SHAPE THEOREM FOR THE SPREAD OF EPIDEMICS

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To my nephews Vicky, Alice and Jesus Eduardo.

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Chapter 1

Introduction

It is not a mystery that humans, we are characterized as curious, and this is undoubtedly what led the pioneers of the study of the epidemic models to try to answer if it is possible to estimate at a certain time, after the epidemic starts, who are infected?, who are immune? or How quickly does the epidemic spread?.

In the present work "Shape theorem for the spread of epidemics", an epidemic model is in such a way individual can only be in 3 states: $1 =$healthy, $i =$infected or $0 =$immune (see [4] and [14]); so that a healthy individual goes to the infected state when one of his neighbors, who is infected, sends him a germ. An individual in infected state emits germs for some time (this time is called lifetime) and then moves to the state immune, and, being immune, it never changes again. An example to what was exposed, would be the current epidemic known as smallpox. Of course, we will focus on the oriented epidemic model, that is, the event that x infects y, is different from the event that y infects x. Later we will give a formal definition of these events. It should be noted that this epidemic model has another interpretation, for a forest fire: $1 =$a live tree, $i =$on fire, and $0 =$burned (see [16] and [17]).

This dissertation is mainly based on the study of Cox and Durrett result in [7], where they worked the epidemic model on $\mathbb{Z}^2$, for individuals whose lifetime is random and only subject to a condition of integrability, and for contamination taking place between nearest neighbors. They showed that if the epidemic spreads with positive probability (or that there is percolation), and assuming that initially only the origin is infected and the rest of the individuals are healthy, then the set of infected sites is linearly asymptotic to the boundary of a convex set that contains the set of points in immune state. This result is known as the "Shape Theorem". It was further extended by Zhang (see [19]), Chabot (see [5]) and Andjel, Chabot and Saada (see [2]). The most recent result so far is that of [2], who demonstrated the shape theorem for general lifetimes, on $\mathbb{Z}^d$ with $d \geq 3$. This result is not studied here because more time is required to study it in detail, but it should be noted that it is up to now the up to now most general result in this context.

The two results subsequent to that of Cox and Durrett are that of Zhang (in [19]) who worked an epidemic model on $\mathbb{Z}^2$ with finite range interactions, and of Chabot (in [5]) who demonstrated the shape theorem on $\mathbb{Z}^d$ with $d \geq 3$, with respect to the nearest neighbor, but with a constant lifetime. These last two, in addition to the case of Cox and Durrett, are the ones studied in this work. Throughout this work, we will refer to the models in [7], [19] and [5] respectively, although these models have been presented before.

Formalizing what was discussed above, let $G_1 = (\mathbb{Z}^2, \mathbb{E})$, $G_2 = (\mathbb{Z}^2, \mathbb{E}_R)$ and $G_3 = (\mathbb{Z}^d, \mathbb{E}^d)$ be the graphs corresponding to the models of Cox and Durrett, Zhang, and Chabot respectively, where $R = 2, 3, 4, ...,$ $d \geq 3$ and:

$$\begin{aligned}(x,z) \in \mathbb{E} &\Leftrightarrow x, z \in \mathbb{Z}^2 \text{ and } \|x - z\|_1 = 1 \\ (x,z) \in \mathbb{E}_R &\Leftrightarrow x, z \in \mathbb{Z}^2 \text{ and } 1 \leq \|x - z\|_\infty \leq R \\ (x,z) \in \mathbb{E}^d &\Leftrightarrow x, z \in \mathbb{Z}^d \text{ and } \|x - z\|_1 = 1\end{aligned}$$

In these graphs, each vertex represents an individual, and each oriented edge (x, z) is where the germ will pass from x to z, in case of x is infected and has chosen z to infect it. We will specify the graph to which we refer when necessary. Each site z can be in the one of three states; $1, i$ or 0. In the epidemic interpretation $1 =$ healthy, $i =$ infected and $0 =$ immune. Let η_s denote the process, $\eta_s(x)$ is the state of site x at time s. An infected individual emits germs according to a Poisson process with rate α. Specifying the germ emission process of each model:

- *Model of Cox and Durrett:* A germed emitted from x goes to one of the four nearest neighbors $x + (1, 0)$, $x + (0, 1)$, $x + (-1, 0)$, $x + (0, -1)$ chosen at random (with equal probabilities). Let T_x, $x \in \mathbb{Z}^2$ be independent identically distributed random variables with distribution F and for $x, z \in \mathbb{Z}^2$ with $\|x - z\|_1 = 1$ let $e(x, z)$ be independent identically distributed random variables with $P(e(x, z) > s) = exp(-\alpha s/4)$. We assume that F is concentrated on the nonnegative half line and is not the unit point mass at zero. To complete the description we declare that the infection periods and Poisson processes of germs associated with different sites are independent. T_x is the amount of time x will stay infected (if it ever becomes infected) and $e(x, z)$ is the time lag from the infection of x until the first germ from x is sent to z. We let

$$\tau(x,z) = \begin{cases} e(x,z) & if \quad T_x > e(x,z) \\ \infty & if \quad T_x \leq e(x,z) \end{cases}$$

and say the oriented bound (x,z) is *open* if $\tau(x,z)<\infty$ and *closed* otherwise. Given the definition of T_x and $e(x,z)$ it should be clear that bond (x,z) is open if x tries to infect z during its "lifetime" and $\tau(x,z)$ gives the time lag from the infection of x until it tries to infect z, with $\tau(x,z)=\infty$ if this never happens. We consider the density of open bonds

$$p = P((x,z)\text{is open}) = 1 - \int_0^\infty \exp(-s\alpha/4)dF(s).$$

- *Model of Zhang:* A germ emitted from x goes to a point z in

$$N_x = \left\{y \in \mathbb{Z}^2 : 1 \le \|z-x\|_\infty \le R\right\}$$

for some finite number $R>1$ at rate $\alpha g(z-x)$, where g is a function from $\mathbb{Z}^2 \to [0,1)$ such that

$$g(z) = 0 \text{ if } z \notin N_0, \quad g(z) > 0 \text{ otherwise,}$$

$$g(z) = g(-z)$$

and,

$$\sum_{z\in N_0} g(z) = 1.$$

In the same way as the previous model, let T_x, $x \in \mathbb{Z}^2$, be independent random variables with distribution F. Let $e(x,z)$, for all $x,z \in \mathbb{Z}^2$ and $1 \le \|z-x\|_\infty \le R$ be independent random variables with

$$P(e(x,z) > s) = \exp(-s\alpha g(z-x)).$$

We define $\tau(x,z)$ exactly like in the model previous model (only this time for $1 \le \|z-x\|_\infty \le R$). We say that the oriented edge (x,z) is *open* if $\tau(x,z)<\infty$ and closed otherwise, thus:

$$p = P((x,z) \text{ is open}) = 1 - \int_0^\infty \exp(-s\alpha g(z-x))dF(s). \tag{1.1}$$

- *Model of Chabot:* In this case, each individual is on $\mathbb{Z}^d$ (with $d \ge 3$) and will be denoted as above, by the variables x, y or z. A germ emitted from x will reach any of these $2d$ nearest neighbors with uniform distribution. In this model, an infected individual has a constant lifetime T during which it emits germs. If $(x,z) \in \mathbb{E}^d$, let $e(x,z)$ be a random variable of exponential distribution of parameter $\alpha/(2d)$ which represents the first moment of passage of a germ from x to z after the infection of x. The $(e(x,z))$ are chosen independent. Let

$$\tau(x,z)=\begin{cases} e(x,z) & if \quad T>e(x,z)\\ \infty & if \quad T\le e(x,z)\end{cases}$$

The oriented edge (x,z) is *open* if $\tau(x,z)<\infty$ and *closed* otherwise, that is, if:

$$p=p(\alpha)=1-\exp(-\alpha T/(2d)) \tag{1.2}$$

then, for $x,z\in\mathbb{Z}^d$, with $\|x-z\|_1=1$:

$$P((x,z)\text{ is open})=p=1-P((x,z)\text{ is closed}) \tag{1.3}$$

In all three models, we are interested only in the first moment of emission of a germ from x to z, because only the first time a germ passes, it spreads the epidemic.

With the definitions above in mind, we define a *path*, in any of the three graphs: $(\mathbb{Z},\mathbb{E})$, $(\mathbb{Z},\mathbb{E}_R)$ or $(\mathbb{Z}^d,\mathbb{E}^d)$, as a finite sequence of points $(x_0,x_1,...,x_n)$ such that (x_{i-1},x_i) is an edge of the respective graph. Thus, we say that this path starts at x_0 and ends at x_n, and as each edge that composes it is oriented, we say that it is oriented.

We say that an oriented path $(x_0,x_1,...,x_n)$ is open if the bond (x_{i-1},x_i) is open for each $i=1,2,...n$. We will write "$x\to z$" when there is a open path from x to z. We are also going to define the first passage time from x to z, as follows:

$$t(x,z)=\inf\left\{\sum_{i=1}^{m}\tau(x_{i-1},x_i):(x=x_0,x_1,...,x_m=z)\text{ is a path from }x\text{ to }z\right\}$$

Note that if $t(x,z)=\infty$ then there is no open path from x to z. Let z be a given site, we define:

$$C_z=\{x:\ z\to x\},$$
$$\overline{C}_z=\{x:\ x\to z\}.$$

In percolation language C_0 and $\overline{C}_0$ are the outgoing and incoming cluster containing the origin 0 (of $\mathbb{Z}^2$ or $\mathbb{Z}^d$, when it is the case). Cox and Durrett have shown that C_0 is also the set of sites that can be joined from the origin by a path of open edges. From Kuulasmaa's work in [14], Cox and Durrett show the existence of two critical parameters, $\alpha_c^0,\ \alpha_c^i\in\mathbb{R}_+^*$, where:

$$\alpha_c^0=\inf\{\alpha:P(|C_0|=\infty)>0\}\ \text{ and }\ \alpha_c^i=\inf\left\{\alpha:P(\left|\overline{C}_0\right|=\infty)>0\right\}.$$

In the next chapters we will see the equality of these parameters. Also, note that the parameter α_c^0 in the first two models depends on the distribution function F, respectively, and in the third model depends only on the constant T. Analogously it happens for α_c^i

Returning to our main objective, the shape theorem in the three models, define ζ_s as the set of immune sites at time s e ξ_s the set of infected sites at time s, that is,

$$\zeta_s = \{x : \eta_s(x) = 0\} \quad \text{and} \quad \xi_s(x) = \{x : \eta_s(x) = i\} \tag{1.4}$$

We assume that initially the origin is infected and all other sites healthy,

$$\eta_0(x) = \begin{cases} i & if \quad x = 0 \\ 1 & if \quad x \neq 0 \end{cases}$$

The shape theorem is:

Teorema 1.1. *Assume that* $\int_0^\infty s^2 dF(s) < \infty$ *and* $\alpha > \alpha_c^0$. *Then there is a convex set* D *such that for any* $\epsilon > 0$,
i) $P(C_0 \cap s(1-\epsilon)D \subset \zeta_s \subset s(1+\epsilon)D$ *for all sufficiently large* $s) = 1$
ii)$P(\xi_s \subset s(1+\epsilon)D - s(1-\epsilon)D$ *for all sufficiently large* $s) = 1$

The second moment assumption is necessary for (ii) to hold.

Since we do not have guaranteed the integrability of $t(x, z)$, following [7], we will approximate this passage time by a new passage time $\hat{t}(x, z)$ which will be such that $E(\hat{t}(x, z))^r < \infty$, for all $r \in \mathbb{N}$. To do this, first we will build a *neighborhood of* x (set of individuals, containing x) for each individual x, which guarantee certain conditions imposed, in a certain way, by Cox and Durrett (see the next three chapters). After the neighborhoods have been constructed, for each pair of sites x and z, the time $t(x, z)$ will be approximate by $\hat{t}(x, z)$, where $\hat{t}(x, z)$ is the minimum of the first passage times from an individual in the neighborhood of x to another individual in the neighborhood of z. Thus, we demonstrate the existence of radial limits and the asymptotic shape, in all three cases, for $\hat{t}$. To conclude in the Chapter 6 with the proof of Shape Theorem. The last two chapters mentioned will be following the work of Cox and Durrett in [7]. An interesting thing to illustrate in this work is how we can apply the work of Cox and Durrett to prove the shape theorem after having built the neighborhoods.

To illustrate the model in [7] we include two images with a simulation in R. On the left hand side, the lifetimes have uniform distribution in the interval [0,10]. On the right hand

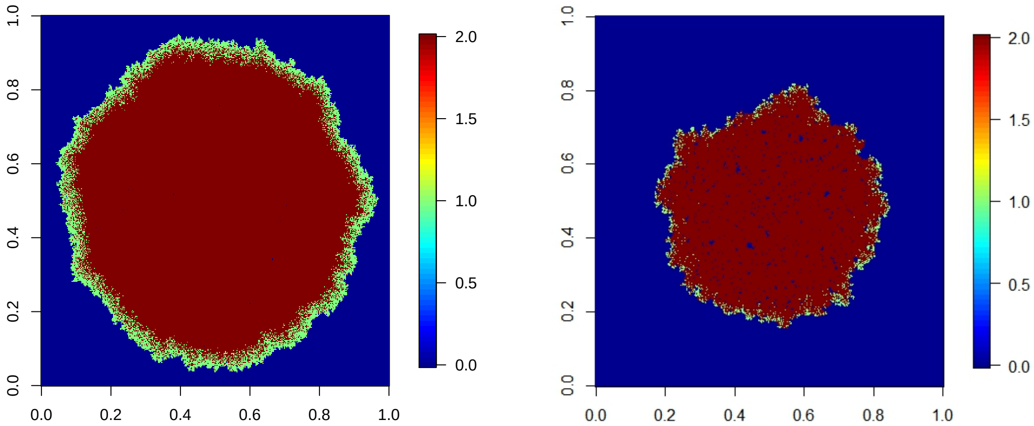


Figure 1.1: The states are 0 =healthy, 1 =infected and 2 =immune

side, the lifetimes have exponential distribution of rate $\alpha = 0.5$. The images illustrate the states of the individuals, 80 units of time from the initial condition η_0.

Chapter 2

Construction of neighborhoods and approximation of t in the model of Cox and Durrett

We start this chapter by studying crossings of the rectangles in $\mathbb{Z}^2$. Let $R_{J,K} = R(J,K)$ be the probability that there is a right-left crossing of the rectangle $(0,J) \times (0,K)$ by open bonds. We will derive inequalities relating the $R_{kL,L}$ for various values of k. These results will imply that if we define the sponge crossing critical value α_s by

$$\alpha_s = \inf \left\{ \alpha : \limsup_{L\to\infty} R_{L,L} = 1 \right\}$$

,

then $\alpha_c^0 = \alpha_s$, and the epidemic dies out at the critical value. The following Lemma is shown in [7], following the ideas of Russo in [18]:

Lemma 2.1. *(RSW)* $R_{3L/2,L} \geq (1-(1-R_{L,L})^{1/2})^3$

With Lemma 2.1 in hand, the next step is to prove:

Lemma 2.2. *1-*$R_{kL,L} \leq 4(1 - R_{(k+1)L/2,L})$ *for* $k \geq 1$.

Proof: To prove this we draw a picture (see figura 2.1) and observe that if all 4 paths exist then there is a crossing. The inequality above results from

$$P(\bigcup_{i=1}^{4} A_i^c) \leq \sum_{i=1}^{4} P(A_i^c).$$

Since, $P(A_1) = P(A_4) = R_{(k+1)L/2,L}$ and $P(A_3) = P(A_2) = R_{L,L} \geq R_{(k+1)L/2,L}$. □

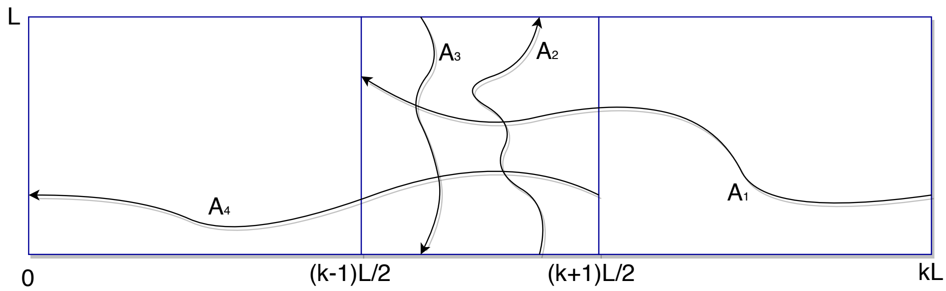


Figure 2.1: The events A_1, A_2, A_3 *and* A_4

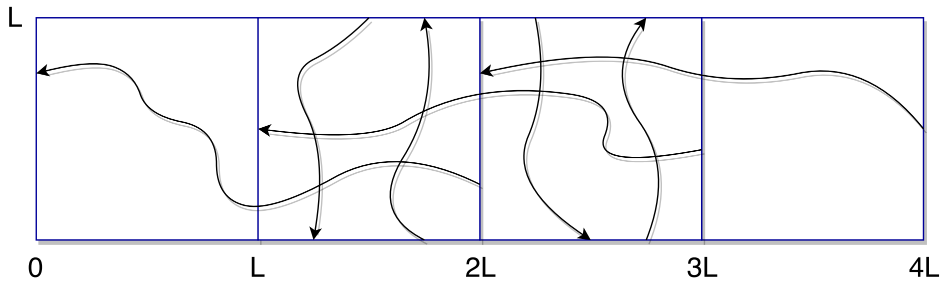


Figure 2.2: Seven paths

Using the lemmas 2.1 and 2.2, we obtain

$$\begin{aligned}
R_{3L/2,L} &\geq (1-(1-R_{L,L})^{1/2})^3, \\
R_{2L,L} &\geq 1-4(1-R_{3L/2,L}), \\
R_{3L,L} &\geq (1-4(1-R_{2L,L}),
\end{aligned}$$

and so on. The point is that once $R_{L,L}$ is close to 1 all the $R_{kL,L}$ are.

The next two inequalities (due to Aizenman, Chayes, Chayes, Frohlich, and Russo (1983)) will allow us to conclude that if $R_{L,L}$ is close enough to one for some L, then $R_{L,L} \to 1$ as $L \to \infty$.

$$1 - R_{4L,L} \leq 7(1-R_{2L,L}) \qquad\qquad R_{4L,2L} \geq 1-(1-R_{4L,L})^2. \tag{2.1}$$

Proof: For the inequality on the left, we draw another picture (see figure 2.2), observe that if all seven paths exist then there is a crossing, and then argue as in the proof of Lemma 2.2. To prove the inequality on the right, we observe that the existence of open crossings in $(0,4L)\times(0,L)$ and in $(0,4L)\times(L,2L)$ are independent. This is reason for using open rectangles in the definition of $R_{K,L}$. $\square$

Combining the inequalities (2.1) gives

$$R_{4L,2L} \geq 1 - 49(1 - R_{2L,L})^2. \tag{2.2}$$

If we iterate (2.2) assuming that $R_{2L,L} = 1 - \lambda/49$ for some $\lambda < 1$ we get

$$R_{4L,2L} \geq 1 - \lambda^2/49,$$
$$R_{8L,4L} \geq 1 - \lambda^4/49.$$

and by induction

$$R_{2^k L, 2^{k-1} L} \geq 1 - \frac{1}{49}\exp(2^{k-1} log\lambda).$$

Combining this with (2.1) we also obtain

$$R_{2^{k+1} L, 2^{k-1} L} \geq 1 - \frac{1}{7}\exp(2^{k-1} log\lambda). \tag{2.3}$$

By this result it follows that if R_{2L_0,L_0} is close enough to one then $R_{2^{k+1}L_0,2^{k-1}L_0} \to 1$ as $k \to \infty$, and $R_{2L,L} \to 1$ as $L \to \infty$.

The development above motivates defining

$$L_0(\alpha) = \inf \{L : R_{2L,L}(\alpha) \geq 0.99\}, \tag{2.4}$$

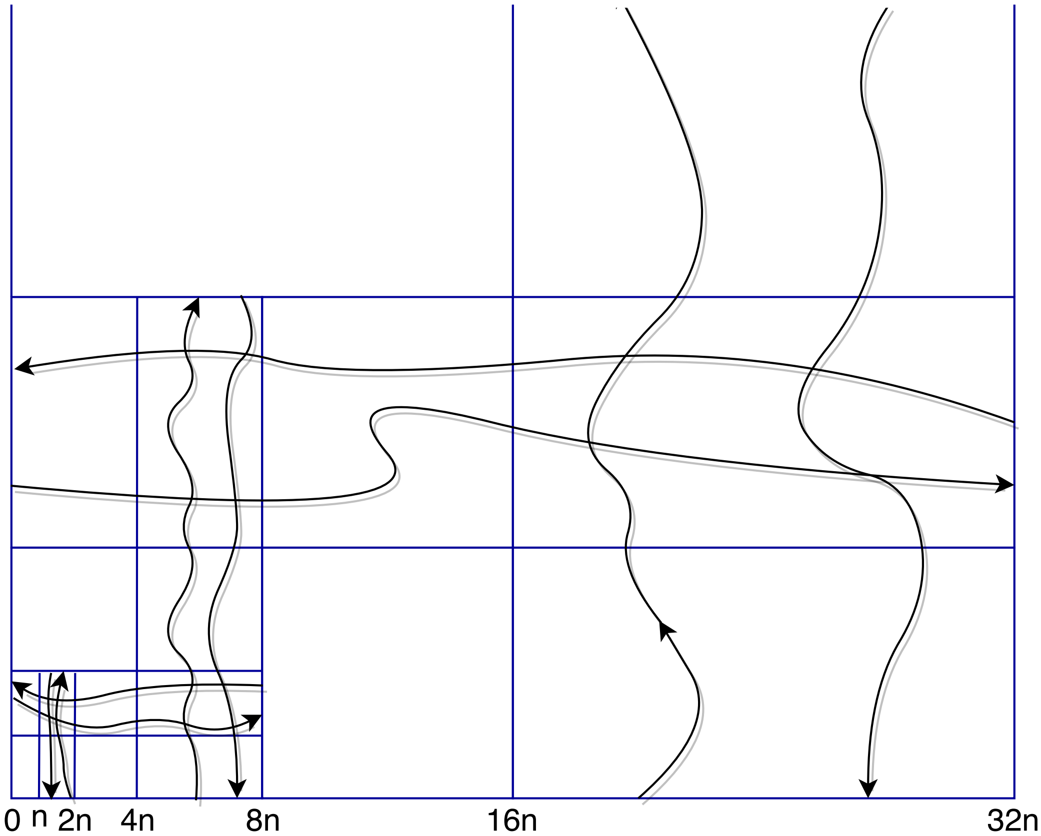
which must be defined for $\alpha > \alpha_s$. The next result shows that all three critical values are the same, and the epidemic dies out at the critical value.

Teorema 2.3. $\alpha_s = \alpha_c^i = \alpha_c^0$ *and*

$$P_{\alpha_s}(|C_0| = \infty) = P_{\alpha_s}(|\overline{C}_0| = \infty) = 0. \tag{2.5}$$

Proof: To prove this theorem consider first $\alpha > \alpha_s$, $n = L_0(\alpha)$, and for $j \geq 1$

$$B_{2j-1} = (2^{2j-2}n, 2^{2j-1}n) \times (0, 2^{2j}n),$$
$$B_{2j} = (0, 2^{2j+1}n) \times (2^{2j-1}n, 2^{2j}n),$$
$$A_{2j-1} = \{\text{there are top-bottom and bottom-top crossings } B_{2j-1}\},$$
$$A_{2j} = \{\text{there are left-right and right-left crossings } B_{2j}\},$$



(see fig. 2.3). By Harris-FKG inequality and (2.3):

$$P(\bigcap_{k=1}^{\infty} A_k) \geq \prod_{k=1}^{\infty} R(2^{k+1}n, 2^{k-1}n)^2 > 0. \tag{2.6}$$

Since there is positive probability that all the bonds on the segment from $(0,0)$ to $(2n,0)$ are open it follows that both $P_\alpha(|C_0| = \infty)$ and $P_\alpha(|\overline{C}_0| = \infty)$ are strictly positive.

The construction above shows $\alpha \geq \max(\alpha_c^i, \alpha_c^0)$. To prove the other inequality observe that if $\alpha < \alpha_s$ then $R_{2L,L} \leq \frac{48}{49}$ for all L, or else (2.3) would imply $R(2^{k+1}L, 2^{k-1}L) \to 1$ and hence $\alpha > \alpha_s$. So it follows from 2.1 and (2.2) that there is an $\epsilon_0 > 0$ so that $R_{L,L}(\alpha) \leq 1 - \epsilon_0$ for all L and $\alpha < \alpha_s$. By continuity, the last conclusion implies $R_{L,L}(\alpha_s) \leq 1 - \epsilon_0$. With the probabilities of sponge crossing bounded away from 1, we can now use the original argument of Harris in [10] to show there is no percolation:
Introduce the dual percolation process with sites in $\mathbb{Y}^2 = (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$, and call be bond (u, v) between neighboring points in $\mathbb{Y}^2$ open (closed) if the bond on the original lattice obtained by rotating it 90 counterclockwise around its midpoint is closed (respectively open).

This duality is the natural generalization to oriented percolation of the duality used in the ordinary case (see Chapter 2 of [11]) and has many of the same properties. In particular, we have:

(*) *Either there is right-left crossing of* $(0,L)\times(0,L)$ *or a top-bottom crossing of* $(\frac{1}{2}, L-\frac{1}{2})\times(\frac{1}{2}, L-\frac{1}{2})$ *on the dual, but not both.*

To prove this result suppose that there is a top-bottom crossing on the dual, then there is a self-avoiding one (i.e. each site appears at most once in the path). If we call the path σ, an application of the Jordan curve theorems shows that σ divides the interior of the square into 2 parts–one well call T_1 which lies to the right of σ, and one we call T_2 which lies to the left of σ. If we move along σ in the direction of the orientation, then T_1 is always on our left and T_2 is always on our right. From this, we see that if there is a path of open bonds from right to left then any time is crosses from T_1 to T_2, it does so along a bond which is a 90 clockwise rotation of a bond on σ. But such bonds are closed so no open path exist.

Now, we will suppose there is no right-left crossing and construct a top-bottom one, to conclude the other direction of (*). Let C be the set of points which can be reached from the right edge by a path of open bounds. Let $D=\left\{(a,b)\in\mathbb{R}^2 : |a|,|b|\le \frac{1}{2}\right\}$, and orient the boundary of D in a counterclockwise fashion. Finally, let $W=\bigcup_{z\in C}(z+D)$. If we combine the boundaries of the $z+D$ with $z\in C$, and let oppositely directed segments cancel, then the boundaries which remain are closed paths on the dual.

One of them,

$$\Gamma = \text{the boundary of the component of } (\frac{1}{2}, L-\frac{1}{2})\times(\frac{1}{2}, L-\frac{1}{2})\backslash W,$$

which contains the left side of the box, is the path that we want (see fig. 2.4). The reader should note that a similar construction can be used to prove that there is a lowest right-left crossing.

Having established (*), we can conclude that the probability of a top-bottom crossing of $(\frac{1}{2}, L-\frac{1}{2})\times(\frac{1}{2}, L-\frac{1}{2})$ is bounded away from 0 when $\alpha=\alpha_s$. If we let $\overline{R}_{L,L}$ denote this probability then applying the Harris-FKG inequality generalized to the model under consideration we have:

$$\overline{R}_{kL,L} \ge (1-(1-\overline{R}_{L,L})^{1/2})^3. \tag{2.7}$$

To see that this is legitimate, recall the proof of Lemma 2.1 works for a slightly different models from the original model with positively correlated edges, see the note after (2.2) in [7]. Although the dual bonds $(x,y)\to(x+1,y)\to(x+1,y+1)\to(x,y+1)\to(x,y)$ are dependent, since they depend on the edges with the same initial point $(x+1/2, y+1/2)$ in the original model. The bonds which go counterclockwise around different squares are

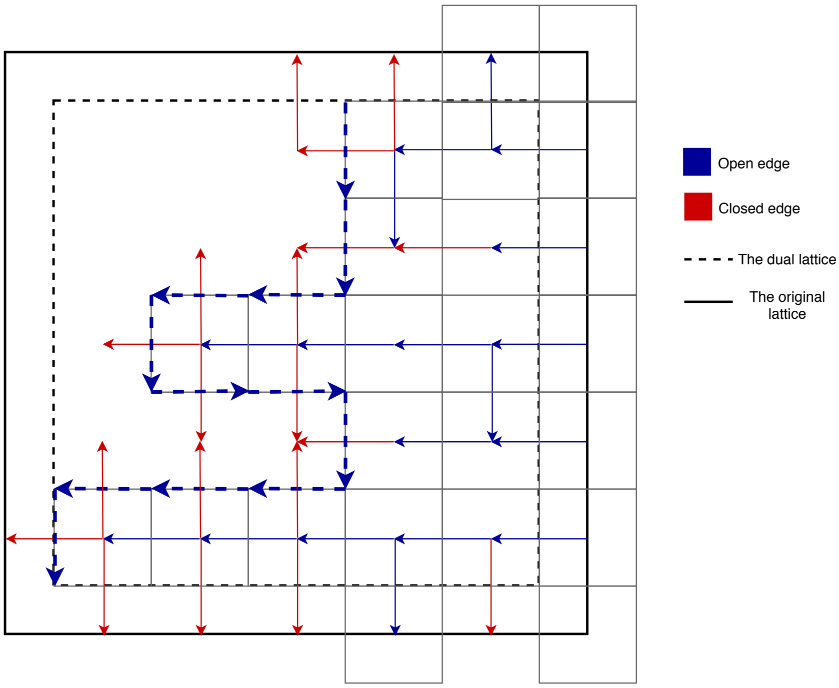


Figure 2.3: Construction of a top-bottom crossing on the dual lattice.

independent. From the last observations we see that if there is a right-left crossing σ then all the bonds above σ are independent of it and the previous argument works.

After (2.7), using the construction in the proof of Lemma 2.2 but making a different estimate shows

$$\overline{R}_{kL,L} \geq (\overline{R}_{(k-1)L/2,L})^4, \tag{2.8}$$

so we have

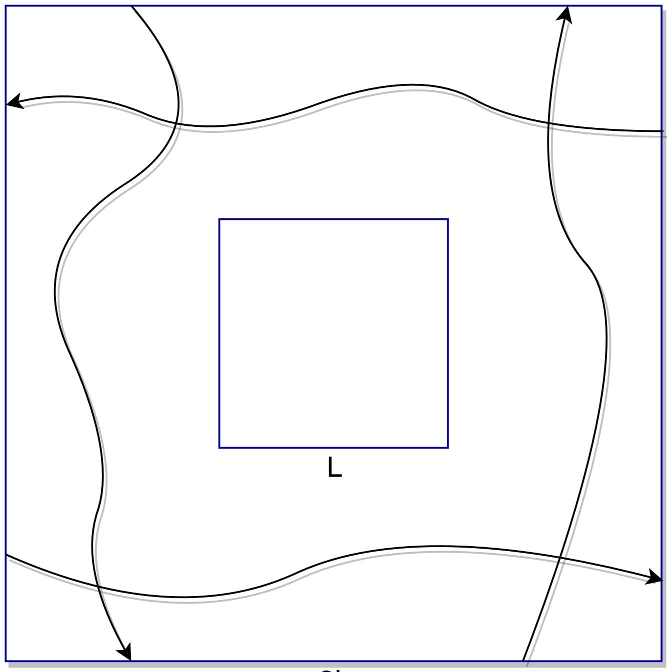
$$\overline{R}_{2L,L} \geq (\overline{R}_{3L/2,L})^4, \tag{2.9}$$

and

$$\overline{R}_{3L,L} \geq (\overline{R}_{(2L,L)})^4. \tag{2.10}$$

By combining crossing of $3L \times L$ rectangles, we get a circuit in an annulus (see figure 2.5). Since $\overline{R}_{L,L}$ is bounded away from 0, we have a ridiculously small but, nonetheless, positive lower bound on the occurrence of a dual circuit in an annulus. By considering an infinite disjoint sequence of annuli we see there is no percolation out or in when $\alpha = \alpha_s$. $\square$

We will need one more estimate for the next section. Let $\Gamma(k)$ be the event that there is an open circuit around the square $[-k, k]^2$ that is contained in the square $[-2k, 2k]^2$



and that there are infinite open paths starting and ending on $[-k,k]^2$. If $n = L_0(\alpha)$ then there is a finite positive constant γ' such that

$$P(\Gamma(2^k n)) \geq 1 - \gamma' \exp(2^k log\lambda). \tag{2.11}$$

The circuit in the annulus can be constructed by constructing four paths each at cost $R(2^{k+2}n, 2^k n)$ and paths to and from infinity can be constructed as in the proof of Theorem 2.3. The relevant probability estimates are (2.3) and (2.6).

Now, as we had promised, we will define for each $z \in \mathbb{Z}^2$ a neighborhood, and we will approximate the time of passage from a site x to another site y, for the time of passage from the neighborhood of x to the neighborhood of y. We start with some notation. For each $z \in \mathbb{Z}^2$ let $\kappa(z)$ be the smallest $k \geq 1$ such that:

i) there are infinite open paths to and from the square $z + [-k,k]^2$ and,

ii) there is an open circuit around $z + [-k,k]^2$ contained in $z + [-2k,2k]^2$.

Let $\Delta(z)$ be the minimal open circuit around z, where "minimal" means if $\kappa(z) = k$ then $\Delta(z)$ is the first open circuit in our ordering.

Having defined $\Delta(z)$, let $\widetilde{\Delta}(z)$ be the union of $\Delta(z)$ and all open bonds inside $\Delta(z)$ which are part of the infinite connected set of open bonds. Let $\hat{t}(x,y)$ be the minimum passage time from a site of $\widetilde{\Delta}(x)$ to a site of $\widetilde{\Delta}(y)$. Observe that

$$\text{If } t(x,y) < \infty, \text{ then } \hat{t}(x,y) \leq t(x,y) \leq \hat{t}(x,y) + u(x) + u(y). \tag{2.12}$$

Following the development in section 2 of [6], here are some facts needed to prove Theorem 1.1.

$$P(\kappa(z) \geq n) \to 0 \text{ exponentially fast as } n \to \infty. \tag{2.13}$$

This is an immediate consequence of (2.11).

$$E(|\widetilde{\Delta}(z)|^m) < \infty \text{ for } m = 1, 2, ... \tag{2.14}$$

This immediate consequence of (2.13) and the crude estimate

$$P(|\widetilde{\Delta}(z)| > 4 \times 2k(2k+1)) \leq P(\kappa(z) > k).$$

$$E(u(z)^m) < \infty \text{ for } m = 1, 2, ... \tag{2.15}$$

This fact uses (2.13) and the observation that if $\tau(x,y) < \infty$, then $\tau(x,y)$ is bounded by an exponential random variable with mean $4/\alpha$ (see Section 2 of [6]).

Lemma 2.4. *If $z_0, z_1, ..., z_k$ is a path from $z_0 = x$ to $z_k = y$, then there is a path from $\widetilde{\Delta}(x)$ to $\widetilde{\Delta}(y)$ contained in $\bigcup_{i=0}^{k} \widetilde{\Delta}(z_i)$.*

Proof: Let $\overline{\Delta}(z)$ be the set of all points strictly surrounded by the circuit $\Delta(z)$ and let $I = \{i : \overline{\Delta}(z_i)) \not\subset \overline{\Delta}(z_j)) \text{ for all } j \neq i\}$. It is easy to see that all the z_i are contained in the some $\overline{\Delta}(z_j)$ with $j \in I$. We will show that there is an $I' \subset I$ so that $\bigcup_{i \in I'} \widetilde{\Delta}(z_i)$ is connected and intersects $\widetilde{\Delta}(x)$ and $\widetilde{\Delta}(y)$.

To construct I' start with the smallest $i_0 \in I$ so that $x \in \overline{\Delta}(z_{i_0})$. To pick the next index, find the smallest j so that $z_j \notin \overline{\Delta}(z_{i_0})$ and then pick the smallest $i_1 \in I$ so that $z_j \in \overline{\Delta}(z_{i_1})$ (observe that this guarantees $\overline{\Delta}(z_{i_0}) \cap \overline{\Delta}(z_{i_1}) \neq \emptyset$ and that neither of the sets $\overline{\Delta}(z_{i_0})$ and $\overline{\Delta}(z_{i_1})$ contains the other). We can continue this procedure, each time picking the point with least index which is not contained in the interior of any previously chosen circuit, until the point z is contained in the interior of some circuit. At this point we stop and let $I' = \{i_0, ..., i_m\}$ be the set of indexes generated.

It is easy to see that $\widetilde{R} = \bigcup_{j=0}^{m} \widetilde{\Delta}(z_{i_j})$ is connected. To prove this, observe that if $1 \leq j \leq m$ it follows from the construction that $\overline{\Delta}(z_{i_j})) \not\subset \overline{\Delta}(z_{i_{j-1}}), \overline{\Delta}(z_{i_{j-1}})) \not\supset \overline{\Delta}(z_{i_j})$ and $\overline{\Delta}(z_{i_j}) \cap \overline{\Delta}(z_{i_{j-1}}) \neq \emptyset$ with which we will show that $\widetilde{\Delta}(z_{i_j}) \cap \widetilde{\Delta}(z_{i_{j-1}}) \neq \emptyset$. Indeed, there are points a and b in $\overline{\Delta}(z_{i_j}))$ so that $a \in \overline{\Delta}(z_{i_{j-1}})$ and $b \notin \overline{\Delta}(z_{i_{j-1}})$. Since,

$a, b \in \overline{\Delta}(z_{i_j})$ there is a polygonal curve entirely contained in $\overline{\Delta}(z_{i_j})$ which connects a and b. Since $a \in \overline{\Delta}(z_{i_{j-1}})$ and $b \notin \overline{\Delta}(z_{i_{j-1}})$, the curve must intersect $\Delta(z_{i_{j-1}})$ at some point c which it implies, $c \in e$ for some $e \in \widetilde{\Delta}(z_{i_{j-1}})$. By the choice of c, $e \subset \widetilde{\Delta}(z_{i_j})$; so $\{e\} \subset \widetilde{\Delta}(z_{i_j}) \cap \widetilde{\Delta}(z_{i_{j-1}})$ for $j = 1, ..., m$. Since each $\widetilde{\Delta}(z_{i_j})$ is connected this shows that $\widetilde{R}$ is connected. To complete the proof it suffices to show that $\widetilde{R}$ intersects $\widetilde{\Delta}(x)$ and $\widetilde{\Delta}(y)$ but this is trivial. If $\widetilde{\Delta}(z_{i_0}) \cap \widetilde{\Delta}(x) = \emptyset$ then $i_0 \neq 0$ and $\overline{\Delta}(z_{i_0}) \subset \overline{\Delta}(x)$ which contradicts the definition of I. $\square$

Applying the Lemma above, $\hat{t}(x, y) \leq \sum_{i=0}^{k} u(z_i)$. Now apply (2.15) to obtain:

$$E(\hat{t}(x, t)^m) < \infty \text{ for } m = 1, 2, ... \tag{2.16}$$

We will use (2.15) and (2.16) in Chapter 5, to demonstrate the existence of radial limits.

Chapter 3

Construction of neighborhoods and approximation of t in the model of Zhang

In this case, the difficulty is that the open circuit method does not work since a path can pass through without meeting it.

Definition 3.1. *Let us consider a cover $\mathbb{Z}^2$ by some blocks $\{[in,(i+1)n]\times[jn,(j+1)n]\}$ for $i,j\in\mathbb{Z}$. Each block $[in,(i+1)n]\times[jn,(j+1)n]$ is called the renormalized site (i,j).*

Denote by V_n all the renormalized sites. Therefore, V_n and the edges between (i,j) and $(i,j+1)$ or $(i+1,j)$ form a standard planar graph. For each renormalized site (i,j), let $A_n(i,j)$ be the event that all three of the following hold:

i) In the block $[in,(i+1)n]\times[jn,(j+2)n]$ there are bottom-top and top-bottom open paths.

ii) For any bottom-top or top-bottom open paths γ_1 and left-right or right-left open paths γ_2 of block $[in,(i+1)n]\times[jn,(j+2)n]$, then they are connected by some open paths from γ_1 to γ_2 and from γ_2 to γ_1 with edges in the block.

iii) For any two bottom-top or two top-bottom open paths γ_1 and γ_2 of $[in,(i+1)n]\times[jn,(j+1)n]$, they are connected by some open paths from γ_1 to γ_2 and from γ_2 to γ_1 with the edges in $[in,(i+1)n]\times[jn,(j+1)n]$ (see figure 3.1)

Similarly, let $B_n(i,j)$ be the event that all three of the following hold:

i) In the block $[in,(i+2)n]\times[jn,(j+1)n]$ there are left-right and right-left open paths.

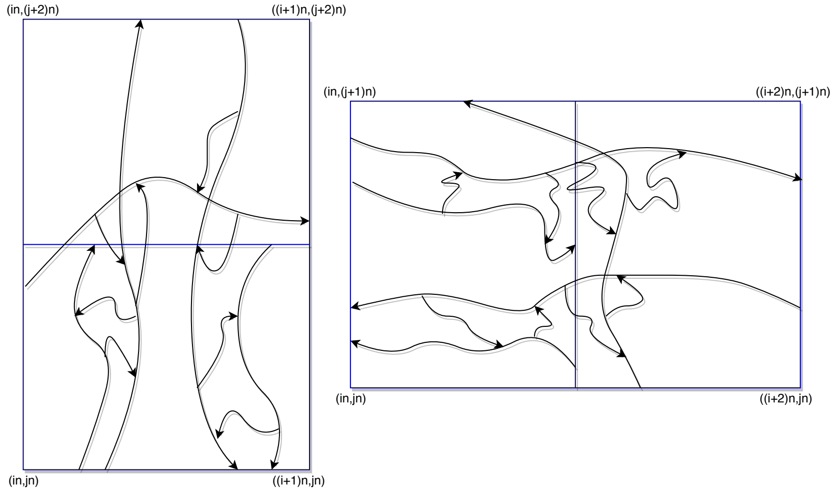


Figure 3.1: The events $A_n(i,j)$ and $B_n(i,j)$

ii) For any left-right or right-left open paths γ_1 and bottom-top or top-bottom open paths γ_2 of block $[in,(i+2)n] \times [jn,(j+1)n]$ they are connected by some open paths from γ_1 to γ_2 and from γ_2 to γ_1 with edges on the block.

iii) For any two left-right or two right-left of open paths γ_1 and γ_2 of $[in,(i+1)n] \times [jn,(j+1)n]$, they are connected by some open paths from γ_1 to γ_2 and from γ_2 to γ_1 with edges in $[in,(i+1)n] \times [jn,(j+1)n]$ (see figure 3.1).

Definition 3.2. *The renormalized site (i,j) is said to be occupied if $A_n(i,j) \cap B_n(i,j)$ occurs.*

With this occupied site in mind, next we construct an occupied circuit. Denote $\Gamma(u,v)$ the event that there is an occupied circuit of the renormalized sites which surrounds u, separating it from v, for $u,v \in \mathbb{Z}^2$. By the definition of renormalized site we can see if $\Gamma(u,v)$ occurs, then:

a) There is an open clockwise circuit C with edges in $(\mathbb{Z}^2, \mathbb{E}_R)$ surrounding u, separating it from v.

b) Each open path in $(\mathbb{Z}^2, \mathbb{E}_R)$ from u to v or from v to u has to be connected to C by some open paths in both directions (see figure 3.2).

Hence our renormalized circuit does not have the problem of connectedness described before.

Replacing Cox and Durrett's circuit by this renormalized occupied circuit also requires a probability estimate corresponding to (2.13). Therefore, we need to show the following lemmas.

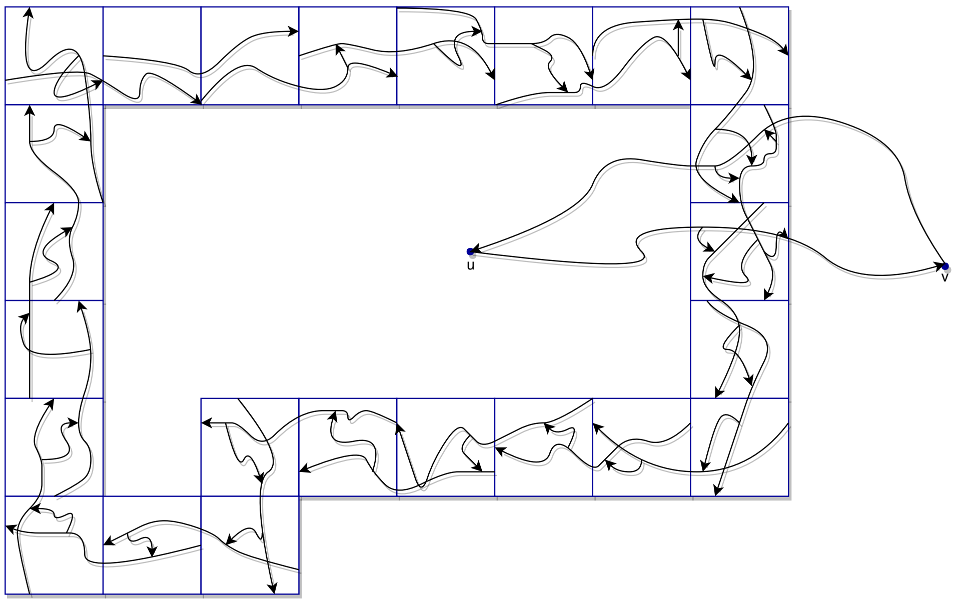


Figure 3.2: The event $\Gamma(u, v)$

Lemma 3.3. *If $\alpha > \alpha_c$, then for some suitably large n there is a positive constant $\kappa = \kappa(\alpha, n) > 0$ such that for each pair of sites $u, v \in \mathbb{Z}^2$,*

$$P_\alpha(\Gamma(u, v)) \geq 1 - exp[-\kappa\,|u - v|]. \tag{3.1}$$

The main step in the proof of Lemma 3.3 is the following proposition, which is based on the methods of [8] and [13].

Proposition 3.4. *Given $\epsilon > 0$, there exist N such that*

$$P_\alpha((0,0) \text{ is occupied}) = P_\alpha(A_n(0,0) \cap B_n(0,0)) \geq 1 - \epsilon, \tag{3.2}$$

when $n \geq N$.

The proof of this proposition is long (see appendix of [19]), at the end of this chapter we demonstrate a particular case.

Proof of Lemma 3.3 from the Proposition: We can build an open clockwise circuit surrounding u, separating it from v, with center u, $2\,\||u_1 - v_1|\,/n\|$ blocks vertically and $2\,\||u_2 - v_2|\,/n\|$ horizontally, all these blocks of size $n \times n$ (if you have $|u_2 - v_2| = 0$ then you consider $2\,\||u_1 - v_1|\,/n\|$ blocks horizontally, or vice versa). To this circuit we relate $k_1\,|u - v|$ occupied sites, with $k_1 = 2, 4$ (see figure 3.3). Then if $\epsilon = \exp[-c]$ for some $c > 0$, the above proposition and the FKG inequality imply:

$$P_\alpha(\Gamma(u, v)) \geq 1 - \exp[-ck_1\,|u - v|]. \qquad \square$$

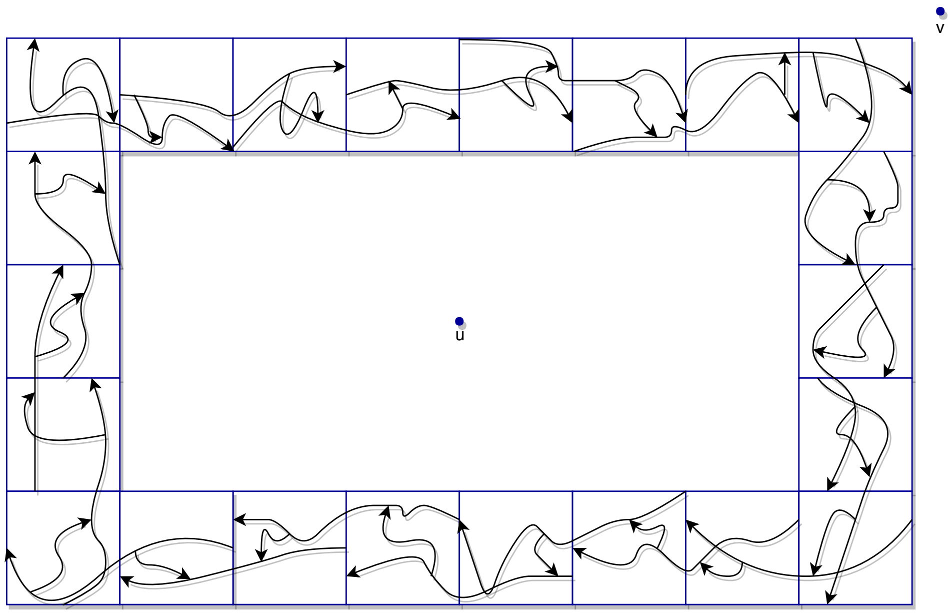


Figure 3.3: Open clockwise circuit surrounding u, separating it from v.

Let $B(l) = [-l, l]^2$. We denote by $d(l) \uparrow$ or $(d(l) \downarrow)$ the event that there is an open path in $(\mathbb{Z}^2, \mathbb{E}_R)$ from (or to) $\partial B(l)$ to (or from) ∞ in $\mathbb{Z}^2 \backslash B(l)$, where ∂A is the boundary of the set A of vertices.

Lemma 3.5. *If $\alpha > \alpha_c$, then there is a constant $c(\alpha)$ which only depend on α such that*

$$P_\alpha(d(l) \downarrow) = P_\alpha(d(l) \uparrow) \geq 1 - exp\left\{-c(\alpha)l\right\}. \tag{3.3}$$

Proof: By the symmetry of $g(z)$, the first equation is obvious. By the definition of $d(l) \uparrow$,

$$P_\alpha(d(l) \uparrow) \leq \sum_{k=l+1}^{\infty} 2k^2 \max_{y \in B(k) \backslash B(\max\{k-R, l\})} P_\alpha(a(k, y)), \tag{3.4}$$

where,

$$\begin{aligned} a(m, y) = \{ & \text{there is an open path in } \mathbb{Z}^2(R) \text{ from } \partial B(l) \\ & \text{to the point } y \in B(i) \text{ for } m - R \leq i \leq m \text{ and no} \\ & \text{open path in } (\mathbb{Z}^2, \mathbb{E}_R) \text{ from } \partial B(l) \text{ to } \partial B(j) \text{ for } j > i\}. \end{aligned}$$

If $a(m, y)$ occurs, there is no occupied circuit with renormalized sites in V_n which encircles

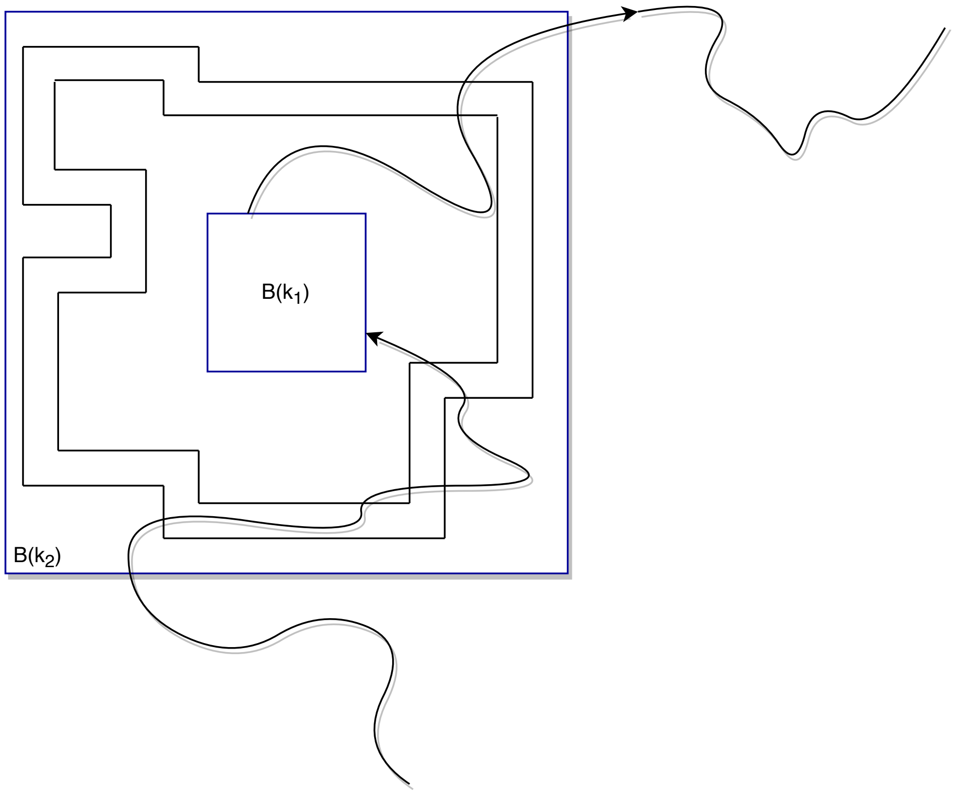


Figure 3.4: The definition of κ_1 and κ_2

the point y separating some point in $\partial B(l)$. Then

$$
\begin{aligned}
&P_\alpha(d(l)\uparrow) && (3.5)\\
&\leq \sum_{k+1}^{\infty}(2kl)^2\left(\exists y\in\mathbb{Z}^2, \|y\|_\infty > \frac{\max(k-R,l)}{n} \textit{ such that } \{\Gamma(y,0)\}^c\right) && (3.6)\\
&\sum_{k+1}^{\infty}(2kl)^2\exp\left\{-\kappa(\alpha)\left(\frac{\max(k-R,l)}{n}\right\}\right) \text{ by Lemma 3.3} && (3.7)\\
&\leq \exp\left\{-c(\alpha)l\right\}, && (3.8)
\end{aligned}
$$

for some constant $c(\alpha)$. □

For any $z\in\mathbb{Z}^2(R)$, we denote by $\kappa_1(z)$ the smallest $k>1$ such that there is an open path in $\mathbb{Z}^2(R)$ from the boundary of $z+[-k,k]^2$ to ∞ and there is another open path in $\mathbb{Z}^2(R)$ from ∞ to the boundary of $z+[-k,k]^2$. We also denote by $\kappa_2(z)$ the smallest $m>\kappa_1(z)$ such that there exist an occupied circuit with the renormalized sites of V_n in the annulus

$$\left\{z+[-m,m]^2\right\}\setminus\left\{z+[-\kappa_1(z),\kappa_1(z)]^2\right\}$$

(see figure 3.4). Then we have the following probability estimate:

Lemma 3.6. *If $\alpha > \alpha_c$, there is a constant $c(\alpha)$ such that*

$$P_\alpha(\kappa_2(0) > l) \leq exp(-c_1(\alpha)l). \tag{3.9}$$

Proof: Clearly,

$$\begin{aligned} &P_\alpha(\kappa_2(0) > l) \\ &= P_\alpha\left(\kappa_2(0) > l, \kappa_1(0) < \frac{l}{2}\right) + P\left(\kappa_2(0) > l, \kappa_1(0) \geq \frac{l}{2}\right) \\ &\leq P_\alpha\left(\kappa_2(0) > l, \kappa_1(0) < \frac{l}{2}\right) + \exp\left(-c(\alpha)\frac{l}{2}\right) \quad \text{by Lemma 3.5.} \end{aligned} \tag{3.10}$$

Now by using the same proof of Lemma 3.5, we can show

$$\begin{aligned} P_\alpha\left(\kappa_2(0) > l, \kappa_1(0) < \frac{l}{2}\right) \leq\ & P_\alpha(\text{there is no occupied circuit with} \\ & \text{the renormalized sites surrounding} \\ & [-\frac{l}{2}, \frac{l}{2}] \text{ in } [-l, l]\backslash[-\frac{l}{2}, \frac{l}{2}]) \\ & \leq \exp(-c_2(\alpha)\frac{l}{2}). \end{aligned} \tag{3.11}$$

The proof follows from (3.10) and (3.11). □

Lemma 3.6 prepares us to show the existence of radial limits. Let $\Delta(z)$ be the set $z+[-\kappa_1(z), \kappa_1(z)]^2$ and $\hat{t}(x, y)$ be the minimum passage time from a site of $\Delta(z)$ to a site of $\Delta(y)$. Let $u(z)$ be the sum of all $\tau\{x, y\} < \infty$ for $\{x, y\} \in z + [-\kappa_2(z), \kappa_2(z)]$.
If $t(x, y) < \infty$, then

$$\hat{t}(x, y) \leq t(x, y) \leq \hat{t}(x, y) + u(x) + u(y). \tag{3.12}$$

By Lemma 3.6, and an argument in Section 3 of [7], it can be seen that $Eu^m(z) < \infty$ and $E\hat{t}(x, y)^m < \infty$, for all $m \geq 1$. The latter, as for the other two models will be used to demonstrate the existence of radial limits, as we will see in the chapter 5.

3.1 Proof of Proposition 3.4 (a particular case).

Let N be a positive integer. We define the configuration $\omega \in \Gamma = \{0, 1\}^{\mathbb{E}_R}$ as For $e \in \mathbb{E}_R$ $\omega(e) = 1$ if e is open and $\omega(e) = 0$ otherwise. For any configuration $\omega \in \Gamma = \{0, 1\}^{\mathbb{E}_R}$, we define the sphere with radius N and center at ω by

$$S^n(\omega) = \left\{ \omega' \in \Gamma : \sum_{e \in \mathbb{E}_R} |\omega_e - \omega'_e| \leq N \right\}. \tag{3.13}$$

$S^n(\omega)$ is the collection of configurations which differ from ω on at most N edges. If $A \subset \Gamma$ is an event, we define the interior and exterior of A by

$$I^N A = \left\{ \omega \in \Gamma : S^N(\omega) \subset A \right\} \text{ and } E^N A = \left\{ \omega \in \Gamma : S^N(\omega) \bigcap A \neq \emptyset \right\}. \tag{3.14}$$

If $\alpha' \leq \alpha$, then by (1.1) and the definition of $g(z)$,

$$0 \leq P_{\alpha'}((x,y) \textit{ is open}) \leq P_\alpha((x,y) \text{ is open}). \tag{3.15}$$

We denote

$$\begin{aligned} m(\alpha) &= \min_{0 < \|y\|_\infty \leq M} \{(P_\alpha \omega(0,y) = 1)\}, \\ m(\alpha', \alpha) &= \min_{0 < \|y\|_\infty \leq M} \{(P_\alpha \omega(0,y) = 1) - (P_{\alpha'} \omega(0,y) = 1)\}. \end{aligned}$$

By the definition of F and (2.2), we can see that $m(\alpha) > 0$ and $m(\alpha', \alpha) > 0$ if $\alpha > \alpha' > \alpha_c$. With the definitions above, we establish the following lemma (see Theorem 2.45 in [9]):

Lemma 3.7. *For any increasing event A and $\alpha' < \alpha$,*

$$P_\alpha(A) \geq m(\alpha', \alpha) P_{\alpha'}(E^N A), \tag{3.16}$$

$$P_\alpha(I^N A) \geq 1 - (1 - P_{\alpha'}(A))/(m(\alpha', \alpha)^N). \tag{3.17}$$

Next we introduce some sets and events necessary for construction of the renormalized site lattice. If S, F and T are three sets, we denote by $S \to T$ in F the event that there is an open path from S to T with edges in F and $S \leftrightarrow T$ in F the event that $S \to T$ in F and $T \to S$ in F. We also denote by $S \nrightarrow T$ in F the event that there is no open path from S to T in F.

Lemma 3.8. *Given $\epsilon > 0$ and integer i, there exists N such that, when $\alpha > \alpha_c$ and $n \geq 2N + M$,*

$$P_\alpha([-n, 2n] \times \{0\} \to [-n, 2n] \times \{in\}) \geq 1 - \epsilon.$$

Proof: Zhang demonstrated this lemma with the use of two other lemmas, which he also demonstrated in [19]. We will only show a particular case. Suppose, for $z, z' \in N_0$:

$$\|z\|_\infty \leq \|z'\|_\infty \Rightarrow g(z) \geq g(z'), \tag{3.18}$$

which means that it is more likely to infect the neighbor closer. Let $\alpha_c < \alpha_1 < \alpha$. Since $\theta(\alpha_1) = P_{\alpha_1}(|C_0| = \infty) > 0$, by a standard ergodic theorem, we can take N large enough that

$$P_{\alpha_1}([0, N] \longleftarrow \infty \text{ in } \mathbb{E}_R) \geq 1 - \epsilon^4. \tag{3.19}$$

Let A_1, A_2, A_3 and A_4 be the events:

$$\begin{aligned}
[[\frac{n}{2} - in, \frac{n}{2} + in] \times [0, R] \to [\frac{n-N}{2}, \frac{n+N}{2}] \times \{in\} \text{ in } \hat{B}],\\
[[\frac{n}{2} - in, \frac{n}{2} + in] \times [2in - R, 2in] \to [\frac{n-N}{2}, \frac{n+N}{2}] \times \{in\} \text{ in } \hat{B}],\\
[[\frac{n}{2} + in - R, \frac{n}{2} + in[\times[0, 2in] \to [\frac{n-N}{2}, \frac{n+N}{2}] \times \{in\} \text{ in } \hat{B}] \text{ and}\\
[[\frac{n}{2} - in, \frac{n}{2} - in + R] \times [0, 2in] \to [\frac{n-N}{2}, \frac{n+N}{2}] \times \{in\} \text{ in } \hat{B}],
\end{aligned}$$

respectively, where $\hat{B} = [\frac{n}{2} - in, \frac{n}{2} + in] \times [0, 2in)$ (see figure 3.5). Then it follows from the square root trick, 3.19, and the symmetry and the translation invariance of $\mathbb{E}_R$ that

$$P_{\alpha_1}(A_1) \geq 1 - \epsilon.$$

We note that, given a path γ from $[\frac{n}{2} - in, \frac{n}{2} + in] \times [0, R]$ to $[\frac{n-N}{2}, \frac{n+N}{2}] \times \{in\}$ in $\hat{B}$, we can reflect $\gamma \cap [2n, n/2 + in] \times [0, 2in]$ through $\{2n\} \times [0, 2in]$ as follows: if γ' is a connected segment of γ in $\gamma \cap [2n+1, n/2 + in] \times [0, 2in]$ with starting point z and end point w, such that, if γ had a point before of γ' it is in $[2n - R, 2n] \times [0, 2in]$ and the same for a point after γ', then

- if γ starts in z, let $w' = (w'_1, w'_2) \in \gamma$ be the point after γ', then we reflect $\gamma' \cup \{w'\}$ through $\{w'_1\} \times [0, 2in]$,
- if z is not the starting point of γ, let $z' = (z'_1, z'_2), w' = (w'_1, w'_2) \in \gamma$ be the previous point and the subsequent point to γ', respectively. Then:
  - if $z'_1 \leq w'_1$, we reflect γ' through $\{w'_1\} \times [0, 2in]$, and replace the edge (z', z) with a new one with less or equal length and in opposite direction (see figure 3.6), to keep a path from $[\frac{n}{2} - in, \frac{n}{2} + in] \times [0, R]$ to $[\frac{n-N}{2}, \frac{n+N}{2}] \times \{in\}$
  - if $w'_1 \leq z'_1$, we reflect γ' through $\{z'_1\} \times [0, 2in]$, and replace the edge (w, w') with a new one with less or equal length and in opposite direction.

Thus, we get a new path from $[\frac{n}{2} - in, \frac{n}{2} + in] \times [0, R]$ to $[\frac{n-N}{2}, \frac{n+N}{2}] \times \{in\}$ in $[n/2 - in, 2n] \times [0, 2in]$, which intersected with $[n/2 - in, -n] \times [0, 2in]$ we are going to reflect in the same

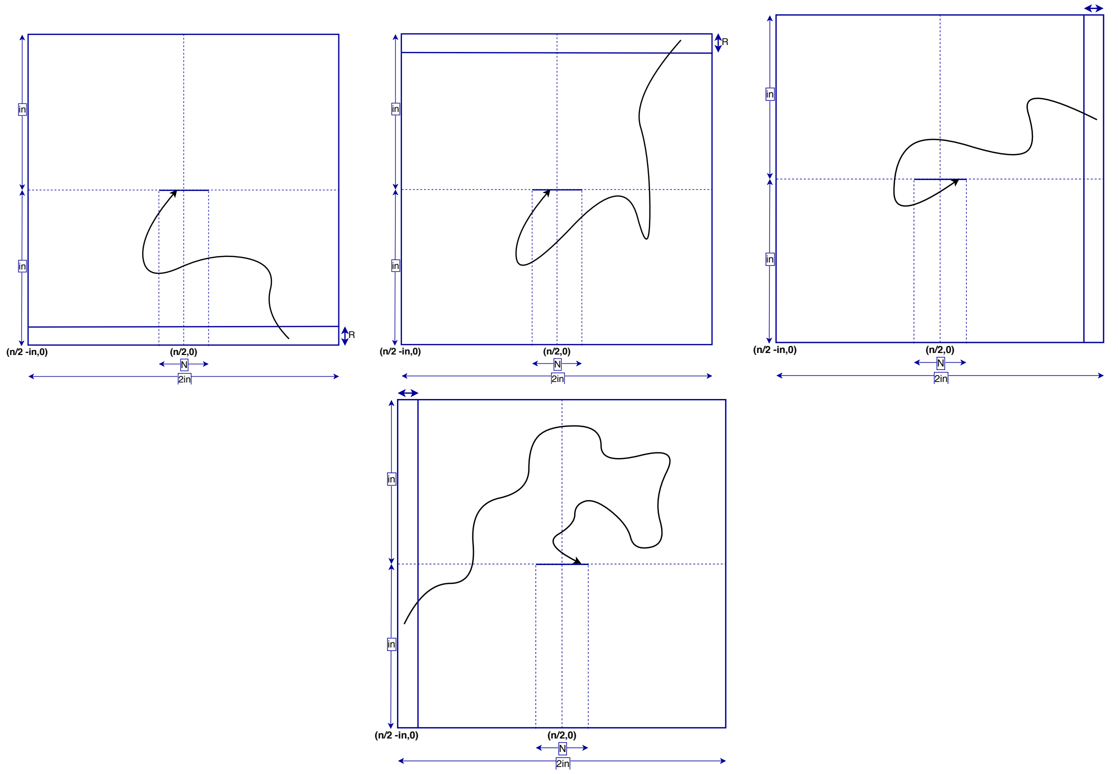


Figure 3.5:

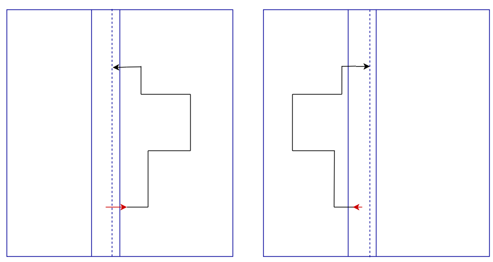


Figure 3.6:

way to the previous reflection, but this time through $\{-n\} \times [0, 2in]$ with what we get a path from $[\frac{n}{2} - in, \frac{n}{2} + in] \times [0, R]$ to $[\frac{n-N}{2}, \frac{n+N}{2}] \times \{in\}$ in $[-n, n/2 + (i-3)n] \times [0, 2in]$, and this last one intersected with $[2n, n/2 + in] \times [0, 2in]$ we reflect through $\{2n\} \times [0, 2in]$. We do this, over and over again, until we get a path $\bar{\gamma}$ in $[-n, 2n] \times [0, in]$. Note that the number of reflections depends only on i, and not on n. Also, by the condition (3.18) we have

$$P_{\alpha_1}(\bar{\gamma} \text{ is open }) \geq P_{\alpha_1}(\gamma \text{ is open }),$$

then,

$$P_{\alpha_1}([-n, 2n] \times [0, R] \to [-n, 2n] \times [in - R, in] \text{ in } [-n, 2n] \times [0, in]) \geq 1 - \epsilon.$$

Then by Lemma A.2, for some large number k,

$$\begin{aligned} 1 - & \frac{1 - P_{\alpha_1}([-n, 2n] \times [0, R] \to [-n, 2n] \times [in - R, in] \text{ in } [-n, 2n] \times [0, in])}{m^k(\alpha, \alpha_1)} \\ & \leq P_\alpha(I^k \{[-n, 2n] \times [0, R] \to [-n, 2n] \times [in - R, in] \text{ in } [-n, 2n] \times [0, in]\}) \\ & \leq P_\alpha(\exists\ k \text{ disjoint open paths from} [-n, 2n] \times [0, R] \text{ to } [-n, 2n] \times [in - R, in]), \end{aligned}$$

which implies:

$$\begin{aligned} P_\alpha(\exists\ k \text{ disjoint open paths from } [-n, 2n] \times [0, R] \text{ to } [-n, 2n] \times [in - R, in]) \\ \geq 1 - \frac{\epsilon}{m^k(\alpha, \alpha_1)}. \end{aligned}$$

Now, let H be the largest $j, 0 \leq j \leq R$, such that there exist at least k/R open paths from $[-n, 2n] \times [0, R]$ to $[-n, 2n] \times [in - R, in]$. Since each such path can be connected to $[-n, 2n] \times \{in\}$ by a single edge and $H = l$ only depends on the edges in $[-n, 2n] \times [0, in - l]$, we can pick k large such that at least one of these k/R open paths can be connected to $[-n, 2n] \times \{in\}$ by a single open edge with a high probability. Note also that $H = l$ and $H = l'$ are disjoint events if $l \neq l'$. Also we have:

$$\begin{aligned} P_\alpha(\exists\ k \text{ disjoint open paths from} [-n, 2n] \times [0, R] \text{ to } [-n, 2n] \times [in - R, in]) \\ \leq \sum_{0 \leq l \leq R} P_{\alpha*}(H = l). \end{aligned}$$

Then, $P_\alpha([-n, 2n] \times [0, R] \to [-n, 2n] \times \{in\}) \to 1$ by taking k,ϵ and N, respectively. Using this argument to connected some vertices, which belong to the open paths from $[-n, 2n] \times [0, R]$ to $[-n, 2n] \times \{in\}$ in $[-n, 2n] \times [0, R]$, to $[-n, 2n] \times \{0\}$ by a single open edge, we can see that Lemma 3.8 hols. $\square$

Chapter 4

Construction of neighborhoods and approximation of t in the model of Chabot

This Chapter is based in the work of Chabot in [5]. The fact that facilitates this case is the independence between the state of one edge and the state of another. This fact and the symmetry of the model imply the following theorem.

Teorema 4.1. $\alpha_c^0 = \alpha_c^i$.

Proof: Define the reversed model as the model obtained by: an edge (y, x) is open if and only if, (x, y) is open in the original model, with $x, y \in \mathbb{Z}^d$ and $(x, y) \in \mathbb{E}^d$. This is, reverses the state of an edge with that of the opposite orientation edge. Now, let P' be the probability measure on the reversed model, then (see (1.2) and (1.3)):

$$P'((y, x) \text{ is open}) = P((x, y) \text{ is open}) = p$$

We call ϕ and ϕ' the null functions with respect to the original model and the reversed model respectively, the functions which, at any subset A of the set of outgoing edges from the origin, associate:

$$\phi(A) = P\left\{\text{every edge in } A \text{ is closed}\right\}$$

and

$$\phi'(A) = P'\left\{\text{every edge in } A \text{ is closed}\right\}.$$

Due to the properties of the original model: independence of the states of the edges and the symmetry, reflection and translation of the model. We have:

$$\phi(A) = \phi'(A) = |1-p|^{|A|},$$

where $|A|$ is the number of edges in A. If B is a set of infinite paths in $\mathbb{Z}^d$, and $\mathscr{B}$ is the event that some path in B is open, then from the result in [14] (see pages 749-750), we have:

$$P(\mathscr{B}) \leq P'(\mathscr{B}) \text{ and } P(\mathscr{B}) \geq P'(\mathscr{B})$$

since $\phi(A) \geq \phi'(A)$ and $\phi(A) \leq \phi'(A)$. Then, $P(\mathscr{B}) = P'(\mathscr{B})$, with which we conclude the proof. $\square$

Now, we will see an algorithmic demonstration that the distribution of the sites of the outgoing cluster, like that of the sites of the returning cluster, at the origin, is identical to that of an independent percolation cluster of classical Bernoulli on $\mathbb{Z}^d$ non-oriented.

Teorema 4.2. *The distribution of the outgoing cluster sites on $\mathbb{Z}^d$ is the same as the distribution of the sites of a Bernoulli conventional percolation cluster on the edges.*

Proof: It is assumed given an order on the edges of $\widetilde{\mathbb{E}}^d$ (set of non-oriented edges). The algorithm below is an algorithm for dynamic construction of the cluster at the origin on the non-oriented model. We use the following recursive method of construction of the cluster of the origin: we obtain an increasing sequence $(\widetilde{S}_n)_{n\in\mathbb{N}}$ of triplets of $\mathbb{Z}^d \times \widetilde{\mathbb{E}}^d \times \widetilde{\mathbb{E}}^d$. We start:

$$\widetilde{S}_0 = (\{0\}, \emptyset, \emptyset).$$

If $\widetilde{\omega} : \widetilde{\mathbb{E}}^d \to \{0, 1\}$ is given, and if we assume that:

$$\widetilde{S}_0 = (X_n, \widetilde{A}_n, \widetilde{B}_n),$$

we define $\widetilde{S}_{n+1}$ as follows:

- We take $\widetilde{e}_n$, the smallest edge in the sense of the order that comes from X_n. which is not in $\widetilde{A}_n \cup \widetilde{B}_n$ and whose second extreme, x_{n+1}, is not in X_n.
  - if $\widetilde{\omega}(\widetilde{e}_n) = 1$ then

$$[X_{n+1} = X_n \cup \{x_{n+1}\}, \widetilde{A}_{n+1} = \widetilde{A}_n \cup \{\widetilde{e}_n\}, \widetilde{B}_{n+1} = \widetilde{B}_n].$$

  - if $\widetilde{\omega}(\widetilde{e}_n) = 0$ then

$$[X_{n+1} = X_n, \widetilde{A}_{n+1} = \widetilde{A}_n, \widetilde{B}_{n+1} = \widetilde{B}_n \cup \{\widetilde{e}_n\}].$$

- if such a edge does not exist, then

$$[X_{n+1} = X_n, \widetilde{A}_{n+1} = \widetilde{A}_n, \widetilde{B}_{n+1} = \widetilde{B}_n].$$

We will name $\mathfrak{A}_0$ this algorithm, which, at a configuration $\widetilde{\omega}$ of $\widetilde{\Omega}$ associates the sequence $(\widetilde{S}_n)_{n\in\mathbb{N}}$.

Suppose given an order on the edges of $\mathbb{E}^d$ that respects the previous order on the edges of $\widetilde{\mathbb{E}}^d$, i.e. such that the index of an edge oriented in one direction differs only from 1 of the index of the same edge orintended in the other direction, and that this induces on $\mathbb{Z}^d$ the previous order on the non-oriented edges: if $\widetilde{e}$ and $\widetilde{f}$ are two distinct unoriented edges, then "$\widetilde{e} \leq \widetilde{f}$" if and only if "$\vec{e} \leq \vec{f}$", this whatever the choice of orientation of $\vec{e}$ and $\vec{f}$.

If e is an oriented edge, we will note $\widetilde{e}$ the non-oriented edge corresponding to it. The following recursive method of construction of the outgoing cluster is then used on the oriented model and it will be observed that it is dynamically constructed like the cluster on the non-oriented model: an increasing sequence $(S_n)_{n\in\mathbb{N}}$ of triplets of $\mathbb{Z}^d \times \mathbb{E}^d \times \mathbb{E}^d$. We start:

$$S_0 = (\{0\}, \emptyset, \emptyset).$$

If $\omega : \mathbb{E}^d \to \{0,1\}$ is given, and if we assume that:

$$S_0 = (X_n, A_n, B_n),$$

we define S_{n+1} as follows:

- We take e_n, the smallest edge in the sense of the order that comes from X_n. which is not in $A_n \cup B_n$ and whose second extreme, x_{n+1}, is not in X_n.
  - if $\omega(e_n) = 1$ then

$$[X_{n+1} = X_n \cup \{x_{n+1}\}, A_{n+1} = A_n \cup \{e_n\}, B_{n+1} = B_n].$$

  - if $\omega(e_n) = 0$ then

$$[X_{n+1} = X_n, A_{n+1} = A_n, B_{n+1} = B_n \cup \{e_n\}].$$

- if such a edge does not exist, then

$$[X_{n+1} = X_n, A_{n+1} = A_n, B_{n+1} = B_n].$$

This algorithm yields an increasing sequence $(S_n)_{n\in\mathbb{N}}$. Thus, the limit as $n \to \infty$ and we denote it by:

$$S_\infty = (X_\infty, A_\infty, B_\infty).$$

We will name $\mathfrak{A}_1$ this algorithm, which, at a configuration ω of Ω, associates the sequence $(S_n)_{n\in\mathbb{Z}}$. We have built a cluster $C_0 = X_\infty$ on $(\mathbb{Z}^d, \mathbb{E}^d)$, such that:

- B_∞ contains all closed oriented edges from a point of X_∞ to a point outside X_∞ and in particular contains only one orientation of the edge between any two points of $\mathbb{Z}^d$,

- A_∞ is a subgraph of $(\mathbb{Z}^d, \mathbb{E}^d)$ which contains only a orientation of the edge between any two points of $\mathbb{Z}^d$, and whose structure, without worrying about orientation, is that of a connected graph with no cycle,

- finally, if we erase the orientation of the edges of A_∞ and B_∞, we obtain the same sets as those obtained by the similar algorithm of dynamic construction of the cluster at the origin of the non-oriented model, which in other words is written:

$$\widetilde{A_\infty} = \widetilde{A}_\infty,$$

$$\widetilde{B_\infty} = \widetilde{B}_\infty.$$

We must give a more precise meaning to the last statement. Suppose that, for k from 0 to n, for every $\Gamma_k = (X_k, A_k, B_k)$ that can be obtained with non-zero probability by $\mathfrak{A}_1$ at step k:

$$P(S_k = \Gamma_k) = \widetilde{P}(\widetilde{S}_k = \widetilde{\Gamma}_k),$$

where, **the application** $\tilde{}$ is simply the application that erases the orientation of a directed graph. Note that Γ_k is obtained by $\mathfrak{A}_1$ in step k if and only if $\widetilde{\Gamma}_k$ is obtained by $\mathfrak{A}_0$ in step k, according to the preceding remarks, since there is a bijection between the (Γ_k) and $(\widetilde{\Gamma}_k)$. Because of the particular order chosen, in step n, e_n, if it exists, is the edge chosen by $\mathfrak{A}_1$, if and only if $\widetilde{e_n}$ is chosen by $\mathfrak{A}_0$. The probability that e_n is open is e, as for $\widetilde{e_n}$.Si $\Gamma_{n+1} = (X_{n+1}, A_{n+1}, B_{n+1})$ is an element of $\mathbb{Z}^d \times \mathbb{E}^d \times \mathbb{E}^d$ which can be obtained by $\mathfrak{A}_1$ with non-zero probability at step $n+1$, then:

$$\begin{aligned} P(S_{n+1} = \Gamma_{n+1}) &= \Sigma_{\Gamma_n} P(S_{n+1} = \Gamma_{n+1} | S_n = \Gamma_n) P(S_n = \Gamma_n) \\ &= \Sigma_{\widetilde{\Gamma}_n} P(\widetilde{S}_{n+1} = \widetilde{\Gamma}_{n+1} | \widetilde{S}_n = \widetilde{\Gamma}_n) P(\widetilde{S}_n = \widetilde{\Gamma}_n) \\ &= \widetilde{P}(\widetilde{S}_{n+1} = \widetilde{\Gamma}_{n+1}). \end{aligned} \qquad \square$$

Thus, the distribution of the outgoing cluster sites on $\mathbb{Z}^d$ is the same as the distribution of the sites of a Bernoulli conventional percolation cluster on the edges. Of course, a similar construction can be done to show that the distribution of the sites of the cluster returning to $\mathbb{Z}^d$ is also identical to the distribution of the sites of a Bernoulli percolation cluster on the edges.

Then, for the epidemic model, for $\alpha > \alpha_c$, C_0 is infinite with a non-zero probability, of even $\overline{C}_0$ is infinite with the same non-zero probability, hence, if $\widetilde{C} = \{x \in \mathbb{Z}^d : \ x \to \infty, \ x \longleftarrow \infty\}$ by FKG inequality for events increasing for the parameter α, we have that

$0 \in \widetilde{C}$ with a non-zero probability, therefore, by ergodicity, $\widetilde{C}$ is a.s. not empty.

In all the following, it will be assumed that $\alpha > \alpha_c$. In fact, for $\alpha < \alpha_c$, the epidemic dies out a.s., and the result of asymptotic form is then of no interest.

Antal and Pisztora in [3] prove large deviation estimates at the correct order for the graph distance of two sites lying in the same cluster of an independent percolation process, see Section A.4. We will now establish for $\alpha > \alpha_c$, that the result in [3] is also valid in this oriented case, which will make it possible to control the lengths of the opens paths. For the non-oriented model, for $p > p_c$, there is $\rho = \rho(p, d) \in [1, \infty)$ such that:

$$\limsup_{|y|\to\infty} \frac{1}{\|y\|_1} \log \widetilde{P}(0 \leftrightarrow y, D(0, y) > \rho \|y\|_1) < 0,$$

where $D(0, y)$ is the minimum number of edges of an open path from 0 to y. Since these probabilities are all strictly smaller than 1, there is $\beta > 0$ such that:

$$\widetilde{P}(0 \leftrightarrow y, D(0, y) > \rho \|y\|_1) \leq \exp(-\beta \|y\|_1), \quad \forall y \in \mathbb{Z}^d.$$

To justify that this result is valid for the oriented framework, we impose a special order, adapted to the problem, for the generator algorithm of the outgoing cluster of 0, to guarantee the existence of $\beta > 0$.

Teorema 4.3. *There is $\beta > 0$ such that:*

$$P(0 \to y, D(0, y) > \rho \|y\|_1) \leq exp(-\beta \|y\|_1), \quad \forall y \in \mathbb{Z}^d.$$

Proof: It is always assumed given an order on the edges of $\mathbb{E}^d$ respecting the order at the edges of $\widetilde{\mathbb{E}}^d$. We use the following recursive method of constructing the outgoing cluster on the oriented model, which is built dynamically like the cluster on the non-oriented model: we obtain a increasing sequence $(S_n)_{n\in\mathbb{N}}$ of triplets of $\mathbb{Z}^d \times \mathbb{E}^d \times \mathbb{E}^d$ which corresponds to the set of points that can be touched in less than n edges. We start:

$$S_0 = (\{0\}, \emptyset, \emptyset).$$

For $\omega \in \Omega = \{0, 1\}^{\mathbb{E}^d}$ given, and if $S_n = (X_n, A_n, B_n)$, we define S_{n+1} as follows:

- We take the set E_0^n of outgoing edges of X_n that are not in $A_n \cup B_n$. We pose: $Y_0^n = X_n$, $C_0^n = A_n$, $D_0^n = B_n$. Then, The following method is iterated, for k varying from 0 to $k_n - 1$, where $k_n = \text{card}(E_0^n)$:

- let e_k^n be, smaller edge of E_k^n, in the sense of the order.

- if the second end of e_k^n, x_{k+1}^n, is in Y_k^n then $Y_{k+1}^n = Y_k^n$, $E_{k+1}^n = E_k^n \backslash \{e_k^n\}$, $C_{k+1}^n = C_k^n$, D_{k+1}^n.

- if the second end of e_k^n, x_{k+1}^n, is not in Y_k^n then $E_{k+1}^n = E_k^n \{e_k^n\}$ and:

  - if $\omega(e_k^n) = 1$ then

$$[Y_{k+1}^n = Y_k^n \cup \{x_{k+1}^n\}, C_{k+1}^n = C_k^n \cup \{e_k^n\}, D_{k+1}^n = D_k^n].$$

  - if $\omega(e_k^n) = 0$ then

$$[Y_{k+1}^n = Y_k^n, C_{k+1}^n = C_k^n, D_{k+1}^n = D_k^n \cup \{e_k^n\}].$$

- Finally, $\mathrm{S}_{n+1} = (Y_{k_n}^n, C_k^n, D_{k_n}^n)$ This is an algorithm that builds the cluster $C_0 = X_\infty$ on $(\mathbb{Z}^d, \mathbb{E}^d)$, in a way that X_n is the set of sites that we have reached since the origin in less than n not. A similar algorithm on the non-oriented model would give the same probability of having X_n, in step n. Then the result in [3] works in the oriented case, that is to say that there is $\beta > 0$ such that:

$$P(0 \to y, D(0,y) > \rho \left\|y\right\|_1) \leq \exp(-\beta \left\|y\right\|_1), \quad \forall y \in \mathbb{Z}^d.$$

$\square$

Now, we will build a neighborhood for each site of $\mathbb{Z}^d$ consisting of points of $\widetilde{C}$. To justify this choice, it is necessary to establish the properties of connections on $\widetilde{C}$.

The main idea is that $\widetilde{C}$ is connected, in the sense that any two points of $\widetilde{C}$ are connected by open paths, and that if an open path connects z to y, for z and y distant, then this path meets $\mathcal{V}(x)$ and $\mathcal{V}(y)$. This last property is linked to the "re-entrant" and "outgoing" roots of a site and will be mentioned later.

Thus, in the case where the epidemic survives, it will spend most of its time on the open paths between the neighborhoods, or more exactly on the paths between points of $\widetilde{C}$, and will spend only negligible time in the neighborhoods, which will justify that $\hat{t}$, the time of passage between two neighborhoods is a good approximation of the time of passage of the epidemic.

Teorema 4.4. *For x and y two different sites of $\mathbb{Z}^d$, we then have:*

$$P(x \to y : |C(x)| = \left|\overline{C}(y)\right| = \infty) = 1.$$

Indeed, if $C(x)$ and $\overline{C}(y)$ are infinite, Chabot in [5] demonstrated with an algorithm

that the probability that $x \nrightarrow y$ is less than to have two infinite cluster disjoint in Bernoulli non-oriented percolation on $\mathbb{Z}^d$. Thus, as the probability to observe two cluster infinite disjoint in Bernoulli non-oriented percolation is zero, the result is established.
Thus we have a single connected component a.s., in the sense that if x and y are in $\widetilde{C}$, then $x \to y$ and $y \to x$.

Now, we are interested in all sites that are connected to x without meeting $\widetilde{C}$. For $x \in \mathbb{Z}^d$ let:

$$R(x) = \{y \in \mathbb{Z}^d : x \to y \text{ out of } \widetilde{C}\} \text{ and}$$

$$\overline{R}(x) = \{y \in \mathbb{Z}^d : x \leftarrow y \text{ out of } \widetilde{C}\},$$

where, "$x \to y$ out of $\widetilde{C}$" means that there is a path of open edges $\Gamma = (t_0 = y, t_1, ..., t_{n-1}, t_n = z)$ from y to z such that none of the $(t_i)_{i=0}^n$ is not in $\widetilde{C}$. We will agree that x belongs to $R(x)$ and $\overline{R}(x)$ if and only if $x \notin \widetilde{C}$.

Lemma 4.5. *There is $\sigma_1 = \sigma_1(\alpha, d)$ such that:*

$$P\left(R(0) \cup \overline{R}(0)\right) \cap \partial B(0,n) \neq \emptyset \leq exp(-\sigma_1 n), \forall n \in \mathbb{N}.$$

Proof: For $n \in \mathbb{N}^*$, $R(0) \cap \partial B(0, 2n) \neq \emptyset$ means that there is a path from 0 to $\partial B(0, 2n)$ which avoids the points of $\widetilde{C}$. $R(0) \cap \partial B(0, 2n) \neq \emptyset$ implies that there is a point x of $\partial B(0, n)$ who checks $0 \to x \to \partial B(0, 2n)$ out of $\widetilde{C}$. Then, this point is in particular a point from which the incoming cluster or the outgoing cluster is finite. This finite cluster is of radius greater than n. However, the distribution of this cluster is identical to the distribution of a cluster of finite percolation on the non-oriented model. So there is $\sigma_0 > 0$ such that (see theorem (9,1) in [9]):

$$P(C(x) \cap \partial B(x, n) \neq \emptyset, C(x) \text{ is finite}) \leq \exp(-\sigma_0 n),$$

$$P(\overline{C}(x) \cap \partial B(x, n) \neq \emptyset, \overline{C}(x) \text{ is finite}) \leq \exp(-\sigma_0 n).$$

So we have:

$$\begin{aligned}
P\left(\left(R(0) \cup \overline{R}(0)\right) \cap \partial B(0, 2n) \neq \emptyset\right) &\leq 2P\left(R(0) \cap \partial B(0, 2n) \neq \emptyset\right) \\
&\leq 2 \sum_{x \in \partial B(0,n)} P\left(|C(x)| < \infty, x \to \partial B(x, n)\right) \\
&\quad + 2 \sum_{x \in \partial B(0,n)} P\left(\left|\overline{C}(x)\right| < \infty, x \leftarrow \partial B(x, n)\right) \\
&\leq 4\left|\partial B(0, n)\right| \exp(-\sigma_0 n),
\end{aligned}$$

then,

$$P\left(\left(R(0)\cup\overline{R}(0)\right)\cap\partial B(0,2n)\neq\emptyset\right)\leq 4(2n+1)^d\exp(-\sigma_0 n).$$

So we have a distribution of the radius of $R(0)\cup\overline{R}(0)$ which is exponentially decreasing from a certain rank, but as

$$\{\left(R(0)\cup\overline{R}(0)\right)\cap\partial B(0,n)\neq\emptyset\}$$

is of probability strictly smaller than 1 for all n, we conclude the proof. □

Now, we can define the neighborhood on $\widetilde{C}$ of a site x. For this, we will examine the smallest set that contains strictly $R(x)$ and $\overline{R}(x)$, which contains points of $\widetilde{C}$, and such that two points of $\widetilde{C}$ in this box are connected by an open path that does not come out of a box a little larger. For this last condition, which allows to increase the crossing time of $\mathcal{V}(x)$, the neighborhood of x, we will use ρ, the parameter defined in [3].

Definition 4.6. *Let $k(x)$ the infimum of the $l\in\mathbb{N}^*$ such that:*

i) $\partial B(x,l)\cap\left(R(x)\cup\overline{R}(x)\right)=\emptyset$

ii) $B(x,l)\cap\widetilde{C}\neq\emptyset$

iii) $\forall(y,z)\in\left(B(x,l)\cap\widetilde{C}\right)^2\quad y\to z$ *in* $B(x,2l(\rho d+1))$

Indeed, not only $k(x)$ is finite a.s., but it's even sub-exponential:

Lemma 4.7. *There is $\sigma=\sigma(\alpha,d)$ such that, for all $n\in\mathbb{N}$:*

$$P(k(x)\geq n)\leq 3\exp(-\sigma n).$$

Proof: We show that the probability that any one of these conditions is not realized for n is exponentially decreasing:

i) $P\left(\partial B(x,n)\cap\left(R(x)\cup\overline{R}(x)\right)\neq\emptyset\right)\leq\exp(-\sigma_1 n)$

ii) there is m integer such that

$$P\left(B(x,n)\cap\widetilde{C}=\emptyset\right)\leq\exp(-\sigma_2[n/(m+1)]).$$

Indeed, Grimmett and Maarstrand [8] have shown that there is $m=m(\alpha)$ such that $p(\alpha)>p_c(S_m)$, where $S_m=[0,m]\times\mathbb{Z}^{d-1}$. Then, for $e_1=(0,0+(1,0,...,0))$, we have:

$$P\left(B(x,n)\cap\widetilde{C}=\emptyset\right)\leq P\left(\forall z\in\{je_1:0\leq j\leq n\},\quad z\notin\widetilde{C}\right),$$

where,

$$\{\forall z \in \{je_1 : 0 \leq j \leq n\},\ \ z \notin \widetilde{C}\} = \{\forall z \in \{je_1 : 0 \leq j \leq n\},\ C(z) \text{ or } \overline{C}(z) \text{ is finite}\}.$$

If $C_m(z)$ and $\overline{C}(z)$ are respectively the clusters incoming and outgoing z in the slice of space

$$S_m(l) = [l(m+1), (l+1)(m+1) - 1] \times \mathbb{Z}^{d-1},$$

to which z belongs. If $C(z)$ (or $\overline{C}(z)$) is finite then $C_m(z) = C(z) \cap S_m(l)$ (or $\overline{C}(z)$) is finite. Events from two different slices are independent, so:

$$\begin{aligned} P\left(\forall z \in \{je_1 : 0 \leq j \leq n\},\ \ z \notin \widetilde{C}\right. &\leq P\left(\forall z \in \{je_1 : 0 \leq j \leq n\},\ C_m(z) \text{ or } \overline{C}_m(z) \text{ is finite}\right) \\ &\leq P\left(\forall z \in \{je_1 : 0 \leq j \leq m\},\ C_m(z) \text{ or } \overline{C}_m(z) \text{ is finite}\right)^{[n/m+1]} \\ &\leq \exp(-\sigma_2[n/(m+1)]), \end{aligned}$$

with $\sigma_2 = \sigma_(\alpha, d) > 0$, independent of n. Indeed, if $z_0 = [m/2]e_1$ we have:

$$\begin{aligned} P\left(\exists z \in \{je_1 : 0 \leq j \leq m\},\ C_m(z) \text{ and } \overline{C}_m(z) \text{ are infinite}\right. & \\ &\geq P\left(C_m(z_0) \text{ and } \overline{C}_m(z_0) \text{ are infinite}\right) \\ &\geq P\left(C_m(z_0) \text{ is infinite}\right) P\left(\overline{C}_m(z_0) \text{ is infinite}\right) \\ &> 0, \end{aligned}$$

because of the FKG inequality and that p is strictly greater than the probability of percolation on the slice S_m.

iii) $P\left(\exists (y,z) \in \left(B(x,n) \cap \widetilde{C}\right)^2 y \nrightarrow z \text{ in } B(x, 2n(\rho d + 1))\right)$
$\leq \sum_{y,z \in \partial B(x,n)} P\left(y, z \in \widetilde{C},\ y \nrightarrow z \text{ in } B(x, 2n(\rho d + 1))\right)$

Thus, if $y \to z$ and $y \nrightarrow x$ in $B(x, 2n(\rho d + 1))$, there is $y' \in \partial B(x, 2n)$ such that

$$y \to z \text{ and } y \nrightarrow x \text{ in } B(x, 2n(\rho d + 1)).$$

But if $z \in \partial B(x,n)$, $y' \in \partial B(x, 2n)$ and $y' \nrightarrow z$ in $B(x, 2n(\rho d + 1))$ then $D(y', z) \geq 4n\rho d + n$ and $\|z - y'\|_1 \leq 3dn$ imply:

$$D(y', z) > \rho \|z - y'\|_1 .$$

From where, for $z \in \partial B(x,n)$ and $y' \in \partial B(x,2n)$:

$$\begin{aligned} P\left(y' \to z,\ y \nrightarrow z \text{ in } B(x,2n(\rho d+1))\right) &\leq P\left(y' \to z,\ D(y',z) > \rho \left\|z-y'\right\|_1\right) \\ &\leq \exp(-\beta \left\|z-y'\right\|_1) \\ &\leq \exp(-\beta n). \end{aligned}$$

Consequently:

$$P\left(\exists (y,z) \in \left(B(x,n) \cap \widetilde{C}\right)^2 y \nrightarrow z \text{ in } B(x,2n(\rho d+1))\right)$$

$$\leq (2n+1)^{2d}(4n+1)^d \exp(-\beta n).$$

Finally:

$$P\left(\forall (y,z) \in \left(B(x,n) \cap \widetilde{C}\right)^2 y \to z \text{ in } B(x,2n(\rho d+1))\right)$$

$$\geq P\left(\text{all the edges of } B(x,n) \text{ are open }\right) > 0.$$

Thus we have just proved the existence of $\sigma_3 = \sigma_3(\alpha,d) > 0$ such that, for $n \in \mathbb{N}$:

$$P\left(\exists (y,z) \in \left(B(x,n) \cap \widetilde{C}\right)^2 y \nrightarrow z \text{ in } B(x,2n(\rho d+1))\right)$$

$$\leq \exp(-\sigma_3 n).$$

which completes showing that the probability that any one of these three properties is not satisfied is sub-exponential: we have the existence of $\sigma = \sigma(\alpha,d) > 0$ such that, for all $n \in \mathbb{N}$:

$$P(k(x) \geq n) \leq 3\exp(-\sigma n).$$

□

We define the neighborhood of x on $\widetilde{C}$ as:

$$\mathcal{V}(x) = B(x,k(x)) \cap \widetilde{C}.$$

For all y and z of $\mathcal{V}(x)$, there is a more path, in number of edges, then in lexicographic order on the edges used according to the order previously fixed, connecting y to z that we will note $\Gamma(y,z)$. This path is particularly included in $B(x,2k(x)(\rho d+1))$, according to the condition iii). Thus, we can define a neighborhood "in edges" of x:

$$\vec{\Gamma} = \{e \subset B(x, k(x)),\ e \text{ open }\} \cup \{e \in \Gamma(y, z),\ y, z \in \mathcal{V}(x)\},$$

also note,

$$\vec{\Gamma} \subset B(x, 2k(x)(\rho d + 1)).$$

We approach $t(x, y)$ by:

$$\hat{t}(x, y) = \inf_{x' \in \mathcal{V}(x),\ y' \in \mathcal{V}(y) t(x', y'),}$$

$$u(x) = \sum_{e \in \vec{\Gamma}} \tau(e).$$

$\hat{t}(x, y)$ is an approximation of $t(x, y)$ bacause, if $y \in C_x \backslash R(x)$ there is a path from x to y, and any path of this type necessarily passes through $\widetilde{C}$. It is not difficult to conclude that:

$$\hat{t}(x, y) \leq t(x, y) \leq u(x) + \hat{t}(x, y) + u(y).$$

The main interest of this appoximation is that $\hat{t}$ is a.s. finite. In fact, $\hat{t}$ and u checking the following regularity condition:

Lemma 4.8. *For x and y of $\mathbb{Z}^d$ and $r \in \mathbb{N}$ we have:*

$$E(u(x))^r < \infty \qquad \textit{and} \qquad E(\hat{t}(x, y))^r < \infty$$

Proof: For $x \in \mathbb{Z}^d$, we note that:

$$u(x) \leq |B(x, k(x))(\rho d + 1)|^2 \, T,$$

this is suffices to prove that for every $r \in \mathbb{N}$, $u(x)$ is r-integrable.

Let us show that for x and y of $\mathbb{Z}^d$, $\hat{t}(x, y)$ is r-integrable for every $r \in \mathbb{N}$. We call medium of (x, y) in $\mathbb{Z}^d$

$$m(x, y) = x + \left\lfloor \frac{x + y}{2} \right\rfloor .$$

For the definition of $k(x)$, we build a box centered in $m(x, y)$ of radius $k(x, y)$, which has the following three properties: contains $B(x, k(x))$, $B(y, k(y))$ and the shortest open path (in number of edges) from $\mathcal{V}(x)$ to $\mathcal{V}(y)$. With these three conditions, we garantee that the shortest path, in number of edges, from $\mathcal{V}(x)$ to $\mathcal{V}(y)$, is in $B(m(x,y), 2k(x,y)(\rho d+1))$:

$$\begin{aligned} k(x,y) = \inf\{l \in \mathbb{N} : & B(x,k(x)) \subset B(m(x,y),l), \\ & B(y,k(y)) \subset B(m(x,y),l), \\ & \mathcal{V}(x) \to \mathcal{V}(y) \text{ in } B(m(x,y), 2l(\rho d+1))\}. \end{aligned}$$

Thus, for $n \in \mathbb{N}$:

$$\begin{aligned} P(k(x,y) \geq n) \leq\ & P(k(x) \geq n - \|x - m(x,y)\|_\infty) + P(k(y) \geq n - \|y - m(x,y)\|_\infty) \\ & + P(\exists z, z' \in B(m(x,y), 2n(\rho d+1)) \cap \tilde{C},\ z \nrightarrow z' \text{ in } B(m(x,y), 2n(\rho d+1))) \\ \leq\ & 3\exp(-\sigma(n - \|x - m(x,y)\|_\infty)) + 3\exp(-\sigma(n - \|y - m(x,y)\|_\infty)) + \exp(-\sigma_3 n) \\ \leq\ & 7\exp(-\sigma(n - \|x - y\|_\infty)). \end{aligned}$$

Then, we have:

$$\hat{t}(x,y) \leq |B(m(x,y), 2k(x,y)(\rho d+1)|^2\, T,$$

which justifies that $\hat{t}(x,y)$ is r-integrable, for all $r \in \mathbb{N}$. □

It should be noted that the distribution of $u(x)$ as that of $t(x,y)$ are in fact of exponential tail at $t^{\frac{1}{4d}}$ at least, which largely justifies the regularity necessary for the establishment of the desired asymptotic result.

Chapter 5

Radial limits and behavior of $\hat{t}$

In this Chapter we write $\mathbb{Z}^j$ with $j = 2$ or $j = d$ for . Now we will prove the existence of radial limits with the following theorem

Teorema 5.1. *If $\theta \in \mathbb{Z}^j$, with $j = 2, d$, then there is a constant $\mu(\theta)$ so that, as $n \to \infty$,*

$$\left[\tfrac{1}{n} t(0, n\theta) - \mu(\theta)\right] \mathbb{1}_{\{n\theta \in C_0\}} \to 0 \ a.s.$$

Before proving this theorem, let $\xi(x, y) = \hat{t}(x, y) + u(y)$, then

$$\xi(x, z) \le \xi(x, y) + \xi(y, z). \tag{5.1}$$

Proof of Theorem 5.1: For fixed $\theta \in \mathbb{Z}^j$ let $\xi_{m,n} = \xi(m\theta, n\theta)$, $0 < n < \infty$. Then, by (5.1) and that for any of the three cases (see chapters 2, 3 and 4 respectively), we have:

$$E(u(z))^m < \infty \quad \text{for} \quad m = 1, 2, 3, ... \quad \text{and} \quad z \in \mathbb{Z}^j,$$

$$E(\hat{t}(x, y))^m < \infty \quad \text{for} \quad m = 1, 2, 3, ... \quad \text{and} \quad x, y \in \mathbb{Z}^j,$$

$\xi_{m,n}$ is subbaditive in the sense of Kingman, and and by the subadditive ergodic theorem (see appendix), the limit $\mu(\theta) := \lim_{n\to\infty} \frac{\xi_{0,n}}{n}$ exists a.s. in L_1 and is constant. To finish the proof, observe that:

$$\left| \frac{\hat{t}(0, n\theta) - t(0, n\theta)}{n} \right| \mathbb{1}_{\{n\theta \in C_0\}} \le \frac{u(0) + u(n\theta)}{n}.$$

Then,

$$\left| \frac{\hat{t}(0, n\theta) - \xi_{0,n}}{n} \right| \mathbb{1}_{\{n\theta \in C_0\}} \le \frac{u(0)}{n},$$

which implies,

$$\left| \frac{\hat{t}(0, n\theta) - \xi_{0,n}}{n} \right| \mathbb{1}_{\{n\theta \in C_0\}} \to 0 \text{ a.s.}. \qquad \square$$

Now, established in each direction of $\mathbb{Z}^j$ with $j = 2, d$ the existence of a linear velocity of propagation; however, in order to establish an asymptotic shape result, especially for the passage times $(\hat{t}(x, y))$, it is necessary to control this propagation velocity uniformly in all directions. We will now study $u(\theta)$ for $\theta \in \mathbb{Z}^j$, or more precisely, following the construction of Cox and Durrett [7], an extension of μ to $\mathbb{R}^j$ denoted φ. The asymptotic form of the epidemic will be the border of the convex:

$$D = \{x \in \mathbb{R}^j : \varphi(x) \leq 1\}.$$

We will see that this extension of μ to $\mathbb{R}^j$ is Lipchtzian, convex and homogeneous, that is, for all $x \in \mathbb{R}^j$ and $\lambda > 0$:

$$\varphi(\lambda x) = \lambda \varphi(x).$$

In particular, if $x \in \mathbb{Q}^j$, for $N = \min\{m \geq 1 : mx \in \mathbb{Z}^j\}$:

$$\varphi(x) = \varphi(Nx)/N = \mu(Nx)/N,$$

which is an expected property for the asymptotic linear velocity of propagation in the direction of x. This is done according to the standard construction of [7]. First, if for every $z \in \mathbb{Z}^j$ $(j = 2, d)$:

$$g(z) = E(\xi(0, z)),$$

then, by the subadditive theorem, we have:

$$\lim_{n \to \infty} \xi(0, nz)/n = \inf_n E(\xi(0, nz)/n) = \inf_n g(nz)/n = \mu(z).$$

- If $j = 2$, extend the domain of g to all $\mathbb{R}^2$ by making it linear on triangles of the form (x, y), $(x, y + 1)$, $(x + 1, y)$ and $(x, y + 1)$, $(x + 1, y)$, $(x + 1, y + 1)$.
- If $j = d$, extend the domain of g to all $\mathbb{R}^d$ in barycentric manner to a continuous function (see [5]).

If we define the sequence of functions $(g_n)_{n \in \mathbb{N}^*}$ by:

$$g_n(x) = g(nx)/n, \quad \forall x \in \mathbb{R}^j.$$

Then, there is a function φ on $\mathbb{R}^j$, Lipschitz, such that $\varphi = \mu$ on $\mathbb{Z}^j$, which is limit uniform throughout compact in $\mathbb{R}^j$, of $(g_n)_{n\in\mathbb{N}^*}$. This is a direct consequence of theorem of Ascoli and Lemma following:

Lemma 5.2. *If $(g_n)_{n\in\mathbb{N}^*}$ is a sequence of λ-Lipschitz functions, then this sequence of functions is equicontinuous.*

Indeed, it is sufficient, of course, to check that g is Lipschitz, which is obtained from the following three inequalities, for $x,\ y \in \mathbb{Z}^j$:

$$\begin{aligned} g(x)+g(y) &\geq g(x+y), \\ g(x+y)+g(-y) &\geq g(x) \text{ and} \\ g(-y)=g(y) &\leq \|y\|_1\, g(e_1). \end{aligned}$$

These three inequalities are result of subaditivity and symmetry of ξ. The demonstration of Lemma 5.2 is based on the development of these inequalities and the convex combination, depending on the case, if the extension is triangular or barycentric, see [7] and [5].

Now, we will establish the properties of homogeneity and convexity of φ. For $x \in \mathbb{Q}^j$, we have:

$$\lim_{n\to\infty} g_n(x) = \varphi(x).$$

Indeed, if $N = \min\{m \geq 1 : mx \in \mathbb{Z}^j\}$, remember $j=2$ or $j=d$, then $(g_n(Nx))_{n\in\mathbb{N}^*}$ is a subadditive sequence. Thus,

$$\lim_{n\to\infty} \frac{g(nNx)}{nN} = \frac{\varphi(Nx)}{N}.$$

For $i \in \{0, ..., N-1\}$, and all $n \in \mathbb{N}$, since g is Lipschitzian, we have that $|g(nNx+jx)-g(nNx)|$ $\lambda^* \|ix\|_1$, which is sufficient to justify the convergence on $\mathbb{Q}^j$. We then have the convergence on $\mathbb{R}^d$, since every subsequence of $(g_n)_{n\in\mathbb{N}^*}$, admits a subsequence which converges uniformly, on any compact, to a continuous function which is equal to φ on $\mathbb{Q}^j$.

Then the homogeneity on $\mathbb{Q}^j$ is obvious, and the convexity of φ deduces: for all $x, y \in \mathbb{Q}^j$, $\lambda \in \mathbb{Q}^+$ and $\lambda' \in \mathbb{Q}^+ \cap [0,1]$, there is $N \in \mathbb{N}^*$ such that Nx, Ny, $N\lambda$ and $N\lambda'$ are integers. Then:

$$g(n\lambda Nx) \leq N\lambda g(Nx).$$

$$\begin{aligned} g(N\lambda' Nx + N(1-\lambda')Ny) &\leq g(N\lambda' Nx) + g(N(1-\lambda')Ny) \\ &\leq N\lambda' g(Nx) + N(1-\lambda')g(Ny), \end{aligned}$$

by subadditivity of ξ on $\mathbb{Z}^j$. The properties are therefore true on $\mathbb{Q}^j$, and by continuity of φ, on $\mathbb{R}^j$.

Finally, $\{x : \varphi(x) \leq \beta\}$ is bounded, for all $\beta > 0$. Indeed, the passage times following the edges are reduced by passage times of exponential law, so the epidemic can not go faster than this first-pass percolation process, whose passage times are exponential law of parameter $\alpha/2d$, and which, according to the theorem (1.15) of [12], moves linearly like the boundary of a convex.

Now, we extend $\hat{t}(0, z)$, $u(z)$ and $\xi(0, z)$ to the points on $\mathbb{R}^j$ just like the extension of the function g, and we check that $\hat{t}(0, kx)/k$ tends a.s. to $\varphi(x)$ when k tends to infinity, this for all $x \in \mathbb{Q}^j$. The existence of a maximum speed of propagation of the epidemic will be deduced from the regularity properties established above, for φ.

If $x \in \mathbb{Q}^j$, then there is $N = \min\{m \geq 1 : mx \in \mathbb{Z}^j\}$. Let $a_{m,n} = \xi(mNx, nNx)$, which is a subadditive process that satisfies the assumptions of the subadditive theorem, and therefore a.s.:

$$\lim_{n\to\infty} \frac{a_{0,n}}{nN} \frac{\varphi(Nx)}{N} = \varphi(x).$$

Also, since $Eu^2(n\theta) = Eu^2(0) < \infty$ for all $\theta \in \mathbb{Z}^j$, the distribution of u being invariant by translation, then by Chebychev's inequality:

$$\sum_{n=0}^{\infty} P(u(n\theta) > n\epsilon) < \infty,$$

for all $\epsilon > 0$, consequently by the Lemma of Borel-Cantelli, we have:

$$\frac{1}{n} u(n\theta) \to_{n\to\infty} 0 \quad \text{a.s.}$$

then,

$$\frac{u(nNx)}{nN} \to_{n\to\infty} 0 \quad \text{a.s.}$$

thus,

$$\frac{\hat{t}(0, nNx)}{nN} \to_{n\to\infty} \varphi(x) \quad \text{a.s..}$$

It remains to verify that this result is still correct for $(\hat{t}(0, kx)/k,\ k \in \mathbb{Z}^*)$. By invariance, symmetry and integer translations of the distributions of $(\xi(x, y))$ and $(u(z))$, for $\epsilon > 0$ and for $0 \leq i < N$, we obtain:

$$\sum_{n=1}^{\infty} P(\left|\hat{t}(0, (i + nN)x) - \hat{t}(0, nNx)\right| > \epsilon) \leq \sum_{n=1}^{\infty} P(u(0) + u(ix) + \xi(0, ix) > \epsilon n) < \infty.$$

This according to Chebychev's inequality. The Lemma of Borel-Cantelli thus makes it possible to conclude:

$$\frac{\hat{t}(0,kx)}{k} \to \varphi(x) \quad \text{a.s.}, \quad \forall x \in \mathbb{Q}^j.$$

Thus, we have the almost sure convergence of $\hat{t}(0,kx)/k$ to $\varphi(x)$ on $\mathbb{Q}^j$, as $k \to \infty$.

Now, we will highlight the fact that the growth of $\hat{t}(o,z)$ can not be more than linear in $\|z\|_\infty$, from a certain rank for $z \in \mathbb{Z}^j$. This is the main argument that allows to establish, together the existence of radial limits, the theorem of asymptotic shape for $\hat{t}$.

Teorema 5.3. *There is $K = K(\alpha, d)$ such that:*

$$\sum_{z\in\mathbb{Z}^j} P(\hat{t}(0,z) > K\,\|z\|_\infty) < \infty.$$

Cox and Durrett [7] used the minimal circuit method to show Theorem 5.3 for the nearest neighbor case on $\mathbb{Z}^2$. However, this does not work for the finite range since we can not even define the minimal circuit in $\mathbb{E}_R$. Fortunately, we can define the minimal circuit with the renormalized sites since V_n is a planar graph (see [19]), then use the same arguments used in [7]. In the case of Chabot, to prove the Theorem 5.3, the definition of $k(z)$ and $\mathcal{V}(x)$ are used (see [5]).

Now, since we have established that the general velocity of propagation for the passage times $(\hat{t}(x,y))$ is a.s. at most linear, in the next chapter we will see an asymptotic result for:

$$\hat{A}_s = \{z \in \mathbb{Z}^j : \hat{t}(0,z) \leq s\} \quad \text{with} \quad j = 2, d. \tag{5.2}$$

Chapter 6

Proof of the shape theorem

6.1 Asymptotic shape for $\hat{t}$

We will prove for $\epsilon > 0$, a.s. for all sufficiently large $s \in \mathbb{R}^+$,

$$(1-\epsilon)sD \cap \mathbb{Z}^j \subset \hat{A}_s \subset (1+\epsilon)sD \cap \mathbb{Z}^j \quad \text{for} \quad j = 2, d.$$

This is, a.s. for all sufficiently large $s \in \mathbb{R}^+$ and $z \in \mathbb{Z}^j$:

- $z \in \hat{A}_s \Rightarrow s^{-1}z \in (1+\epsilon)D$, and

- $s^{-1}z \in (1-\epsilon)D \Rightarrow \hat{t}(0,z) \leq s$

with $D = \{x \in \mathbb{R}^j : \varphi(x) \leq 1\}$. For this, we will follow the construction of Cox and Durrett exhibited in [7].
We will show, by compactness arguments, that for every x for which $\varphi(x) < 1$, there is $\delta(x) > 0$ such that:

$$P(\hat{A}_s \supset sB(x, \delta(x)) \cap \mathbb{Z}^j \text{ for all sufficiently large } s) = 1,$$

and for every x for which $\varphi(x) > 1$ there is $\delta(x) > 0$ such that:

$$P(s^{-1}\hat{A}_s \cap B(x, \delta(x)) = \emptyset \text{ for all sufficiently large } s) = 1.$$

Indeed, we will cover $\mathbb{Z}^j$ by a finite number of cones ($\bigcup_{s \geq 0} B(xs, \delta s,\ x \in \mathbb{R}^j$) in which the propagation is linear as the boundary of D from a certain rank. In fact, we will prove this property for $x \in \mathbb{Q}^j$ and δ rational, which suffices. If the existence of δ for the origin is demonstrated obviously, for $x \neq 0$ and $\delta < \|x\|_\infty$, we will approach all z of cone

$$D(x, \delta) = \mathbb{Z}^j \cap \bigcup_{s \geq 0} B(xs, \delta s),$$

by the center of a ball of type $B(xs, \delta s)$, $s \geq 0$, which contains z. Then, we will show that the time taken to reach z is close to the time to reach the center of this ball and which is in the order of $\|z\|\,\varphi(x)$. To do this, let:

$$\sigma_z = \inf\{s : \; z \in B(xs, \delta s)\}.$$

Now, we show that for a δ satisfactory, $\hat{t}(\sigma_z x, z)$ is growing a.s. at most linearly with z on $D(x, \delta)$, which will allow to establish a result of asymptotic shape for $\hat{t}$. Let $x \in \mathbb{Q}^j$ be different from the origin. We take $0 < \delta < \|x\|_\infty$ then, for $z \in D(x, \delta)$, we have that $\sigma_z x$ is the center of the first ball of the form $B(sx, s\delta)$ that contains z, and thus:

$$\|z - \sigma_z x\|_\infty = \delta \sigma_z.$$

It should be noted in particular that there is then $i \in \{1, ..., j\}$ (remember $j = 2$ or $j = d$) such that:

$$|z_i - \sigma_z x_i| = \sigma_z \delta,$$

thus,

$$\sigma_z = \frac{|z_i|}{|x_i| + \delta}.$$

For $i \in \{1, ..., j\}$, let:

$$\lambda_i = \frac{1}{|x_i| + \delta}.$$

The centers $(\sigma_z x, \; z \in D(x, \delta))$ are therefore all of the form ny_i, for $i \in \{1, ..., j\}$, where $y_i = \lambda_x$. In particular, this justifies the subsequent use of the result of radial limits for $\hat{t}(0, \sigma_z x)$. It establishes the following result:

Lemma 6.1. *Let $x \in \mathbb{Q}^j$ be different from the origin, for $a > 0$, there is $\delta > 0$ such that a.s. there is only a finite number of $z \in D(x, \delta)$ such that:*

$$\hat{t}(\sigma_z x, z) > a\sigma_z.$$

Proof: Let $\delta \in \mathbb{Q}_+^*$ such that $a/\delta > K$ where K was determined in Theorem 5.3. If $z \in D(x, \delta)$, there is $i \in \{1, ..., j\}$ and $n \in \mathbb{N}$ such that $\sigma_z = n\lambda_i$ and thus $z \in \partial B(n\lambda_i x, n\lambda_i \delta)$.

We then have:

$$\begin{aligned}
\sum_{z\in D(x,\delta)} P(\hat{t}(\sigma_z x, z) > a\sigma_z) &\leq \sum_{i\in\{1,\dots,j\}} \sum_{n\geq 0} \sum_{z\in D(x,\delta):\sigma_z = n\lambda_i} P(\hat{t}(\sigma_z x, z) > a\sigma_z) \\
&\leq \sum_{i\in\{1,\dots,j\}} \sum_{n\geq 0} \sum_{z\in D(x,\delta):\sigma_z = n\lambda_i} P(\hat{t}(\sigma_z x, z) > K\delta\sigma_z) \\
&\leq \sum_{i\in\{1,\dots,j\}} \sum_{n\geq 0} \sum_{z\in D(x,\delta):\sigma_z = n\lambda_i} P(\hat{t}(0, z - \sigma_z x) > K\delta\sigma_z) \\
&\leq \sum_{i\in\{1,\dots,j\}} \sum_{n\geq 0} (2n\lambda_i\delta + 1)^j \exp(-n\sigma\delta\lambda_i) < \infty.
\end{aligned}$$

□

Existence of $\delta(x)$ for $\varphi(x) < 1$. First, for $x = 0$, there is $\delta(0) \in \mathbb{Q}$ such that:

$$0 < \delta(0) < 1/K.$$

We know that there exists a.s. a finite number of $y \in \mathbb{Z}^j$ such that:

$$\hat{t}(0, y) > K \|y\|_\infty .$$

Let Y be all these sites. If $s_0 = \max(\hat{t}(0, y),\ y \in Y)$ then, for $s > s_0$, we know that $\hat{t}(0, y) < s$ for all $y \in Y$, and for $z \in \mathbb{Z}^j \backslash Y$ such that $s^{-1}z \in B(0, \delta(0)$, then:

$$\hat{t}(0, z) \leq K \|z\|_\infty = sK \left\|s^{-1}z\right\|_\infty \leq s.$$

Thus, $\delta(0)$ is a radius that satisfies:

$$P(s^{-1}\hat{A}_s \supset B(0, \delta(0)) \text{ for all sufficiently large } s) = 1.$$

Now, we assume that $x \in \mathbb{Q}^j$, $x \neq 0$, such that $\varphi(z) < 1$. Let ϵ be such that $0 < \epsilon < (1 - \varphi(x))/2$. Then, we take $\delta \in \mathbb{Q}$, $\delta > 0$ such that $\delta < \|x\|_\infty$, which verifies:

$$(1 - \varphi(x) - 2\epsilon)/\delta > K.$$

From the existence of radial limits, we have, for all $i \in \{1, ...j\}$ and $\delta \in \mathbb{Q}^+$:

$$P(\hat{t}(0, n\lambda_i x) > (\varphi(x) + \epsilon)n\lambda_i \text{ i.o.}) = 0.$$

Indeed, from the previous chapter, $\lambda_i \in \mathbb{Q}^+$ and:

$$\varphi(x) = \lambda_i^{-1}\varphi(\lambda_i x).$$

We know, from the previous lemma, that there exists a.s. a finite number of $z \in (x, \delta)$, such that:

$$\hat{t}(\sigma_z x, z) > (1 - \varphi(x) - 2\epsilon)\sigma_z.$$

It has already been established, that there exists only a finite number of $n \in \mathbb{N}$ such that, for all $i \in \{1, ...j\}$:

$$\frac{u(n\lambda_i x)}{n\lambda_i} > \epsilon.$$

Finally, we have:

$$\hat{t}(0, z) \leq \hat{t}(0, \sigma_z x) + u(\sigma_z x) + \hat{t}(\sigma_z x, z).$$

Thus, a.s., with the possible exception of a finite number of z, we have:

$$\hat{t}(0, z) \leq \sigma_z,$$

which means $\hat{t}(0, z) \leq s$, if $z \in B(xs, \delta s)$.

We can thus prove the first of the two inclusions justifying the asymptotic shape for the passage times $\hat{t}$: for $\epsilon > 0$, since $\{x : \varphi(x) \leq 1-\epsilon\} = (1-\epsilon)D$ is compact, it is enough to take a finite recovery of this compact by the $(B(x, \delta(x)))$ for $x \in \mathbb{Q}^j$ and $\varphi(x) \leq 1 - \epsilon$. Then:

$$P(\hat{A}_s \supset s\{x : \varphi(x) \leq 1 - \epsilon\} \cap \mathbb{Z}^j \text{ for all sufficiently large } s) = 1.$$

Existence of $\delta(x)$ for $\varphi(x) > 1$. Now, we want to show that:

$$P(s^{-1}\hat{A}_s \cap \{x : \varphi(x) \geq 1 + \epsilon\} = \emptyset \text{ for all sufficiently large } s) = 1.$$

If $\epsilon < 1$ and $s^{-1}\hat{A}_s \cap \{x : \varphi(x) \geq 1 + \epsilon\} \neq \emptyset$, observe that:

$$s^{-1}\hat{A}_s \cap \{x : 2 \geq \varphi(x) \geq 1 + \epsilon\} \neq \emptyset,$$

thus, by compactness, just check that for any x such that

$$2 \geq \varphi(x) > 1,$$

there is $\delta(x) > 0$ such that:

$$P(s^{-1}\hat{A}_s \cap B(x, \delta(x)) = \emptyset \text{ for all sufficiently large } s) = 1.$$

From the existence a.s. of radial limits, we have, for all $\epsilon > 0$, $i \in \{1, ...j\}$ and for all $\delta \in \mathbb{Q}^+$:

$$P(\hat{t}(0, n\lambda_i x) < (\varphi(x) - \epsilon/2)n\lambda_i \text{ i.o. }) = 0.$$

Similarly, we have:

$$P(u(z) > \sigma_z \epsilon/2 \text{ i.o.}) = 0.$$

According to the previous lemma, we have seen that:

$$P(\hat{t}(\sigma_z x, z) > (\varphi(x) - 1 - 2\epsilon)\sigma_z \text{ i.o.}).$$

From the subadditivity, we have the following inequality:

$$\hat{t}(0, z) \geq \hat{t}(0, \sigma_z x) - \hat{t}(z, \sigma_z x) - u(z).$$

Then, for all sufficiently large z a.s.:

$$\hat{t} > (1 + \epsilon)\sigma_z.$$

Let $\sigma'_z = \sup\{s \geq 0 : \; z \in B(sx, s\delta\}$. For all s such that $z \in B(sx, s\delta\}$:

$$\frac{\|x\|_\infty}{\|x\|_\infty + \delta} \leq \frac{s\,\|x\|_\infty}{\|z\|_\infty} \leq \frac{\|x\|_\infty}{\|x\|_\infty - \delta},$$

then,

$$\sigma'_z \leq \frac{\|x\|_\infty + \delta}{\|x\|_\infty - \delta}\sigma_z.$$

Thus, if δ is sufficiently small, we have $\sigma'_z \leq (1 + \epsilon)\sigma_z$. We have proved the existence of $\delta > 0$, such that a.s.:

$$z \in B(sx, s\delta) \Longrightarrow z \notin \hat{A}_s.$$

Then, for all $\epsilon > 0$, from the compactness of $\{x : 1 + \epsilon \leq \varphi(x) \leq 2\}$, the second of the two inclusions justifying the asymptotic shape for the passage times $\hat{t}$, this is:

$$P(s^{-1}\hat{A}_s \cap \{x : \varphi(x) \geq 1 + \epsilon\} = \emptyset \text{ for all sufficiently large } s) = 1.$$

Thus, we have established an asymptotic result for $\hat{t}$:

Teorema 6.2. *Let $\epsilon > 0$. For $\hat{A}_s = \{z \in \mathbb{Z}^j : \hat{t}(0, z) \leq t\}$ (j=2,d), and $D = \{x \in \mathbb{R}^j : \varphi(x) \leq 1\}$, we have the following asymptotic result:*

$$P((1 - \epsilon)sD \cap \mathbb{Z}^j \subset \hat{A}_s \subset (1 + \epsilon)sD \cap \mathbb{Z}^j \;\; \textit{for all sufficiently large } s) = 1.$$

6.2 Asymptotic shape of the epidemic

Now, it can be established that the epidemic will propagate linearly as the boundary of a convex set. First, remember (1.4):

$$\zeta_s = \{x : \eta_s(x) = 0\} \quad \text{and} \quad \xi_s(x) = \{x : \eta_s(x) = i\}.$$

Also, we need:

$$\text{If } \epsilon > 0 \text{ then } P(u(z) > \epsilon \|z\| \text{ i.o. }) = P(T_z > \epsilon \|z\| \text{ i.o. }) = 0. \tag{6.1}$$

This is a consequence of $Eu(z)^2 < \infty$ and $ET_z^2 < \infty$.

To establish the desired result of the asymptotic shape, it must be shown that the infected region does not get stuck inside D. More precisely, it must be shown that, if $\epsilon > 0$:

$$P(\zeta_s \cap (1-\epsilon)sD = \emptyset \text{ for all sufficiently large } s) = 1. \tag{6.2}$$

Indeed, we already know that, for $\epsilon > 0$, a.s. for s sufficiently large, if $z \in (1-\epsilon)sD$, then:

$$\hat{t} \leq (1-\epsilon/2)s.$$

We add $u(0) + u(z) + T_z$ (in the case of Chabot, $T_z = T$ for all $z \in \mathbb{Z}^d$) to both members of this inequality and using that:

$$\text{If } t(x,y) < \infty, \text{ then } \hat{t}(x,y) \leq t(x,y) \leq \hat{t}(x,y) + u(x) + u(y), \ \forall x, y \in \mathbb{Z}^j \ \text{ (in all three cases)},$$

to obtain:

$$t(0,z) + T_z \leq (1-\epsilon/2)s + u(0) + T_z + u(z).$$

With $d = \sup_{x \in D} \|x\|$, we have from 6.1 that a.s., for all s:

$$u(0) + u(z) + T_z \leq \frac{\epsilon}{3d}\|z\| \leq \frac{\epsilon}{3}(1-\epsilon)s$$

(since $z \in (1-\epsilon)sD$). Combining this with the previous inequality gives us:

$$t(0,z) + T_z \leq (1-\epsilon/6)t,$$

and so z belongs to ζ_s, not ξ_s. This proves 6.2 and

$$P(t(1-\epsilon)D \cap C_0 \subset \zeta_s \text{ for all sufficiently large } s) = 1.$$

On the other hand, if $z \in \xi_s$ or $z \in \zeta_s$, then $t(0,z) \leq s$ so certainly $\hat{t}(0,z) \leq s$, and by Theorem 6.2, $z \in (1+\epsilon)sD$. That is:

$$P(\xi_s \subset (1+\epsilon)sD \ \text{ for all sufficiently large } s) = 1,$$

and

$$P(\zeta_s \subset (1+\epsilon)sD \ \text{ for all sufficiently large } s) = 1.$$

Bibliography

[1] M. Aizenman, J. Chayes, L. Chayes, J. Frohlich and L. Russo, *On a sharp transition area law to perimeter law in system of random surfaces*, Comm. Math. Phys. (92) 19-69, 1983.

[2] E. D. Andjel, N. Chabot and E. Saada *A shape theorem for an epidemic model in dimension* $d \geq 3$, 2015.

[3] P. Antal and A. Pisztora,*On the chemical distance for supercritical Bernoulli percolation*,The Annals of Probability. Vol. 24, No. 2, 1036-1048 ,1996.

[4] N.T.J. Bailey, *The simulation of stochastic epidemics in two dimensions*. Fifth Berkeley Symp., Vol. IV 237-257, 1965.

[5] N. Chabot, *Forme asymptotique pour un modèle épidemique en dimension supérieure à trois*, 1998. Thèse de doctorat, Université de Provence.

[6] J.T. Cox and R. Durrett, *Some limit theorems for percolation processes with necessary and sufficient conditions*, Ann. Probab. (9) 583-603, 1981.

[7] J.T. Cox and R. Durrett, *Limit theorems for the spread of epidemics an forest fires*, Stochastic Processes and their Applications (30) 171-191, 1988.

[8] G. Grimmett and J.M. Marstand, *The supercritical phase of percolation is well behaved* Proc. Roy. Soc. London Ser (430) 439-459, 1991.

[9] G. Grimmett, *Percolation*, Springer. Vol. 321, 2.ed., 1999.

[10] T. Harris, *A lower bound for the critical probability in a certain percolation process*, Proceedings of the Cambridge Philosophical Society (56) 13-20, 1960.

[11] H. Kesten, *Percolation Theory for Mathematicians*, Birkhauser, Boston, 1982.

[12] H. Kesten, *Aspects of first passage percolation.* Lecture Notes in Math., 1180, Springer, Berlin (1986). Ecole d'été de probabilités de Saint-Flour XIV, 1984.

[13] H. Kesten and Y. Zhang, *The probability of a large finite cluster in supercritical Bernoulli percolation* Ann. Probab. (18) 537-555, 1990.

[14] K. Kuulasmaa, *The spatial general epidemic and locally dependent random graphs*. J. Appl. Prob., (19) 745-758, 1982.

[15] T.M. Liggett, *Interacting particle systems*, Springer. Vol. 276, reprint of the 1985 edition, 2005.

[16] G. McKay and N. Jan, *Forest fires as critical phenomena.* J. Phys., A 17, L757-L760, 1984.

[17] T. Ohtsuki and T. Keyes, *Biased percolation: forest fires with wind.* J. Phys., A19, L281-L287, 1986.

[18] L. Russo, *On the critical percolation probabilities*, Z. Wahrsch. Verw. Geb.(56) 229-237, 1981.

[19] Yu Zhang, *A shape theorem for epidemics and forest fires with finite range interactions*, The Annals of Probability. Vol. 21, No. 4, 1755-1781,1993.

Appendix A

Useful theorems

A.1 Harris-FKG inequality

On the spae $\Omega = \{0,1\}^{\mathbb{E}^d}$ we consider the partial orden given by $\omega \leq \omega'$ if $\omega(e) \leq \omega'(e)$ for all edges e, i.e. whenever and edge is open in ω it is also open in ω'.

A random variable X on Ω is said to be *increasing* if $X(\omega) \leq X(\omega')$ whenever $\omega \leq \omega'$. If $-X$ is increasing we say that X is *decreasing*. We say that an event $A \in \mathbb{A}$ is increasing (decreasing) if its indicatir function 1_A is increasing (decreasing, resp), where $1_A(\omega) = 1$ if $\omega \in A$ and ,$\{x \leftrightarrow y\}, \{|C(0)| = \infty\}, \{\exists\ an\ infinite\ open\ path\}$.

Let $P_p(\omega(e) = 1) = p$ be for each $e \in \mathbb{E}^d$. A very useful property held by the measures P_p is the Harris-FKG inequality

Teorema A.1. *(a) If X and Y are bounded increasing random variables in $(\Omega, \mathbb{A})$, then*

$$E_p(XY) \geq E_P(X)E_P(Y), \tag{A.1}$$

(b) If A and B are increasing events in $(\Omega, \mathbb{A})$, then

$$P_p(AB) \geq P_p(A)P_p(B). \tag{A.2}$$

The above theorem says that increrasing events (variables) are positively correlated under the measures P_p. In particular in (A.2) can be stated in terms of the conditional probability:

$$P_p(A|B) \geq P_p(A). \tag{A.3}$$

i.e. Knowing that B occurs increases the chance to A occur. For product measures this was first proven by Harri's 1960. The propertie was later investigated for amore general classof measures of special importance in statistical mechanical models for ferromagnetic interactions. The extension is due to Fortuin, Kasteleyn and Ginibre 1971, and it is usually named simply FKG inequality (or FKG property).

A.2 Ergodic Theorem

Now, we will discuss a little about ergodic theory. For $x \in \mathbb{Z}^n$, define the shift transformation τ_x on X by

$$(\tau_x \eta)(y) = \eta(y - x)$$

where $\eta \in \{0,1\}^{\mathbb{Z}^d}$. These induce in a natural way shift transformation on the space of all functions on X via

$$\tau_x f(\eta) = f(\tau_x \eta),$$

and if μ is a probability measure on X, then

$$\int f d(\tau_x \mu) = \int (\tau_x f) d\mu.$$

Let $\mathcal{L}$ be the set of all the probability measures μ, such that $\tau_x \mu = \mu$, for all $x \in \mathbb{Z}^n$.

We will need to use the following version of the multiparameter ergodic theorem. Its proof can be found in Chapter VIII of Dunford and Schwartz (1958). The inequalities in the statement are to be interpreted componentwise, and $x \longrightarrow \infty$ is interpreted as saying that each component of x tends to ∞.

Teorema A.2. *If $\mu \in \mathcal{L}$ and f is a bounded measurable function on X, then*

$$\lim_{x \longrightarrow \infty} \frac{\sum_{0 \leq y \leq x} \tau_y f}{|\{y \in \mathbb{Z}^n : \ 0 \leq y \leq x\}|}$$

exists a.s. and in L_1 relative to μ. If $n = 1$, it suffices that f be in $L_1(\mu)$

Definition A.3. *A $\mu \in \mathcal{L}$ is said to be ergodic if whenever $\tau_x f = f$ for all $x \in \mathbb{Z}^n$ and f is measurable on X, it follows that f is constant a.s. relative to μ.*

A.3 Subadditive Ergodic Theorem

In the demonstration of the existence of radial limits we use the following general result, which is know as the Subadditive Ergodic Theorem (see Theorem 2.6 in [15].

Teorema A.4. *Suppose $\{X_{m,n}, \ m \leq n\}$ are random variables which satisfy the following properties:*

a) $X_{0,0} = 0,\ X_{0,n} \le X_{0,m} + X_{m,n}$, *for* $0 \le m \le n$.

b) $\{X_{(n-1)k,nk},\ n \ge 1\}$ *is a stationary process for each* $k \ge 1$.

c) $\{X_{m,m+k},\ k \ge 0\} = \{X_{m+1,m+k+1},\ k \ge 0\}$ *in distribution for each* m.

d) $EX_{0,1}^{+} < \infty$.

Let $\alpha_n = EX_{0,n}$, *which is well defined by a), b), and d). Then*

$$\alpha = \lim_{n\to\infty} \frac{\alpha_n}{n} = \inf_{n\ge 1} \frac{\alpha_n}{n} \in [-\infty, \infty), \quad \textit{and} \tag{A.4}$$

$$X_\infty = \lim_{n\to\infty} \frac{X_{0,n}}{n} \ \textit{exists a.s., with} \ -\infty \le X_\infty < \infty \tag{A.5}$$

Furthermore, $EX_\infty = \alpha$. *If* $\alpha > -\infty$, *then*

$$\lim_{n\longrightarrow\infty} E\left|\frac{X_{0,n}}{n} - X_\infty\right| = 0$$

If the stationary processes in b) are ergodic, then $X_\infty = \alpha$ *a.s.*

A.4 Connection Length

Finally we present a result of non-oriented percolation (is used in the third model), established in [3], for controlling the length of an open path between two connected sites:

Teorema A.5. *If* $p > p_c$, *there is a constant* $\rho = \rho(p,d) \in [1,\infty)$ *such that*

$$\limsup_{|y|\to\infty} \frac{1}{\|y\|_1} \log \widetilde{P}_p(0 \leftrightarrow y, D(0,y) > \rho \|y\|_1) < 0$$

where "$0 \longleftrightarrow y$" means that there is a path of open unoriented edges from 0 to y, and $D(0,y)$ is the minimum number of open unoriented edges of a open path from 0 to y.