq {}^\omega\omega \text{ is unbounded in } ({}^\omega\omega, \leqslant^*)\}$.

For each $\alpha < \theta$ define the function h_α as

$$\begin{aligned} h_\alpha &: \omega \to \omega \\ n &\mapsto \ \lg(f_\alpha(n)) + \sum f_\alpha(n)(j)[j < \lg(f_\alpha(n))] \end{aligned}$$

So what it does is it takes the sum of all the elements of the sequence $f_\alpha(n)$ and the size of the domain of $f_\alpha(n)$. This is simply the most trivial method to guarantee that, for $f_\alpha(n)$, we get an upper bound for each individual element and the size of the domain.

Let H be the set of all h_α for $\alpha < \theta$. Since H has size $\theta < \mathfrak{t} \leqslant \mathfrak{b}$, there exists an element g of ${}^\omega\omega$ such that $h_\alpha \leqslant^* g$ for all $\alpha < \theta$.

Define

$$s_n := {}^{g(n)}g(n) = \{\eta : \eta \text{ is a sequence of length} \leqslant g(n) \text{ of numbers} < g(n)\}.$$

We have that:

1. Each s_n is a finite, nonempty subtree of ${}^\omega\omega$;
2. if $\alpha < \theta$ then $(\forall^\infty n)(f_\alpha(n) \in s_n)$, making $Y'_\alpha \cap (\{n\} \times s_n)$ nonempty for all but finitely many indexes.

As desired. □

Definition 4.2.7. Let κ_1, κ_2 be infinite cardinals. A (κ_1, κ_2)-*peculiar cut* in ${}^\omega\omega$ is a pair $(\langle g_i : i < \kappa_1 \rangle, \langle f_j : j < \kappa_2 \rangle)$ of sequences of functions in ${}^\omega\omega$ such that

1. $(\forall i < j < \kappa_1)(g_i <^* g_j)$;
2. $(\forall i < j < \kappa_2)(f_j <^* f_i)$;

3. $(\forall i < \kappa_1)(\forall j < \kappa_2)(g_i <^* f_j)$;

4. if $f : \omega \to \omega$ is such that $(\forall j < \kappa_2)(f \leqslant^* f_j)$, then $f \leqslant^* g_i$ for some $i < \kappa_1$;

5. if $f : \omega \to \omega$ is such that $(\forall i < \kappa_1)(g_i \leqslant^* f)$, then $f_j \leqslant^* f$ for some $j < \kappa_2$.

Theorem 4.2.8 ([30], Theorem 1.12)**.** *Assume $\mathfrak{p} < \mathfrak{t}$. Then for some regular cardinal κ there exists a $(\kappa, \mathfrak{p})$-peculiar cut in ${}^\omega\omega$ with $\aleph_1 \leqslant \kappa < \mathfrak{p}$.*

Lemma 4.2.9. *Working in $\mathcal{V}[G]$, suppose $\mathfrak{p} < \mathfrak{t}$ and let $\mathcal{N}$, $\boldsymbol{s}$ be as before. Then for some regular κ_1 with $\aleph_1 \leqslant \kappa_1 < \mathfrak{p}$, we have that $(\kappa_1, \mathfrak{p}) \in \mathcal{C}(\boldsymbol{s}, \mathfrak{t}_{\boldsymbol{s}})$.*

Proof. Let $(\langle g_i : i < \kappa_1\rangle, \langle f_j : j < \kappa_2\rangle)$ be a (κ_1, κ_2)-peculiar cut in ${}^\omega\omega$ with $\kappa_2 = \mathfrak{p}$, that Theorem 4.2.8 gives us.

For each n let $I_n := [0, f_0(n)]$. In this peculiar cut, f_0 is the greatest element among fs and gs, in the $<^*$ order. We can use these sets to construct a pseudo-finite linearly ordered set, I, to work within the CSP. For each $n < \omega, (I_n, <)$ is finite and linearly ordered, then

$$I := (\prod_{n<\omega} [0, f_0(n)])/G$$

is pseudo-finite and linearly ordered, with the induced order, so it is an order $\mathbf{a}$ of the CSP $\mathbf{s}$.

Since G is a non-principal ultrafilter over $\mathcal{P}(\mathbb{N})^{\mathcal{V}}$, it contains all cofinite sets and by definition $g_i <^* f_0$ means that the set $\{n \in \omega : g_i(n) < f_0(n)\}$ is cofinite, so for each $i < \kappa_1$ we have that $g_i/G \in I$, with $g_i/G <_{\mathbf{a}} g_j/G$ when $i < j < \kappa_1$, since $g_i <^* g_j$.

For the fs, we also have that $f_i/G \in I$, and $f_j/G <_{\mathbf{a}} f_i/G$ when $i < j < \kappa_1$, because $f_j <^* f_i$ when $i < j < \kappa_2$.

Now, for $i < \kappa_1, j < \kappa_2$ we have $g_i/G <_{\mathbf{a}} f_j/G$, since $g_i <^* f_j$. So $(\langle g_i/G : i < \kappa_1\rangle, \langle f_j/G : j < \kappa_2\rangle)$ represents a pre-cut in I.

Now we need to show that it is a cut. Let $h/G \in I$ be such that $h/G < f_j/G$ for all $i < \kappa_1, j < \kappa_2$. Let $h \in {}^\omega\omega$ be a representative for h/G. By the truth lemma (2.4.8), there exists $B \in G$ where

$$B \Vdash_{\mathbf{Q}} \text{“}h/\tilde{G} < f_j/\tilde{G}\text{” for all } j < \kappa_2.$$

The fact that $B \in \mathbf{Q}$ implies that B is infinite, and $B \in G$ implies that, for all $j < \kappa_2$, the set $B_{f_j} := \{n \in B : h(n) \geqslant f_j(n)\}$ is finite, as otherwise there would be an $j' < \kappa_2$ and an infinite $B_{f_{j'}}$ stronger then B ($B_{f_{j'}} \subseteq^* B$) where $B_{f_{j'}} \Vdash_{\mathbf{Q}}$ “$h/\tilde{G} \geqslant f_j/\tilde{G}$”, a contradiction, because $B_{f_{j'}}$ being stronger then B implies that $B_{f_{j'}} \Vdash$ “$h/\tilde{G} < f_j/\tilde{G}$” for all $j < \kappa_2$ as well.

Now define $h' \in {}^\omega\omega$ as

$$h'(n) = \begin{cases} h(n) & n \in B; \\ g_0(n) & n \notin B. \end{cases}$$

We conclude that $h' <^* f_j$ for all $j < \kappa_2$. This implies that there exists a $i < \kappa_1$ where $h' \leqslant^* g_i$.

By our construction of h', it is equal to h for all $n \in B \in G$, so $h'/G = h/G$. With $h' \leqslant^* g_i$ we conclude that $h/G = h'/G \leqslant g_i/G$. So there isn't an element of I that is below the sequence $\langle f_j/G : j < \kappa_2 \rangle$ and above the sequence $\langle g_i/G : i < \kappa_1 \rangle$ at the same time, in other words $(\langle g_i/G : i < \kappa_1 \rangle, \langle f_j/G : j < \kappa_2 \rangle)$ is a cut. □

Theorem 4.2.10. $\mathfrak{p} = \mathfrak{t}$.

Proof. Suppose that $\mathfrak{p} < \mathfrak{t}$. Given that our forcing poset $\mathbf{Q}$ does not change the value of $\mathfrak{t}$ we have that $\mathfrak{p}^{\mathcal{V}} < \mathfrak{t}^{\mathcal{V}}$ implies $\mathfrak{p}^{\mathcal{V}[G]} < \mathfrak{t}^{\mathcal{V}[G]}$, as any new set added can only decrease the value of $\mathfrak{p}$.

Let $\mathbf{s}$ be the cofinality spectrum problem given by 4.2.5. By Lemma 4.2.9, we know that there exists in $\mathcal{V}[G]$ a $(\kappa_1, \mathfrak{p}^{\mathcal{V}[G]})$-cut, for some $\aleph_1 \leqslant \kappa_1 < \mathfrak{p}^{\mathcal{V}[G]}$, so $(\kappa_1, \mathfrak{p}^{\mathcal{V}[G]}) \in \mathcal{C}(\mathbf{s}, \mathfrak{t}^{\mathcal{V}[G]})$. By Lemma 4.2.6, we conclude that $(\kappa_1, \mathfrak{p}^{\mathcal{V}[G]}) \in \mathcal{C}(\mathbf{s}, \mathfrak{t}_{\mathbf{s}}^{\mathcal{V}[G]})$; however, by the main result about CSPs, Theorem 3.0.15, we know that, in $\mathcal{V}[G]$, the set $\mathcal{C}(\mathbf{s}, \mathfrak{t}_{\mathbf{s}}^{\mathcal{V}[G]})$ is empty, a contradiction. □

5 Cuts in Pseudo-finite Structures

5.1 Basic definitions and properties

Throughout this section, λ will be an infinite regular cardinal and D a non-principal ultrafilter over λ.

Definition 5.1.1. Denote by $\mathbb{L}(D)$ the class of all ultraproducts on λ modulo D of finite linear orders with cardinality not uniformly bounded, that is, ultraproducts $\prod_{i\in\lambda} L_i/D$ such that

$$\{i \in \lambda : |L_i| > n\} \in D$$

for all $n \in \mathbb{N}$. Whenever we abuse the notation by writing $L \in \mathbb{L}(D)$ it is implied that there is an order $\leqslant_L$ and $(L, \leqslant_L) \in \mathbb{L}(D)$.

Definition 5.1.2. Let $\mathbb{T}(D)$ be the class of ultraproducts $\prod_{i\in\lambda}(T_i, \subseteq)/D$, where for all $i \in \lambda$ the pair $(T_i, \subseteq)$ is a finite tree (cf. Definition 3.0.4) and the cardinality of the sets is not uniformly bounded. Whenever we abuse the notation by writing $T \in \mathbb{T}(D)$ it is implied that there is an order $\trianglelefteq$ and $(T, \trianglelefteq) \in \mathbb{T}(D)$.

The hypothesis of uniformly boundedness guarantees that elements of $\mathbb{L}(D)$ and $\mathbb{T}(D)$ are not finite. Making a connection with cofinality spectrum problems, this is, in a way, the same as removing all the trivial orders from $\mathrm{Or}(\mathbf{s})$.

By Łoś's theorem every L in $\mathbb{L}(D)$ is a discrete linear order with a minimum and a maximum. We will denote the minimum element of L by 0_L and the maximum element by d_L. For any element $x \in L$ different from d_L and $y \in L$ different from 0_L we will denote by $S(x)$, or $x + 1$, the successor of x and $S^{-1}(y)$, or $y - 1$, the predecessor of y.

We will use the same definition of cuts introduced in chapter 3, namely Definition 3.0.1.

We can now define similar objects as the ones used for CSPs.

$$\mathcal{C}(D) := \{(\kappa_1, \kappa_2) : \text{ there exists a } (\kappa_1, \kappa_2)\text{-cut in some linear order } L \in \mathbb{L}(D)\}.$$

The cardinal $\mathfrak{p}_D$ is the minimum of the set

$$\{\kappa : \text{ there exists } (\kappa_1, \kappa_2) \in \mathcal{C}(D) \text{ such that } \kappa_1 + \kappa_2 = \kappa\},$$

and the cardinal $\mathfrak{t}_D$ is the minimum of the set

$$\{\kappa \geqslant \aleph_0 : \kappa \text{ is regular and there is an increasing unbounded sequence}$$
$$\langle x_\alpha : \alpha \in \kappa\rangle \text{ in some } T \text{ of } \mathbb{T}(D)\}.$$

For a regular cardinal λ, the set of cuts in $\mathcal{C}(D)$ with size less than λ is denoted by

$$\mathcal{C}(D,\lambda) = \{(\kappa_1,\kappa_2) \in \mathcal{C}(D) : \kappa_1 + \kappa_2 < \lambda\}.$$

Definition 5.1.3. Let $\mathcal{N} = \prod_{i\in\lambda} \mathcal{M}_i/D$ be an ultraproduct. We say that a subset X of N is *internal* if there exists a sequence $\langle X_i : i \in \lambda\rangle$ such that the following holds for all $n \in N$ and $i \in \lambda$:

1. $X_i \subseteq M_i$;
2. $n \in X$ if and only if $\{i \in \lambda : n(i) \in X_i\} \in D$.

We say that a function $f : N^t \to N$ is *internal* if there exists a sequence $\langle f_i : i \in \lambda\rangle$ such that the following holds for all $\bar{n} \in N^t$ and $i \in \lambda$:

1. $f_i : N_i^t \to N_i$;
2. $f(\bar{n}) = y$ if and only if $\{i \in \lambda : f_i(\bar{n}(i)) = y_i\} \in D$.

We say that a tree $(T, \trianglelefteq)$ is *internal* if all of it's elements are internal functions.

By Łoś's Theorem every nonempty internal subset X of $L \in \mathbb{L}(D)$ has a minimum and a maximum.

Lemma 5.1.4. *Let $\mathcal{N} = \prod_{i\in\lambda} \mathcal{M}_i/D$ be an ultraproduct. The set of all internal subsets of N is closed under finite unions, finite intersections and complements. Moreover, every definable subset of N is internal.*

Proof. Let A, B be two internal subsets of N, where A_i and B_i are the projections on M_i for all $i \in \lambda$. Then, for $x \in N$, we have that

$$\begin{aligned} x \in N \setminus A &\Leftrightarrow x \notin A \\ &\Leftrightarrow \{i \in \lambda : x[i] \in A_i\} \notin D \\ &\Leftrightarrow \lambda \setminus \{i \in \lambda : x[i] \in A_i\} \in D \\ &\Leftrightarrow \{i \in \lambda : x[i] \notin A_i\} \in D, \end{aligned}$$

where the third equivalence comes from the fact that D is an ultrafilter,

$$\begin{aligned} x \in A \cup B &\Leftrightarrow x \in A \vee x \in B \\ &\Leftrightarrow \{i \in \lambda : x[i] \in A_i\} \in D \vee \{i \in \lambda : x[i] \in B_i\} \in D \\ &\Leftrightarrow \{i \in \lambda : x[i] \in A_i\} \cup \{i \in \lambda : x[i] \in B_i\} \in D \\ &\Leftrightarrow \{i \in \lambda : x[i] \in A_i \cup B_i\} \in D, \end{aligned}$$

where in the third equivalence, the forward part is given by the fact the filters are upwards closed, and the backwards part comes from the fact that D is an ultrafilter, because suppose by contradiction that $\{i \in \lambda : x[i] \in A_i\} \notin D$ and $\{i \in \lambda : x[i] \in B_i\} \notin D$, then $\{i \in \lambda : x[i] \notin A_i\} \in D$ and $\{i \in \lambda : x[i] \notin B_i\} \in D$, implying that

$$\{i \in \lambda : x[i] \notin A_i\} \cap \{i \in \lambda : x[i] \notin B_i\} \in D,$$

so its complement, that is $\{i \in \lambda : x[i] \in A_i\} \cup \{i \in \lambda : x[i] \in B_i\}$, is not in D, given that D in non-principal. Continuing the proof in the case of intersections,

$$\begin{aligned} x \in A \cap B &\Leftrightarrow x \in A \wedge x \in B \\ &\Leftrightarrow \{i \in \lambda : x[i] \in A_i\} \in D \wedge \{i \in \lambda : x[i] \in B_i\} \in D \\ &\Leftrightarrow \{i \in \lambda : x[i] \in A_i\} \cap \{i \in \lambda : x[i] \in B_i\} \in D \\ &\Leftrightarrow \{i \in \lambda : x[i] \in A_i \cap B_i\} \in D, \end{aligned}$$

where in the third equivalence, the forward part is given by the fact that filters are closed under intersection, and the backwards part comes from the fact that the filters are upwards closed.

Lastly for the definable sets, let $\varphi(x; \bar{y})$ be a formula in the language of $\mathcal{N}$ and $\bar{c}$ a tuple of parameters in N. Let A be the subset of N defined by $\varphi(x; \bar{c})$, that is $\{a \in N : \mathcal{N} \models \varphi(a, \bar{c})\}$, and A_i the subset of M_i defined by $\varphi(x; \bar{c}[i])$, that is $\{a \in M_i : \mathcal{M}_i \models \varphi(a, \bar{c}[i])\}$, for all $i \in \lambda$. By Łoś's Theorem we have that

$$\mathcal{N} \models \varphi(x, \bar{c}) \Leftrightarrow \{i \in \lambda : \mathcal{M}_i \models \varphi(x[i], \bar{c}[i])\} \in D,$$

concluding that

$$x \in A \Leftrightarrow \{i \in \lambda : x[i] \in A_i\} \in D.$$

□

Now we can say that every definable subset of L has a minimum and a maximum, given that every internal subset has this property. This is exactly what the pseudo-finite property inside of the definition of cofinality spectrum problem is trying to emulate.

Before continuing there is an important remark that needs to be made.

Remark 5.1.5. In Chapter 3 when working with functions f in $\mathcal{M}^+$, we were able to, when going to the ultraproduct $\mathcal{N}^+$, talk about how the element f^λ/D is internally seen as a function by $\mathcal{N}^+$, given that $\mathcal{M}^+$ was constructed to be powerful enough to talk about functions and trees. What happened was that we had a sentence in $\mathcal{M}^+$ that stated that f is a function, and by Łoś's Theorem this property was carried to f^λ/D in $\mathcal{N}^+$. Given this, we were able to use f^λ/D as a function, like $f^\lambda/D(x) = y$ and $\mathrm{dom}(f^\lambda/D) = A$ (this because $\mathcal{N}^+$ was also rich enough to define dom), without a problem. Now that we are looking at the ultraproduct of functions from an external perspective, we need to look at it for what it really is, an equivalence class of λ-tuples of functions. Of course we do not want to lose the intuition of looking at then as functions in the ultraproduct, but we need to make sure we are formally correct first. Lets define the auxiliary functions necessary to use the ultraproduct as a function.

Let $(T, \trianglelefteq) \in \mathbb{T}(D)$ with $(T, \trianglelefteq) = \prod_{i\in\lambda}(T_i, \subseteq)/D$, and $T_i \subseteq {}^{<X_i}Y_i$. Let $f_D \in T$ and f be a representative of f_D. For each $i \in \lambda$, the element $f[i]$ is in T_i and is a function. We define $\mathrm{dom}(f)$ as the set of all $x = \prod_{i\in\lambda} x_i$ such that $\{i \in \lambda : x_i \in \mathrm{dom}(f[i])\} \in D$. For each $i \in \lambda$ fix an element y_j of Y_j. For each $x \in \mathrm{dom}(f)$ we define $\mathrm{val}(f, x)$ as the λ-tuple with $f[i](x[i])$ when $x[i]$ is in the domain of $f[i]$ and with y_i when it is not, that is,

$$\mathrm{val}(f, x) := \{(i, f[i](x[i])) : i \in A\} \cup \{(j, y_j) : j \in \lambda \setminus A\}$$

where $A = \{i \in \lambda : x[i] \in \mathrm{dom}(f[i])\}$, and note that $\mathrm{val}(f, x) \in Y := \prod_{i\in\lambda} Y_i$. We will denote $\mathrm{val}(f, x)$ as $f(x)$. Define $\mathrm{dom}(f_D) := \mathrm{dom}(f)_D = \{x_D : x \in \mathrm{dom}(f)\}$ and for any $x_D \in \mathrm{dom}(f_D)$ define $\mathrm{val}(f_D, x_D) := f(x)_D$ where x is any representative of x_D. It is easy to see that these definitions do not depend on the choice of representative f for f_D or x for x_D. We will denote $\mathrm{val}(f_D, x_D)$ as $f_D(x_D)$. Note that $\mathrm{dom}(f_D)$ is an initial segment of $X = \prod_{i\in\lambda} X_i$, because let $m_i := \max(\mathrm{dom}(f[i]))$ for each $i \in \lambda$, each X_i is finite by definition, so m_i is well defined, and let $m_D = \prod_{i\in\lambda} m_i/D$. On one hand, for $x_D \in \mathrm{dom}(f_D)$ the set $\{i \in \lambda : x[i] \leqslant m_i\}$ has to be in the filter D, because otherwise $x \notin \mathrm{dom}(f)$, at the other hand, for $x_D \in \{a_D : a_D \leqslant m_D\}$ we have that the set $\{i \in \lambda : x[i] \leqslant m_i\}$ is in the filter D, and it is the same as $\{i \in \lambda : x[i] \in \mathrm{dom}(f[i])\}$, so $x \in \mathrm{dom}(f)$, concluding that $x_D \in \mathrm{dom}(f_D)$. What we got from this little detour is that we can indeed work with the elements of $(T, \trianglelefteq) \in \mathbb{T}(D)$ as though they are function from initial segments of X/D to Y/D. With this we can also see $(T, \trianglelefteq)$ as a tree composed of the internal functions in ${}^{<X/D}Y/D$, we denote it by $I({}^{<X/D}Y/D)$.

Let $(L, \leqslant_L) \in \mathbb{L}(D)$, with $L = \prod_{i\in\lambda}(L_i, \leqslant_i)/D$. Let $(T, \trianglelefteq) \in \mathbb{T}(D)$ be such that $T \subseteq {}^{<L}X$ for some set X. We say that $(T, \trianglelefteq)$ is a *tree indexed by* L. We also define the function lg from T to L where $\mathrm{lg}(x) = \min(L \setminus \mathrm{dom}(x))$ which is well defined because we can use the lemma that we just proved together with the fact that an initial segment of L, in this case $\mathrm{dom}(x)$, is a $(L, \leqslant_L)$-definable subset.

Let $(L, \leqslant_L) \in \mathbb{L}$ and $(T, \trianglelefteq) \in \mathbb{T}(D)$ indexed by L. We say that $c \in T$ is *below the ceiling* if $S^k(\mathrm{lg}(c)) <_L d_L$ for all $k < \omega$. **An element having this property will allow us to construct desired successor elements freely until we hit a limit element.**

For $(L, \leqslant_L) \in \mathbb{L}$, define $(T_L, \trianglelefteq_L) \in \mathbb{T}(D)$ as the set ${}^{<L}L$.

Lemma 5.1.6. *Let $L \in \mathbb{L}(D)$ and $T \in \mathbb{T}(D)$ a tree indexed by L. For $\kappa <$min$\{\mathfrak{p}_D, \mathfrak{t}_D\}$, let $T^* \subseteq T$ be a definable subtree and $\bar{c} = \langle c_\alpha : \alpha < \kappa\rangle$ a strictly $\trianglelefteq$-increasing sequence of* *elements of* T^*. *Then there exists* $c_{**} \in T^*$ *such that* $\alpha < \kappa$ *implies* $c_\alpha \trianglelefteq c_{**}$ *and* c_{**} *is below the ceiling.*

Proof. Since $\kappa < \mathfrak{t}_D$ there is an element $c \in T$ such that for all α we have that $c_\alpha \trianglelefteq c$. The set $\{\lg(c') : c' \trianglelefteq c \text{ and } c' \in T^*\}$ is nonempty and definable, hence contains a last element $a_* \in L$, let $c_* = c \restriction a_*$, if c_* is below the ceiling it's done, otherwise the pair

$$(\{\lg(c_\alpha) : \alpha < \kappa\}, \{S^{-k}(\lg(c_*)) : k < \omega\})$$

is a pre-cut, given that $\kappa < \mathfrak{p}_D$. Choose $a_{**} \in L$ realizing the pre-cut and $c_{**} = c_* \restriction a_{**}$. □

5.2 The function lcf is well defined below $\min\{\mathfrak{p}_D^+, \mathfrak{t}_D\}$

In this section we will show that for any regular cardinal κ below $\min\{\mathfrak{p}_D^+, \mathfrak{t}_D\}$ there exists a unique regular cardinal θ such that $(\kappa, \theta) \in \mathcal{C}(D)$. We will say that this cardinal is the *lower cofinality* of κ. With this we will be able to define the function $\mathrm{lcf}(\kappa, D)$ that goes from $\lambda \restriction_{\min\{\mathfrak{p}_D^+, \mathfrak{t}_D\}}$ to λ where $\mathrm{lcf}(\kappa, D) = \theta$.

Lemma 5.2.1. *Let* D *be a regular ultrafilter. If* $L \in \mathbb{L}(D)$, *then for each infinite regular* $\kappa \leqslant \mathfrak{p}_D$:

1. *there is a strictly increasing* κ*-indexed sequence* $\bar{a} = \langle a_\alpha : \alpha < \kappa \rangle$ *of elements of* L *such that*
$$(\{a_\alpha : \alpha < \kappa\}, \{S^{-k}(d_L) : k < \omega\})$$
represents a pre-cut in L;

2. *there is a strictly decreasing* κ*-indexed sequence* $\bar{a} = \langle a_\alpha : \alpha < \kappa \rangle$ *of elements of* L *such that*
$$(\{S^k(0_L) : k < \omega\}, \{a_\alpha : \alpha < \kappa\})$$
represents a pre-cut in L.

Proof. By induction. The fact that $\alpha < \kappa \leqslant \mathfrak{p}_D$ will guarantee the existence of the limit α element, by picking any realization of the pre-cut $(\{a_\beta : \beta < \alpha\}, \{S^{-k}(d_L) : k < \omega\})$, because otherwise we would have a $(\mathrm{cf}(\alpha), \aleph_0)$-cut. □

Theorem 5.2.2 (Existence)**.** *Let* D *be a regular ultrafilter and* $L \in \mathbb{L}(D)$. *For each infinite regular* $\kappa \leqslant \mathfrak{p}_D$, *there exists infinite regular cardinals* θ *and* θ' *such that* $(\kappa, \theta) \in \mathcal{C}(D)$ *and* $(\theta', \kappa) \in \mathcal{C}(D)$, *witnessed by a* (κ, θ)*-cut in* L *and a* (θ', κ)*-cut in* L *respectively.*

Proof. Pick a sequence given by lemma 5.2.1 and let $B = \{b \in L : (\forall \alpha < \kappa) a_\alpha <_L b\}$. Let θ be the coinitiality of B. θ is not finite because if it were it would be equal to 1, and there is no $b \in B$ such that

$$(\{a_\alpha : \alpha < \kappa\}, \{b\})$$

represents a cut, because $a_\alpha <_{\mathbf{a}} S^{-1}(b) <_L b$. For θ' its the same but reversed and with cofinality. $\square$

Note that this theorem guarantees us that if a cut exists in $\mathcal{C}(D)$, where one of the sides has cofinality less than or equal to $\mathfrak{p}_D$, it is witnessed in all orders $L \in \mathbb{L}(D)$.

Theorem 5.2.3 (Uniqueness)**.** *Let D be a regular ultrafilter and $L \in \mathbb{L}(D)$. For each regular $\kappa < min\{\mathfrak{p}_D^+, \mathfrak{t}_D\}$:*

1. *there is one and only one λ such that $(\kappa, \lambda) \in \mathcal{C}(D)$;*
2. *$(\kappa, \lambda) \in \mathcal{C}(D)$ if and only if $(\lambda, \kappa) \in \mathcal{C}(D)$.*

Proof. Let $(L_1, \leqslant_1), (L_2, \leqslant_2) \in \mathbb{L}$, with a (κ, θ)-cut in L_1 and a (θ', κ)-cut in L_2, represented by the following pairs

- $(\langle a^1_\alpha : \alpha \in \kappa\rangle, \langle b^1_\epsilon : \epsilon \in \theta\rangle)$;
- $(\langle b^2_\epsilon : \epsilon \in \theta'\rangle, \langle a^2_\alpha : \alpha \in \kappa\rangle)$.

So the a^1_α and a^2_α represent the κ part of the (κ, θ)-cut and the (θ', κ)-cut respectively, while the b^1_ϵ and b^2_ϵ represent the θ part of the (κ, θ)-cut and the θ' part of the (θ', κ)-cut respectively.

We want to show that $\theta = \theta'$. For this we will define the following elements: Let $(L_3, \leqslant_3)$ be any element of $\mathbb{L}(D)$ with cardinality greater than or equal to the cardinality of $L_1 \times L_2$. Let $(T, \trianglelefteq) \in \mathbb{T}(D)$ be such that $T = {}^{<L_3}L_1 \times L_2$. Let $T_* \subseteq T$ be the subtree consisting of all elements strictly increasing in the first coordinate and strictly decreasing in the second coordinate, that is, T^* is the subtree defined by the formula

$$(\forall n \in L_3)(\forall n' \in L_3) n' <_3 n <_3 \lg(x) \\ \rightarrow ((x(n')(0) <_1 x(n)(0)) \wedge (x(n')(1) >_2 x(n)(1))).$$

Similar to lemma 3.0.8, we'll construct sequences c_α and n_α by induction on $\alpha < \kappa$ such that:

1. $c_\alpha \in T_*$ and $n_\alpha \in L_3$;
2. $\beta < \alpha$ implies $c_\beta \trianglelefteq c_\alpha$;
3. c_α is below the ceiling;
4. $n_\alpha = S^{-1}(\lg(c_\alpha)) = \max(\mathrm{dom}(c_\alpha))$;
5. $c_\alpha(n_\alpha)(0) = a^1_\alpha$ and $c_\alpha(n_\alpha)(1) = a^2_\alpha$.

For the base case let $c_0 := \langle(a_0^1, a_0^2)\rangle$ and $n_0 = 0_3$.

When $\alpha = \beta + 1$, let $c_\alpha := c_\beta {}^\frown \langle(a_\alpha^1, a_\alpha^2)\rangle$ and $n_\alpha = S(n_\beta)$. Remember that since c_β is below the ceiling we can concatenate without problems as long as the new element is still in T^*.

When α is a limit ordinal, given that $\alpha < \kappa < \mathfrak{t}_D$ we can use lemma 5.1.6 to conclude that there is a $c_* \in T^*$ such that c_* is below the ceiling and $c_\beta \trianglelefteq c_*$ for all $\beta < \alpha$. Let $n_* = \max(\mathrm{dom}(c_*))$. By the definition of T^* we know that the element c_* retains the property of being increasing in the first coordinate and decreasing in the second, but it can the case that c_* increases "too much" in the first coordinate and surpasses a_α^1, with $c_*(n_*)(0) >_1 a_\alpha^1$, or it decreases "too much" in the second, with $c_*(n_*)(1) <_2 a_\alpha^2$. So we need to restrict our c_* to before a point where any of these cases happens. For this we use the following set

$$A = \{n \leqslant_3 n_* : c_*(n)(0) <_1 a_\alpha^1 \wedge a_\alpha^2 <_2 c_*(n)(1)\}.$$

The set A is nonempty, because n_β is in it for any $\beta < \alpha$, bounded, trivially by n_*, and definable with c_* as a parameter, so it has a maximum m_*. As $n_\beta \leqslant m_*$ for any $\beta < \alpha$, let $c_{**} := c_* \restriction_{S(m_*)}$. Now let $c_\alpha := c_{**} {}^\frown \langle(a_\alpha^1, a_\alpha^2)\rangle$ and $n_\alpha := S(m_*)$.

Now we have the sequences $\langle c_\alpha : \alpha \in \kappa\rangle$ and $\langle n_\alpha : \alpha \in \kappa\rangle$ with the properties stated before. Again given that $\kappa < \mathfrak{t}_D$ we can use lemma 5.1.6 to conclude that there is a $c_* \in T^*$ such that $c_\alpha \trianglelefteq c_*$ for all $\alpha < \kappa$. Let $n_* = \max(\mathrm{dom}(c_*))$. We will construct new sequences $\langle m_\epsilon^1 : \epsilon \in \theta\rangle$ and $\langle m_\epsilon^2 : \epsilon \in \theta'\rangle$ in L_3 such that each element m_ϵ^1 is the maximum index $n \leqslant_3 n_*$ where the element $c_*(n)(0)$ is still less than b_ϵ^1, and each m_ϵ^2 is the maximum index $n \leqslant_3 n_*$ where the element $c_*(n)(1)$ is still greater than b_ϵ^2, that is

$$m_\epsilon^1 = \max\{n \leqslant n_* : c_*(n)(0) <_1 b_\epsilon^1\};$$
$$m_\epsilon^2 = \max\{n \leqslant n_* : b_\epsilon^2 <_2 c_*(n)(1)\}.$$

Note that what we did is we essentially "transported" the sequence $\langle b_\epsilon^1 : \epsilon \in \theta\rangle$ in L_1 to the sequence $\langle m_\epsilon^1 : \epsilon \in \theta\rangle$ in L_3 by doing a "pullback" by $c_*(.)(0)$, the same can be said about b_ϵ^2 to m_ϵ^2 by $c_*(.)(1)$. At the same time, thinking in this way, the sequence $\langle n_\alpha : \alpha \in \kappa\rangle$ is the "pullback" of $\langle a_\alpha^1 : \alpha < \kappa\rangle$ by $c_*(.)(0)$, and it is also the "pullback" of $\langle a_\alpha^2 : \alpha < \kappa\rangle$ by $c_*(.)(1)$.

Now we have the sequences $\langle n_\alpha : \alpha \in \kappa\rangle$, $\langle m_\epsilon^1 : \epsilon \in \theta\rangle$ and $\langle m_\epsilon^2 : \epsilon \in \theta'\rangle$. Where both pairs

$$(\langle n_\alpha : \alpha \in \kappa\rangle, \langle m_\epsilon^1 : \epsilon \in \theta\rangle),$$
$$(\langle n_\alpha : \alpha \in \kappa\rangle, \langle m_\epsilon^2 : \epsilon \in \theta'\rangle)$$

are cuts in L_3, because if there were elements x and y in between, then the elements $c_*(x)(0)$ and $c_*(y)(1)$ would realize our original (κ, θ)-cut and (θ', κ)-cut, respectively. To conclude the proof note that, by the fact that the left side is the same in both cases, then there is a subset $M \subseteq L_3$ such that both sequences $\langle m^1_\epsilon : \epsilon \in \theta \rangle$ and $\langle m^2_\epsilon : \epsilon \in \theta' \rangle$ are coinitial in M, this together with the hypothesis that both θ and θ' are regular, we conclude that $\theta = \theta'$, as otherwise we would have a contradiction of the definition of being coinitial. □

Definition 5.2.4 (The lower cofinality of κ, $\mathrm{lcf}(\kappa, D)$)**.** Let D be a regular ultrafilter. For any regular $\kappa < \min\{\mathfrak{p}^+_{\mathbf{s}}, \mathfrak{t}_{\mathbf{s}}\}$, we define $\mathrm{lcf}(\kappa, D)$ to be the unique θ such that $(\kappa, \theta) \in \mathcal{C}(D)$.

To finish this section I compile the idea behind the proofs in this section and lemma 3.0.8 that uses the connection between treetops and cuts to construct desired elements by induction. I will try to flesh out the core elements of it. As we have seen it at least three times by now the reader may be familiar with the pattern already, these ideas will continue to be useful until the end of the chapter and may also be useful for future applications involving cuts and trees.

Remark 5.2.5 (Principle of the proofs (kinda))**.** Let κ be regular with $\kappa < \min(\{\mathfrak{p}^+_D, \mathfrak{t}_D\})$ and $L \in \mathbb{L}(D)$. Let $T \in \mathbb{T}(D)$ be a subset of ${}^{<L_2}L^*$ where L^* is the space with the desired elements for the construction, for example $L^* = L \times L$ as in 3.0.8, and L_2 is an element of $\mathbb{L}(D)$ with cardinality greater or equal to that of L^*. Construct two sequences, $\langle c_\alpha : \alpha < \kappa \rangle$ of elements of a definable subtree $T^* \subseteq T$, and $\langle n_\alpha : \alpha < \kappa \rangle$ of elements of L_2 by induction on $\alpha < \kappa$ where:

1. $n_\alpha = S^{-1}(\lg(c_\alpha))$, that is, n_α is the index of the last element of the branch c_α;
2. each node c_α extends the nodes before it;
3. c_α is below the ceiling (to guarantee that we can concatenate finitely new elements without problems);
4. the element $c_\alpha(n_\alpha)$ represents the properties that we want to prove for each α (kinda);
5. an extra inductive property depending on the problem that will generally be imposed by the subtree T^*.

For the successor case $\alpha = \beta + 1$, we simply concatenate c_β with b_α where b_α has the desired property. For α limit, since $\alpha < \mathfrak{t}_D$, by lemma 5.1.6 choose a c_* that is an upper bound and is below the ceiling. Let $n_* = S^{-1}(\lg(c_*))$, and restrict c_* to make sure it respects the induction. To select a restriction first we pick, for each property that we want, the greatest index $m \leqslant_{\mathbf{a}} n_*$ for which the property holds, then we want n_α to be less than or equal to all the ms and greater than all the n_β with $\beta < \alpha$. If the number of properties is finite, just pick the smallest m, as we have done until now. If the number of properties is infinite we can look at the pre-cut $(\langle n_\beta : \beta < \alpha \rangle, \langle M_j : j \in \mathrm{dcf}(\theta) \rangle)$, where $\langle M_j : j \in \mathrm{dcf}(\theta) \rangle$

is a decreasing sequence coinitial in the set $\{m_i : i \in \theta\}$. If the coinitiality of the set of properties is less than $\mathfrak{p}_D$, together with the fact that also $\alpha < \mathfrak{p}_D$, we can show that the pre-cut above defined is not a cut given that it has cofinality less than $\mathfrak{p}_D$ and then pick an element realizing it to be the n_α. This more general case will be used in the main theorem of this chapter, namely Theorem 5.4.5.

5.3 On symmetric cuts

We now prove, in a similar way to Lemma 3.0.8, that as long as $\kappa < \min\{\mathfrak{p}_D^+, \mathfrak{t}_D\}$ there are no (κ, κ)-cuts in $\mathcal{C}(D)$. Nonetheless, we will show that there is a $(\mathfrak{t}_D, \mathfrak{t}_D)$-cut, so $\mathfrak{t}_D$ is a dividing line in the existence of symmetric cuts.

Theorem 5.3.1. *Let D be an ultrafilter over λ. Let $\kappa < \min\{\mathfrak{p}_D^+, \mathfrak{t}_D\}$ be regular. Then there are no (κ, κ)-cuts in $\mathcal{C}(D)$.*

Proof. The proof of this theorem is the same as the one for Lemma 3.0.8 with just minor adjustments in notation.

Let $(L, \leqslant) \in \mathbb{L}(D)$ be a linear order that witnesses a (κ, κ)-cut with the pair $(\langle a_\alpha : \alpha < \kappa\rangle, \langle b_\beta : \beta < \kappa\rangle)$. Let $(L_2, \leqslant_2)$ be any element of $\mathbb{L}(D)$ with cardinality greater than or equal to the cardinality of $L \times L$. Let $(T, \trianglelefteq) \in \mathbb{T}(D)$ be such that $T = {}^{<L_2}L \times L$.

Let $T_* \subseteq T$ be the subtree consisting of all elements strictly increasing in the first coordinate, strictly decreasing in the second coordinate and such that the second coordinate is always greater then the first, that is, T^* is the subtree defined by the formula

$$\begin{aligned}(\forall n \in L_2)(\forall n' \in L_2) n' <_2 n <_2 \lg(x)\\ \to (x(n)(0) < x(n')(0) < x(n')(1) < x(n)(1))\end{aligned}$$

Now we construct by induction two κ-sequences, one of elements c_α of T_* and one of elements n_α of L_2 where:

1. for all $\alpha < \kappa$, we have that $n_\alpha = \max(\mathrm{dom}(c_\alpha))$;
2. for all $\beta < \alpha < \kappa$, we have that $c_\beta \trianglelefteq c_\alpha$;
3. for all $\alpha < \kappa$, we have that c_α is below the ceiling;
4. for all $\alpha < \kappa$, we have that $c_\alpha(n_\alpha)(0) = a_\alpha$ and $c_\alpha(n_\alpha)(1) = b_\alpha$.

For the base case, let $c_0 = \langle(a_0, b_0)\rangle$. When $\alpha = \beta + 1$, let $c_\alpha = c_\beta{}^\frown\langle(a_\alpha, b_\alpha)\rangle$ and $n_\alpha = n_\beta + 1$. When α is a limit ordinal, by Lemma 5.1.6 there is $c_* \in T_*$ such that $c_\beta \trianglelefteq c_*$ for all $\beta < \alpha$ and c_* is below the ceiling. Let $n_* = \max(\mathrm{dom}(c_*))$. By the definition of T_* and the fact that the order $\trianglelefteqslant$ is by initial segment we have that, for $\beta < \alpha$, $c_\beta(n_\beta, 0) = c_*(n_\beta, 0) < c_*(n_*, 0) < c_*(n_*, 1) < c_*(n_\beta, 1) = c_\beta(n_\beta, 1)$, that is, $a_\beta < c_*(n_*, 0) < c_*(n_*, 1) < b_\beta$ for all $\beta < \alpha$, but it may also be the case that $a_\alpha < c_*(n_*, 0) < c_*(n_*, 1) < b_\alpha$, so we need to restrict our element c_* to before a point where that happens. We can then concatenate it with the element $\langle(a_\alpha, b_\alpha)\rangle$ without failing the definition of T_*. For this we use the following set

$$A = \{n \leqslant n_* : c_*(n)(0) < a_\alpha \wedge b_\alpha < c_*(n)(1)\}.$$

The set A is nonempty, because n_β is in it for any $\beta < \alpha$, bounded, trivially by n_*, and definable with c_* as a parameter, so it has a maximum m_*. Necessarily $c_\beta \trianglelefteqslant c_* \restriction_{m_*}$ for each $\beta < \alpha$, because $n_\beta \in A$ for each $\beta < \alpha$, so $n_\beta \leqslant m_*$. Now we can concatenate without problems, so $c_\alpha := (c_* \restriction_{m_*})^\frown\langle(a_\alpha, b_\alpha)\rangle$ and $n_\alpha := m_*$.

Now we have the sequence $\bar{c} = \langle c_\alpha : \alpha < \kappa\rangle$ that is a non-decreasing path in a branch of T_*, where each c_α is an element with (a_α, b_α) in its node. Again by Lemma 5.1.6, there exists an element $c_* \in T_*$ with $c_\alpha \trianglelefteqslant c_*$ for all $\alpha < \kappa$. Let $n_* = \max(\mathrm{dom}(c_*))$. Then, for each $\alpha < \kappa$, by definition of T_* we have

$$a_\alpha = c_\alpha(n_\alpha)(0) = c_*(n_\alpha)(0) < c_*(n_*)(0) < c_*(n_*)(1) < c_*(n_\alpha)(1) = c_\alpha(n_\alpha)(1) = b_\alpha.$$

This implies that both $c_*(n_*)(0)$ and $c_*(n_*)(1)$ realize the cut $(\langle a_\alpha : \alpha < \kappa\rangle, \langle b_\beta : \beta < \kappa\rangle)$, contradiction the hypothesis that $(\langle a_\alpha : \alpha < \kappa\rangle, \langle b_\beta : \beta < \kappa\rangle)$ represents a cut. □

But it is important to remember that this theorem is only valid when $\kappa < \min\{\mathfrak{p}_D^+, \mathfrak{t}_D\}$. We can actually construct a symmetric cut of size $\mathfrak{t}_D$.

Lemma 5.3.2. *Let D be an ultrafilter over λ, then $(\mathfrak{t}_D, \mathfrak{t}_D) \in \mathcal{C}(D)$.*

Proof. Let $\kappa = \mathfrak{t}_D$. By the definition of $\mathfrak{t}_D$ there is some $(T, \trianglelefteqslant) = \prod_{i\in\lambda}(T_i, \trianglelefteqslant_i)/D \in \mathbb{T}(D)$, with $T_i \subseteq {}^{<L_i}Y_i$ which contains a $\trianglelefteqslant$-increasing sequence $\langle c_\alpha : \alpha \in \kappa\rangle$ with no upper bound. Let $Y = \prod_{i\in\lambda} Y_i/D$. For $c, d \in T_i$, for any $i \in \lambda$, define $\mathrm{cis}(c, d)$ as the maximal common initial segment between c and d, that is, $\mathrm{cis}(c, d) = b \in T_i$ where $b \trianglelefteqslant c$, $b \trianglelefteqslant d$ and there are $y_c \neq y_d \in Y_i$ such that $b^\frown y_c \trianglelefteqslant c$ and $b^\frown y_d \trianglelefteqslant d$.

Let X_i be the set $T_i \times \{0_{L_i}, 1_{L_i}\}$. Let $<_{X_i}$ be the linear order on X_i defined as follows:

1. For any $c \in T_i$, we have $(c, 0) <_{X_i} (c, 1)$;
2. If $c \trianglelefteqslant_i d$ and $c \neq d$, then $(c, 0) <_{X_i} (d, 0) <_{X_i} (d, 1) <_{X_i} (c, 1)$;
3. if c, d are $\trianglelefteqslant_i$-incompatible, then let $b \in T_i$ and $y_c, y_d \in Y_i$ be such that $b = \mathrm{cis}(c, d)$, $b^\frown y_c \trianglelefteqslant_i c$, and $b^\frown y_d \trianglelefteqslant_i d$. Given that by definition of $\mathrm{cis}(c, d)$ we have that $y_c \neq y_d$,

we define, for $s, t \in \{0,1\}$,

$$(c,s) <_{X_i} (d,t) \iff y_c <_{Y_i} y_d.$$

Let $(X, <_X) = \prod_{i\in\lambda}(X_i, <_{X_i})$. Given that $(T, \trianglelefteq) \in \mathbb{T}(D)$ we have that the set of sizes of T_i is not uniformly bounded, so the set of sizes of X_i is also not uniformly bounded, concluding that $X \in \mathbb{L}(D)$.

Now we can use the sequence $\langle c_\alpha : \alpha \in \kappa\rangle$ without an upper bound in T to construct the following cut in $(X, <_X)$:

$$(\langle (c_\alpha, 0) : \alpha \in \kappa\rangle, \langle (c_\alpha, 1) : \alpha \in \kappa\rangle).$$

This pair represents a (κ,κ)-cut, because suppose that there is a $d \in T$ such that (d, s) realizes it. If there were a $\beta \in \kappa$ where c_β is $\trianglelefteq$-incompatible with d, then $(d, s) <_X (c_\beta, 0)$ or $(c_\beta, 1) <_X (d, s)$ for some $s \in \{0, 1\}$, by definition of the order $<_X$, so $c_\alpha \trianglelefteq d$ for all $\alpha \in \kappa$, contradicting the hypothesis about the sequence $\langle c_\alpha : \alpha \in \kappa\rangle$. □

This lemma shows us that $\mathfrak{p}_D \leqslant \mathfrak{t}_D$.

5.4 On asymmetric cuts

In this section we will prove the main result relating trees and cuts, namely that $\mathfrak{t}_D \leqslant \mathfrak{p}_D$, equivalently $\mathcal{C}(\mathfrak{t}_D, D) = \emptyset$. For the proof we suppose that $\mathfrak{p}_D < \mathfrak{t}_D$, then pick a (κ,θ)-cut, represented by $(\langle \ell_\xi : \xi \in \kappa\rangle, \langle e_\eta : \eta \in \theta\rangle)$ in some $(X, \leqslant) \in \mathbb{L}(D)$ with $\theta < \kappa$ and $\kappa = \mathfrak{p}_D$. With this, the idea behind the proof is as follows. We use an internal tree, $(T, \trianglelefteq) \in \mathbb{T}(D)$, where its elements are pairs (f^1, f^2) such that $f^2(n)$ is an auxiliary "distance estimate" internal function that takes pairs of objects in X and returns a value in X, and has a lower bound of $f^1(n)$. We will construct a sequence $\langle c_\alpha : \alpha \in \kappa\rangle$ in this tree, as in 5.2.5, in a way that $f^2_\alpha(n_\alpha)$ will be an extension of all the ones beforehand with lower bound $f^1_\alpha(n_\alpha) = \ell_\alpha + 1$. The step of imposing the lower bound will guarantee us that any element c of the tree extending the elements c_α for all $\alpha < \kappa$ will have an associated "distance estimate" functions $f^2(m)$, where for any $m < \lg(c)$ with $m > n_\alpha$ for all $\alpha < \kappa$, it measure distances greater than ℓ_ξ for all $\xi \in \kappa$, so we will be able to pick an element e_γ of the right side of the cut where $e_\gamma < f^1(m)$, so it is a lower bound for $f^2(m)$. However, we will construct these function in a special way that will guarantees us that, if there is an element of the tree c that extends all of the elements in our sequence, then we can find a $m < \lg(c)$ and subset a A of X such that for any $\eta < \theta$ we can find two elements $y_1, y_2 \in A$ where $f^2(m)(y_1, y_2) < e_\eta$, but this will be a contradiction given that the lower bound is some e_γ, concluding that the sequence we constructed does not have an upper bound, contradicting the fact that $\theta < \kappa = \mathfrak{p}_D < \mathfrak{t}_D$.

Firstly we need two lemmas that will guarantee the existence of some needed internal sets.

Lemma 5.4.1. *Let $\{(X_i, \leqslant_i)\}_{i\in\lambda}$ be a family of linear orders and D a non-principal ultrafilter over λ. Define*

$$(X, \leqslant) := \prod_{i\in\lambda}(X_i, \leqslant_i)/D.$$

If there exists an infinite set $U \subseteq X$ and a family $\mathcal{Z}$ of internal sets of X such that $|U|, |\mathcal{Z}| < \min(\{\mathfrak{p}_D, \mathfrak{t}_D\})$ and $U \subseteq Z$ for all $Z \in \mathcal{Z}$. Then there is an internal set Y such that $U \subseteq Y \subseteq \bigcap \mathcal{Z}$.

Proof. With $\kappa = |\mathcal{Z}|$, let $\mathcal{Z} = (Z_\xi)_{\xi\in\kappa}$ be an enumeration of the family $\mathcal{Z}$. Let $(T_i, \subseteq)$ be the tree composed of the functions f with the following properties:

1. $\mathrm{dom}(f)$ is an initial segment of X_i;
2. $\mathrm{range}(f) \subseteq \mathcal{P}(X_i)$;
3. $f(y) \subseteq f(x)$, whenever $x \leqslant_i y$.

Let $(T, \trianglelefteq) := \prod_{i\in\lambda}(T_i, \subseteq)/D$. Given that $\kappa < \min(\{\mathfrak{p}_D, \mathfrak{t}_D\})$, we can, as in 5.2.5, construct two sequences, $(c_\alpha)_{\alpha\in\kappa}$ in T and $(n_\alpha)_{\alpha\in\kappa}$ in X, by induction on $\alpha < \kappa$ such that the following hold:

1. $n_\alpha = \max(\mathrm{dom}(c_\alpha))$;
2. if $\beta < \alpha$ then $c_\beta \trianglelefteq c_\alpha$;
3. c_α is below the ceiling;
4. $c_\alpha(n_\alpha) \subseteq Z_\alpha$;
5. $U \subseteq c_\alpha(n)$ for all $n \leqslant n_\alpha$.

For the base case let $n_0 = 0_X$ and $c_0 = (n_0, Z_0)$.

For the successor case $\alpha = \beta + 1$, let $c_\alpha = c_\beta{}^\frown(n_\beta + 1, Z_\beta \cap c_\beta(n_\beta))$, which is an element of T since the set Z_α is internal, and an element of T_* because $U \subseteq Z_\alpha$.

For the limit case, given that $\alpha < \mathfrak{t}_D$ we have that, by 5.1.6, there exists a $c_* \in T_*$ such that $c_\beta \trianglelefteq c_*$ for all $\beta < \alpha$ and c_* is below the ceiling. Let $n_* = \max(\mathrm{dom}(c_*))$. Now we prune the element c_* to ensure that it respects the induction. For each $u \in U$ let

$$m_u = \max(\{n \leqslant n_* : \forall z \leqslant n(u \in c_*(z))\}).$$

For each $u \in U$ we have that $n_\beta \leqslant m_u$ for all $\beta < \alpha$, given that they are in the above set. Let $\langle M_i : i \in \xi\rangle$ be a coinitial sequence in $\{m_u : u \in U\}$. Given that the number of properties $|U|$ is less than $\mathfrak{p}_D$, and α is also less than $\mathfrak{p}_D$, we conclude that the pair $(\langle n_\beta : \beta \in \alpha\rangle, \langle M_i : i \in \xi\rangle)$ is not a cut. Pick an element n_{**} that realizes this pre-cut. Let $n_\alpha = n_{**} + 1$ and $c_\alpha = c_* \restriction_{n_\alpha} {}^\frown (n_\alpha, Z_\alpha \cap c_*(n_{**}))$. Note that the set $c_*(n_{**})$ is internal because $c_* \in T$ and T is an internal tree.

Now with the sequences $(c_\alpha)_{\alpha\in\kappa}, (n_\alpha)_{\alpha\in\kappa}$ we can use the fact that $\kappa < \mathfrak{t}_D$ to apply the same method that we used in the limit case to construct c_κ and n_κ where $U \subseteq c_\kappa(z)$ for all $z \leqslant n_\kappa$. By property (3) of our tree T we have that $c_\kappa(n_\kappa) \subseteq c_\kappa(n_\alpha) = c_\alpha(n_\alpha) \subseteq Z_\alpha$ for all $\alpha \in \kappa$, concluding that the set $Y := c_\kappa(n_\kappa)$ has the desired property. □

Lemma 5.4.2. *Let $\{(X_i, \leqslant_i)\}_{i\in\lambda}$ be a family of finite linear orders, D be an ultrafilter over λ, $\kappa < \min(\{\mathfrak{p}_D, \mathfrak{t}_D\})$ and $N = \{n_\alpha\}_{\alpha\in\kappa}$ be a decreasing chain in*

$$(X, \leqslant) := \prod_{i\in\lambda}(X_i, \leqslant_i)/D.$$

For any $F : N^2 \to X$ there exists an internal function $H : X^2 \to X$ such that $F \subseteq H$.

Proof. Firstly we prove the one-dimensional case, and later use it to prove the two-dimensional one. Let $F : D \to X$. For each $i \in \lambda$, let $(T_i, \subseteq)$ be the tree composed of all the functions $f \in {}^{<(X_i,\geqslant_i)}X_i$, that is, the functions from a final segment of X_i (the initial segment in the reverse order) into X_i. Let $(T, \trianglelefteq) := \prod_{i\in\lambda}(T_i, \subseteq)/D$. We construct inductively an increasing sequence $\langle c_\alpha : \alpha < \kappa\rangle$ such that:

1. $\lg(c_\alpha) = n_\alpha - 1$ (remember that the domain is in the reverse order, so $\mathrm{dom}(c_\alpha) = \{x \in X : n_\alpha \leqslant x\}$);
2. for all $\beta \leqslant \alpha$ we have that $c_\alpha(n_\beta) = F(n_\beta)$.

Let c_0^* be any element of T with $\lg(c_0^*) = n_0$, define $c_0 := c_0^* {}^\frown \langle F(n_0)\rangle$. For the successor case $\alpha = \beta + 1$, let c_α be defined as follows:

$$c_\alpha(x) = \begin{cases} c_\beta(x), & \text{if } x \in \mathrm{dom}(c_\beta); \\ F(n_\alpha), & \text{if } n_\alpha \leqslant x < n_\beta. \end{cases}$$

For the limit case α, by Lemma 5.1.6 there is a $c_* \in T$ below the ceiling such that $c_\beta \trianglelefteq c_*$ for all $\beta < \alpha$. If $n_\alpha \in \mathrm{dom}(c_*)$ then let $c_{**} := c_* \restriction_{n_\alpha}$ and $c_\alpha := c_{**} {}^\frown \langle F(n_\alpha)\rangle$; otherwise, define c_α as follows:

$$c_\alpha(x) = \begin{cases} c_*(x), & \text{if } x \in \mathrm{dom}(c_*); \\ F(n_\alpha), & \text{if } n_\alpha \leqslant x \leqslant \lg(c_*). \end{cases}$$

This completes the induction.

Now that we constructed our sequence $\langle c_\alpha : \alpha < \kappa\rangle$ with $\kappa < \min(\{\mathfrak{p}_D, \mathfrak{t}_D\})$ we can use again Lemma 5.1.6 to pick a $c \in T$ such that $c_\alpha \trianglelefteq c$ for all $\alpha < \kappa$. By the definition of T, the function c is internal. Extend c on X if necessary.

Now, we use the one-dimensional case to construct the two-dimensional case. For each $i \in \lambda$, let P_i be the set of all function f such that $\mathrm{dom}(f) = \{x \in X_i : b \leqslant_i x\}^2$ for some $b \in X$ and $\mathrm{range}(f) \subseteq X_i$. Let $P := \prod_{i\in\lambda} P_i/D$. Let $(T_i, \subseteq)$ be the tree composed of all the functions $f \in {}^{<(X_i,\geqslant)}P_i$, that is, the functions from a final segment of X_i into P_i. Let $(T, \trianglelefteq) := \prod_{i\in\lambda}(T_i, \subseteq)/D$. Let $T^* \subseteq T$ be the subtree composed of elements $c \in T$ such that for all $y > \lg(c)$ we have $\mathrm{dom}(c(y)) = \{x \in X : y \leqslant x\}^2$. We construct inductively an $\trianglelefteq$-increasing sequence $\langle c_\alpha : \alpha < \kappa\rangle$ in T^* such that:

1. $\lg(c_\alpha) = n_\alpha - 1$;

2. for all $y \geqslant n_\alpha$ and $\beta, \gamma \leqslant \alpha$ with $n_\beta, n_\gamma \geqslant y$, we have that $c_\alpha(y)(n_\beta, n_\gamma) = F(n_\beta, n_\gamma)$.

Let c_0^* be any element of T^* with $\lg(c_0^*) = n_0$. The successor case and the base case c_0 will be built in similar ways, so lets look at the general successor case first. For the successor case $\alpha = \beta + 1$, let $F_1(x) := F(n_\alpha, x)$ and $F_2(x) := F(x, n_\alpha)$. We can use the one-dimensional case to construct internal functions g_1 and g_2 such that $g_1(n_\beta) = F_1(n_\beta)$ and $g_2(n_\beta) = F_2(n_\beta)$ for all $n_\beta < n_\alpha$. Let $f_\alpha : \{x \in X : n_\alpha \leqslant x\} \to X$ be an element of P defined as follows:

$$f_\alpha(x, y) = \begin{cases} c_\beta(n_\beta)(x, y), & \text{if } (x, y) \in \mathrm{dom}(c_\beta(n_\beta)); \\ g_1(y), & \text{if } x = n_\alpha; \\ g_2(x), & \text{otherwise.} \end{cases}$$

Let $c_\alpha(y) := c_\beta(y)$ if $n_\beta \leqslant y$, and $c_\alpha(y) := f_\alpha \restriction_{\{x\in X: y\leqslant x\}^2}$ if $n_\alpha \leqslant y < n_\beta$. For the base case we do almost the same with a minor correction. Using $\alpha = 0$, let f_0 be defined as follows:

$$f_0(x, y) = \begin{cases} c_0^*(n_0+1)(x, y), & \text{if } (x, y) \in \mathrm{dom}(c_0^*(n_0+1)); \\ g_1(y), & \text{if } x = n_0; \\ g_2(x), & \text{otherwise.} \end{cases}$$

Let $c_0(y) := c_0^*(y)$ if $n_0 < y$, and $c_\alpha(n_0) := f_0$.

For the limit case α, by Lemma 5.1.6 there is a $c_* \in T^*$ below the ceiling such that $c_\beta \trianglelefteq c_*$ for all $\beta < \alpha$. Let A be the set of $m > \lg(c_*)$ such that property (2) of the induction still holds, that it,

$$A := \{m > \lg(c_*) : \forall y \geqslant m[\forall \beta, \gamma \leqslant \alpha[n_\beta, n_\gamma \geqslant y \to c_*(y)(n_\beta, n_\gamma) = F(n_\beta, n_\gamma)]]\}.$$

We know that $n_\beta \in A$ for all $\beta < \alpha$, so A is nonempty, and A is bounded (given the order used) by $\lg(c_*)$. Given that A is definable, nonempty, and bounded, it has a maximum element m_* (in the inverse order, so a minimum element in the usual order) such that $m_* \leqslant n_\beta$ for all $\beta < \alpha$. If $m_* \leqslant n_\alpha$, let $c_{**} := c_* \restriction_{n_\alpha}$, otherwise let $c_{**} := c_* \restriction_{m_*}$. Let

$f_\alpha : \{x \in X : n_\alpha \leqslant x\} \to X$ be an element of P defined as follows:

$$f_\alpha(x,y) = \begin{cases} c_{**}(\lg(c_{**})+1)(x,y), & \text{if } (x,y) \in \operatorname{dom}(c_{**}(\lg(c_{**})+1)); \\ g_1(y), & \text{if } x = n_\alpha; \\ g_2(x), & \text{otherwise.} \end{cases}$$

Let $c_\alpha(y) := c_{**}(y)$ if $\lg(c_{**}) < y$, and $c_\alpha(y) := f_\alpha \restriction_{\{x \in X : y \leqslant x\}^2}$ if $n_\alpha \leqslant y \leqslant \lg(c_{**})$. This completes the induction.

Now that we constructed our sequence $\langle c_\alpha : \alpha < \kappa \rangle$ with $\kappa < \min(\{\mathfrak{p}_D, \mathfrak{t}_D\})$ we can use again Lemma 5.1.6 to pick a $c_* \in T$ such that $c_\alpha \trianglelefteq c_*$ for all $\alpha < \kappa$. Construct the set A again for c_* and pick the m_* maximum (minimum in the usual order) of it. The function $H := c_*(m_*)$ is an internal extension of F. Extend H on X if necessary. □

Both of these lemmas together gives us the necessary tools to prove the following lemma that will enables us to, during the induction on $\alpha < \kappa$, present in the proof of Theorem 5.4.5, extend our "distance estimate" function to a new internal function that is able to compare the new y_α. Remember that $\kappa < \mathfrak{t}_D$ only guarantees us that there are no unbounded increasing sequences, of size κ, in any element of $\mathbb{T}(D)$, that is, on internal trees, that is why we need to make sure we are working with internal objects if we want to use the hypothesis $\kappa < \mathfrak{t}_D$ to our advantage.

Lemma 5.4.3. *Let D be an ultrafilter over λ, and $X = \prod_{i\in\lambda} X_i/D \in \mathbb{L}(D)$. Let P_i be the set of all partial functions with domain W^2, for some $W \subseteq X_i$, and codomain X_i. Let $P = \prod_{i\in\lambda} P_i/D$. Now let*

- *$U \subseteq X$ be an infinite subset that is well-ordered by the inverse order of X, that is every subset of U has a maximum element in the order $\leqslant$;*
- *$F : U^2 \to \{x \in X : x > w\}$ be a function;*
- *$\bar{\rho} \in P$ be such that $\bar{\rho}(u_1,u_2) = F(u_1,u_2)$ for all u_1, u_2 such that $(u_1,u_2) \in \operatorname{dom}(\bar{\rho})$ holds.*

Then there exists $\rho \in P$ such that:

1. *$\rho(u_1,u_2) = F(u_1,u_2)$ for all $u_1,u_2 \in U$;*
2. *$\rho(x,y) \geqslant w$ for all $(x,y) \in \operatorname{dom}(\rho)$;*
3. *if $(x,y) \in \operatorname{dom}(\rho) \cap \operatorname{dom}(\bar{\rho})$, then $\bar{\rho}(x,y) = \rho(x,y)$.*

Proof. By lemma 5.4.2 there exists an internal function $\rho_1 : X^2 \to X$ that extends F. For $u \in U$, let $Z_u := \{x \in X : \rho_1(x,u) > w\}$. We have that $U \subseteq Z_u$, by definition of $\bar{\rho}$ and F, and Z_u is an internal set, because it is definable by an internal function. More precisely, since the function ρ_1 is internal, we have that $\rho_1 = \prod_{i\in\lambda} \rho_1^i/D$, where $\rho_1^i \in P_i$ for each $i \in \lambda$. Let

$$Z_u^i = \{x_i \in X_i : \rho_1^i(x_i, u[i]) > w[i]\},$$

for each $i \in \lambda$, we obtain that Z_u is internal, since

$$x \in Z_u \Longleftrightarrow x[i] \in Z_u^i.$$

By lemma 5.4.1, there exists an internal set Y such that $U \subseteq Y \subseteq Z_u$ for all $u \in U$. Let

$$Y^* = Y \setminus \{y \in Y : \exists y' \in Y(\rho_1(y, y') < w)\}$$

and observe that $U \subseteq Y^*$, because $Y \subseteq Z_u$ for all $u \in U$, and $\rho_1(x, y) \geqslant w$ for any $x, y \in Y^*$. Then $\rho = \rho_1 \restriction_{(Y^*)^2}$ has all the desired properties. □

This next fact is essential for the proof, as it enables us to construct the "distance estimate" function in such a way to guarantee that for any upper bound in the tree of our sequence $\langle c_\alpha : \alpha \in \kappa\rangle$ we can find a subset A of X where the distance between them can be as small as the size of any element in the right side of the cut.

Fact 5.4.4 ([20], Fact 8.4)**.** If κ is a regular infinite cardinal, then there exists a symmetric function

$$g : \kappa^+ \times \kappa^+ \to \kappa$$

such that $g(W \times W)$ is a cofinal subset of κ for all W cofinal subset of κ^+.

Theorem 5.4.5. *If D is an ultrafilter over λ, then $\mathfrak{t}_D \leqslant \mathfrak{p}_D$ holds.*

Proof. Suppose that $(X, \leqslant) := \prod_{i\in\lambda}(X_i, \leqslant_i)/D$ is a linear order in $\mathbb{L}(D)$ such that there exists a (κ, θ)-cut, where $\theta \leqslant \kappa = \mathfrak{p}_D$. We know by 5.3.1 that it cannot be the case that $\theta = \kappa$, so $\theta < \kappa$.

Suppose that $\mathfrak{p}_D < \mathfrak{t}_D$, and the pair $(\langle \ell_\xi : \xi \in \kappa\rangle, \langle e_\eta : \eta \in \theta\rangle)$ witnesses a (κ, θ)-cut in X. Let P_i be the set of all partial functions with domain W^2, for some $W \subseteq X_i$, and range included in X_i. Let $(T_i, \subseteq)$ be the tree composed of the functions f with the following properties:

1. The domain of f is an initial segment of X_i;
2. the range of f is a subset of $X_i \times P_i$ and $f(z) = (f^1(z), f^2(z))$ ($f^1(z)$ is an element of X_i while $f^2(z)$ is a "distance estimate" function);
3. for any $z \in \mathrm{dom}(f)$ and $(a, b) \in \mathrm{dom}(f^2(z))$ we have that $f^2(z)(a, b) \geqslant_i f^1(z)$ ($f^1(z)$ is a lower bound for the distance measured by $f^2(z)$);

4. if $z \leqslant_i z' \leqslant_i x$ then $f^1(z) \leqslant_i f^1(z')$ (this lower bound increases as we "go up" in the tree);

5. if $z \leqslant_i z' \leqslant_i x$ and $\{a, b\}$ is a subset of $\mathrm{dom}(f^2(w))$ for any $z \leqslant_i w \leqslant_i z'$, then $f^2(z)(a, b) = f^2(w)(a, b) = f^2(z')(a, b)$ for all such $z \leqslant_i w \leqslant_i z'$ (the "distance estimate" between a and b does not change as long as both stay continuously in the domain).

Let $(T, \trianglelefteq) := \prod_{i\in\lambda}(T_i, \subseteq)/D$. By Fact 5.4.4, there exists a function $g_0 : {\theta^+}^2 \to \theta$ such that, if $A \subseteq \theta^+$ is cofinal in θ^+, then $g_0(A^2)$ is cofinal in θ. We can extend g_0 to a function $g : \kappa^2 \to \theta$ such that $g \restriction_{{\theta^+}^2} = g_0$ and outside of ${\theta^+}^2$ it assumes any value. For all $f \in T$ let $W_f(z)$ be the set such that $(W_f(z))^2 = \mathrm{dom}(f^2(z))$. Our objective is to show that given our assumption that $\mathfrak{p}_D < \mathfrak{t}_D$ we can construct an increasing sequence in T of size $\kappa = \mathfrak{p}_D$ that is unbounded, a contradiction.

For this, given that $\kappa = \mathfrak{p}_D < \min\{\mathfrak{p}_D^+, \mathfrak{t}_D\}$, we can, again as in 5.2.5, construct two sequences $(c_\alpha)_{\alpha\in\kappa}$ in T and $(n_\alpha)_{\alpha\in\kappa}$ in X, by induction on α such that the following hold:

1. $n_\alpha = \max(\mathrm{dom}(c_\alpha))$;

2. if $\beta < \alpha$ then $c_\beta \trianglelefteq c_\alpha$;

3. c_α is below the ceiling;

4. there exists $y_\beta \in W_{c_\beta}(n_\beta)$, such that if $\beta \in \alpha \in \kappa$, then:

   a) $y_\alpha < y_\beta$;

   b) $c_\alpha^2(n_\alpha)(y_\beta, y_\alpha) = e_{g(\alpha,\beta)}$;

   c) if $n_\beta \leqslant z \leqslant n_\alpha$ then $y_\beta \in W_{c_\alpha}(z)$.

5. $c_\alpha^1(n_\alpha) = \ell_\alpha + 1$.

First lets show that if such a sequence $(c_\alpha)_{\alpha\in\kappa}$ exists then it is indeed unbounded. Suppose there exists a c where $c_\alpha \trianglelefteq c$ for all $\alpha \in \kappa$. For each $\eta \in \theta^+$, let m_η be the maximum of the set

$$A_\eta = \{m \in \mathrm{dom}(c) : \forall m'(n_\eta \leqslant m' \leqslant m \to y_\eta \in W_c(m'))\},$$

that is, the greatest index m of c where the element y_η is in the domain continuously from n_η to m. By property (4) of the induction we know that for each $\alpha \in \kappa$ with $\eta \leqslant \alpha$, the element y_η is in $W_{c_\alpha}(z)$ for all $n_\eta \leqslant z \leqslant n_\alpha$. By the definition of $(T, \trianglelefteq)$, having $c_\alpha \trianglelefteq c$ implies that $W_c(n_\alpha) = W_{c_\alpha}(n_\alpha)$. Both of the facts stated before gives us that, for each $\eta \in \theta^+$ and each $\alpha \in \kappa$, we have that $n_\alpha \leqslant m_\eta$.

By property (4) of the tree T together with property (5) of the induction we know that, for each $\eta \in \theta^+$, we have $c^1(m_\eta) \geqslant c^1(n_\alpha) = c^1_\alpha(n_\alpha) > \ell_\alpha$ for all $\alpha \in \kappa$. So we can uniformly select, using the axiom of choice, for each $\eta \in \theta^+$ an element $F(\eta) \in \theta$ such that $e_{F(\eta)} < c^1(m_\eta)$, because $(\langle \ell_\xi : \xi \in \kappa\rangle, \langle e_\eta : \eta \in \theta\rangle)$ witnesses a cut. Using this new function F we can represent θ^+ as

$$\theta^+ = \bigcup\{F^{-1}(\alpha) : \alpha \in \theta\},$$

and since θ^+ is a regular cardinal, this implies that there exists $\gamma \in \theta$ such that $A := F^{-1}(\gamma)$ is cofinal in θ^+. By the property of g, we have that $g(A^2)$ is cofinal in θ, so for any $\eta \in \theta$ there are $\zeta, \xi \in A$ such that $g(\zeta, \xi) > \eta$. In particular there are $\zeta, \xi \in A$ such that $g(\zeta, \xi) > \gamma$. Note that this is a problem given how we defined the "distance estimate" of the elements y_α, because $\gamma < g(\zeta, \xi)$ implies that $e_{g(\zeta,\xi)} < e_\gamma$, breaking the lower bound stipulated by c^1. More clearly, let $m^* = \min\{m_\zeta, m_\xi\}$, we have that $\{y_\zeta, y_\xi\} \subseteq D_c(m^*)$. Let $\mu = \max\{\zeta, \xi\}$, then we have that $\{y_\zeta, y_\xi\} \subseteq D_c(z)$ for all $n_\mu \leqslant z \leqslant m^*$. Hence we have

$$c^2(m^*)(y_\zeta, y_\xi) = c^2(n_\mu)(y_\zeta, y_\xi) = e_{g(\zeta,\xi)} < e_\gamma,$$

and at the same time

$$c^2(m^*)(y_\zeta, y_\xi) \geqslant c^1(m^*) > e_{F(\mu)} = e_\gamma,$$

a contradiction.

So, now that we know that the existence of such a sequence is indeed a problem, lets construct it using $\mathfrak{p}_D < \mathfrak{t}_D$. To construct the sequences $(c_\alpha)_{\alpha\in\kappa}, (n_\alpha)_{\alpha\in\kappa}$ by induction we do as follows:

For the successor case $\alpha = \beta + 1$: First choose $y_\alpha = y_\beta - 1$. Now we will use lemma 5.4.3 to extend our "distance estimate" function to be able to compare this new y_α with the old ones. To do this we define the objects to be used in the lemma as follows: Let $U := \{y_\gamma\}_{\gamma\in\alpha+1}$, let $F(y_\gamma, y_\delta) := e_{g(\gamma,\delta)}$, let $w = \ell_\alpha$ and let $\bar{\rho}(x, y) := c^2_\beta(n_\beta)(x, y)$. The lemma gives us a function ρ such that:

1. $\rho(y_\gamma, y_\delta) = e_{g(\gamma,\delta)}$, if $\gamma \leqslant \delta \leqslant \alpha$;
2. $\rho(x, y) \geqslant \ell_\alpha + 1$ for all $x, y \in \mathrm{dom}(\rho)$;
3. if $x, y \in \mathrm{dom}(\rho) \cap \mathrm{dom}(c^2_\beta)$, then $\rho(x, y) = c^2_\beta(x, y)$.

Then let $n_\alpha = n_\beta + 1$ and $c_\alpha = c_\beta{}^\frown(n_\alpha, (\ell_\alpha + 1, \rho))$.

For the limit case α: given that $\alpha < \kappa = \mathfrak{p}_D < \mathfrak{t}_D$ we can use lemma 5.1.6 to find a $c_* \in T$ such that $c_\beta \trianglelefteq c_*$ for all $\beta \in \alpha$ and c_* is below the ceiling. For each $\beta \in \alpha$, let m_β be the maximum of the set

$$A_\beta = \{m \in \mathrm{dom}(c) : \forall m'(n_\beta \leqslant m' \leqslant m \to y_\beta \in W_c(m'))\},$$

that is, the greatest index m of c where the element y_β is in the domain continuously from n_η to m. By the same arguments that we used when supposing that the sequence already exists, we conclude that for each $\beta \in \alpha$, we have that $n_\gamma < m_\beta$ for all $\gamma \in \alpha$. Let $\langle M_\beta : \beta \in \xi \rangle$ be a decreasing coinitial sequence in $\{m_\beta : \beta \in \alpha\}$. Now the difference here is that $\alpha < \mathfrak{p}_D$, as opposed to being equal to $\mathfrak{p}_D$, so we can guarantee that the pair $(\langle n_\beta : \beta \in \alpha\rangle, \langle M_\beta : \beta \in \xi\rangle)$ is not a cut, so there exists an element n_{**} in between both sequences. Let $n_\alpha = n_{**} + 1$ and $c_{**} = c_* \restriction_{n_\alpha}$. Choose a y_α that is below y_β for all $\beta \in \alpha$ and is not finite. Such a y_α exists because $(\langle S^\kappa(0_X) : \kappa \in \omega\rangle, \langle y_\beta : \beta \in \alpha\rangle)$ represents a pre-cut but cannot be a cut, given that $\alpha < \kappa < \mathfrak{p}_D$. Now we use lemma 5.4.3 again to extend our "distance estimate" function. Let $U := \{y_\beta\}_{\beta \in \alpha+1}$, let $F(y_\gamma, y_\delta) := e_{g(\gamma,\delta)}$ and let $\bar{\rho}(x, y) := {c_{**}}^2(n_{**})(x, y)$. The lemma gives us a suited ρ the same way as for the successor case. We can then let $c_\alpha = c_{**}{}^\frown(\ell_\alpha + 1, \rho)$. This concludes the induction, that together with the fact that the constructed sequence does not have an upper bound while having size less than $\mathfrak{t}_D$ gives us a contradiction, concluding that $\mathfrak{t}_D \leqslant \mathfrak{p}_D$. $\square$

The conclusion that $\mathfrak{t}_D \leqslant \mathfrak{p}_D$ is equivalent to $\mathcal{C}(D, \mathfrak{t}_D) = \emptyset$. Connecting with cofinality spectrum problems, this is the special case of the main result of [20] as stated in 3.0.15, namely that $\mathcal{C}(\mathbf{s}, \mathfrak{t}_\mathbf{s}) = \emptyset$.

5.5 $\mathfrak{p}_D = \mathfrak{t}_D$

Theorem 5.5.1. *Let D be a non-principal ultrafilter over λ, then $\mathfrak{p}_D = \mathfrak{t}_D$.*

Proof. Theorem 5.4.5 together with lemma 5.3.2 gives us that $\mathfrak{p}_D = \mathfrak{t}_D$. $\square$

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