On the Growth Rate of Periodic Orbits for Flows

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Orientador: Alexander Eduardo Arbieto Mendoza

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On the Growth Rate of Periodic Orbits

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Resumo

O objetivo deste trabalho é estudar a relação entre a taxa de crescimento dos pontos periódicos e a entropia topológica. Em 1970, Bowen [Bow70] provou que para sistemas Axioma A, a taxa de crescimento dos pontos periódicos coincide com a entropia topológica. Dez anos depois, Katok [Kat80] mostrou que se f é um difeomorfismo de classe $C^{1+\alpha}$ $(\alpha > 0)$ numa variedade compacta, então para qualquer medida hiperbólica f-invariante a taxa de crescimento dos pontos periódicos é maior ou igual que a entropia métrica. Em particular, se f é um difeomorfismo de classe $C^{1+\alpha}$ em uma superfície, então a taxa de crescimento dos pontos periódicos é maior ou igual que a entropia topológica. O teorema que apresentaremos nessa dissertação é uma extensão desse resultado de Katok para o caso de campos de vetores C^1 genéricos.

On the Growth Rate of Periodic Points

Walter Britto Peçanha Alves

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Abstract

Our goal in this thesis is to study the relationship between the growth rate of the periodic orbits and the topological entropy. In the early 70's, Bowen [Bow70] proved that for Axiom A systems, the growth rate of the periodic orbits equals to the topological entropy. A decade later, Katok [Kat80] showed that if f is a $C^{1+\alpha}$ $(\alpha > 0)$ diffeomorphism on a compact manifold, then for any f-invariant hyperbolic measure the growth rate of periodic points is larger than or equal to its metric entropy. In particular, if f is a $C^{1+\alpha}$ surface diffeomorphism, then the growth rate of periodic points is larger than or equal to its topological entropy. The theorem presented here extends Katok's result for C^1 generic vector fields of any dimension.

Contents

Chapter 1

Introduction

1.1 History and basic concepts

1.1.1 History

What is a dynamical system?

A dynamical system is all about the evolution of "something" over time. To create a dynamical system we simply need to decide what is the "something" that will evolve over time and what is the rule that specifies how that "something" evolve with time. In this way, a dynamical system is simply a model describing the temporal evolution of a system. As examples of dynamical systems, we have the mathematical models that describe the swinging of a clock pendulum, the flow of water in a pipe, and population growth.

A little bit of history

The concept of a dynamical system has its origins in Newtonian Mechanics, but a lot of what is considered dynamical systems today was developed by the French mathematician Henri Poincaré. Poincaré published two classic monographs, one called "New Methods of Celestial Mechanics"(1892-1899) and the other called "Lectures on Celestial Mechanics"(1905-1910). In them, he successfully applied the results of his research to the problem of the motion of three bodies and studied in detail the behaviour of the solutions(frequency, stability, asymptotic, and so on). These papers included the so called Poincaré Recurrence Theorem, which states that certain system will, after a sufficiently long but finite time, return to a state very close of the initial state.

Another very important person to the development of dynamical systems is the Russian mathematician Aleksandr Lyapunov, he developed many important approximations methods. His methods, which he developed in 1899, make it possible to define the stability of sets of ordinary differential equations. He created the modern theory of the stability of a dynamical system.

In 1913, the American mathematician George David Birkhoff prove the Poincaré's "Last Geometric Problem", the special case of the three-body problem, a result that made him world famous. In 1927, he published his *Dynamical Systems*. Birkhoff's most acclaimed result has been his 1931 discovery of what is now called the ergodic theorem. Combining insights from physics on the ergodic hypothesis with measure theory, this theorem solved, at least in principle, a fundamental problem of statistical mechanics. The ergodic theorem is also of great importance in dynamics.

The American mathematician Stephen Smale also made significant advances, one of his most famous contributions is the Smale horseshoe that jumpstarted important researches in dynamical systems. He also outlined a research program carried out by many others.

1.1.2 Basic concepts

Let X be a compact metric space and $f : X \to X$ a homeomorphsm. This generates a family of homeomorphisms, called iterates of f, written as

$$
\begin{aligned}
f^n &= f \circ f \cdots \circ f \\
f^0 &= id \\
f^{-n} &= (f^n)^{-1}
\end{aligned}
$$

For any $x \in X$, the set $\{f^n(x), x \in \mathbb{Z}\}$ is called the orbit of x under f, denoted by $\mathscr{O}_f(x)$, or simply by $\mathscr{O}(x)$. Any two orbits are either identical or else disjoint. A point $x \in X$ is called periodic if there is $n \geq 1$ such that $f^n(x) = x$. The minimal positive integer that satisfies this equality is called the period of x. The orbit of a periodic point is called a periodic orbit. Periodic points of period 1 are just fixed points.

A subset $\Lambda \subset X$ is called invariant under f if $f(\Lambda) = \Lambda$.

Given $x \in X$, the positive orbit $x, f(x), f^2(x), \cdots$ generally do not converge. Nevertheless many subsequences of it do. A point y is called an ω-limit of x if there exists a subsequence $n_i \to +\infty$ such that $f^{n_i}(x) \to y$. The set of ω-limit points of x is called the ω-limit set of x, denoted by $\omega(x)$. Reversing time defines the α-limit set of x. A simply proof shows that the ω- limit is a nonempty, compact, and invariant set.

A point $x \in X$ is called positively recurrent if $x \in \omega(x)$. In other words, x is positively recurrent if its positive orbit accumulates on x itself. Analogously we define negatively recurrent. Positively or negatively recurrent points are both called recurrent.

1.2 Topological entropy

In this section, we define topological entropy as a non negative real number representing the asymptotic average of the exponential growth of the number of distinguishable orbit segments. This concept will be studied more carefully in chapter 4.

Consider a homeomorphism $f : X \to X$ of a compact metric space.

Let d be a metric on X. It induces a family of metrics $\{d_n\}$ on X given by

$$d_n(x, y) := \max_{0 \le i \le n-1} d(f^i(x), f^i(y))$$

where each metric d_n measures the distance between the orbit arcs $\{x, \cdots, f^{n-1}(x)\}$ and $\{y, \cdots, f^{n-1}(y)\}$.

Definition 1.2.1 *Let $n \in \mathbb{N}$, and $\varepsilon > 0$. A subset $E \subset X$ is said to be (n, ε)-separated with respect to f if $x, y \in E$, $x \neq y$, implies $d_n(x, y) = \max_{0 \le i \le n-1} d(f^i(x), f^i(y)) > \varepsilon$.*

We denote by $S_n(\varepsilon)$ the cardinality of the (n, ε)-separated set with respect to f of maximum cardinality. It is easy to see that if $\varepsilon_1 < \varepsilon_2$, then $S_n(\varepsilon_2) < S_n(\varepsilon_1)$.

Later in chapter 4, we will see that

$$S_n(\varepsilon) < \infty \tag{1.1}$$

Definition 1.2.2 *Let X be a compact metric space. The topological entropy of $f : X \to X$ is the number*

$$h_{top}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S_n(\varepsilon)$$

1.3 Examples

In this section, we present some examples in order to understand the relation between the topological entropy and the growth rate of the periodic points.

1.3.1 The shift map

Let $\Sigma_2^+ = \{0,1\}^{\mathbb{N}}$ be the set of all sequences formed by the symbols 0 and 1, that is

$$\Sigma_2^+ = \{\{x_i\}_{i=0}^{\infty}, x_i \in \{0,1\}\}$$

We define the shift map as

$$\begin{aligned} \sigma : \Sigma_2^+ &\to \Sigma_2^+ \\ \{x_i\}_{i=0}^{\infty} &\mapsto \sigma(\{x_i\}_{i=0}^{\infty}) = \{x_{i+1}\}_{i=0}^{\infty} \end{aligned}$$

That is, the image of the sequence $\{x_i\}_{i=0}^{\infty}$ is obtained omitting the first digit and shifting the other digits to the left.

Given $x = x_0 x_1 \cdots$ and $y = y_1 y_2 \cdots$ sequences in Σ_2^+, we define

$$d(x,y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}$$

Observe that, since $|x_i - y_i|$ always equals to 0 or 1, the above series converges. The function $d : \Sigma_2^+ \to \mathbb{R}$ defines a metric in Σ_2^+.

Note that the maximum distance is attained in the points $\{0\} = \{0, 0, \cdots\}$ and $\{1\} = \{1, 1 \cdots\}$. The maximum distance is

$$d(1,0) = \sum_{i=0}^{\infty} \frac{1}{2^i} = 2$$

In the space Σ_2^+ two sequences are close if, and only if, the first n entries of the sequences coincide, more precisely, let $x, y \in \Sigma_2^+$

$$x_i = y_i, \forall i \leq n \Leftrightarrow d(x,y) \leq \frac{1}{2^n}$$

Indeed, if $x_i = y_i, \forall i \leq n$, then

$$\begin{aligned} d(x,y) &= \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i} \\ &= \sum_{i=n+1}^{\infty} \frac{|x_i - y_i|}{2^i} \\ &\leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} \\ &= \frac{1}{2^n} \end{aligned}$$

Conversely, if $x_i \neq y_i$ for some $i \leq n$, then

$$\begin{aligned} d(x,y) &= \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i} \\ &\geq \frac{1}{2^i} \\ &\geq \frac{1}{2^n} \end{aligned}$$

Since $n \geq i$.

Denote by $Fix(\sigma^n)$ the set of the points with period n. That is,

$$Fix(\sigma^n) = \{x \in \Sigma_2^+ : \sigma^n(x) = x\}$$

Observe that, if a point $x \in \Sigma_2^+$, then it has the form

$$x = x_0x_1x_2x_3 \cdots x_{n-1}x_0x_1x_2x_3 \cdots x_{n-1}x_0x_1x_2x_3 \cdots x_{n-1} \cdots$$

Therefore, $\#Fix(\sigma^n) = 2^n$.

A natural question is: what is the growth rate of the periodic points? The growth rate is given by

$$\lim_{n\to\infty} \frac{1}{n} \log \#Fix_n(\sigma) = \lim_{n\to\infty} \frac{1}{n} \log 2^n = \log 2$$

Is well known from the ergodic theory, that the topological entropy of the 2 symbols shift map is given by $h_{top}(\sigma) = \log 2$.

Thus, in the shift case we have $h_{top}(\sigma) = \lim_{n\to\infty} \frac{1}{n} \log \#Fix(\sigma^n)$. In other words, the topological entropy and the growth rate of the periodic points coincide.

1.3.2 The tent map

Now, let us take a look at another example.

Consider the map $f_2 : [0,1] \to [0,1]$ given by

$$f_2(x) = \begin{cases} 2x, & \text{if } 0 \le x \le \frac{1}{2} \\ 2(1-x), & \text{if } \frac{1}{2} < x \le 1 \end{cases}$$

This map is called the tent map, it is an example of an expansor map in a interval. The graph of f_2 appears in figure 1.1.

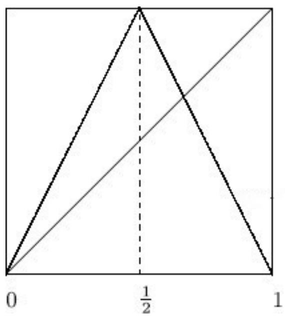

Figure 1.1: The Tent Map

The tent map stretch the interval $[0, \frac{1}{2}]$ over the intire interval $[0, 1]$, and folds the interval $\left(\frac{1}{2}, 1\right]$ back over the interval $[0, 1]$. The fixed points of f_2 are the points $x = 0$ and $x = \frac{2}{3}$. Figure 1.2 indicates that f_2^2 and f_2^3 have, respectively, four and eight fixed points.

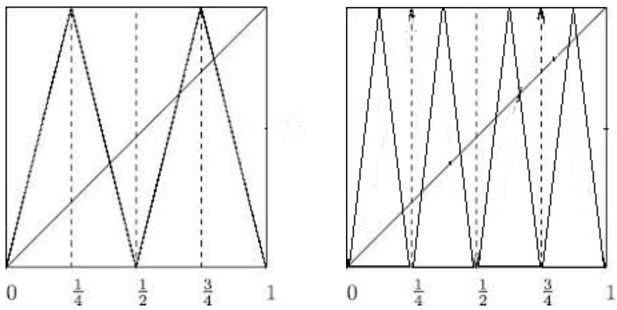

Figure 1.2: Iterates of the tend map

That is, f_2 has four points with period two and eight points with period four. Proceeding by induction, we obtain $\#Fix(f_2^n) = 2^n$. As in the shift case, we have $h_{top}(f_2) = \log 2$. Therefore,

$$h_{top}(f_2) = \lim_{n\to\infty} \frac{1}{n} \log \#Fix(f_2^n)$$

1.3.3 Anosov diffeomorphism

Suppose $A \in GL_n(\mathbb{R})$, where $GL_n(\mathbb{R})$ is the set of all n x n invertible matrices with real entries. We say that A is hyperbolic if each of its eigenvalues $\lambda_i \in \mathbb{C}$ satisfy $|\lambda_i| \neq 1$. We call a eigenvalue λ_i contracting if $|\lambda_i| < 1$ or expanding if $|\lambda_i| > 1$. Similarly, a matrix A is called contracting(expanding) if its eigenvalues are contracting(expanding). Given a hyperbolic matriz $A \in GL_n(\mathbb{R})$, we can split the domain of A into the direct sum of two A-invariant subspaces E^s and E^u, i.e., $\mathbb{R}^n = E^s \oplus E^u$, where E^s and E^u are the generalized eigenspaces corresponding to the contracting and expanding eigenvalues of A respectively. It follows that A restricted to E^s is contracting, and A restricted to E^u is expanding. This gives us a direction on which A is contracting and another on which A is expanding.

Now consider a hyperbolic $A \in GL_n(\mathbb{Z})$. We have that $A(\mathbb{Z}^n) = \mathbb{Z}^n$. Quotienting $\mathbb{R}^n$ by $\mathbb{Z}^n$, A induces a map

$$\widetilde{A} : x + \mathbb{Z} \mapsto A(x) + \mathbb{Z}$$

on the torus $\mathbb{T}^n = \mathbb{R}^n \setminus \mathbb{Z}^n$ to itself. Note that $\widetilde{A}$ is a diffeomorphism of $\mathbb{T}^n$.

Given a diffeomorphism $f : M \to M$, we say that a compact invariant subset $\Lambda \subset M$ is hyperbolic, if there is a continuous splitting, invariant under the action of the derivative, that is, $Df(x)|_{E^s_x} = E^s_{f(x)}$ and $Df(x)|_{E^u_x} = E^u_{f(x)}$, for every $x \in \Lambda$, and there are constants $C > 0$ and $\lambda \in (0, 1)$ such that for every $n \geq 0$

$$||Df^n(x)|_{E^s_x}|| \leq C\lambda^n \quad and \quad ||Df^{-n}(x)|_{E^u_x}|| \leq C\lambda^n$$

Now, let us return our attention to the diffeomorphism $\widetilde{A} : \mathbb{T}^n \to \mathbb{T}^n$ defined above.

We Claim that this is hyperbolic. Indeed, the space $\mathbb{R}^n$ is hyperbolic with respect to the matrix $A \in GL_n(\mathbb{Z})$. Thus, we can consider a splitting of $\mathbb{R}^n$ into subspaces E^s_x and E^u_x. Since the tangent space of $\mathbb{R}^n$ is naturally identified with $\mathbb{R}^n$ for all $x \in \mathbb{R}^n$, we can pass this splitting to the tangent space of the coset of x in $\mathbb{T}^n$, giving a new splitting under $\widetilde{A}$.

The diffeomorphism induced by a hyperbolic matrix $A \in GL_n(\mathbb{Z})$ is called hyperbolic toral automorphism.

Let $x \in \mathbb{T}^n$. If x is a fixed point for $\widetilde{A}$, then it satisfies

$$\widetilde{A}(x) = A(x) + \mathbb{Z} = x$$

That is, if

$$(A - I)(x) \in \mathbb{Z}$$

where I is the identity map.

For obtaining the cardinality of the set of fixed points of $\widetilde{A}$, we need to check how many points in $(A-I)(\mathbb{T}^n)$ lie on the lattice $\mathbb{Z}^n$. Since there is exactly one in the fundamental domain $[0,1)^n$ of the n-torus, one can see that this just corresponds to the volume of the parallelepiped $(A-I)([0,1)^n)$, i.e, $\#Fix(\widetilde{A}) = |det(A-I)|$. Since the fixed points of $\widetilde{A}^n$ are the periodic points of $\widetilde{A}$ with period n, one has $\#Fix(\widetilde{A}^n) = |det(A^n - I)|$.

In truth, hyperbolic toral automorphism fall into a more general class of objects called Anosov diffeomorphism. Now, we give the precise definition of Anosov diffeomorphism.

Definition 1.3.1 *A diffeomorphism $f : M \to M$ on a compact manifold is called Anosov if M is hyperbolic.*

Example (Arnold's Cat Map)

Consider the following matrix in $GL_2(\mathbb{Z})$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

One can easily see that $det(A) = 1$.

A number $\lambda \in \mathbb{C}$ is an eigenvalue of $A \Leftrightarrow det(A - \lambda I) = 0$. A simple calculation shows that its eigenvalues are $\frac{3 \pm \sqrt{5}}{2}$. Therefore, A is a hyperbolic matrix, and according to what we saw previously, A induces an Anosov toral automorphism $\widetilde{A}$, and in particular $\widetilde{A}$ is an Anosov diffeomorphism.

In next chapter we will see that

$$h_{top}(\widetilde{A}) = \lim_{n \to \infty} \frac{1}{n} \log \#Per_n(\widetilde{A})$$

One may ask if the the topological entropy always is equal to the the growth rate of the periodic points. The next example will answer this question.

1.3.4 The identity map

Consider the identity map $I : [0, 1] \to [0, 1]$

Since any point is periodic, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \#Per_n(I) = \infty$$

Observe that for any pair of points $x, y \in [0, 1]$

$$\begin{aligned} d_1(x, y) &= d(x, y) \\ d_2(x, y) &= \max\{d(x, y), d(I(x)), d(I(y)\} = d(x, y) \\ &. \\ &. \\ &. \\ d_n(x, y) &= \max_{0\le i\le n-1}\{d(I^i(x), I^i(y))\} = d(x, y) \end{aligned}$$

Therefore, given $\varepsilon > 0$

$$S_1(\varepsilon) = S_2(\varepsilon) = \cdots = S_n(\varepsilon)$$

for any $n \in \mathbb{N}$. Thus

$$\begin{aligned} h_{top}(I) &= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \#S_n(\varepsilon) \\ h_{top}(I) &= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \#S_1(\varepsilon) \\ &= 0 \end{aligned}$$

Since $\log \#S_1(\varepsilon)$ is constant.

Then, for the identity map we have

$$h_{top}(I) < \lim_{n\to\infty} \frac{1}{n} \log \#Per_n(I)$$

So, a reasonable question is: Under which condition does the equality holds? Is it still holding for continuous time?

Our purpose in this work is to give a satisfactory answer to these questions.

In the discrete case, under certain conditions, one has that the growth rate of the periodic orbits is indeed equal to its topological entropy. We have the following:

Theorem A: *Let $f : X \to X$ be an expansive homeomorphism on a compact metric space. If f has the shadowing property, then $\#Per_n(f) \leq \infty$ and*

$$h_{top}(f) = \limsup_{n\to\infty} \frac{1}{n} \log(\#Per_n(f))$$

However, the same result is not valid for continuous time dynamical systems. In this case one can only guaranties that the growth rate of the periodic points is larger than or equal to the topological entropy.

Let M be a compact manifold. Next theorem is the main result of this work.

Theorem B [WYZ19]: *There exists a residual set $\mathscr{R} \subset \mathscr{X}^1(M)$ such that for any $X \in \mathscr{R}$, one has*

$$h_{top}(X) := h_{top}(X_1) \leq \limsup_{T\to\infty} \frac{1}{T} \log(\#P_T(X))$$

In the next chapters, we will define all the objects that are necessary for the proof of the above theorems.

Chapter 2

Presentation of the main theorem and proof of Theorem A

In the previous chapter, we see that the growth rate of the periodic points can differ of the topological entropy. We are interested in finding conditions to have equality. The following theorem give us such conditions.

Theorem A: *Let $f : X \to X$ be an expansive homeomorphism on a compact metric space. If f has the shadowing property, then $\#Per_n(f) \leq \infty$ and*

$$h_{top}(f) = \limsup_{n\to\infty} \frac{1}{n} \log(\#Per_n(f))$$

The idea behind the proof of Theorem A is the following: For the inequality $h_{top}(f) > \limsup_{n\to\infty} \frac{1}{n} \log(\#Per_n(f))$, the expansivity will force the set $Per_n(f)$ to be (n, ε)-separated, and then, the result will follow from the fact that $S_n(\varepsilon)$ is the (n, ε)- separated set with maximum cardinality. For the inequality $h_{top}(f) < \limsup_{n\to\infty} \frac{1}{n} \log(\#Per_n(f))$, we cover X by dynamical balls with radius smaller than the expansivity constant. By the pigeon principle together with the shadowing property, we will obtain periodic shadows, and espansivity plus separability will imply that the period is smaller than n, and then, the result will follow.

Before proving the above theorem, we give some important definitions.

Definition 2.0.1 *A dynamical system $f : X \to X$ is expansive, if there is a constant $\gamma > 0$ such that, for every pair of different points $x \neq y$ in X, there is an integer m such that $d(f^m(x), f^m(y)) \geq \delta$. The number $\delta > 0$ is called an expansive constant of f.*

As an example of an expansive system, we have the hyperbolic diffeomorphisms. Indeed, let Λ be a hyperbolic set for a diffeomorphism g. Let $\beta > 0$ be given by the theorem of the stable manifold (see for instance [Wen16]), and suppose $d(f^j(x)), f^j(y)) < \gamma$, $\forall\ j \in \mathbb{Z}$. Thus, $x \in W^s_\beta(y)$ and $x \in W^u_\beta(y)$. This implies $x = y$.

Let $\delta > 0$. We call a sequence $\{x_n\}_{-\infty}^{\infty}$ in X a δ-pseudo-orbit of f if, for every $n \in \mathbb{Z}$,

$$d(f(x_n), x_{n+1}) < \delta$$

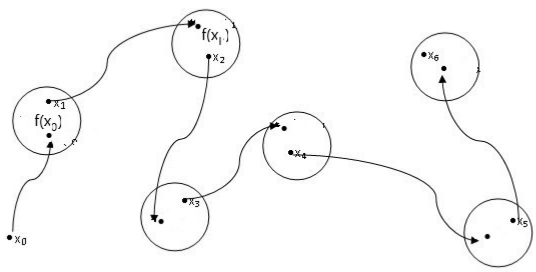

Figure 2.1: δ-Pseudo Orbit

We say a point $y \in X$ ε-shadows a pseudo-orbit $\{x_n\}_{-\infty}^{\infty}$ if, for every $n \in \mathbb{Z}$,

$$d(f^n(y), x_n) < \varepsilon$$

Definition 2.0.2 *A dynamical system $f : X \to X$ is said to have the shadowing property if, given $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit can be ε-shadowed.*

Observe that if we also assume that f is expansive with expansive constant γ , then the shadow is unique. Indeed, Let $0 < \varepsilon \leq \gamma$ and consider the δ-pseudo-orbit $\{x_n\}_{-\infty}^{\infty}$, where δ is given by the shadowing property. Suppose that z_1 and z_2 are shadows, then by definition

$$d(f^n(z_1), x_n) < \frac{\varepsilon}{2}, \quad \forall n \in \mathbb{Z}$$

and

$$d(f^n(z_2), x_n) < \frac{\varepsilon}{2}, \quad \forall n \in \mathbb{Z}$$

By the triangle inequality

$$d(f^n(z_1), f^n(z_2)) \leq d(f^n(z_1), x_n) + d(f^n(z_2), x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon \leq \delta, \quad \forall n \in \mathbb{Z}$$

and it follows from expansivity that $z_1 = z_2$.

Now, let us return our attention to the example of the subsection 1.3.3 (Arnold's Cat Map).

We have seen that the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

induces a hyperbolic diffeomorphism $\widetilde{A} : \mathbb{T}^2 \to \mathbb{T}^2$

Is well known by the hyperbolic theory that hyperbolic sets have the shadowing property, and since it also is expansive, it follows from Theorem A that

$$h_{top}(\widetilde{A}) = \lim_{n\to\infty} \frac{1}{n} \log \#Per_n(\widetilde{A})$$

More generally, for any Anosov diffeomorphism, the topological entropy equals to the growth rate of the periodic orbis.

Let $\varepsilon > 0$ be given, and suppose that f is expansive with the shadowing property. Let $\{x_n\}_{-\infty}^{\infty}$ be a periodic δ-pseudo-orbit, where δ is given by the shadowing. There is $m > 0$ such that $x_n = x_{n+m}$ for all $n \in \mathbb{Z}$. Then, $\{x_n\}$ is ε-shadowed by a point z, that is

$$d(f^n(z), x_n) < \varepsilon, \quad \forall n \in \mathbb{Z}$$

that is

$$d(f^{n+m}(z), x_{n+m}) < \varepsilon, \quad \forall n \in \mathbb{Z}$$

But since $x_n = x_{n+m}$

$$d(f^{n+m}(z), x_n) = d(f^n(f^m(z)), x_n) < \varepsilon, \quad \forall n \in \mathbb{Z}$$

In other words, $f^m(z)$ also ε-shadows $\{x_n\}$. By the uniqueness of the shadow, $f^m(z) = z$, meaning z is periodic.

We just proof that a periodic-pseudo-orbit can be shadowed by a periodic orbit. Moreover, the shadow has the same period.

Now, we will give the proof of the Theorem A.

Proof of Theorem A: Let δ be an expansive constant for f. Let $\varepsilon \leq \frac{\delta}{2}$, we will show that the set $Per_n(f)$ is (n, ε)-separated. First, observe that the set $Per_n(f)$ is (n, ε)-separated, and since $S_n(\varepsilon)$ is the (n, ε)-separated set of maximum cardinality, we have

$$S_n(\varepsilon) \geq Per_n(f)$$

This implies

$$\frac{1}{n}\log(S_n(\varepsilon) \geq \frac{1}{n}\log(\#Per_n(f))$$

taking limits on both sides we obtain

$$h_{top}(f) \geq \limsup_{n\to\infty} \frac{1}{n}\log(\#Per_n(f))$$

For the reverse inequality, let $\varepsilon < \delta$, and denote by $E_n(\varepsilon)$ the (n, ε)-separated set with maximum cardinality. We have that

$$X = \bigcup_{x_i \in E_n(\varepsilon)} B(x_i, n, \varepsilon)$$

Pick a point $x \in X$. Then, $x \in B(x_i, n, \varepsilon)$ for some i. If x is not a fixed point, expansivity implies that there is $j_1 \in \mathbb{N}$ such that $f^{j_1}(x) \notin B(x_i, n, \varepsilon)$. That is $f^{j_1}(x) \in B(x_{i_{j_1}}, n, \varepsilon)$, for some $x_{i_{j_1}} \neq x_i$. Analogously, there is $j_2 \in \mathbb{N}$ such that $f^{j_2}(f^{j_1}(x)) \in B(x_{i_{j_2}}, n, \varepsilon)$, for some $x_{i_{j_2}} \neq x_{j_1}$. Thus, if x is not a periodic point, we can construct an infinite sequence $s_f = \{f^{j_k}(x), k \in \mathbb{N}\} \subset X$. By the pigeonhole principle, there is some $B(x_i, n, \varepsilon)$ with infinite points of s_f. By compacity, s_f has a accumulation point belonging to $B(x_i, n, \varepsilon)$. Then, for every $\delta > 0$ one can construct a periodic δ-pseudo-orbit $\{x_j\}$ through the limit point of s_f. By the shadowing property, since f is an expansive map, there is a unique periodic point p_x such that $d(x_j, f^j(p_x)) \leq \varepsilon$, for all $j \in \mathbb{Z}$. Moreover, the separability implies that $\pi(p_x) \leq n$, where $\pi(p_x)$ is the period of the point p_x. Therefore, $Per_n(f) \geq S_n(\varepsilon)$. This implies

$$h_{top}(f) \leq \limsup_{n\to\infty} \frac{1}{n}\log(\#Per_n(f))$$

□

This result was originally proved by Bowen [Bow70] in the setting of Axiom A systems . For a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism f on a compact manifold, and any f-invariant Borel probability measure with non-zero Lyapunov exponents, Katok [Kat80] showed that the upper limit of the growth rate of the periodic points is larger than or equal to its metric entropy, i.e.,

$$h_\mu(f) \leq \limsup_{n\to\infty} \frac{1}{n}\log(\#Per_n(f))$$

where μ is a hyperbolic measure.

In this work, we are concerned about the case of vector fields. Assume that M is a boundaryless compact smooth manifold and let $\mathscr{X}^1(M)$ be the space of all C^1 vector fields on M with the C^1 norm. Note that $\mathscr{X}^1(M)$ is a banach space. A vector field $X \in \mathscr{X}^1(M)$ generates a flow X_t. Let

$$\#P_T(X) = \sum_{x \in Per_T(X)} \pi(x)$$

where $\pi(x)$ is the minimum period of x and $Per_T(X) = \{x \in M : 0 \leq \pi(x) \leq T\}$

Now, we will state the main result of this text

Theorem B [WYZ19]: *There exists a residual set $\mathscr{R} \subset \mathscr{X}^1(M)$ such that for any $X \in \mathscr{R}$, one has*

$$h_{top}(X) := h_{top}(X_1) \leq \limsup_{T \to \infty} \frac{1}{T} \log(\#P_T(X))$$

Strategy for the proof

First, we give the precise construction of the residual set $\mathscr{R}$. So, the prove is divided in two cases:

Case 1: The generic vector field is not star. In this case, since it can be approximated by vector fields, each one them having a non-hyperbolic periodic orbit, we emulate Example 1.3.4 (identity map). To do so, we apply the Franks Lemma together with the definition of the residual set $\mathscr{R}$ to prove that the growth rate of the periodic orbits is infinite, and then, larger or equal to the topological entropy.

Case 2: The generic vector field is star. In this case, we emulate the proof of Theorem A. If X is star, Shi-Gan-Wen [SGW14] proved that every X_t-invariant ergodic measure μ is hyperbolic, then we prove that the Oseledec splitting with respect to μ is a dominated splitting. After that, we apply Liao's Shadowing Lemma [Lia81a] to prove that if the hyperbolic Oseledec splitting is a dominated splitting, then the growth rate of the periodic orbits is larger than or equal to the topological entropy, and then the result follows from the variational principle.

Chapter 3

Star Flows

A vector field $X \in \mathscr{X}^1(M)$ is called a star vector field or a star flow, if it satisfies the star condition, i.e., there exists a C^1 neighborhood $\mathscr{U}$ of X such that every critical element of every $Y \in \mathscr{U}$ is hyperbolic. The set of C^1 star vector fields in M is denoted by $\mathscr{X}^*(M)$.

The notion of star systems came up from the study of the stability conjecture. A classical theorem of Smale [Sma70] (for diffeomorphism) and Pugh-Shub [PS70b] (for flows) states that Axiom A plus the no-cycle condition implies the Ω-stability. Palis and Smale [PS70a] conjectured that the converse also holds, which has been known as the Ω-stability conjecture. In the study of the conjecture , Pliss, Liao and Mañé noticed an important condition called by Liao "the star condition". As defined above, the star condition looks quite weak because, though it is a robust property, it is only concerned with critical elements, and the hyperbolicity considered is not in a uniform way. Indeed, the Ω-stability implies the star condition [Fra71, Lia79]. Thus whether the star condition could give back Axiom A plus the no-cycle condition became a striking problem, raised by Liao [Lia81b] and Mañé [Mañ82]. An affirmative answer to the problem would contain the Ω-stability conjecture. For diffeomorphsm, Aoki [Aok92] and Hayashi [Hay92] proved that the star condition indeed implies Axiom A plus the no-cycle condition. For flows, there are counterexamples if the flow has singularities. For instance, the geometric Lorenz attractor [Guc76], which has a singularity is a star flow, but it fails to satisfy Axiom A. In fact, Liao [Lia81b] and Mañé [Mañ82] raised this problem for nonsingular star flows, and hence it was solved by Gan-Wen [GW06] proving that nonsigular star flows do satisfy Axiom A plus no-cycle condition.

An important feature of star vector fields, is that they are in a certain way source of hyperbolicity, as we will see in subsection 3.5.2. Moreover, the hyperbolicity is in a uniform way.

3.1 General Definitions

Let X be a C^1 vector field on a compact boundaryless Riemannian manifold M, and denote by $X_t : M \to M$ the flow generated by it, that is $\frac{d}{dt}\mid_{t=0} X_t(x) = X(x)$ for every $x \in M$. Since M is compact this flow can be defined for every $t \in \mathbb{R}$. The flow X_t has the following properties:

- $X_0(x) = x$ for every $x \in M$.
- $X_{t+s}(x) = X_t \circ X_s(x)$ for every $t, s \in \mathbb{R}$ and every $x \in M$.
- $(X_t)^{-1}(x) = X_{-t}(x)$ for every $t \in \mathbb{R}$ and every $x \in M$.
- For every $t \in \mathbb{R}$ the application $X_t : M \to M$ is a C^2-diffeomorphism.

These properties tell us that the application

$$\begin{aligned} \mathbb{R} \times M &\to M \\ (t, x) &\mapsto X_t(x) \end{aligned}$$

defines an action of the group $(\mathbb{R}, +)$ on M. Just as in group actions, we define the orbit of a point $x \in M$ as the set of points $\mathscr{O}(x) = \{X_t(x) : t \in \mathbb{R}\}$.

A point $\sigma \in M$ is called a singularity of X, if $X(\sigma) = 0$. Note that a singularity is just a fixed point for the flow X_t generated by the vector field X.

An orbit $\mathscr{O}(x)$ is periodic, if there exists $p \in \mathscr{O}(x)$ and $T > 0$ such that $X_T(p) = p$. In this case $\mathscr{O}(x) = \{X_t(p) : 0 \leq t \leq T\}$ and p is said to be a periodic point. The smaller $T > 0$ that satisfies $X_T(p) = p$ is called the period of p, and denoted by $\pi(p)$. The set of periodic points is denoted by $Per(X)$. The set of critical points of X is defined by

$$Crit(X) = Per(X) \cup Sing(X)$$

3.2 Hyperbolicity

Recall that a periodic point p is hyperbolic for a diffeomorphism $f : M \to M$, there is a continuous splitting $T_pM = E^s_p \oplus E^u_p$, of the tangent bundle over p, invariant under the action of the derivative, that is, $Df(p)|_{E^s_p} = E^s_{f(p)}$ and $Df(p)|_{E^u_p} = E^u_{f(p)}$, and there are constants $C > 0$ and $\lambda \in (0, 1)$ such that for every $n \geq 0$

$$||Df^n(p)|_{E^s_p}|| \leq C\lambda^n \quad and \quad ||Df^{-n}(p)|_{E^u_p}|| \leq C\lambda^n$$

Next, we define the concept of hyperbolicity for critical points of a vector field.

Definition 3.2.1 *A singularity σ is hyperbolic if for every eigenvalue λ of $DX(\sigma)$, we have $Re(\lambda) \neq 0$.*

For periodic orbits, we need the concept of Poincaré map associated to such orbit.

Let $\mathscr{O}(p)$ be a periodic orbit for $X \in \mathscr{X}^1(M)$. Consider a cross section Σ through p. The orbit of p crosses Σ again at the time $\pi(p)$. By continuity of the flow X_t, the orbit through $x \in \Sigma$ sufficiently close to p also returns to Σ after a time close to $\pi(p)$. Thus, in a sufficiently small neighborhood $V \subset \Sigma$ of p, one can define a map $P : V \to \Sigma$ that associates each point $x \in V$ to a point $P(x)$, where $P(x)$ is the first point of the orbit of x to return to Σ. This map is called *the Poincaré map* associated to Σ and x.

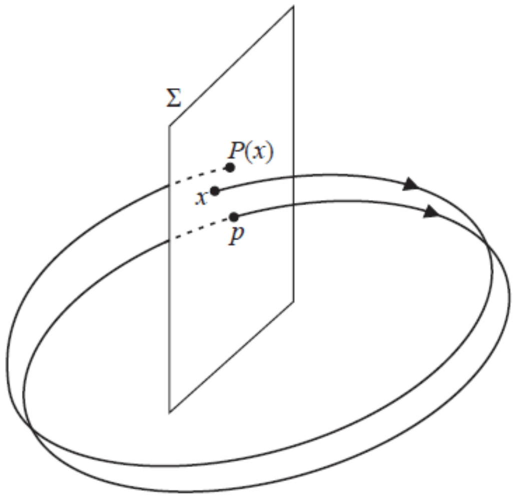

Figure 3.1: Poincaré Map

Definition 3.2.2 *A periodic point $p \in M$ is hyperbolic with respect to $X \in \mathscr{X}^1(M)$ if every eigenvalue λ of $DP(p)$ satisfies $|\lambda| \neq 1$.*

3.3 The linear Poincaré flow

Let $X \in \mathscr{X}^1(M^d)$ and denote the normal bundle of X by

$$N = \bigcup_{x \in M \setminus Sing(X)} N_x$$

where $N_x = \{v \in T_xM : v \perp X(x)\}$. That is, N is the $(d-1)$- dimensional subbundle over $M \setminus Sing(X)$ orthogonal to the vector field direction.

For the flow X_t generated by X, its derivative $DX_t : TM \to TM$ is called the tangent flow, that can be described as

$$DX_t(x, v) = (X_t(x), DX_t(x)(v))$$

In other words, we have a dynamics X_t that acts on the base space M and we have the derivative that acts on the fibers T_xM.

Obviously N is not invariant by the action of the derivative DX_t, but we can force this invariance taking the orthogonal projections $\pi_x = T_xM \to N_x$, and defining the *the linear Poincaré flow*

$$\begin{aligned} P_t(x) : N_x &\to N_{X_t(x)} \\ v &\mapsto \pi_x(DX_t(x)(v)) \end{aligned}$$

That is, it is the orthogonal projection of $DX_t(x)(v)$ on $N_{X_t(x)}$ along the flow direction $X(X_t(x))$.

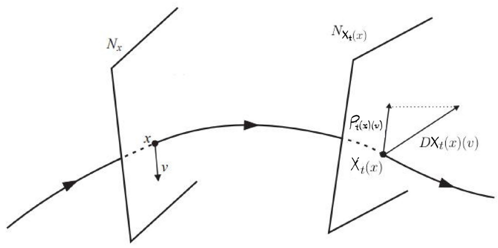

Figure 3.2: Linear Poincaré Flow

The linear Poincaré flow can also be written as

$$P_t(x)(v) = DX_t(x)(v) - \frac{\langle DX_t(x)(v), X(X_t(x))\rangle}{||X(X_t(x))||^2} X(X_t(x))$$

One can also define the rescaled linear Poincaré flow

$$P_t^*(x)(v) = \frac{||X(x)||}{||X(X_t(x))||} P_t(x)(v)$$

Now, let us now investigate some properties of the linear Poincaré flow. First observe that

$$P_0(x)(v) = DX_0(x)(v) - \frac{\langle DX_0(x)(v), X(X_0(x))\rangle}{||X(X_0(x))||^2} X(X_0(x))$$

by the flow property, one has

$$P_0(x)(v) = v - \frac{\langle v, X(x)\rangle}{||X(x)||^2} X(x)$$

and since v and $X(x)$ are orthogonal, we have

$$P_0(x)(v) = v$$

that is, $P_0 = Id$. Also observe that

$$P_{t+s}(x)v = \pi(DX_{t+s}(x)v) = \pi(DX_t(X_s(x))v)$$

by the chain rule,

$$DX_t(X_s(x))v = DX_t(X_s(x))DX_s(x)v$$

but we can write $DX_s(x)v = \pi(DX_s(x)v) + (DX_s(x)v)^X$

where $(DX_s(x)v)^X$ is the component of $DX_s(x)v$ along the flow direction. Thus

$$\begin{aligned} DX_t(X_s(x))v &= DX_t(X_s(x))[\pi(DX_s(x)v) + (DX_s(x)v)^X] \\ &= DX_t(X_s(x))\pi(DX_s(x)v) + DX_t(X_s(x))(DX_s(x)v)^X \end{aligned}$$

Then

$$\begin{aligned} P_{t+s}(x)v &= \pi(DX_t(X_s(x))\pi(DX_s(x)v) + DX_t(X_s(x))(DX_s(x)v)^X) \\ &= \pi(DX_t(X_s(x))\pi(DX_s(x)v)) + \pi(DX_t(X_s(x))(DX_s(x)v)^X) \end{aligned}$$

and since the flow direction is invariant under the action of the derivative, we have

$$\pi(DX_t(X_s(x))(DX_s(x)v)^X) = 0$$

thus,

$$\begin{aligned} P_{t+s}(x)v &= \pi(DX_t(X_s(x))\pi(DX_s(x)v)) \\ &= P_t(X_s(x))P_s(x)v \\ &= P_t \circ P_s(x)v \end{aligned}$$

Now, since $P_{-t} \circ P_t = P_0 = Id = P_t \circ P_{-t}$ it follows that

$$P_{-t} = (P_t)^{-1}$$

Since P_t satisfies this group property it can be regarded as a flow, which justifies its name.

Now, let us see what is the relation between the linear Poincaré flow, and the Poincaré map. First, observe that for a non periodic point, we can also define the Poincaré map. Indeed, let $x \in M \setminus Sing(X)$, and fix a time $t \in \mathbb{R}$, so that $X_t(x) \neq x$. Considering small cross sections Σ_x through x and $\Sigma_{X_t(x)}$ through $X_t(x)$, it follows from the tubular flow theorem, that the map $P : \Sigma_x \to \Sigma_{X_t(x)}$, which for each $y \in \Sigma_x$ associates the first point that the orbit of y hits $\Sigma_{X_t(x)}$ is a C^1 diffeomorphism. This map is called the Poincaré map from x to X_t.

An important observation is, if the cross sections are chosen conveniently, then the derivative of the Poincaré map coincides with the linear Poincaré flow. More precisely:

For each regular value $x \in M$, and any $\delta > 0$, denote

$$N_x(\delta) = N_x \cap B(0, \delta) = \{v \in N_x : ||v|| \leq \delta\}$$

where $B(0, \delta)$ is the open ball centred at origin with radius δ in T_xM.

Proposition 3.3.1 *Let $x \in M \setminus Sing(X)$ and $X_t(x)$ as above. There is $\delta > 0$ such that, taking $\Sigma_x = exp_x(N_x(\delta))$ and $\Sigma_{X_t(x)} = exp_{X_t(x)}(N_{X_t(x)}(\delta))$ as cross sections, one has that the derivative of the Poincaré map $P : \Sigma_x \to \Sigma_{X_t(x)}$ at 0 equals to the liner Poincaré flow $P_t : N_x \to N_{X_t(x)}$. That is*

$$DP(0) = P_t(x) \tag{3.1}$$

Proof. Take $\delta > 0$ small enough so that the exponential map exp_x is a diffeomorphism under its image, and consider the map $h : N_{X_t(x)}(\delta) \times \mathbb{R} \to M$, given by $h(v, s) = X_s(\exp_{X_t(x)}(v))$. In exponential coordinates, reducing δ if necessary, one can represents the Poincaré map by

$$P = \pi \circ h^{-1} \circ X_t \circ exp_x|_{N_x(\delta)} : N_x(\delta) \to N_{X_t(x)}(\delta)$$

where $\pi : N_{X_t(x)} \times \mathbb{R} \to N_{X_t(x)}$ is the canonical projection. Thus, by the chain rule

$$DP(0) = D(\pi \circ h^{-1}) \circ DX_t(exp_x(0))Dexp_x(0)$$

Since, $exp_x(0) = x$ and $Dexp_x(0) = I$, where I is the identity. Therefore, for any $w \in N_x(\delta)$

$$DP(0)w = D(\pi \circ h^{-1}) \circ DX_t(x)w \in N_{X_t(x)}(\delta)$$

That is, $DP(0)w$ is the orthogonal projection of the vector $DX_t(x)w \in T_{X_t(x)}M$. that is, $DP(0) = P_t(x)$.

□

3.4 Generic Dynamics

One of the main purpose in dynamics is to try to obtain properties that are valid for most systems. But what do we mean by most systems? Let $O_n \subset \mathscr{X}^1(M)$ be open and dense in the C^1-topology. By the Baire's theorem, we have that $\mathscr{R} = \bigcap_{n\in\mathbb{N}} O_n$ is dense in $\mathscr{X}^1(M)$. We call such set a residual subset of $\mathscr{X}^1(M)$, in other words, a residual is an intersection of open and dense subsets. We say that a property is generic if it is valid on residual subset of $\mathscr{X}^1(M)$. The great advantage of obtaining generic properties is that the intersection of residual subsets is a residual.

As an example of a C^1-generic dynamical system, we have the family of Kupka-Smale systems. Recall that a system is said to be Kupka-Smale, if every critical element is hyperbolic and if $W^s(orb(x_1))$ is transversal to $W^u(orb(x_2))$ for every $x_1, x_2 \in Crit(X)$.

Theorem 3.4.1 *(Kupka-Smale) There exits a residual set $KM \in \mathscr{X}^1(M)$ such that every $X \in KM$ is a Kupka-Smale vector field.*

The Kupka-Smale systems have a great importance for the theory. By definition they are source of local hyperbolicity, since every critical element is hyperbolic, it follows from the Hartman-Grobman theorem that in a small neighborhod of every critical element, the dynamics is linear. This also leads us to the local stability of the system near to these points. Moreover, the above theorem tells us that the family of the Kupka-Smale systems is in a certain sense big.

3.5 Definition of star flow

The Kupka-Smale theorem states the the hyperbolicity of every critical element occurs in a residual set. So, is natural to study such systems whose all critical elements are hyperbolic robustly. This lead us to the following definition:

Definition 3.5.1 *We say that a vector field $X \in \mathscr{X}^1(M)$ is star if there exists a neighborhood $\mathscr{U}$ of X such that if $Y \in \mathscr{U}$ and $y \in Crit(Y)$, then y is hyperbolic.*

The reason behind the definition of star flows is not just as an extension of the Kupka-Smale systems, its definition is motivated by the structural stability theory.

An important class of examples of star systems is generated by Axiom A systems without cycles. Before defining the axiom A systems, recall that a compact invariant set Λ is hyperbolic for the vector field $X \in \mathscr{X}^1(M)$, if there exists a continuous splitting $T_\Lambda M = E^s \oplus \langle X \rangle \oplus E^u$ invariant by the action of DX_t and constants $C > 0$ and $\lambda > 0$ such that for every $x \in \Lambda$, and every $t \geq 0$

$$||DX_t(x)|_{E^s_x}|| \leq Ce^{-\lambda t} \quad and \quad ||DX_{-t}(x)|_{E^u_x}|| \leq Ce^{-\lambda t}$$

Remark: The definition of hyperbolicity for flows implies that every singularity contained in Λ must be isolated in Λ. It is due to the continuity of the splitting, which implies in the continuity of the dimension of the fibers. The dimension of each fiber has to be locally constant, and this is clearly false if we have a singularity accumulated by periodic orbits (or regular orbits) in the hyperbolic set.

We say that a vector field is Axiom A if the non-wandering set $\Omega(X)$ is hyperbolic, and satisfies

$$\Omega(X) = \overline{Crit(X)}$$

We say that a vector field $X \in \mathscr{X}^1(M)$ is structurally stable if there exists a C^1 neighborhood $\mathscr{U}$ of X such that for each $Y \in \mathscr{U}$, there is a homeomorphism $h : M \to M$ that takes orbits of X in orbits of Y preserving orientation, that is, if $p \in M$ and $\delta > 0$, there exists $\varepsilon > 0$, such that for $0 < t < \delta$, $h(X_t(p)) = Y_{t'}(h(p))$ for some $0 < t' < \varepsilon$.

One can consider only the asymptotic part of the system.

Definition 3.5.2 *We say that a vector field $X \in \mathscr{X}^1(M)$ is Ω-stable if there exists a C^1 neighborhood $\mathscr{U}$ of X such that for each $Y \in \mathscr{U}$, there is a homeomorphism $h : \Omega(X) \to \Omega(Y)$ that takes orbits of X in orbits of Y preserving orientation, that is, if $p \in \Omega(X)$ and $\delta > 0$, there exists $\varepsilon > 0$, such that for $0 < t < \delta$, $h(X_t(p)) = Y_{t'}(h(p))$ for some $0 < t' < \varepsilon$.*

On both cases, the map h is called a conjugation. In other words, a vector field is structurally stable (Ω-stable) if there is a neighborhood on which every vector field is conjugated to it.

Next, we will state the Franks lemma, this lemma shows us that the star condition is intimately related to the Ω-stability. In [Fra71], Franks proved in the setting of diffeomorphism that Ω-stability implies the star condition. For that, Franks created a lemma for obtaining a perturbation of the original diffeomorphism by means of its derivative. Moreover, the perturbation obtained is linear in exponential coordinates. Here, we state it for singularities and for periodic orbits in the flow setting.

Theorem 3.5.3 *(Frank's lemma for singularities) Let $X \in \mathscr{X}^1(M)$ and $\sigma \in Sing(X)$. Then, for every C^1-neighborhood $\mathscr{U}$ of X, there exist $\delta > 0$ and $\varepsilon > 0$ such that if $L : T_pM \to T_pM$ is a linear map satisfying $||DX(\sigma) - L|| < \delta$, then there exists $Y \in \mathscr{U}$ and $r > \varepsilon$ such that*

$$Y(x) = (D_{exp^{-1}_{\sigma(x)}} exp_\sigma) \circ L \circ exp^{-1}_\sigma(x), \quad x \in B_\varepsilon(\sigma)$$
$$Y(x) = X(x), \quad x \in M \setminus B_r(\sigma)$$

Now, consider a regular point $x \in M$. Let Σ be a cross section at p. A *tube of radius ε centred at p* is the image of $B_\varepsilon(p) \cap \Sigma$ by the flow action.

Theorem 3.5.4 *(Frank's lemma for periodic orbits) Let $X \in \mathscr{X}^1(M)$, $p \in Per(X)$, and $P : \Sigma \to \Sigma$ be the Poincaré map associated to p, where Σ is a suitable cross section. Consider a C^1-neighborhood $\mathscr{U}$ of X. Then, given $\varepsilon > 0$ there exist $\delta > 0$ such that if $L : N_p \to N_p$ is a linear isomorphism satisfying $||DP(p) - L|| < \delta$, then there exists $Y \in \mathscr{U}$ such that*

- *$Y(x) = X(x)$, if x does not belong to the tube centred at p with radius ε.*
- *$p \in Per(X)$.*
- *If $P_Y : \Sigma \to \Sigma$ is the Poincaré map for Y, then*

$P_Y(x) = exp_p \circ L \circ exp^{-1}_p(x)$, if $x \in B_\varepsilon(p) \cap \Sigma$.

$P_Y(x) = P(x)$, if $x \notin B_r(p) \cap \Sigma$, for $r > \varepsilon$ sufficiently close to ε.

With this version of the Frank's lemma, one can prove the following:

Proposition 3.5.5 *If $X \in \mathscr{X}^1(M)$ has a non-hyperbolic critical element, then X is not structurally stable.*

For the proof, we recommend the reader to see [ASS]. Similarly, one can prove Frank's Lemma for the case on which the critical point is a singularity.

The above proposition tells us that a structurally stable vector field can only have hyperbolic critical elements. Since structural stability is an open property, this implies that all structurally stable vector field has the star property. However, during the proof we only use the conjugation with sufficiently small vector fields to prove that a vector field with infinitely periodic orbits of a certain period can not be conjugated to a vector field that possesses only finitely many periodic orbits with such a period. However, this prevents these vector fields to be Ω-conjugated. Thus, we have the following result:

Theorem 3.5.6 *Every Ω-stable vector field has the star property.*

Now, let us return our attention to the stability conjecture. Recall, that is is concerned about to answer the following question:

$$\Omega\text{-}stability \Rightarrow Axiom\ A\ plus\ no\ cycle\ condition?$$

By theorem 3.5.6, one can try to solve the conjecture by using star flows. Next theorem, due to Gan-Wen [GW06] give a positive answer when the flow has no singularities.

Theorem 3.5.7 *Every non-singular star flow is Axiom A without cycles.*

However, if the flow has singularities, then not necessarily the conjecture holds. Next, we give some examples to illustrate that. All examples mentioned below are pictorial, for more details see [ASS]

3.5.1 Some Examples

Exemple 1: Loss of Hyperbolicity

Our first example, is the most famous example of star flow that fails to be Axiom A, the Lorentz attractor. Since it has two singularities that can be approximated by periodic orbits, and since for a hyperbolic set the singularities must be isolated, we have that the non-wandering set is not hyperbolic.

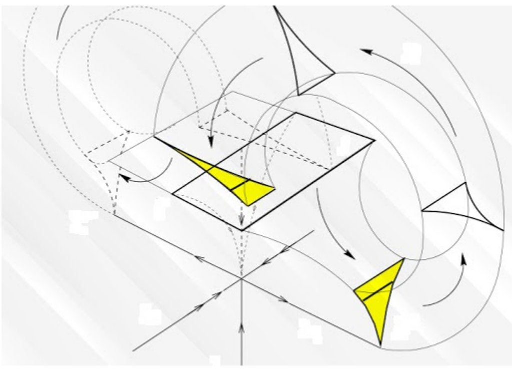

Figure 3.3: Lorentz Attractor

Example 2: Existence of cycles

This first example describes a star flow that is Axiom A with cycles. Figure 3.4 shows the example. Here, we are considering a 3-manifold, and the vector field is Morse-Smale far from the part represented in the picture. We have three singularities $\sigma_1, \sigma_2, \sigma_3$, all of them with index 2. We also have three sinks, p_1, p_2, p_3. The cycle is formed so that the unstable manifold of one of the singularities goes to the stable manifold of the other, according to the figure. The torsion together with the sinks p_i are used to prove that the whole cycle is wandering. For instance, note what happens when we evolve the ball B in the figure. The ball is divided in three parts, one of them goes to σ_1, since is contained on its unstable manifold. The part B_1, goes to p_1, and due to the λ-lemma, b_2 follows the unstable manifold of σ_1, and due o the torsion, it goes to the sink p_2. This shows that there is no recurrence in B. A similar argument can be used in the other connections. Thus, this example is Axiom A with cycles. For this example to be star, one just need that the singularities are Lorentz-like. Then, the maximal invariant set that contains the cycle sectional hyperbolic, and since we are in dimension three, this implies the star property inside the maximal invariant. Finally, since outside of the open set that contains the cycle the vector field is Morse-Smale, we also have the star property. The same argument can also be done in higher dimension. For dimension two, see the example of Li and Wen [LW95].

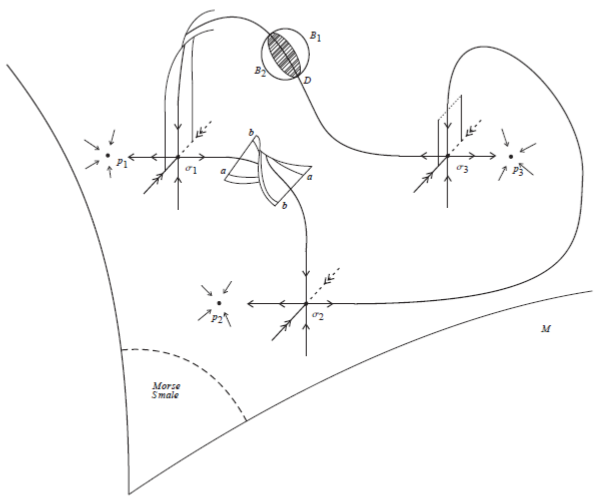

Figure 3.4: Existence of Cycles

Example 3: $\Omega(X) \neq \overline{Per(X)}$

Here, we have an example of a star flow on which the closure of the periodic points differs of the non-wandering set. As the previous one, far from the figure the dynamics is Morse-Smale on a 3-manifold. Now, we have a singularity σ_1 with index 2, a periodic orbit $\mathscr{O}(q)$ that s saddle type, and two singularities σ_1 and σ_2 that are sinks. Again, we have a torsion in the unstable manifold of σ_1. Also note that the unstable manifold of σ_1 goes to the stable manifold of the periodic and vice-versa. Similarly to the previous example, one can prove that the closure of the recurrent points differs of the non-wandering set. In particular $\Omega(X) \neq \overline{Per(X)}$. By the same argument used in the previous example, we have that the system is star.

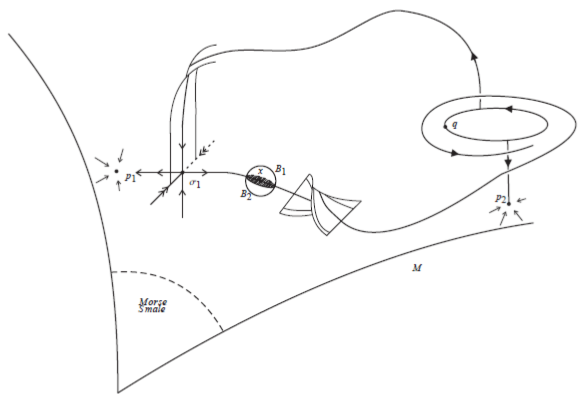

Figure 3.5: $\Omega(X) \neq \overline{Per(X)}$

3.5.2 Domination

A weaker form of hyperbolicity is the domination property. Now we will present some results about domination which will be important for this work.

Definition 3.5.8 *Let X be a C^1-vector field on M and Λ be a compact invariant set. We say that a DX_t-invariant splitting*

$$T_\Lambda M = E \oplus F$$

on Λ is a dominated splitting, if there are constants $C \geq 1, \lambda > 0$ such that

$$||DX_t \mid_{E_x} ||.||DX_{-t} \mid_{F_{X_t(x)}} || \leq Ce^{-\lambda t} \quad \forall x \in \Lambda, \quad \forall t \geq 0 \tag{3.2}$$

Next, we present another way to write equation (3.2) . Recall, if A is a linear map, then $m(A) := ||A^{-1}||^{-1}$ denotes the mini-norm of A.

Since $X_{-t}(x) = (X_t(x))^{-1}$, it follows that (3.2) is equivalent to

$$||DX_t \mid_{E_x} ||.||(DX_t \mid_{F_x})^{-1}|| \leq Ce^{-\lambda t} \quad \forall x \in \Lambda, \quad \forall t \geq 0$$

and since

$$||(DX_t\mid_{F_x})^{-1}|| = \frac{1}{m(DX_t|_{F_x})}$$

we have that (3.2) is equivalent to

$$\frac{||DX_t\mid_{E_x}||}{m(DX_t|_{F_x})} \leq Ce^{-\lambda t} \quad \forall x \in \Lambda, \quad \forall t \geq 0$$

Definition 3.5.8 is equivalent to say that there exists $T > 0$ such that

$$||DX_T\mid_{E_x}||.||DX_{-T}\mid_{F_{X_T(x)}}|| \leq \frac{1}{2} \quad \forall x \in \Lambda \tag{3.3}$$

Indeed, if (3.2) holds, since $Ce^{-\lambda t} \to 0$ as $t \to \infty$, there exists $t_0 > 0$ such that for every $T \geq t_0$, we have $||DX_T\mid_{E_x}||.||DX_{-T}\mid_{F_{X_T(x)}}|| \leq Ce^{-\lambda T} \leq 1/2$. To see that (3.3) implies (3.2) we only have to consider

$$\lambda = \frac{1}{T}\log(2) \quad and \quad C = \left(\sup_{t\in[0,T]} ||DX_t\mid_E||\right).\left(\sup_{t\in[0,T]} ||DX_{-t}\mid_F||\right)$$

Now, let us see what does domination means. Consider a vector $v \in T_xM$ with $x \in \Lambda$. Then v can be written as $v = v_{E_x} + v_{F_x}$, where $v_{E_x} \in E_x$ and $v_{F_x} \in F_x$. Suppose both coordinates of v are nonzero. By the chain rule, one has

$$||DX_{kT}(x)v_{E_x}||.||DX_{-kT}(x)v_{F_x}|| = \left|\left|\prod_{j=1}^{k} DX_T(X_{(k-j)T}(x))v_{Ex}\right|\right|.\left|\left|\prod_{j=2}^{k+1} DX_{-T}(X_{-(k-j)T}(x))v_{Fx}\right|\right|$$

by domination

$$\left|\left|\prod_{j=1}^{k} DX_T(X_{(k-j)T}(x))v_{Ex}\right|\right|.\left|\left|\prod_{j=2}^{k+1} DX_{-T}(X_{-(k-j)T}(x))v_{Fx}\right|\right| \leqslant \frac{1}{2^k}$$

Then, since $||DX_{-t}\mid_{F_{X_t(x)}}|| = \frac{1}{m(DX_t|_{F_x})}$, we have

$$\frac{||DX_{kT}(x)v_{F_x}||}{||DX_{kT}(x)v_{E_x}||} \geqslant 2^k$$

In other words, it means that the coordinate v_F grows much faster than the coordinate v_E, which means that the vector $DX_{kT}(x)v$ is converging to the direction F in the future. If we did the same for the past we obtain that that the vector is converging to E. Thus domination has the property to take any vector to F in the future and to E in the past.

In the beginning of this subsection, we presented domination as a weaker form of hyperbolicity. To see that, let Λ be a compact invariant set for $X \in \mathscr{X}^1(M)$. By definition of hyperbolic set, there is a splitting $T_\Lambda M = E^s \oplus \langle X \rangle \oplus E^u$ and two constants $C > 0$, $\lambda > 0$ such that

$$||DX_t|_{E_x^s}|| \leq Ce^{-\lambda t} \quad and \quad ||DX_{-t}|_{E_x^u}|| \leq Ce^{-\lambda t}$$

Let $E^c = \langle X \rangle \oplus E^u$. If a vector v belongs to $T_\Lambda M$, then it can be written as $v = v^s + v^c$, where $v^s \in E^s$, $v^c \in E^c$, and $v^c = v^X + v^u$. Since M is compact, there is $K > 0$ such that $||\langle X(x) \rangle|| \leq K \; \forall \; x \in M$. Thus

$$||DX_t(x)v^s|| \leq Ce^{-\lambda t}||v^s||$$

and

$$\begin{aligned} ||DX_{-t}(X_t(x))v^c|| &= ||DX_{-t}(X_t(x))(v^X + v^u)|| \\ &= ||DX_{-t}(X_t(x))v^X|| + ||DX_{-t}(X_t(x))v^u|| \\ &= ||X(x)|| + ||DX_{-t}(X_t(x))v^u|| \\ &\leq K + Ce^{-\lambda t}||v^u|| \end{aligned}$$

Hence

$$||DX_t(x)|_{E_x^s}||.||DX_{-t}(X_t(x))|_{E^c_{X_t(x)}}|| \leq Ce^{-\lambda t}(K + Ce^{-\lambda t}) \leq \widetilde{C}e^{-\lambda t}$$

For some constant $\widetilde{C} > 0$. Therefore, hyperbolicity implies domination.

Definition 3.5.9 *Let $\Lambda \subset M \setminus Sing(X)$ be a compact X_t-invariant set. We say that the linear Poincaré flow P_t is hyperbolic over Λ if there are a P_t-invariant continuous splitting $N_\Lambda = E^s \oplus E^u$ of the normal bundle and constants $C > 0$, $\lambda > 0$ such that*

$$||P_t(x)|_{E^s_x}|| \leq Ce^{-\lambda t} \quad and \quad ||P_{-t}(x)|_{E^u_x}|| \leq Ce^{-\lambda t} \quad \forall x \in \Lambda, \quad t \geq 0$$

Now, we introduce the concept of domination for the linear Poincaré flow.

Definition 3.5.10 *Let $\Lambda \subset M \setminus Sing(X)$ be an X_t-invariant set. We say that the linear Poincaré flow P_t has a dominated splitting over Λ, if there exist a P_t-invariant splitting $N = E \oplus F$ of the normal bundle, and constants $C \geq 1$ and $\lambda > 0$ such that for all $t \geq 0$ and for all $x \in \Lambda$, one has*

$$||P_t|_{E_x}||.||P_{-t}|_{F_{X_t(x)}}|| \leq Ce^{-\lambda t}$$

As in the case of the actual flow, the above relation is equivalent to say that there exists $T > 0$ such that

$$||P_T|_{E_x}||.||P_{-T}|_{F_{X_T(x)}}|| \leq 1/2$$

Next, we give two criteriums of hyperbolicity for the linear Poincaré flow. First, assuming that Λ is compact, we have the following

Lemma 3.5.11 *Suppose that $\Lambda \subset M \setminus Sing(X)$ is a compact set. Let $N_\Lambda = E \oplus F$ be an P_t-invariant splitting. If there is $T_0 > 0$ such that for all $x \in \Lambda$ there are $1 < t(x), s(x) \leq T_0$ such that*

$$||P_{t(x)}|_E|| \leq \frac{1}{2} \quad and \quad ||P_{-s(x)}|_F|| \leq \frac{1}{2}$$

then, the linear Poincaré flow is hyperbolic over Λ.

Proof. For every T, it follows from the chain rule that

$$\begin{aligned}||P_T(x)|_E|| &= ||P_{(T-t(x))+t(x)}(x)|_E|| \\ &= ||P_{T-t(x)}(X_{t(x)}(x)).P_{t(x)}(x)|_E|| \\ &\leq ||P_{T-t(x)}(X_{t(x)}(x))|_E||.||P_{t(x)}(x)|_E|| \\ &\leq \frac{1}{2}||P_{T-t(x)}(X_{t(x)}(x))|_E||\end{aligned}$$

Proceeding by induction, we obtain

$$||P_T(x)|_E|| \leq \left(\frac{1}{2}\right)^{\left[\frac{T}{T_0}\right]} ||P_{T-l}(X_l(x))|_E||$$

where $l > 0, T - l < T_0$ and [.] denotes the entire part. By compacity, one has

$$C := \max\{||P_s(x)|_E|| : x \in \Lambda, s \in [0, T_0]\} < \infty$$

Therefore,

$$||P_t(x)|_E|| \leq C\left(\frac{1}{2}\right)^{\left[\frac{T}{T_0}\right]} = Ce^{-\lambda t}$$

For some $\lambda > 0$. Similarly we obtain an estimative for the subbundle F.

□

The following theorem asserts the equivalence between the hyperbolicity of Λ as a nonsingular compact invarian set of X and the hyperbolicity of the linear Poincaré flow over Λ.

Theorem 3.5.12 *(Hyperbolic Lemma) Let $X \in \mathscr{X}^1(X)$ and Λ be a nonsingular compact invariant set. Then Λ is hyperbolic if and only if Λ is hyperbolic with respect to the linear Poincaré flow.*

For a proof, see for instance [BM]

3.5.3 Liao's Inequalities

Let us return our attention to the star flows again. Recall that, by definition there exists a neighborhood $\mathscr{U}$ of X such that all periodic orbits of a vector field $Y \in \mathscr{U}$ are hyperbolic. A priori it does not seem to imply any kind of uniform strength of contraction and expansion on the periodic orbits. However, the star property is sufficient to guarantee this. Moreover, it guaranties domination on the closure of the periodic orbits.

Denote by N_x^j the normal projection of the subspace E_x^j for $j = s, u$ where x is a point on a hyperbolic periodic orbit. For a given subspace $A \subset N_x$, where $N_x = \langle X(x) \rangle^{\perp}$. Define

$$\eta_-(X, A, t) = \sup_{u \in A, ||A||=1} \log ||P_{X,t}(u)|| \quad \textit{and} \quad \eta_+(X, A, t) = \inf_{u \in A, ||A||=1} \log ||P_{X,t}(u)||$$

Theorem 3.5.13 *(Liao's Inequalities) Let $X \in \mathscr{X}^*(M)$. Then there exists a C^1-neighborhood $\mathscr{U}$ of X, together with two uniform constants $\eta > 0$ and $T_0 > 1$, such that for every $Y \in \mathscr{U}$ one has:*

1. Whenever x is a point on a periodic orbit of Y in $\mathscr{U}$ and $T_0 < t < \infty$, then

$$\frac{1}{t}[\eta_+(Y, N_x^u, t) - \eta_-(Y, N_x^s, t)] \geq 2\eta \tag{3.4}$$

2. Whenever $\mathscr{O}$ is a periodic orbit of Y with period $\pi(\mathscr{O})$, $x \in \mathscr{O}$, and whenever an integer $m \geq 1$ and a partition $0 = t_0 < \cdots < t_l = m\pi(\mathscr{O})$ of $[0, m\pi(\mathscr{O})]$ are given, with $t_k - t_{k-1} \geq \pi(\mathscr{O})$ for $k = 1, 2, \cdots, l$. Then

$$\frac{1}{m\pi(\mathscr{O})}\sum_{k=0}^{l-1}\eta_-(Y, N^s_{X_{t_k(x)}}, t_{k+1-t_k}) \leq -\eta \quad \textit{and} \quad \frac{1}{m\pi(\mathscr{O})}\sum_{k=0}^{l-1}\eta_+(Y, N^u_{X_{t_k(x)}}, t_{k+1-t_k}) \geq \eta \tag{3.5}$$

Let us explain the meaning of these inequalities.

We have that if the linear Poincaré flow is dominated, then there are constants $C > 0$ and $\lambda > 0$ such that

$$||P_t|_{N_x^s}||.||P_{-t}|_{N^u_{X_{t(x)}}}|| \leq Ce^{-\lambda t}$$

but

$$||P_{-t}|_{N^u}|| = \frac{1}{m(P_t|_{N^u})}$$

Then from the domination inequality, assuming $C = 1$, we obtain

$$||P_t|N^s||.\frac{1}{m(P_t|_{N^u})} \leq e^{-\lambda t}$$

which implies

$$m(P_t|_{N^u}).||P_t|N^s||^{-1} \geq e^{\lambda t}$$

Dividing both sides by $1/t$ and taking logarithms,

$$\frac{1}{t}\log[m(P_t|_{N^u}).||P_t|_{N^s}||^{-1}] \geq \frac{1}{t}\log(e^{\lambda t})$$

By logarithm properties and the definition of η_+ and η_-,

$$\frac{1}{t}[\log(m(P_t|_{N^u})) - \log(||P_t|_{N^s}||)] = \frac{1}{t}[\eta_+(X, N^u_x, t) - \eta_-(X, N^s_x, t)] \geq \lambda$$

By taking $\lambda = 2\eta$ we obtain the first item of the theorem. Indeed, everything we did can be done in the reverse direction, that is, from the first item of the theorem we can obtain uniform domination for the linear Poincaré flow on periodic orbits of star flows. Moreover, we obtain uniform domination in a neighborhood of the system.

Now, let us see what the second inequality of the theorem means. Suppose that x is a periodic point satisfying the inequalities of the second item. Then we have

$$\frac{1}{m\pi(\mathscr{O})}\sum_{k=0}^{l-1}\eta_-(Y, N^s_{X_{t_k(x)}}, t_{k+1-t_k}) \leq -\eta$$

Which implies

$$\sum_{k=0}^{l-1}\eta_-(Y, N^s_{X_{t_k(x)}}, t_{k+1-t_k}) \leq -\eta m\pi(\mathscr{O})$$

By taking the exponential on both sides and using the definition of η_- we obtain

$$||P_{Y,(t_{k+1}-t_k)}|_{N^s_{Y_{t_k}(x)}}|| \cdots ||P_{t_1}|_{N^s_x}|| \leq e^{-\eta m\pi(\mathscr{O})}$$

By applying the chain rule we obtain

$$||P_{m\pi(\mathscr{O})}|_{N_x^s}|| \le ||P_{Y,(t_{k+1}-t_k)}|_{N^s_{Y_{t_k}(x)}}|| \cdots ||P_{t_1}|_{N_x^s}|| \le e^{-\eta m \pi(\mathscr{O})}$$

Thus, we can see that the inequality of the second item of the theorem actually implies that there is some uniform contraction for the linear Poincaré flow along periodic orbits. Similarly one can see that there is some uniform expansion for the linear Poincaré flow.

By the above considerations, taking $t_k - t_{k-1} = \pi(\mathscr{O})$, one has that the inequalities (3.4) and (3.5) can be rewritten as

$$\frac{||P_t^Y|_{N_x^s}||}{m(P_t^Y|_{N_x^u})} \le e^{-2\eta t}, \quad \forall t \ge T_0, \forall x \in \mathscr{O} \tag{3.6}$$

$$\prod_{i=0}^{\left[\frac{\pi(\mathscr{O})}{T_0}\right]-1} ||P_{T_0}^Y|_{N^s(X^Y_{iT_0}(x))}|| \le e^{-\eta\pi(\mathscr{O})}, \qquad \prod_{i=0}^{\left[\frac{\pi(\mathscr{O})}{T_0}\right]-1} m(P_{T_0}^Y|_{N^u(X^Y_{iT_0}(x))}) \ge e^{\eta\pi(\mathscr{O})}, \quad \forall x \in \mathscr{O} \tag{3.7}$$

3.6 The Extended Linear Poincaré Flow

One of the main difficulties of studying flows is the existence of singularities. In this section, we introduce a tool that allows us to study hyperbolic like properties of sets with singularities. in the previous section, was introduced a key tool to study hyperbolic like properties of nonsingular flows, the linear Poincaré flow. But as we saw, it is only defined on regular orbits. Using ideas from Liao [Lia96] and Li-Gan-Wen [LGW05] we define the extended linear Poincaré flow which allow us the better understanding of the dynamical properties near singularities.

Denote by $SM = \{e \in TM : ||e|| = 1\}$ the unit sphere bundle of M and let $\pi : SM \to M$ be the bundle projection defined by $\pi(e) = x$ if $e \in SM \cap T_xM = S_xM$. The tangent flow DX_t induces a flow $\Phi_t : SM \to SM$ given by

$$\Phi_t(x)(e) = \frac{DX_t(x)(e)}{||DX_t(x)(e)||}$$

For every $e \in S_xM$. Let

$$N_e = \{v \in T_{\pi(e)}M : v \perp e\}$$

be the normal space of e. Denote

$$\mathscr{N} = \mathscr{N}_{SM} = \bigcup_{e \in SM} N_e$$

Then $\mathscr{N}$ is a dim(M)-1 vector bundle over the basis space SM.

Definition 3.6.1 *We define the extended linear Poincaré flow* $\Psi_t : \mathscr{N}_{S_x M} \to \mathscr{N}_{S_{X_t(x)} M}$ *by*

$$\Psi_t(e_x, v) = (\Phi_t(e_x), \Pi_{\Phi_t(e_x)}(DX_t(x)v))$$

Here, $\Pi_{\Phi_t(e_x)}(DX_t(x)v))$ denotes the orthogonal projection of the vector $DX_t(x)v$ over the unit vector $\Phi_t(e_x)$.

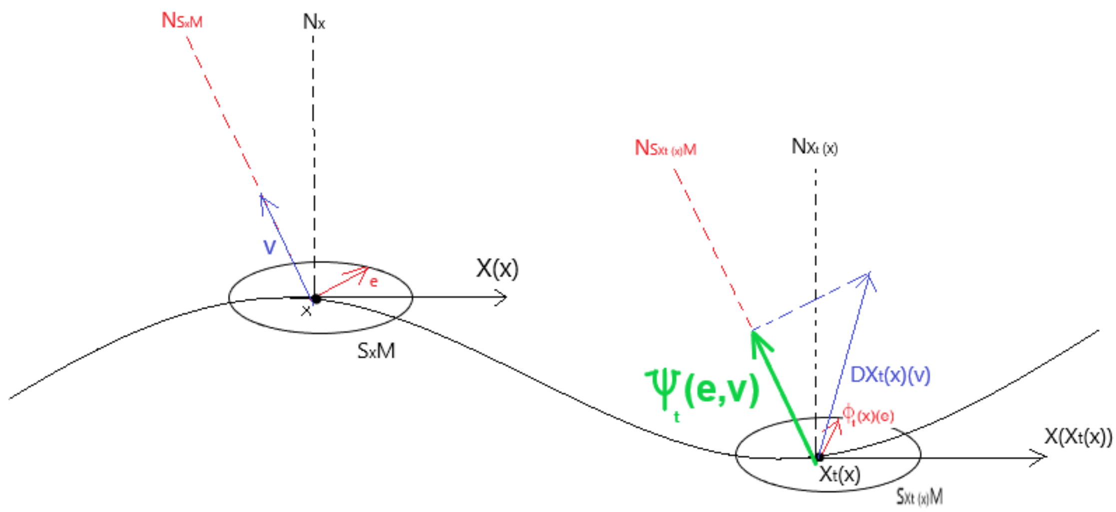

Figure 3.6: Extended Linear Poincaré Flow

Since Ψ_t is an orthogonal projection, it is given by the following formula

$$\Psi_t(e)(v) = DX_t(x)(v) - \frac{\langle DX_t(x)(v), \Phi_t(x)(e)\rangle}{||\Phi_t(x)(e)||^2}\Phi_t(x)(e) \quad v \in N_e, \quad e \in S_x M$$

Now, let Λ be a compact invariant set. We define its transgression by

$$\widetilde{\Lambda} = \overline{\{X(x)/||X(x)|| : x \in \Lambda \setminus Sing(X)\}}$$

Remark 1: $\widetilde{\Lambda}$ is a compact Φ_t-invariant set.

Indeed, Since $\widetilde{\Lambda}$ is a closed subset of a compact set, it is a compact set. For invariance, we shall prove $\Phi_t(\widetilde{\Lambda}) = \widetilde{\Lambda}$. Let $e \in \Phi_t(\widetilde{\Lambda})$, there exists $\widetilde{e} \in \widetilde{\Lambda}$ such that $e = \Phi_t(\widetilde{e})$. By the definition of $\widetilde{\Lambda}$, there is $\{x_n\} \subset \Lambda \setminus Sing(X)$ such that $\lim_{n\to\infty} \frac{X(x_n)}{||X(x_n)||} = \widetilde{e}$. Then,

$$
\begin{aligned}
e &= \Phi_t\left(\lim_{n\to\infty} \frac{X(x_n)}{||X(x_n)||}\right) \\
&= \lim_{n\to\infty} \Phi_t\left(\frac{X(x_n)}{||X(x_n)||}\right) \\
&= \lim_{n\to\infty} \frac{DX_t(x_n)\left(\frac{X(x_n)}{||X(x_n)||}\right)}{\left|\left|DX_t(x_n)\left(\frac{X(x_n)}{||X(x_n)||}\right)\right|\right|} \\
&= \lim_{n\to\infty} \frac{\frac{1}{||X(x_n)||}}{\frac{1}{||X(x_n)||}} \frac{DX_t(x_n)(X(x_n))}{||DX_t(x_n)(X(x_n))||} \\
&= \lim_{n\to\infty} \frac{X(X_t(x_n))}{||X(X_t(x_n))||}
\end{aligned}
$$

Thus, $e \in \widetilde{\Lambda}$. This proves $\Phi_t(\widetilde{\Lambda}) \subset \widetilde{\Lambda}$. For the other inclusion, let $e \in \widetilde{\Lambda}$. We shall prove that there is $\widetilde{e} \in \widetilde{\Lambda}$ such that $e = \Phi_t(\widetilde{e})$. Since $e \in \widetilde{\Lambda}$, we have that there is a sequence $\{x_n\} \subset \Lambda \setminus Sing(X)$ such that $\lim_{n\to\infty} \frac{X(x_n)}{||X(x_n)||} = e$. Taking subsequences if necessary, we may assume that $\lim_{n\to\infty} \frac{X(X_{-t}(x_n))}{||X(X_{-t}(x_n))||}$ converges to a unit vector $\widetilde{e} \in \widetilde{\Lambda}$. Thus

$$
\begin{aligned}
\Phi_t(\widetilde{e}) &= \Phi_t\left(\lim_{n\to\infty} \frac{X(X_{-t}(x_n))}{||X(X_{-t}(x_n))||}\right) \\
&= \lim_{n\to\infty} \Phi_t\left(\frac{X(X_{-t}(x_n))}{||X(X_{-t}(x_n))||}\right) \\
&= \lim_{n\to\infty} \frac{DX_t(X_{-t}(x_n)))\left(\frac{X(X_{-t}(x_n)))}{||X(X_{-t}(x_n)))||}\right)}{\left|\left|DX_t(X_{-t}(x_n)))\left(\frac{X(X_{-t}(x_n)))}{||X(X_{-t}(x_n)))||}\right)\right|\right|} \\
&= \lim_{n\to\infty} \frac{X(x_n))}{||X(x_n))||} \\
&= e
\end{aligned}
$$

Therefore, $\Phi_t(\widetilde{\Lambda}) = \widetilde{\Lambda}$.

Remark 2: If $x \in \Lambda \setminus Sing(X)$, then by the continuity of X, for any sequence $\{x_n\} \subset \Lambda \setminus Sing(X)$ converging to x we have $\frac{X(x_n)}{||X(x_n)||} \to \frac{X(x)}{||X(x)||}$ as $n \to \infty$. In other words, on $\widetilde{\Lambda}$

there is only one unit vector associated to a regular value, the unit vector along the vector field direction.

If $x \in \Lambda \setminus Sing(X)$, then it follows from remark 2 that

$S_xM \cap \widetilde{\Lambda} = \frac{X(x)}{||X(x)||}$, $N_{S_xM} = N_x$ and $\Phi_t(x)\left(\frac{X(x)}{||X(x)||}\right) = \frac{X(\phi_t(x))}{||X(\phi_t(x))||}$

Thus, for any $v \in N_{S_xM}$ one has

$$\Psi_t(x)(v) = DX_t(x)(v) - \frac{\left\langle DX_t(x)(v), \frac{X(X_t(x))}{||X(X_t(x))||}\right\rangle}{\left|\left|\frac{X(X_t(x))}{||X(X_t(x))||}\right|\right|^2} \frac{X(X_t(x))}{||X(X_t(x))||} = P_t(x)(v)$$

In other words, the extended linear Poincaré flow Ψ_t over the subset $\{X(x)/||X(x)|| : x \in \Lambda \setminus Sing(X)\} \subset SM$ coincides with the linear Poincaré flow P_t over $\Lambda \setminus Sing(X)$.

Hereafter, we only consider the extended linear Poincaré flow restricted to the set $\widetilde{\Lambda}$.

Lemma 3.6.2 *The extended linear Poincaré flow Ψ_t varies continuously with respect to the vector field X, the time t and the vector v.*

Lemma 3.6.3 *If $N_{\Lambda \setminus Sing(X)} = E \oplus F$ is a dominated splitting with respect to the linear Poincaré flow P_t on an invariant set Λ, then the extended linear Poincaré flow admits a dominated splitting $\mathscr{N}_{\widetilde{\Lambda} \setminus Sing(X)} SM = \widetilde{E} \oplus \widetilde{F}$, where $\widetilde{E}$ and $\widetilde{F}$ are the lifts of E and F respectively.*

Proof. By the definition of dominated splitting, there are constants $C \geq 1$ and $\lambda > 0$ such that for any $x \in \Lambda \setminus Sing(X)$ and any fixed $t > 0$, one has

$$||P_t|_{E_x}||.||P_{-t}|_{F_{X_t(x)}}|| \leq Ce^{-\lambda t}$$

Thus, by the previous lemma, we have that on the set $\Gamma = \{X(x)/||X(x)|| : x \in \Lambda \setminus Sing(X)\} \subset SM$ the above inequality is still holding for the lifts $\widetilde{E}$ and $\widetilde{F}$. Thus, the bundles E and F can be extended on the closure of Γ, which is $\widetilde{\Lambda}$.

□

Chapter 4

Ergodic Theory

As we mentioned in the introduction, one of the main historic motivations for the dynamical systems theory was the n-body problem. This is an example of ordinary differential equations with the following remarkable property: If one considers the flow generated by the equations solutions and consider, by instance, the evolution of a cube through this flow, then its volume will not change through time, though its shape may change. using this property, Poincaré proved his well-known recurrence theorem, which asserts that to most of the initial data, the system returns to a condition arbitrarily close to the initial one.

In this chapter we review some crucial results on ergodic theory.

4.1 Ergodic Theory

Let $\mathscr{B}$ be the Borel σ-algebra of M and $X \in \mathscr{X}^1(M)$. A probability measure μ on $(M, \mathscr{B})$ is called a X_t-invariant measure if for any $A \in \mathscr{B}$ we have $\mu(A) = \mu(X_{-t}(A))$, for all $t \in \mathbb{R}$. Denote by $\mathscr{M}$ the set of all X_t-invariant borelian probability measures. It is known from the theory that $\mathscr{M}$ is a nonempty compact set for the *weak**-topology.

Let δ_x be the Dirac measure on $(M, \mathscr{B})$ associated with x. that is

$$\delta_x(A) = \begin{cases} 1, & \text{se } x \in \text{A} \\ 0, & \text{se } x \notin \text{A} \end{cases}$$

As consequence of the compactness of $\mathscr{M}$ we have the following theorem.

Theorem 4.1.1 *Let $x \in M$. Then any accumulation point for the weak*-topology of the set of the probability measures*

$$\left\{ \frac{1}{T} \int_0^T \delta_{X_s(x)}\, ds \right\}_{T>0}$$

is a X_t-invariant measure. In particular we have that $\mathscr{M}(X)$ is nonempty.

Similarly, a continuous map $f : M \to M$ is said to be f-invariant with respect to μ if $\mu(A) = \mu(f^{-1}(A))$ for any $A \in \mathscr{B}$. such as in the continuous case, $\mathscr{M}$ is a nonempty compact set for the *weak**-topology.

One property that is enjoyed by all measure-preserving transformations is recurrence:

Theorem 4.1.2 *(Poincaré's Recurrence Theorem) Let $f : M \to M$ be a measure preserving transformation of a probability space $(X, \mathscr{B}, \mu)$. Let $E \in \mathscr{B}$ with $m(E) > 0$. Then almost all points E return infinitely many often to E under positive iteration by f (i.e there exists $F \subset E$ with $m(E) = m(F)$ such that for each $x \in F$ there is a sequence $n_1 < n_2 < \cdots$ of natural numbers with $f^{n_i}(x) \in F$ for each i).*

Remark: The above theorem is false if a measure space of infinite measure is used. An example is given by the map $f(x) = x + 1$ defined on $\mathbb{R}$ with the Lebesgue measure m. In this case, there is no subset of positive measure such that its positive iterates return to itself infinitely often times, since the set of recurrent points is just $\{0\}$ and $m(\{0\}) = 0$.

Ergodicity

Let $(X, \mathscr{B}, \mu)$ be a probability space and $f : X \to X$ be a measure-preserving transformation. If $f^{-1}(B) = B$ for $B \in \mathscr{B}$, then also $f^{-1}(X \setminus B) = X \setminus B$ and we could study f by studying the two simpler transformations $f|_B$ and $f|_{X \setminus B}$. If $0 < \mu(B) < 1$ this has simplified the study of f. If $\mu(B) = 0$ (or $\mu(X \setminus B) = 0$) we can ignore B or $(X \setminus B)$ and we have not significantly simplified f since neglecting a set of zero measure is allowed in measure theory. This raises the idea of studying those sets that cannot be decomposed as above and of trying to express every measure-preserving transformation in terms of these indecomposable ones. The indecomposable transformations are called ergodic.

Definition 4.1.3 *Let $(X, \mathscr{B}, \mu)$ be a probability space. A measure-preserving transformation f of $(X, \mathscr{B}, \mu)$ is called ergodic if the only members B of $\mathscr{B}$ with $f^{-1}(B) = B$ satisfy $\mu(B) = 0$ or $\mu(B) = 1$.*

Remark 4.1: If $(X, \mathscr{B}, \mu)$ is probability space and $f : X \to X$ is a measurable-preserving, then

f is ergodic $\Leftrightarrow$ whenever φ is measurable and $(\varphi \circ f)(x) = f(x)\ \forall\, x \in X$ then f is constant a.e.

For a proof, see for instance [Wal82].

Analogously, if μ is an X_t-invariant probability measure, we say that μ is ergodic if for every $B \in \mathscr{B}$ such that $X_{-t}(B) = B$ then $\mu(B) = 0$ or 1.

As mentioned in the introduction, the first major result in ergodic theory was proved by G.D. Birkhoff. Now we will state it.

Theorem 4.1.4 *(Birkhoff's Ergodic Theorem). Suppose μ is a f-invariant probability measure. Then, μ-a.e. $x \in M$ and for every $\varphi \in L^1(\mu)$ we have that the limit*

$$\varphi^*(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))$$

always exist. Moreover, $\varphi^* \in L^1(\mu)$ and

$$\int_M \varphi^* \, d\mu = \int_M \varphi \, d\mu$$

As an application of the theorem we have the following:

If μ is ergodic, then

$$\varphi^* = \int_M \varphi \, d\mu$$

Indeed, by the theorem we have

$$\varphi^*(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))$$

$$\varphi^*(f(x)) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^{j+1}(x))$$

Subtracting one equation from the other, we obtain

$$\varphi^*(x) - \varphi^*(f(x)) = \lim_{n\to\infty} \frac{1}{n}(\varphi(x) - \varphi(f^n(x)) = 0$$

Since $\varphi \in L^1(\mu)$. This implies $\varphi^*(x) = \varphi^*(f(x))$. Therefore, it follows from [Remark 4.1] that φ^* is constant. Thus,

$$\varphi^*\mu(M) = \varphi^* \int_M d\mu = \int_M \varphi^* \, d\mu = \int_M \varphi \, d\mu$$

and since $\mu(M) = 1$, one has

$$\varphi^* = \int_M \varphi \, d\mu$$

Since in this work we are dealing with flows, we will use the ergodic theorem when $f = X_T$ for some fixed $T > 0$.

4.2 Entropy of a Measure-Preserving Transformation

In 1958 Kolmogorov introduced the concept of entropy into ergodic theory, and this has been the most successful invariant so far. The definition of entropy of a measure preserving transformation f of $(X, \mathscr{B}, \mu)$ is in three stages: the entropy of a finite sub-σ-algebra of $\mathscr{B}$, the entropy of the trasnformation f relative to a finite sub-σ-algebra, and, finally, the entropy of f.

Thoughout this section $(X, \mathscr{B}, \mu)$ will denote a probability space.

Definition 4.2.1 *A partition of* $(X, \mathscr{B}, \mu)$ *is a disjoint collection of elements of* $\mathscr{B}$ *whose union is* X*.*

Given a partition $\xi = \{A_1, \cdots, A_k\}$, define the set

$$f^{-1}\xi = \{f^{-1}A_1, \cdots, f^{-1}A_k\}$$

Further, given two partitions $\xi = \{A_1, \cdots, A_k\}$ and $\eta = \{B_1, \cdots, B_m\}$, we define their refinement as

$$\xi \vee \eta = \{A_i \cap B_j : i = 1 \cdots, k, j = 1, \cdots, m, \mu(A_i \cap B_j) > 0\}$$

With these two constructions, we may define the following refinement:

$$\bigvee_{i=0}^{n-1} f^{-i}\xi = \xi \vee f^{-1}\xi \vee \cdots \vee f^{-(n-1)}\xi$$

The entropy of a partition ξ is defined as the number

$$H(\xi) = -\sum_{i=1}^{k} \mu(A_i) \log \mu(A_i)$$

The measure theoretic entropy with respect to a partition ξ is defined as

$$h_\mu(f, \xi) = \lim_{n\to\infty} \frac{1}{n} H\Big(\bigvee_{i=0}^{n-1} f^{-i}\xi \Big)$$

Finally, the measure-theoretic entropy is defined as

$$h_\mu(\xi) = \sup_{\xi} h_\mu(f, \xi)$$

4.3 Topological Entropy

Adler, Konheim, and McAndrew [AKM65] introduced topological entropy as an invariant of topological conjugation. To each continuous transformation $f : X \to X$ of a compact topological space a non-negative real number or ∞, denoted by $h_{top}(f)$ is assigned. Later Dinaburg and Bowen gave a new but equivalent definition. In this section we give the definition of topological entropy using separated sets. This was done by Dinaburg and Bowen, but Bowen also gave the definition when the space is not compact. we will give the definition when X is a compact metric space.

Let d be a metric on X. It induces a family of metrics $\{d_n\}$ on X given by

$$d_n(x, y) := \max_{0\le i\le n-1} d(f^i(x), f^i(y))$$

where each metric d_n measures the distance between the orbit arcs $\{x, \cdots, f^{n-1}(x)\}$ and $\{y, \cdots, f^{n-1}(y)\}$. So, for each fixed $n \geq 1$ we define the *dynamical ball* with center x and radius r as

$$B(x, n, r) = \{y \in X : d_n(x, y) < r\}$$

that is, it is the open ball in the metric d_n

Definition 4.3.1 *Let $n \in \mathbb{N}$, and $\varepsilon > 0$. A subset $E \subset X$ is said to be (n, ε)-separated with respect to f if $x, y \in E$, $x \neq y$, implies $d_n(x, y) =: \max_{0\le i\le n-1} d(f^i(x), f^i(y)) > \varepsilon$.*

We denote by $S_n(\varepsilon)$ the cardinality of the (n,ε)-separated set with respect to f of maximum cardinality.

Remarks

- $S_n(\varepsilon) < \infty$.
- If $\varepsilon_1 < \varepsilon_2$ then $S_n(\varepsilon_1) \geq S_n(\varepsilon_2)$.

The second item of the above remark is easy to see. For the first item, denote by $E_n(\varepsilon)$ the (n,ε)-separated set of maximum cardinality. We claim that

$$\bigcup_{x \in E_n(\varepsilon)} B(x,n,\varepsilon)$$

covers X. Indeed, if not, one can find a point $y \in X$ such that $y \notin B(x,n,\varepsilon)$ for every $x \in E_n(\varepsilon)$. Hence, by the definition of $E_n(\varepsilon)$, for each $x \in E_n(\varepsilon)$ there is $0 \leq j \leq n-1$ such that $d(f^j(y), f^j(x)) > \varepsilon$, that is, $y \in E_n(\varepsilon)$, which is a contradiction. The claim is then proved. Now, suppose $S_n(\varepsilon) = \infty$. Since X is compact, there are points $x_1, \cdots, x_k \in E_n(\varepsilon)$ such that $X = \bigcup_{i=1}^{k} B(x_i,n,\varepsilon)$. For the pigeonhole principle, this i such that $B(x_i,n,\varepsilon)$ contains more than one point of $E_n(\varepsilon)$. Contradicting the separability of $E_n(\varepsilon)$. Therefore, $S_n(\varepsilon) < \infty$.

Definition 4.3.2 *We define the topological entropy of f as the quantity*

$$h_{top}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S_n(\varepsilon)$$

As an example of the calculation of the topological entropy, consider an isometry $f : X \to X$. Then, $d_n = d_1$ for all $n \in \mathbb{N}$ so that $S_n(\varepsilon) = S_1(\varepsilon)$ and then $h_{top}(f) = 0$.

Remark: Katok defined the metric entropy $h_\mu(f)$ of an f-invariant ergodic measure μ as

$$h_\mu(f) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log S_n(\varepsilon, \delta)$$

where $S_n(\varepsilon,\delta)$ is the minimal number of ε-balls in the d_n metric covering the set of measure larger than or equal to $1-\delta$.

Theorem 4.3.3 *(Variational Principle): Let $f : X \to X$ be a continuous map of a compact metric space. Then $h_{top}(T) = \sup\{h_\mu(T) : \mu \in \mathscr{M}(X)\}$.*

Remarks:

1. $h(f) = \sup\{h_\mu(f) : \mu \in \mathscr{E}(X)\}$, where $\mathscr{E}$ denotes the set ergodic measures.

2. $h(f) = h(f|_{\Omega(f)})$.

For a proof of the remarks, see, for instance [Wal82].

For the flow X_t generated by $X \in \mathscr{X}^1(M)$, we define its topological entropy as the entropy of the time $t = 1$, that is, $h(X) = h(X_1)$.

4.4 Differentiable Ergodic Theory

In this section we will study the notion of hyperbolicity in a more general way, on which the contraction and the expansion rates are not necessarily constants.

4.4.1 Lyapunov Exponents

Consider a diffeomorphism $f : M \to M$. A point $x \in M$ is said to be regular if there exist real numbers $\lambda_1(x) < \lambda_2(x) < \cdots < \lambda_k(x)$ and a splitting

$$T_xM = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_k(x)$$

such that

$$\lim_{n\to\infty} \frac{1}{n} \log ||Df^n(x)v|| = \lambda_j(x)$$

for every $0 \neq v \in E_j(x)$, with $j = 1, 2, \cdots, k$. The numbers $\lambda_j(x)$ are called the Lyapunov exponents of f on x and the above splitting is called the Oseledec splitting of f on x. It is possible to prove that the Lyapunov exponents and the Oseledec splitting are uniquely determined on a regular point x, for instance see [Mañ87].

Let Λ be the set of regular points of M. A natural question, is if Λ is always non-empty. The following theorem gives us a satisfactory answer.

Theorem 4.4.1 *(Oseledec) If M is compact, the set of regular points of a diffeomorphism $f : M \to M$ has total measure.*

Moreover, if μ is an invariant ergodic measure, one can prove that μ-a.e. x, the Lyapunov exponents are constants.

Theorem 4.4.2 *(Margulis-Ruelle's inequality) Let $f : M \to M$ be a C^1 diffeomorphism on a compact Riemannian manifold M, $\mu \in \mathscr{M}(M, f)$. Then*

$$h_\mu(f) \leq \int \sum_{i=0}^{k} \lambda_i^+(x) dim E_i(x)\, d\mu$$

where λ_i^+ are the positive Lyapunov exponents and $E_i(x)$ is the eigenspace associated to $\lambda_i(x)$.

Remark:

If μ is ergodic, then the Lyapunov exponents are constants, therefore

$$\begin{aligned} h_\mu(f) \leq \int \sum_{i=0}^{k} \lambda_i^+(x) dim E_i(x)\, d\mu &= \sum_{i=0}^{k} \lambda_i^+(x) dim E_i(x) \int d\mu \\ &= \sum_{i=0}^{k} \lambda_i^+(x) dim E_i(x) \mu(M) \\ &= \sum_{i=0}^{k} \lambda_i^+(x) dim E_i(x) \end{aligned}$$

When we are dealing with flows, , we define the Lyapunov exponents by means of the linear Poincaré flow P_t. More precisely, let $X \in \mathscr{X}^1(M)$. A point $x \in M$ is said to be regular if there are numbers $\lambda_1(x) < \lambda_2(x) < \cdots < \lambda_k(x)$ and a splitting $N_x = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_k(x)$ of the normal bundle such that

$$\lim_{t \to \infty} \frac{1}{t} \log ||P_t(x)v|| = \lambda_j(x), \quad \forall v \in E_j(x) \setminus \{0\}$$

Next, we give two important definitions about ergodic measures.

Definition 4.4.3 *We say that an invariant ergodic measure μ is regular, if it is not supported on a singularity.*

Definition 4.4.4 *We say that an invariant ergodic measure μ is hyperbolic, if for μ-a.e. x, the Lyapunov exponents of the linear Poincaré flow P_t are non-zero.*

For a regular hyperbolic ergodic measure μ we can rewrite the splitting $N = \bigoplus_{i=1}^{k} E_i$ as $N = E^s \oplus E^u$, where all Lyapunov exponents along E^s are negative and all Lyapunov exponents along E^u are positive. We call the splitting $N = E^s \oplus E^u$ the hyperbolic Oseledec splitting with respect to the hyperbolic ergodic measure μ.

We could also define the Lyapunov exponents of a flow by using the tangent flow DX_t as usual. However, for ergodic measures that are not supported on singularities, there will always be one zero Lyapunov exponent for the tangent flow along the flow direction. Indeed, if the Lyapunov exponents ae defined by using the tangent flow DX_t, then we have to consider the direction generated by the flow on its Oseledec splitting, that is

$$T_xM = E_1(x) \oplus \cdots E_k(x) \oplus \langle X(x) \rangle$$

The Lyapunov exponent along the flow direction is

$$\lambda_X = \lim_{t\to\infty} \frac{1}{t} \log ||DX_t(x)(X(x))|| = \lim_{t\to\infty} \frac{1}{t} \log ||X(X_t(x))||$$

Since M is compact, there is $C > 0$ such that $||X(x)|| \leq C$, $\forall\ x \in M$. This implies that there exists constant $C' > 0$ such that $-C' \leq \log ||X(x)|| \leq C'$ $\forall\ x \in M$. Thus

$$-\lim_{t\to\infty} \frac{1}{t} C' \leq \lim_{t\to\infty} \frac{1}{t} \log ||X(X_t(x))|| \leq \lim_{t\to\infty} \frac{1}{t} C'$$

Therefore, $\lambda_X = 0$.

Lemma 4.4.5 *(Definition of the transgression of a measure) If μ is an ergodic X_t-invariant measure on M with $\mu(Sing(X)) = 0$, then there exists an ergodic Φ_t-invariant measure $\widetilde{\mu}$ on $\widetilde{\Lambda}$, the transgression of $\Lambda = Supp(\mu)$, such that the Lyapunov exponents of the extended linear Poincaré flow Ψ_t with respect to the measure $\widetilde{\mu}$ are the same as the Lyapunov exponents of the linear Poincaré flow with respect to the measure μ. The measure $\widetilde{\mu}$ is called the transgression of μ.*

Proof. Let $\pi : SM \to M$ be the projection, which is a continuous surjection with $\pi(v) = x$ for any $x \in S_xM$. Take the measure $\widetilde{\mu} = \mu \circ \pi$ on $\widetilde{\Lambda}$. Let $\widetilde{A} \subset \widetilde{\Lambda}$ be a Borel subset and $A = \pi(\widetilde{A})$. Since μ is X_t-invariant, for every $t \in \mathbb{R}$, we have $\widetilde{\mu}(\Phi_{-t}(\widetilde{A})) = \mu(\pi(\Phi_{-t}(\widetilde{A}))) = \mu(X_{-t}(A)) = \mu(A) = \mu(\pi(\widetilde{A})) = \widetilde{\mu}(\widetilde{A})$, that is, $\widetilde{\mu}$ is Φ_t-invariant. Now, we will prove that $\widetilde{\mu}$ is ergodic. For, suppose $\Phi_{-t}(\widetilde{A}) = \widetilde{A}$ for every $t \in \mathbb{R}$, we shall prove that $\widetilde{\mu}(\widetilde{A}) = 0$ or 1 . The equality $\Phi_{-t}(\widetilde{A}) = \widetilde{A}$ implies that $\pi(\Phi_{-t}(\widetilde{A})) = \pi(\widetilde{A})$, that is, $X_{-t}(A) = A$ for every $t \in \mathbb{R}$. Since μ is an ergodic X_t-invariant measure, $\widetilde{\mu}(\widetilde{A}) = \mu(\pi(\widetilde{A})) = \mu(A) = 0$ or 1. It means that $\widetilde{\mu}$ is an ergodic Φ_t-invariant measure.

Applying the Oseledec theorem to the linear Poincaré flow P_t, for μ-a.e.x, there is a splitting $N_x = E_1 \oplus E_2 \oplus \cdots \oplus E_m$ and numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ such that

$$\lim_{t\to\infty} \frac{1}{t} \log ||P_t(x)(v)|| = \lambda_i, \quad \forall v \in E_i \setminus \{0\}, \quad i = 0, 1, \cdots, m$$

Since the extended linear Poincaré flow coincides with the usual linear Poincaré flow on regular points, one has

$$\lim_{t\to\infty} \frac{1}{t} \log ||\Psi_t(x)(v)|| = \lim_{t\to\infty} \frac{1}{t} \log ||P_t(x)(v)|| = \lambda_i$$

Therefore, the Lyapunov exponents of Ψ_t with respect to the measure $\widetilde{\mu}$ are the same as the Lyapunov exponents of the linear Poincaré flow with respect to the measure μ.

□

Definition 4.4.6 *Let $X \in \mathscr{X}^1(M)$. A point $x \in M \setminus Sing(X)$ is said to be strongly closable if for any C^1-neighborhood $\mathscr{U}$ of X and any $\delta > 0$, there are $Y \in \mathscr{U}$, $z \in M$, $\tau > 0$, and $T > 0$ such that the following conditions are satisfied:*

1. $Y_\tau(z) = z$
2. $d(X_t(x), Y_t(z)) < \delta$ *for any* $t \in [0, \tau]$
3. $X = Y$ *on* $M \setminus \bigcup_{t\in[-T,0]} B(X_t(x), \delta)$

The set of all strongly closable points for X will be denoted by $\Sigma(X)$.

Theorem 4.4.7 *(Ergodic Closing Lemma). Let $X \in \mathscr{X}^1(M)$. Then for any X_t-invariant probability measure μ one has that*

$$\mu(Sing(X) \cup \Sigma(X)) = 1$$

Theorem 4.4.8 *(Shi-Gan-Wen) If μ is an ergodic measure of a star flow, then μ is a hyperbolic measure.*

Proof. If μ is supported on a critical element, then from the definition of star flow, it should be hyperbolic. So for the rest of the proof, we may assume that μ does not support on any critical element. We will use the ergodic closing lemma to show that μ is hyperbolic. Applying Lemma 4.4.7, we may assume $x \in B(\mu) \cap supp(\mu) \cap \Sigma(X)$. By Definition 4.4.6 there are $X_n \in \mathscr{X}^1(M)$, $x_n \in M$, $\tau_n > 0$ such that

- $X^{X_n}_{\tau_n}(x_n) = x_n$, where τ_n is the minimal period of x_n.

- $d(X^X_t(x), X^{X_n}_t(x_n)) < \frac{1}{n}$, $\forall\ 0 < t < \tau_n$

- $||X_n - X||_{C^1} < \frac{1}{n}$

Here, $B(\mu)$ denotes the set of generic points of μ. Recall that x is a generic point of μ if for any continuous function $f : M \to \mathbb{R}$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X_t(x))\, dt = \int f\, d\mu$$

That is, is the set of points that satisfies the Birkhoff's ergodic theorem.

Consider the ergodic measure μ_n which is supported on the orbit of x_n. Since x is strongly closable, for any continuous function f, one has

$$\lim_{n\to\infty} \int f\, d\mu_n = \lim_{n\to\infty} \frac{1}{\tau_n}\int_0^{\tau_n} f(X^{X_n}_t(x_n))\, dt = \lim_{n\to\infty} \frac{1}{\tau_n}\int_0^{\tau_n} f(X_t(x))\, dt = \int f\, d\mu$$

Since μ is not supported on any critical element, one has $\mu_n \to \mu$ and $\tau_n \to \infty$ as $n \to \infty$.

From Lemma 3.5.13, we know that for any $x \in \mathscr{O}(x_n)$, there are constants $\eta > 0$ and $T_0 > 0$ such that for sufficiently large $n \in \mathbb{N}$, and for the natural hyperbolic splitting $N_{\mathscr{O}(x_n)} = N^s \oplus N^u$ with respect to the linear Poincaré flow, one has

$$\frac{||P^{X_n}_t|_{N^s_x}||}{m(P^{X_n}_t|_{N^u_x})} \le e^{-2\eta t}, \quad \forall t \ge T_0, \forall x \in \mathscr{O}(x_n) \tag{4.1}$$

$$\prod_{i=0}^{\left[\frac{\tau_n}{T_0}\right]-1} ||P^{X_n}_{T_0}|_{N^s(X^{X_n}_{iT_0}(x_n))}|| \le e^{-\eta\tau_n}, \qquad \prod_{i=0}^{\left[\frac{\tau_n}{T_0}\right]-1} m(P^{X_n}_{T_0}|_{N^u(X^{X_n}_{iT_0}(x_n))}) \ge e^{\eta\tau_n}, \quad \forall x \in \mathscr{O}(x_n) \tag{4.2}$$

Taking logarithm in (4.2), we obtain

$$\sum_{i=0}^{\left[\frac{\tau_n}{T_0}\right]-1} \log ||P^{X_n}_{T_0}|_{N^s(X^{X_n}_{iT_0}(x_n))}|| \le -\eta\tau_n \quad \textit{and} \quad \sum_{i=0}^{\left[\frac{\tau_n}{T_0}\right]-1} \log m(P^{X_n}_{T_0}|_{N^u(X^{X_n}_{iT_0}(x_n))}) \ge \eta\tau_n$$

Dividing both sides by τ_n

$$\frac{1}{\tau_n}\sum_{i=0}^{\left[\frac{\tau_n}{T_0}\right]-1}\log||P_{T_0}^{X_n}|_{N^s(X_{iT_0}^{X_n}(x_n))}|| \leq -\eta \quad and \quad \frac{1}{\tau_n}\sum_{i=0}^{\left[\frac{\tau_n}{T_0}\right]-1}\log m(P_{T_0}^{X_n}|_{N^u(X_{iT_0}^{X_n}(x_n))}) \geq \eta$$

By the definition of the extended linear Poincaré flow, we have

$$\frac{1}{\tau_n}\sum_{i=0}^{\left[\frac{\tau_n}{T_0}\right]-1}\log||\Psi_{T_0}^{X_n}|_{N^s(X_{iT_0}^{X_n}(x_n))}|| \leq -\eta \quad and \quad \frac{1}{\tau_n}\sum_{i=0}^{\left[\frac{\tau_n}{T_0}\right]-1}\log m(\Psi_{T_0}^{X_n}|_{N^u(X_{iT_0}^{X_n}(x_n))}) \geq \eta$$

Since the extended linear Poincaré flow varies continuously with respect to t, X and v, we have

$$\frac{1}{\tau_n}\sum_{i=0}^{\left[\frac{\tau_n}{T_0}\right]-1}\log||\Psi_{T_0}^{X}|_{N^s(X_{iT_0}^{X}(x))}|| \leq -\eta \quad and \quad \frac{1}{\tau_n}\sum_{i=0}^{\left[\frac{\tau_n}{T_0}\right]-1}\log m(\Psi_{T_0}^{X}|_{N^u(X_{iT_0}^{X}(x))}) \geq \eta$$

Then,

$$\int \log||\Psi_{T_0}^{X}|_{N_x^s}||\, d\widetilde{\mu} \leq -\eta \quad and \quad \int \log m(\Psi_{T_0}^{X}|_{N_x^u})\, d\widetilde{\mu} \geq \eta$$

Where $\widetilde{\mu}$ is the transgression of μ. This proves that μ is hyperbolic for X.

□

Chapter 5

The Main Theorem

Now, we recall the statement of the main result of this work. Let M be a boundaryless compact smooth Riemannian manifold.

Theorem B [WYZ19]: *There is a residual set $\mathscr{R} \subset \mathscr{X}^1(M)$ such that for any $X \in \mathscr{R}$, one has*

$$\limsup_{T\to\infty} \frac{1}{T} \log \#P_T(X) \geq h_{top}(X) := h_{top}(X_1)$$

One of the main difficulties for proving the above theorem is the presence of singularities. Flows with singularities have rich and complicated dynamics such as the *Lorenz attractor*. At singularities, one can not define the linear Poincaré flow. Hence we lose some compactness properties. Even there is no singularities, we are not able to use the usual Pesin theory, since the vector field is only C^1. Adittionaly, one may have "shear" for flows. This is sharp in this work since we have to control the periods by the nature of this text. In the proof of the main theorem, we consider two cases. When the vector field is star, and when it is not.

The next lemma give us the precise construction of such residual set $\mathscr{R} \subset \mathscr{X}^1(M)$.

Lemma 5.0.1 *There is a residual $\mathscr{R} \subset \mathscr{X}^1(X)$ such that for given $T, k \in \mathbb{N}$, if for every C^1 neighborhood $\mathscr{U}$ of $X \in \mathscr{R}$, there is $Y \in \mathscr{U}$ having k periodic orbits whose periods belong to $(\frac{T}{2}, \frac{3T}{2})$, then X has k periodic orbits whose periods belong to $(\frac{T}{2}, \frac{3T}{2})$.*

Proof. Fix a countable basis $\mathscr{B} = \{B_1, B_2, \cdots, B_i, \cdots\}$ of M. Let $\{U_1, U_2, \cdots, U_n, \cdots\}$ be the family of finite unions of the elements of $\mathscr{B}$. We define

$\mathscr{H}^k_{n,T} := \Big\{X \in \mathscr{X}^1(M) : X$ has k hyperbolic periodic orbits with period belonging to $\left(\frac{T}{2}, \frac{3T}{2}\right)$ in $U_n\Big\}$

$\mathscr{N}^k_{n,T} := \Big\{ X \in \mathscr{X}^1(M) : \exists\ C^1$ neighborhood $\mathscr{U}$ of X, such that for any $Y \in \mathscr{U}$, either Y has no k periodic orbits with periods belonging to $\left(\frac{T}{2}, \frac{3T}{2}\right)$ or all k periodic orbits with period belonging to $\left(\frac{T}{2}, \frac{3T}{2}\right)$ of Y are not in U_n $\Big\}$

By definition, the set $\mathscr{N}^k_{n,T}$ is open. By the stability of the hyperbolicity, $\mathscr{H}^k_{n,T}$ is open.

We claim that $\overline{\mathscr{H}^k_{n,T} \cup \mathscr{N}^k_{n,T}} = \mathscr{X}^1(M)$

Indeed, one only needs to prove that $\overline{\mathscr{H}^k_{n,T} \cup \mathscr{N}^k_{n,T}} \supset \mathscr{X}^1(M)$. Let $X \in \mathscr{X}^1(M)$. We shall prove that there exists a sequence in $\mathscr{H}^k_{n,T} \cup \mathscr{N}^k_{n,T}$ converging to X. If $X \in \mathscr{N}^k_{n,T}$, then $X \in \mathscr{N}^k_{n,T} \subset \mathscr{H}^k_{n,T} \cup \mathscr{N}^k_{n,T} \subset \overline{\mathscr{H}^k_{n,T} \cup \mathscr{N}^k_{n,T}}$ and therefore the Claim is proved. If $X \notin \mathscr{N}^k_{n,T}$, then by definition of $\mathscr{N}^k_{n,T}$ we have that for any C^1 neighborhood $\mathscr{U}$ of X , there is $Y \in \mathscr{U}$ which has k periodic orbits whose periods belong to $\left(\frac{T}{2}, \frac{3T}{2}\right)$ belonging to U_n. Thus, there is a sequence of C^1 vector fields $\{X_m\}_{m\in\mathbb{N}} \subset \mathscr{H}^k_{n,T} \subset \mathscr{H}^k_{n,T} \cup \mathscr{N}^k_{n,T}$ such that $X_m \overset{m\to\infty}{\longrightarrow} X$. Therefore, $X \in \overline{\mathscr{H}^k_{n,T} \cup \mathscr{N}^k_{n,T}}$. This proves the Claim.

Consequently, $\mathscr{H}^k_{n,T} \cup \mathscr{N}^k_{n,T}$ is open and dense in $\mathscr{X}^1(M)$. Let

$$\mathscr{R} = \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcap_{T=1}^{\infty} (\mathscr{H}^k_{n,T} \cup \mathscr{N}^k_{n,T})$$

It is clear that $\mathscr{R}$ is a residual subset of $\mathscr{X}^1(M)$. We will prove that $\mathscr{R}$ is the desired residual set.

Given $T > 0$ and $k \in \mathbb{N}$, let $X \in \mathscr{R}$ arbitrary. By definition of $\mathscr{R}$, $X \in \mathscr{H}^k_{n,T} \cup \mathscr{N}^k_{n,T}$ for all $n \in \mathbb{N}$. That is, or $X \in \mathscr{H}^k_{n,T}$ or $X \in \mathscr{N}^k_{n,T}$. If $X \in \mathscr{H}^k_{n,T}$, by definition, X has k hyperbolic periodic orbits whose periods belong to $\left(\frac{T}{2}, \frac{3T}{2}\right)$. Now, it only remains to show that $X \in \mathscr{N}^k_{n,T}$ implies that X has k hyperbolic periodic orbits whose periods belong to $\left(\frac{T}{2}, \frac{3T}{2}\right)$. If not, there exists $n_0 \in \mathbb{N}$ such that $X \notin \mathscr{N}^k_{n_0,T}$. Therefore, $X \in \mathscr{H}^k_{n_0,T}$, i.e, X has k periodic orbits whose period belong to $\left(\frac{T}{2}, \frac{3T}{2}\right)$.

□

5.1 The non star case

If the generic vector field is not star, based on the fact that the vector field can be approximated by periodic orbits whose periods turn arbitrarily large, it will follow from the Frank's Lemma and from the definition of the residual $\mathscr{R}$ that the upper limit of the growth rate of periodic orbits is infinity. Since we are concerned about proving that the growth rate is larger than or equal to the entropy, we may assume that $h_\mu(X) > 0$, since otherwise the inequality would be trivially satisfied.

Theorem 5.1.1 *For a residual set $\mathscr{R} \subset \mathscr{X}^1(X)$ as in Lemma 5.0.2, if $X \in \mathscr{R}$ is not star, then the growth rate of the periodic orbits is infinity, that is*

$$\limsup_{T\to\infty} \frac{1}{T} \log \# P_T(X) = +\infty$$

Proof. Consider the residual $\mathscr{R} \subset \mathscr{X}^1(M)$ as in Lemma 5.0.2. If any $X \in \mathscr{R}$ is not star, then for any C^1 neighborhood $\mathscr{U}_n$ of X, there is $X_n \in \mathscr{U}_n$, such that X_n has a non-hyperbolic periodic orbit x_n. That is, there are sequences $\{X_n\} \to X$, $\{x_n\} \subset M$, and $\{\tau_n : \tau_n > 0\}$ with $X^{X_n}_{\tau_n}(x_n) = x_n$. Proceeding as in Theorem 4.4.8, one has that $\tau_n \to \infty$ as $n \to \infty$. Given $\varepsilon > 0$, consider $\delta = \delta(x_n) > 0$ given by the Frank's Lemma. For each $n \in \mathbb{N}$, since x_n is non-hyperbolic, one has that $P^{X_n}_{\tau_n}$ has an eigenvalue on the complex unit circle. Thus, there is a linear map $L_n : N_{x_n} \to N_{x_n}$, δ-close to $P^{X_n}_{\tau_n}$ having an eigenvalue λ such that it is an j-th root of unit, for some $j \in \mathbb{N}$. By Frank's Lemma, there is a vector field $Y_n \in \mathscr{U}_n$ such that $x_n \in Per(Y_n)$ whose Poincaré map $P_{Y_n} : \Sigma \to \Sigma$ is conjugated to the linear map L_n by the exponential map on $B_\varepsilon(x_n) \cap \Sigma$. Pick an eigenvector v of $L_n : N_{x_n} \to N_{x_n}$ associated to λ, taking $|s|$ small enough so that $exp_{x_n}(sv) \in B_\varepsilon(x_n) \cap \Sigma$, we have

$$\begin{aligned} P^j_{Y_n}(exp_{x_n}(sv)) &= exp_{x_n} \circ L^j_n \circ exp^{-1}_{x_n}(exp_{x_n}(sv)) \\ &= exp_{x_n} \circ L^j_n(sv) \\ &= exp_{x_n}(s\lambda^j v) \\ &= exp_{x_n}(sv) \end{aligned}$$

Therefore, the Poincaré map of Y_n, and consequently Y_n itself has infinite periodic points with period j. Since the neighborhood $B_\varepsilon(x_n) \cap \Sigma$ can be taken arbitrarily small, by the continuity of the flow we may assume that Y_n has infinite periodic points with period τ_n. In particular, Y_n has at least $e^{n.2\tau_n}$ periodic orbits whose period belongs to $\left(\frac{[\tau_n]}{2}, \frac{3[\tau_n]}{2}\right)$. By Lemma 5.0.2, X has at least $e^{n.2\tau_n}$ periodic orbits whose period belongs to $\left(\frac{[\tau_n]}{2}, \frac{3[\tau_n]}{2}\right)$. That is, for n large $\# P_{2\tau_n}(X) \geq e^{n.2\tau_n}$. Consequently, $\frac{1}{2\tau_n} \log \# P_{2\tau_n}(X) \geq n$. Taking limits we obtain

$$\limsup_{n\to\infty} \frac{1}{2\tau_n} \log \# P_{2\tau_n}(X) = +\infty$$

Therefore,

$$\limsup_{T\to\infty} \frac{1}{T} \log \# P_T(X) = +\infty$$

□

5.2 The star case

For star vector fields, we have to steps. First based on that any regular ergodic measure of a star vector field is hyperbolic (Theorem 4.4.8), we show that the hyperbolic Oseledec splitting is a dominated splitting (Theorem 5.2.1). Secondly, we prove that if the hyperbolic Oseledec splitting with respect to a regular hyperbolic measure is a dominated splitting, then the growth rate of the periodic orbits is larger than or equal to the metric entropy (Theorem 5.2.2).

Theorem 5.2.1 *If μ is a regular ergodic invariant measure of a C^1 star vector field X with $h_\mu(X) > 0$, then its Oseledec splitting $N = E^s \oplus E^u$ is a dominated splitting.*

First we will give the outline of the proof.

Outline of the proof. By Theorem 4.4.8, the measure μ is hyperbolic. Since the metric entropy is positive, we may assume that the measure μ is not supported on a any critical element. Then, it follows from the ergodic closing lemma that μ-a.e. $x \in M$ is strongly closable. By definition of strongly closable x can be approximated by periodic orbits. We will then show that these periodic orbits are hyperbolic of saddle type, since otherwise it would follow from the Ruelle's inequality that the metric entropy is zero, a contradiction. So, based on that these periodic orbits orbits are hyperbolic saddles, we will apply Lemma 4.4.5 (transgression of a measure) to obtain the dominated splitting.

Proof of theorem 5.2.1. According to theorem 4.4.8, μ is hyperbolic. Let $x \in B(\mu) \cap Supp(\mu) \cap \Sigma(X)$ and let x_n as in the previous theorem. We claim that there are only finite sinks or sources among $\{orb(x_n)\}$. Indeed, if not, we may assume that $orb(x_n)$ are sinks, then we only have

$$\prod_{i=0}^{\left[\frac{\tau_n}{T_0}\right]-1} ||P_{T_0}^{X_n}|_{N^s(X_{iT_0}^{X_n}(x_n))}|| \le e^{-\eta\tau_n}$$

Thus, as we saw before, it implies that the Lyapunov exponents of the linear Poincaré flow P_t are negative. By the Ruelle inequality we get that $h_\mu(X_{T_0}) = 0$. Since μ is an ergodic measure, $h_\mu(X_{T_0}) = |T_0|h_\mu(X_1) = |T_0|h_\mu(X) > 0$. This is a contradiction. The claim is thus proved.

It follows from the above claim and from Theorem 3.5.13, that for the non-trivial hyperbolic splitting $N_{orb(x_n)} = E^s \oplus E^u$ with respect to the linear Poincaré flow $P_t^{X_n}$,

$$\prod_{i=0}^{\left[\frac{\tau_n}{T_0}\right]-1} ||P_{T_0}^{X_n}|_{N^s(X_{iT_0}^{X_n}(y))}|| \le e^{-\eta\tau_n}, \quad \prod_{i=0}^{\left[\frac{\tau_n}{T_0}\right]-1} m(P_{T_0}^{X_n}|_{N^u(X_{iT_0}^{X_n}(y))}) \ge e^{\eta\tau_n}, \quad \forall y \in orb(x_n)$$

We may assume that the indices of $orb(x_n)$ are the same. Then, there is a dominated splitting $N_x = F^s_x \oplus F^u_x$ on $x = \lim_{n\to\infty} x_n$, where $F^s_x = \lim_{n\to\infty} E^s_{x_n}$ and $F^u_x = \lim_{n\to\infty} E^u_{x_n}$. We shall prove that $F^s_x = E^s_x$ and $F^u_x = E^u_x$. As we saw before, the above inequalities means that

$$\int \log ||\Psi^X_{T_0}|_{N^s_x}||\, d\widetilde{\mu} \le -\eta \quad and \quad \int \log m(\Psi^X_{T_0}|_{N^u_x})\, d\widetilde{\mu} \ge \eta$$

According to the Birkhoff ergodic theorem and Lemma 4.4.5, one has

$$\lim_{m\to\infty} \frac{1}{m}\sum_{i=0}^{m-1} \log ||P^X_{T_0}|_{F^s_{X_{iT_0}(x)}}|| = \lim_{m\to\infty} \frac{1}{m}\sum_{i=0}^{m-1} \log ||\Psi^X_{T_0}|_{F^s_{X_{iT_0}(x)}}|| = \int \log ||\Psi^{X_n}_{T_0}|_{F^s_x}||\, d\widetilde{\mu} \le -\eta < 0$$

Since E^s is the finest subbundle where the Lyapunov exponents are negative, it follows that $F^s_x \subset E^s_x$. Similarly one can prove that $F^u_x \subset E^u_x$. To prove the reverse inclusion we proceed by contradiction. So, if $E^s_x \nsubseteq F^s_x$, then there exists a non-zero vector v belonging to E^s_x but not belonging to F^s_x. Since $N_x = F^s_x \oplus F^u_x$, one has that the vector v can be written as $v = v_1 + v_2$, where $v_2 \neq 0$. By domination, for t sufficiently large, one has that the coordinate v_1 is insignificant in comparison with the coordinate v_2, therefore

$$\lim_{t\to\infty} \frac{1}{t} \log ||P^X_t(v)|| = \lim_{t\to\infty} \frac{1}{t} \log ||P^X_t(v_2)|| = \lim_{t\to\infty} \frac{1}{t} \log ||\Psi^X_t(v_2)|| \ge \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} \log m(\Psi^X_{T_0}|_{F^u_{X_{iT_0}(x)}}) > 0$$

This contradicts the fact that the Lyapunov exponents along E^s are negative. Consequently one has that $E^s_x \subset F^s_x$. Therefore, $E^s_x = F^s_x$. Similarly we can prove that $E^u_x = F^u_x$.

□

Theorem 5.2.2 *Let μ be a regular invariant ergodic measure of $X \in \mathscr{X}^1(M)$. If the hyperbolic Oseledec splitting $N = E^s \oplus E^u$ is a dominated splitting, then*

$$\limsup_{T\to\infty} \frac{1}{T} \log \#P_T(X) \ge h_\mu(X) := h_\mu(X_1)$$

For this theorem, we have to deal with the re-parametrization problem. In Liao's shadowing lemma (Theorem 5.3.2), the period of the periodic point which shadows the recurrent point is a re-parametrization of the recurrent time. For our goal, we have to estimate the difference between the recurrent time and its re-parametrization (Proposition 5.3.3).

5.3 A shadowing lemma with time control

For the linear Poincaré flow, one has the shadowing lemma of Liao for some quasi hyperbolic orbit segments.

Definition 5.3.1 *Assume that $\Lambda \subset M \setminus Sing(X)$ is an invariant (not necessarily compact) set having a dominated splitting $N_\Lambda = E \oplus F$ with respect to the linear Poincaré flow. Given $\eta > 0$ and $T_0 > 0$, an orbit arc $X_{[0,T]}(x) \subset \Lambda$ with $T > T_0$ is (η, T_0)-quasi hyperbolic (associated to Λ) if there is a time partition $0 = t_0, t_1 < t_2 < \cdots < t_l = T$ with $t_{i+1} - t_i \leqslant T_0$, $i = 0, \cdots, l-1$ such that for $k = 1, \cdots, l-1$, one has*

$$\prod_{i=0}^{k-1} ||P^*_{t_{i+1}-t_i}|_{E_{X_{t_i}(x)}}|| \leq e^{-\eta t_k} \quad and \quad \prod_{i=k}^{l-1} m(P^*_{t_{i+1}-t_i}|_{F_{X_{t_i}(x)}}) \geq e^{\eta(T-t_k)} \tag{5.1}$$

Theorem 5.3.2 *(Liao's shadowing lemma) Suppose $\Lambda \subset M \setminus Sing(X)$ is a compact invariant set with a dominated splitting $N_\Lambda = E \oplus F$. Given ε_0, $\eta > 0$, $T_0 \geq 1$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any orbit segment $X_{[0,T]}(x) \subset \Lambda$ with the following properties:*

- *$d(x, Sing(X)) \geq \varepsilon_0$ and $d(X_T(x), Sing(X)) \geq \varepsilon_0$*
- *$X_{[0,T]}(x)$ is η-quasi hyperbolic*
- *$d(x, X_T(x)) < \delta$*

Then there exists a C^1 increasing homeomorphism $\theta : [0,T] \to \mathbb{R}$ and a periodic point $p \in M$, $X_{\theta(T)}(p) = p$ such that:

1. *$1 - \varepsilon < \theta'(t) < 1 + \varepsilon$, $\forall\ t \in [0,T]$*
2. *$d(X_t(x), X_{\theta(t)}(p)) \leq \varepsilon ||X(X_t(x))||$, $\forall\ t \in [0,T]$*

As mentioned before, the following proposition is one of the most important steps on the proof of Theorem 5.2.2.

Proposition 5.3.3 *Under the setting of Liao's shadowing lemma, if $T = mT_0$ for some $m \in \mathbb{R}^+$, then there exists $N = N(\eta, T_0)$ such that*

$$|\theta(t) - t| \leq N d(x, X_T(x)), \quad \forall t \in \mathbb{N} \cap [0,T]$$

The proof of proposition 5.3.3 consists in defining a Poincaré map for the vector field X, and then we will show that the time on which each point on the domain of the Poincaré map takes to hit the contra-domain satisfies some Lipschitz estimative (Lemma 5.3.5), we also show that for fixed $\eta > 0$ and $T_0 > 1$, the distance between the (η, T_0)-quasi hyperbolic orbit $X_{[0,mT_0]}(x)$ and its shadowing periodic orbit can be controlled by the distance between the starting point end the ending point of this quasi hyperbolic orbit (Lemma 5.3.6). The Proposition 5.3.3 will then follow from these two lemmas.

The Exponential Map

It is well known from the Riemannian Geometry, that given any initial point $p \in M$ and any initial velocity vector $v \in T_pM$, they determine a unique maximal geodesic γ_v, that is, the unique geodesic through p in the direction of the vector v. This implicitly defines a map from the tangent bundle to the set of geodesics in M. More importantly, it allows us to define a map from (a subset of) the tangent bundle to M itself, by sending the vector to the point obtained by fallowing γ_v for time 1. To be more precise, define a subset $\mathscr{E} \subset TM$, by

$\mathscr{E} := \{V = (p, v) \in TM : \gamma_V \text{ is defined on an interval containing } [0, 1]\}$

Then, define the *exponential map* $\exp : \mathscr{E} \to M$ by

$$\exp(V) = \gamma_V(1)$$

For each $p \in M$, the restricted exponential map $\exp_p$ is the restriction of the exponential map to the set $\mathscr{E}_p = \mathscr{E} \cap T_pM$

Remark: For each $V \in TM$, γ_V is given by $\gamma_V(t) = \exp(tV)$, for $t \in \mathbb{R}$ such that $tV \in \mathscr{E}$. For the proof, see for instance [Lee97].

A important result about the exponential map is the following:

Normal Neighborhood Lemma: *For any $p \in M$, there is a neighborhood V of the origin in T_pM and a neighborhood U of p in M such that $\exp_p : V \to U$ is a diffeomorphism.*

Proof. This follows immediately from the inverse function theorem, once we show that $D\exp_p$ is invertible at 0. Since T_pM is a vector space, there is a natural identification $T_0(T_pM) = T_pM$. Under this identification, we will show that $D\exp_p(0) : T_pM \to T_pM$ has a particularly simple expression, it is the identity map.

To compute $D\exp_p(0)v$ for an arbitrary vector $v \in T_pM$, we just need to choose a curve α in T_pM starting at 0 whose initial vector is v, that is, $\alpha(0) = 0$ and $\alpha'(0) = v$, and compute the initial tangent vector of the composite curve $\exp_p \circ \alpha(t)$. We can take $\alpha(t) = tv$ as such curve. Thus

$$D\exp_p(0)v = \frac{d}{dt}\Big|_{t=0} (\exp_p \circ \alpha)(t) = \frac{d}{dt}\Big|_{t=0} \exp_p(tv) = \frac{d}{dt}\Big|_{t=0} \gamma_v(t) = v$$

The above lemma allows us to define a coordinate system on M. By the definition of manifold, an orthonormal basis $\{E_i\}$ for T_pM gives an isomorphism $E : \mathbb{R}^n \to T_pM$. If U is a neighborhood of $p \in M$ as in the previous result, we can combine this isomorphism with the exponential map to get a coordinate chart $\varphi = E^{-1} \circ exp_p^{-1} : U \to \mathbb{R}^n$.

Now let us return our attention to the Poincaré map defined in section 3.2. Given $\beta > 0$ small enough and a regular point $y \in M$, one has that the image of the normal ball $N_y(\beta) = \{v \in N_y : ||v|| \leq \beta\}$ under the exponential map is a diffeomorphism from $N_y(\beta)$ to $\Sigma_y(\beta)$, where

$$\Sigma_y(\beta) = \exp_y(N_y(\beta))$$

As we saw before, to study the dynamics of a periodic orbit of a vector field, Poincaré defined the sectional return map of a cross section of a periodic point. By generalizing this idea to every regular point, one can define the Poincaré map for any two points in the same regular orbit.

The next lemma says that for any vector field $X \in \mathscr{X}^1(M)$, there exists $0 < \delta \leq \beta$ such that for any regular point $x \in M$, the Poincaré map $P_{x,X_t(x)} : \Sigma_x(\delta||X(x)||) \to \Sigma_{X_t(x)}(\delta||X(X_t(x))||)$ is well defined for certain values of $t \in \mathbb{R}$.

Lemma 5.3.4 *For any $X \in \mathscr{X}^1(M)$, there exists $0 < \delta \leq \beta$ such that for any regular point $y \in \Sigma_x(\delta||X(x)||)$, and for any $t \in \left[\frac{\delta}{3}, \frac{2\delta}{3}\right]$, there is a unique $s = s(t,y) \in [0,\delta]$ such that $X_s(y) \in \Sigma_{X_t(x)}(\delta||X(X_t(x))||)$.*

Proof. Let $\varepsilon_0 > 0$ be the number such that the exponential map $\exp_x$ is a diffeomorphism on the ball $T_xM(\varepsilon_0)$, where $T_xM(\varepsilon_0)$ is the ball on T_xM centered at the origin with radius ε_0. For any $x \in M$ and any y close to x we can lift the local orbit of y to T_xM and define the local flow

$$\tilde{X}_t(v) = exp_x^{-1} \circ X_t \circ exp_x(v)$$

Since the derivative of the flow equals to the vector field that generates the flow, using the fact that $\frac{d}{dt}(X_t(exp_x(v)) = X(exp_x(v))$ it follows from the chain rule that the flow $\widetilde{X}_t$ is generated by the C^1 vector field on T_xM

$$\widetilde{X}_x(v) = D(exp_x)^{-1} \circ X(exp_x(v))$$

Since $\widetilde{X}_x \in \mathscr{X}^1(T_xM)$, M is compact, and $T_xM(\varepsilon_0)$ is bounded, one has

$$K := \sup_{x \in M, v \in T_xM(\varepsilon_0)} \{||\widetilde{X}_x(v)||, ||D\widetilde{X}_x(v)||\} < +\infty$$

Since the derivative of the exponential map at $v = 0$ is the identity map, given $\varepsilon > 0$, by reducing ε_0 if necessary we may assume that the map $Dexp_x(v)$ is ε-close to the identity map for any $x \in M$ and for any $v \in T_xM(\varepsilon_0)$.

Claim. There exists $\delta > 0$ such that for every regular point $x \in M$, one has

$$exp_x N_x(\delta||X(x)||) \cap Sing(X) = \emptyset$$

Proof of the Claim. Consider $\delta < \frac{\varepsilon}{K}$. For any $v \in N_x(\delta||X(x)||)$ it follows from the Mean Value Theorem and from the second triangle inequality that

$$||\widetilde{X}_x(0)|| - ||\widetilde{X}_x(v)|| \leq ||\widetilde{X}_x(v) - \widetilde{X}_x(0)|| \leq \max_{\xi \in T_xM(\varepsilon_0)} ||D\widetilde{X}_x(\xi)||.||v - 0||$$

then

$$\begin{aligned} ||\widetilde{X}_x(v)|| &\geq ||\widetilde{X}_x(0)|| - \max_{\xi \in T_xM(\varepsilon_0)} ||D\widetilde{X}_x(\xi)||.||v|| \\ &\geq ||X(x)|| - K\delta||X(x)|| \\ &\geq ||X(x)|| - \varepsilon||X(x)|| \\ &= (1-\varepsilon)||X(x)|| \\ &> 0 \end{aligned}$$

Since the map $Dexp_x$ is ε-close to identity, we have that $||X(exp_x(v)|| > 0$, for all $v \in N_x(\delta||X(x)||)$. This proves the claim.

By reducing δ if necessary, by continuity one has

$$\sup_{t\in(-\delta,\delta)} \frac{||\widetilde{X}_x(v)||}{||\widetilde{X}_x(\widetilde{X}_t(v)||} < 1 + \frac{\varepsilon}{K} \quad and \quad \sup_{t\in(-\delta,\delta)} \angle(||\widetilde{X}_x(v)||, ||\widetilde{X}_x(\widetilde{X}_t(v)||) < 1 + \frac{\varepsilon}{K} \tag{5.2}$$

For any $v \in N_x(\delta||X(x)||/3)$, let $\delta \geq t_0 > 0$ be the time satisfying

$$||\widetilde{X}_{t_0}(v)|| = \delta||\widetilde{X}_x(\widetilde{X}_{t_0}(0))|| \quad and \quad ||\widetilde{X}_s(v)|| \leq \delta||\widetilde{X}_x(\widetilde{X}_s(0))||, \quad \forall s \in [0, t_0)$$

Observe that $||\widetilde{X}_x(\widetilde{X}_s(0))|| = ||X(X_s(x))||$. Since M is compact and $X \in \mathscr{X}^1(M)$, there is $C > 0$ such that $||X(x)|| \leq C$, $\forall\ x \in M$.

Consider the integral equation

$$\widetilde{X}_{t_0}(v) = v + \int_0^{t_0} \widetilde{X}_x(\widetilde{X}_s(v))\, ds$$

Then,

$$\begin{aligned}
||\widetilde{X}_{t_0}(v)|| &= ||v + \int_0^{t_0} \widetilde{X}_x(\widetilde{X}_s(v))\, ds|| \\
&\leq ||v|| + \int_0^{t_0} ||\widetilde{X}_x(\widetilde{X}_s(v))||\, ds \\
&\leq \frac{\delta}{3}||\widetilde{X}_x(0)|| + \int_0^{t_0} (||\widetilde{X}_x(0)|| + K||\widetilde{X}_s(v)||)\, ds \\
&\leq \frac{\delta}{3}||\widetilde{X}_x(0)|| + t_0||\widetilde{X}_x(0)|| + \int_0^{t_0} K||\widetilde{X}_s(v)||)\, ds \\
&\leq \frac{\delta}{3}||\widetilde{X}_x(0)|| + t_0||\widetilde{X}_x(0)|| + \int_0^{t_0} K\delta||\widetilde{X}_x(\widetilde{X}_s(0))||)\, ds \\
&\leq \frac{\delta}{3}||\widetilde{X}_x(0)|| + t_0||\widetilde{X}_x(0)|| + \int_0^{t_0} \varepsilon||X(X_s(x))||)\, ds \\
&\leq \frac{\delta}{3}||\widetilde{X}_x(0)|| + t_0||\widetilde{X}_x(0)|| + \varepsilon C \int_0^{t_0} ds \\
&\leq \frac{\delta}{3}||\widetilde{X}_x(0)|| + t_0||\widetilde{X}_x(0)|| + \varepsilon' t_0 \\
&\leq \frac{\delta}{3}||\widetilde{X}_x(0)|| + (1+\varepsilon')t_0||\widetilde{X}_x(0)||
\end{aligned}$$

By the other hand, (5.2) implies

$$\begin{aligned}
||\widetilde{X}_{t_0}(v)|| &= \delta||\widetilde{X}_x(\widetilde{X}_{t_0}(0))|| \\
&\geq \delta \frac{||\widetilde{X}_{t_0}(v)||}{1+\frac{\varepsilon}{K}}
\end{aligned}$$

Thus,

$$\delta \frac{||\widetilde{X}_x(0)||}{1+\frac{\varepsilon}{K}} \leq ||\widetilde{X}_{t_0}(v)|| \leq \frac{\delta}{3}||\widetilde{X}_x(0)|| + (1+\varepsilon')t_0||\widetilde{X}_x(0)||$$

This implies $\frac{\delta}{3} \leq t_0 \leq \frac{2\delta}{3}$.

□

The above lemma allows us to define the Poinvaré map from $N_x(\delta||X(x)||)$ to $N_{X_t(x)}(\delta||X(X_t(x))||)$ by the equation $\mathscr{P}_{x,X_T(x)} = \exp^{-1}_{X_T(x)} \circ P_{x,X_T(x)} \circ \exp_x$.

To eliminate the dependence of the vector field norm, we consider the rescaled Poincaré map

$$\begin{aligned} \mathscr{P}^*{}_t : N_x(\delta) &\to N_{X_t(x)}(\delta) \\ v &\mapsto \frac{\mathscr{P}_t(||X(x)||v)}{||X(X_t(x))||} \end{aligned}$$

Let us represent $\mathscr{P}^*{}_t$ in normal coordinates. Define the map $\tau : N_x(\delta/2) \to \mathbb{R}$ such that $\widetilde{X}_{\tau(v)} \circ \frac{\widetilde{X}_t(||\widetilde{X}_x(x)||v)}{||\widetilde{X}_x(\widetilde{X}_t(x))||} \in N_{X_t(x)}(\delta)$ for any $v \in N_x(\delta/2)$. By the above lemma, one has that the map τ is injective. In coordinates, we can represent $\mathscr{P}^*{}_t$ by

$$\mathscr{P}^*_t(v) = \widetilde{X}_{\tau(v)} \circ \frac{\widetilde{X}_t(||\widetilde{X}_x(x)||v)}{||\widetilde{X}_x(\widetilde{X}_t(x))||}$$

Now, we will estimate the function τ. For $t \in \left[\frac{\delta}{3}, \frac{2\delta}{3}\right]$, consider the function

$$H(x,t,y,\tau) = \left\langle \widetilde{X}_\tau \circ \frac{\widetilde{X}_t(||\widetilde{X}_x(x)||y)}{||\widetilde{X}_x(\widetilde{X}_t(x))||}, \frac{\widetilde{X}_x(\widetilde{X}_t(x))}{||\widetilde{X}_x(\widetilde{X}_t(x))||} \right\rangle$$

where $\langle .,. \rangle$ denotes the inner product in the local euclidean coordinate. Fix $x = x_0$ and $t = t_0$. We can consider $H(x_0, t_0, y, \tau(y))$ as a map in the variables y and τ. From the definition of the flow $\widetilde{X}_t$ and by the chain rule, one has

$$\begin{aligned} \left.\frac{\partial H}{\partial \tau}\right|_{y=0,\tau=0} &= \frac{\partial}{\partial \tau}\left(\left\langle \widetilde{X}_\tau \circ \frac{\widetilde{X}_{t_0}(||\widetilde{X}_{x_0}(x_0)||y)}{||\widetilde{X}_{x_0}(\widetilde{X}_{t_0}(x_0))||}, \frac{\widetilde{X}_{x_0}(\widetilde{X}_{t_0}(x_0))}{||\widetilde{X}_{x_0}(\widetilde{X}_{t_0}(x_0))||} \right\rangle\right) \\ &= \left\langle \frac{\partial}{\partial \tau}\left(\widetilde{X}_\tau \circ \frac{\widetilde{X}_{t_0}(||\widetilde{X}_{x_0}(x_0)||y)}{||\widetilde{X}_{x_0}(\widetilde{X}_{t_0}(x_0))||}\right), \frac{\widetilde{X}_{x_0}(\widetilde{X}_{t_0}(x_0))}{||\widetilde{X}_{x_0}(\widetilde{X}_{t_0}(x_0))||} \right\rangle \\ &+ \left\langle \left(\widetilde{X}_\tau \circ \frac{\widetilde{X}_{t_0}(||\widetilde{X}_{x_0}(x_0)||y)}{||\widetilde{X}_{x_0}(\widetilde{X}_{t_0}(x_0))||}\right), \frac{\partial}{\partial \tau}\frac{\widetilde{X}_{x_0}(\widetilde{X}_{t_0}(x_0))}{||\widetilde{X}_{x_0}(\widetilde{X}_{t_0}(x_0))||} \right\rangle \\ &= \left\langle \frac{\partial}{\partial \tau}\left(\widetilde{X}_\tau \circ \frac{\widetilde{X}_{t_0}(||\widetilde{X}_{x_0}(x_0)||y)}{||\widetilde{X}_{x_0}(\widetilde{X}_{t_0}(x_0))||}\right), \frac{\widetilde{X}_{x_0}(\widetilde{X}_{t_0}(x_0))}{||\widetilde{X}_{x_0}(\widetilde{X}_{t_0}(x_0))||} \right\rangle \\ &= \left\langle \frac{\widetilde{X}_{x_0}(\widetilde{X}_{t_0}(x_0))}{||\widetilde{X}_{x_0}(\widetilde{X}_{t_0}(x_0))||}, \frac{\widetilde{X}_{x_0}(\widetilde{X}_{t_0}(x_0))}{||\widetilde{X}_{x_0}(\widetilde{X}_{t_0}(x_0))||} \right\rangle = 1 \end{aligned}$$

Then, it follows from the implicit function theorem that the map τ is differentiable and

$$\frac{\partial \tau}{\partial y} = -\frac{\frac{\partial H}{\partial y}}{\frac{\partial H}{\partial \tau}} \tag{5.3}$$

Lemma 5.3.5 *For the flow X_t generated by $X \in \mathscr{X}^1(M)$, there are constants $C > 0$ and $\delta > 0$ such that if $y \in \exp_x N_x(\delta)$, then there exists a unique $s = s(y)$ such that $X_s(y) \in \exp_{X_t(x)} N(\delta)$ and $|s(y) - t(x)| \le Cd(y, x)$.*

Proof. Since X_t is C^1 and $\widetilde{X}_t(y) \in N_x(\beta_*)$, $\frac{\partial H}{\partial y}$ is uniformly bounded and for the above computation $\frac{\partial H}{\partial \tau}$ is uniformly bounded away from zero. Thus, by equation (5.3) one has that $\frac{\partial \tau}{\partial y}$ is uniformly bounded with respect to y. This means that there is a constant $C > 0$ such that $|\tau(y)| = |s(y) - t(x)| \le Cd(y, x)$.

□

Fixed $\eta > 0$ and $T_0 > 1$, the following lemma will show that the distance between the (η, T_0)-quasi hyperbolic orbit $X_{[0,mT_0]}(x)$ and its shadowing periodic orbit can be controlled by the distance between the starting point end the ending point of this quasi hyperbolic orbit.

Lemma 5.3.6 *Under the assumption of Liao's shadowing lemma, taking $\alpha = e^{-\eta/2}$, if $T = mT_0$ for some $m \in \mathbb{N}$, then there exists a constant $C > 0$ such that for the (η, T_0)-quasi hyperbolic orbit $X_{[0,mT_0]}(x)$ and the shadowing orbit $X_{[0,\theta(mT_0)]}(p)$, one has*

$$d(X_{iT_0}(x), X_{\theta(iT_0)}(p)) \le C\alpha^{min\{i,m-i\}} d(x, X_{mT_0}(x)), \quad \forall i \in \mathbb{N} \cap [0, m]$$

Proof. Consider $T_0 > 0$ such that the sequence of Poincaré maps

$$\{\mathscr{P}^*_{X_{iT_0}(x), X_{(i+1)T_0}(x)} : N_{X_{iT_0}(x)} \to N_{X_{(i+1)T_0}(x)}\}_{i=0}^{m-1}$$

are well defined. Since $X_{[0,mT_0]}(x)$ is a (η, T_0)-quasi hyperbolic orbit, by Definition 5.3.1, for the dominated splitting $N_\Lambda = E \oplus F$ with respect to the linear Poincaré flow, one has

$$\prod_{i=0}^{k-1} ||P^*_{T_0}|_{E_{X_{iT_0}(x)}}|| \le e^{-\eta k} \quad and \quad \prod_{i=k}^{m-1} m(P^*_{T_0}|_{F_{X_{iT_0}(x)}}) \ge e^{\eta(m-k)}, fork = 0, 1, \cdots, m \tag{5.4}$$

By Liao's shadowing lemma, one has $d(X_t(x), X_{\theta(t)}(p)) \le \varepsilon ||X(X_t(x))||$, $\forall\ t \in [0, T]$. Since ε is arbitrary, for suitable δ we may assume $X_{\theta(t)}(p) \in \Sigma_{X_t(x)}(\delta ||X(X_t(x))||)$ for any $t \in [0, T]$. Then for the periodic point p, one can also define the sequence of rescale sectional Poincaré maps

$\{\mathscr{P}^*_{X_{iT_0}(p),X_{(i+1)T_0}(p)} : N_{X_{iT_0}(p)} \to N_{X_{(i+1)T_0}(p)}\}_{i=0}^{m-1}$. We also denote by d the distance function in the normal bundle $N = E \oplus F$. Let d_E and d_F be the induced distances in the subbundles E and F respectively. There is a constant $C \geq 1$ such that $d_E(x,p) \leq Cd(x,p)$, $d_F(x,p) \leq Cd(x,p)$ and $d(x,p) \leq d_E(x,p) + d_F(x,p)$. Since the derivative of the rescaled sectional Poincaré map equals to the rescaled linear Poincaré flow, by the estimates (5.4) , for each $i = 0, \cdots m$,

$$\begin{aligned} d_E(X_{iT_0}(x), X_{\theta(iT_0)}(p)) &\leq ||P^*_{T_0}(X_{iT_0}(x))||d_E(X_{(i-1)T_0}(x), X_{\theta((i-1)T_0)}(p)) \\ &\leq ||P^*_{T_0}(X_{iT_0}(x))||.||P^*_{T_0}(X_{(i-1)T_0}(x))||d_E(X_{(i-2)T_0}(x), X_{\theta((i-2)T_0)}(p)) \\ &\leq \cdots \leq \prod_{j=0}^{i} ||P^*_{T_0}(X_{jT_0}(x))||d_E(x,p) \leq e^{-\eta k}d_E(x,p) \leq \alpha^i d_E(x,p) \end{aligned}$$

Analogously, $d_F(X_{iT_0}(x), X_{\theta(iT_0)}(p)) \leq \alpha^{m-i} d_F(X_{mT_0}(x), X_{mT_0}(p))$. Therefore,

$$\begin{aligned} d(X_{iT_0}(x), X_{\theta(iT_0)}(p)) &\leq d_E(X_{iT_0}(x), X_{\theta(iT_0)}(p)) + d_F(X_{iT_0}(x), X_{\theta(iT_0)}(p)) \\ &\leq \alpha^i d_E(x,p) + \alpha^{m-i} d_F(X_{mT_0}(x), X_{mT_0}(p)) \\ &\leq \alpha^i Cd(x,p) + \alpha^{m-i} Cd(X_{mT_0}(x), X_{mT_0}(p)) \\ &\leq C\alpha^{\min\{i,m-i\}}(d(x,p) + d(X_{mT_0}(x), X_{mT_0}(p))) \end{aligned}$$

By Theorem 1.1 of [Gan02], enlarging C if necessary, we have

$$d(x,p) \leq Cd(x, X_{mT_0}(x)) \quad and \quad d(X_{mT_0}(x), X_{\theta(mT_0)(p))}) \leq Cd(x, X_{mT_0}(x))$$

Therefore,

$$d(X_{iT_0}(x), X_{\theta(iT_0)}(p)) \leq C^2\alpha^{\min\{i,m-i\}} d(x, X_{mT_0}(x)), \quad \forall i \in \mathbb{N} \cap [0,m]$$

□

Proposition 5.3.7 *Under the setting of Liao's shadowing lemma, if $T = mT_0$ for some $m \in \mathbb{N}$, then there exists $N = N(\eta, T_0)$ such that*

$$|\theta(t) - t| \leq Nd(x, X_T(x)), \quad \forall t \in \mathbb{N} \cap [0,T]$$

Proof. By Liao's shadowing lemma, given $\varepsilon_0 > 0$, $\eta > 0$ and $T_0 \leq 1$, for every $\varepsilon > 0$, there is $\delta > 0$ such that for any (η, T_0)-quasi hyperbolic orbit segment $X_{[0,mT_0]}(x) \subset \Lambda$ with $d(x, Sing(X)) \geq \varepsilon_0$ and $d(X_{mT_0}(x), Sing(X)) \geq \varepsilon_0$ and $d(x, X_{mT_0}(x)) < \delta$, there is a periodic point $p \in M$ and a C^1 strictly increasing function θ such that $X_{\theta(mT_0)}(p) = p$ and $d(X_t(x), X_{\theta(t)}(p)) \leq \varepsilon||X(X_t(x))||$ for any $t \in [0, mT_0]$. Consider a time partition $0 = t_0 < t_1 < t_2 < \cdots < mT_0 = T = t_m$ with $t_{i+1} - t_i = T_0$. Taking $\alpha = e^{-\eta/2}$, by Lemma 5.3.6 , there exists $C_1 > 0$ such that

$$d(X_{t_i}(x), X_{\theta(t_i)}(p)) \leq C_1 \alpha^{min\{i,m-i\}} d(x, X_T(x)), \quad \forall i = 0, \cdots, m$$

Since $d(X_t(x), X_{\theta(t)}(p)) \leq \varepsilon||X(X_t(x))||$, one may assume $X_{\theta(t)}(p) \in \mathscr{N}_{X_t(x)}(N_{X_t(x)}(\delta||X(X_t(x)||))$, for some $\delta = \delta(\varepsilon) > 0$ and for every $t \in [0, T]$. then, by Lemma 5.3.5, there is $C_2 = C_2(T_0) > 0$ such that for $i = 1, \cdots, m$, one has

$$|\theta(t_{i+1}) - \theta(t_i) - (t_{i+1} - t_i)| \leq C_2 d(X_{\theta(t_i)}(p), X_{t_i}(x)), \quad \forall i = 1 \cdots, m$$

Let $N = N(\eta, T_0) = \frac{C_2.C_1}{1-\alpha}$, by applying the above inequalities, it follows from the triangle inequality that

$$\begin{aligned} |\theta(T) - T| &\leq \sum_{i=0}^{m-1} |\theta(t_{i+1}) - \theta(t_i) - (t_{i+1} - t_i)| \\ &\leq C_2 \sum_{i=0}^{m-1} d(X_{\theta(t_i)}(p), X_{t_i}(x)) \\ &\leq C_2.C_1.d(X_T(x), x) \sum_{i=0}^{m-1} \alpha^{min\{i,m-i\}} \end{aligned}$$

Since $\alpha = e^{-\eta/2} \in (0, 1)$, $\sum_{i=0}^{m-1} \alpha^{min\{i,m-i\}} \leq \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}$. Therefore,

$$|\theta(T) - T| \leq \frac{C_2.C_1}{1-\alpha} d(X_T(x), x) = N d(X_T(x), x)$$

By the above discussion, for every $t \in \mathbb{N} \cap [0, T]$

$$|\theta(t) - t| \leq \sum_{i=0}^{m-1} |\theta(t_{i+1}) - \theta(t_i) - (t_{i+1} - t_i)| \leq N d(X_T(x), x)$$

□

5.4 Pesin Block of vector fields

For a regular ergodic hyperbolic measure μ and its Oseledec splitting $N = E^s \oplus E^u$, by the definition of the P_t^*, for μ-a.e., one has

$$\lambda^-(\mu) = \lim_{t\to\infty} \frac{1}{t} \log ||P_t^*|_{E_x^s}|| < 0 \quad and \quad \lambda^+(\mu) = \lim_{t\to\infty} \frac{1}{t} \log m(P_t^*|_{E_x^u}) > 0 \tag{5.5}$$

Lemma 5.4.1 *If the hyperbolic Oseledec splitting of a regular hyperbolic ergodic measure μ is a dominated spitting, then for any $\varepsilon > 0$, there exists $T_0 = T(\varepsilon) \in \mathbb{R}$ such that for μ-a.e. $x \in M$ and every $T > T_0$, one has*

$$\lim_{k\to\infty} \frac{1}{kT} \sum_{i=0}^{k-1} \log ||P_T^*|_{E^s_{X_{iT}(x)}}||$$

exists and is contained in $[\lambda^-(\mu), \lambda^-(\mu) + \epsilon)$, *and*

$$\lim_{k\to\infty} \frac{1}{kT} \sum_{i=0}^{k-1} \log ||P_{-T}^*|_{E^u_{X_{-iT}(x)}}||$$

exists and is contained in $(-\lambda^+(\mu) - \epsilon, -\lambda^+(\mu)]$.

Proof. Let R be the support of μ and $\widetilde{R}$ its transgression. By Lemma 3.6.3, $\widetilde{R}$ admits a dominated splitting $N_{\widetilde{R}}SM = \widetilde{E}^s \oplus \widetilde{E}^u$ with respect to the extended linear Poincaré flow. By 4.4.6, one has

$$\lambda^-(\mu) = \lambda^-(\widetilde{\mu}) = \lim_{t\to\infty} \frac{1}{t} \int \log ||\Psi_t|_{E^s}|| \, d\widetilde{\mu}$$

where $\widetilde{\mu}$ is the transgression of μ. Therefore, for any $\varepsilon > 0$, there is $T_0 > 0$ such that for $T \geq T_0$, one has $|\frac{1}{T} \int \log ||\Psi_T|_{E_x^s}|| \, d\widetilde{\mu} - \lambda^-(\mu)| < \varepsilon$, This is equivalent to $\lambda^-(\mu) - \varepsilon < \frac{1}{T} \int \log ||\Psi_T|_{E_x^s}|| \, d\widetilde{\mu} < \lambda^-(\mu) + \varepsilon$. Thus, it follows from the Birkhoff ergodic theorem that

$$\lim_{k\to\infty} \frac{1}{kT} \sum_{i=0}^{k-1} \log ||P_T^*|_{E^s_{X_{iT}(x)}}|| = \lim_{k\to\infty} \frac{1}{kT} \sum_{i=0}^{k-1} \log ||\Psi_T|_{E^s_{X_{iT}(x)}}|| = \frac{1}{T} \int \log ||\Psi_T|_{E_x^s}|| \, d\widetilde{\mu} < \lambda^-(\mu) + \varepsilon$$

By the other hand, let $t \in \mathbb{R}$. By the Euclidean algorithm, there are $k \in \mathbb{N}$ and $0 < r < T-1$ such that $t = kT + r$. Observe that $t \to \infty$ implies $kT + r \to \infty$. Thus

$$\frac{1}{t}\log||P_t^*|_{E_x^s}|| = \frac{1}{kT+r}\log||P_{kT+r}^*|_{E_x^s}||$$

Since the norm is sub-multiplicative, it follows from the chain rule that

$$||P_{kT+r}^*|_{E_x^s}|| = ||P_T^*|_{E_{X_{T(k-1)}(X_r(x))}} \cdots P_T^*|_{E_{X_r(x)}} \cdot P_r^*|_{E_x}|| \leq ||P_T^*|_{E_{X_{T(k-1)}(X_r(x))}}|| \cdots ||P_T^*|_{E_{X_r(x)}}|| \cdot ||P_r^*|_{E_x}||$$

This implies

$$\begin{aligned}\frac{1}{kT+r}\log||P_{kT+r}^*|_{E_x^s}|| &\leq \frac{1}{kT+r}\log(||P_T^*|_{E_{X_{T(k-1)}(X_r(x))}}|| \cdots ||P_T^*|_{E_{X_r(x)}}|| \cdot ||P_r^*|_{E_x}||) \\ &= \frac{1}{kT+r}\left(\sum_{i=0}^{k-1}\log||P_{X_{iT}(X_r(x))}^*|| + \log||P_r^*|_{E_x}||\right)\end{aligned}$$

Thus,

$$\begin{aligned}\lambda^-(\mu) &= \lim_{t\to\infty}\frac{1}{t}\log||P_t^*|_{E_x^s}|| \\ &= \lim_{k\to\infty}\frac{1}{kT+r}\log||P_{kT+r}^*|_{E_x^s}|| \\ &\leq \lim_{k\to\infty}\frac{1}{kT+r}\left(\sum_{i=0}^{k-1}\log||P_{X_{iT}(x)}^*|| + \log||P_r^*|_{E_x}||\right) \\ &= \lim_{k\to\infty}\left(\frac{1}{kT+r}\sum_{i=0}^{k-1}\log||P_{X_{iT}(x)}^*|| + \frac{1}{kT+r}\log||P_r^*|_{E_x}||\right)\end{aligned}$$

Since $\log||P_r^*|_{E_x}||$ is bounded, we have

$$\lim_{k\to\infty}\frac{1}{kT+r}\log||P_r^*|_{E_x}|| = 0$$

Then,

$$\lambda^-(\mu) = \lim_{t\to\infty}\frac{1}{t}\log||P_t^*|_{E_x^s}|| \leq \lim_{k\to\infty}\frac{1}{kT+r}\sum_{i=0}^{k-1}\log||P_{X_{iT}(x)}^*|| \leq \lim_{k\to\infty}\frac{1}{kT}\sum_{i=0}^{k-1}\log||P_{X_{iT}(x)}^*||$$

This proves that $\lim_{k\to\infty}\frac{1}{kT}\sum_{i=0}^{k-1}\log||P_T^*|_{E_{X_{iT}(x)}^s}|| \subset [\lambda^-(\mu), \lambda^-(\mu)+\epsilon)$.

The conclusion for the subbundle E^u is obtained similarly.

□

Definition 5.4.2 *(Pesin Block) Let μ be a regular hyperbolic ergodic measure of $X \in \mathscr{X}^1(M)$, and $N_\Lambda = E^s \oplus E^u$ its Oseledec splitting, where Λ is a Borel set with μ-total measure. Given $\lambda \in (0,1)$, $L > 0$ and $k \geq 0$, the Pesin block $\Lambda^L_\lambda(k)$ is defined as:*

$$\Lambda^L_\lambda(k) := \left\{ x \in \Lambda : \prod_{i=0}^{n-1} ||P^*_L|_{E^s_{X_{iL(x)}}}|| \leq k\lambda^n, \prod_{i=0}^{n-1} ||P^*_{-L}|_{E^u_{X_{-iL(x)}}}|| \leq k\lambda^n, \forall n \geq 1, d(x, Sing(X)) \geq \frac{1}{k} \right\}$$

Proposition 5.4.3 *If the hyperbolic Oseledec splitting of a regular hyperbolic measure μ is a dominated splitting, then the Pesin block $\Lambda^L_\lambda(k)$ is a compact set such that*

$\mu(\Lambda^L_\lambda(k)) \to 1$ *as* $k \to \infty$,

where $\lambda = e^{-\eta}$, $0 < \eta < min\{-\lambda^-, \lambda^+\}$, $L \geq T(min\{-\lambda^-, \lambda^+\}) - \eta)$ *as in Lemma 5.4.1.*

Proof. By definition, the Pesin block $\Lambda^L_\lambda(k)$ is a compact set. By Lemma 5.4.1 given $\varepsilon > 0$, for L and n sufficiently large , for μ-a.e. $x \in M$, one has

$$\frac{1}{nL} \sum_{i=0}^{n-1} \log ||P^*_L|_{E^s_{X_{iL(x)}}}|| \leq \lambda^- + \varepsilon$$

This is equivalent to

$$\sum_{i=0}^{n-1} \log ||P^*_L|_{E^s_{X_{iL(x)}}}|| \leq nL(\lambda^- + \varepsilon)$$

By the logarithm property, one has

$$\log(\prod_{i=0}^{n-1} ||P^*_L|_{E^s_{X_{iL(x)}}}||) \leq kL(\lambda^- + \varepsilon)$$

Applying the exponential on both sides, we obtain

$$\begin{aligned}
\prod_{i=0}^{n-1} ||P^*_L|_{E^s_{X_{iL(x)}}}|| &\leq e^{nL(\lambda^- + \varepsilon)} \\
&= e^{nL(\lambda^- + min\{-\lambda^-, \lambda^+\} - \eta)} \\
&= \begin{cases} e^{-nL\eta} & \text{if } min\{-\lambda^-, \lambda^+\} = -\lambda^- \\ e^{nL(-\lambda^- + \lambda^+ - \eta)} & \text{if } min\{-\lambda^-, \lambda^+\} = \lambda^+ \end{cases}
\end{aligned}$$

If $min\{-\lambda^-, \lambda^+\} = \lambda^+$, then $\lambda^- + \lambda^+ < 0$. Therefore $e^{nL(-\lambda^- + \lambda^+ - \eta)} < e^{-nL\eta}$.

So, on both cases we have

$$\begin{aligned}\prod_{i=0}^{n-1} ||P_L^*|_{E^s_{X_{iL}(x)}}|| &\leq e^{nL(\lambda^- + min\{-\lambda^-, \lambda^+\} - \eta)} \\ &\leq e^{-nL\eta} \\ &= (e^{-\eta})^{nL} = \lambda^{nL} \leq C\lambda^n\end{aligned}$$

For some constant $C = C(x) > 0$.

Let $\Gamma_\lambda^L(k) := \{x \in \Lambda : \prod_{i=0}^{n-1} ||P_L^*|_{E^s_{X_{iL}(x)}}|| \leq k\lambda^n, \prod_{i=0}^{n-1} ||P_{-L}^*|_{E^u_{X_{-iL}(x)}}|| \leq k\lambda^n, \forall n \geq 1\}$

By the above discussion, for k large, we have that the set $\Gamma_\lambda^L(k)$ is non-empty. Also, Since the rescaled linear Poincaré flow is uniformly bounded, one has

$$\mu(\bigcup_{k>0} \Gamma_\lambda^L(k)) = 1$$

For any two real numbers $0 < k_1 < k_2$, one has $\Gamma_\lambda^L(k_1) \subset \Gamma_\lambda^L(k_2)$. Consequently,

$$\mu(\Gamma_\lambda^L(k)) \overset{k\to\infty}{\longrightarrow} 1$$

According to the facts that $\Lambda_\lambda^L(k) \subset \Gamma_\lambda^L(k)$ and $\mu(Sing(X)) = 0$, for any $\varepsilon > 0$, there is $K = K(\varepsilon) \in \mathbb{N}$ such that $|\mu(\Gamma_\lambda^L(k)) - \mu(\Lambda_\lambda^L(k))| < \varepsilon$, $\forall\ k \geq K$. Then

$$\mu(\Lambda_\lambda^L(k)) \overset{k\to\infty}{\longrightarrow} 1$$

□

5.5 Constructing many periodic orbits: proof of theorem 5.2.2

Now we will state the version of Poincaré Recurrence Theorem for flows. Since it can be obtained by the case for diffeomorphism, the proof is omitted.

Proposition 5.5.1 *Let μ be an X_t-invariant measure. For any fixed time t_0 and any set B with positive μ-measure, for μ-a.e. $x \in B$ there is a sequence of integers $0 < n_1 < n_2 < \cdots < n_i < \cdots$ such that*

1. $X_{n_i t_0}(x) \in B \; , \forall \; i \in \mathbb{N}$

2. $d(x, X_{n_i t_0}(x)) \stackrel{i\to\infty}{\longrightarrow} 0$

Proposition 5.5.2 *Assume that $f : M \to M$ is a homeomorphism on a compact metric space. Let μ be an ergodic f-invariant measure. If Λ is a set with positive μ-measure, the given $\delta > 0$ and $\varepsilon > 0$, we have that*

$$\lim_{n\to\infty} \mu(\Lambda_n) = \mu(\Lambda)$$

where

$$\Lambda_n = \{x \in \Lambda : \exists m \in [n, (1+\varepsilon)n], f^m(x) \in \Lambda, d(f^m(x), x) < \delta\}$$

Proof. Given $\delta > 0$ and $\varepsilon > 0$, take a finite measurable partition $\mathscr{P} = \{P_i\}_{i=i}^{l}$ such that

$$diam(P_i) \leq \delta, \quad P_i \subset \Lambda \quad or \quad P_i \cap \Lambda = \emptyset, \quad i = 1, 2, \cdots, l$$

Consider the set

$\Lambda_n(\mathscr{P}) := \{x \in \Lambda : \exists\; i \in [0, l]$ and $m \in [n, (1+\varepsilon)n]$ with $f^m(x) \in \Lambda$ and $x, f^m(x) \in P_i\}$

Fix $P_i \subset \Lambda$ and define

$$\mathscr{P}^i_{n,\varepsilon} := \left\{ x \in P_i : \sum_{j=0}^{n-1} \chi_{P_i}(f^j(x)) \leq n\mu(P_i)\left(1 + \frac{\varepsilon}{3}\right), \sum_{j=0}^{[n(1+\varepsilon)]} \chi_{P_i}(f^j(x)) \geq n\mu(P_i)\left(1 + \frac{2\varepsilon}{3}\right) \right\}$$

where χ_{P_i} is the characteristic function of the set P_i.

It follows from the definition of the set $\mathscr{P}^i_{n,\varepsilon}$ that $\mathscr{P}^i_{n,\varepsilon} \subset P_i \cap \Lambda_n(\mathscr{P})$. We shall prove that $\mu(P_i \backslash \mathscr{P}^i_{n,\varepsilon}) \stackrel{n\to\infty}{\longrightarrow} 0$, that is $\lim_{n\to\infty} \mu(\mathscr{P}^i_{n,\varepsilon}) = \mu(P_i)$. Since $\mathscr{P}^i_{n,\varepsilon} \subset P_i$, one has $\mu(\mathscr{P}^i_{n,\varepsilon}) \leq \mu(P_i)$, for all $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} \mu(\mathscr{P}^i_{n,\varepsilon}) \leq \mu(P_i)$. To the reverse inequality, we apply the Birkhoff ergodic theorem. For μ a.e. $x \in P_i$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_{P_i}(f^j(x)) = \int \chi_{P_i}(x)\, d\mu = \mu(P_i) \leq \mu(P_i)\left(1 + \frac{\varepsilon}{3}\right)$$

and

$$\begin{aligned}\sum_{j=0}^{[n(1+\varepsilon)]} \chi_{P_i}(f^j(x)) &= \frac{[n(1+\varepsilon)]+1}{[n(1+\varepsilon)]+1} \sum_{j=0}^{[n(1+\varepsilon)]} \chi_{P_i}(f^j(x)) \\ &= ([n(1+\varepsilon)]+1). \frac{1}{[n(1+\varepsilon)]+1} \sum_{j=0}^{[n(1+\varepsilon)]} \chi_{P_i}(f^j(x)) \\ &\geq n\mu(P_i)\left(1+\frac{2\varepsilon}{3}\right)\end{aligned}$$

this implies

$$\frac{[n(1+\varepsilon)]+1}{n}. \frac{1}{[n(1+\varepsilon)]+1} \sum_{j=0}^{[n(1+\varepsilon)]} \chi_{P_i}(f^j(x)) \geq \mu(P_i)\left(1+\frac{2\varepsilon}{3}\right)$$

taking limits, we obtain

$$\lim_{n\to\infty} \frac{[n(1+\varepsilon)]+1}{n}. \lim_{n\to\infty} \frac{1}{[n(1+\varepsilon)]+1} \sum_{j=0}^{[n(1+\varepsilon)]} \chi_{P_i}(f^j(x)) \geq \mu(P_i)\left(1+\frac{2\varepsilon}{3}\right)$$

Then

$$(1+\varepsilon)\int \chi_{P_i}(x)\, d\mu = (1+\varepsilon)\mu(P_i) \geq \mu(P_i)\left(1+\frac{2\varepsilon}{3}\right)$$

Therefore, $\lim_{n\to\infty} \mu(\mathscr{P}^i_{n,\varepsilon}) = \mu(P_i)$, and the proposition is then proved.

□

Choosing a Pesin block

We choose $\lambda \in (0,1)$ and $L > 0$ as in Lemma 5.4.1 to define the Pesin block $\Lambda^L_\lambda(k)$:

$$\Lambda^L_\lambda(k) := \left\{ x \in \Lambda : \prod_{i=1}^{n-1} ||P^*_L|_{E^s_{X_{iL(x)}}}|| \leq k\lambda^n, \prod_{i=1}^{n-1} ||P^*_{-L}|_{E^u_{X_{-iL(x)}}}|| \leq k\lambda^n, \forall n \geq 1, d(x, Sing(X) \geq \frac{1}{k} \right\}$$

By Proposition 5.4.3, since $\mu(\Lambda^L_\lambda(k)) \stackrel{k\to\infty}{\longrightarrow} 1$, for $k > 0$ large enough we have that $\mu(\Lambda^L_\lambda(k)) > 0$. Fix $\lambda_0 \in (\lambda, 1)$. Then

$$\prod_{i=1}^{n-1} ||P^*_L|_{E^s_{X_{iL(x)}}}|| \leq k\lambda_0^n$$

Since, for each $i \in \{0, \cdots, n-1\}$ the subbundle $E^s_{X_{iL(x)}}$ is contracted, we have that there exists $j_1 = j_1(k) \in \mathbb{N}$ such that

$$\prod_{i=1}^{n-1} ||P^*_{j_1L}|_{E^s_{X_{ij_1L(x)}}}|| \leq \lambda_0^n$$

For each $x \in \Lambda^L_\lambda(k)$ and any $n \geq 1$.

Similarly, since $E^u_{X_{-iL(x)}}$ is expanded, one has that there is $j_2 \in \mathbb{N}$ such that

$$\prod_{i=1}^{n-1} ||P^*_{-j_2L}|_{E^u_{X_{-ij_2L(x)}}}|| \leq \lambda_0^n$$

For each $x \in \Lambda^L_\lambda(k)$ and any $n \geq 1$.

Taking $j = j(k) = max\{j_1, j_2\}$, for each $x \in \Lambda^L_\lambda(k)$, one has

$$\prod_{i=1}^{n-1} ||P^*_{jL}|_{E^s_{X_{ijL(x)}}}|| \leq \lambda_0^n \quad and \quad \prod_{i=1}^{n-1} ||P^*_{-jL}|_{E^u_{X_{-ijL(x)}}}|| \leq \lambda_0^n, \quad \forall n \geq 1$$

Consider the set $\Lambda^{L_0}_{\lambda_0}(k)$ defined by

$$\Lambda^{L_0}_{\lambda_0}(k) := \left\{ x \in \Lambda = \prod_{i=1}^{n-1} ||P^*_{jL}|_{E^s_{X_{ijL(x)}}}|| \leq \lambda_0^n, \prod_{i=1}^{n-1} ||P^*_{-jL}|_{E^u_{X_{-ijL(x)}}}|| \leq \lambda_0^n, \forall n \geq 1, d(x, Sing(X) \cap \Lambda) \geq \frac{1}{k} \right\}$$

where $L_0 = jL$. Since μ is a regular measure and $\Lambda^L_\lambda(k) \subset \Lambda^{L_0}_{\lambda_0}(k)$, one has $\mu(\Lambda^{L_0}_{\lambda_0}(k)) \geq \mu(\Lambda^L_\lambda(k))$. Hereafter we fix this k.

By the Poincaré recurrence theorem for flows, since $\mu(\Lambda^{L_0}_{\lambda_0}(k)) > 0$, we have that for μ-a.e. $x \in \Lambda^{L_0}_{\lambda_0}(k)$, the forward orbit of x will return infinitely many times to $\Lambda^{L_0}_{\lambda_0}(k)$ and will be arbitrarily close to x. Let $\eta_0 = -\frac{1}{L_0}\log(\lambda_0)$. If $X_{nL_0}(x) \in \Lambda^{L_0}_{\lambda_0}(k)$ for some $n \in \mathbb{N}$, then $X_{[0,nL_0]}(x)$ is a (η_0, L_0)-quasi hyperbolic orbit arc. Indeed, by the definition of $\Lambda^{L_0}_{\lambda_0}(k)$, one has that $X_{[0,nL_0]}(x) \subset \Lambda^{L_0}_{\lambda_0}(k)$. Taking the partition $0 = t_0 < t_1 = L_0 < t_2 = 2L_0 < t_3 = 3L_0 < \cdots < nL_0 = t_l = T$, we have $t_{i+1} - t_i = L_0$ for any $0 \leq i \leq l-1$. Moreover,

$$\prod_{i=1}^{n-1} ||P^*_{L_0}|_{E^s_{X_{iL_0(x)}}}|| \leq \lambda_0^n = e^{-\eta_0 n L_0}, \quad n = 0, 1, \cdots, l-1$$

Similarly,

$$\prod_{i=n}^{l-1} m(P^*_{L_0}|_{E^u_{X_{iL_0(x)}}}) \geq e^{\eta_0(T - nL_0)}, \quad n = 0, 1, \cdots, l-1$$

That is, $X_{[0,nL_0]}(x)$ is a (η_0, L_0)-quasi hyperbolic orbit arc.

The shadowing constants

Let $C = \max\left\{1, \max_{x \in M} ||X(x)||\right\}$. Given $\varepsilon_0 = \frac{1}{k}$, $\eta = \eta_0$, $T_0 = L_0$ and $\varepsilon > 0$, for $\varepsilon_1 = \frac{\varepsilon}{3C}$, by Liao's shadowing lemma, there exists $\delta = \delta(\varepsilon)$ much smaller than ε such that for any $x, X_{nL_0}(x) \in \Lambda^{L_0}_{\lambda_0}(k)$, if $d(x, X_{nL_0}(x)) < \delta$, then there is a point $p \in M$ and a C^1-increasing homeomorphism $\theta : [0, nL_0] \to \mathbb{R}$ such that $X_{\theta(nL_0)}(p) = p$ and $d(X_t(x), X_{\theta(t)}(p)) < \varepsilon_1 ||X(X_t(x))|| = \frac{\varepsilon}{3C} ||X(X_t(x))|| \leq \frac{\varepsilon}{3C} C = \frac{\varepsilon}{3}$ for all $t \in [0, nL_0]$. Moreover, by Proposition 5.3.3, one has $|\theta(t) - t| \leq N d(x, X_{nL_0}(x)) \leq N\delta$ for any integer $t \in [0, nL_0]$, where N is constant independent of x and n. One can also assume that $N\delta$ is much smaller than ε.

A separation set K_n

For $\varepsilon > 0$ and $n \in \mathbb{N}$, we claim that there exists a finite set $K_n = K_n(k, \varepsilon) \subset \Lambda^{L_0}_{\lambda_0}(k)$ with the following properties:

1. For any two points $x, y \in K_n$, there is $t \in \mathbb{N} \cap [0, nL_0]$ such that $d(X_t(x), X_t(y)) > \varepsilon$.

2. For any $x \in K_n$, there exists an integer $m = m(n)$ with $n < m \leq (1+\varepsilon)n$ such that $X_{mL_0}(x) \in \Lambda^{L_0}_{\lambda_0}(k)$ and $d(x, X_{mL_0}(x)) < \delta(\varepsilon)$.

3. $\liminf_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{nL_0} \log \#K_n \geq h_\mu(X_1)$.

The construction of K_n

Now, we give the precise construction of K_n. Consider the following set:

$$\Lambda^{L_0}_{\lambda_0}(k, n) = \{x \in \Lambda^{L_0}_{\lambda_0}(k) : \exists m \in [n, (1+\varepsilon)n], X_{mL_0}(x) \in \Lambda^{L_0}_{\lambda_0}(k), d(x, X_{mL_0}(x)) < \delta\}$$

Since $\mu(\Lambda_{\lambda_0}^{L_0}(k)) > 0$, taking $f = X_{L_0}$, by Proposition 5.5.2 we have

$$\lim_{n\to\infty} \mu(\Lambda_{\lambda_0}^{L_0}(k,n)) = \mu(\Lambda_{\lambda_0}^{L_0}(k))$$

We take a maximal choice of $K_n = K_n(k,\varepsilon) \subset \Lambda_{\lambda_0}^{L_0}(k,n)$ such that item 1 is satisfied. By definition of $\Lambda_{\lambda_0}^{L_0}(k,n)$, item 2 is satisfied.

For item 3, we use Katok's metric entropy. By maximality of K_n, one has

$$\#K_n \geq S_{X_1}(nL_0, \varepsilon, 1 - \mu(\Lambda_{\lambda_0}^{L_0}(k,n)))$$

Thus,

$$\liminf_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{nL_0} \log \#K_n \geq \liminf_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{nL_0} \log S_{X_1}(nL_0, \varepsilon, 1 - \mu(\Lambda_{\lambda_0}^{L_0}(k,n))) \geq h_\mu(X_1)$$

The construction of K_n is hence complete.

Estimating the growth rate of the periodic orbits

Now, we can complete the proof of Theorem 5.2.2. For every point $x \in K_n$, by item 2 of the construction of K_n, there is m_x with $n < m_x \leq (1+\varepsilon)n$ such that $X_{m_x L_0}(x) \in \Lambda_{\lambda_0}^{L_0}(k)$ and $d(x, X_{m_x L_0}(x)) < \delta(\varepsilon)$. By Liao's shadowing lemma, there exists a C^1-strictly increasing homeomorphism $\theta_x : [0, m_x L_0] \to \mathbb{R}$ and a periodic point $p = p_x$ of period $\theta_x(m_x L_0)$ such that

$$d(X_t(x), X_{\theta_x(t)}(p)) = \varepsilon_1 ||X(X_t(x))|| < \frac{\varepsilon}{3}, \quad \forall t \in [0, m_x L_0]$$

By Proposition 5.3.3, one has that

$$|\theta_x(t) - t| \leq N d(x, X_{m_x L_0}(x)) \leq N\delta, \quad \forall t \in \mathbb{N} \cap [0, m_x L_0]$$

For any two different points $x, y \in K_n$, by item 1 of the construction of K_n, there exists $j \in \mathbb{N} \cap [0, nL_0]$ such that $d(X_j(x), X_j(y)) > \varepsilon$. Thus, by the triangle inequality, we have

$$d(X_{\theta_x(j)}(p_x), X_{\theta_y(j)}(p_y)) \geq d(X_{\theta_x(j)}(P_x), X_j(y)) - d(X_{\theta_y(j)}(P_y), X_j(y))$$

and

$$d(X_{\theta_x(j)}(P_x), X_j(y)) \geq d(X_j(x), X_j(y)) - d(X_{\theta_x(j)}(P_x), X_j(x))$$

Therefore,

$$\begin{aligned} d(X_{\theta_x(j)}(P_x), X_{\theta_y(j)}(P_y)) &\geq d(X_j(x), X_j(y)) - d(X_{\theta_x(j)}(P_x), X_j(x)) - d(X_{\theta_y(j)}(P_y), X_j(y)) \\ &> \varepsilon - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} \\ &= \frac{\varepsilon}{3} \end{aligned}$$

We claim that

$$X_{(\theta_x(j)-\frac{\varepsilon}{32C}, \theta_x(j)+\frac{\varepsilon}{32C})}(p_x) \cap X_{(\theta_y(j)-\frac{\varepsilon}{32C}, \theta_y(j)+\frac{\varepsilon}{32C})}(p_y) = \emptyset$$

where $C = \sup_{z \in M} ||X(z)|| < \infty$.

Indeed, by Proposition 5.3.3, taking $\delta \in \big(0, \frac{\varepsilon}{64CN}\big)$, one has $|\theta_x(j) - j| \leq N\delta$ and $|\theta_y(j) - j| \leq N\delta$. Therefore, by the triangle inequality, one has

$$|\theta_x(j) - \theta_y(j)| \leq |\theta_x(j) - j| + |\theta_y(j) - j| \leq N\delta + N\delta = 2N\delta < 2N\frac{\varepsilon}{64CN} = \frac{\varepsilon}{32C}$$

If $t \in \big(-\frac{\varepsilon}{32C}, \frac{\varepsilon}{32C}\big)$, using the fact the derivative of the flow equals to the vector field, it follows from the mean value theorem that

$$d(X_{\theta_x(j)+t}(p_x), X_{\theta_x(j)}(p_x)) \leq \sup_{\tau \in \mathbb{R}}\{||DX_\tau(p_x)||\}|\theta_x(j) + t - \theta_x(j)| \leq C|t| \leq 2C\frac{\varepsilon}{32C} = \frac{\varepsilon}{16}$$

Analogously,

$$d(X_{\theta_y(j)+t}(p_y), X_{\theta_y(j)}(p_y)) \leq \frac{\varepsilon}{16}$$

Consequently, for any $t, s \in (-\frac{\varepsilon}{32C}, \frac{\varepsilon}{32C})$, by the triangle inequality,

$$d(X_{\theta_x(j)+t}(p_x), X_{\theta_y(j)+s}(p_y)) \geq d(X_{\theta_x(j)}(p_x), X_{\theta_y(j)+s}(p_y)) - d(X_{\theta_x(j)}(p_x), X_{\theta_x(j)+t}(p_x))$$

and

$$d(X_{\theta_x(j)}(p_x), X_{\theta_y(j)+s}(p_y)) \geq d(X_{\theta_x(j)}(p_x), X_{\theta_y(j)}(p_y)) - d(X_{\theta_y(j)}(p_y), X_{\theta_y(j)+s}(p_y))$$

Therefore,

$$\begin{aligned} d(X_{\theta_x(j)+t}(p_x), X_{\theta_y(j)+s}(p_y)) &\geq d(X_{\theta_x(j)}(p_x), X_{\theta_y(j)}(p_y)) - d(X_{\theta_x(j)}(p_x), X_{\theta_x(j)+t}(p_x)) \\ &\quad - d(X_{\theta_y(j)}(p_y), X_{\theta_y(j)+s}(p_y)) \\ &> \frac{\varepsilon}{3} - \frac{\varepsilon}{16} - \frac{\varepsilon}{16} \\ &= \frac{\varepsilon}{3} - \frac{\varepsilon}{4} \\ &= \frac{\varepsilon}{12} \\ &> 0 \end{aligned}$$

This prove the claim.

From the claim, for $z \in orb(p_x)$, any orbit segment $X_{[0,1]}(z)$ contains at most $\frac{32C}{\varepsilon}$ points in the set $\{p_x\}_{x \in K_n}$. Consequently, we have that

$$\sum_{x \in P_T(X), nL_0(1-\varepsilon)-N\delta \leq \pi(x) \leq nL_0(1+\varepsilon)+N\delta} \pi(x) \geq \frac{\varepsilon}{32C} \# K_n$$

Thus,

$$\# P_{nL_0(1+\varepsilon)+N\delta}(X) \geq \frac{\varepsilon}{32C} \# K_n$$

Therefore,

$$\begin{aligned} \limsup_{T \to \infty} \frac{1}{T} \log \# P_T(X) &\geq \limsup_{n \to \infty} \frac{1}{nL_0(1+\varepsilon)+N\delta} \log \# P_{nL_0(1+\varepsilon)+N\delta}(X) \\ &= \limsup_{n \to \infty} \frac{1}{nL_0} \frac{nL_0}{nL_0(1+\varepsilon)+N\delta} \log \# P_{nL_0(1+\varepsilon)+N\delta}(X) \\ &= \lim_{n \to \infty} \frac{nL_0}{nL_0(1+\varepsilon)+N\delta} \limsup_{n \to \infty} \frac{1}{nL_0} \log \# P_{nL_0(1+\varepsilon)+N\delta}(X) \end{aligned}$$

Since $\lim_{n \to \infty} \frac{nL_0}{nL_0(1+\varepsilon)+N\delta} = \frac{1}{1+\varepsilon}$, we have

$$
\begin{aligned}
\limsup_{T\to\infty}\frac{1}{T}\log\#P_T(X) &\geq \frac{1}{1+\varepsilon}\limsup_{n\to\infty}\frac{1}{nL_0}\log\#P_{nL_0(1+\varepsilon)+N\delta}(X)\\
&\geq \frac{1}{1+\varepsilon}\liminf_{n\to\infty}\frac{1}{nL_0}\log\left(\frac{\varepsilon}{32C}\#K_n\right)\\
&\geq \frac{1}{1+\varepsilon}\liminf_{n\to\infty}\frac{1}{nL_0}\log(\#K_n)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\liminf_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{1}{T}\log\#P_T(X) &\geq \liminf_{\varepsilon\to 0}\frac{1}{1+\varepsilon}\liminf_{n\to\infty}\frac{1}{nL_0}\log(\#K_n)\\
&= \lim_{\varepsilon\to 0}\frac{1}{1+\varepsilon}\liminf_{n\to\infty}\frac{1}{nL_0}\log(\#K_n)\\
&= \liminf_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{1}{nL_0}\log(\#K_n)\\
&\geq h_\mu(X)
\end{aligned}
$$

Since μ is arbitrary and $\liminf_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{1}{T}\log\#P_T(X)$ does not depends on ε, one has

$$
\limsup_{T\to\infty}\frac{1}{T}\log\#P_T(X)\geq h_{top}(X)
$$

□

5.6 Proof of Theorem B

Proof of the main theorem (Theorem B). Take a residual set $\mathscr{R}\subset\mathscr{X}^1(M)$ as in Theorem 5.0.2. For any $X\in\mathscr{R}$, if X is not star, by Theorem 5.1.1, one has

$$
\limsup_{T\to\infty}\frac{1}{T}\log Per_T(X)=\infty > h_{top}(X)
$$

If X is star, then any ergodic invariant measure μ is hyperbolic by Lemma 4.4.8. We can assume that $h_{top}(X)>0$, since if $h_{top}(X)=0$, the inequality is true. According to Theorem 5.2.1, the hyperbolic Oseledec splitting $N=E^s\oplus E^u$ with respect to μ is a dominated splitting. By Theorem 5.2.2, one has

$$
\limsup_{T\to\infty}\frac{1}{T}\log Per_T(X) > h_\mu(X)
$$

By the variational principle,

$h_{top}(X) = h_{top}(X_1) = \sup\{h_\mu(X_1) : \mu$ is an ergodic measure of $X\}$.

Thus, we have

$$\limsup_{T\to\infty} \frac{1}{T} \log Per_T(X) > h_{top}(X)$$

The proof of the main theorem is hence complete.

□

Bibliography

[AKM65] R. L. Adler, A. G. Konheim, and M. H. McAndrew. Topological entropy. *Trans. Amer. Math. Soc.*, (114):309–319, 1965.

[Aok92] N. Aoki. The set of Axiom A diffeomorphisms with no cycle. *Bol. Soc. Brasil Mat.*, (23):21–65, 1992.

[ASS] A. Arbieto, B. Santiago, and T. Sodero. *Fluxos estrela.* Publicações Matemáticas (28° Colóquio Brasileiro de Matemática). IMPA. In Portuguese.

[BM] S. Bautista and C. Morales. Lectures on Anosov-flows. Monograph.

[Bow70] R. Bowen. Topological entropy and Axiom A. *Proceedings of Symposia in Pure Mathmatics*, (14):23–41, 1970.

[Fra71] J. Franks. Necessary conditions for stability of diffeomorphisms. *Trans. Amer. Math. Soc.*, (158):301–308, 1971.

[Gan02] S. Gan. A generalized shadowing lemma. *Discrete and Continuous Dynamical Systems*, (8):627–632, 2002.

[Guc76] J. Guchenheimer. *A Strange Strange Attractor. The Hopf Biffurcation Theorems and Its Applications (Applied Mathematical Series, 19)*, pages 368–381. Springer-Verlag, 1976.

[GW06] S. Gan and L. Wen. Nonsingular star flows satisfy Axiom A and the no-cycle condition. *Invent. Math*, (164):279–365, 2006.

[Hay92] S. Hayashi. Diffeomorphisms in $\mathscr{F}^1(M)$ satisfy Axiom A. *Ergodic Theory & Dynamical Systems*, (12):233–253, 1992.

[Kat80] A. Katok. Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. *Publications Mathématiques de l'I.H.É.S*, (51):137–173, 1980.

[Lee97] J. Lee. *Riemannian manifolds: an introduction to curvature*, volume 176 of *Graduate Texts in Mathmatics.* Springer-Verlag, 1997.

[LGW05] M. Li, S. Gan, and L. Wen. Robustly transitive singular sets via approch of the extended linear Poincaré flow. *Discrete Contin. Dyn. Syst.*, (13):239–269, 2005.

[Lia79] S. Liao. A basic property of a certain class of differential systems. *Acta Math. Sinica*, (22):316–343, 1979. in chinese.

[Lia81a] S. Liao. Certain uniform properties of differential systems and a generalization of an existence theorem for periodic orbits. *Acta Sci. Natur. Univ. Pekinensis*, (2):1–19, 1981. in chinese.

[Lia81b] S. Liao. Obstruction sets II. *Acta Sci. Natur. Univ. Pekinensis*, (2):1–36, 1981. in chinese.

[Lia96] S. Liao. *Qualitative theory of differentiable dynamical systems*. Science Press (Distributed by American Mathematical Society), 1996. Translated from the Chinese.

[LW95] C. Li and L. Wen. $\mathscr{X}^*$ plus Axiom A does not imply no cycle. *J. Differential Equations*, (2):395–400, 1995.

[Mañ82] R. Mañé. An ergodic closing lemma. *Annals of Math.*, (116):503–540, 1982.

[Mañ87] R. Mañé. *Ergodic theory and differentiable dynamics*. Springer-Verlag, 1987.

[PS70a] J. Palis and S. Smale. Structural stability theorems. *Proc. Symp. Pure Math.*, (14):223–232, 1970.

[PS70b] C. Pugh and M. Shub. Ω-stability for flows. *Invent. Math.*, (11):150–158, 1970.

[SGW14] Y. Shi, S. Gan, and L. Wen. On the singular hyperbolicity of star flows. *Journal of modern dynamics*, (8):191–219, 2014.

[Sma70] S. Smale. The Ω-stability theorem. *Proceedings of Symposia in Pure Mathmatics*, (14):289–297, 1970.

[Wal82] P. Walters. *An introduction to ergodic theory*, volume 79 of *Graduate texts in mathematics*. Springer-Verlag, 1982.

[Wen16] L. Wen. *Differentiable dynamical systems: an introduction to structural stability and hyperbolicity*, volume 173 of *Graduate studies in mathematics*. American Mathematical Society, 2016.

[WYZ19] W. Wu, D. Yang, and Y. Zhang. On the growth rate of periodic orbits for vector fields. *Advances in Mathmatics*, (346):170–193, 2019.