Universidade Federal do Rio de Janeiro

An overview on the Boltzmann equation and renormalized solutions

Tiago dos Santos Domingues

Rio de Janeiro

Agosto de 2017

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Dissertação de Mestrado apresentada ao Programa de Pós-graduação em Matemática Aplicada, Instituto de Matemática da Universidade Federal do Rio de Janeiro (UFRJ), como parte dos requisitos necessários à obtenção do título de Mestre em Matemática Aplicada.

Orientador: Prof. Wladimir Augusto das Neves.

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This is the only part of my dissertation written in my native language.

Essa lista certamente não será exaustiva, e se você foi esquecido, peço desculpas.

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Resumo

An overview on the Boltzmann equation and renormalized solutions

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Resumo da dissertação de Mestrado submetida ao Programa de Pós-graduação em Matemática Aplicada, Instituto de Matemática da Universidade Federal do Rio de Janeiro (UFRJ), como parte dos requisitos necessários à obtenção do título de Mestre em Matemática Aplicada.

Resumo:

Apesar do mundo ser constituído por átomos, no mundo macroscópico, temos a impressão de que o mundo é contínuo. A área que serve de ponte entre essas duas descrições é por excelência a mecânica estatística. A dedução do comportamento macroscópico da matéria a partir de seus constituintes microscópicos fundamentais permanece sendo um dos grandes desafios da física, com já algum sucesso, especialmente na descrição de materiais em equilíbrio termodinâmico. Entretanto, no tratamento de objetos fora de equilíbrio, ainda há lacunas tanto do ponto de vista teórico quanto do ponto de vista prático que dificultam uma descrição mais detalhada, que leve em conta informações microscópicas em vez de apenas fenomenológicas.

Uma das primeiras modelagens de sistemas fora de equilíbrio foi a equação de Boltzmann, de 1872, que conseguiu com sucesso reproduzir as equações da mecânica do contínuo e modelar a irreversibilidade macroscópica de certos fenômenos, partindo de uma descrição probabilística de interações microscópicas.

Embora não seja um modelo novo, a equação de Boltzmann ainda serve de partida para modelos mais sofisticados em mecânica estatística, e ainda é um tópico ativo de análise matemática, tendo inspirado técnicas de uso corrente em sistemas de leis de conservação. Nesse texto, são discutidas tanto a origem física desse modelo quanto a teoria de existência de solução da equação de Boltzmann, com um foco especial na teoria de renormalização de Ron Diperna e Pierre L. Lions.

Palavras–chave. Análise, Análise Funcional, Equações Diferenciais Parciais, Teoria da Medida, Mecânica estatística, Teoria Cinética, Equação de Boltzmann, Mecânica dos Fluidos, Mecânica do Contínuo.

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Abstract

An overview on the Boltzmann equation and renormalized solutions

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Abstract da dissertaçao de Mestrado submetida ao Programa de Pós-graduação em Matemática Aplicada, Instituto de Matemática da Universidade Federal do Rio de Janeiro (UFRJ), como parte dos requisitos necessários à obtenção do título de Mestre em Matemática Aplicada.

Abstract:

Despite the fact that the world is made of atmos, in the macroscopic world, we have the impression that the world is a continuum. The area that serves as a bridge between these two description is prominently statistical mechanics. The derivation of the macroscopic behavior of matter from its microscopic constituents remains one of the great challenges of physics, with considerable success, in the description of materials in thermodynamic equilirbium. However, in the treatment of objects outside of equilibrium, there are still important gaps in our undestanding from both theoretical and practical perspectives which hinder any description that takes into account not only phenomenological, but microscopic information.

One of the first attempts at modelling systems out of equilibrium was Boltzmann's equation of 1872k which successfully reproduced macroscopic balance laws of continuum mechanics modelling some of the macroscopic irreversibility, while starting from a probabilistic description of microscopic interactions.

Although this modelling is not new, Boltzmann's equation still serves as a starting point for more sophisticated models in statistical mechanics, and still is an active topic in mathematical analysis, having inspired current techniques in the theory of conservation laws. In this text, borh the physical origins as well as the existence thory for the Boltzmann equation are discussed, with special focus o the theory of renormalization of Ron Diperna and Pierre. L. Lions.

Keywords. Analysis, Functional Analysis, Partial Differential Equations, Meaure theory, Statistical mechanics, Kinetic theory, Boltzmann equation, Fluid mechanics, Continuum mechanics.

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Contents

Chapter 1

Introduction and Physical motivation

> If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generations of creatures, what statement would contain the most information in the fewest words? I believe it is the atomic hypothesis (or the atomic fact, or whatever you wish to call it) that all things are made of atoms—little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another. In that one sentence, you will see, there is an enormous amount of information about the world, if just a little imagination and thinking are applied... *Richard P. Feynman in [16]*

Usually in mechanics and engineering, when one models the static or dynamic behavior of a material subject to stress and external forces, it is treated as a continuum. This means that some portion of the material is identified with a subset Ω of $\mathbb{R}^3$, and we assign functions $f : \Omega \longrightarrow \mathbb{R}^k$ that return, for each point, the value of measurable properties of the material (e.g. pressure, velocity) around it.

Then, continuum mechanics provides balance laws, which predict the evolution of these properties over time. In order to effectively solve the equations corresponding to each balance law, one chooses constitutive relations to "close" our system of equations, by defining how the values of different macroscopic properties are linked to each other.

Constitutive relations may respect some universal requirements, such as the principle of material frame-indifference (that is, their structure must not depend on the particular inertial reference frame we use), symmetry and thermodynamic considerations, however, the latter are rarely, if ever, sufficient to uniquely determine the former. Rather, experiments must be made to determine their exact form, and so constitutive relations act as a property intrinsic to the material, making the dynamics of each material (e.g. water and steel) work differently.

The continuum mechanics description is elegant and highly successful, but as is the case with any model, it ignores part of the physical setting, as it is common knowledge that at the microscopic level, no material is truly a continuum. The task of obtaining material properties from atomic and molecular interactions remains a challenge to this day, with both fundamental and practical interest.

The field of equilibrium statistical mechanics, created by Josiah W. Gibbs (1839-1903), James Clerk Maxwell (1831-1879) and Ludwig Boltzmann (1844-1906) was a major milestone in that direction. It is a staple subject for any physics student, providing a detailed description of the thermodynamic behavior of gases, conductance-band electrons in a metal, Bose-Einstein condensates, phase transitions, and many other phenomena [24].

Whereas in the equilibrium setting we have a single framework describing all the above situations, with few unanswered questions regarding its interpretation (e.g. the role of entropy maximization in statistical mechanics, the role of the Ergodic hypothesis, whether the obtained probabilities are Bayesian or frequentist), in the non-equilibrium setting most bets are off. There are a number of successful models in non-equilibrium statistical mechanics (e.g. Langevin equations for Brownian motion, Master equations, Ising models for magnetisation, Spin glasses, etc), but there isn't yet a unified framework from which to derive these results.

Historically, the first attempt to model a material system out of equilibrium taking into account its microscopic behavior was the Kinetic theory of gases from Maxwell and Boltzmann. Its most important idea (in fact, the central idea behind statistical mechanics in general) was that in order to include microscopic information into the macroscopic description of a material, it is not necessary to know the full dynamics of each atom within the material, which is practically impossible to obtain even in a classical description.

To see why this is the case, consider the (classical) equations of motion for a system composed of $N \approx 10^{23}$ particles (the atoms in a gas) encased in a solid vessel, which we identify with $\Omega \subseteq \mathbb{R}^3$ open with smooth boundary and compact closure. Since we assume our particles are modeled according to classical mechanics, where the state of a particle is uniquely determined by its position and velocity, define $\Gamma^N = \{\gamma \in \Omega^N \times \mathbb{R}^{3N}\}$ as the phase space for the system. Then each $i = 1, \ldots, N$ we have

$$\begin{cases} \dot{x}_i(t) = \mathbf{v}_i = \nabla_{\mathbf{v}_i} H, \\ \dot{\mathbf{v}}_i(t) = F_i(x) = -\nabla_{x_i} H, \end{cases} \tag{1.0.1}$$

supplemented by the initial data $x(0) = x_0$, $\mathbf{v}(0) = \mathbf{v}_0$, where $x_i = (x_{i,1}, x_{i,2}, x_{i,3})$, $\mathbf{v}_i$ are respectively the positions and velocities of each particle, $H : \mathbb{R}^{6N} \longrightarrow \mathbb{R}$ is the system's hamiltonian, and F is the total force per unit mass acting on particle i, defined as

$$F(x_i) = -\nabla_{x_i} H = -\nabla_{x_i} \left(V(x_i) \ + \sum_{i \neq j} \phi_{ij}(x_i, x_j) \right). \tag{1.0.2}$$

Where V describes an external potential, and ϕ_{ij} the potential energy contribution from particles i and j (if all particles in this system are of the same kind, then ϕ_{ij} will be independent of the indices) . To introduce some notation, define the function $U : \mathbb{N} \times \{1, 2, 3\} \to \mathbb{N}$, given by $U(i, j) = 3(i - 1) + j$.Since U is a bijection, We can introduce then state vector in Γ^N

$$\gamma_k = \begin{cases} x_{U^{-1}(k)} & k \leq 3N \\ \mathbf{v}_{U^{-1}(k-3N)} & k > 3N \end{cases},$$

we see that the previous equations can be regarded as

$$\frac{d}{dt}\gamma = A \nabla H(\gamma(t)), \quad \gamma(0) = \gamma_0 \in \Gamma^N = \Omega^N \times \mathbb{R}^{3N}, \tag{1.0.3}$$

where

$$A_{ij} = \begin{cases} 1 & i \leq 3N, j > 3N, \\ -1 & i > 3N, j \leq 3N, \\ 0 & \text{otherwise.} \end{cases}$$

The solution for this system, if it exists, can be written as

$$\gamma(t) = X(t, \gamma_0)^1, \tag{1.0.4}$$

where $X(t, y) : \mathbb{R}_+ \times \Gamma^N \longrightarrow \mathbb{R}^{6N}$ is the flow associated with equation 1.0.3. Summarizing the argument given by by [7], we can say that for this system:

(1) Acquiring initial data for 10^{23} particles simultaneously is impossible in practice. Similarly, writing down this initial data on any numerical solver you choose to approximately solve this system is also impossible (since there are less than 10^8 seconds in a year, assuming that one can write the six data for each particle in one second, the total time need to write down the initial data would be greater than 10^{15} years).

(2) Initial data cannot be infinitely accurate. By choosing to work with a certain number of decimal

[1] $X(t, \gamma_0)$ induces the semi-group: $\gamma(t + s) = X(s, \gamma(t))$

places, one indroduces truncation errors in the initial data, even in the absence of measurement errors, which can be magnified by the numerical methods used to solve these equations.

(3) Even if we could work with infinitely many decimal places and supposing measurements were perfect, unless we include all forces and particles in the universe in our simulation, there will always be round-off and truncation errors, which will be important due to the number of equations we need to solve simultaneously. To quote [7], if we consider the total force acting on a closed system of particles: "the displacement of 1 gramme of matter by 1cm on a not too distant star (say, Sirius) would produce a change of force larger than 10^{-100} times a typical force acting on a molecule". Therefore the error of not including very distant particles manifests itself if we work with many decimal places, and is magnified by numerical solutions; since the solution is strongly sensitive to changes in the initial conditions.

The above argument suggests that we would need to solve (1.0.1) for an ensemble of systems, each with a slightly different initial condition, compatible with the amount of errors we are introducing by simplifying our description and the amount of uncertainty in the initial data [9]. All this can be swiftly accomplished by assigning a probability distribution for the initial data of our system. The domain of this distribution will be the phase space Γ^N.

The idea is that, given the overwhelming magnitude of the number of atoms inside any macroscopic object, the motions of individual atoms have little influence to any observations we make. Only the collective behavior as measured by statistics should matter for any material property, and most material properties would emerge as averages of corresponding microscopic properties.

A central equation in the Kinetic theory of gases (considering gases out of equilibrium) is Boltzmann's transport equation, of 1872:

$$\begin{cases} \frac{\partial f}{\partial t} + v \cdot \nabla_x f + W \cdot \nabla_v f = Q(f,f), \\ \left[Q(f,f)\right](t,x,v) = \int_{\mathbb{R}^3} \iint_{\mathbb{S}^2} (f' f'_* - f f_*)\, B(\omega, v_* - v)\, d\mathcal{H}^2(\alpha)\, dv_*, \end{cases}$$

where: $f : \Omega \times \mathbb{R}^3 \to \mathbb{R}_+$ is a probability density over the phase space of a single particle; $B(\omega, v_* - v)$ is called the collision kernel, which takes into account interparticle interaction; $d\mathcal{H}^{d-1}$ is the Hausdorff measure of the surface, denoting a surface integral over the unit sphere; f', f'_*, f_* are shorthand for $f(t,x,v'), f(t,x,v'_*), f(t,x,v_*)$ respectively; and the pairs (v, v_*) and (v', v'_*) are respectively the velocities of 2 particles before and after a binary collision, which are related by the expression

$$\begin{cases} v' = v + \langle w, \alpha \rangle\, \alpha, \\ v'_* = v_* - \langle w, \alpha \rangle\, \alpha. \end{cases}$$

Many further models of non-equilibrium statistical mechanics were inspired by this equation, and it still finds applications today in the fields of aerodynamics, fluid mechanics, cosmology, and nuclear reactor design [28],[7]. The goal of this dissertation is to be an introduction to the mathematical theory of the Boltzmann equation, focusing in its existence theorems. Specifically, we cover in detail the theory of renormalized solutions to the Boltzmann equation, as defined by [12].

1.1 A brief chronology of the kinetic theory

The forefather of the kinetic theory of gases was the prominent swiss mathematician Daniel Bernoulli (1700-1782). In his modelling, gases were made of hard spheres colliding according to classical mechanics. He identified the sphere's kinetic energy with the notion of macroscopic tempreature, and the momentum transfer of particles colliding with the wall as the gases's pressure, obtaining an approiximate derivation to the empirical law of Boyle-Mariotte for dilute gases, which had been demonstrated a century before, [17], i.e.

$$PV = g(T)$$

From conservation of energy he deduced that temperature increases at constant volume would lead to pressure increases, and defined an equivalent expression for the mean free path of a gas a century before

Rudolph Clausius made the definition popular. Daniel Bernoulli also gave major contributions to other areas, such as fluid mechanics, writing the first book on the subject (inventing the term 'hydrodynamics').

Sadly, his work on kinetic theory was largely ignored by the scientific community of his time [17]. The reason for this was that, at the time, most scientists believed in the so-called 'calloric hypothesis', according to which heat was a fluid, flowing between solid bodies. The 'density' of that fluid would amount to macroscopic temperature, with warmer bodies having a denser calloric fluid, and heat would flow from warm to cold bodies due to differences in the calloric density. It was believed at the time that dynamic theories of heat (where heat is a subproduct of motion) couldn't explain the latent heat of boiling of materials, as well as how the heat generated by the sun could reach the earth traversing the vacuum of space [30]. Among notable adherents of the calloric hypothesis, we find Lord Kelvin (1824-1907) and Pierre Simon de Laplace (1749-1827).

It was only after the death of Laplace, and with a better understanding of heat transfer by radiation, that the calloric hypothesis started to lose momentum. In 1820, John Herapath derived the Boyle-Mariotte relationship based on the kinetic theory of gases, which was not accepted for publishing at the Royal Academy of Science [17], but which was read by James Prescott Joule (1818-1889) [17] . In 1847, Joule publicou published his celebrated result establishing the equivalence of mechanical energy and thermal energy, strengthening the idea that heat would be just another form of energy, paving the way for the definition of the First Law of thermodynamicss. Since the fundamental work of Sadi Carnot (1796-1832) was written considering the calloric hypothesis, the convertibility between heat and mechanical energy was not immediately accepted. Reconciliation between the works of Joule and Carnot marked the downfall of the calloric hypothesis [30].

Kinetic theory gained credibility when Clausius (1822-1888), famous for his definition of the Second Law of thermodynamics, started publishing on the subject citing te works of Herapath and Joule, in 1858.

James Clerk Maxwell (1831-1879), widely known for his synthesis of equations of classical electromagnetism, published 3 major papers on the subject of kinetic theory. In 1860, considering a monoatomic gas in a closed system in thermodynamic equilibrium, derived using only symmetry arguments the equilibrium velocity distribution of a gas

$$f : \Omega \times \mathbb{R}^3 \to \mathbb{R}^+;\ f(x,v) = \frac{n(x)}{V}\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} e^{-\frac{m||v||^2}{2k_B T}},$$

where n is the number density of particles, m, is a particles's mass, T its temperature in kelvin and k_B the Boltzmann constant (using modern notation).

Ludwig Boltzmann (1844-1906) earned his PhD under the supervision of Josef Stefan in 1866, and from the outset of his career (already in his first paper), his avowed goal was to derive the Second Law of thermodynamic from mechanical considerations [38]. He wrestled with this problem through his entire career, and his ideas on the precise relationship between thermodynamical properties of macroscopic bodies and their microscopic constitution, and the role of probability in this relationship changed sharply over time [38]. Today there are a few different approaches and conflicting positions regarding the foundations of statistical physics (e.g. the role of the Ergodic hypothesis), but all of today's approaches were pioneered in some way in the works of Boltzmann [38], due to the many different lines of reasoning he used to tackle the problem of the Second law. His equation of 1872, which is the central object of this dissertation, is one of these approaches.

1.2 A Heuristic derivation of Boltzmann's equation

This heuristic relies on 4 assumptions, which we outline below:

1. External forces are negligible when compared to interaction between particles in the moment of collisions (that is, $\|F_{external}(t_{col})|| \ll \|F_{internal}(t_{col})\|$), and the opposite is true when particles are streaming.

2. The time scale of a typical collision is much smaller than the average time elapsed between collisions $\tau_{collision} \ll \tau_{streaming}$. This also means that changes in position of the order of a characteristic distance σ leave the value of the distribution f approximately unchanged, where σ measures how far apart can 2 particles be while still interacting . This establishes a separation of time scales in the problem.

3. The gas is considered to be rarefied. This means that *only binary collisions will be considered (all collisions involving more particles are considered rare and statistically irrelevant).*

4. (*Stoßzahlansatz*) The state of each particle is an independent identically distributed random variable, with density function f, i.e. $\gamma \sim f(t, x, v)$. Likewise, if we select s particles at random from our system; their joint distribution in the phase space Γ^N will have density $P^{(s)}(t, x_{ij}, v_{ij}) = \prod_{i=1}^{s} f(t, x_{ij}, v_{ij})$. *This allows us to construct an evolution equation only for the phase space of a single particle.* This condition enforces a certain independence between particles, which can hold approximately only if particles interact very weakly. Therefore, this modelling will be justified only for gases, where attractive forces play a small role since the average distance between particles is large.

   From Liouville's equation, it is also possible to show that any conservative system of classical particles in equilibrium satisfies this property in the limit when $N \to \infty$ [7]. What distinguishes the *Stoßzahlansatz* is that we assume that this condition holds out of equilibrium at all times for a time dependent probability density, and for a finite number of particles N.

The argument goes as follows [23]:

If there were no collisions (or interaction) between particles, they would only be subject to external force fields, and their trajectories would be solutions to problem (1.0.1), considering only external forcing. Therefore, their distributions, as functions defined in the phase space; should be "carried" with the flow. This amounts to say

$$\frac{d}{dt} f(X(t, z_0)) = 0 \quad (\forall z_0 \in \Gamma),$$

where $\Gamma = \Omega \times \mathbb{R}^d$ is the phase space of a single particle, $\Omega \subseteq \mathbb{R}^d$ is an open set, and $X(t, \mathbf{z}_0)$ is the flow in phase space given by equation (1.0.4). In the absence of inter-particle forces, the trajectories generated by the flow $X(t, z_0)$ in phase space are called extrinsic trajectories.

However, according to hypothesis (1), interactions are significant only in the moment of a collision (e.g. potential energy greater than a given threshold, which means both particles are very close to each other); and according to (2), collisions appear instantaneous when compared to the time scale of streaming.

Therefore, collisions (or interaction between particles) cannot continuously alter the extrinsic trajectory that particles follow. Collisions can only *instantantly change* the velocity of the pair of particles involved (hypothesis (3)); generating new initial conditions for the streaming problem (I). That is, collisions make particles *swap trajectories, but do not generate new trajectories.*

If we restrict ourselves to one extrinsic trajectory, the perceived effect of collisions will be an increase/decrease in the number of particles following it. Defining the rate of change in probability due to creation of particles in state (t,x,v) as $Q(f, f)(t, x, v)$; this corresponds to the equation

$$\frac{d}{dt} f(X(t, z_0)) = Q(f, f)(X(t, z_0)), \quad \forall z_0 \in \Gamma$$

Formally, we can use the chain rule to obtain:

$$\begin{aligned}\frac{d}{dt}f(X(t,z_0)) &= \nabla f \cdot \frac{d}{dt}X(t,z_0) = \nabla f \cdot (1, A\nabla H)\\ &= \frac{\partial f}{\partial t} + \nabla_v H \cdot \nabla_x f - \frac{1}{m}\nabla_x H \cdot \nabla_v f.\end{aligned}$$

Then, since our hamiltonian tracks only the external forces, we define $W(t,x,v) = -\frac{1}{m}\nabla_x H$ as field of acceleration induced by external forces (e.g. gravitational acceleration). With this notation,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + W(t,x,v) \cdot \nabla_v f = Q(f)(t,x,v), \tag{1.2.1}$$

(1.2.1) **is the classical Boltzmann equation**, if we define the operator on the right-hand side accordingly (aptly named the collision operator), which requires us to understand the effect that collisions (or interactions) have on the probability distribution f.

1.3 Basic definition and properties

Representations and the Encounter problem

In order to reach a closed expression of the collision operator, we must be able express how collisions affect the velocities of individual particles, and then translate this into changes in the function f. In Appendix A.3, we treat in detail the problem of a binary collision in classical mechanics and obtain the results shown in this section. Denoting x^1, x^2 as the positions of the two particles, we reach then the conclusion that the following limits exist:

$$\lim_{t\longrightarrow\infty} \frac{dx_1}{dt} := v', \lim_{t\longrightarrow-\infty} \frac{dx_1}{dt} := v, \lim_{t\longrightarrow\infty} \frac{dx_2}{dt} := v'_* \lim_{t\longrightarrow-\infty} \frac{dx_2}{dt} := v_*.$$

These velocities are called **asymptotic velocities** of particles 1 and 2. Assuming the interparticle force to be conservative, these velocities must satisfy

$$(B) \begin{cases} m_1 v + m_2 v_* = m_1 v' + m_2 v'_* & \text{(momentum conservation)} \\ m_1\|v\|^2 + m_2\|v_*\|^2 = m_1\|v'\|^2 + m_2\|v'_*\|^2 & \text{(energy conservation)} \end{cases}$$

Since $\tau_{collision} \ll \tau_{streaming}$ was among our assumptions, all that matters to us are the asymptotic velocities, as collisions are assumed to be instantaneous.

We notice that if $x^1, x^2 \in \mathbb{R}^d$, the above system contains $d+1$ equations, and $2d$ unknowns (since we assume v, v_* as known), so we cannot guarantee uniqueness of solutions. Rather, define $G : \mathbb{R}^{2d} \longrightarrow \mathbb{R}^{d+1}$ as $G(v', v'_*) = (m_1 v' + m_2 v'_*, m_1\|v'\|^2 + m_2\|v'_*\|^2)$. Using the Inverse Function Theorem, the pre-image of $(m_1 v + m_2 v_*, m_1\|v\|^2 + m_2\|v_*\|^2)$ will be a $d-1$ dimensional manifold $\mathcal{M}_{v,v_*}$ in $\mathbb{R}^d$. For fixed v, v_*, any function that takes arguments in $\mathcal{M}_{v,v_*}$ and associates them to the corresponding output velocities v', v'_* is said to be a **representation of a solution to (B)**. In the case of equal masses $m_1 = m_2$ and $d = 3$, two popular representations are the α representation,

$$\begin{cases} v' = v + \langle w, \alpha\rangle \alpha, \\ v'_* = v_* - \langle w, \alpha\rangle \alpha, \end{cases}$$

and the σ representation

$$\begin{cases} v' = \frac{v+v_*}{2} + \frac{\|v_*-v\|}{2}\sigma, \\ v'_* = \frac{v+v_*}{2} + \frac{\|v_*-v\|}{2}\sigma, \end{cases}$$

where α, σ are unit vectors in $\mathbb{R}^3$, $w = v_* - v$ is the relative velocity between particles. The first one will be prefered throughout this text, and a derivation for it can be found in Appendix A.3. Although the second representation has the advantage of showing clearly that in general $\mathcal{M}_{v,v_*}$ is a sphere in $\mathbb{R}^d$ with diameter $\|v - v_*\|$ and centered at $\frac{v+v_*}{2}$.

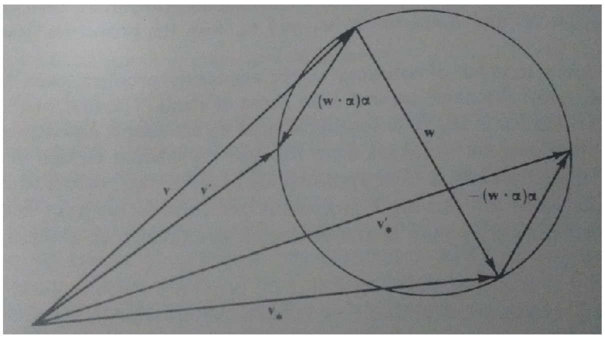


Figure 1.3.1: This diagram shows how pre and post-collisional velocities are related, as well as the manifold $\mathcal{M}_{v,v_*}$. This image was taken from [36]

Since α is a unit vector, it will be parametrized by spherical coordinates

$\alpha = e(\theta, \varphi) = (\cos\varphi \sin\theta)\,\xi + (\sin\varphi \sin\theta)\,\eta + (\cos\theta)\,\hat{w}$, where $\hat{w} = w/\|w\|$, and $\{\eta, \xi, \hat{w}\}$ form a positively oriented basis in $\mathbb{R}^3$ and $(\theta, \phi) \in [0, \pi/2) \times [0, 2\pi)$. The physical significance of α is that knowing the velocities v, v_* does not determine uniquely v', v'_* if we don't know the orientation of the collision (e.g. thinking in terms of billiards, the final velocities after a collision depend not only on the initial velocities, but also on the angle with which the collision occured). Physically the vector α is a unit vector in the direction $x^1 - x^2$ when the particles are at their minimal distance during a collision (if they interact as hard spheres, that is precisely when they collide).

1.3.1 The Collision operator

Here we use the notation and properties of a binary collision problem described in Appendix A.3. Consider again 2 particles in the moment of a collision, with velocities (v, v_*), with a reference frame centered in particle 1. Define $w = v_* - v$ as their relative velocity, and define Π_2 as the plane orthogonal to $\hat{w} = \frac{w}{||w||}$.

As was shown in the Appendix, all motion of one particle relative to the other happens on a fixed plane Π_1. If particle 2 passes sufficiently close to 1 in Π_2, interaction takes place. So there is an open set $\mathcal{A} \subseteq \Pi_2$ containing the origin that accounts for the range of interaction for both particles. Choose ξ, η in Π_2 so as to render $\{\xi, \eta, \hat{w}\}$ a positively oriented orthonormal basis set in $\mathbb{R}^3$. Then, parametrize $\partial\mathcal{A}$ by: $r_e = b(\varphi)\,[cos(\varphi)\xi + sin(\varphi)\eta]\,;\ \varphi \in (0, 2\pi)$.

The above definitions can be summarized in the picture below.

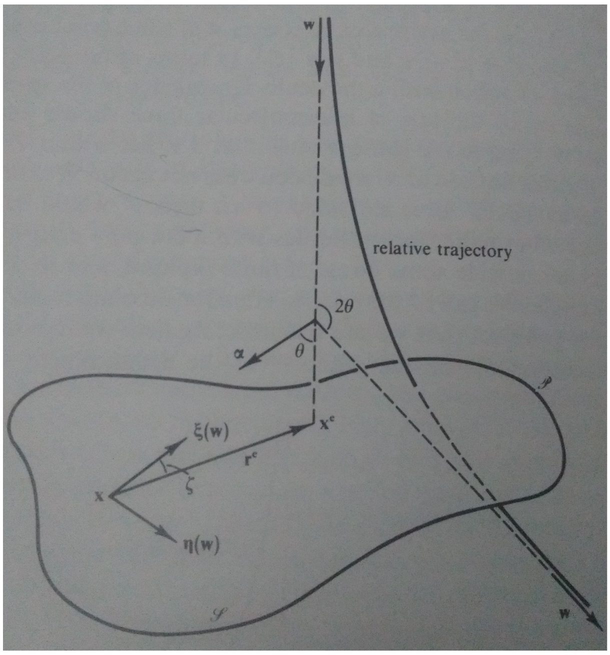


Figure 1.3.2: A picture detailing the relative motion during a binary collision. In this picture, the plane $\mathcal{P}$ corresponds to our definition of Π_2. This image was taken from [36]

In our case, since the potential for interaction is assumed to be spherically symmetric, the border of $\mathcal{A}$ should not depend on φ. Therefore, $b(\varphi) = R$, and so $\mathcal{A} \subseteq \Pi_2$ is a disk centered on particle 1.

We will factor the rate of change of probability associated with the state $(t, x, v) \in \Gamma$ as:

$$Q(f,f) = \big(Rate\,of\,increase \equiv Q^{+}(f,f)\big) - \big(Rate\,of\,decrease \equiv Q^{-}(f,f)\big)$$

- Rate of Decrease: Assuming w fixed, the volume (in the phase space) that crosses the cross-section A per unit time is given by:

$$R = \iint\limits_{\mathcal{A}} w dS.$$

It is clear that the rate of decrease $Q^{-}(f,f)$ will be given by the flux of probability across $\mathcal{A}$. The joint probability distribution of 2 particles with states $(x,v),(x_*,v_*)$ is defined as the **pair distribution**, and denoted by $P^{(2)}(t,x,x_*,v,v_*)$; Since we want the rate for a given state (x,v) and any (x_*,v_*), the correct expression for a given w will be

$$Q^{-}(f,f)(t,x,v) = \iint\limits_{\mathcal{A}} P^{(2)}(t,x,x_*,v,v_*)\,\|w\| dS_*.$$

Invoking the second part of hypothesis (2), we see that σ can be taken as the radius of the disk $\mathcal{A}$, such that when $|x - x_*| < \sigma$, $P^{(2)}(t,x,x_*,v,v_*) = P^{(2)}(t,x,x,v,v_*)$. Letting the incoming velocity of the second particle vary, and using the *Stosszahlansatz*, we have

$$P^{(2)}(t,x,x_*,v,v_*) = f(t,x,v)f(t,x,v_*), \tag{1.3.1}$$

$$Q^{-}(f,f)(t,x,v) = \iint\limits_{\mathbb{R}^3\times\mathcal{A}} P^{(2)}(t,x,x,v,v_*)\,\|w\| dS_* dv_* = \iint\limits_{\mathbb{R}^3\times\mathcal{A}} f(t,x,v)f(t,x,v_*)\,\|w\| dS_* dv_*. \tag{1.3.2}$$

- Rate of Increase: We apply the same idea, except that in this case we must consider particles whose collision will *generate* the state (t,x,v). Since collisions are reversible, we see that those are precisely the states resulting from a collision between particles where one of them was at state (t,x,v). Therefore, we expect that:

$$Q^+(f,f)(t,x,v) = \iint\limits_{\mathbb{R}^3\times\mathcal{A}} P^{(2)}(t,x,x,v',v'_*)\,\|w\|dS_*dv_* = \iint\limits_{\mathbb{R}^3\times\mathcal{A}} f(t,x,v')f(t,x,v'_*)\,\|w\|dS_*dv_* \tag{1.3.3}$$

In many cases, both of these integrals are divergent. However, one can combine them into a single integral:

$$Q(f,f)(t,x,v) = \iint\limits_{\mathbb{R}^3\times\mathcal{A}} \left(f(t,x,v')f(t,x,v'_*) - f(t,x,v)f(t,x,v_*)\right)\,\|w\|dS_*dv_*$$

which can be shown to be finite for some choices of interaction potential (e.g. for inverse power law potentials of the form $1/r^k$, $\kappa > 3$) [36]. This is the full collision operator, whose properties will be further examined in the next section. Meanwhile, writing the integral over $\mathcal{A}$ in polar coordinates, we see that the distance in Π_2 between particles 1 and 2 (The offset between trajectories once their relative velocity is already set) is exactly the definition of the impact parameter. Then, writing in terms of polar coordinates, we have

$$Q(f,f)(t,x,v) =$$

$$\int\limits_{\mathbb{R}^3}\int\limits_0^{2\pi}\int\limits_0^{\sigma} \left(f\left(t,x,v'(\theta,\varphi)\right)f\left(t,x,v'_*(\theta,\varphi)\right) - f\left(t,x,v\right)f\left(t,x,v_*\right)\right)\,\|v_*-v\|b\,db\,d\varphi dv_*. \tag{1.3.4}$$

Here, a cutoff assumption is used, meaning we assumed ϕ to have a compact support. If the support of ϕ is unbounded, we postulate the same expression for $Q(f,f)(t,x,v)$, only passing the limit of $\sigma\to\infty$. Thus, if we can both correctly parametrize θ in terms of (b,φ) and express post-collisional velocities starting from pre-collisional ones, then $Q(f,f)$ will be well defined. One can show (see Appendix A.3) that the correct way to relate the deviation angle in a collision and the impact parameters is given by the **Orbital equation**

$$\theta = \int\limits_{\frac{b}{\sigma}}^{smax} \frac{1}{\sqrt{1-s^2-\frac{2\phi(b/s)}{\mu\|w\|^2}}}ds + \arcsin\left(\frac{b}{\sigma}\right). \tag{1.3.5}$$

This completes our description of the collision operator. For shorthand, we use the notation

$$Q(f,f)(t,x,v) = \int\limits_{\mathbb{R}^3}\iint\limits_{\mathcal{A}} \left(f'f'_* - ff_*\right)\,\|w\|dA\,d^3v_*.$$

There are a few cases in which the expression for the potential ϕ is simple enough to allow analytical calculations for the kernel B. These are:

- The hard sphere potential, where ϕ is a dirac mass. Then one can show explicitly that the kernel B is simply a contant times the relative velocity $\|w\| = \|v_*-v\|$.
- The coulomb potential $\phi = \frac{1}{r}$, for which one obtains the rutherford scattering kernel: $B(|v-v*|,p) = K\frac{1}{\|v-v*\|^3 sin^4(\mathrm{p}/2).}$, where $K = \frac{e^2}{4\pi\epsilon_0 m}$.
- The inverse power law potential, where

$$\phi = \frac{1}{r^{s-1}}.$$

Although explicit expressions are not available for this model, one can factorize B in this case as $B(\|v_* - v\|, p) = f(cos(p))\,\|v_* - v\|^{\gamma}$, where $\gamma = \frac{s-(2N-1)}{s-1.}$. In $\mathbb{R}^3$, where the definition for gamma reads $\gamma = \frac{s-5}{s-1}$, the following terminology is commonly used:

$$\begin{cases} s > 5 \Rightarrow \gamma > 0, & \phi\, is\, called\, a\, \textbf{hard potential} \\ s = 5 \Rightarrow \gamma = 0 & the\, particles\, are\, said\, to\, be\, Maxwellian \\ s < 5 \Rightarrow \gamma < 0 & \phi\, is\, called\, a\, \textbf{soft potential} \end{cases}$$

Spherical description

Geometrically, the angle θ corresponds to the "reflection angle" for the relative velocity w, as seen from the frame of the particle with velocity v. Therefore, the deflection angle can be easily obtained as $p = \pi - 2\theta$. The orbital equation can be seen as a function, such that $\theta = f(b)$, and therefore a relationship between the deflection angle and the impact parameter. Since $p \in [0, \pi]$, one can use the orbital equation to define a mapping $\psi : [0, \pi] \times [0, 2\pi] \longrightarrow [0, \infty) \times [0, 2\pi]$, such that $\psi(p, \varphi) = (b, \varphi)$, where $b = f^{-1}(\theta)$, the jacobian of this transformation being simply

$$\frac{db}{dp} = \frac{1}{-2\frac{d\theta}{db}}.$$

Then, take $\phi_1$as the parametrization of the shock cross section $\mathcal{A}$ using cylindrical coordinates, and ϕ_2 as the parametrization of the unit sphere using spherical coordinates. Define $\Psi : \mathbb{S}^2 \longrightarrow \mathcal{A};\ \Psi = \phi_1 \circ \psi \circ \phi_2^{-1}$. Pulling back by Ψ our area form, we can obtain an integral over the unit sphere

$$\iint\limits_{\mathcal{A}} \left(f'f'_* - ff_*\right) \|w\| dA = \iint\limits_{\ominus(\mathbb{S}^2)} \left(f'f'_* - ff_*\right) \|w\| dA = \iint\limits_{\mathbb{S}^2} \Psi^* \left[\left(f'f'_* - ff_*\right) \|w\| dA\right],$$

which by the chain rule will be given by

$$\iint\limits_{\mathbb{S}^2} \Psi^* \left(f'f'_* - ff_*\right) \frac{\|w\| b}{\sin(p)} \frac{db}{dp} dA',$$

where dA' is the area form of the unit sphere. Then, one may define $B(p, \|w\|) = \frac{\|w\| b}{\sin(p)} \frac{db}{dp}$ as the **collision kernel**[2], and rewrite the collision operator as

$$Q(f,f)(t,x,v) = \int\limits_{\mathbb{R}^3} \iint\limits_{\mathbb{S}^2} \left(f'f'_* - ff_*\right) B(p, \|w\|)\, dA\, dv_*.$$

Now, no reference has to be made to the orbital equation, as its information is contained in the collision kernel B. This form of the equation is best suited for the analysis of properties of the collision operator, and will be used for most proofs in this thesis.

[2]In most physics textbooks of classical scattering theory, the term $\frac{b}{\sin(p)} \frac{db}{dp}$ is named the **differential scattering cross-section**. In the study of the Boltzmann equation, however, since the scattering kernel usually appears multiplied by $\|w\|$, we define this term as the 'collision kernel', a nomenclature frequently used in the mathematics community.

1.3.2 Summational invariants

The equation of transfer

Once a solution $f(t,x,v)$ to the Boltzmann equation is available, we can make a few definitions. First, taking the marginal of f with respect to the variable v, we obtain

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)dv \geq 0. \tag{1.3.6}$$

Multiplying ρ by the total mass of gas in a vessel and integrating in position over a bounded open set $B \subseteq \Omega$, one obtains the mass of gas in B at a given time. Therefore, ρ is indeed the macroscopic density of the gas, up to a constant multiplicative factor. Define the random variables V and X as the velocity and position of a particle, respectively; with $(X,V) \sim f_t(x,v)$, $X \sim \rho$. Let $\psi_x(v,t)$ denote a conditional probability density such that $V|X \sim \psi_{x,t}(v)$:

$$\psi_{t,x}(v) = \frac{f(t,x,v)}{\int_{\mathbb{R}^3} f(t,x,v)dv} = \frac{f(t,x,v)}{\rho(t,x)}, \tag{1.3.7}$$

which is well defined for x and t belonging to the essential support of ρ. The conditional expectation of V is what we define as the macroscopic velocity field u:

$$u(x,t) := \mathbb{E}[V|X=x] = \int_{\mathbb{R}^3} v\psi_x(t,v)dv = \frac{1}{\rho(t,x)}\int_{\mathbb{R}^3} vf(t,x,v)dv \tag{1.3.8}$$

Therefore, if $g : \Gamma \longrightarrow \mathbb{R}^n$ is a property associated with a particle, its average value the point (x,t) will generally be given by:

$$G(x,t) = \int\limits_{\mathbb{R}^3} g(x,v)f(t,x,v)dv = \rho(t,x)\int\limits_{\mathbb{R}^3} g(x,v)\psi_{t,x}(v)dv = \rho(t,x)\overline{g(x,t)}, \tag{1.3.9}$$

Where $G(t,x), \overline{g(x,t)}$ are the macroscopic properties corresponding to g, where the former is per unit volume, and the latter is written per unit mass.

If we multiply both sides of Boltzmann's equation by g and perform the above integration, we will have the following:

$$\int_{\mathbb{R}^3} \frac{\partial}{\partial t}[g(x,v)f]\,dv + \int_{\mathbb{R}^3} v\cdot\nabla_x(f)\,g(x,v)\,dv + \int_{\mathbb{R}^3} W(t,x,v)\cdot\nabla_v f g(x,v)dv =$$
$$\int\limits_{\mathbb{R}^3} g(x,v)Q(f,f)(t,x,v)dv \tag{1.3.10}$$

Notice that the first term of this equation can be cast as

$$\frac{\partial}{\partial t}\int f_t g(x,v)dv = \frac{\partial}{\partial t}G(t,x),$$

so that this equation (after further simplifications) may provide an evolution equation for the macroscopic property G. Define $\int_{\mathbb{R}^3} g(x,v)Q(f,f)(t,x,v)dv \equiv \overline{Q_f}g(t,x)$ (the right hand side of this expression), which is named **the total collision operator** [3].

Now performing the following splitting: $c(t,x,v) = v - u(t,x)$ and using some vector calculus identities,

[3] total in the sense that it averages the effect of collisions for all possible velocities; taking into account a reference distribution f

we can say that

$$\begin{aligned} v \cdot \nabla_x (f)\, g(x,v) &= v \cdot \nabla_x (f g(x,v)) - v \cdot \nabla_x (g(x,v))\, f \\ &= \nabla_x \cdot (g(x,v) \otimes v f) - v \cdot \nabla_x (g(x,v))\, f \\ &= \nabla_x \cdot (g(x,v) \otimes u(t,x) f) + \nabla_x \cdot (g(x,v) \otimes c f) - v \cdot \nabla_x (g(x,v))\, f, \end{aligned}$$

(where $\otimes$ denotes the Kroenecker or tensor product). Therefore

$$\int_{\mathbb{R}^3} v \cdot \nabla_x (f)\, g(x,v)\, dv = \nabla_x \cdot \left[\int_{\mathbb{R}^3} f g(x,v)\, dv \otimes u(t,x) \right] + \nabla_x \cdot \int_{\mathbb{R}^3} g(x,v) \otimes c f\, dv$$
$$- \int_{\mathbb{R}^3} v \cdot \nabla_x (g(x,v))\, f\, dv,$$

and likewise,

$$\int_{\mathbb{R}^3} (W(t,x,v) \cdot \nabla_v f) g(x,v) dv = \int_{\mathbb{R}^3} W(t,x,v) \cdot \nabla_v (f g(x,v)) dv - \int_{\mathbb{R}^3} (W(t,x,v) \cdot \nabla_v g(x,v)) f\, dv.$$

If we further assume that the external force field W is independent of the particle's velocity (e.g. not a magnetic force field) and use the divergence theorem, this can be simplified even further to give us

$$\int_{\mathbb{R}^3} W(t,x,v) \cdot (\nabla_v f) g(x,v) dv = \int_{\mathbb{R}^3} \nabla_v \cdot (f g(x,v) \otimes W(t,x)) dv - \left[\int_{\mathbb{R}^3} f \nabla_v g(x,v) dv \right] W(t,x)$$
$$= \lim_{R \to \infty} \oint_{\partial B_R} (f(\cdot,\cdot,v) g(\cdot,v) \otimes W(\cdot,\cdot,v)) \cdot \hat{n}(v)\, d\mathcal{H}^2(v) - \left[\int_{\mathbb{R}^3} f \nabla_v g(x,v) dv \right] W(t,x),$$

where $B_R \subseteq \mathbb{R}^3$ is a ball centered at the origin with radius $R > 0$. With sufficient decay conditions on $f(t,x,v), W(t,x,v), g(x,v)$ when $v \to \infty$, we may hope that the first term of the right hand side disappears. Inserting all of the above identities on equation (1.3.10), we find the following expressions:

$$\frac{\partial}{\partial t} \left[\int_{\mathbb{R}^3} g(x,v) f\, dv \right] + \nabla_x \cdot \left[\int_{\mathbb{R}^3} f g(x,v)\, dv \otimes u(t,x) \right]$$
$$= -\nabla_x \cdot \int_{\mathbb{R}^3} g(x,v) \otimes c f\, dv + \int_{\mathbb{R}^3} W(t,x,v) \cdot (\nabla_v f) g(x,v) - v \cdot \nabla_x (g(x,v))\, f dv + \overline{Q_f} g(t,x) \quad (1.3.11)$$

using the following definitions

$$\begin{cases} \sigma = \int_{\mathbb{R}^3} c \otimes g(x,v) f(t,x,v)\, dv, \\ F = \int_{\mathbb{R}^3} W(t,x,v) \cdot (\nabla_v f) g(x,v) - v \cdot \nabla_x (g(x,v))\, f dv, \end{cases}$$

And recalling our definitions of $G(t,x)$ and $\overline{g}(t,x)$, the above can be rewritten as

$$\frac{\partial}{\partial t} G(t,x) + \nabla_x \cdot [G \otimes u(t,x)] = -\nabla_x \cdot \sigma - F + \overline{Q_f} g(t,x), \quad (1.3.12)$$

in the general case. And, if the force term is velocity independent, then we find

$$\frac{\partial}{\partial t} G(t,x) + \nabla_x \cdot [G \otimes u(t,x)] = -\nabla_x \cdot \sigma + F_W + \overline{Q_f} g(t,x) + r, \quad (1.3.13)$$

where we used the following definitions

$$\begin{cases} \sigma = \int_{\mathbb{R}^3} c \otimes g(x,v) f(t,x,v)\, dv, \\ F_W = \int_{\mathbb{R}^3} \left[W(t,x) \cdot \nabla_v g(x,v) + v \cdot \nabla_x g(x,v) \right] f(t,x,v) dv, \\ r = \lim_{R\to\infty} \oint_{\partial B_R} (f(\cdot,\cdot,v) g(\cdot,v) \otimes W(\cdot,\cdot)) \cdot \hat{n}(v)\, d\mathcal{H}(v). \end{cases}$$

If we write the equation in terms of $\overline{g}$, then we find that

$$\frac{\partial}{\partial t} [\rho(t,x)\overline{g}] + \nabla_x \cdot [\rho(t,x) u(t,x) \otimes \overline{g}] = -\nabla_x \cdot \sigma + F_W + \overline{Q_f} g(t,x) + r \tag{1.3.14}$$

Further simplifications can be made if we assume that the property $g(t,x,v)$ depends only on the velocity. Equations (1.3.12), (1.3.13), (1.3.14) are called **equations of transfer**, and they can be seen as macroscopic balance laws for the properties $G, \overline{g}$ (if the remainder term r is well defined). It roughly states that the rate of change of property $\overline{g}$ is a sum of 4 components: convective transport of $\overline{g}$ by the fluid's bulk velocity (second term on the left hand side), a convective transport term involving the deviation of the velocity v from the bulk velocity u (σ), production of $\overline{g}$ by external forces (F_W), and $\overline{Q_f} g(t,x)$, which by employing the collision operator is the production of $\overline{g}$ by internal forces (collisions).

However, for that approach to be fruitful in obataining macroscopic equations, we need to be able to give definite results for $\overline{Q_f} g$. In order to simplify the total collisions operator, define the Bilinear form[4]

$$Q[f,h](t,x,v) = \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{S}^2} (f'h'_* + h'f'_* - fh_* - hf_*)\ B(w,\alpha) d\mathcal{H}^2(\alpha) dv_*. \tag{1.3.15}$$

We see that trivially $Q[f,f](t,x,v) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} (f'f'_* - ff_*)\ B(w,\alpha) d\mathcal{H}^2(\alpha)$, and that it is symmetric, but not positive definite (therefore not exactly an inner product). We define the total bilinear form in the same fashion

$$\int_{\mathbb{R}^3} g(x,v) Q[f,h](t,x,v) dv \equiv \overline{Q_{f,h}} g(t,x).$$

Denote by g_*, g'_*, g' the function $g(x,v)$ when the v variable has been replaced by v_*, v'_*, v' respectively. Expanding the definition of $\overline{Q_{f,h}} g(t,x)$, we have:

$$\overline{Q_{f,h}} g(t,x) = \frac{1}{2} \int_{\mathbb{R}^3} \iint_{\mathbb{R}^3 \times \mathbb{S}^2} (f'h'_* + h'f'_* - fh_* - hf_*)\, g\, B(w,\alpha) d\mathcal{H}^2(\alpha) dv_*\, dv \tag{1.3.16}$$

The changes of variables $G_1(v, v_*, v', v'_*) = (v_*, v, v'_*, v')$ and $(v, v_*, v', v'_*) = (v', v'_*, v, v_*)$ have unit jacobian, and so does $G_2 \circ G_1$. Also, elastic collisions preserve the absolute value of the relative velocity $\|v - v_*\| = \|v' - v'_*\|$. From these simple observations, performing the above change of variable induced by G_2 one can show that

$$\begin{aligned} \overline{Q_{f,h}} g(t,x) &= \frac{1}{2} \int_{\mathbb{R}^3} \iint_{\mathbb{R}^3 \times \mathbb{S}^2} (fh_* + hf_* - f'h'_* - h'f'_*)\, g'\, B(w,\alpha) d\mathcal{H}^2(\alpha) dv_*\, dv \\ &= -\frac{1}{2} \int_{\mathbb{R}^3} \iint_{\mathbb{R}^3 \times \mathbb{S}^2} (f'h'_* + h'f'_* - fh_* - hf_*)\, g'\, B(w,\alpha) d\mathcal{H}^2(\alpha) dv_*\, dv \\ &= -\overline{Q_{f,h}} g'(t,x) \end{aligned}$$

[4]The bilinear form is useful when generalizing the Boltzmann equation to describe a gas with multiple components. In this case, f, h can be seen as the probability densities of 2 gases, and the bilinear form would be the way to generalize the collision operator, to account for collisions between different particles.

For $G_2 \circ G_1$ we also find:

$$\begin{aligned}\overline{Q_{f,h}}g(t,x) &= \frac{1}{2}\int\limits_{\mathbb{R}^3}\iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} (f_*h + h_*f - f'_*h' - h'_*f')\, g_*\, B(w,\alpha) d\mathcal{H}^2(\alpha) dv_*\, dv \\ &= \overline{Q_{f,h}}g_*(t,x) = -\overline{Q_{f,h}}g'_*(t,x)\end{aligned}$$

Since we have many ways to represent the total collision $\overline{Q_{f,h}}g(t,x)$, we can do the following trick:

$$\begin{aligned}\overline{Q_{f,h}}g(t,x) &= \frac{\overline{Q_{f,h}}g(t,x) + \overline{Q_{f,h}}g_*(t,x) - \overline{Q_{f,h}}g'(t,x) - \overline{Q_{f,h}}g'_*(t,x)}{4} \\ &= \frac{1}{4}\overline{Q_{f,h}}\left[g + g_* - g' - g'_*\right](t,x),\end{aligned}$$

or expanding it,

$$\overline{Q_{f,h}}g(t,x) = \frac{1}{8}\int\limits_{\mathbb{R}^3}\iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} (f'h'_* + h'f'_* - fh_* - hf_*)\left[g + g_* - g' - g'_*\right] B(w,\alpha) d\mathcal{H}^2(\alpha) dv_*\, dv. \qquad (1.3.17)$$

This can be separated into two integrals:

$$\begin{aligned}\overline{Q_{f,h}}g(t,x) = &\frac{1}{8}\int\limits_{\mathbb{R}^3}\iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} (f'h'_* + h'f'_*)\left[g + g_* - g' - g'_*\right] B(w,\alpha) d\mathcal{H}^2(\alpha) dv_*\, dv \\ &-\frac{1}{8}\int\limits_{\mathbb{R}^3}\iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} (fh_* + hf_*)\left[g + g_* - g' - g'_*\right] B(w,\alpha) d\mathcal{H}^2(\alpha) dv_*\, dv\end{aligned}$$

Performing the same procedure on the second integral, of changing variables using G_2 (unpriming primed functions and vice-versa, changing sign) actually shows both are equal, so we end with:

$$\overline{Q_{f,h}}g(t,x) = \frac{1}{4}\int\limits_{\mathbb{R}^3}\iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} (fh_* + hf_*)\,\|w\|\left[g' + g'_* - g - g_*\right] B(w,\alpha) d\mathcal{H}^2(\alpha) dv_*\, dv \qquad (1.3.18)$$

Remark 1.3.1. This last step is only possible if both of the above integrals are convergent, which does not generally follow even if our original integral for the total collision operator did. Here we present a few of the classical results that guarantee that the collision operator and the total collision operator are well defined. All of the results below are sufficient, but not necessary conditions for Q to be well-defined.

For particles interacting through a potential with cutoff, we have the

Theorem 1.3.1. *Let $\mathcal{A}$ be a disk. If $(1+\|v\|)f, (1+\|v\|)h \in L^1(\Omega\times\mathbb{R}^d)$, then $Q[f,h](t,x,v)$ is well defined, and is integrable ($Q[f,h] \in L^1_{x,v}$ for a.e. $t > 0$). As a corollary, $Q^+(f,f), Q^-(f,f)$ are well defined.*

Theorem 1.3.2. *Let $\mathcal{A}$ be a disk, and $g(v) : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a function satisfying the condition*

$$|g| \leq C(1 + \|v\|)^n$$

If $(1+\|v\|)^{n+1}f, (1+\|v\|)^{n+1}h \in L^1(\Omega\times\mathbb{R}^d)$, then $\overline{Q_{f,h}}g(t,x)$ is well defined, and equations 1.3.17 and 1.3.18 are valid.

For particles interacting through a potential without cutoff, the relevant condition is

$$\int_{\mathbb{S}^2}\left(\frac{\pi}{2} - p\right) B(p, \|w\|)\|w\|\, d\mathcal{H} < C(1+\|v\|)^2(1+\|v_*\|)^2 \qquad (1.3.19)$$

Theorem 1.3.3. *Let B be a collision kernel satisfying equation 1.3.19. If f, h are C^1 with respect to the* *velocity variable, and* $(1+\|v\|)^2 K \in L^1(\Omega\times\mathbb{R}^d)$ *for* $K = f, h, \|\nabla_v f\|, \|\nabla_v h\|$, *then* $Q[f,h](t,x,v)$ *is well defined, and is integrable* ($Q[f,h] \in L^1_{x,v}$ *for a.e.* $t>0$).

Theorem 1.3.4. *Let* B *be a collision kernel satisfying equation 1.3.19, and* $g(v) : \mathbb{R}^d \longrightarrow \mathbb{R}$ *be a* C^1 *function satisfying*

$$|g| \leq C_1(1+\|v\|)^n, \quad \|\nabla g\| \leq C_1(1+\|v\|)^{n-1}.$$

If f,h *are* C^1 *with respect to the velocity variable, and* $(1+\|v\|)^{n+2}K \in L^1(\Omega\times\mathbb{R}^d)$ *for* $K = f, h, \|\nabla_v f\|, \|\nabla_v h\|$, *then* $\overline{Q_{f,h}}g(t,x)$ *is well defined, and equations 1.3.17 and 1.3.18 are valid.*

Theorem 1.3.2 is proven in the Appendix A.1; for the demonstration of the other theorems, refer to [36] page 309.

Then, we see that if the property g satisfies the equation

$$g' + g'_* = g + g_* \tag{1.3.20}$$

The total effect of collisions is zero. These functions are called **summational invariants**. It follows from (1.3.18)[5] that any scalar continuous $g(x,v)$ is a summational invariant $\Leftrightarrow \overline{Q_{f,h}}g(t,x) = 0$ for all f,h probability densities on the phase space Γ of a single particle. The first examples of summational invariants that come to mind are energy and momentum; indeed by their conservation laws:

$$(B)\ \begin{cases} v + v_* = v' + v'_* \\ \|v\|^2 + \|v_*\|^2 = \|v'\|^2 + \|v'_*\|^2 \end{cases} \tag{1.3.21}$$

Another trivial (but not so immediate) summational invariant is any constant function of (x,v). We can thus ask ourselves: are there any other physically significant summational invariants of the collision operator? The answer for this question is, in a sense, negative:

Theorem 1.3.5 (Boltzmann-Gronwall)**.** *Let* $g : \mathbb{R}_+ \times \Gamma \longrightarrow \mathbb{R}^n$ *be a measurable function with respect to the lebesgue measure* ($\mathcal{L}$). *g satisfies condition 1.3.20, where* v', v'_* *are solutions to the system 1.3.21 of algebraic equations for a given pair* (v, v_*), *if and only if g is a second order polynomial in* v:

$$g(x,v) = a(x)\|v\|^2 + \langle b(x), v\rangle + c(x),$$

where $a(x), c(x) : \Omega \longrightarrow \mathbb{R}^n$; $b(x) : \Omega \longrightarrow GL(\mathbb{R}^n;\mathbb{R}^n)$ *are continuous functions.*

Corollary 1.3.1. *If* $g(t,x,v)$ *satisfies the above conditions, but also belongs in* $C^0_{t,x}$, *then the functions* $a(x), c(x) : \Omega \longrightarrow \mathbb{R}^n$; $b(x) : \Omega \longrightarrow GL(\mathbb{R}^n;\mathbb{R}^n)$ *in the above theorem will be continuous.*

Proof. See Appendix A.2.1 □

In classical mechanics, it is known that systems with n particles in 3 dimensions can have at most $2n+1$ constants of motion (conserved quantities). However, it can be shown that the only time-independent additive conserved quantities are mass, energy, momentum and angular momentum [24], [31], and this theorem is a manisfestation of that fact.

Summational invariants are essential in the theory of the boltzmann equation, for they allow us to obtain macroscopic balance laws that do not depend on the complicated structure of the collision operator. This is precisely what we'll do in the next section.

1.3.3 Macroscopic balance laws

We recall now equation (1.3.14) (the equation of transfer), in the case where the external force field W does not depend on the particles's velocities:

$$\frac{\partial}{\partial t}\left[\rho(t,x)\overline{g}\right] + \nabla_x \cdot \left[\rho(t,x)\overline{g} \otimes u(t,x)\right] = -\nabla_x \cdot \sigma + F_W + \overline{Q_f}g(t,x) + r \tag{1.3.22}$$

[5]See [36] page 100

Where we use the notation

$$\begin{cases} \sigma(t,x) = \int_{\mathbb{R}^3} c \otimes g(x,v) f(t,x,v)\, dv, \\ F_W(t,x) = \int_{\mathbb{R}^3} \left[W(t,x)\cdot\nabla_v g(x,v) + v\cdot\nabla_x g(x,v)\right] f(t,x,v) dv, \\ r(t,x) = \lim_{R\to\infty} \oint_{\partial B_R} (f(t,x,v)g(x,v)) \left[W(t,x)\cdot\hat{n}(v)\right]\, d\mathcal{H}(v), \\ G(x,t) = \int_{\mathbb{R}^3} g(x,v) f(t,x,v) dv = \rho(t,x)\int_{\mathbb{R}^3} g(x,v)\psi_{t,x}(v) dv = \rho(t,x)\overline{g(x,t)}, \end{cases}$$

and u, ρ are given by equations 1.3.6, 1.3.8. Then, if we restrict ourselves to the case when $g(x,v)$ is a summational invariant, the term $\overline{Q_f} g(t,x)$ will be identically 0. Taking $g(x,v) = 1$, for instance, we find that: $\sigma = F_W = \vec{0}$, $G = \rho(t,x) \Rightarrow \overline{g} = 1$. So our equation then reads

$$\frac{\partial}{\partial t}\left[\rho(t,x)\right] + \nabla_x \cdot \left[\rho(t,x)u(t,x)\right] = r. \tag{1.3.23}$$

Finally, if we analyze the error term r

$$\begin{aligned} \|r\| &= \left\| \lim_{R\to\infty} \oint_{\partial B_R} (f(t,x,v)g(x,v)) \left[W(t,x)\cdot\hat{n}(v)\right]\, d\mathcal{H}(v) \right\| \\ &= \left\| \left[\lim_{R\to\infty} \oint_{\partial B_R} f(t,x,v)(g(x,v)\otimes\hat{n}(v))\, d\mathcal{H}(v) \right] W(t,x) \right\|. \end{aligned}$$

Since we are interested in estimating r for the choices $g = 1$, $g = v$ and $g = \|v\|^2$, we see that in these 3 cases it is sufficient that f be an even function of velocity (radially symmetric) or $W \equiv 0$ for this term to vanish. If neither of these conditions is met, one can use the Cauchy-Schwartz and triangle inequalities to conclude that

$$\|r\| \leq \|W(t,x)\| \lim_{R\to\infty} \oint_{\partial B_R} f(t,x,v)\|g(x,v)\|\, d\mathcal{H}(v). \tag{1.3.24}$$

In the case when $g = 1$, we have that

$$0 \leq |r| \leq \|W(t,x)\| \lim_{R\to\infty} \oint_{\partial B_R} f(t,x,v)\, d\mathcal{H}(v)$$

Then If $f \leq A/\|v\|^{2+\varepsilon}$ for some $A \geq 0$ and any $\epsilon > 0$, then $r = 0$ (if $v, x \in \mathbb{R}^d$, then the restriction becomes $f \leq A/\|v\|^{d-1+\varepsilon}$). in these conditions, it follows that the gas obeys the **continuity equation** from continuum mechanics,

$$\frac{\partial}{\partial t}\left[\rho(t,x)\right] + \nabla_x \cdot \left[\rho(t,x)u(t,x)\right] = 0. \tag{1.3.25}$$

Next, choose $g(x,v) = v$. Then, we get that

$$\begin{cases} \sigma(t,x) = \int_{\mathbb{R}^3} c \otimes v f(t,x,v)\, dv, \\ F_W(t,x) = \int_{\mathbb{R}^3} \left[W(t,x)\cdot\nabla_v v + v\cdot\nabla_x v\right] f(t,x,v) dv = \int_{\mathbb{R}^3} W(t,x) f(t,x,v)\, dv = \rho(t,x)W(t,x) \\ r(t,x) = \lim_{R\to\infty} \oint_{\partial B_R} (f(t,x,v)v) \left[W(t,x)\cdot\hat{n}(v)\right]\, d\mathcal{H}(v), \\ G(x,t) = \int_{\mathbb{R}^3} v f(t,x,v) dv = \rho(t,x)\int_{\mathbb{R}^3} v\psi_{t,x}(v) dv = \rho(t,x)u(t,x). \end{cases}$$

From which we get $\overline{g} = u(t,x)$, and we can further decompose σ as

$$\sigma(t,x) = \int_{\mathbb{R}^3} (c\otimes v) f(t,x,v)\, dv$$

$$= \int_{\mathbb{R}^3} (c\otimes c) f(t,x,v)\, dv + \rho\left[\int_{\mathbb{R}^3} (v - u(t,x))\psi_x(t,v)\, dv\right] \otimes u(t,x).$$

In the last equation, by definition of average velocity, we can say that the second term on the right hand side vanishes. Denote the first term in the above decompostion of σ as $M(t,x) = \int_{\mathbb{R}^3} (c \otimes c) f(t,x,v)\, dv$. Inserting the above expressions, equation 1.3.14 then becomes

$$\frac{\partial}{\partial t}\left[\rho(t,x)u(t,x)\right] + \nabla_x \cdot \left[\rho(t,x)u(t,x) \otimes u(t,x)\right] = -\nabla \cdot M + \rho(t,x)W(t,x) + r. \tag{1.3.26}$$

We proceed then to estimate r using 1.3.24:

$$0 \leq |r| \leq \|W(t,x)\| \lim_{R\to\infty} \oint_{\partial B_R} f(t,x,v)\|v\|\, d\mathcal{H}(v)$$

If f is neither even, nor $W \equiv 0$, it is sufficient to demand that $f \leq A/\|v\|^{3+\varepsilon}$, for some choice of $A, \epsilon > 0$ for the last term to vanish (in the d dimensional case, the restriction for f becomes $f \leq A/\|v\|^{d+\varepsilon}$). Then we are left with

$$\frac{\partial}{\partial t}\left[\rho(t,x)u(t,x)\right] + \nabla_x \cdot \left[\rho(t,x)u(t,x) \otimes u(t,x)\right] = -\nabla \cdot M + \rho(t,x)W(t,x), \tag{1.3.27}$$

which is simply the **balance of linear momentum** for the gas, where M plays the role of the Cauchy stress tensor. Thus, the mechanical pressure can be found by taking

$$p = \frac{1}{3}\text{Trace}(M) = \int_{\mathbb{R}^3} \frac{\|c\|^2}{3} f(t,x,v)\, dv.$$

Finally, notice that the following identity holds

$$\int_{\mathbb{R}^3} \|v\|^2 f(t,x,v)dv = \rho \int_{\mathbb{R}^3} \|v\|^2 \psi_x(t,v)dv = \rho\mathbb{E}(\|V\|^2) = \rho[\text{Trace}(\text{Cov}(V)) + \|\mathbb{E}(V)\|^2]$$

$$= \rho\left[\int_{\mathbb{R}^3} \|c\|^2 \psi_x(t,v)dv + \|u(t,x)\|^2\right].$$

So if we choose $g(x,v) = \frac{\|v\|^2}{2}$, we will have that

$$G(x,t) = \int_{\mathbb{R}^3} \frac{\|v\|^2}{2} f(t,x,v)dv = \rho \int_{\mathbb{R}^3} \frac{\|c\|^2}{2} \psi_x(t,v)dv + \frac{\rho\|u(t,x)\|^2}{2} \tag{1.3.28}$$

for this choice of g, G will be the total kinetic energy density (per unit volume) for the gas. We see that it can be decomposed into 2 terms: the second term is the kinetic energy associated with the average drift velocity, and the second is a kinetic energy associated with a deviation from the average velocity, which we define as the internal energy U at that point:

$$U(x,t) = \frac{\rho(x,t)}{2} \int_{\mathbb{R}^3} \|v-u\|^2 \psi_{t,x}(v)dv.$$

U is an extensive quantity measuring energy per unit volume. We now define an intensive quantity ϵ by the equation $\rho(x,t)\epsilon(t,x) = U(x,t)$, measuring the internal energy per particle (or per mass):

$$\epsilon(t,x) = \frac{1}{2} \int_{\mathbb{R}^3} \|v-u\|^2 \psi_{t,x}(v)dv.$$

With the above notation in place, this means that $\overline{g} = \frac{\|u(t,x)\|^2}{2} + \epsilon(t,x)$. Inserting $g = \frac{v^2}{2}$ into our definitions, we find

$$\begin{cases} \sigma(t,x) = \int_{\mathbb{R}^3} c\frac{\|v\|^2}{2} f(t,x,v)\,dv, \\ F_W(t,x) = \int_{\mathbb{R}^3} \left[W(t,x)\cdot\nabla_v \frac{\|v\|^2}{2} + v\cdot\nabla_x\frac{\|v\|^2}{2}\right] f(t,x,v)dv, \\ r(t,x) = \lim_{R\to\infty}\oint_{\partial B_R}(f(t,x,v)\frac{\|v\|^2}{2})\,[W(t,x)\cdot\hat{n}(v)]\;d\mathcal{H}(v), \end{cases}$$

Then, the following simplifications can be made, separating $v = c(t,x,v) + u(t,x)$ whenever possible:

$$F_W(t,x) = \int_{\mathbb{R}^3} W(t,x)\cdot vf(t,x,v)\,dv = \rho u(t,x)\cdot W(t,x),$$

$$\sigma(t,x) = \int_{\mathbb{R}^3} c\frac{\|c\|^2}{2} f(t,x,v)\,dv + \frac{\|u\|^2}{2}\int_{\mathbb{R}^3} cf(t,x,v)\,dv + \int_{\mathbb{R}^3} c(c\cdot u)f(t,x,v)\,dv$$
$$= \int_{\mathbb{R}^3} c\frac{\|c\|^2}{2} f(t,x,v)\,dv + \left[\int_{\mathbb{R}^3} c\otimes cf(t,x,v)\,dv\right] u(t,x)$$

The first term in the previous equation our expression is a vector along the direction of c, the random velocity deviation from the mean, multiplied by a factor $\|c\|^2\psi_{t,x}(v)$ which gives e upon integration. We can interpret it then as a 'diffusion' of internal energy; which we'll denote as a heat flux $q(t,x)$. So our previous equation reads

$$\sigma(t,x) = q(t,x) + [M(t,x)]u(t,x)$$

And equation 1.3.14 becomes

$$\frac{\partial}{\partial t}\left[\rho(t,x)\left(\frac{\|u(t,x)\|^2}{2} + \epsilon(t,x)\right)\right] + \nabla_x\cdot\left[\left(\frac{\|u(t,x)\|^2}{2} + \epsilon(t,x)\right)u(t,x)\right]$$
$$= -\nabla\cdot(q + Mu(t,x)) + \rho(t,x)u(t,x)\cdot W(t,x) + r. \tag{1.3.29}$$

Finally, we can bound the error term r by using again equation 1.3.24

$$0 \le |r| \le \frac{1}{2}\|W(t,x)\| \lim_{R\to\infty}\oint_{\partial B_R} f(t,x,v)\|v\|^2\,d\mathcal{H}(v),$$

and so r vanishes if f is neither rarially symmetric nor $W \equiv 0$, r will still vanish if we guarantee that $f \le A/\|v\|^{4+\varepsilon}$, for some choice of $A, \epsilon > 0$ (in the d dimensional case, the restriction for f becomes $f \le A/\|v\|^{d+1+\varepsilon}$). We can thus write

$$\frac{\partial}{\partial t}\left[\rho(x,t)\left(\epsilon + \frac{\|u\|^2}{2}\right)\right] + \nabla\cdot\left[\rho(x,t)u(t,x)\left(\epsilon + \frac{\|u\|^2}{2}\right)\right]$$
$$= \rho(x,t)\,\langle W(x,t),u\rangle - \nabla\cdot(q + Mu)\,, \tag{1.3.30}$$

which is a macroscopic **energy balance** for our gas.

With this, we have just derived the balance laws of continuum mechanics from the Boltzmann equation (we have restricted ourselves to the physical case where $x, v \in \mathbb{R}^3$, however we can reach the exact same expressions if $x, v \in \mathbb{R}^d$, $d > 3 \in \mathbb{N}$). For the case with external forcing, we made assumptions on the decay of $f(t,x,v)$ as $\|(x,v)\| \to \infty$ in order to derive the previous equations, but these assumptions are generally satisfied if the distribution f is close to equilibrium, and therefore are not very restrictive. However, these equations do not form a closed system yet, due to the absence of constitutive relations to define the relationship between quantities such as the Cauchy stress tensor and the heat flux vector. So far, we can only determine their form if we solve the Boltzmann equation first, which would make them unnecessary. Therefore these equations serve only as a plausibility argument for the Boltzmann equation, confirming that it is consistent with macroscopic continuum mechanics [36],[7].

The problem of determining which constitutive relations are obeyed by a gas described by special classes of solutions to the Boltzmann equation, and potentially recover well-known equations (e.g. Navier-Stokes-Fourier system) from the Boltzmann equation is called the 'hydrodynamic limit' problem.

Remark 1.3.2. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set. Any Boltzmann gas whose probability distribution $f : \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}_+$ satisfies

$$\int_{\Omega\times\mathbb{R}^d} f(t,x,v)(1+\|v\|^2)\,dxdv < \infty \quad \text{for a.e. } t>0$$

must also satisfy the following relation

$$p(t,x) = \frac{2}{3}\rho(t,x)\epsilon(t,x). \tag{1.3.31}$$

since, by definition

$$p(t,x) = \frac{\rho(x,t)}{3}\int_{\mathbb{R}^3} \|v-u\|^2\psi_{t,x}(v)dv = \frac{2}{3}U(t,x) = \frac{2}{3}\rho(t,x)\epsilon(t,x).$$

Now, for p to be a real measure of pressure of the gas (and for it to be dimensionally consistent), we must multiply its definition by the total mass of gas in the vessel, $M = Nm$, where N is the total number of particles and m the mass of a single particle. We define then: $\tilde{p} = pNm$, $\tilde{e} = m\epsilon$, $\tilde{\rho} = N\rho$. Multiplication by mN on both sides of the previous equation yields

$$\tilde{p}(t,x) = \frac{2}{3}\tilde{\rho}(t,x)\tilde{e}(t,x),$$

where $\tilde{\rho}$ now denotes the number density of the gas. Integrating over the position variable, and defining the mean values

$$\begin{cases} |\Omega| = V, \\ \bar{P}(t) = \frac{1}{V}\int_\Omega \tilde{p}(t,x)\,dx, \\ \bar{e}(t) = \frac{1}{\int_\Omega \tilde{\rho}(t,x)\,dx}\int_\Omega \tilde{\rho}\tilde{e}(t,x)\,dx = \frac{1}{N}\int_\Omega \tilde{\rho}\tilde{e}(t,x)\,dx, \end{cases}$$

we arrive at

$$P(t)V = \frac{2N}{3}\bar{e}(t). \tag{1.3.32}$$

If the particles that compose the gas are assumed to be point masses with no internal degrees of freedom, and the system is assumed to be in equilibrium (meaning we omit the time dependence in the above equation), then by the equipartition theorem from classical statistical mechanics, $\bar{e} = \frac{3}{2}k_BT$ (actually, this can be applied to $\tilde{e}(x,t)$ in order to have a local definition of temperature). Since $N = nN_{av}$ (where N_{av}=6.022$\times 10^{23}$ is the avogadro number, n the number of moles of particles) and the constant of idea gase R is related to the Boltzmann constant by $R = N_{av}k_B$, we finally arrive at

$$PV = \frac{2N}{3}\frac{3}{2}k_BT = nN_{av}k_BT = nRT. \tag{1.3.33}$$

Therefore, the Boltzmann equation when restricted to a gas of point particles describes a classical ideal gas, with the advantage of modelling its behavior of convergence to equilibrium. We notice however, that the equipartition theorem makes restrictions on the form of the hamiltonian of this system. If these restrictions are not met, the internal energy may depend on other factors aside from temperature, and so the gas described by Boltzmann equation may not be necessarily ideal in this case. The only restriction imposed by Boltzmann equation is 1.3.32.

The balance laws just derived can be cast in an integral form, becoming

$$\begin{cases} \frac{d}{dt}\int_{\Omega\times\mathbb{R}^d} f(t,x,v)\,dxdv = -\int_{\partial\Omega}\rho(t,x)u(t,x)\cdot\hat{n},d\mathcal{H}^{d-1}(x) \\ \frac{d}{dt}\int_{\Omega\times\mathbb{R}^d} vf(t,x,v)\,dxdv = \int_{\Omega\times\mathbb{R}^d} W(t,x)f(t,x,v)\,dxdv - \int_{\partial\Omega}\left[\rho u\otimes u + M\right]\hat{n}\,d\mathcal{H}^{d-1}(x), \\ \frac{d}{dt}\int_{\Omega\times\mathbb{R}^d}\frac{\|v\|^2}{2}f(t,x,v)\,dxdv = \int_{\Omega\times\mathbb{R}^d}(W(t,x)\cdot v)f(t,x,v)\,dxdv \\ -\int_{\partial\Omega}\left[q + Mu + \rho u\left(\epsilon + \frac{\|u\|^2}{2}\right)\right]\cdot\hat{n}\,d\mathcal{H}^{d-1}(x). \end{cases} \tag{1.3.34}$$

In the case when $W\equiv 0$ and the domain $\Omega=\mathbb{R}^d$, the right hand side of all of hte above equations vanishes, so that we have:

$$\begin{cases} \int_{\Omega\times\mathbb{R}^d} f(t,x,v)\,dxdv = \int_{\Omega\times\mathbb{R}^d} f_0(x,v)\,dxdv \\ \int_{\Omega\times\mathbb{R}^d} vf_0(x,v)\,dxdv = \int_{\Omega\times\mathbb{R}^d} vf(t,x,v)\,dxdv, \\ \int_{\Omega\times\mathbb{R}^d}\frac{\|v\|^2}{2}f(t,x,v)\,dxdv = \int_{\Omega\times\mathbb{R}^d}\frac{\|v\|^2}{2}f_0(x,v)\,dxdv. \end{cases} \tag{1.3.35}$$

This last form of the system of balance laws will be useful for us in the next chapter.

1.3.4 Boundary conditions in kinetic theory

For completeness, we will briefly review important kinds of boundary conditions in kinetic theory. Since in kinetic theory all of the material's behavior is encoded in the probability distribution $f(t,x,v)$, any imposed boundary condition should be a particular form of the distribution at the wall. Specifically, the boundary condition should take the form of a relationship between the distribution of particles coming outward from the wall and incoming particles before interacting with the wall.

This can be realized by splitting the distribution function in the following manner:

$$f(t,x,v) = \begin{cases} f^i(t,x,v) & x\in\partial\Omega,\ \ (v-u_w(x,t))\cdot\hat{n}(x,t) > 0, \\ f^o(t,x,v) & x\in\partial\Omega,\ \ (v-u_w(x,t))\cdot\hat{n}(x,t) \le 0, \end{cases} \tag{1.3.36}$$

where $\hat{n}(x,t)$ is the unit inward-pointing normal vetor to the surface $\partial\Omega$, and u_w is the velocity of the boundary if it is not fixed (we also assume Ω is a bounded open set with an orientable Lipschitz boundary, and that f has enough regularity so that the restriction to the boundary is well defined). The superscripts 'i' and 'o' mean incoming and outbound relative to the wall, respectively.

Therefore boundary conditions should generally be of the form

$$f^o(t,x,v) = G(f^i(t,x,v)), \tag{1.3.37}$$

where G is some operator, to be determined and properly defined later. The simplest boundary condition in kinetic theory is the specular reflection. It assumes that colliding particles are reflected as light in a mirror, reversing the velocity component normal to the wall. This means that

$$f^o(t,x,v) = f^i(t,x,v-2\hat{n}(\hat{n}\cdot v)). \tag{1.3.38}$$

This condition assumes that the wall is ideally smooth, so that particles experience no friction as they interact with the wall. This can be seen if we attempt to compute the stress at the wall (following the argument made in [36])

$$p = M\hat{n} = m\int f(t,x,v)(c\otimes c)\hat{n}\,dv = m\int f(t,x,v)c(c\cdot\hat{n})\,dv$$

Defining $(1-\hat{n}\otimes\hat{n})c \doteq c_t$, $c = (1-\hat{n}\otimes\hat{n}+\hat{n}\otimes\hat{n})c = (\hat{n}\cdot c)\hat{n} + c_t$, so we have that

$$m\int f(t,x,v)c(c\cdot\hat{n})\,dv = m\left[\int f(t,x,v)(c\cdot\hat{n})^2\,dv\right]\hat{n} + m\int f(t,x,v)c_t(c\cdot\hat{n})\,dv.$$

The last integral can be split as

$$m\int f(t,x,v)c_t(c\cdot\hat{n})\,dv = m\int_{v\cdot\hat{n}>0} f(t,x,v)c_t(c\cdot\hat{n})\,dv + m\int_{v\cdot\hat{n}\leq 0} f(t,x,v)c_t(c\cdot\hat{n})\,dv \tag{1.3.39}$$

Performing the change of variables $r = v - 2\hat{n}(\hat{n}\cdot v)$ on the first term of the right hand side, one can show that

$$m\int f(t,x,v)c_t(c\cdot\hat{n})\,dv = 0 \Rightarrow p = m\left[\int f(t,x,v)(c\cdot\hat{n})^2\,dv\right]\hat{n}$$

In this way, the specular reflection boundary condition only allows stresses to be perpendicular to the the wall's surface [36]. Since real gases can cause stresses that are not perpendicular to the wall's surface, this boundary condition is not used frequently; it is used mainly to impose 'full-slip' boundary conditions on the gas. This can be seen in a simple calculation

$$\hat{n}\cdot\rho u = m\hat{n}\cdot\int f(t,x,v)v\,dv = m\int f(t,x,v)(\hat{n}\cdot v)\,dv,$$

and performing the same splitting as in equation 1.3.39 we see that the last integral is 0, so that the gas may have a velocity component tangent to the wall.

A variation of the above is the bounceback boundary condition, which can be written as

$$f^o(t,x,v) = f^i(t,x,-v). \tag{1.3.40}$$

It is a non-physical boundary condition in which particle velocities are reversed as they reach the wall. Despite that, if we denote $u(t,x)$ as the bulk fluid velocity, this boundary condition implies that $u(t,x) = 0$ if $x\in\partial\Omega$, and therefore serves as a 'no-slip' Dirichlet boundary condition for the gas flow.

Another significant boundary condition, first proposed by Maxwell, is the so called ideally rough wall: he considered the wall as an array of packed hard spheres agains which gas particles collided [36], and reasoned that the outgoing distribution should not be equal or even similar to the distribution of the incoming particles. Rather, since the impact with the wall will produce scattering, he proposed the following expression

$$f^o(t,x,v) = f_w(t,x,v) = a_w(t,x)e^{-\frac{3\|v-u_w(t,x)\|^2}{4e_w(t,x)}}, \tag{1.3.41}$$

where a_w is a normalization constant for the probability distribution, e_w is directly proportional to the local temperature at the wall T_w, and u_w is the wall's velocity if it is not fixed. The definition of e_w allows us to enforce a wall with fixed temperature, which can be different from the temperature of the fluid at that point of the wall (if not in equilibrium). Finally, an intermediate condition between specular reflection (ideally smooth wall) and an ideally rough wall is the Maxwell's boundary condition, which is simply a convex combination of the above conditions:

$$f^o(t,x,v) = mf^o_{\text{ideally smooth}}(t,x,v) + (1-m)f^o_{\text{ideally rough}}(t,x,v),$$

where $m\in[0,1]$ is the accomodation coefficient. This boundary condition is sufficient for a qualitative treatment of gases and interpretation of experiments, but does not yield accurate results [36].

All previous conditions can be regarded as special cases of the so called linear boundary conditions, in which the mapping between the inbound and outbound parts of the distribution are related by a linear map, which usually takes the form

$$-f^o(t,x,v)(\hat{n}\cdot c) = \int_{\{\hat{n}\cdot c'>0\}} f^i(t,x,v')(\hat{n}\cdot c')\,d\mu_v(v'). \tag{1.3.42}$$

where the measure μ must satisfy a few generic properties:

- $\mu\geq 0$ for almost every v, v',
- $\int_{\{c\cdot\hat{n}<0\}} d\mu_v(t,v')dv = dv'$ for a.e. $t>0$,

- there is a distribution which is left unchanged by the operator G, which is defined as the distribution of the wall. This means that if the gas has the wall distribution it is left unchanged by interacting with the wall:

$$-f_w(t,x,v)(\hat{n}\cdot c) = \int_{\{\hat{n}\cdot c'>0\}} f_w(t,x,v')(\hat{n}\cdot c')\,d\mu_v(v'). \tag{1.3.43}$$

Notice that the condition $\hat{n}\cdot c' > 0$ is equivalent to $\hat{n}\cdot v > \hat{n}\cdot u(t,x)$, and that the fluid's average velocity at the wall should match the wall's velocity, and so $\hat{n}\cdot c' > 0 \Leftrightarrow \hat{n}\cdot v > \hat{n}\cdot u_w(t,x)$. This means we restrict our domain in order to integrate only the incoming distribution.

Linear boundary conditions are suitable descriptions for a gas in case there is no adsorption of gas particles on the wall and the gas is sufficiently dilute, so that there is no gas-gas interaction at the wall [9]. Other restrictions will depend on the particular modelling of the wall-fluid interaction for any given problem. Comments about the meaning of this last property will be made in the next section: for now, we use the above properties to prove a lemma which will be very useful for us in the near future:

Lemma 1.3.1. *Let $F:[0,\infty)\longrightarrow\mathbb{R}$ be a convex function, $f(t,x,v):\mathbb{R}^+\times\Omega\times\mathbb{R}^d\longrightarrow\mathbb{R}^+$ be a classical solution to the Boltzmann equation, and $\Omega\subseteq\mathbb{R}^d$ a bounded open set with smooth boundary. If $x\in\partial\Omega$, and $f(t,x,v)$ has linear boundary conditions at the wall, then*

$$\int_{\mathbb{R}^d} (c\cdot\hat{n}) f_w(t,x,v) F\left(\frac{f(t,x,v)}{f_w(t,x,v)}\right) dv \geq 0$$

Proof. The main tool in this proof will be Jensen's inequality. In order to use it, we will construct a suitable probability measure ν and a non-negative measurable function g from our hypotheses. Define

$$g(\cdot,\cdot,v) = \frac{f(\cdot,\cdot,v)}{f_w(\cdot,\cdot,v)} = \begin{cases} f^i(\cdot,\cdot,v)/f_w(\cdot,\cdot,v) & \text{if } c\cdot\hat{n} > 0,\\ f^o(\cdot,\cdot,v)/f_w(\cdot,\cdot,v) & \text{if } c\cdot\hat{n}\leq 0,\end{cases}$$

$$d\nu_v(\cdot,\cdot,v') = \begin{cases} \frac{-(c'\cdot\hat{n})f_w(\cdot,\cdot,v')}{(c\cdot\hat{n})f_w(\cdot,\cdot,v)}\,d\mu_v(v') & \text{if } c'\cdot\hat{n} > 0,\\ 0 & \text{if } c'\cdot\hat{n}\leq 0.\end{cases}$$

Then, g is clearly non negative, as is the measure ν. finally from equation 1.3.43, dividing through by the right hand side, one can show:

$$\int_{\mathbb{R}^d} d\nu_v(v') = \int_{\{c\cdot\hat{n}>0\}} d\nu_v(v') = 1$$

So that ν is indeed a probability measure. Also, if we multiply and divide equation 1.3.42 by f_w:

$$-(\hat{n}\cdot c)f_w(t,x,v)\frac{f^o(t,x,v)}{f_w(t,x,v)} = \int_{\{\hat{n}\cdot c'>0\}} (\hat{n}\cdot c')f_w(t,x,v')\frac{f^i(t,x,v')}{f_w(t,x,v')}\,d\mu_v(v').$$

$$\Rightarrow g(\cdot,\cdot,v) = \int_{\{c\cdot\hat{n}>0\}} g(\cdot,\cdot,v')\,d\nu_v(v') = \int_{\mathbb{R}^d} g(\cdot,\cdot,v')\,d\nu_v(v').$$

So that g is integrable. By Jensen's inequality:

$$F\left(\int_{\mathbb{R}^d} g(\cdot,\cdot,v')\,d\nu_v(v')\right) \leq \int_{\mathbb{R}^d} F(g(\cdot,\cdot,v'))\,d\nu_v(v') = \int_{\{c'\cdot\hat{n}>0\}} F(g(\cdot,\cdot,v'))\,d\nu_v(v')$$

Multiplying through by $(c\cdot\hat{n})f_w(t,x,v)$, restricting ourselves to the domain where $(c\cdot\hat{n})<0$, we have that

$$(c\cdot\hat{n})f_w(t,x,v)F(g(\cdot,\cdot,v)) \geq -\int_{\{c'\cdot\hat{n}>0\}} (c'\cdot\hat{n})f_w(t,x,v')F(g(\cdot,\cdot,v'))\,d\mu_v(v').$$

Integrating over $\{c\cdot\hat{n}<0\}$ on the v variable, and using Fubini's theorem and the second property of the measure μ:

$$\begin{aligned}&\int_{\{c\cdot\hat{n}<0\}}(c\cdot\hat{n})f_w(t,x,v)F(g(\cdot,\cdot,v))\,dv\\ \geq &-\int_{\{c\cdot\hat{n}<0\}}\int_{\{c'\cdot\hat{n}>0\}}(c'\cdot\hat{n})f_w(t,x,v')F(g(\cdot,\cdot,v'))\,d\mu_v(v')dv\\ = &-\int_{\{c'\cdot\hat{n}>0\}}\int_{\{c\cdot\hat{n}<0\}}(c'\cdot\hat{n})f_w(t,x,v')F(g(\cdot,\cdot,v'))\,d\mu_v(v')dv\\ = &-\int_{\{c'\cdot\hat{n}>0\}}(c'\cdot\hat{n})f_w(t,x,v')F(g(\cdot,\cdot,v'))\,dv'.\end{aligned}$$

Finally,

$$\int_{\{c\cdot\hat{n}<0\}}(c\cdot\hat{n})f_w(t,x,v)F(g(\cdot,\cdot,v))\,dv \geq -\int_{\{c'\cdot\hat{n}>0\}}(c'\cdot\hat{n})f_w(t,x,v')F(g(\cdot,\cdot,v'))\,dv',$$

$$\int_{\mathbb{R}^d}(c\cdot\hat{n})f_w(t,x,v)F(g(\cdot,\cdot,v))\,dv = \int_{\mathbb{R}^d}(c\cdot\hat{n})f_w(t,x,v)F\left(\frac{f(t,x,v)}{f_w(t,x,v)}\right)\,dv \geq 0,$$

and the proof is complete. □

For further details on this topic, we recommend the reading of [7], [9] and [36].

1.4 The H-Theorem and its consequences

Consider now the microscopic property

$$g(t,x,v) = \begin{cases}\log(f(t,x,v)) & \text{if } (t,x,v)\in \operatorname{ess\,supp}(f)\\ 0 & \text{otherwise}\end{cases}$$

We wish now to derive a macroscopic balance equation for a macroscopic property associated with g, as we did in subsection 1.3.3. We have 2 problems to consider in this case: first, g is time dependent, so there will be extra terms in our transfer equation, and chiefly, g is not continuous at $\partial\operatorname{supp}(f)$. This makes some of the manipulations we made in subsection 1.3.2 incorrect, so a little more care must be taken in order to derive this particular transfer equation. The procedure multiplying the Boltzmann equation by the property g and integrating remains unchanged, which gives us

$$\int_{\mathbb{R}^3}\frac{\partial}{\partial t}(f)g(t,x,v)\,dv + \int_{\mathbb{R}^3} v\cdot\nabla_x\,(f)\,g(t,x,v)\,dv + \int_{\mathbb{R}^3} W(t,x,v)\cdot\nabla_v(f)g(t,x,v)dv$$
$$= \int_{\mathbb{R}^3} g(x,v)Q(f,f)(t,x,v)dv$$

In this entire subsection, we consider only points (t,x,v) where $(x,t)\in\operatorname{ess\,supp}(\rho)$. Notice that, when outside $\operatorname{ess\,supp}(f)$, these integrals are all 0 since g is 0, therefore we can restrict the domains of the above integrals accordingly (here we also assume that $\partial\operatorname{supp}(f)$ has measure 0). We can derive an equation of the form:

$$\frac{\partial}{\partial t}G(t,x) + \nabla_x\cdot[u(t,x)G] = -\nabla_x\cdot\sigma + F_W + \overline{Q_f}g(t,x) + r,$$

where now the definitions read

$$\begin{cases} \sigma = \int_{\{f>0\}} c\log(f(t,x,v))f(t,x,v)\,dv, \\ F_W = \int_{\{f>0\}} \left[W(t,x)\cdot\nabla_v g(x,v) + v\cdot\nabla_x g(x,v) + \frac{\partial}{\partial t}g\right] f(t,x,v)dv \\ r = \lim_{R\to\infty} \oint_{\partial(B_R\cap\{f>0\})} (f(\cdot,\cdot,v)g(\cdot,v)W(\cdot,\cdot))\cdot\hat{n}(v)\,d\mathcal{H}^2(v). \end{cases}$$

Define $\mathfrak{D}$ as the material derivative operator, given by

$$\mathfrak{D}g = \frac{\partial}{\partial t}g + v\cdot\nabla_x g(x,v) + W(t,x)\cdot\nabla_v g(x,v).$$

Then, or definition of F_W can be written as

$$F_W = \int\limits_{\mathbb{R}^3} f\mathfrak{D}g\,dv.$$

Notice that, in the support of f, we have that $\mathfrak{D}g = \frac{1}{f}\mathfrak{D}f$, so that in general $f\mathfrak{D}g = \mathfrak{D}f$. On the other hand, we also derived in subsection 1.3.3 that

$$\begin{aligned} \int\limits_{\{f>0\}} \mathfrak{D}f dv = \int\limits_{\mathbb{R}^3} \mathfrak{D}f dv &= \int_{\mathbb{R}^3} \frac{\partial}{\partial t} f\,dv + \int_{\mathbb{R}^3} v\cdot\nabla_x f\,dv + \int_{\mathbb{R}^3} W(t,x,v)\cdot\nabla_v f dv \\ &= \overline{Q_f}1 = 0\ (Summational\,Invariant!) \end{aligned}$$

Now, restricting to the case when the external potential is 0, we have that $r=0$. In order to introduce a more standard notation for the subject, we will change the names of a few variables: for this particular choice of $g(t,x)$, define

$$\begin{cases} h(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\log(f(t,x,v))\,dv, = G(t,x) \\ \mathfrak{H}(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\log(f(t,x,v))c(t,x,v)\,dv = \sigma(t,x). \end{cases} \tag{1.4.1}$$

Then, equation 1.3.13 becomes:

$$\frac{\partial}{\partial t}h(t,x) + \nabla_x\cdot(u(t,x)h(t,x)) = -\nabla_x\cdot\mathfrak{H}(t,x) + \overline{Q_f}\log(f). \tag{1.4.2}$$

Recalling a property we established in section 1.3.2 (equation 1.3.17)

$$\overline{Q_{f,h}}g(t,x) = \frac{1}{8}\int\limits_{\mathbb{R}^3}\iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} (f'h'_* + h'f'_* - fh_* - hf_*)\,[g + g_* - g' - g'_*]\ B(w,\alpha)d\mathcal{H}^2(\alpha)dv_*\,dv,$$

we can apply this to our particular choice of g, which gives us the following expression

$$\begin{aligned} &\overline{Q_f}\log(f)(t,x) = \\ &\frac{-1}{4}\int\limits_{\mathbb{R}^3}\iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} (f'f'_* - ff_*)\,[\log(f') + \log(f'_*) - \log(f) - \log(f_*)]\ B(w,\alpha)d\mathcal{H}^2(\alpha)dv_*\,dv \\ &= \frac{-1}{4}\int\limits_{\mathbb{R}^3}\iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} (f'f'_* - ff_*)\log\left(\frac{f'f'_*}{ff_*}\right)\ B(w,\alpha)d\mathcal{H}^2(\alpha)dv_*\,dv. \end{aligned} \tag{1.4.3}$$

For simplicity of notation, from now on we will use the following definition

$$\begin{cases} D[f] = \dfrac{-1}{4} \displaystyle\iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} (f'f'_* - ff_*)\log\left(\frac{f'f'_*}{ff_*}\right) B(w,\alpha)d\mathcal{H}^2(\alpha)dv_*, & (1.4.4a) \\ \overline{Q_f}\log(f)(t,x) = \displaystyle\int\limits_{\mathbb{R}^3} D[f]\,dv. & (1.4.4b) \end{cases}$$

Or, if $x, v \in \mathbb{R}^d$,

$$\begin{cases} D[f] = \dfrac{-1}{4} \displaystyle\iint\limits_{\mathbb{R}^d\times\mathbb{S}^{d-1}} (f'f'_* - ff_*)\log\left(\frac{f'f'_*}{ff_*}\right) B(w,\alpha)d\mathcal{H}^{d-1}(\alpha)dv_*, & (1.4.5a) \\ \overline{Q_f}\log(f)(t,x) = \displaystyle\int\limits_{\mathbb{R}^d} D[f]\,dv. & (1.4.5b) \end{cases}$$

From the structure of $D[f]$ we can derive the following

Proposition 1.4.1. *Let $f \in L^1_+(\Omega \times \mathbb{R}^d)$ be a probability density function such that*

$$\iint\limits_{\Omega\times\mathbb{R}^d} D[f]\,dxdv < \infty.$$

Then,

1.
$$D[f] \leq 0.$$

2.
$$\int_{\mathbb{R}^d} D[f](t,x,v)\,dv = 0 \quad \Leftrightarrow \quad f(t,x,v) = \frac{\rho(t,x)}{\sqrt{\frac{4\pi}{d}\epsilon(t,x)}^{\,d}} e^{-\frac{d\|v-u(t,x)\|^2}{4\epsilon(t,x)}}$$

where $u(t,x) : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $\rho(t,x) : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^+$, $\epsilon(t,x) : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^+$ are respectively: the bulk velocity field, the density, and internal energy per particle of the gas, as defined in subsection 1.3.3.

Proof. For the first item, notice that for any 2 positive numbers $x, y \in \mathbb{R}$ the following inequality holds trivially

$$(x-y)\log\left(\frac{x}{y}\right) \geq 0.$$

Therefore, choosing $x = f'f'_*$, $y = ff_*$, we see that the integrand in the definition of $D[f]$ is non-negative, and the minus factor in equation 1.4.5a means that $D[f] \leq 0$.

For the second item, notice that

$$\int_{\mathbb{R}^d} D[f](t,x,v)\,dv = \overline{Q_f}\log(f)(t,x) = 0$$

$$\Leftrightarrow \quad \log(f)(t,x,v) \text{ is a summational invariant!}$$

Then, invoking theorem 1.3.5, this happens if and only if

$$\log(f)(t,x,v) = a(t,x)\|v\|^2 + b(t,x)\cdot v + c(t,x)$$

$$\Leftrightarrow f(t,x,v) = e^{a(t,x)\|v\|^2 + b(t,x)\cdot v + c(t,x)}$$

Finally, a, b, c can be written in terms of the first and second moments of the distribution of $\psi_{t,x}(v)$, the conditional distribution obtained from f:

$$a(t,x)\|v\|^2 + b(t,x)\cdot v + c(t,x) = a\left\|v + \frac{b}{2a}\right\|^2 + c - \frac{\|b\|^2}{4a},$$

$$\Rightarrow f(t,x,v) = e^{c-\frac{\|b\|^2}{4a}} e^{a\left\|v+\frac{b}{2a}\right\|^2}$$

$$\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,dv = e^{c-\frac{\|b\|^2}{4a}}\sqrt{\frac{\pi}{|a|}}^{-d} \Rightarrow e^{c-\frac{\|b\|^2}{4a}} = \rho(t,x)\sqrt{\frac{\pi}{|a|}}^{-d}$$

$$\rho(t,x)u(t,x) = \int_{\mathbb{R}^d} vf(t,x,v)\,dv = \rho(t,x)\int_{\mathbb{R}^d} v\sqrt{\frac{\pi}{|a|}}^{-d} e^{a\left\|v+\frac{b}{2a}\right\|^2}\,dv = -\rho\frac{b}{2a} \Rightarrow -\frac{b}{2a} = u(t,x)$$

Finally,

$$\rho(t,x)\epsilon(t,x) = \frac{d}{2}\int_{\mathbb{R}^d}\|v-u\|^2 f(t,x,v)\,dv = \rho(t,x)a \Rightarrow a = \frac{2\epsilon(t,x)}{d}$$

Replacing a, b, c by $\rho, \epsilon(t,x), u(t,x)$ in the above expression completes the proof. □

Distributions of the form

$$f(t,x,v) = \frac{\rho(t,x)}{\sqrt{\frac{4\pi}{d}\epsilon(t,x)}^{d}} e^{-\frac{d\|v-u(t,x)\|^2}{4\epsilon(t,x)}} \tag{1.4.6}$$

are called **local Maxwellian distributions**, and will be denoted by $M_{\rho,u,\epsilon}(t,x,v)$ (in standard probability notation, $V|X \sim N\left(u(t,x), \frac{2\epsilon(t,x)}{d}\right)$). If we impose the additional requirement that $f(t,x,v) = f(t,x,-v)$, then $\rho = \frac{1}{|\Omega|}$, $\epsilon(t,x) = Const.$ and $u(t,x) = 0$, such that f becomes time-independent, and identical to the classical Maxwell-Boltzmann dustribution of ideal gases in equilibrium.

With this definition in hand, we can go back to the topic of the previous section: when discussing the boundary condition of linear walls, we said that a probability distribution was associated to the wall (the so-called wall distribution), and the linear operator relating incoming and outbound distributions had this distribution as a fixed point (equation 1.3.43). This distribution is usually set as a maxwellian distribution:

$$f_w(t,x,v) = \frac{\rho(t,x)}{\sqrt{\frac{4\pi}{d}e_w(t,x)}^{d}} e^{-\frac{d\|v-u_w(t,x)\|^2}{4e_w(t,x)}}. \tag{1.4.7}$$

Since the second moment of the probability distribution $f(t,x,v)$ was associated with internal energy and temperature in subsection 1.3.3, this allows us to prescribe a temperature to the wall, and consider problems in which the gas is subjected to a heat bath. Likewise, u_w is identified as the wall's velocity.

Integrating equation 1.4.2 with respect to the position variable over the domain Ω, we find

$$\frac{d}{dt}\int_\Omega h(t,x)\,dx + \int_\Omega \nabla_x\cdot(u(t,x)h(t,x))\,dx + \int_\Omega \nabla_x\cdot\mathfrak{H}(t,x)\,dx = \iint\limits_{\Omega\times\mathbb{R}^d} D[f]\,dxdv. \tag{1.4.8}$$

Define

$$H(t) = \int_\Omega h(t,x)\,dx. \tag{1.4.9}$$

The previous remarks and definitions allowed Boltzmann to prove his celebrated

Theorem 1.4.1 (Formal H-Theorem). *Let $f(t,x,v) \in L^\infty_{loc}([0,\infty), L^1(\Omega\times\mathbb{R}^d)$ be a smooth solution to* *the initial value problem 2.1.1 (Boltzmann equation), which satisfies*

$$\int_{\Omega\times\mathbb{R}^d} f(t,x,v)\left(1+\|v\|^2+\log(f(t,x,v))\right)\,dxdv<\infty$$

and

$$\int_{\Omega}\mathfrak{H}\,dx<\infty.$$

Then, using the definitions for $h(t,x)$, $\mathfrak{H}(t,x)$ *as stated in equation 1.4.1, it follows that for every* $t\in[0,\infty)$*:*

1. *if* $\Omega\subseteq\mathbb{R}^d$ *is an open set with a smooth and orientable boundary*

$$\frac{d}{dt}\iint\limits_{\Omega\times\mathbb{R}^{d-1}} f(t,x,v)\log(f(t,x,v))\,dxdv=\frac{d}{dt}H(t)\le-\int_{\partial\Omega}\mathfrak{H}(t,x)\cdot\hat{n}\,d\mathcal{H}^{d-1}(x), \tag{1.4.10}$$

   where $\hat{n}$ *denotes the outward-pointing unit normal vector at* $\partial\Omega$.

2. *if* $\Omega=\mathbb{R}^d$*, then*

$$\frac{d}{dt}\iint\limits_{\Omega\times\mathbb{R}^{d-1}} f(t,x,v)\log(f(t,x,v))\,dxdv=\frac{d}{dt}H(t)\le 0. \tag{1.4.11}$$

3. *Inequalities 1.4.10 and 1.4.11 will become equalities if and only if*

$$f(t,x,v)=M_{\rho,u,\epsilon}(t,x,v),$$

   for some choice of $u(t,x):\Omega\times\mathbb{R}^d\longrightarrow\mathbb{R}^d$, $\rho(t,x):\Omega\times\mathbb{R}^d\longrightarrow\mathbb{R}^+$, $\epsilon(t,x):\Omega\times\mathbb{R}^d\longrightarrow\mathbb{R}^+$ *compatible with the initial data* $f_0(x,v)$.

Proof. This is only formal, as we haven't yet proven existence of solutions for the Boltzmann equation, nor defined how smooth do we need the solution to be. Starting from equation 1.4.8:

$$\frac{d}{dt}H(t)=-\int_{\Omega}\nabla_x\cdot\mathfrak{H}(t,x)\,dx+\iint\limits_{\Omega\times\mathbb{R}^d}D[f]\,dxdv\le-\int_{\Omega}\nabla_x\cdot\mathfrak{H}(t,x)\,dx,$$

where we used the fact that $D\le 0$ (item 1 of proposition 1.4.1). For item 1, we use the divergence theorem, and equation 1.4.10 is satisfied. For item 2, simply write

$$-\int_{\mathbb{R}^d}\nabla_x\cdot\mathfrak{H}(t,x)\,dx=\lim_{R\to\infty}-\int_{B_R}\nabla_x\cdot\mathfrak{H}(t,x)\,dx=\lim_{R\to\infty}-\int_{\partial B_R}\mathfrak{H}(t,x)\cdot\hat{n}\,d\mathcal{H}^{d-1}(x)=0$$

where we used again the divergence theorem, and the integrability assumption on $\mathfrak{H}(t,x)$. For item 3, we simply apply item 3 of proposition 1.4.1. □

For the unbounded case, when $\Omega=\mathbb{R}^d$, we see in the demonstration above that one can write that

$$\frac{d}{dt}H(t)=\iint\limits_{R^{2d}}D[f]\,dxdv,$$

or integrating in the variable t

$$\int_{\Omega\times\mathbb{R}^d} f(t,x,v)\log(f(t,x,v))\,dvdx=\int_{\Omega\times\mathbb{R}^d} f_0(x,v)\log(f_0(x,v))\,dvdx+\int_0^t\iint\limits_{R^{2d}}D[f]\,dxdv\,ds \tag{1.4.12}$$

Now in the bounded case, we would like to see if the boundary term $\int_{\partial\Omega} \mathfrak{H}(t,x)\cdot\hat{n}\, d\mathcal{H}^2(x)$ can be simplified somehow, or related to other, more physically significant expressions. In the case of linear walls, we can prove the following important corollary

Corollary 1.4.1. *Suppose that $f(t,x,v) \in L^\infty([0,\infty)_{loc}, L^1(\Omega\times\mathbb{R}^d)$ satisfies a linear boundary condition aside from the other conditions imposed in theorem 1.4.1. Then, defining the heat flux vector, internal energy and bulk velocity as we defined in subsection 1.3.3:*

$$q(t,x) = \rho(x,t)\int_{\mathbb{R}^d} \frac{\|v-u\|^2}{2}(v-u)\psi_x(v,t)\, dv, \quad U(x,t) = \frac{\rho(x,t)}{2}\int_{\mathbb{R}^3} \|v-u\|^2\psi_{t,x}(v)dv.$$

$$u(t,x) = \int_{\mathbb{R}^3} v\psi_{t,x}(v)dv.$$

Then, the following inequality holds

$$\mathfrak{H}(t,x)\cdot\hat{n} \geq -\frac{q(t,x)\cdot\hat{n}}{2e_w(t,x)} \tag{1.4.13}$$

with equality if and only if $\mathfrak{H}(t,x) = q(t,x) = 0$.

Proof. we apply lemma 1.3.1 in the case when f_w is a maxwellian distribution (equation 1.4.7). For any convex function $F : [0,\infty) \longrightarrow \mathbb{R}$, we have that

$$\int_{\mathbb{R}^d} (c\cdot\hat{n}) f_w(t,x,v) F\left(\frac{f(t,x,v)}{f_w(t,x,v)}\right) dv \geq 0.$$

Choose

$$F(x) = \begin{cases} x\log(x) & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The above equation then becomes

$$\int_{\mathbb{R}^d} (c\cdot\hat{n}) f(t,x,v)\log\left(\frac{f(t,x,v)}{f_w(t,x,v)}\right) dv \geq 0.$$

Separating the logarithm,

$$\int_{\mathbb{R}^d} (c\cdot\hat{n}) f(t,x,v)\log\left(f(t,x,v)\right)\, dv \geq \int_{\mathbb{R}^d} (c\cdot\hat{n}) f(t,x,v)\log\left(f_w(t,x,v)\right)\, dv$$

$$= \int_{\mathbb{R}^d} (c\cdot\hat{n}) f(t,x,v)\left[\log\left(\frac{\rho(t,x)}{\sqrt{4\pi e_w(t,x)}^d}\right) - \frac{\|v-u_w(t,x)\|^2}{4e_w(t,x)}\right] dv$$

$$= \frac{-1}{2e_w(t,x)}\hat{n}\cdot\left[\int_{\mathbb{R}^d} f(t,x,v)\frac{\|c\|^2}{2}c\, dv\right] = -\frac{q(t,x)\cdot\hat{n}}{2e_w(t,x)},$$

therefore,

$$\int_{\mathbb{R}^d} (c\cdot\hat{n}) f(t,x,v)\log\left(f(t,x,v)\right)\, dv = \mathfrak{H}(t,x)\cdot\hat{n} \geq -\frac{q(t,x)\cdot\hat{n}}{2e_w(t,x)}.$$

□

Corollary 1.4.2 (Formal narrow H-Theorem)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set with an orientable smooth boundary. Suppose that $f(t,x,v) \in L^\infty([0,\infty)_{loc}, L^1(\Omega\times\mathbb{R}^d)$ satisfies a linear boundary condition* *aside from the other conditions imposed in theorem 1.4.1. Then, it follows that*

$$\frac{d}{dt}H(t) \leq \int_{\partial\Omega} \frac{q(t,x)\cdot\hat{n}}{2e_w(t,x)}\, d\mathcal{H}^{d-1}(x). \tag{1.4.14}$$

If there is no heat flux through the wall, then $\frac{d}{dt}H(t) \leq 0$.

Proof. Simply take the equation 1.4.10 and use the previous corollary. One finds

$$\frac{d}{dt}H(t) \leq -\int_{\partial\Omega} \mathfrak{H}(t,x)\cdot\hat{n}\, dS \leq \int_{\partial\Omega} \frac{q(t,x)\cdot\hat{n}}{2e_w(t,x)}\, d\mathcal{H}^2(x).$$

□

The results of these theorems are striking and have a strong analogy with classical thermodynamics: If we recall that e_w is proportional to the temperature at the wall, equation 1.4.14[6] is a kinetic analogue of the Clausius-Duhem inequality of classical thermodynamics, which states that the entropy change of a gas in a vessel is related to the heat flux at the boundary by an inequality. And, by this interpretation, H should be a measure of entropy of the gas!

Remarks on the H-theorems

Boltzmann's paper in 1872 marks the first explicit mathematical connection between entropy and probability. In light of the information theory developed by Claude Shannon in the XXth century, some of the previous equations have another interesting interpretation: H is the symmetric of the information entropy of a continuous probability distribution, and if we look at equation 1.4.8

$$\frac{d}{dt}\int_\Omega h(t,x)\, dx = -\int_\Omega \nabla_x\cdot(u(t,x)h(t,x))\, dx - \int_\Omega \nabla_x\cdot\mathfrak{H}(t,x)\, dx + \iint_{\Omega\times\mathbb{R}^d} D[f]\, dxdv,$$

we see that: the first and second terms on the left hand side amount to rates of change in entropy due to convection (the first from material entering through the boundary and the second from heat coming through the boundary). The only term unrelated to convection is the last term, which can be negative even if the system is isolated, and must be related to entropy production inside the region Ω. If we inspect it further

$$\begin{aligned}\iint_{\Omega\times\mathbb{R}^d} D[f]\, dxdv &= \frac{-1}{4}\iiint_{\Omega\times\mathbb{R}^d}\int_{\mathbb{R}^3\times\mathbb{S}^2} (f'f'_* - ff_*)\log\left(\frac{f'f'_*}{ff_*}\right) B(w,\alpha)d\mathcal{H}^{d-1}(\alpha)\, dv_*\, dxdv \\ &= \frac{-1}{4}\iiint_{\Omega\times\mathbb{R}^d}\int_{\mathbb{R}^3\times\mathbb{S}^2} f'f'_*\log\left(\frac{f'f'_*}{ff_*}\right) B(w,\alpha)d\mathcal{H}^{d-1}(\alpha)dv_*\, dxdv \\ &+ \frac{-1}{4}\iint_{\Omega\times\mathbb{R}^d}\int_{\mathbb{R}^3\times\mathbb{S}^2} ff_*\log\left(\frac{ff_*}{f'f'_*}\right) B(w,\alpha)d\mathcal{H}^{d-1}(\alpha)dv_*\, dxdv\end{aligned}$$

In order to have a qualitative understanding of this term, we make the following definition:

Definition 1.4.1. Let $(\Omega \subseteq \mathbb{R}^N, \Sigma)$ be a measurable space, and let P, Q be two random variables defined over Ω with probability density functions $p(x), q(x)$ respectively. We define the Kullback-Leibler

[6]The Clausius-Duhem inequality is one of the classical statements of the second law of thermodynamics: it states that the entropy increase of a closed system is related to the heat flux at the boundary, such that

$$dS \geq \frac{d^*Q}{T},$$

with equality if the system undergoes a reversible transformation.

divergence or relative entropy between the distributions as

$$D(p\|q) = \int_{\Omega} p(x) \log\left(\frac{p(x)}{q(x)}\right) dx.$$

The Kullback-Leibler divergence can almost be regarded as a distance function from the theory of metric spaces, (it satisfies the triangle inequality and is non-negative, being 0 if and only if $f = g$ a.e.), but is not symmetric. However, the expression

$$d(p,q) = [D(p\|q) + D(q\|p)] \tag{1.4.15}$$

is clearly symmetric, defining a true distance between distributions p, q. with this definition, we can go back to our expression and say that

$$\iint_{\Omega\times\mathbb{R}^d} D[f]\, dxdv = \frac{-1}{4}\int_{\mathbb{S}^{d-1}} [D_\alpha(f'f'_*\|ff_*) + D_\alpha(ff_*\|f'f'_*)]\; d\mathcal{H}^{d-1}(\alpha), \tag{1.4.16}$$

where D_α is a weighted Kullback-leibler divergence, given by

$$D_\alpha(p\|q)(\cdot) = \int_{\mathbb{R}^{2d}} p(\cdot,x,v,v_*,\alpha) \log\left(\frac{p(\cdot,x,v,v_*,\alpha)}{q(\cdot,x,v,v_*,\alpha)}\right) B(v_*-v,\alpha)\, dxdvdv_*.$$

We recall from equation 1.3.1 that ff_* and $f'f'_*$ are the pair distributions of particles before and after a collision, respectively. Then, equation 1.4.16 states that the rate of entropy production is a negative constant times the distance between the pair distributions of particles pre- and -post collision, integrated over all possible collision orientations; hence a non-positive quantity. The rate of production can then be 0 if and only if

$$f'f'_* = ff_* \text{ almost everywhere.}$$

Item 3 of theorem 1.4.10 then implies the above expression holds if and only if f is a local Maxwellian.

Local Maxwellians can also be viewed in a new light after the H-theorem: since they make the entropy constant, yet they may not be equilibrium distributions (may be time dependent and inhomogeneous), they fit precisely with the concept of **local** thermodynamic equilibrium. If we inspect them further to see what is the macroscopic behavior of a gas whose distribution is $M_{\rho(t,x)\;u(t,x)\;\epsilon(t,x)}$, we find that, using the notation from subsection 1.3.3:

$$\begin{cases} M(t,x) = \int_{\mathbb{R}^3} (c\otimes c)\frac{\rho(t,x)}{\sqrt{4\pi\epsilon(t,x)}^3} e^{-\frac{\|c\|^2}{4\epsilon(t,x)}}\, dv = I\frac{2}{3}\rho\epsilon(t,x) = Ip(t,x) \\ q(t,x) = \int_{\mathbb{R}^3} c\frac{\|c\|^2}{2}\frac{\rho(t,x)}{\sqrt{\frac{4\pi}{3}\epsilon(t,x)}^3} e^{-\frac{3\|c\|^2}{4\epsilon(t,x)}}\, dv = 0 \end{cases}$$

where I is the identity matrix. Then, the 'parameters' $\{\rho(t,x)\; u(t,x)\; \epsilon(t,x)\}$ for our local maxwellian will satisfy the system of balance laws derived in subsection 1.3.3, which then becomes:

$$\begin{cases} \frac{\partial}{\partial t}[\rho(t,x)] + \nabla_x\cdot[\rho(t,x)u(t,x)] = 0, \\ \frac{\partial}{\partial t}[\rho(t,x)u(t,x)] + \nabla_x\cdot[\rho(t,x)u(t,x)\otimes u(t,x)] = -\nabla p(t,x) + \rho(t,x)W(t,x), \\ \frac{\partial}{\partial t}\left[\rho(x,t)\left(\epsilon + \frac{\|u\|^2}{2}\right)\right] + \nabla\cdot\rho(x,t)\left(\epsilon+\frac{\|u\|^2}{2}\right) = \rho(x,t)\langle W(x,t),u\rangle - \nabla\cdot(p(t,x)u)\,. \end{cases}$$

These are the compressible Euler equations for a perfect fluid in continuum mechanics! Therefore, at least for a monoatomic gas (i.e. where each particle has no internal degrees of freedom) of classical particles described by the Boltzmann equation, local thermodynamic equilibrium implies the macroscopic behavior of a perfect fluid. Any deviation from this behavior (e.g. viscous dissipation) implies then deviations from the local equilibrium assumption for the distribution f.

Since H is dimensionless and entropy has units of $\frac{\text{energy}}{\text{temperature}}$, Boltzmann proposed that

$$S \propto H$$

with a negative proportionality constant bearing the appropriate dimensions, so that S grows when H diminishes. This proportionality constant is precisely $-k_B$, where k_B is the Boltzmann constant of the gas.

In this way, the Clausius-Duhem inequality would be now 'proven', and H is non-increasing for isolated systems, which would imply S is increasing. Finally, by theorem 1.4.10 item 3, if S stops increasing, the distribution f must be a local Maxwellian, closely related to the Maxwell-Boltzmann distribution of a gas at equilibrium. All of these conclusions seem to point at a proof of the Second law of thermodynamics, with the Boltzmann equation as a starting point (indeed, Boltzmann's avowed goal when introducing his equation was to find a proof of the Second law of thermodynamics starting from microscopic considerations).

Among historians of science, there is a discussion about what was Boltzmann's original interpretation on the H-theorem as it was first published in 1872.

Boltzmann proved the theorem first for homogeneous gases (i.e. gases where the probability ditribution f was a function of (t, v) alone), and then extended the proof to the inhomogeneous case. It thus seem that although he imagined there could be exceptions to his theorem for inhomogeneous gases, he considered his theorem a valid justification for the second law of thermodynamics for homogeneous ideal gases, stating that: "(...)because of the atomic movement in systems consisting of arbitrarily many material points, there always exists a quantity which, due to these atomic movements, cannot increase" .

The desired interpretation would then be: From any initial configuration of a system of N classical particles, one can construct an initial distribution $f_0(t, x, v)$ accounting for uncertainty on their position and momentum. Then, letting it evolve according to Boltzmann equation with a linear boundary condition, its functional H will decrease if the system is isolated. After a sufficiently long time has passed, H will be close to its absolute minimum, and this would imply that its instantaneous distribution $f(t, x, v)$ becomes arbitrarily close to the equilibrium Maxwell-Boltzmann distribution.

There are, however, many physical and mathematical problems with this claim. From the mathematical viewpoint, the main issues are related to convergence to equilibrium:

- We have yet to provide an existence theorem for the Boltzmann equation (we will do so in the next chapter). We need the solution to be sufficiently regular so that the above calculations make sense, and have a sufficiently fast decay as $\|(x, y)\| \to \infty$ so that moments of order 0,1, and 2 of the distribution are finite, and $H(t)$ is finite. The solution must be defined globally in time in order to have any result of convergence to equilibrium.

- Let $f(t, x, v)$ be a solution of the Boltzmann equation describing a gas an adiabatic vessel with a linear boundary condition. Local Maxwellians are stationary points for $H(t)$, where $H(t)$ hasn't yet reached its minimum value. For convergence to equilibrium to happen, H must reach its global minimum, and not become 'stuck' at these stationary points. If f becomes a local Maxwellian at any time and remains a local maxwellian, $H(t)$ will become constant (therefore not reaching its minimum value) and f would not reach equilibrium. This can be avoided with a uniqueness theorem. Nevertheless, as noted first by H. Grad [36], in order to have convergence to equilibrium, one has to guarantee that all local Maxwellian solutions are not "attractors" except for the equilibrium Maxwel-Boltzmann distribution (in other words, f must be sufficiently "far" from any local maxwellian at all times, unless it started off as a local Maxwellian).

- Define the operator $H[f_t] = \iint_{\Omega \times \mathbb{R}^3} f_t(x, v) \log(f_t(x, v))\, dxdv$. It does **not** follow in general that

$$\lim_{t \to \infty} H[f_t] = H(g(x, v)) \Rightarrow \lim_{t \to \infty} f_t = g(x, v).$$

  in any mode of convergence.

- It is possible to choose initial and boundary conditions such that a gas exhibits behaviors forbidden by continuum mechanics (e.g. a gas being deformed and giving out work instead of consuming it) [36].

All of the above problems suggest that the conclusions of theorems (1.4.10),(1.4.11) may still be valid, but only for a special class of solutions. The physical objections are more subtle. The H-theorem caused great debate among physicists of his time, and was met with 2 famous objections, labeled in the literature as 'paradoxes'.

The first one comes from a letter written by Loschimdt in 1877, in response to a paper from Boltzmann in 1875 further extending his results. Loschimdt argued that, for any configuration of gas molecules which started with a low value for H_0 and evolved in time to another configuration with $H' < H$, one can reverse the velocities (essentially "play the film backwards") and have H grow as a consequence of dynamics.

The second one was given by Zermelo in 1896, and was known as the recurrence paradox. Zermelo's argument is based on a theorem by Henri Poincaré about dynamical systems, which we state below:

Theorem 1.4.2 (Recurrence theorem)**.** *Let (Γ, Σ, μ) be a measure space with $\mu(\Gamma) < \infty$ and $T : \mathbb{R}_+ \times \Gamma \to \Gamma$ be a flow on Γ (inducing a continuous-time dynamical system). Let $A \in \Sigma$ be any measurable set and $\tau > 0$ be an arbitrary time. Let*

$$R_{A,\tau} := \{x \in A \subseteq \Gamma : \forall t \geq \tau, T_t(x) \notin A\}. \tag{1.4.17}$$

Then,

$$\mu(R_{A,\tau}) = 0.$$

The hamiltonian equations of motion induce a flow phase space 1.0.4, which is itself unbounded, but if the system has a definite energy then it lies on a bounded submanifold in $\mathbb{R}^{6N}$ with finite Lebesgue measure, and so the theorem applies to a gas of classical particles. This theorem implies that aside from a set of measure 0, all points from the domain Γ (in our case a submanifold in phase space) will return arbitrarily close to their starting positions after a finite time as a consequence of dynamics.

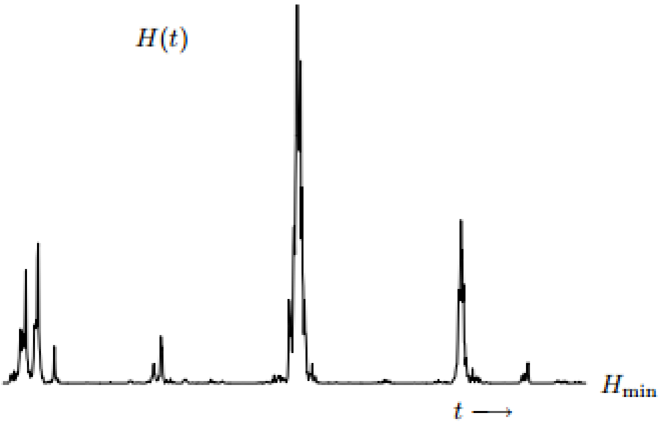
This, Zermelo argued, would mean that H would have to return arbitrarily close to its initial value after a finite time, thus creating a fundamental conflict between the H-theorem and classical mechanics. His argument is similar to the one given by Poincaré when discussing other kinetic theories. Poincaré stated that

> [t]he world tends at first towards a state where it remains for a long time without apparent change; and this is consistent with experience; but it does not remain that way forever, it the theorem cited above is not violated; it merely stays there for an enormously long time, a time which is longer the more numerous are the molecules. This state will not be the final death of the universe but a sort of slumber, from which it will awake after millions and millions of centuries. According to this theory, to see heat pass from a cold body into a warm one, it will not be necessary to have the acute vision, the intelligence and the dexterity of Maxwell""́ s demon; it will suffice to have a little patience (Brush 2003, p.380).

Boltzmann's reply to Loschmidt was based on the idea that Loschmidt's initial state was not typical. He conceded that any attempt to prove the second law of thermodynamics from microscopic interactions for arbitrary initial conditions would be "in vain" 1.4, but argued that there are 'infinitely many' more initial states in which H diminishes than states in which H increases for a real gas (though he did not provide a direct proof to this claim).

This can be seen in the definition of the *Stosszahlansatz*: in the heuristic derivation of the Boltzmann equation, we assumed velocities of particles before a collision were uncorrelated. Loschmidt's initial condition corresponds conversely to a particular correlation between the particle's initial velocities (as they were obtained by reversing post-collisional velocities), and therefore the Boltzmann equation (and the H-theorem) will not correctly describe this dynamics.

The reply to Zermelo even included a diagram of a typical 'H-curve' for an isolated gas contained in a vessel (we reproduce here a version similar to the original):



Consistently with the recurrence theorem (1.4.2), Boltzmann concedes the gas will spontaneously fluctuate out of equilibrium, and in this situation corresponds to 'peaks' in the H-curve (where H is far from its minimum). He argues then that the higher the peak, the less likely likely it is to happen (or to observe the system in this state), and at a given time instant the gas can be either

1. with H close to H_{min}, which will be by far the most common situation,
2. at a maximum, which means H will decrease,
3. at one of the ascending or descending sides of a 'peak', with equal probability.

Only one out of these 3 situations (being at the ascending side of a 'peak') actually contradicts the H-theorem, so most of the time it is satisfied (or putting it differently, for most initial conditions that can be chosen from this curve, H will decrease for a finite time interval shortly afterwards).

Finally, although recurrence does happen, Boltzmann argued that the recurrence time τ is too large to be observed in practice. In a back-of-the-envelope calculation, Boltzmann estimated that, for a sample of 10^9 molecules in a reservoir of about $1cm^3$, there would be at least 10^{10^9} possible ways to arrange these molecules in macrostates (since the number of states that a system can occupy in classical mechanics is infinite, he employed a discretization of phase space in order to have a definite answer). If the system has to pass through all other macrostates before recurring, assuming it visits 10^{27} macrostates per second, it would take $10^{10^9-27} \approx 10^{10^9}$ seconds for it to exhibit recurrence behavior, a time scale much greater than the age of the universe and beyond the validity of his equation [37].

He did not, however, provide a justification for assuming the system would pass through all other macrostates before recurring. If we assume the Ergodic hypothesis is valid, then it is conceivable that the system could exhibit this behavior, although it is not required to do so.

These arguments seem to point out that Boltzmann may have changed his interpretation of the H-theorem and his equation over time [37]. Rather than being fundamental, they would reflect only the most probable behavior to be observed in a gas, and their validity should be restricted to certain time-scales and initial conditions [7]. However, no formal mathematical proof of this 'statistical interpretation of the H-theorem' was given [36].

It would take nearly a century for a more complete resolution for this apparent conflict between classical mechanics and the H-theorem to appear.

1.5 The Liouville Equation

Since our goal in this chapter is to highlight the physical origin of the Boltzmann equation before presenting analytical results, we will not present proofs for lemmas and theorems in this section.

It is clear that, in order to have a better grasp at the limitations and meaning of the Boltzmann equation and theorem 1.4.10, one would need to understand what are the implications of the hypotheses made in subsection 1.2. Here, we present a formal argument and a theorem to achieve this goal.

Since the underlying mechanical system evolves in time according to classical mechanics, it seems reasonable that an evolution law for a probability density could be derived directly from it. Define a probability density P^N over Γ_N (the phase space of all particles). Given $B \subseteq \Gamma_N$, we consider the following balance law for P^N

$$\frac{d}{dt}\int_B P^N(t,x,v)\,dxdv = -\int_{\partial B} P^N(t,x,v)\,\dot{\gamma}(t)\cdot dS,$$

where $\dot{\gamma} = (\dot{x}, \dot{\mathbf{v}})$ is the drift term in phase space, and the dynamics of the system carries probability along the phase space much as a fluid carries momentum or heat. In writing the balance law, we implicitly assume that no probability is produced, in the sense that no state in the phase space can be created - every state must come from a pre-existing one, which means probability must be transported across Γ_N. In differential form, this yields the classical continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\dot{\gamma}\rho) = 0. \tag{1.5.1}$$

Taking into account the fact that our dynamics is classical, we can use Hamilton's equations of motion for the drift term:

$$\dot{\gamma}_{ij} = \begin{bmatrix}\dot{x}_{3i+j}\\ \dot{p}_{3i+j}\end{bmatrix} = \begin{bmatrix}\frac{\partial H}{\partial p_{3i+j}}\\ -\frac{\partial H}{\partial x_{3i+j}}\end{bmatrix} = \begin{bmatrix}\dot{p}_{3i+j}\\ F_j(x_{3i+1}, x_{3i+2}, x_{3i+2})\end{bmatrix} = j$$

Hence

$$\nabla\cdot j = \sum_{i=1}^{N}\sum_{j=1}^{3}\frac{\partial}{\partial x_{3i+j}}\frac{\partial H}{\partial p_{3i+j}} - \frac{\partial}{\partial p_{3i+j}}\frac{\partial H}{\partial x_{3i+j}} = 0$$

Where i and j run over all the particles and each particle's coordinate components, respectively. This is proves the Liouville Theorem for classical mechanics, which states that volume in the phase space is preserved by the dynamics (the 'flow' of probability in phase space is incompressible). Our equation now reads

$$\frac{\partial P^N}{\partial t} + \nabla P^N\cdot\dot{\gamma} = 0 \tag{1.5.2}$$

Or, inserting our expression for $\dot{\gamma}$:

$$\nabla P^N\cdot\dot{\gamma} = \sum_{i=1}^{N}\sum_{j=1}^{3}\frac{\partial P^N}{\partial x_{3i+j}}\dot{x}_{3i+j} + \frac{\partial P^N}{\partial p_{3i+j}}\dot{p}_{3i+j}$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{3}\frac{\partial P^N}{\partial x_{3i+j}}\frac{\partial H}{\partial p_{3i+j}} - \frac{\partial P^N}{\partial p_{3i+j}}\frac{\partial H}{\partial x_{3i+j}} \equiv \{P^N, H\} \tag{1.5.3}$$

where $\{P^N, H\}$ is what's commonly defined as the Poisson Bracket operator. Our equation then reads

$$\frac{\partial P^N}{\partial t} + \sum_{i=1}^{N}\sum_{j=1}^{3}\frac{\partial P^N}{\partial x_{3i+j}}\frac{\partial H}{\partial p_{3i+j}} - \frac{\partial P^N}{\partial p_{3i+j}}\frac{\partial H}{\partial x_{3i+j}} = \frac{\partial P^N}{\partial t} + \{P^N, H\} = 0$$

Mathematically, this corresponds to the companion transport equation to the ODE system 1.0.3.

In this second form, the equation is usually called the Liouville equation; and it provides the correct way to let P^N evolve in time classicaly. Though certainly better than solving the problem of the motion of individual particles, the solution to the Liouville equation will be a function of $N \sim 10^{24}$ coordinates, which is still impractical to solve.

1.5.1 The BBGKY hierarchy

It is possible to find a more rigorous derivation to the Boltzmann Equation than the one presented previously, taking the Liouville Equation as a starting point. First, define for $s \leqslant N$

$$P^{(s)}(x_1, v_1, x_2, v_2, ..., x_s, v_s) = \iint\limits_{(\Omega\times\mathbb{R}^3)^{N-s}} P^N(x_1, v_1, x_2, v_2, ..., x_N, v_N) \prod_{i=s+1, j=1}^{i=N, j=3} dx_{3i+j} dp_{3i+j}$$

$P^{(s)} : \Gamma_s \longrightarrow \mathbb{R}_+$ is called a **reduced density of order s,** and is obtained simply by marginalizing the positions and velocities of particles having with an index greater than s [7].

Definition 1.5.1. We define the operator $E_s : L^1(\Gamma_N) \longrightarrow L^1(\Gamma_s)$ as the operation connecting P^N and $P^{(s)}$, that is, given $\phi(x_1, v_1, x_2, v_2, ..., x_N, v_N) \in L^1(\Gamma_N)$:

$$E_s[\phi](x_1, v_1, x_2, v_2, ..., x_s, v_s) = \iint\limits_{(\Omega\times\mathbb{R}^3)^{N-s}} \phi(x_1, v_1, x_2, v_2, ..., x_N, v_N) \prod_{i=s+1, j=1}^{i=N, j=3} dx_{3i+j} dp_{3i+j},$$

$$E_s[P^N] = P^{(s)}(x_1, v_1, x_2, v_2, ..., x_s, v_s).$$

We see that, if we take $s = 1$, this becomes the probability density distribution of a single typical particle in the gas, which is what the Boltzmann equation attempts to model. If this connection is indeed valid (that we can replace $P^{(1)} = f$, where f is the unknown in Boltzmann's equation), then we have two evolution laws at our disposal: the Liouville equation for P^N, and the Boltzmann equation for $P^{(1)}$.

Now, assume we are given an initial condition P_0^N to Liouville's equation in $\Gamma_N = \mathbb{R}^{2dN}$, and an initial condition f_0 to the Boltzmann equation in $\mathbb{R}^{2d}$ (in the physical case, $d = 3$). Suppose further that both initial conditions are compatible: $E_1[P_0^N] = f_0$. We can let both initial conditions evolve according to their respective equations up to some time t, yielding P_t^N and f_t, respectively. Now we pose ourselves the question that motivated Lanford's original investigation: are they still compatible after time elapses, i.e. $E_1[P_t^N] = f_t$? This would mean that the following diagram commutes:

$$\begin{array}{ccc} P_0^N & \xrightarrow{E_1} & f_0 \\ \big\downarrow{\scriptstyle \text{Liouville}} & & \big\downarrow{\scriptstyle \text{Boltzmann}} \\ P_t^N & \xrightarrow{E_1} & f_t \end{array}$$

This is, as with most optimistic guesses, false in general. Understanding which conditions would make this diagram commute is our goal in this section. What Lanford proposed was a weaker version of the above diagram: if we introduce a distance d in the set of probability measures, assume that $d(f_0, E_1(P_0^N))$ is small. In this case, can we conclude that $d(f_t, E_1(P_t^N))$ is small in a sense? this question too has a negative answer, as a consequence of Loschmidt's paradox [38]. What Lanford discovered is the following

Proposition 1.5.1. *(Idea) Let $P_0^N \in L^1(\mathbb{R}^{2dN}; \mathbb{R}_+)$ be an initial condition to the Liouville equation, and $f_0 \in L^1(\mathbb{R}^{2d}; \mathbb{R}_+)$ an initial condition to the Boltzmann equation. Then, given a certain distance in $L^1(\mathbb{R}^{2d}; \mathbb{R}_+)$, if f_0 and P_0^N are compatible with respect to that distance (in a sense which will be*

[7] we assume our probability density P^N is symmetric with respect to permutation of particle indices,i.e., that particles are indistinguishable. Therefore, when integrating over all indices greater than s, we do not need to consider the combinatorial problem of choosing s particles among the total N.

made precise later), then there is a constant $\tau > 0$ such that for each $t \in [0, \tau]$, ***with overwhelming probability*** *P_t^N and f_t are compatible (in the sense that $d(E_1(P_t^N), f_t)$ is small).*

We now wish to make the above proposition mathematically precise. From the Liouville equation, we can find an evolution equation for the reduced density $P^{(s)}$, which could reduce the complexity of solving the full Liouville equation by giving us a reduced description of the behavior of s typical particles in our gas, moving through a system of N particles.

If we apply this integral operator E_s to both sides of the Liouville equation, the result is the following system of coupled partial differential equations:

$$\frac{\partial P^{(s)}}{\partial t} + \{P^{(s)}, H\}_s = -(N-s) \sum_{j=1}^{3} \frac{\partial}{\partial p_{3(s+1)+j}} \iint\limits_{\Omega \times \mathbb{R}^3} F_j^{i\, s+1}(\mathbf{x}_i, \mathbf{x}_{s+1}) P^{(s+1)}\, dx_{3(s+1)+j} dp_{3(s+1)+j} \quad (1.5.4)$$

for $s \leqslant N$ (see [23] for a derivation of this result). This is known as the **BBGKY Hierarchy of equations**[8]. Its connection to the Boltzmann equation is more readily seen in the case when the particle interaction is modelled as elastic collisions between hard spheres. In this case, the above hierarchy becomes,

$$\frac{\partial P^{(s)}}{\partial t} + \{P^{(s)}, H\}_s$$
$$= (N-s)\sigma^2 \sum_{i=1}^{s} \int\limits_{\mathbb{S}^2 \times \mathbb{R}^3} \left[P^{(s+1)'} - P^{(s+1)} \right] |(v_i - v_{s+1}) \cdot \alpha| d\mathcal{H}^2(\alpha) dv_*; \quad (1.5.5)$$

which, for $s = 1$ has an uncanny resemblance to the Boltzmann equation, if we recall that one of our assumptions when establishing the heuristic derivation of Boltzmann's equation was the *Stosszahlansatz*, that $P^{(2)}(t, x, v, x, v_*) = f(t, x, v) f(t, x, v_*)$ (see section 1.2). Notice that in this case, the phase space occupied by the particles is given by

$$\Gamma_N^{\sigma \neq} = \{\gamma \in (\Omega \times \mathbb{R}^d)^N \ s.t. \ \|x_i - x_j\| \geq \sigma \ \text{ for each } 1 \leq i, j \leq N\}.$$

Therefore, the naive connection that $E_1(P^N)$ is the unknown in Boltzmann's equation is false, as the evolution equation $E_1(P^N)$ obeys is not (1.2.1), but rather a system of coupled PDEs, and in the case the *Stosszahlansatz* holds, the first equation of this system 'decouples' from the rest, becoming the Boltzmann equation.

Instead of simply imposing the *Stosszahlansatz* in the system (1.5.5), we would like to present a more physical condition implying this factorization. In the case of a system with negligible internal forces in thermodynamic equilibrium, in the limit of large N, the *Stosszahlansatz* can be verified to hold, as done in Appendix A.5. This signals that the right condition to impose should come from an analysis of the behavior of the $BBGKY$ hierarchy in the limit when $N \to \infty$, in which case we may hope that the first equation in this hierarchy approximates the Boltzmann equation.

Another way to justify the thermodynamic limit $N \to \infty$ would be the following: the Boltzmann equation has a built-in irreversibility (manifested in the H-theorem) that is incompatible with the reversibility of Liouville equation. The latter reversibility appears in the fact that the Hamiltonian flow in Γ_N preserves the Lebesgue measure (Liouville's theorem). So in order to obtain Boltzmann's equation from the hierarchy 1.5.5 (which is equivalent to Liouville's equation), one has to go to a limiting regime in which Liouville's theorem 'fails', and one way to accomplish this is to take the limit $N \to \infty$ of $P^{(1)}$(notice that in this limit, phase-space volume loses meaning [7]).

In this limit, however, for the volume occupied by the particles to be finite, we must also have the particle diameter $\sigma \to 0$. Then, we can break this limit in a few cases:

[8]named after 5 scientists: Nikolay Nikolaevich Bogolyubov (1909–1996) Russian mathematician and physicist; Max Born (1882 – 1970) German physicist; Herbert Sydney Green (1920 – 1999) British physicist; John Gamble Kirkwood (1907 – 1959) American chemist and physicist; and Jacques Yvon, French scientist.

- If If $N\sigma^3 \nrightarrow 0$, then the total volume occupied by the particles is not negligible, so the gas cannot be considered ideal (consider for instance the Wan der Waals equation of state, which describes a non-ideal gas by including in its expression an "exclusion volume" occupied by the molecules).

- From the Maxwellian distribution for gases in equilibrium, we can calculate the average thermal relative velocity $|w_T|$ from the standard deviation of the distribution. Then, if we construct a cylinder of radius σ and height $|w_T|$, this will describe the volume a particle traverses on average per unit time relative to the others. By multiplying this by the number density of particles, one finds the average number of particles a single particle collides with per unit time, which is the collision frequency ν_T[23]:
$$\nu_T = |w_T|\pi\sigma^2\overline{n}$$
The average relative velocity $|w_T|$ is related to the average thermal speed $|v_t|$ by $|w_T| = \sqrt{2}|v_t|$, so we can find the mean free path of the particles as
$$\lambda_T = \frac{|v_T|}{\nu_T} = \frac{|v_T|}{|w_T|\pi\sigma^2\overline{n}} = \frac{V}{\sqrt{2}\pi\sigma^2 N}.$$
If $N\sigma^2 \to 0$, then collisions are negligible, since that would make the mean free path between collisions $\lambda_T \to \infty$.

Therfore, if we hope to describe a gas in which the occupied volume is negligible (so that it can still be considered ideal), but collisions are not negligible as a limit from the BBGKY hierarchy, we have to consider the case in which
$$N\sigma^2 = \kappa, \quad N \to \infty, \tag{1.5.6}$$
this limit is called the **Boltzmann-Grad limit**.

Definition 1.5.2. Let $\{f_0^{(n)} \in L^1(\mathbb{R}^{2n}; R_+)\}_{\{n\in\mathbb{N}\}}$ be a sequence of functions. The sequence $\{f^{(n)} \in L^1([0,\tau] \times \mathbb{R}^{2n}; R_+)\}_{\{n\in\mathbb{N}\}}$ is said to be a weak solution to the Boltzmann Hierarchy with initial condition $\{f_0^{(n)}\}_{\{n\in\mathbb{N}\}}$ if for each $n \in \mathbb{N}$ $f^{(n)}(0,\cdot) = f_0^{(n)}(\cdot)$, and $\{f^{(n)}\}_{\{n\in\mathbb{N}\}}$ solves the following (infinite) system of differential equations in the weak sense:
$$\frac{\partial f^{(s)}}{\partial t} + \{f^{(s)}, H\}_s$$
$$= \kappa \sum_{i=1}^{s} \int_{\mathbb{S}^2\times\mathbb{R}^3} \left[f_i^{(s+1)'} - f^{(s+1)}\right] |(v_i - v_{s+1}) \cdot \alpha| d\mathcal{H}^2(\alpha) dv_*; \tag{1.5.7}$$
where
$$\begin{cases} f_i^{(s+1)'} = f(x,v_1, x_2, v_2, ..., x_i, v_i', ..., x_{s+1}, v_{s+1}'), \\ f^{(s+1)} = f(x,v_1, x_2, v_2, ..., x_i, v_i, ..., x_{s+1}, v_{s+1}), \end{cases}$$
and primed (post-collisional) velocities are to unpimed (pre-collisional) velocities by
$$\begin{cases} v_i' = v_i + \langle v_{s+1} - v_i, \alpha\rangle\, \alpha, \\ v_{s+1}' = v_{s+1} - \langle w, \alpha\rangle\, \alpha, \end{cases}$$

Lemma 1.5.1. *Let $f(t, x, v) \in C((0,T) \times L^1(\mathbb{R}^{2d})$ be a weak solution to the Boltzmann equation 2.1.1 with initial datum $f_0 \in L^1(\mathbb{R}^{2d})$, for the hard-sphere collision kernel ($B(v_* - v, \alpha) = |\langle v_* - v, \alpha\rangle|$). Then, the functions*
$$f^{(s)}(t, x_1, x_2, ..., x_s, v_1, v_2, ..., v_s) = \prod_{i=1}^{s} f(t, x_i, v_i)$$

are weak solutions to the Boltzmann hierarchy with initial data

$$f_0^{(s)}(x_1, x_2, ..., x_s, v_1, v_2, ..., v_s) = \prod_{i=1}^{s} f_0(x_i, v_i)$$

and $\kappa = 1$.

In the next chapter, we will focus on the existence theory for the Boltzmann equation, and we will prove a theorem (2.1.1) that guarantees the existence of weak solutions. Therefore, by the above lemma, we will also have weak solutions to the Boltzmann hierarchy defined above. If we can prove that in the Boltzmann-Grad limit, solutions to the BBGKY hierarchy are 'close' to solutions to the Boltzmann Hierarchy, proposition 1.5.1 will be complete, and we will have a robust justification for using the Boltzmann equation.

1.5.2 Lanford's Theorem

The derivation of the theorems in this subsection is delicate, and exceeds the scope of this dissertation. For a comprehensive review on this topic, we recommend the reading of [18] and [40]. We start by presenting some definitions:

Definition 1.5.3. The following norms will be used in this subsection: given $\epsilon \geq 0, \beta > 0$, for $g_s(z) : \Gamma_s^{\sigma \neq} \longrightarrow \mathbb{R}_+$, define

$$\|g_s\|_{\epsilon,\beta} = \sup_{z \in \Gamma_s^{\sigma \neq}} |g_s(z) e^{\beta H_\epsilon(z)}|,$$

where $H_\epsilon(z)$ is the system's hamiltonian, given by

$$H_\epsilon(x_1, v_1, x_2, v_2, ..., x_s, v_s) = \sum_{i=1}^{s} \frac{\|v_i\|^2}{2} + \frac{1}{\epsilon} \sum_{i=1}^{s} \sum_{j \neq i} F\left(\frac{x_i - x_j}{\epsilon}\right).$$

The case when $\epsilon = 0$ corresponds to the free hamiltonian case.

Definition 1.5.4. For $p \in \mathbb{R}^{2d}$, let $\Delta(p) = \{x \in \mathbb{R}^{2d} : x_i \leq p_i, \, i \in (1, 2, ..., 2d)\}$. Given a probability measure ν defined over $B \times \mathbb{R}^d$, where $B \subseteq \mathbb{R}^d$ is an open set, the cumulative distribution of ν is given by $F_\nu : B \times \mathbb{R}^d \longrightarrow [0, 1]$, $F_\nu(p) = \nu(\Delta(p))$.

Let ν, μ be 2 probability measures over the measurable set $(B \times \mathbb{R}^d, \mathcal{A})$ (where $\mathcal{A}$ denotes the Borel σ-algebra). The **discrepancy** between ν, μ is a metric over the set of probability measures, given by

$$d(\nu, \mu) = \sup_{p \in B \times \mathbb{R}^d} |F_\nu(p) - F_\mu(p)| = \|F_\mu - F_\nu\|_{L^\infty(B \times \mathbb{R}^d)}$$

content

Lemma 1.5.2. *With the same notation as in the above definition, let* $\{\mu_n\}_{n \in \mathbb{N}}$ *be a sequence of probability measures over* $B \times \mathbb{R}^d$*, and* μ *be an absolutely continuous probability measure with respect to the Lebesgue measure over* $B \times \mathbb{R}^d$*. Then*

$$\mu_n \overset{*}{\rightharpoonup} \mu \Leftrightarrow \lim_{n \to \infty} d(\mu_n, \mu) = 0$$

Definition 1.5.5. A sequence of probability measures $\{P^{(N)}\}_{N \in \mathbb{N}}$ over $\Gamma_N^{\sigma \neq}$ approximates $f \in L^1(\mathbb{R}^{2d}; \mathbb{R}_+)$ **in discrepancy** if for each $\epsilon > 0$

$$\lim_{\substack{N \to \infty \\ BG}} P^N \left(z \in \Gamma_N^{\sigma \neq} : \, d(\omega^N(z), f) > \epsilon \right) = 0,$$

where

$$\omega^N(z) = \frac{1}{N} \sum_{i=1}^{N} \delta\{x_i, v_i\}$$

is the empirical distribution at the point $z = (x_1, v_1, ..., x_N, v_N)$.

The empirical distribution has its name for the fact that it would be the real one-particle distribution of a gas, were the positions and momenta of the total N particles exactly known. Its cumulative distribution is piecewise constant, so that it is a discrete probability distribution, with distribution function

$$F_{\omega^N(z)}(x) = \frac{1}{N}\sum_{i=1}^{N} \chi_{\{z_i \leq x_i\}}.$$

combining lemma 1.5.2 with the previous definition, one can conclude that

Lemma 1.5.3. *The family $\{P^{(N)}\}_{N\in\mathbb{N}}$ of probability measures over $\Gamma_N^{\sigma\neq}$ approximates $f \in L^1(\mathbb{R}^{2d};\mathbb{R}_+)$ in discrepancy if and only if*

$$P^{(s)}(x_1, v_1, x_2, v_2, ..., x_s, v_s)\prod_{i=1}^{s} dx_i dv_i = E_s[P^N]\prod_{i=1}^{s} dx_i dv_i \underset{N \underset{BG}{\to} \infty}{\overset{*}{\rightharpoonup}} \prod_{i=1}^{s} f(x_i, v_i) dx_i dv_i \quad \text{for all } s \in \mathbb{N}$$

(weak convergence of in the sense of measures).

Theorem 1.5.1 (Lanford (1975))**.** *Let $\{P_0^{(N)}\}_{N\in\mathbb{N}}$ be a sequence of probability densities over $\Gamma_N^{\sigma\neq}$, and $\{f_0^{(s)}(x_1, v_1, ..., x_s, v_s)\}_{s\in\mathbb{N}} = \prod_{i=1}^{s} f_0(x_i, v_i)$ a sequence of factored initial data for the Boltzmann hierarchy, where $f_0 \in L^1(\mathbb{R}^{2d}; R_+)$. Assume that the following holds*

1. *for some $b, C, \beta > 0$:*
$$\|P_0^N\|_{0,\beta} \leq Cb^N$$
2. *$\{P_0^{(N)}\}_{N\in\mathbb{N}}$ is continuous, and*
$$P_0^{(s)} \to f_0^{(s)}$$
as $N \to \infty$ in $L^\infty_{loc}(\Gamma_s^{\sigma\neq})$ for each $s \in \mathbb{N}$, $\sigma > 0$.

Then, if $\{P_0^{(N)}\}_{N\in\mathbb{N}}$ approximates f_0 in discrepancy, there is a constant $\tau > 0$ such that, for each $t \in (0, \tau)$, the weak solution to the BBGKY hierarchy $\{P_t^{(s)}\}_{s\in\mathbb{N}}$ exists and is unique, and approximates a weak solution to the Boltzmann equation f_t in discrepancy.

The (much) more intrincate case of when particles interact through a short range potential instead of hard collisions, known as King's theorem (1983) is done in great detail in [18]. Currently there is no similar result for long range potentials (potentials which are not truncated at a finite distance from the origin). This theorem guarantees that if the initial data for the BBGKY hierarchy converges weakly to a factorized distribution, there is a short time-scale in which the time evolution of the BBGKY hierarchy (which is the true system of equations followed by the system) still factorizes in the Boltzmann-Grad limit, and this limit is exactly the time evolution of the Boltzmann equation. This means that the factorization property of the distribution (*Stosszahlansatz*) is preserved under the time evolution for short times (a property known as 'propagation of chaos' in the literature).

Since the $N \to \infty$ limit is never attained, the weak convergence can be translated as such: Take an initial condition to the Boltzmann equation f_0, and a sequence of initial conditions to the Liouville equation $\{P^N\}_{\{N\in\mathbb{N}\}}$. Assume that the sequence satisfies the following property:

- for each $\epsilon > 0$, there is an N_0 such that, if $N > N_0$, for any set A_s in the phase space Γ_s, the difference $P_0^{(s)}(A_s) - f_0^{(s)}(A_s)$ can be no larger than ϵ in absolute value. $P_0^{(s)} = E_s(P^N)$ and E_s and $f_0^{(s)}$ are defined as previously, in 1.5.1 and 1.5.1.

Then, there is a short time scale $\tau > 0$ such that the same property holds between f_t and P_t^N (which are the time evolutions of f_0 by the Boltzmann equation and P_0^N by the Liouville equation, respectivelly).

The theorem can be equivalently phrased in terms of the empirical distribution function: given that the initial data follows the above property, let $d(\omega_z^N, f_t)$ be the discrepancy between the solution to the Boltzmann equation and an empirical one-particle distribution, where the particles are located at the state z, and z is distributed according to P_t^N. For each $\delta > 0$ and $\epsilon > 0$, there is $N_0 \in \mathbb{N}$ such that, the probability of the event ($d(\omega_z^N, f_t)$ is larger than ϵ) is bounded by δ. In other words, it is possible to replace the true empirical distribution by f_t, and the error of doing so (discrepancy) will be small with high probability, if the same error was small in the initial condition, and the total number of particles N is large enough. Thus, the theorem captures Boltzmann's intuition that his equation measures the 'most probable' behavior observed in the gas.

This theorem shows that rather than being a toy model used to describe gases, the Boltzmann equation is indeed a limiting behavior of the time evolution of the one particle distribution, whose evolution can be found exactly from the Liouville equation. Therefore, conclusions taken from the Boltzmann equation, such as the H-theorem and the macroscopic balance laws, hold approximately for $P^{(1)}(t, x, v)$ for short times, vindicating Boltzmann's statistical reading of the H-theorem [37].

The theorem has several limitations, however, the main ones being:

- The lack of control of the limit: N is usually a fixed quantity rather than a variable parameter. Although large, it would be necessary to know how fast the limit converges as $N \to \infty$, in order to know if typical system sizes are enough for this approximation to be reasonable,

- the restriction to short range interparticle potentials,

- the restriction for short times: it is posible to estimate the time scale τ during which the theorem holds, as being roughly $\frac{2}{5}$ of the mean free time of the gas [37]. For gases at standard pressure and room temperature, this time scale is of the order of microseconds.

With this, we conclude our survey on the physical meaning and properties of the Boltzmann equation, and move to the mathematical study of its existence theorems.

Chapter 2

Solvability of the Cauchy problem for the Boltzmann equation

Boundary conditions, like field equations, are proposed by theorists who dare to represent nature by mathematical hypotheses (...) In framing boundary conditions, just as in framing field equations, the theorist outlines Nature as best he can from what little of herself she lets him see through the fogs with which she covers her sincerity. To do so, he follows the forms and practices that his masters, the great theorists of old, have taught him by example (...) Like his great forebears, he runs the risk that solutions of the kind he analyses may not exist: that all his labor may be spent on describing one of the countless attributes of the null set. *Clifford Truesdell in [36]*

2.1 Diperna-Lions Theory

In this chapter, we will review some a known result about the existence theory of the Boltzmann equation by [12]. Although there are many other existence results for the Boltzmann equation, the class of renormalized solutions defined by [12] is the only result which guarantees existence of solutions globally in time and for a broad class of initial conditions, which sets this result apart in the theory [39]. Given its importance, we will give a proof of this theorem below, and further results will be mentioned at the end of the text. Consider the following initial value problem (Boltzmann equation without external forcing) with the unknown $f : \mathbb{R}^+ \times \mathbb{R}^{2d} \longrightarrow \mathbb{R}^+$:

$$\begin{cases} \frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^{2d} \\ f(0,x,v) = f_0(x,v) & \text{in} \quad \mathbb{R}^{2d} \end{cases} \tag{2.1.1}$$

where $Q(f,f)(t,x,v) = \iint_{\mathbb{R}^d \times \mathbb{S}^{d-1}} (f' f'_* - f f_*)\, B(\alpha, v_* - v)\, d\mathcal{H}^{d-1}(\alpha)\, dv_*$. f', f'_*, f_* are shorthand for $f(t,x,v')$, $f(t,x,v'_*)$, $f(t,x,v_*)$ respectively, and the pairs (v, v_*) and (v', v'_*) are respectively the velocities of 2 particles before and after a binary collision, which are related by the expression

$$\begin{cases} v' = v + \langle w, \alpha \rangle\, \alpha, \\ v'_* = v_* - \langle w, \alpha \rangle\, \alpha, \end{cases}$$

$\alpha \in \mathbb{S}^{d-1}$. The collision Kernel $B(\alpha, v_* - v)$ is a smooth non-negative function, which takes into account the interparticle interaction potential. Restrictions on the integrability of B are needed in order to derive any existence results for this problem.

2.1.1 Solution types and main theorem

First we wish to define and classify a few concepts of solution to the above initial value problem.

Definition 2.1.1. A function f is said to be a weak (or distributional) solution to equation 2.1.1 with initial datum $f_0 \in L^1(\mathbb{R}^{2d})$ if, for every $\phi \in C_0^\infty([0,\infty) \times \mathbb{R}^d)$ the following holds:

$$\int_0^T \int_{\mathbb{R}^{2d}} -f\left[\frac{\partial \phi}{\partial t} + v \cdot \nabla_x \phi\right] dxdvdt - \int_{\mathbb{R}^{2d}} f_0(x,v)\phi(0,x,v)\,dxdv = \int_0^T \int_{\mathbb{R}^{2d}} \phi Q(f,f)\,dxdvdt, \quad (2.1.2)$$

where we also require $Q(f,f) \in L^1_{loc}([0,\infty) \times \mathbb{R}^{2d})$ so that it is well defined.

This is not, however, the main type of solution we'll concern ourselves with. A very useful type of solution, described by the first time in [12], is the following

Definition 2.1.2 (Renormalized solution)**.** Let $\beta \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, $\beta(0) = 0$ be a function satisfying, for any $C > 0$,

$$0 < \beta'(x) \leq \frac{C}{1+x}.$$

$f \in L^1_{loc}([0,\infty); L^1(\mathbb{R}^{2d}))$ is called a *renormalized solution* to problem 2.1.1 with initial condition $f_0 \in L^1(\mathbb{R}^{2d})$ if it solves the following equation in the sense of distributions

$$\frac{\partial \beta(f)}{\partial t} + v \cdot \nabla_x \beta(f) = \beta'(f)Q(f,f), \quad (2.1.3)$$

$$\frac{Q(f,f)}{1+f} \in L^1_{loc}([0,\infty) \times \mathbb{R}^{2d}).$$

The prototypical choices for β are $\beta_\delta(t) = \frac{1}{\delta}\log(1+\delta t)$ and $\beta_\delta(t) = \frac{t}{1+\delta t}$, for some $\delta > 0$.

The next solution concept comes from Duhamel's principle: from the characteristic map $w = x - vt$, one can define for any $g : \mathbb{R} \times \mathbb{R}^{2d} \longrightarrow \mathbb{R}$ the composition $g^\#(t,w,v) = g(t, w + vt, v)$, and notice that

$$\frac{d}{dt} f^\#(t,w,v) = \left[\frac{\partial}{\partial t} + v \cdot \nabla_x\right] f(t, w + vt, v).$$

As we remarked in section 1.2, this means that along characteristics the equation reads

$$\frac{d}{dt} f^\#(t,w,v) = Q^\#(f,f)(r,x,v),$$

and so we define

Definition 2.1.3 (Mild solution)**.** Let $f : \mathbb{R}_+ \times \mathbb{R}^{2d} \longrightarrow \mathbb{R}^+, f_0 : \mathbb{R}^{2d} \longrightarrow \mathbb{R}^+$ be measurable functions. f is a *mild solution* if for almost every (x,ξ)

$$f^\#(t,x,\xi) - f_0^\#(x,\xi) = \int_0^t Q^\#(f,f)(r,x,\xi)\,dr \quad \text{(for each } 0 < t < \infty), \quad (2.1.4)$$

$$\text{where} \quad Q^\#(f,f)(r,x,\xi) \in L^1(0,T) \quad (\forall T > 0).$$

In order to justify the next definition, we now perform a formal calculation. The collision operator has a natural decomposition into a 'gain' term and a 'loss' term (recall equations 1.3.2, 1.3.3), denoted respectively by $Q^+(f,f)$ and $Q^-(f,f)$, and defined by

$$\begin{cases} Q^+(f,f)(t,x,v) = \displaystyle\iint_{\mathbb{R}^d \times \mathbb{S}^{d-1}} f' f'_* B(\alpha, v_* - v)\,d\mathcal{H}^{d-1}(\alpha)\,dv_*. & (2.1.5a) \\ Q^-(f,f)(t,x,v) = \displaystyle\iint_{\mathbb{R}^d \times \mathbb{S}^{d-1}} f f_* B(\alpha, v_* - v)\,d\mathcal{H}^{d-1}(\alpha)\,dv_*. & (2.1.5b) \end{cases}$$

In the loss term, since none of the integrations is performed over the v variable, we can take f outside the integral, which gives us:

$$Q^-(f,f)(t,x,v) = f\int_{\mathbb{R}^d} f_* \left[\int_{\mathbb{S}^{d-1}} B(\alpha, v_* - v)\, d\mathcal{H}^{d-1}(\alpha)\right] dv_*.$$

Defining $A(v)$ as the term in square brackets, that is

$$A(v) = \int_{\mathbb{S}^{d-1}} B(\alpha, -v)\, d\mathcal{H}^{d-1}(\alpha), \tag{2.1.6}$$

This can be arranged as

$$Q^-(f,f)(t,x,v) = f\int_{\mathbb{R}^d} f(t,x,v_*)A(v-v_*)\, dv_* = f(A*f). \tag{2.1.7}$$

This can only be done if B has sufficient integrability conditions. Inserting this expression in the Boltzmann equation and

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f = Q(f,f) = Q^+(f,f) - Q^-(f,f),$$

i.e.

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f + f(A*f) = Q^+(f,f).$$

Using the characteristic map $w = x - vt$,

$$\frac{d}{dt} f^\#(t,w,v) + f^\#(A*f)^\#(t,w,v) = Q^+(f,f)^\#(t,w,v).$$

By the product rule

$$\frac{d}{dt} f^\# e^{\int_0^t (A*f)^\#(r,w,v)\, dr} = Q^+(f,f)^\# e^{\int_0^t (A*f)^\#(r,w,v)\, dr},$$

where $e^{\int_0^t (A*f)^\#(r,w,v)\, dr}$ plays the role of an integrating factor. Integrating on both sides, we reach the expression

$$\begin{aligned} f^\#(t,w,v)e^{\int_0^t (A*f)^\#(r,w,v)\, dr} &- f^\#(s,w,v)e^{\int_0^s (A*f)^\#(r,w,v)\, dr} \\ &= \int_s^t Q^+(f,f)^\#(l,w,v)e^{\int_0^l (A*f)^\#(r,w,v)\, dr}\, dl. \end{aligned} \tag{2.1.8}$$

For simplicity of notation, we define

$$O_f(g)(t,w,v) = g_e(x,w,v) = g^\#(t,w,v)e^{F^\#(t,w,v)},$$

where $F^\#(t,x,v) = \int_0^t (A*f)^{\#t}(r,x,v)\, dr$. Then from equation 2.1.8 we can make the following

Definition 2.1.4. Let $f : \mathbb{R}_+ \times \mathbb{R}^{2d} \to \mathbb{R}_+$ be a measurable function, such that $A*f$, $Q_e^+(f,f) \in L^1(0,T)$ for a.e. $(w,v) \in \mathbb{R}^{2d}$. Then, f is said to be an exponentially mild solution if the following equation holds

$$f_e(t,w,v) - f_e(s,w,v) = \int_s^t Q_e^+(f,f)(l,w,v)\, dl, \tag{2.1.9}$$

for a.e. $(w,v) \in \mathbb{R}^{2d}$ and for each $t,s \in \mathbb{R}$ such that $0 \le s < t < T$ (recall that for $s = 0$, $f_e(0,w,v) = f_0(w,v)$).

This is defined as a **exponential multiplier** form of the Boltzmann equation, and the 'exponentially mild' nomenclature comes from the similarity of this definition with that of a mild solution.

We have the following lemma to connect these definitions:

Lemma 2.1.1. *Let $f \in L^1_{loc}((0,\infty)\times\mathbb{R}^{2d})$ be a non-negative function. Then the following conditions hold:*

(i) If $Q^{\pm}(f,f) \in L^1_{loc}((0,\infty)\times\mathbb{R}^{2d})$, then f is a weak (or distributional) solution to the Boltzmann equation if and only if f is a mild solution.

(ii) If f satisfies the definition 2.2.4 at least for the particular choice of $\beta(x) = \log(1+x)$, $x > 0$, then it is a mild solution.

(iii) If f is a mild solution and $Q^{\pm}(f,f)\beta'(f)$ are both in $L^1_{loc}((0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$, then f is a renormalized solution.

(iv)If $Q^-(f,f) \in L^1_t(0,T)$ for a.e. x, v, and f is an exponentially mild solution, then f is a mild solution.

Proof. See Appendix B.1. □

This lemma shows that, in the important case when $Q^{\pm}(f,f) \in L^1_{loc}((0,T)\times\mathbb{R}^{2d})$, renormalized solutions are weak solutions.

Although mild and weak solutions are common in the study of conservation laws, the need for a different solution concept (renormalized solution) might not be apparent at first. The reason for defining renormalized solutions comes from observing that the collision operator has no integrals in the position variable [19]. This means that in the position variable, it acts as a pointwise multiplication of ff_*, $f'f'_*$, and as we will see in a short while, the Boltzmann equation has no a priori estimates that guarantee that $f \in L^2$ with respect to position. If $f \in L^1$ (as is our case), f^2 may fail to be locally integrable[33], and so we cannot guarantee that $Q(f,f) \in L^1_{loc}(\mathbb{R}^{2d}_{x,v})$, a necessary condition for the weak solution defined previously to be meaningful. However, formally if we divide both sides of the Boltzmann equation by $1+f$, we see that the negative part of the collision operator becomes

$$\frac{f}{1+f} A * f \leq A * f,$$

which is better suitable for analysis with the estimates at our disposition. We present now the following estimate, due to [33], in order to clarify the above reasoning.

Lemma 2.1.2. *Assume that $f : \mathbb{R}_+ \times \mathbb{R}^{2d} \to \mathbb{R}_+$ is a measurable function such that*

- $$\iint_{\mathbb{R}^{2d}} f(t,x,v)(1+\|v\|^2)\, dxdv < C_1,$$

- $$\int_0^T \iint_{\mathbb{R}^{2d}} |D[f]|(t,x,v)\, dxdv\, dt < C_2(T),$$

where we use the definition

$$D[f] = \frac{-1}{4} \iint_{\mathbb{R}^3\times\mathbb{S}^2} (f'f'_* - ff_*) \log\left(\frac{f'f'_*}{ff_*}\right) B(w,\alpha) d\mathcal{H}^2(\alpha) dv_*.$$

If $B(w,\alpha)$ satisfies

$$0 \leq A(w) \leq C_3(1+\|w\|^2),$$

Then it follows that

$$\int_{(0,T)\times\mathbb{R}^d\times\{\|v\|\leq R\}} \frac{|Q(f,f)|}{1+f}\, dtdxdv < \infty,$$

and an upper bound can be found explicitly in terms of C_1, C_2, C_3*.*

Therefore, if the above requirements are met, we can expect that the equation

$$\frac{\partial \log(f)}{\partial t} + v \cdot \nabla_x \log(f) = \frac{Q(f,f)}{1+f}$$

may have a well defined weak solution, which in turn would imply that f was a renormalized solution.

Proof. Notice that

$$\begin{aligned} |f'f'_* - ff_*| &= \left|\sqrt{f'f'_*} - \sqrt{ff_*}\right| \left(\sqrt{f'f'_*} + \sqrt{ff_*}\right) \\ &\leq \left|\sqrt{f'f'_*} - \sqrt{ff_*}\right|^2 + 2\sqrt{ff_*}\left|\sqrt{f'f'_*} - \sqrt{ff_*}\right| \end{aligned}$$

So that,

$$\int_{(0,T)\times\mathbb{R}^d\times\{\|v\|\leq R\}} \frac{|Q(f,f)|}{1+f}\, dtdxdv \leq \int_{(0,T)\times\mathbb{R}^d\times\{\|v\|\leq R\}} \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \frac{1}{1+f} |f'f'_* - ff_*| B(v-v_*,\alpha)\, d\mathcal{H}^{d-1}(\alpha) dv_* dvdxdt$$

Using the fact that $\frac{1}{1+f} \leq 1$ and the previous inequality, we find

$$\begin{aligned} &\leq \int_{(0,T)\times\mathbb{R}^{2d}} \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \left|\sqrt{f'f'_*} - \sqrt{ff_*}\right|^2 B(v-v_*,\alpha)\, d\mathcal{H}^{d-1}(\alpha) dv_* dvdxdt \\ &+ \int_{(0,T)\times\mathbb{R}^d\times\{\|v\|\leq R\}} \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \frac{2\sqrt{ff_*}}{1+f} \left|\sqrt{f'f'_*} - \sqrt{ff_*}\right| B(v-v_*,\alpha)\, d\mathcal{H}^{d-1}(\alpha) dv_* dvdxdt. \end{aligned}$$

Using the fact that $\frac{\sqrt{f}}{1+f} \leq 1$ and factoring $B = \sqrt{B}\sqrt{B}$, we can say that

$$\begin{aligned} &\leq \int_{(0,T)\times\mathbb{R}^{2d}} \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \left|\sqrt{f'f'_*} - \sqrt{ff_*}\right|^2 B(v-v_*,\alpha)\, d\mathcal{H}^{d-1}(\alpha) dv_* dvdxdt \\ &+ \int_{(0,T)\times\mathbb{R}^d\times\{\|v\|\leq R\}} \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \left(2\sqrt{f_*}\sqrt{B(v-v_*,\alpha)}\right) \left(\left|\sqrt{f'f'_*} - \sqrt{ff_*}\right| \sqrt{B(v-v_*,\alpha)}\right) d\mathcal{H}^{d-1}(\alpha) dv_* dvdxdt. \end{aligned}$$

Finally, using the Cauchy-Schwartz inequality,

$$\begin{aligned} &\leq M(R,T) + 2\sqrt{M(R,T)} \left(\int_{(0,T)\times\mathbb{R}^d\times\{\|v\|\leq R\}} \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} f_* B(v-v_*,\alpha)\, d\mathcal{H}^{d-1}(\alpha) dv_* dvdxdt \right)^{\frac{1}{2}} \\ &= M(R,T) + 2\sqrt{M(R,T)} \left(\int_{(0,T)\times\mathbb{R}^d\times\{\|v\|\leq R\}} \int_{\mathbb{R}^d} f_* A(v-v_*)\, dv_* dvdxdt \right)^{\frac{1}{2}} \\ &\leq M(R,T) + 2\sqrt{C_3 M(R,T)} \left(\int_{(0,T)\times\mathbb{R}^d\times\{\|v\|\leq R\}} \int_{\mathbb{R}^d} f_* \left(1 + \|v-v_*\|^2\right) dv_* dvdxdt \right)^{\frac{1}{2}} \end{aligned}$$

where $M(R,T)$ is given by

$$M(R,T) = \int_{(0,T)\times\mathbb{R}^{2d}} \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \left|\sqrt{f'f'_*} - \sqrt{ff_*}\right|^2 B(v-v_*,\alpha)\, d\mathcal{H}^{d-1}(\alpha) dv_* dvdxdt.$$

We also see that since $\|v-v_*\|^2 \leq 2\|v\|^2 + 2\|v_*\|^2$,

$$\iint_{\mathbb{R}^{2d}} f_* \left(1+\|v-v_*\|^2\right) dv_* dx \leq 2 \iint_{\mathbb{R}^{2d}} f_* \|v\|^2 \, dv_* dx + 2 \iint_{\mathbb{R}^{2d}} f_* \left(1+\|v_*\|^2\right) dv_* dx \leq 2C_1(1+\|v\|^2).$$

Therefore, coming back to our main argument, we can say that

$$\int_{(0,T)\times\mathbb{R}^d\times\{\|v\|\leq R\}} \frac{|Q(f,f)|}{1+f} \, dt dx dv \leq M(R,T) + 2\sqrt{C_3 M(R,T)} \left(\int_{(0,T)\times\{\|v\|\leq R\}} 2C_1(1+\|v\|^2) \, dv dt \right)^{\frac{1}{2}}.$$

Finally, notice that the following inequality holds for any $a, b > 0$:

$$|\sqrt{a} - \sqrt{b}|^2 \leq \frac{1}{4}(a-b)(\log(a) - \log(b)),$$

since, if $a > b$, one can use the Cauchy-Schwartz inequality, finding

$$|\sqrt{a} - \sqrt{b}|^2 = \left(\int_b^a \frac{1}{2\sqrt{x}} \, dx \right)^2 \leq \frac{1}{4} \left(\int_b^a 1 \, dx \right) \left(\int_b^a \frac{1}{x} \, dx \right).$$

So choosing $a = f' f'_*$, $b = f f_*$, we get that

$$\left| \sqrt{f' f'_*} - \sqrt{f f_*} \right|^2 \leq \frac{1}{4} (f' f'_* - f f_*)(\log(f' f'_*) - \log(f f_*)),$$

and also the estimate

$$\begin{aligned} M(R,T) &\leq \frac{1}{4} \int_{(0,T)\times\mathbb{R}^{2d}} \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} (f' f'_* - f f_*)(\log(f' f'_*) - \log(f f_*)) B(v-v_*,\alpha) \, d\mathcal{H}^{d-1}(\alpha) dv_* dv dx dt \\ &= \int_0^T \iint_{\mathbb{R}^{2d}} |D[f]|(t,x,v) \, dx dv \, dt < C_2(T). \end{aligned}$$

Therefore,

$$\int_{(0,T)\times\mathbb{R}^d\times\{\|v\|\leq R\}} \frac{|Q(f,f)|}{1+f} \, dt dx dv < C_2(T) + 2\sqrt{2TC_1C_2(T)C_3} \left(\int_{\{\|v\|\leq R\}} (1+\|v\|^2) \, dv \right)^{\frac{1}{2}} < \infty.$$

□

From the above, we can already see some of the requirements we will need for our main existence theorem, so as to make renormalized solutions viable. With all definitions set, we can now state the main results about global existence of weak solutions to the Boltzmann equation. The theorem we wish to prove is the following:

Theorem 2.1.1 (Main Theorem). *Let $B : \mathbb{R}^d \times \mathbb{S}^{d-1} \longrightarrow \mathbb{R}_+$ be a smooth function, and $f_0 : \mathbb{R}^{2d} \longrightarrow \mathbb{R}_+$ be a measurable function such that*

- $$\int_{\mathbb{R}^{2d}} f_0(1+|v|^2+|x|^2+|\ln f_0|) \, dx dv < \infty, \tag{2.1.10}$$

- *B depends only on $|z|, |\langle z, w\rangle|$,*

- $$A \in L^1_{loc}(\mathbb{R}^d), \quad \lim_{|v|\to\infty} \frac{1}{(1+v^2)} \int_{|v_*|<R} A(v_* - v)\, dv_* = 0 \quad \forall R > 0. \tag{2.1.11}$$

where $A(w)$ is defined as in equation (2.1.6)*. Then, for all $T, R > 0$, there exists $f \in C([0,T]; L^1(\mathbb{R}^d \times \mathbb{R}^d))$ that is a renormalized solution to problem 2.1.1, such that $f|_{t=0} = f_0$ and the following conditions hold*

$$\begin{cases} A * f \in L^\infty(0,\infty; L^1(\mathbb{R}^d \times B_R)), & (2.1.12a) \\ \dfrac{1}{1+f} Q^-(f,f) \in L^\infty(0,\infty; L^1(\mathbb{R}^d \times B_R)), & (2.1.12b) \\ \dfrac{1}{1+f} Q^+(f,f) \in L^1(0,\infty; L^1(\mathbb{R}^d \times B_R)), & (2.1.12c) \\ f \geq 0; \ \sup_{t\geq 0} \iint_{\mathbb{R}^{2d}} f(1 + |x - vt|^2 + |v|^2 + |\ln f|)\, dv dx < \infty. & (2.1.12d) \end{cases}$$

The proof will be divided in 4 steps. In make the proof more easily readable, we provide the following flowchart, listing the the main ingredients and milestones of each step of the proof.

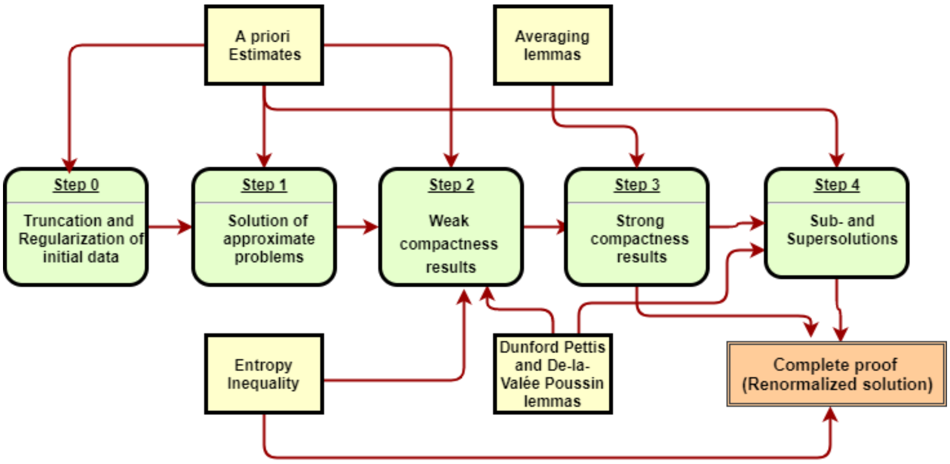


Figure 2.1.1: Steps of the proof and main tools used.

In **Step 1**, we construct a sequence of problems that approximate the original Boltzmann equation and we know how to solve, producing a sequence of approximate solutions $\{f_n\}_{n\in\mathbb{N}}$. In **Step 2**, we prove that the weak limit of this sequence of solutions exists, $f_n \rightharpoonup f$. Our goal then is to prove that this limit f is a renormalized solution for the Boltzmann equation as specified in the theorem. In **Step 3** we prove some important lemmas and estimates, and then in **Step 4** we use those estimates to accomplish our goal, finishing the proof. In this proof, the decay of $A(v)$ as $\|v\| \to \infty$ will be decisive to establish some estimates; in the next section we will mention how to weaken this hypothesis. Most of the steps are done following derivations contained in either [21], [33], or [12].

2.1.2 Step 1 -Approximate solutions

Unless our gas is composed of maxwellian molecules, the collision kernel has sigularities for: (a) large relative velocities and (b) small deviation angles in the unit sphere. Another difficulty with the original Boltzmann equation is that the collision operator is quadratic in the function f, and for the method we'll use to prove existence, we need a 'slower growth' of the right hand side with f. To solve problem (a), we begin by truncating and smoothing our initial condition, by defining

- a monotonically decreasing sequence $\{\delta_n\}$ with $0 \leq \delta_n \leq 1$, and $\delta_n \to 0$.

- A smoothed sequence $\tilde{f_0^n}(x,v)$, constructed by mollifying our initial data after truncating it appropriately:
$$\tilde{f_0^n}(x,v) = \frac{1}{\delta_n^{2d}} \rho\left(\frac{x}{\delta_n}, \frac{v}{\delta_n}\right) * \left[\chi_{\{|x|<\frac{1}{\delta_n}\}} f_0(x,v)\right],$$
where ρ is a standard mollifier in $\mathbb{R}^{2d}$ (e.g. $\rho \in C_0^\infty$ and nonnegative). By construction, $\tilde{f_0^n}(x,v)$ is non-negative, and $\tilde{f_0^n} \in C_0^\infty(\mathbb{R}^{2d})$.

Define
$$f_0^n(x,v) = \tilde{f_0^n}(x,v) + \delta_n e^{\frac{-1}{2}(|x|^2+|v|^2)},$$
such that now $f_0^n > 0$. This will provide us with a control on how fast the solution decays to 0 as $\|(x,v)\| \to \infty$, which will be useful later since we will need to make estimates involving $\log(f(x,v))$. It can be shown that, for each $n \in \mathbb{N}$

- $$f_0^n \in C^\infty(\mathbb{R}^{2d}), \quad \sup_{(x,v)\in\mathbb{R}^d\times\mathbb{R}^d} \|(x,v)\|^\alpha (D^\beta f_0^n) < \infty,$$
for any pair of multi-indices α, β (This defines f_0^n as a **Schwartz function**. In fact $f \in \mathcal{S}_{x,v}(\mathbb{R}^{2d})$ if and only if the above conditions are met).

- $$\sup_{n>0} \int_{\mathbb{R}^{2d}} f_0^n |\log(f_0^n)|\, dxdv < \infty.$$

Then, we introduce a truncation of the original collision operator, to avoid the angular singularity (b). Define, for each $n \in \mathbb{N}$
$$B_n(v_* - v, \alpha) := \begin{cases} 0 & \text{if } \langle (v - v_*),\, \alpha\rangle < \delta_n, \\ B(v_* - v, \alpha), & \text{otherwise.} \end{cases}$$

Recall from subsection 1.3 that $\alpha \in \mathbb{R}^d$ is a unit vector in the direction of the displacement vector $x^2 - x^1$ in the moment of a collision, and therefore the above restriction forbids "grazing" collisions, in which the relative velocity between the particles doesn't change significantly in direction.

Finally, for the last difficulty, define a normalized collision operator as
$$Q_n(f,f) = \frac{1}{1 + \delta_n \int_{\mathbb{R}^n} f dv} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} (f' f'_* - f f_*)\, B_n(\alpha, v_* - v)\, d\mathcal{H}^{d-1}(\alpha)\, dv_*, \tag{2.1.13}$$
where $d\mathcal{H}^{d-1}$ denotes the Hausdorff measure of the surface, which, since the surface is smooth, will become simply the Lebesgue measure in $d-1$ dimensions after introducing a parametrization of $\mathbb{S}^2$. Then, the approximate problem we will solve is
$$\frac{\partial f_n}{\partial t} + v \cdot \nabla_x f_n = Q_n(f_n, f_n). \tag{2.1.14}$$

Remark 2.1.1. Notice that, for each fixed $n \in \mathbb{N}$, a solution f_n to the Boltzmann equation with initial condition f_0^n would have to be a solution to the equation
$$\frac{\partial f_n}{\partial t} + v \cdot \nabla_x f_n = Q(f_n, f_n),$$
in the appropriate sense (weak, renormalized, mild or classical), instead of equation 2.1.14. What we are

doing essentially is to say that the Boltzmann equation is equivalent to

$$\frac{\partial f_n}{\partial t} + v \cdot \nabla_x f_n + W \cdot \nabla_v f_n = Q_n(f_n, f_n) + [Q(f_n, f_n) - Q_n(f_n, f_n)],$$

and treating the term in square brackets as an 'error' term $r_n = [Q(f_n, f_n) - Q_n(f_n, f_n)]$, which will later be shown to converge to 0.

We see that equation 2.1.14 can be cast as

$$\frac{d}{dt} f_n(t, w + vt, v) = Q_n(f_n, f_n),$$

where we introduce $w = x - vt$ as the characteristic equation for the flow in phase space with drift v. Integrating from 0 to t, we get an integral equation similar to a solution based on Duhamel's principle:

$$f_n(t, w, v)^{\#} = f_0(w, v) + \int_0^t [Q_n(f_n, f_n)]^{\#}(s, w, v) ds, \tag{2.1.15}$$

where $g(s, x, v)^{\#} = g(s, x + vt, v)$ is just g restricted to a characteristic. Reverting the original x variable, this means:

$$f_n(t, x, v) = f_0(x - vt, v) + \int_0^t [Q_n(f_n, f_n)]^{\#}(s, x - v(t - s), v) ds.$$

If $Q_n(f_n, f_n)$ were replaced by a prescribed function $g(t, x, v)$ in the right hand side of equation 2.1.15, it would be the correct solution for a transport equation with a source term, and our problem would be finished. However, since the righ-hand side depends on f_n itself, equation 2.1.15 instead is an integral equation for f_n, which we will not assume a priori that has a solution. Define then the operator $G_n : L^{\infty}_{loc}(\mathbb{R}^+; L^1_{x,v}(\mathbb{R}^{2d})) \longrightarrow L^{\infty}_{loc}(\mathbb{R}^+; L^1_{x,v}(\mathbb{R}^{2d}))$ such that

$$G_n[f^{\#}](t, x, v) = f_0^n(x, v) + \int_0^t [Q_n(f, f)]^{\#}(s, x, v) ds.$$

We claim that this operator is a contraction on the L^1 norm for $t \in [0, \tau]$ for some choice of $\tau > 0$. To prove that, we use the following

Lemma 2.1.3. *Given $f, g \in L^1_v(\mathbb{R}^d)$, for almost every t, x in $\mathbb{R} \times \mathbb{R}^d$, there exists a constant $C(n, d) > 0$ such that*

$$\|Q_n(f, f) - Q_n(g, g)\|_{L^1(\mathbb{R}^d_v)} \leq C(n, d) \|f - g\|_{L^1(\mathbb{R}^d_v)}.$$

Whose proof is detailed in the Appendix B.2. Using this result, we see that for each $t \in (0, \tau)$,

$$\begin{aligned}
\|G_n[f^{\#}](t) - G_n[g](t)\|_{L^1_{x,v}(\mathbb{R}^{2d})} &= \left\| \int_0^t [Q_n(f, f)]^{\#}(s, x, v) - [Q_n(g, g)]^{\#}(s, x, v)\, ds \right\|_{L^1(\mathbb{R}^{2d}_{x,v})} \\
&\leq \int_0^t \left\| [Q_n(f, f)]^{\#}(s, x, v) - [Q_n(g, g)]^{\#}(s, x, v) \right\|_{L^1(\mathbb{R}^{2d}_{x,v})} ds \\
&\leq C(n, d) \int_0^t \left\| f^{\#}(s, x, v) - g^{\#}(s, x, v) \right\|_{L^1(\mathbb{R}^{2d}_{x,v})} ds \\
&\leq C(n, d) \tau \sup_{s \in [0, \tau]} \|f^{\#}(s, \cdot, \cdot) - g^{\#}(s, \cdot, \cdot)\|_{L^1(\mathbb{R}^{2d}_{x,v})}.
\end{aligned}$$

Therefore,

$$\sup_{t \in [0, \tau]} \|G_n[f^{\#}](t) - G_n[g^{\#}](t)\|_{L^1(\mathbb{R}^{2d}_{x,v})} \leq C(n, d) \tau \sup_{s \in [0, \tau]} \|f^{\#}(s, x, v) - g^{\#}(s, x, v)\|_{L^1(\mathbb{R}^{2d}_{x,v})}.$$

Thus, we can choose τ such that $C(n,d)\tau < 1$, and G_n is a contraction, and finally, by Banach's Fixed Point Theorem, for every fixed n there is a unique fixed point $f_n^\#$ for $G_n[f^\#]$.

We now claim that if $f \in \mathcal{S}_{x,v}(\mathbb{R}^{2d})$, then $G_n[f] \in \mathcal{S}_{x,v}(\mathbb{R}^{2d})$. Since the collision operator has no integrals over the position variable x, we see that

- Since we assume $f \in \mathcal{S}_{x,v}(\mathbb{R}^{2d})$, we can use the Leibniz rule for differentiating 'under the integral', and conclude that $x^\alpha D_x^\beta G_n[f]$ is still integrable for any multi-indices β, α (restricting our derivatives to the x variable).

- For the v variable, we have to be more cautious, since the collision kernel depends on v. By theorem 1.3.2, $v^\alpha G_n[f]$ is integrable for any multi-index α if $f \in \mathcal{S}_{x,v}(\mathbb{R}^{2d})$. Also, since the collision Kernel is piecewise smooth and has no singularities, we can again use the Leibniz rule and conclude that $D_v^\beta G_n[f]$ is integrable for any multi-index β.

- in all of the above, it is necessary that $f_0 \in \mathcal{S}_{x,v}(\mathbb{R}^{2d})$.

Also, because of the integration with respect to the time variable, we see that $[G_n[f](t,x,v)$ is continuous with respect to the variable t.

Since f_n is the fixed point, it solves our approximate problem for $t \in [0,\tau]$. However, since none of our estimates depended on the initial condition f_0, we can choose a time in $[0,\tau]$, say $t = \tau/2$ and define $f_1(x,v) = f_n(\tau/2, x, v)$. Then we can repeat the process above using $f_1(x,v)$ as an initial condition, and guarantee existence for another time interval of length τ, such that our approximate problem with initial condition f_0 now has a solution for $t \in [0, \frac{3\tau}{2}]$. Recursively, we can show that this problem has a solution for $t \in [0, \tau + \frac{n\tau}{2}]$ for each $n \in \mathbb{N}$, which means that the solution exists for all $t \in [0,T]$ $\forall T > 0$.

A priori estimates

We now establish some properties that this solution must posess.

Proposition 2.1.1. *For any $T>0$, Let $f \in C\left([0,T]; \mathcal{S}_{x,v}(\mathbb{R}^2 d)\right)$ be solution to the Boltzmann equation (2.1.1)with initial datum $f_0(x,v)$. Also assume that $|\log(f)|$ grows at most polynomially as a function of (x,v), and $B \in L^\infty_{loc}(\mathbb{R}^d \times \mathbb{R}^d \times S^{d-1})$ is a non-negative, measurable function. If $f_0 \in \mathcal{S}_{x,v}(\mathbb{R}^{2d})$, and satisfies*

$$\int_{\mathbb{R}^{2d}} f_0(1 + |x|^2 + |v|^2 + |\log(f_0)|)\, dxdv < \infty,$$

then the following hold:

(1)

$$\int_{\mathbb{R}^{2d}} f(t,x,v)\, dxdv = \int_{\mathbb{R}^{2d}} f_0(x,v)\, dxdv \qquad \textit{(mass conservation)},$$

(2)

$$\int_{\mathbb{R}^{2d}} |v|^2 f(t,x,v)\, dxdv = \int_{\mathbb{R}^{2d}} |v|^2 f_0(x,v)\, dxdv \qquad \textit{(energy conservation)},$$

(3)

$$\int_{\mathbb{R}^{2d}} |x - tv|^2 f(t,x,v)\, dxdv = \int_{\mathbb{R}^{2d}} |x|^2 f_0(x,v)\, dxdv,$$

(4)

$$\int_{\mathbb{R}^{2d}} \log(f) f(t,x,v)\, dxdv - \int_0^t \int_{\mathbb{R}^{2d}} D[f](s,x,v)\, dxdv\, ds = \int_{\mathbb{R}^{2d}} \log(f_0) f_0(x,v)\, dxdv.$$

where D is defined as in 1.4.5a.

Corollary 2.1.1. *Let* $\{f_n\}_{n\in\mathbb{N}} \in C\left([0,T];\mathcal{S}_{x,v}(\mathbb{R}^2 d)\right)$ *be a a sequence of solutions to the approximate problem (2.1.14), with corresponding initial data* $f_0^n(x,v)$*, satisfying*

$$\int_{\mathbb{R}^{2d}} f_0^n(1+|x|^2+|v|^2+|\log(f_0^n)|)\,dxdv < \infty \qquad \forall n\in\mathbb{N},$$

and any other conditions in the previous proposition, replacing f *with* f_n*,* B *with* $B_n(\alpha,w)$ *and* f_0 *with* f_0^n*. Then, the same conclusions hold.*

Proof. (Corollary) Notice that for fixed n, the only difference between f_n being a solution to 2.1.1 and being a solution to the aproximate problem 2.1.14 is in the collision kernel, which does not play a role in the derivation of these equations. Also notice that, since $f_0^n \geq \delta_n e^{\frac{-1}{2}(|x|^2+|v|^2)}$, we Therefore, all results derived in proposition 2.1.1 will be useful to our case. □

Proof. For the proof of of the proposition itself, Items (1) and (2) were already proven in equation 1.3.35, subsection 1.3.3, and item (4) was proven in equation 1.4.12, subsection 1.4. For item (3), the idea is to show that $|x-tv|^2$ is a summational invariant. In fact,

$$|x-tv_*'|^2+|x-tv'|^2 = 2|x|^2 - 2tx\cdot(v_*'+v') + t^2(|v_*'|^2+|v'|^2)$$
$$= 2|x|^2 - 2tx\cdot(v_*+v) + t^2(|v_*|^2+|v|^2) = |x-tv_*|^2+|x-tv|^2$$

Therefore, $\int_{\mathbb{R}^{2d}} |x-tv|^2 f(t,x,v)\,dxdv$ is a conserved quantity, and $\int_{\mathbb{R}^d} |x-tv|^2 f(t,x,v)\,dv$ obeys a conservation law that can be derived from the Boltzmann equation in a similar way as the conservation of energy and momentum.

□

With this proposition in hand, one can derive a few estimates. Using the fact that $|x|^2 = |x-vt+vt|^2 \leq (|x-vt|+t|v|)^2 \leq 2(|x-vt|^2+t^2|v|^2)$, and combining with items 1,2,3 gives us the following estimate

$$\int_{\mathbb{R}^{2d}} |x|^2 f(t,x,v)\,dxdv \leq \int_{\mathbb{R}^{2d}} 2(|x-vt|^2+t^2|v|^2) f(t,x,v)\,dxdv$$
$$= 2\int_{\mathbb{R}^{2d}} (|x|^2+t^2|v|^2) f_0(x,v)\,dxdv. \tag{2.1.16}$$

Finally, from item (4) we can derive another inequality using the following

Lemma 2.1.4. *Let* $g\in L^1(\mathbb{R}^{2d})$ *be a positive function, such that* $\int_{\mathbb{R}^{2d}} g|\log(g)|\,dxdv < \infty$*. Then*

$$\int_{\mathbb{R}^{2d}} g|\log(g)|\,dxdv = \int_{\mathbb{R}^{2d}} g\log(g)\,dxdv + 2\int_{\mathbb{R}^{2d}} g(|x|^2+|v|^2)\,dxdv + C(d).$$

Using the above lemma for $g = f(t,x+vt,v)$, we get

$$\int_{\mathbb{R}^{2d}} f(t,x+vt,v)|\log(f(t,x+vt,v))|\,dxdv = \int_{\mathbb{R}^{2d}} f(t,x+vt,v)\log(f(t,x+vt,v))\,dxdv$$
$$+2\int_{\mathbb{R}^{2d}} f(t,x+vt,v)(|x|^2+|v|^2)\,dxdv + C(d),$$

making the change of variables $w = x+vt$:

$$\int_{\mathbb{R}^{2d}} f(t,w,v)|\log(f(t,w,v))|\,dwdv = \int_{\mathbb{R}^{2d}} f(t,w,v)\log(f(t,w,v))\,dwdv$$
$$+2\int_{\mathbb{R}^{2d}} f(t,w,v)(|w-vt|^2+|v|^2)\,dwdv + C(d).$$

Now, using estimates 3 and 4, the previous expression can be rewritten as

$$\int_0^t \int_{\mathbb{R}^{2d}} D[f](s,x,v)\,dxdv\,ds + \int_{\mathbb{R}^{2d}} \log(f_0)f_0\,dxdv + 2\int_{\mathbb{R}^{2d}} f_0(w,v)(|w|^2+|v|^2)\,dwdv + C(d),$$

therefore

$$\int_{\mathbb{R}^{2d}} f(t,w,v)|\log(f(t,w,v))|\,dwdv - \int_0^t \int_{\mathbb{R}^{2d}} D[f](s,x,v)\,dxdv\,ds \\ \leq \int_{\mathbb{R}^{2d}} (2|w|^2 + 2|v|^2 + |\log(f_0)|)f_0(w,v)\,dwdv + C(d). \tag{2.1.17}$$

These last inequalities can be combined, which gives us

$$\int_{\mathbb{R}^{2d}} f(t,x,v)(|\log(f(t,x,v))| + |x|^2 + |v|^2 + 1)\,dxdv - \int_0^t \int_{\mathbb{R}^{2d}} D[f](s,x,v)\,dxdv\,ds \leq$$

$$\int_{\mathbb{R}^{2d}} (4|x|^2 + (2t^2+3)|v|^2 + |\log(f_0)| + 1)f_0(x,v)\,dxdv + C(d), \tag{2.1.18}$$

$$\leq 4(1+t^2)\int_{\mathbb{R}^{2d}} (|x|^2 + |v|^2 + |\log(f_0)| + 1)f_0(x,v)\,dxdv + C(d).$$

Therefore, it follows that

$$\int_{\mathbb{R}^{2d}} f(t,x,v)(|\log(f(t,x,v))| + |x|^2 + |v|^2 + 1)\,dxdv - \int_0^t \int_{\mathbb{R}^{2d}} D[f](s,x,v)\,dxdv\,ds \leq C_0(1+t^2), \tag{2.1.19}$$

where

$$C_0 = \max\left\{C(d), 4\int_{\mathbb{R}^{2d}} (|x|^2 + |v|^2 + |\log(f_0)| + 1)f_0(x,v)\,dxdv\right\}.$$

Before moving to the next section, we present a proof for lemma 2.1.4

Proof. First, we decompose the domain in regions where the logarithm has a definite sign:

$$\int_{\mathbb{R}^{2d}} g|\log(g)|\,dxdv = \int_{g\leq 1} g|\log(g)|\,dxdv \int_{g\geq 1} g|\log(g)|\,dxdv$$

$$= -\int_{g\leq 1} g\log(g)\,dxdv \int_{g\geq 1} g\log(g)\,dxdv$$

$$= -2\int_{g\leq 1} g\log(g)\,dxdv + \int_{g\leq 1} g\log(g)\,dxdv + \int_{g\geq 1} g\log(g)\,dxdv = -2\int_{g\leq 1} g\log(g)\,dxdv$$

$$+ \int_{\mathbb{R}^{2d}} g\log(g)\,dxdv.$$

The first integral on the right hand side can be further broken into 2 pieces

$$\int_{g\leq 1} g\log(g)\,dxdv = \int_{g\leq e^{-(|x|^2+|v|^2)}} g\log(g)\,dxdv + \int_{e^{-(|x|^2+|v|^2)}\leq g\leq 1} g\log(g)\,dxdv$$

$$= \int_{g\leq e^{-(|x|^2+|v|^2)}} g\log(g)\,dxdv + \int_{-(|x|^2+|v|^2)\leq \log(g)\leq 0} g\log(g)\,dxdv$$

$$\geq \int_{g\leq e^{-(|x|^2+|v|^2)}} g\log(g)\,dxdv - \int_{-(|x|^2+|v|^2)\leq \log(g)\leq 0} g(|x|^2+|v|^2)\,dxdv.$$

Inserting back in the previous expression:

$$\int_{\mathbb{R}^{2d}} g|\log(g)|\,dxdv = -2\int_{g\le 1} g\log(g)\,dxdv + \int_{\mathbb{R}^{2d}} g\log(g)\,dxdv$$

$$\le 2\int_{g\le e^{-(|x|^2+|v|^2)}} -g\log(g)\,dxdv + 2\int_{-(|x|^2+|v|^2)\le\log(g)\le 0} g(|x|^2+|v|^2)\,dxdv$$

$$\le 2\int_{g\le e^{-(|x|^2+|v|^2)}} -g\log(g)\,dxdv + 2\int_{\mathbb{R}^{2d}} g(|x|^2+|v|^2)\,dxdv.$$

Finally, since $-t\log(t)\le A\sqrt{t}\ \ \forall t\in[0,1]$, we can say that

$$\int_{g\le e^{-(|x|^2+|v|^2)}} -g\log(g)\,dxdv \le \int_{g\le e^{-(|x|^2+|v|^2)}} A\sqrt{g}\,dxdv \le \int_{g\le e^{-(|x|^2+|v|^2)}} Ae^{\frac{-(|x|^2+|v|^2)}{2}}\,dxdv$$

$$\le \int_{\mathbb{R}^{2d}} Ae^{\frac{-(|x|^2+|v|^2)}{2}}\,dxdv \equiv C(d),$$

concluding the proof. □

2.1.3 Step 2 - Weak compactness results

We now wish to prove that the sequence of solutions derived from our approximate problems converges as $n\to\infty$. From the inequality 2.1.18 and other estimates from the last section, we see that

$$\int_{\mathbb{R}^{2d}} f_n(|\log(f_n)|+|x|^2+|v|^2+1)\,dxdv < C(t)\ \ \forall n>0.$$

So that a uniform bound can be chosen if $t\in[0,T]$ for a given $T>0$. A uniform bound on the norm of this sequence hints that a compactness argument may allow us to take the limit. However, since $L^1(d\mu)$ is not a reflexive space, we wouldn't expect in principle that common tools for proving compactness would work in this setting. However, the fact that $f\log(f)$ is integrable formally means that our sequence f_n has a slightly stronger integrability than L^1 functions, and so we can hope that a weak convergence result can be found. This result comes from 2 lemmas, whose proofs are detailed in Appendix B.3.

Definition 2.1.5. Let μ be a Radon measure in $\mathbb{R}^N$. A bounded subset $\mathcal{F}\subseteq L^1(d\mu)$ is said to be uniformly integrable if any of the following equivalent conditions is true

- $$\lim_{R\to\infty}\sup_{f\in\mathcal{F}}\int_{f\ge R} f\,d\mu = 0.$$
- for each $\epsilon>0$ there exists $\delta>0$ s.t.
$$\mu(A)<\delta \Rightarrow \sup_{f\in\mathcal{F}}\int_A f\,d\mu<\epsilon.$$

Lemma 2.1.5 (Dunford-Pettis). *Let μ be a Radon measure in $\mathbb{R}^N$. A bounded subset $\mathcal{F}\subseteq L^1(d\mu)$ is weakly relatively compact if and only if $\mathcal{F}$ is uniformly integrable and tight, the last condition meaning that, for each $\epsilon>0$ there is compact set $K_\epsilon\subseteq\mathbb{R}^N$ such that*

$$\sup_{f\in\mathcal{F}}\int_{\mathbb{R}^N-K_\epsilon} f\,d\mu\le\epsilon.$$

Lemma 2.1.6 (De-la-Valeé-Poussin). *Let μ be a Radon measure in $\mathbb{R}^N$ and $\Omega\subseteq\mathbb{R}^N$ be a Borel set. A family $\mathcal{F}\subseteq L^1(\Omega,d\mu)$ is uniformly integrable if and only if there is a nonnegative nondecreasing function*

$G : [0, \infty) \longrightarrow [0, \infty)$ *such that:*

$$\lim_{t\to\infty} \frac{G(t)}{t} = \infty \qquad \sup_{f\in\mathcal{F}} \int G(|f|)\, d\mu(y) < \infty$$

Ee see that our family $\mathcal{F} = f_n$ obeys the conditions of the second lemma. Let $G(t) = t\log^+(t)$, where $\log^+(t) = \max\{\log(t), 0\}$. $G(t)$ is convex, nondecreasing, non-negative, and

$$\lim_{t\to\infty} \frac{t\log^+(t)}{t} = \lim_{t\to\infty} \log^+(t) = \infty.$$

Finally,

$$\sup_{n>0} \int_{\mathbb{R}^{2d}} G(|f_n|)\, dxdv \le \int_{\mathbb{R}^{2d}} f_n(|\log(f_n)| + |x|^2 + |v|^2 + 1)\, dxdv,$$

which is bounded, using the estimates derived in the previous subsection. Next, we show that the sequence f_n is tight. For $R > 0$, notice that $\|x\|^2 + \|v\|^2 > R^2 \Leftrightarrow 1 < \frac{\|x\|^2+\|v\|^2}{R^2}$, so that

$$\begin{aligned}\sup_n \iint_{\{\|x\|^2+\|v\|^2>R^2\}} f_n\, dxdv &\le \sup_n \iint_{\{\|x\|^2+\|v\|^2>R^2\}} f_n \frac{\|x\|^2+\|v\|^2}{R^2}\, dxdv \\ &\le \frac{1}{R^2} \sup_n \iint_{\mathbb{R}^{2d}} f_n(1 + \|x\|^2 + \|v\|^2)\, dxdv \le \frac{C_1}{R^2}\end{aligned}$$

where in the last inequality the estimate 2.1.18 was used. This implies

$$\lim_{R\to\infty} \sup_n \iint_{\{\|x\|^2+\|v\|^2>R^2\}} f_n\, dxdv = 0,$$

as claimed. Therefore, combining the two lemmas above, we verify that f_n is weakly compact, such that up to passing to a subsequence, there exists f satisfying the same bounds, that is:

$$\int_{\mathbb{R}^{2d}} f(|\log(f)| + |x|^2 + |v|^2 + 1)\, dxdv < \infty$$

and such that $f_{n_k} \rightharpoonup f \in L^1_{x,v}$ as $k \to \infty$ (we use the symbol $\rightharpoonup$ to denote weak convergence). The lemmas 2.1.6 and 2.1.5 will be standard tools in the proof of theorem 2.1.1, to prove the weak compactness of many sequences of functions.

Proposition 2.1.2. *The sequence of functions Z_n defined by*

$$Z_n(t, x, v) = \frac{Q_n(f_n, f_n)}{1 + f_n} \qquad (\forall n \ge 1)$$

is also weakly compact in $L^1(\mathbb{R}^{2d}, \mathbb{R})$

Remark 2.1.2. This means that, taking our previous subsequence f_{n_k} which converged weakly, we can pass to another subsequence such that

$$\begin{cases} f_{n_j} \rightharpoonup f, \\ \frac{Q_{n_j}(f_{n_j}, f_{n_j})}{1+f_{n_j}} \rightharpoonup g. \end{cases}$$

One observes that, it does not follow necessarily that $g = \frac{Q(f,f)}{1+f}$, but in the next step of the proof of the main theorem, we will establish this result.

Proof. First, we consider the case in which $\|A_n\|_{L^1} = a_n$ and $\sup_{n>0} a_n < \infty$, then generalize it to more general collision kernels (recall the definition of A in equation 2.1.6). The idea is to show that $\frac{Q_n(f_n,f_n)}{1+f_n}$ satisfies the conditions on the Dunford-Pettis lemma. We will use the standard decomposition on the collision operator into 'gain' and 'loss' terms (recalling equations 2.1.5b, 2.1.5a),

$$Z_n = -\frac{Q_n^-(f_n, f_n)}{1+f_n} + \frac{Q_n^+(f_n, f_n)}{1+f_n},$$

and prove the convergence of each term in the right nad side of the equation above separately. Beginning with the loss term, we have

$$\frac{Q_n^-(f_n, f_n)}{1+f_n} = \frac{f_n}{1+f_n}\frac{1}{1+\delta_n\int_{\mathbb{R}^d} f_n\,dv} A_n * f_n \le A_n * f_n,$$

where $*$ denotes the convolution operator. If we can show that $A_n * f_n$ is

- uniformly bounded in $L^1_{x,v}(\mathbb{R}^{2d})$,
- Tight, as in the definition stated before,
- Uniformly integrable, using any of the 3 equivalent conditions,

then it will be weakly compact, and so will be $\frac{Q_n^-(f_n,f_n)}{1+f_n}$. To show that it is uniformly bounded in L^1, it suffices to use the Young inequality for convolutions:

$$\|A_n * f_n\|_{L^1(\mathbb{R}^{2d})} \le \|A_n\|_{L^1(\mathbb{R}^d)}\|f_n\|_{L^1(\mathbb{R}^{2d})} = a_n\|f_{0,n}\|.$$

Next, we prove uniform integrability using De-la-Valée-Poussin's lemma. Consider the function $\phi(s) = s\log^+(s)$. Notice that, for $a, s > 0$,

$$s\log(s) = s\log\left(\frac{s}{a}a\right) = s\log(a) + s\log\left(\frac{s}{a}\right).$$

Define the positive and negative parts of the logarithm in the usual manner, as $\log^+(x) = \max\{\log(x), 0\}$, $\log^-(x) = \max\{-\log(x), 0\}$. If we split the above equation using this definition, we find:

$$s\log^+(s) - s\log^-(s) = s\log^+(a) - s\log^-(a) + s\log^+\left(\frac{s}{a}\right) - s\log^-\left(\frac{s}{a}\right),$$

then, adding and subtracting $(s\log^-(a))$ to the right hand side,

$$s\log^+(s) = s|\log(a)| + s\log^+\left(\frac{s}{a}\right) - s\left[2\log^-(a) + \log^-\left(\frac{s}{a}\right) - \log^-(s)\right].$$

We claim that the term in square brackets is always non-negative. Thus

$$s\log^+(s) \le s|\log(a)| + s\log^+\left(\frac{s}{a}\right) \Rightarrow \phi(s) \le s|\log(a)| + a\phi\left(\frac{s}{a}\right). \tag{2.1.20}$$

To show this, we consider the 3 possible cases: $s > a, s < 1 < a$ and $s, a < 1$

1. If $s > a$, then $2\log^-(a) - \log^-(s) \ge 0$, since $\log^-(x)$ is non-increasing and non-negative.

2. If $s < a < 1$, then all properties of the logarithm hold, and

$$\left[2\log^-(a) + \log^-\left(\frac{s}{a}\right) - \log^-(s)\right] = \log^-(a) \ge 0.$$

3. If $s < 1 < a$, then $\log^-(a) = 0$ and $\frac{s}{a} < s$, so our term in brackets becomes

$$\left[\log^-\left(\frac{s}{a}\right) - \log^-(s)\right] \geq 0,$$

again since $\log^-(s)$ is non-increasing.

Using inequality 2.1.20 taking $a = a_n = \|A_n\|_{L^1}$ and $s = A_n * f_n$, we get that

$$\int_{\mathbb{R}^{2d}} \phi(A_n * f_n)\, dxdv \leq a_n \int_{\mathbb{R}^{2d}} \phi\left(\frac{A_n * f_n}{a_n}\right) dxdv + \int_{\mathbb{R}^{2d}} A_n * f_n |\log(a_n)|\, dxdv,$$

and using Young's convolution inequality on the second term of the right hand side,

$$\leq a_n \int_{\mathbb{R}^{2d}} \phi\left(\frac{A_n * f_n}{a_n}\right) dxdv + a_n|\log(a_n)| \int_{\mathbb{R}^{2d}} f_n\, dxdv.$$

Finally, since $\phi(x)$ is a convex function and $\frac{A_n}{a_n}$ is a probability density (as $g_n = \frac{A_n}{a_n} \geq 0$ and $\int_{\mathbb{R}^d} g_n\, dv = 1$), we may use Jensen's Inequality to deduce that

$$\phi\left(\frac{A_n * f_n}{a_n}\right) = \phi\left(\int_{\mathbb{R}^d} \frac{A_n(v_*)}{a_n} f_n(v - v_*)\, dv_*\right) \leq \int_{\mathbb{R}^d} \frac{A_n(v_*)}{a_n} \phi(f_n(v - v_*))\, dv_*,$$

$$\Rightarrow \int_{\mathbb{R}^{2d}} \phi(A_n * f_n)\, dxdv \leq a_n \int_{\mathbb{R}^{2d}} \frac{A_n * \phi(f_n)}{a_n}\, dxdv + a_n|\log(a_n)| \int_{\mathbb{R}^{2d}} f_n\, dxdv$$

$$\leq a_n \int_{\mathbb{R}^{2d}} \left[f_n \log^+(f_n) + |\log(a_n)| f_n\right] dxdv,$$

which is bounded, using our estimates for the solution derived in Step 1. Finally, all we need to prove is that the sequence f_n is tight. To show that, for fixed $R > 0$, consider the following expression

$$\int_{\mathbb{R}^d} \int_{|v|>R} A_n * f_n\, dxdv = \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|v|>R} A_n(v - v_*) f_n(t, x, v_*)\, dxdvdv_*$$

$$= \iiint_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} A_n(v - v_*) f_n(t, x, v_*) \chi_{|v|>R}(v) [\chi_{|v_*| \geq R/2}(v_*) + \chi_{|v_*| \leq R/2}(v_*)]\, dxdvdv_*$$

$$\leq \iiint_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} A_n(v - v_*) f_n(t, x, v_*) \chi_{|v_*| \geq R/2}(v_*)\, dxdvdv_*$$

$$+ \iiint_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} A_n(v - v_*) f_n(t, x, v_*) \chi_{|v|>R}(v) \chi_{|v_*| \leq R/2}(v_*)\, dxdvdv_*.$$

For the first integral after the inequality sign, we have that $|v_*| \geq R/2 \Leftrightarrow 1 \leq \frac{4v_*^2}{R^2}$, so we can create this term inside the first integral, and get the following inequality

$$\iiint_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} A_n(v - v_*) f_n(t, x, v_*) \chi_{|v_*| \geq R/2}(v_*)\, dxdvdv_* \leq \int_{\mathbb{R}^{2d}} A_n * \left[\frac{4v^2}{R^2} f_n\right] dxdv$$

$$\leq \frac{4a_n}{R^2} \int_{\mathbb{R}^{2d}} v^2 f_n\, dxdv.$$

For the second integral, define $z = v_* - v$. We remark that, if $|v| \geq R, |v_*| \leq R/2$, then $|v| = |v_* - z| \leq |v_*| + |z| \leq |z| + R/$, thus $|z| \geq R/2$, such that

$$\{(v, v_*) \in \mathbb{R}^{2d} ||v| \geq R, |v_*| \leq R/2\} \subseteq \{(v, v_*) \in \mathbb{R}^{2d} ||v - v_*| = |z| \geq R/2\}.$$

Therefore, since the integrand is non-negative, we can write

$$\iiint\limits_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} A_n(v-v_*)f_n(t,x,v_*)\chi_{|v|>R}(v)\chi_{|v_*|\le R/2}(v_*)\,dxdvdv_*$$

$$\le \iiint\limits_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} A_n(v-v_*)f_n(t,x,v_*)\chi_{|v|>R}(v)\chi_{|z|\ge R/2}(v-v_*)\,dxdzdv_*$$

$$= \|A_n\chi_{|z|\ge R/2} * f_n\|_{L^1(x,v)} \le \|f_n\|_{L^1(x,v)}\int_{|z|\ge R/2} A(z)\,dz.$$

Combining both, we finally arrive at the expression

$$\int_{\mathbb{R}^d}\int_{|v|>R} A_n * f_n\,dxdv \le \|f_n\|_{L^1(x,v)}\int_{|z|\ge R/2} A(z)\,dz + \frac{4a_n}{R^2}\int_{\mathbb{R}^{2d}} v^2 f_n\,dxdv$$

Since we have bounds for $\int_{\mathbb{R}^{2d}}(1+v^2)f_n\,dxdv$ that are uniform in time, from the above inequality we get that

$$\lim_{R\to\infty}\sup_{t\in[0,T]}\int_{\mathbb{R}^d}\int_{|v|>R} A_n * f_n\,dxdv = 0.$$

Thus we get the tightness property, and from it follows that $\frac{Q_n^-(f_n,f_n)}{1+f_n}(t,x,v)$ is weakly compact in $L^\infty((0,T);L^1(\mathbb{R}^{2d}))$.

Remark 2.1.3. Since $(0,T)$ is bounded, for all fixed $0<T<\infty$:

$$\frac{Q_n^-(f_n,f_n)}{1+f_n}(t,x,v)\in L^\infty((0,T);L^1(\mathbb{R}^{2d})) \Rightarrow \frac{Q_n^-(f_n,f_n)}{1+f_n}(t,x,v)\in L^1((0,T)\times L^1(\mathbb{R}^{2d}))$$

And so $\frac{Q_n^-(f_n,f_n)}{1+f_n}(t,x,v)$ is weakly compact in $L^1((0,T)\times L^1(\mathbb{R}^{2d}))$.

Next, we wisth to prove that $\frac{Q_n^+(f_n,f_n)}{1+f_n}(t,x,v)$ is weakly compact in $L^1((0,T)\times\mathbb{R}^{2d})$ for any $0<T<\infty$. For that, we introduce another important estimate of this demonstration, the so called **entropy inequality**:

$$Q_n^\pm(f_n,f_n) \le Q_n^\mp(f_n,f_n) + \frac{1}{\log K}E_n(f_n), \tag{2.1.21}$$

for all $K>1$, where $E_n = -4D_n$, and

$$D_n(f_n) = \frac{-1}{4}\iint\limits_{\mathbb{R}^d\times\mathbb{S}^{d-1}} [f_n' f_{*,n}' - f_n f_{n,*}]\log\left(\frac{f_n' f_{*,n}'}{f_n f_{n,*}}\right) B_n(v-v_*,\alpha)\,d\mathcal{H}^{d-1}(\alpha)dv_*,$$

(Notice that D_n is exactly the entropy production functional $D[f]$ defined in equation 1.4.5a, with the collision kernel B_n).Then, from this remarkable inequality the weak compactness of the 'gain' part of the collision operator (Q^+)follows from the weak compactness of the 'loss' part (Q^-). To see this, notice that it implies uniform L^1 boundedness, as

$$\left\|\frac{Q_n^+(f_n,f_n)}{1+f_n}\right\|_{L^1_{x,v}} \le \left\|\frac{Q_n^-(f_n,f_n)}{1+f_n}\right\|_{L^1_{x,v}} + \frac{1}{\log(K)}\left\|\frac{E_n(f_n)}{1+f_n}\right\|_{L^1_{x,v}}.$$

Notice that $\left\|\frac{E_n(f_n)}{1+f_n}\right\|_{L^1_{x,v}} < 4\|D_n(f_n)\|_{L^1_{x,v}}$ and we have the following estimate derived in proposition 2.1.1, that

$$\int_{\mathbb{R}^{2d}} f(t,w,v)|\log(f(t,w,v))|\,dwdv + \int_0^t \|D[f](s,x,v)\|_{L^1_{x,v}}\,ds$$

$$\leq \int_{\mathbb{R}^{2d}} (2|w|^2 + 2|v|^2 + |\log(f_0)|) f_0(w,v)\, dwdv + C(d),$$

Applying this to our case,

$$\int_0^T \|D_n[f_n](s,x,v)\|_{L^1_{x,v}}\, ds \leq \int_{\mathbb{R}^{2d}} (2|w|^2 + 2|v|^2 + |\log(f_0^n)|) f_0^n(w,v)\, dwdv + C(d)$$

$$\leq M + C(d),$$

which shows that $\|E_n(f_n)\|_{L^1_{x,v}}$ is uniformly bounded in $L^1((0,T)\times\mathbb{R}^{2d})$.Next, we wish to show that for every $\epsilon > 0$ there is a $\delta > 0$ and $M > 0$ such that for any borel set $A \in \mathbb{R}^{2d}$ with $\mathfrak{L}(A) < \delta$ and any $R > M$, we have

$$\sup_n \int_0^T \int_{\mathbb{R}^{2d}} \frac{Q_n^+(f_n,f_n)}{1+f_n} \left[\chi_A(x,v) + \chi_{\|(x,v)\|>R}\right] dxdvdt < \epsilon,$$

which is equivalent to tightness and uniform integrability. To show this, given that we have already proven the above condition for Q^-, for each $K > 1$ we can always choose A and $R > 0$ such that

$$\sup_n \int_0^T \int_{\mathbb{R}^{2d}} \frac{Q_n^-(f_n,f_n)}{1+f_n} \left[\chi_A(x,v) + \chi_{\|(x,v)\|>R}\right] dxdvdt < \frac{\epsilon}{2K}.$$

Then, using the entropy inequality:

$$\sup_n \int_0^T \int_{\mathbb{R}^{2d}} \frac{Q_n^+(f_n,f_n)}{1+f_n} [\chi_A(x,v) + \chi_{\|(x,v)\|>R}]\, dxdvdt$$

$$\leq K \sup_n \int_0^T \int_{\mathbb{R}^{2d}} \frac{Q_n^-(f_n,f_n)}{1+f_n} [\chi_A(x,v) + \chi_{\|(x,v)\|>R}]\, dxdvdt$$

$$+ \frac{1}{\log(K)} \sup_n \int_0^T \int_{\mathbb{R}^{2d}} \frac{E_n(f_n)}{1+f_n} [\chi_A(x,v) + \chi_{\|(x,v)\|>R}]\, dxdvdt$$

$$< \frac{\epsilon}{2} + \frac{1}{\log(K)} \sup_n \int_0^T \int_{\mathbb{R}^{2d}} \frac{E_n(f_n)}{1+f_n} [\chi_A(x,v) + \chi_{\|(x,v)\|>R}]\, dxdvdt$$

$$< \frac{\epsilon}{2} + \frac{4}{\log(K)} \sup_n \int_0^T \int_{\mathbb{R}^{2d}} -D_n[f_n]\, dxdvdt < \frac{\epsilon}{2} + 4\frac{(M + C(d))}{\log(K)},$$

where $M > 0$ is a constant coming from our choice of initial data, and we used again estimate (4) from proposition2.1.1. Finally, since $K > 1$ is arbitrary, for each $\epsilon > 0$ we can pick $K = e^{\frac{2(M+C(d))}{\epsilon}}$, and conclude that, for every $\epsilon > 0$ we always choose A and R such that

$$\sup_n \int_0^T \int_{\mathbb{R}^{2d}} \frac{Q_n^+(f_n,f_n)}{1+f_n} \left[\chi_A(x,v) + \chi_{\|(x,v)\|>R}\right] dxdvdt < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and so we finish the proof. By the Dunford-Pettis lemma, the positive part of the collision operator is also weakly compact in $L^1((0,T)\times\mathbb{R}^{2d})$. All that remains to prove in this case is the entropy inequality (equation 2.1.21).

Proof. (Entropy Inequality) To simplify notation, in the following demonstration $n \in \mathbb{N}$ is fixed, and we drop the n dependence of f_n. Following a proof by [12], define $A_K = \{(v, v_*, \alpha) : f' f'_* \geq K f f_*\}$. Then, in the case when Q^+ is on the left hand side, we can see that

$$Q_n^+(f_n,f_n) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} f' f'_* B_n(\alpha, v_* - v)[\chi_{A_K}(v,v_*,\alpha) + \chi_{A^c_{K,n}}(v,v_*,\alpha)]\, d\mathcal{H}^{d-1}(\alpha)\, dv_*$$

$$= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} f' f'_* B_n(\alpha, v_* - v) \chi_{A^c_{K,n}}(v, v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha)\, dv_*$$
$$+ \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} [f' f'_* - f f_*] B_n(\alpha, v_* - v) \chi_{A_K}(v, v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha)\, dv_*$$
$$+ \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} f f_* B_n(\alpha, v_* - v) \chi_{A_K}(v, v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha)\, dv_*.$$

In the set A_K, by definition

$$\frac{f' f'_*}{f f_*} \geq K \Leftrightarrow 1 \leq \frac{1}{\log(K)} \log\left(\frac{f' f'_*}{f f_*}\right),$$

so we can manipulate the second term of the r.h.s. as follows

$$\int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} [f' f'_* - f f_*] B_n(\alpha, v_* - v) \chi_{A_K}(v, v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha)\, dv_*$$
$$\leq \frac{1}{\log(K)} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} [f' f'_* - f f_*] \log\left(\frac{f' f'_*}{f f_*}\right) B_n(\alpha, v_* - v) \chi_{A_K}(v, v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha)\, dv_*$$
$$\leq \frac{1}{\log(K)} E_n(f),$$

since the integrand is non-negative. Then by definition of $A^c_{K,n}$,

$$\int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} f' f'_* B_n(\alpha, v_* - v) \chi_{A^c_{K,n}}(v, v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha)\, dv_*$$
$$+ \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} [f' f'_* - f f_*] B_n(\alpha, v_* - v) \chi_{A_K}(v, v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha)\, dv_*$$
$$+ \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} f f_* B_n(\alpha, v_* - v) \chi_{A_K}(v, v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha)\, dv_*$$
$$\leq K \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} f f_* B_n(\alpha, v_* - v) \chi_{A^c_{K,n}}(v, v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha)\, dv_* + \frac{1}{\log(K)} E_n(f)$$
$$+K \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} f f_* B_n(\alpha, v_* - v) \chi_{A_K}(v, v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha)\, dv_* = K Q^-_n(f,f) + \frac{1}{\log(K)} E_n(f).$$

In the case we start with Q^- on the right hand side, the argument is entirely analogous. Take $K < 1$ and split Q^- as

$$Q^-_n(f,f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} f' f'_* B_n(\alpha, v_* - v) \chi_{A^c_{K,n}}(v, v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha)\, dv_*$$
$$+ \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} f f_* B_n(\alpha, v_* - v) \chi_{A_K}(v, v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha)\, dv_*$$
$$+ \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} [f f_* - f' f'_*] B_n(\alpha, v_* - v) \chi_{A^c_{K,n}}(v, v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha)\, dv_*.$$

For the second term on the second term of the r.h.s., we have $f f_* \leq \frac{1}{K} f' f'_*$, and for the third term,

$$\frac{f' f'_*}{f f_*} \leq K \Leftrightarrow 1 \leq \frac{1}{\log(1/K)} \log\left(\frac{f f_*}{f' f'_*}\right),$$

so we find that

$$Q_n^-(f_n, f_n) \leq \frac{1}{K}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} f' f'_* B_n(\alpha, v_* - v)\chi_{A^c_{K,n}}(v, v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha)\, dv_*$$

$$+\frac{1}{K}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} f' f'_* B_n(\alpha, v_* - v)\chi_{A_K}(v, v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha)\, dv_*$$

$$+\frac{1}{\log(1/K)}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} [f f_* - f' f'_*]\log\left(\frac{f f_*}{f' f'_*}\right) B_n(\alpha, v_* - v)\chi_{A^c_{K,n}}(v, v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha)\, dv_*,$$

$$= \frac{1}{K}Q^+(f,f) + \frac{1}{\log(1/K)}E_n(f).$$

Then, defining $K' = \frac{1}{K}$, this means

$$Q_n^-(f,f) \leq K' Q^+(f,f) + \frac{1}{\log(K')}E_n(f).$$

□

We have concluded our proof of weak compactness of the collision operator in the case where $A_n \in L^1$. However, this case is highly non-physical: we can understand truncation in the interaction potential φ in the collision operator (B_n) as a plausible approximation, since it will be small if particles are far apart, which allows us to define A_n (as in equation 2.1.6). But there is no reason to assume that $A_n(v)$ is integrable (in fact, $\|A_n\|$ should increase as $\|v\|$ increases), unless we introduce a non-physical cutoff in the velocity variable. Now we wish to relax this hypothesis before going further. The assumption we'll make on the collision kernel will be that

$$\lim_{|v|\to\infty}\frac{1}{(1+|v|^2)}\int_{|v_*|<R} A_n(v - v_*)\, dv_* = 0 \quad \forall R > 0.$$

Define $A_{R,n}(v) = \int_{|v_*|<R} A_n(v - v_*)\, dv_*$. Then the above condition means that for a given $\epsilon > 0$, there is an M such that

$$A_{R,n}(v) \leq \epsilon(1 + |v|^2) \;\text{ if }\; |v| > M.$$

Since the closed ball defined by $|v| \leq R$ is a compact set and $A_{R,n}$ is continuous, there is an upper bound C_ϵ to the value of $A_{R,n}(v)$ in this set, depending on the chosen ϵ. Therefore, one can always say that for any $R, \epsilon > 0$, there is a C_ϵ such that

$$A_n(v) \leq \epsilon(1 + |v|^2) + C_\epsilon \qquad \forall v \in \mathbb{R}^d. \tag{2.1.22}$$

We do not, however, let go on all assumptions of integrability of A_n: we still require that $A_n \in L^1_{loc}$. Let $A_{n,K}(z) = A_n(z)\chi_{|z|<K}$. Then $A_{n,K}$ is compactly supported and therefore integrable, such that all of the above reasoning used to prove weak compactness holds. What we want to show is that

Lemma 2.1.7.

$$\lim_{K\to\infty}\sup_n \|A_{n,K} * f_n - A_n * f_n\|_{L^\infty(0,T;L^1(\mathbb{R}^d\times B_R))} = 0.$$

If this is true, then suppose $A_{n,K} * f_n$ converges weakly to g_K after taking a convenient subsequence, and $\psi \in C_0^\infty(\mathbb{R}^{2d})$. Then

$$\lim_{K\to\infty}\int_{\mathbb{R}^{2d}} g_K \psi\, dx dv = \lim_{K\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^{2d}} A_{n,K} * f_n \psi\, dx dv$$

$$= \lim_{n\to\infty}\int_{\mathbb{R}^{2d}} A_n * f_n \psi\, dx dv + \lim_{K\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^{2d}} [A_{n,K} - A_n] * f_n \psi\, dx dv.$$

But,

$$\lim_{K\to\infty}\lim_{n\to\infty}\left|\int_{\mathbb{R}^{2d}}[A_{n,K}-A_n]*f_n\psi\,dxdv\right| \le \lim_{K\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^{2d}}|A_{n,K}*f_n-A_n*f_n|\psi\,dxdv$$

$$\le \|\psi\|_{L^\infty}\lim_{K\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^d\times B_R}|A_{n,K}*f_n-A_n*f_n|\,dxdv$$

$$\le \|\psi\|_{L^\infty}\lim_{K\to\infty}\sup_n\|A_{n,K}*f_n-A_n*f_n\|_{L^1(\mathbb{R}^d\times B_R)}=0.$$

Therefore, the fact that the proposition specifies a limit uniform in n makes it possible to 'commute' the two limits and conclude that A_n*f_n is weakly compact if $A_{n,K}$ is. Then, one can conclude that $Q_n^-(f_n.f_n)/(1+f_n)$ is also weakly compact, and finally using again the entropy inequality one concludes that $\frac{Q^\pm}{1+f_n}$ is weakly compact (respectively in $L^\infty((0,T);L^1(\mathbb{R}^{2d}))$, and $L^1((0,T)\times\mathbb{R}^{2d})$). With this, our claim of the weak L^1 compactnes of the collision operator is established.

□

Proof. (lemma) To prove the above lemma, writing it down explicitly,

$$\|A_{n,K}*f_n-A_n*f_n\|_{L^1(\mathbb{R}^d\times B_R)}=\int_{B_R}\int_{\mathbb{R}^{2d}}A_n(v-v_*)\chi_{|v-v_*|\ge R}(v_*)f_n(v_*)\,dxdv_*dv.$$

Notice that if $K>R$, then $|v_*|=|v_*-v+v|\ge|v_*-v|-|v|\ge K-R$. Therefore,

$$\int_{B_R}\int_{\mathbb{R}^{2d}}A_n(v-v_*)\chi_{|v-v_*|\ge R}(v_*)f_n(t,x,v_*)\,dxdv_*dv$$

$$\le\int_{B_R}\int_{\mathbb{R}^{2d}}A_n(v-v_*)\chi_{|v_*|\ge K-R}(v_*)f_n(t,x,v_*)\,dxdv_*dv$$

$$=\int_{\mathbb{R}^{2d}}A_{n,R}(v-v_*)\chi_{|v_*|\ge K-R}(v_*)f_n(t,x,v_*)\,dxdv_*$$

$$\le\epsilon\int_{\mathbb{R}^{2d}}\chi_{|v_*|\ge K-R}(v_*)(1+|v_*|^2)f_n(t,x,v_*)\,dxdv_*+C_\epsilon\int_{\mathbb{R}^{2d}}\chi_{|v_*|\ge K-R}(v_*)f_n(t,x,v_*)\,dxdv_*.$$

Finally, since $|v_*|\ge K-R\Leftrightarrow 1\le\frac{|v_*|^2}{(K-R)^2}$,

$$\le\epsilon\int_{\mathbb{R}^{2d}}(1+|v_*|^2)f_n(t,x,v_*)\,dxdv_*+\frac{C_\epsilon}{(K-R)^2}\int_{\mathbb{R}^{2d}}|v_*|^2f_n(t,x,v_*)\,dxdv_*$$

$$\le\left[\epsilon+\frac{C_\epsilon}{(K-R)^2}\right]\int_{\mathbb{R}^{2d}}(1+|v_*|^2)f_n(t,x,v_*)\,dxdv_*\le M_T\left[\epsilon+\frac{C_\epsilon}{(K-R)^2}\right].$$

Notice that our bound is independent of n,t, so we can say

$$\|A_{n,K}*f_n-A_n*f_n\|_{L^\infty(0,T;L^1(\mathbb{R}^d\times B_R))}\le M_T\left[\epsilon+\frac{C_\epsilon}{(K-R)^2}\right],$$

and taking the limit,

$$\lim_{K\to\infty}\sup_n\|A_{n,K}*f_n-A_n*f_n\|_{L^\infty(0,T;L^1(\mathbb{R}^d\times B_R))}\le M_T\epsilon,$$

which can be made arbitrarily small for any value of R. □

2.1.4 Step 3 - Averaging lemmas and strong compactness

The goal of this step of the proof is to use the weak compactness results from the previous section, and obtain strong compactness, allowing us to have L^1 convergence instead of weak convergence. The main tools for this task will be what's known in the literature as 'averaging lemmas'.

Definition 2.1.6. Throught this and the next sections, to reduce notation, we use the symbol $\mathfrak{D}$ to denote the material derivative, as it was defined in section 1.4:

$$\begin{cases} \mathfrak{D} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} v_i \frac{\partial}{\partial x_i}, \quad \mathfrak{D} f(t,x,v) = \frac{d}{dt} f^{\#}(t,w,v), \\ w = x - vt \end{cases} \tag{Characteristic equation}$$

The lemma which we are interested in proving is the following

Proposition 2.1.3. *Let f_n be a weakly compact sequence in $L^1((0,T)\times\mathbb{R}^{2d})$ and f be a limit point of f_n, and let $g_\delta^n = \beta_\delta(f_n)$, where $\beta_\delta(x) = \frac{1}{\delta}\log(1+\delta x)$. For all $\phi \in L^\infty((0,T)\times\mathbb{R}^{2d})$ with compact support, passing to a subsequence if necessary, we have that*

1.
$$\int_{\mathbb{R}^d} f_n\, dv \to \int_{\mathbb{R}^d} f\, dv \ \text{ in } L^1((0,T)\times\mathbb{R}^d), \tag{2.1.23}$$

2.
$$\int_{\mathbb{R}^d} g_\delta^n\, dv \to \int_{\mathbb{R}^d} g_\delta\, dv \ \text{ in } L^1((0,T)\times\mathbb{R}^d), \tag{2.1.24}$$

3.
$$A_n * f_n \to A * f \ \text{ in } L^1((0,T)\times\mathbb{R}^d\times B_R(0)), \tag{2.1.25}$$

4.
$$\int_{\mathbb{R}^d} Q_n^{\pm}(f_n,f_n)\phi\, dv \to \int_{\mathbb{R}^d} Q^{\pm}(f,f)\phi\, dv \ \text{ in } L^1((0,T)\times\mathbb{R}^d). \tag{2.1.26}$$

The proof of this proposition uses mainly the following theorems

Theorem 2.1.2. *Let $(f_n) \subseteq L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ be a weakly compact subset.*

Assume that $\mathfrak{D} f_n$ is also weakly compact in $L^1_{loc}((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$. Let (ψ_n) be a uniformly bounded sequence in $L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ converging almost everywhere to ψ. Then the sequence

$$(\Psi_n(t,x)) = \int_{\mathbb{R}^d} f_n(t,x,v)\psi_n(t,x,v)\, dv$$

forms a strongly compact subset of $L^1((0,T)\times\mathbb{R}^d)$

Corollary 2.1.2. *If $f_n \rightharpoonup f \in L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$, then*

$$\Psi_n(t,x) = \int_{\mathbb{R}^d} f_n\psi_n\, dv \to \int_{\mathbb{R}^d} f\psi\, dv = \Psi(t,x)$$

strongly in $L^1((0,T)\times\mathbb{R}^d)$.

Theorem 2.1.3 (Vector valued averaging lemma). *Let $(f_n) \subseteq L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ be a weakly compact subset.*

Assume that $\mathfrak{D} f_n$ is also weakly compact in $L^1_{loc}((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$, but this time, let (ψ_n) be a uniformly bounded sequence converging almost everywhere to ψ, both in $L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d;L^1(\mathbb{R}^d))$. Then the *sequence*

$$(\Psi_n(t,x)) = \int_{\mathbb{R}^d} f_n(t,x,v)\psi_n(t,x,v)\,dv$$

forms a strongly compact subset of $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$

The second one is a simple extension of the first one to the vector-valued case, whose proof we will omit. We give a proof to the first theorem in Appendix B.4.

We are now ready to prove our main proposition

Proof. (proposition 2.1.3) Proving the first is simply an application of lemma 2.1.2, as we checked beforehand that all hypotheses for them hold. Taking $\psi_n \equiv 1$ gets us the desired result. For the second item,

$$g_\delta^n(t,x,v) = \frac{1}{\delta}\log(1+\delta f_n),$$

and by the chain rule, g_δ^n solves the equation

$$\mathfrak{D}g_\delta^n = \frac{1}{1+\delta f_n}Q_n(f_n,f_n)$$

where the right-hand side is weakly compact in L^1, as was shown in Step 2. So we again apply lemma 2.1.2, and find the desired result. For the third item,

$$\|A_n * f_n - A*f\|_{L^1} \leq \|A_{n,K}*f_n - A_n * f_n\|_{L^1} + \|A_{n,K} - A_K * f\|_{L^1} + \|A_K * f - A*f\|_{L^1}. \quad (2.1.27)$$

For the first term

$$\|A_{n,K}*f_n - A_n*f_n\| = \int_0^T\int\int_{|v|>K} A_n * f_n\,dtdxdv \leq \epsilon\int_0^T\int\int_{|v|>K}(1+|v|^2)f_n\,dtdxdv \leq \epsilon C_T.$$

For the second term, notice that $A_{n,K} = \psi_n(t,x,v,v_*) \in L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d;L^1(B_R))$. So applying the vector valued version of the averaging lemmas 2.1.3 gives us

$$\int_{B_R} A_{n,K}(v-v_*)f_n(v_*)\,dv_* \to \int_{B_R} A_K(v-v_*)f(v_*)\,dv_*$$

strongly in $L^1((0,T)\times\mathbb{R}^d\times B_R)$ $\quad\forall R>0$ upon passing to a subsequence; so the second term on the right hand side of 2.1.27 can become arbitrarily small. Finally, letting $K\to\infty$ allows us to conclude that $\|A_K - A*f\|\to 0$, concluding the proof.

For the fourth item, we'll consider the gain and loss parts of the collision operator separately (recall equations 2.1.5a, 2.1.7). For the loss term, define

$$\psi_n^- = \frac{A_n * f_n}{1+\delta_n\int_{\mathbb{R}^d} f_n\,dv}\varphi,$$

for any compactly supported $\varphi\in L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ whose support is contained in $B_R(0)$ for a given $R>0$. From the L^1 compactness, we know that there are subsequences such that $A_n * f_n \to A*f$ *a.e.*, $\int_{\mathbb{R}^d} f_n\,dv \to \int_{\mathbb{R}^d} f\,dv$ *a.e.*. Therefore we know that

$$\frac{A_n * f_n}{1+\delta_n\int_{\mathbb{R}^d} f_n\,dv}\varphi \to (A*f)\varphi \;\; a.e.$$

Also, by Young's convolution inequality we have that

$$\|A_n * f_n\|_{L^\infty((0,T)\times\mathbb{R}_x^d\times B_R)} \leq \|A_n\|_{L^\infty((0,T)\times\mathbb{R}_x^d\times B_R)}\|f_n\|_{L^\infty((0,T);L^1(\mathbb{R}_x^d\times B_R)} \leq C_{T,R},$$

uniformly in n. Therefore, the sequence ψ_n^- defined this way almost everywhere convergent and $\psi_n^- \in L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$, so we can use again lemma 2.1.2 and say that

$$\int_{\mathbb{R}^d} Q_n^-(f_n,f_n)\varphi\,dv = \int_{\mathbb{R}^d} f_n\psi_n^-\,dv \to \int_{\mathbb{R}^d} f(A*f)\varphi\,dv = \int_{\mathbb{R}^d} Q^-(f,f)\varphi\,dv.$$

For the positive part of the collision operator, we start by using a standard change of variables (labeled previously as G_2), used when studying the properties of the collision operator in subsection 1.3.2. Then, it can be expressed as

$$\int_{\mathbb{R}^d} Q^+(f,f)\varphi\,dv = \iiint\limits_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} ff_*Bn(v-v_*,\alpha)\varphi'\,d\mathcal{H}^{d-1}(\alpha)dvdv_*,$$

where $\varphi'=\varphi(t,x,v')$. Define then the functions

$$\tilde{A}_n(t,x,v,v_*) = \int_{\mathbb{S}^{d-1}} B_n(v-v_*,\alpha)\varphi'\,d\mathcal{H}^{d-1}(\alpha).$$

$$L_n'(f)(t,x,v) = \int_{\mathbb{R}^d} \tilde{A}_n(t,x,v,v_*)f(t,x,v_*)\,dv_*.$$

For the 'gain' term of our approximate collision operator, we can then say

$$\int_{\mathbb{R}^d} Q_n^+(f_n,f_n)\varphi\,dv = \int_{\mathbb{R}^d} f_n\frac{L_n'(f_n)}{1+\delta_n\int_{\mathbb{R}^d} f_n\,dv}dv = \int_{\mathbb{R}^d} f_n\psi_n^+\,dv,$$

where $\psi_n^+ = \frac{L_n'(f_n)}{1+\delta_n\int_{\mathbb{R}^d} f\,dv}$, a structure which is very similar to the previous case. Since $\exists M>0$ such that $\|\varphi'\|_{L^\infty}\le M$, $\tilde{A}_n\le MA_n$, which implies that $\tilde{A}_n\in L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d;L^1_{loc}(\mathbb{R}^d))$ (since ψ' is compactly supported we can actually drop the locality restriction and say $\tilde{A}_n\in L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d;L^1(\mathbb{R}^d)))$. Therefore, again using lemma 2.1.3

$$L_n'(f_n)(t,x,v) = \int_{\mathbb{R}^d} \tilde{A}_n(t,x,v,v_*)f_n(t,x,v_*)\,dv_*$$

$$\underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B(v-v_*,\alpha)\varphi' f(t,x,v_*)\,d\mathcal{H}^{d-1}(\alpha)dv_* = L'(f)$$

strongly in $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ after passing to a subsequence, which implies convergence almost everywhere. Finally, we can say that

$$\psi_n^+ = \frac{L_n'(f_n)}{1+\delta_n\int_{\mathbb{R}^d} f_n\,dv} \in L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d),$$

and converges almost everywhere to $L'(f)$. Finally then, by lemma 2.1.2,

$$\int_{\mathbb{R}^d} Q_n^+(f_n,f_n)\varphi right),dv = \int_{\mathbb{R}^d} f_n\psi_n^+\,dv \to \int_{\mathbb{R}^d} fL'(f)\,dv = \int_{\mathbb{R}^d} Q^+(f,f)\varphi\,dv,$$

which concludes the proof.

□

In fact, we can generalize proposition 4 in the following way:

Proposition 2.1.4. *Let $\beta\in L^\infty([0,T);\mathbb{R})$ be a Lipschitz function, with $|\beta'(x)|\le C/(1+x)$ for some $C>0$, $\beta(0)=0$. Then, for all $T>0$, $\varphi\in L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$, and f_n satisfying the same conditions* *as above. Then:*

$$\int_{\mathbb{R}^N} Q^{\pm}(\beta(f_n),\beta(f_n))\varphi(x,v)\,dv \to \int_{\mathbb{R}^N} Q^{\pm}(\beta(f),\beta(f))\varphi(x,v)\,dv \tag{2.1.28}$$

strongly in $L^1((0,T)\times\mathbb{R}^d)$ *as* $n\to\infty$.

Proof. we simply perform the same steps as before. first we note by using the chain rule that

$$\mathfrak{D}\beta(f_n) = \beta'(f_n)Q_n(f_n,f_n) \le C\frac{Q_n(f_n,f_n)}{1+f_n},$$

and

$$\beta'(x) \le C\frac{1}{1+x} \Rightarrow \int_0^x \beta'(t)\,dt \le C\int_0^x \frac{1}{1+t}\,dt \therefore \beta(f_n) \le C\log(1+f_n) \le Cf_n.$$

Since the right-hand sides of both inequalities $\left(C\frac{Q_n(f_n,f_n)}{1+f_n}, Cf_n\right)$ are weakly convergent in L^1, as we established before, their left hand side will be as well. Let f_β be the weak limit of $\beta(f_n)$. Using the fact that

$$\|A_n * \beta(f_n)\|_{L^\infty((0,T)\times\mathbb{R}^d\times B_R)} \le \|A_n\|_{L^\infty((0,T)\times\mathbb{R}^d\times B_R)}\|\beta(f_n)\|_{L^1((0,T)\times\mathbb{R}^d\times B_R)} \le MC_{T,R},$$

we can define as in the proof of proposition 2.1.3

$$\psi_n^- = \frac{A_n * \beta(f_n)}{1+\delta_n\int_{\mathbb{R}^d}\beta(f_n)\,dv}\varphi \in L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d),$$

$$\psi_n^+ = \frac{L_n'(\beta(f_n))}{1+\delta_n\int_{\mathbb{R}^d}\beta(f_n)\,dv} \in L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d),$$

and finally conclude that

$$\int_{\mathbb{R}^d} Q^{\pm}(\beta(f_n),\beta(f_n))\varphi\,dv = \int_{\mathbb{R}^d}\beta(f_n)\psi_n^{\pm}\,dv \to \int_{\mathbb{R}^d} Q^{\pm}(f_\beta,f_\beta)\varphi\,dv,$$

where in this case we replaced f_n by $\beta(f_n)$ in the definition of L_n' whenever possible.

□

2.1.5 Step 4- Exponentially Mild solutions

In this section, we prove that every limit function f of the sequence $f_{n\,n\in\mathbb{N}}$ is indeed a renormalized solution to the Boltzmann equation. We start by proving that it satisfies an exponential multiplier form

Proposition 2.1.5. *For each* $T>0$, *any limit function* f *satisfies the following equation for almost every* $(x,v)\in\mathbb{R}^{2d}$

$$f_e(t,x,v) = f_0(x,v) + \int_0^t Q_e^+(f,f)^{\#}(l,x,v)\,dl \quad \theta\in(0,T), \tag{2.1.29}$$

with $Q_e^+(f,f)^{\#}$ *and* $A*f$ *in* $L_t^1(0,T)$, *where we use the definitions*

$$g_e(t,w,v) = g^{\#}(t,w,v)e^{F^{\#}(t,w,v)}, \quad F^{\#}(t,x,v) = \int_0^t (A*f)^{\#}(r,x,v)\,dr,$$

$$\textit{and} \quad A(z) = \int_{\mathbb{S}^{n-1}} B(z,\alpha)\,d\mathcal{H}^{d-1}(\alpha).$$

Proof. of lemma 2.1.5 Dropping some of the heavier notation, what we wish to prove is that

$$f^{\#} = e^{-F^{\#}} \left(f_0 + \int_0^t Q^+(f,f)^{\#} e^{F^{\#}} \, dl \right)$$

To do this, we split this equality into

- $$f^{\#} \le e^{-F^{\#}} \left(f_0 + \int_0^t Q^+(f,f)^{\#} e^{F^{\#}} \, dl \right), \tag{2.1.30}$$

- $$f^{\#} \ge e^{-F^{\#}} \left(f_0 + \int_0^t Q^+(f,f)^{\#} e^{F^{\#}} \, dl \right), \tag{2.1.31}$$

and proving each inequality separately gives us the desired result. In the original derivation of the existence theorem by [12], the authors used these inequalities to say that f was both a supersolution and a subsolution to the Boltzmann equation. To prove the first one, we first perform a computation similar to the one used when deriving equation 2.1.9. We consider again the sequence $g_\delta^n = \frac{1}{\delta} \log(1+\delta f_n)$ We know that this function satisfies from the chain rule, for every fixed n

$$\mathfrak{D} g_\delta^n = \frac{1}{1+\delta f_n} Q_n(f_n, f_n).$$

Separate the collision term in 'gain' and 'loss' parts. This gives us

$$\mathfrak{D} g_\delta^n = \frac{1}{1+\delta f_n} Q_n^+(f_n, f_n) - \frac{(A_n * f_n)}{(1+\delta_n \int_{\mathbb{R}^d} f_n \, dv)(1+\delta f_n)} f_n.$$

Adding $\frac{(A_n * f_n)}{(1+\delta_n \int_{\mathbb{R}^d} f_n \, dv)} g_\delta^n$ to both sides of this equation and passing to the characteristic map $x = w + vt$, we get

$$\frac{d}{dt} g_\delta^{n,\#} + \frac{(A_n * f_n)}{(1+\delta_n \int_{\mathbb{R}^d} f_n \, dv)} g_\delta^{n,\#} = \frac{1}{1+\delta f_n^{\#}} Q_n^{+,\#}(f_n, f_n) - \frac{(A_n * f_n)^{\#}}{(1+\delta_n \int_{\mathbb{R}^d} f_n^{\#} \, dv)} \left[g_\delta^{n,\#} - \frac{f_n^{\#}}{(1+\delta f_n^{\#})} \right].$$

Solving this using an integrating factor, we get that

$$\frac{d}{dt} \left[e^{F_n^{\#}} g_\delta^{n,\#} \right] = e^{F_n^{\#}} \left(\frac{1}{1+\delta f_n^{\#}} Q_n^{+,\#}(f_n, f_n) + \frac{(A_n * f_n)^{\#}}{(1+\delta_n \int_{\mathbb{R}^d} f_n^{\#} \, dv)} \left[g_\delta^{n,\#} - \frac{f_n^{\#}}{(1+\delta f_n^{\#})} \right] \right),$$

or in integral form

$$e^{F_n^{\#}} g_\delta^{n,\#} = g_{\delta,0}^{n,\#} + \int_0^t e^{F_n^{\#}} \left(\frac{1}{1+\delta f_n^{\#}} Q_n^{+,\#}(f_n, f_n) + \frac{(A_n * f_n)^{\#}}{(1+\delta_n \int_{\mathbb{R}^d} f_n^{\#} \, dv)} \left[g_\delta^{n,\#} - \frac{f_n^{\#}}{(1+\delta f_n^{\#})} \right] \right) (r,w,v) \, dr, \tag{2.1.32}$$

where $g_{\delta,0}^n = \frac{1}{\delta} \log(1+\delta f_{n,0})$, and F_n is defined as

$$F_n^{\#}(t,w,v) = \int_0^t \frac{(A_n * f_n)^{\#}}{(1+\delta_n \int_{\mathbb{R}^d} f_n^{\#} \, dv)} (r,w,v) \, dr.$$

Thus, if we define $T_{F_n}[g]^{\#}(t,w,v) = e^{-F_n^{\#}} \int_0^t e^{F_n^{\#}} g^{\#}(r,w,v)\,dr$, equation 2.1.32 can be rewritten as

$$g_\delta^{n,\#} = e^{-F_n^{\#}} g_{\delta,0}^{n,\#} + T_{F_n}\left[\frac{1}{1+\delta f_n} Q_n^+(f_n,f_n)\right]^{\#} + T_{F_n}\left[\frac{(A_n * f_n)}{(1+\delta_n \int_{\mathbb{R}^d} f_n\,dv)}\left[g_\delta^n - \frac{f_n}{(1+\delta f_n)}\right]\right]^{\#}. \quad (2.1.33)$$

The exact same reasoning can be applied to the equation

$$\mathfrak{D} f_n = Q_n(f_n,f_n)$$

Leading to a similar (but simpler) expression

$$f_n^{\#} = e^{-F_n^{\#}} f_0^{n,\#} + T_{F_n}\left[Q_n^+(f_n,f_n)\right]^{\#}, \quad (2.1.34)$$

which will be used when proving the inequality 2.1.31.

We wish to let $n \to \infty$ in equation 2.1.33. From previous sections we know that both g_δ^n and $\frac{Q_n(f_n,f_n)}{1+\delta f_n}$ are weakly compact sequences, so that upon passing to a subsequence, $g_\delta^n \rightharpoonup g_\delta$ and $\frac{Q_n(f_n,f_n)}{1+\delta f_n} \rightharpoonup Q_\delta^+$. Consider now the sequence

$$l_\delta^n = \frac{f_n}{1+\delta f_n},$$

which appears in the right-hand side of equation 2.1.33. Clearly $l_\delta^n < f_n\ \forall \delta > 0$, so that by Dunford-pettis lemma, l_δ^n is also weakly convergent, and $l_\delta^n \rightharpoonup l_\delta$. We also remark that the added sequence $\frac{(A_n * f_n)}{(1+\delta_n \int_{\mathbb{R}^d} f_n\,dv)} g_\delta^n$ is weakly compact, since $g_\delta^n \leq f_n$ and $Q_n^-(f_n,f_n)$ is weakly convergent, as shown previously. Therefore The sequence

$$\frac{(A_n * f_n)}{(1+\delta_n \int_{\mathbb{R}^d} f_n\,dv)}\left[g_\delta^n - \frac{f_n}{(1+\delta f_n)}\right]$$

is weakly compact. Then we introduce the following

Lemma 2.1.8. *Fix $T > 0$. for each $t \in (0,T)$ let $g_n(t,x,v)$ be a weakly convergent sequence in $L^1(\mathbb{R}^d; L^1_{loc}(\mathbb{R}^d))$, $g_n \rightharpoonup g$. Then $T_{F_n}[g_n]^{\#} \to T_F[g]^{\#}$ in $L^1(\mathbb{R}^d; L^1_{loc}(\mathbb{R}^d))$, where $T_F[g]^{\#}(t,w,v) = e^{-F^{\#}} \int_0^t e^{F^{\#}} g^{\#}(r,w,v)\,dr$ and*

$$F^{\#}(t,w,v) = \int_0^t (A * f)^{\#}(r,w,v)\,dr.$$

Proof.

$$T_{F_n}[g_n]^{\#}(t,w,v) = \int_0^t e^{-\int_r^t A_n * f_n(l,w+vl,v)\,dl} g_n^{\#}(r,w,v)\,dr.$$

Let $u_n = e^{-\int_r^t A_n * f_n(l,w+vl,v)\,dl}$. Since $0 < r < t$, we know the exponent is always negative, such that $u_n \leq 1$. Also from the previous section, we find that $A_n * f_n$ converges almost everywhere, and so by continuity so does u_n. We use then the following lemma, whose proof is in Appendix B.5

Lemma 2.1.9. *let u_n be uniformly bounded in $L^\infty(\Omega)$ and $u_n \to u$ a.e. Let $v_n \rightharpoonup v \in L^1(\Omega)$. Then*

$$u_n v_n \rightharpoonup uv \in L^1(\Omega).$$

Taking $\Omega = \mathbb{R}^{2d}$, this means

$$T_{F_n}[g_n] \rightharpoonup T_F[g],$$

□

so we are guaranteed that the right hand side of equation 2.1.33 converges.

Remark 2.1.4. Now it is more apparent why the exponential multiplier form of the solution is convenient for the proof of existence: inside the operator $T_{F_n}(g_n)$, the sequence F_n converges strongly, so we have a product between the weakly converging sequences we constructed previously(generically labeled as g_n) and e^{F_n}, and we are able to prove the convergence of this product.

From the velocity averaging results in the previous section, we know that $\frac{(A_n * f_n)}{(1+\delta_n \int_{\mathbb{R}^d} f_n\, dv)} \to A * f$ almost everywhere in $L^1((0,T)\times\mathbb{R}^d\times B_R)$ $\forall R>0$ after passing to a subsequence. We claim that

$$\left\| \frac{(A_n * f_n)}{(1+\delta_n \int_{\mathbb{R}^d} f_n\, dv)} \right\|_{L^\infty((0,T)\times\mathbb{R}^d\times B_R)} \leq C \quad \forall n, R>0.$$

This comes from our main bound on A_n, namely that $\forall\epsilon>0$,

$$A_n(v) \leq \epsilon(1+|v|^2) + C_\epsilon \qquad \forall v\in\mathbb{R}^d.$$

Then, using Jensen's inequality and the triangle inequality, one can conclude that

$$[A_n * f_n](t,w,v) \leq (2|v|^2 + C_\epsilon + 1)\int_{\mathbb{R}^d} f_n(t,w,y)\, dy + 2\int_{\mathbb{R}^d} |y|^2 f_n(t,w,y)\, dy,$$

and the right hand side is bounded $L^\infty((0,T)\times\mathbb{R}^d\times B_R)$ uniformly in n (since we can bound the integrals using the initial data). Finally, using again lemma 2.1.9 taking $\Omega = ((0,T)\times\mathbb{R}^d\times B_R)$, we can conclude that after passing to a subsequence

$$\frac{(A_n * f_n)}{(1+\delta_n \int_{\mathbb{R}^d} f_n\, dv)}\left[g_\delta^n - \frac{f_n}{(1+\delta f_n)}\right] \rightharpoonup A * f\,[g_\delta - l_\delta] \in L^1((0,T)\times\mathbb{R}^d\times B_R),$$

and finally, passing to a subsequence and taking $n\to\infty$ in equation 2.1.33:

$$g_\delta^\# = e^{-F^\#} g_{\delta,0} + T_F[Q_\delta^+]^\# + T_F[A * f\,[g_\delta - l_\delta]]^\#.$$

Next, we want to take the limit of this expression when $\delta\to 0$. For this, we introduce the following

Lemma 2.1.10. *Let f_n be a weakly convergent sequence in $L^1(\mathbb{R}^{2d})$, and take $\beta_\delta(x):\mathbb{R}^+\to\mathbb{R}^+$ to be either $\beta_\delta(x) = \frac{1}{\delta}\log(1+\delta x)$ or $\beta_\delta(x) = \frac{x}{1+\delta x}$. Then, we have*

$$\lim_{\delta\to 0}\sup_n \|\beta_\delta(f_n) - f_n\|_{L^1(\mathbb{R}^{2d})} = 0.$$

Proof. for $\beta_\delta(x) = \frac{x}{1+\delta x}$, we have that

$$\sup_n \|\beta_\delta(f_n) - f_n\|_{L^1(\mathbb{R}^{2d})} = \delta \sup_{n,t\in(0,T)} \int_{\mathbb{R}^{2d}} \frac{f_n^2}{1+\delta f_n}\, dxdv$$

Notice that we can say that, for each $M>0$

$$t\frac{\delta t}{1+\delta t}\chi_{(M,\infty)}(t) + \delta t\frac{t}{1+\delta t}\chi_{[0,M]}(t) \leq t\chi_{(M,\infty)}(t) + \delta t^2\chi_{[0,M]}(t) \leq t\chi_{(M,\infty)}(t) + \delta t M$$

$$\Rightarrow \frac{\delta t^2}{1+\delta t} \leq t\chi_{(M,\infty)}(t) + \delta t M \quad \forall t>0,$$

therefore,

$$\sup_n \int_{\mathbb{R}^{2d}} \frac{\delta f_n^2}{1+\delta f_n}\, dxdv \leq \sup_n \int_{\{f_n>M\}} f_n\, dxdv + M\delta \sup_n \int_{\mathbb{R}^{2d}} f_n\, dxdv.$$

By definition of unifform integrability, for each $\epsilon>0$ one can always find $M>0$ such that the first term is less than $\epsilon/2$. Then, for this particular choice of M, since we have a bound for $\sup_n \int_{\mathbb{R}^{2d}} f_n \, dxdv$ using the initial data, we can say that there is $C > 0$ such that

$$\sup_n \int_{\mathbb{R}^{2d}} \frac{\delta f_n^2}{1+\delta f_n} \, dxdv < \frac{\epsilon}{2} + CM\delta.$$

Then, taking $\delta < \frac{\epsilon}{2CM}$ makes the right hand side less than ϵ, proving the assertion. For $\beta_\delta(x) = \frac{1}{\delta}\log(1+\delta x)$, since $\frac{d}{dx}\beta_\delta(x) = \frac{1}{1+\delta x}$, then $\beta_\delta(x) \geq \frac{x}{1+\delta x} \geq 0$. So for any $M > 0$, we can establish the bound

$$(x - \beta_\delta(x)) = (x - \beta_\delta(x))\chi_{|x| \leq M} + (x - \beta_\delta(x))\chi_{|x|>M}$$

$$\leq (x - \frac{x}{1+\delta x})\chi_{|x|\leq M} + x\chi_{|x|>M} \leq \delta \frac{x^2}{1+\delta x} + x\chi_{|x|>M}.$$

Therefore,

$$\sup_n \|\beta_\delta(f_n) - f_n\|_{L^1(\mathbb{R}^{2d})} \leq \delta \sup_n \int_{\mathbb{R}^{2d}} \frac{f_n^2}{1+\delta f_n} \, dxdv + \sup_n \int_{\mathbb{R}^{2d}} f_n \chi_{f_n > M} \, dxdv,$$

and by uniform integrability, the last term on the right hand side can be made arbitrarily small, such that

$$\leq \epsilon + \delta \sup_n \int_{\mathbb{R}^{2d}} \frac{f_n^2}{1+\delta f_n} \, dxdv.$$

Then the conclusion follows by the same reasoning as in the previous case.

□

Corollary 2.1.3. *The lemma still holds for any $\beta_\delta(x) \in C^\infty(\mathbb{R}^+;\mathbb{R}^+)$ that satisfies $0 \leq \frac{d}{dx}\beta_\delta(x) \leq \frac{A}{1+\delta x}$ for some $A > 0$.*

Proof. For this class of non-linearities, we can still bound $\beta_\delta(x) \leq \frac{Ax}{1+\delta x}$ and use the same splitting as before. □

Notice that in this lemma, the limit is uniform with respect to n, which allows us to say that

$$\lim_{n\to\infty} \lim_{\delta\to 0} \beta_\delta(f_n) = \lim_{\delta\to 0} \lim_{n\to\infty} \beta_\delta(f_n)$$

weakly for both choices of $\beta_\delta(x)$, namely for $\beta_\delta(f_n) = g_\delta^n$, $\beta_\delta(f_n) = l_\delta^n$. This allows us to conclude that $\lim_{\delta\to 0} g_\delta = \lim_{\delta \to 0} l_\delta = f$, and by continuity

$$T_F[A * f\,[g_\delta - l_\delta]]^\# \to T_F[A * f\,[f - f]]^\# = 0,$$

in $L^1(\mathbb{R}^{2d})$. It would be tempting at this point to search for a similar lemma for the sequence $\frac{Q_n(f_n,f_n)}{1+\delta f_n}$ which would allow us to swap the order of the limits. Due to the absence of such lemma, instead we'll bound Q_δ^+ using the averaging lemmas from the previous section. What we would like to show is that

Proposition 2.1.6.

$$Q_\delta^+ \leq Q^+(f,f).$$

Then, it follows that

$$e^{-F^\#} \int_0^t e^{F^\#} Q_\delta^{+,\#}(r,w,v)\, dr \leq e^{-F^\#} \int_0^t e^{F^\#} Q^{+,\#}(f,f)(r,w,v)\, dr,$$

which means one can write

$$\begin{aligned} g_\delta^\# &= e^{-F^\#} g_{\delta,0}^\# + T_F[Q_\delta^+]^\# + T_F[A * f\,[g_\delta - l_\delta]]^\# \\ &\leq e^{-F^\#} g_{\delta,0}^\# + T_F[Q^+(f,f)]^\# + T_F[A * f\,[g_\delta - l_\delta]]^\#. \end{aligned} \tag{2.1.35}$$

Now proving our desired assertion: for any non-negative $\varphi \in L^\infty((0,T)\times\mathbb{R}^{2d})$ with compact support, we can say

$$\int_{\mathbb{R}^d} \frac{Q_n^+(f_n,f_n)}{1+\delta f_n}\varphi\,dv \leq \int_{\mathbb{R}^d} Q_n^+(f_n,f_n)\varphi\,dv \quad \forall\delta > 0.$$

From the averaging lemmas, we know that the right hand side converges strongly in L^1, while the left hand side integrand is weakly convergent, so that

$$\int_{\mathbb{R}^{2d}} Q_\delta^+\varphi\,dxdv \leq \int_{\mathbb{R}^{2d}} Q^+(f,f)\varphi\,dxdv.$$

Since this is true for any φ, we reach our desired conclusion. Returning to equation 2.1.35, taking the limit as $\delta \to 0$,

$$f^\# \leq e^{-F^\#} \lim_{\delta\to 0} g_{\delta,0}^\# + T_F[Q^+(f,f)]^\#,$$

and

$$\lim_{\delta\to 0} g_{\delta,0} = \lim_{\delta\to 0}\log(1+\delta f_0)^{\frac{1}{\delta}} = \log e^{f_0} = f_0.$$

pointwise. Thus we have that

$$f^\# \leq e^{-F^\#}\left(f_0 + \int_0^t Q^+(f,f)^\# e^{F^\#}\,dr\right)$$

as promised.

Remark 2.1.5. The above limit implies that

$$\int_0^t Q^+(f,f)^\#(r,x,v)e^{\left[F^\#(r,x,v)-F^\#(t,x,v)\right]}\,dr < \infty,$$

for a.e. $(x,v)\in\mathbb{R}^{2d}$. $A * f \geq 0 \Rightarrow F(t,x,v) \geq 0$, which means that $e^{F^\#(r,x,v)} \geq 1$, and so we find that

$$e^{-F^\#(t,x,v)}\int_0^t Q^+(f,f)^\#(r,x,v)\,dr < \infty$$

and

$$Q^+(f,f)^\#(r,x,v) \in L^1((0,T)) \text{ for a.e. } (x,v)\in\mathbb{R}^{2d}.$$

For the opposite inequality, consider the sequence $h_\delta^n(t,x,v) = \min\{f_n(t,x,v), \frac{1}{\delta}\}$. Recalling equation 2.1.34,

$$f_n^\# = e^{-F_n^\#} f_0^{n,\#} + T_{F_n}\left[Q_n^+(f_n,f_n)\right]^\# \geq e^{-F_n^\#} f_0^{n,\#} + T_{F_n}\left[Q_n^+(h_\delta^n,h_\delta^n)\right]^\#. \tag{2.1.36}$$

Since $h_\delta^n \leq f_n$, by using lemmas 2.1.5 and 2.1.6 (Dunford-Pettis's and De-la-Valée-Poussin's lemma), we know that h_δ^n is weakly convergent in L^1 to some function h_δ. Now, by using proposition 2.1.4, we have

$$\int_{\mathbb{R}^d} Q_n^+(h_\delta^n,h_\delta^n)\varphi\,dv \to \int_{\mathbb{R}^d} Q^+(h_\delta,h_\delta)\varphi\,dv,$$

for fixed $\delta > 0$ and $\varphi \in L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ with compact support. This also means that $Q_n^+(h_\delta^n,h_\delta^n) \rightharpoonup Q^+(h_\delta,h_\delta)$ in L^1. Therefore, we can safely take the limit $n\to\infty$ (upon passing to a subsequence) on equation 2.1.36, and conclude that

$$f^{\#} \geq e^{-F^{\#}} f_0^{\#} + T_F \left[Q^+(h_\delta, h_\delta) \right]^{\#}.$$

Finally, one can prove a lemma similar to the previous case to allow the interchange of iterated limits on the sequence h_δ^n:

Lemma 2.1.11. *Let f_n be a weakly compact sequence in $L^1(\mathbb{R}^{2d})$. Then, it follows that*

$$\lim_{\delta \to 0} \sup_{n>0} \left\| f_n - \min\left\{ f_n(x,v), \frac{1}{\delta} \right\} \right\|_{L^1(\mathbb{R}^{2d})} = 0.$$

Proof.

$$\begin{aligned} \left\| f_n - \min\left\{ f_n(x,v), \frac{1}{\delta} \right\} \right\|_{L^1(\mathbb{R}^{2d})} &= \int_{\{f_n > \frac{1}{\delta}\}} \left| f_n - \frac{1}{\delta} \right| dxdv \\ &\leq \int_{\{f_n > \frac{1}{\delta}\}} f_n \, dxdv + \frac{1}{\delta} \int_{\{f_n > \frac{1}{\delta}\}} 1 \, dxdv \\ &\leq 2 \int_{\{f_n > \frac{1}{\delta}\}} f_n \, dxdv, \end{aligned}$$

where we used Chebyshev's inequality to estimate the second term on the right hand side. Finally taking the supremum,

$$\sup_{n>0} \left\| f_n - \min\left\{ f_n(x,v), \frac{1}{\delta} \right\} \right\|_{L^1(\mathbb{R}^{2d})} \leq 2 \sup_{n>0} \int_{\{f_n > \frac{1}{\delta}\}} f_n \, dxdv,$$

this term can be made arbitrarily small by uniform integrability, so we have our desired assertion. □

So we have that $h_\delta \rightharpoonup f$. Since this convergence is monotonic, we find that $Q^+(h_\delta, h_\delta) \uparrow Q^+(f,f)$, we can conclude by the monotone convergence theorem, by taking $\delta \to 0$,

$$f^{\#} \geq e^{-F^{\#}} f_0^{\#} + T_F \left[Q^+(f, f) \right]^{\#}.$$

□

Now, what we wish to say is that

Proposition 2.1.7. *f constructed in this way is a renormalized solution satisfying all conditions in theorem 2.1.1*

Proof. Notice that from the above results, $Q_e^+, (A * f) \in L^1(0,T)$, and by which means that f satisfies an exponential multiplier form. We will first collect the estimates that we need, and then by applying the lemma 2.1.1, we will be able to show that f is indeed a renormalized solution. Starting from the entropy inequality (2.1.21) for f_n, for any $K > 1$

$$Q_n^{\pm}(f_n, f_n) \leq K Q_n^{\mp}(f_n, f_n) + \frac{1}{\log K} E_n(f_n).$$

Divide both sides by $1 + \delta \int_{\mathbb{R}^d} f_n \, dv$, and multiply both sides by a non-negative, compactly supported $\varphi \in L^\infty((0,T) \times \mathbb{R}^{2d})$:

$$\frac{Q_n^{\pm}(f_n, f_n)}{1 + \delta \int_{\mathbb{R}^d} f_n \, dv} \varphi \leq K \frac{Q_n^{\mp}(f_n, f_n)}{1 + \delta \int_{\mathbb{R}^d} f_n \, dv} \varphi + \frac{1}{\log K} \frac{E_n(f_n)}{1 + \delta \int_{\mathbb{R}^d} f_n \, dv} \varphi.$$

By the averaging lemmas, we can integrate and take limits, concluding that

$$\frac{\int Q^{\pm}(f,f) \varphi \, dv}{1 + \delta \int_{\mathbb{R}^d} f \, dv} \leq K \frac{\int Q^{\mp}(f,f) \varphi \, dv}{1 + \delta \int_{\mathbb{R}^d} f \, dv} + \frac{1}{\log K} \lim_{n \to \infty} \frac{\int_{\mathbb{R}^d} E_n(f_n) \varphi \, dv}{1 + \delta \int_{\mathbb{R}^d} f_n \, dv}.$$

$E_n(f_n)$ is a uniformly bounded sequence in $L^1((0,T)\times\mathbb{R}^{2d})$. Therefore, it will converge to some measure μ, such that

$$\frac{\int Q^{\pm}(f,f)\varphi\,dv}{1+\delta\int_{\mathbb{R}^d} f\,dv} \le K\frac{\int Q^{\mp}(f,f)\varphi\,dv}{1+\delta\int_{\mathbb{R}^d} f\,dv} + \frac{1}{\log K}\frac{\int_{\mathbb{R}^d}\varphi\,d\mu}{1+\delta\int_{\mathbb{R}^d} f\,dv}.$$

Since $Q^{\pm}dv$ are absolutely continuous measures, we can consider only the absolutely continuous part of the measure μ, and denoting its density by $e\in L^1((0,T)\times\mathbb{R}^{2d})$, this means

$$Q^{\pm}(f,f)\le KQ^{\mp}(f,f)+\frac{1}{\log K}e, \tag{2.1.37}$$

almost everywhere. This allows to prove that

- $Q^-(f,f)\in L^1(0,T)$ for a.e. $x,v\in\mathbb{R}^d$. Using the above inequality, taking the $-$ sign on the left hand side, all we need to show is that $Q^+(f,f)\in L^1(0,T)$ for a.e. $x,v\in\mathbb{R}^d$, which we showed in remark 2.1.5. Therefore, from item (iii) of lemma (2.1.1), f is also a mild solution.
- $\frac{Q^{\pm}(f,f)}{1+f}\in L^1((0,T)\times\mathbb{R}^d;L^1_{loc}(\mathbb{R}^d))$. For $\frac{Q^-(f,f)}{1+f}$, we know that since $A_n*f_n\to A*f$ in $L^1((0,T)\times\mathbb{R}^d\times B_R)$ for each $R>0$,

  $$\frac{Q^-(f,f)}{1+f}=\frac{(A*f)f}{1+f}\le A*f\in L^1((0,T)\times\mathbb{R}^d\times B_R).$$

  Therefore $\frac{Q^-(f,f)}{1+f}\in L^1((0,T)\times\mathbb{R}^d;L^1_{loc}(\mathbb{R}^d))$. For $\frac{Q^+(f,f)}{1+f}$, dividing inequality (2.1.37) by $1+f$ we can say,

  $$\frac{Q^+(f,f)}{1+f}\le K\frac{Q^-(f,f)}{1+f}+\frac{1}{\log K}\frac{e}{1+f}\le K\frac{Q^-(f,f)}{1+f}+\frac{1}{\log K}e,$$

  which implies that $\frac{Q^+(f,f)}{1+f}\in L^1((0,T)\times\mathbb{R}^d;L^1_{loc}(\mathbb{R}^d))$.

Therefore, f is a renormalized solution as promised, using item (ii) of theorem 2.1.1.

2.1.6 Other properties of a renormalized solution

1. *Time continuity* We can show that in fact, the limit $f(\cdot,x,v)$ is continuous a.e. From the fact that f is a mild solution, for a.e. $(x,v)\in\mathbb{R}^{2d}$ and every $h,T\in\mathbb{R}_+$ such that $t,t+h\in(0,T)$, we have

   $$|f(t+h,x,v)-f(t,x,v)|=\left|\int_t^{t+h}Q(f,f)(s,x,v)\,ds\right|\le\int_t^{t+h}|Q(f,f)(s,x,v)|\,ds.$$

   Since $Q(f,f)\in L^1_t((0,T))$, for all $\epsilon>0$ we can always find $h>0$ such that

   $$|f(t+h,x,v)-f(t,x,v)|<\epsilon,$$

   and we have continuity of our solution.

2. *Conservation laws* Firstly, we can assert that the renormalized solution f must also satisfy the same bounds as f_n, namely, for any $T>0$, $t\in(0,T)$,

   $$\int_{\mathbb{R}^d}(1+\|x-vt\|^2+\|v\|^2)f\,dv<C(d,T)$$

   To prove this, from the weak convergence of f_n, we can take $\phi(t,x,v)=(1+\|x-vt\|^2+\|v\|^2)\chi_{B_R(0)}(v)\in L^\infty((0,T)\times\mathbb{R}^{2d})$, and conclude that

   $$\int_{\mathbb{R}^{2d}}f_n\phi(t,x,v)\,dxdv\to\int_{\mathbb{R}^{2d}}f\phi(t,x,v)\,dxdv,$$

and for f_n we have the estimate

$$\int_{\mathbb{R}^{2d}} f_n \phi(t,x,v)\,dv \leq \int_{\mathbb{R}^{2d}} f_n(1+\|x-vt\|^2+\|v\|^2)\,dxdv = \int_{\mathbb{R}^{2d}} f_0^n(1+\|x\|^2+\|v\|^2)\,dxdv$$

$$\underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^{2d}} f_0(1+\|x\|^2+\|v\|^2)\,dxdv$$

(recall that we constructed the sequence f_0^n in order for this last convergence to hold). Finally, this means that for each $R>0$

$$\int_{\mathbb{R}^{2d}} f(1+\|x-vt\|^2+\|v\|^2)\chi_{B_R(0)}(v)\,dxdv \leq \int_{\mathbb{R}^{2d}} f_0(1+\|x\|^2+\|v\|^2)\,dxdv,$$

and since the estimate is independent of R,

$$\int_{\mathbb{R}^{2d}} f(1+\|x-vt\|^2+\|v\|^2)\,dxdv \leq \int_{\mathbb{R}^{2d}} f_0(1+\|x\|^2+\|v\|^2)\,dxdv < \infty, \tag{2.1.38}$$

and proceeding as in the proof of proposition 2.1.1, we find that for some $C>0$,

$$\int_{\mathbb{R}^{2d}} f(1+\|x\|^2+\|v\|^2)\,dxdv \leq \int_{\mathbb{R}^{2d}} f_0(1+\|x\|^2+\|v\|^2)\,dxdv < C(1+t^2). \tag{2.1.39}$$

This also allows us to conclude that renormalized solutions may not conserve energy, although we guarantee that energy cannot increase, that is

$$\int_{\mathbb{R}^{2d}} f(t,x,v)\frac{\|v\|^2}{2}\,dxdv \leq \int_{\mathbb{R}^{2d}} f_0(x,v)\frac{\|v\|^2}{2}\,dxdv, \tag{2.1.40}$$

and a similar reasoning as given above shows that

$$\int_{\mathbb{R}^{2d}} f(t,x,v)v\,dxdv \leq \int_{\mathbb{R}^{2d}} f_0(x,v)v\,dxdv. \tag{2.1.41}$$

In order to understand why there may not be equalities in these 2 cases, we perform the next calculations, following a proof by [32].

For any $\phi \in C_c^1((0,T)\times\mathbb{R}^{2d})$ with subquadratic growth at infinity, i.e.

$$\lim_{\|v\|\to\infty} \frac{\phi(v)}{1+\|v\|^2} = 0$$

we have that

$$\int_{\mathbb{R}^d} g_\delta^n \phi(v)\,dv \to \int_{\mathbb{R}^d} g_\delta \phi(v)\,dv$$

from the averaging lemmas, where $g_\delta^n = \frac{1}{\delta}\log(1+\delta f_n)$ (modulo passing to a subsequence). Then, it is possible to show that [33]

$$\int_{\mathbb{R}^d} g_\delta^n v\,dv \to \int_{\mathbb{R}^d} g_\delta v\,dv,$$

and in the limit when $\delta \to 0^+$ we recover that

$$\int_{\mathbb{R}^d} f_n v\,dv \to \int_{\mathbb{R}^d} f v\,dv.$$

This allows us to conclude that the total momentum is actually conserved, i.e.

$$\int_{\mathbb{R}^{2d}} f(t,x,v)v\,dxdv = \int_{\mathbb{R}^{2d}} f(t,x,v)v\,dxdv. \tag{2.1.42}$$

Remark 2.1.6. Since

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^d} f_n\,dv + \nabla_x \cdot \int_{\mathbb{R}^d} v f_n\,dv = 0$$

holds in the classical sense, it is also a solution in the weak sense, such that for each $\phi \in C_c^1((0,T)\times \mathbb{R}^d)$:

$$\int_0^T \int_{\mathbb{R}^d} \left[-\frac{\partial \phi}{\partial t}\left(\int_{\mathbb{R}^d} f_n\,dv\right) - \left(\int_{\mathbb{R}^d} v f_n\,dv\right)\nabla_x \phi\right] dtdx = 0.$$

So from the above results and item (1) of proposition 2.1.3, taking the limit of a converging subsequence, this converges to

$$\int_0^T \int_{\mathbb{R}^d} \left[-\frac{\partial \phi}{\partial t}\left(\int_{\mathbb{R}^d} f v\,dv\right) - \left(\int_{\mathbb{R}^d} v f\,dv\right)\nabla_x \phi\right] dtdx = 0,$$

which is a weak version of the continuity equation.

In the quadratic case, this does not hold. What we can say in this case is that, for any $i,j \in \{1,2,...,d\}$ and for a.e. $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$:

$$\left|\int_{\mathbb{R}^d} v_i v_j f_n\,dv\right| \leq \int_{\mathbb{R}^d} \frac{1}{2}\|v\|^2 f_n\,dv = \int_{\mathbb{R}^d} \frac{1}{2}\|v\|^2 f_0^n\,dv,$$

which is uniformly bounded. Therefore, we have that

$$\int_{\mathbb{R}^d} v_i v_j f_n\,dv \underset{w-*}{\rightharpoonup} \mu_{ij}$$

however, if we truncate the integrand, by the averaging lemmas, we find

$$\int_{\mathbb{R}^d} v_i v_j \chi_{B_R(0)}(v) f_n\,dv \to \int_{\mathbb{R}^d} v_i v_j \chi_{B_R(0)}(v) f\,dv.$$

Then, define the measure

$$m_{ij}^R = \left[\mu_{ij} - \int_{\mathbb{R}^d} v_i v_j \chi_{B_R(0)}(v) f\,dv\right]. \tag{2.1.43}$$

Notice that both μ_{ij} and m_{ij}^R are symmetric with respect to permutation of indices. For any nonnegative $g(t,x) \in C_c((0,T)\times\mathbb{R}^d)$ and a fixed vector $r \in \mathbb{R}^d$, we have that

$$\sum_{i,j=1}^{i,j=d} \int_0^T \int_{\mathbb{R}^d} g r_i r_j\,dm_{ij}^R = \lim_{n\to\infty} \sum_{i,j=1}^{i,j=d} \int_0^T \int_{\mathbb{R}^d} g \int_{\mathbb{R}^d} \|v\cdot r\|^2 \chi_{\|v\|\geq R}(v)\,dv\,dxdt \geq 0.$$

Using the monotone convergence theorem, we can take the limit of $R \to \infty$ on the right hand side and say that

$$\sum_{i,j=1}^{i,j=d} \int_0^T \int_{\mathbb{R}^d} g r_i r_j\,dm_{ij} \geq 0,$$

where

$$m_{ij} = \left[\mu_{ij} - \int_{\mathbb{R}^d} v_i v_j f\,dv\right] \Leftrightarrow \mu_{ij} = m_{ij} + \int_{\mathbb{R}^d} v_i v_j f\,dv, \tag{2.1.44}$$

and therefore

$$\int_{\mathbb{R}^d} v_i v_j f_n \, dv \underset{w-*}{\rightharpoonup} m_{ij} + \int_{\mathbb{R}^d} v_i v_j f \, dv \tag{2.1.45}$$

where $m_{ij} \in L^\infty(\mathbb{R}_+; \mathcal{M}(\mathbb{R}^d))$ forms a symmetric, nonnegative definite matrix. If this matrix is identically 0, then we will have energy conservation. To see this, simply take the trace of equation (2.1.45) to conclude that

$$\int_{\mathbb{R}^d} \frac{\|v\|^2}{2} f_0 \, dv = \frac{Tr(m)}{2} + \int_{\mathbb{R}^d} \frac{\|v\|^2}{2} f \, dv.$$

Remark 2.1.7. Since

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^d} f_n v \, dv + \nabla_x \cdot \int_{\mathbb{R}^d} v \otimes v f_n \, dv = 0$$

holds in the classical sense, it is also a solution in the weak sense, such that for each $\phi \in C_c^1((0,T) \times \mathbb{R}^d)$:

$$\int_0^T \int_{\mathbb{R}^d} \left[-\frac{\partial \phi}{\partial t} \left(\int_{\mathbb{R}^d} f_n v \, dv \right) - \left(\int_{\mathbb{R}^d} v \otimes v f_n \, dv \right) \nabla_x \phi \right] dt dx = 0$$

then, taking the limit of a converging subsequence, from equation (2.1.45) and item (1) of proposition 2.1.3 this converges to

$$\int_0^T \int_{\mathbb{R}^d} \left[-\frac{\partial \phi}{\partial t} \left(\int_{\mathbb{R}^d} f v \, dv \right) - \left(\int_{\mathbb{R}^d} v \otimes v f \, dv \right) \nabla_x \phi \right] dt dx = \int_0^T \int_{\mathbb{R}^d} \nabla_x \phi \, dm(t,x).$$

In this sense only, we can say that the following balance law holds for momentum

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^d} f v \, dv + \nabla_x \cdot \int_{\mathbb{R}^d} v \otimes v f \, dv + \nabla_x \cdot m = 0.$$

3. *Entropy production* we remark that, since $x \log(x)$ is a convex function, we must have that, under weak convergence

$$\begin{aligned} \int_{\mathbb{R}^{2d}} f \log(f) \, dx dv &\leq \varliminf_{n \to \infty} \int_{\mathbb{R}^{2d}} f_n \log(f_n) \, dx dv \leq \\ = \varliminf_{n \to \infty} \int_0^t \int_{\mathbb{R}^{2d}} D_n[f_n](s,x,v) \, dx dv \, ds &+ \int_{\mathbb{R}^{2d}} \log(f_0^n) f_0^n(x,v) \, dx dv \\ &\leq \varliminf_{n \to \infty} \int_{\mathbb{R}^{2d}} \log(f_0^n) f_0^n(x,v) \, dx dv < M, \end{aligned} \tag{2.1.46}$$

where we used that the entropy of f_0^n was uniformly bounded by assumption. Finally, using lemma 2.1.4, we can conclude that:

$$\begin{aligned} \int_{\mathbb{R}^{2d}} f |\log(f)| \, dx dv &\leq C(d) + 2 \int_{\mathbb{R}^{2d}} f(|x - vt|^2 + |v|^2) \, dx dv + \int_{\mathbb{R}^{2d}} f \log(f) \, dx dv \\ &< C(d) + M + 2 \int_{\mathbb{R}^{2d}} f_0 |v|^2 \, dx dv + 2 \int_{\mathbb{R}^{2d}} f |x - vt|^2 \, dx dv. \end{aligned}$$

Finally, from the weak convergence of f_n (upon passing to a subsequence):

$$\int_{\mathbb{R}^{2d}} f_n |x - vt|^2 \chi_{\|(x,v)\| < R}(x,v) \, dx dv \to \int_{\mathbb{R}^{2d}} f |x - vt|^2 \chi_{\|(x,v)\| < R}(x,v) \, dx dv,$$

$$\int_{\mathbb{R}^{2d}} f_n |x - vt|^2 \chi_{\|(x,v)\| < R}(x,v) \, dx dv \leq \int_{\mathbb{R}^{2d}} f_n |x - vt|^2 \, dx dv = \int_{\mathbb{R}^{2d}} f_0^n |x|^2 \, dx dv$$

From the above results, we find that

$$\int_{\mathbb{R}^{2d}} f|x-vt|^2 \chi_{\|(x,v)\|<R}(x,v)\,dxdv \le \int_{\mathbb{R}^{2d}} f_0|x|^2\,dxdv.$$

Finally, taking the limit as $R \to \infty$, from the monotone convergence theorem, we find

$$\int_{\mathbb{R}^{2d}} f|x-vt|^2\,dxdv \le \int_{\mathbb{R}^{2d}} f_0|x|^2\,dxdv,$$

and therefore,

$$\int_{\mathbb{R}^{2d}} f(t,x,v)|\log(f(t,x,v))|\,dxdv \le C(d) + M + 2\int_{\mathbb{R}^{2d}} f_0(x,v)\left[|x|^2+|v|^2\right]\,dxdv \tag{2.1.47}$$

So that the entropy always remains bounded.

□

Remark 2.1.8. l As was shown in [13], this result can be improved upon, if we recall that the function $f(x,v) : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $f(x,y) = (x-y)\log(x/y)$ is convex. Then, if we establish that

$$f_n(\cdot,\cdot,v)f_n(\cdot,\cdot,v_*)\frac{B_n(v-v_*,\alpha)}{1+\delta_n\int_{\mathbb{R}^d} f_n(\cdot,\cdot,\xi)\,d\xi} \rightharpoonup f(\cdot,\cdot,v)f(\cdot,\cdot,v_*)B(v-v_*,\alpha) \tag{2.1.48}$$

$$f_n(\cdot,\cdot,v')f_n(\cdot,\cdot,v_*')\frac{B_n(v-v_*,\alpha)}{1+\delta_n\int_{\mathbb{R}^d} f_n(\cdot,\cdot,\xi)\,d\xi} \rightharpoonup f(t,x,v')f(t,x,v_*')B(v-v_*,\alpha) \tag{2.1.49}$$

in $L^1_{loc}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1})$ for a.e. $(t,x) \in [0,T] \times \mathbb{R}^d$, we will be able to say that

$$-\int_0^t \int_{\mathbb{R}^{2d}} D[f](s,x,v)\,dxdv\,ds \le \varliminf_{n\to\infty} -\int_0^t \int_{\mathbb{R}^{2d}} D_n[f_n](s,x,v)\,dxdv\,ds$$

where D is defined in equation (1.4.5a). Finally, this implies that

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} f\log(f)\,dxdv - \int_0^t \int_{\mathbb{R}^{2d}} D[f](s,x,v)\,dxdv\,ds \\
&\le \varliminf_{n\to\infty}\left[\int_{\mathbb{R}^{2d}} f_n\log(f_n)\,dxdv - \int_0^t \int_{\mathbb{R}^{2D}} D_n[f_n](s,x,v)\,dxdv\,ds\right] \\
&= \varliminf_{n\to\infty}\int_{\mathbb{R}^{2d}} \log(f_0^n)f_0^n(x,v)\,dxdv = \int_{\mathbb{R}^{2d}} \log(f_0)f_0(x,v)\,dxdv,
\end{aligned}$$

and therefore, f satisfies estimate (4) from proposition 2.1.1:

$$\int_{\mathbb{R}^{2d}} f\log(f)\,dxdv - \int_0^t \int_{\mathbb{R}^{2d}} D[f](s,x,v)\,dxdv\,ds \le \int_{\mathbb{R}^{2d}} \log(f_0)f_0(x,v)\,dxdv. \tag{2.1.50}$$

and from this, we conclude that

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^{2d}} D[f](s,x,v)\,dxdv\,ds &\le \int_{\mathbb{R}^{2d}} \log(f_0)f_0(x,v)\,dxdv - \int_{\mathbb{R}^{2d}} f\log(f)\,dxdv \\
&\le \int_{\mathbb{R}^{2d}} \log(f_0)f_0(x,v)\,dxdv + \int_{\mathbb{R}^{2d}} f|\log(f)|\,dxdv
\end{aligned}$$

$$\leq C(d) + 2M + 2\int_{\mathbb{R}^{2d}} f_0(x,v)\left[|x|^2 + |v|^2\right] dxdv = B(d)$$

$$\Rightarrow \int_0^T \int_{\mathbb{R}^{2d}} D[f](s,x,v)\, dxdv\, ds < B(d) \quad (\forall T > 0), \tag{2.1.51}$$

so that the entropy production always remains finite. Finally, we prove the desired convergence results. Starting from (2.1.48), taking $R \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1})$, what we wish to say is that

$$\iiint_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} f_n(\cdot,\cdot,v) f_n(\cdot,\cdot,v_*) \frac{B_n(v-v_*,\alpha)}{1+\delta_n \int_{\mathbb{R}^d} f_n(\cdot,\cdot,\xi)\, d\xi} R(v,v_*,\alpha)\, dvdv_* d\mathcal{H}^{d-1}(\alpha)$$

Converges upon passing to a subsequence. Denoting the above integral as I_n, we see that it must be finite, as if $R < C$ for some $C > 0$, then $I_n \leq C Q_n^-(f_n, f_n)$. Then, we are able to apply Fubini's theorem, and define

$$\begin{cases} A_{R,n}(v,v_*) = \int_{\mathbb{S}^{d-1}} B_n(v-v_*,\alpha) R(v,v_*,\alpha)\, d\mathcal{H}^{d-1}(\alpha), \\ L_{R,n}(f_n) = \int_{\mathbb{R}^d} A_{R,n}(v,v_*) f_n(t,x,v)\, dv_*. \end{cases}$$

Then, $A_{R,n}(v,v_*) \leq C A_n(v_* - v)$, which implies that

$$|L_{R,n}(t,x,v)| \leq \int_{\mathbb{R}^d} |A_{R,n}(v,v_*)| f_n(t,x,v)\, dv_* \leq C A_n * f_n,$$

and we can write our expression as

$$\iiint_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} f_n(\cdot,\cdot,v) f_n(\cdot,\cdot,v_*) \frac{B_n(v-v_*,\alpha)}{1+\delta_n \int_{\mathbb{R}^d} f_n(\cdot,\cdot,\xi)\, d\xi} R(v,v_*,\alpha)\, dvdv_* d\mathcal{H}^{d-1}(\alpha)$$

$$= \int_{\mathbb{R}^d} f_n \frac{L_{R,n}(f_n)}{1+\delta_n \int_{\mathbb{R}^d} f_n(\cdot,\cdot,\xi)\, d\xi}\, dv.$$

Define

$$\psi_{R,n}^- = \frac{L_{R,n}(f_n)}{1+\delta_n \int_{\mathbb{R}^d} f_n(\cdot,\cdot,\xi)\, d\xi}.$$

Again from the averaging lemmas we find that $\int_{\mathbb{R}^d} f_n \psi_{R,n}^-\, dv$ converges as desired. Applying the exact same reasoning to the other case allows us to prove 2.1.49

2.2 Extensions and other remarks

Here we wish to draw a few conclusions from the existence result derived in the previous chapter, and we would like to mention, even if briefly, recent results about renormalized solutions.

1. Even though theorem 2.1.1 guarantees that under special conditions renormalized and weak solutions are equivalent, the renormalized formulation has the advantage of having to solve equation 2.1.3

   $$\frac{\partial \beta(f)}{\partial t} + v \cdot \nabla_x \beta(f) = \beta'(f) Q(f,f),$$

   instead of the original Boltzmann equation. Crucial to this proof was the fact that we were able to choose $\beta'(f) = \frac{1}{1+f}$, so that our right hand side became $\frac{Q(f,f)}{1+f}$. Replacing then f by a constructed sequence f_n, the sequence $Z_n = \frac{Q(f_n,f_n)}{1+f_n}$ was shown to be weakly compact in L^1, a non-trivial result since weak convergence is generally not preserved under non-linear operators.

2. The existence theorem is proved in the previous section is not the only celebrated result from the

Diperna-Lions theory of renormalized solutions: in fact, this demonstration can be adapted in order to prove a stability result which is more important than the existence result. The theorem goes as follows.

Theorem 2.2.1. *Let (f_0^n) and B be a sequence of initial datum and a collision kernel, respectively, such that they satisfy hypotheses* (2.1.10),(2.1.1),(2.1.11) *as in theorem 2.1.1. Let f_n be the sequence of renormalized solutions associated with the initial data f_0^n, and assume without loss of generality that $(f^n) \rightharpoonup f \in L^p((0,T) \times L^1(\mathbb{R}^{2d}))$, $(1 \le p < \infty)$. Then, the following holds*

(a) f is a renormalized solution.

(b) $f_n \to f$ strongly in $L^p((0,T) \times L^1(\mathbb{R}^{2d}))$, if and only if $f_0^n \to f_0$ strongly in $L^1(\mathbb{R}^{2d})$.

Proof. The proof of item (2) relies crucially in an estimate on the collision operator, called the **collision smoothing property**:

Lemma 2.2.1. *Define $Q^+(f,g)$ as the following operator:*

$$Q^+(f,g) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} f(v')g(v'_*)B(v_* - v, \alpha)\, d\alpha dv_*,$$

where $B(z,\alpha) \in C^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})$, $f \in L^1(\mathbb{R}^d_v)$ and $g \in L^2(\mathbb{R}^d_v)$. If B satisfies the hypotheses in theorem 2.1.1 along with

- $\operatorname{supp} B \subseteq \{(z,\alpha)$ *s.t.* $r_1 < \|z\|, k_1 < \|\alpha \cdot z\| < k_2\}$ *for some choice of $r_1 > 0$ and $k_1, k_2 \in (0, \|z\|)$,*
- $\lim_{\|z\|\to\infty} B(z,\alpha) = 0$ *uniformly in α,*

Then we find that,

$$\|Q^+(f,g)\|_{H^{\frac{d-1}{2}}(\mathbb{R}^d_v)} \le \|f\|_{L^1(\mathbb{R}^d_v)} \|g\|_{L^2(\mathbb{R}^d_v)}. \tag{2.2.1}$$

Proofs of this lemma generally use the theory of pseudo-differential operators [25] or properties of generalized Radon transforms [41], both of which fall beyond the scope of this dissertation. We direct the reader to the sources [26], [25].

For item (1), the proof strategy is the same as in theorem 2.1.1: for each initial data f_0^n we are guaranteed that exists a renormalized solution, obeying the estimates (2.1.47), (2.1.50),(2.1.38),(2.1.45). Then, for step (2) of the demonstration, again from lemmas 2.1.5 and 2.1.6 we can conclude that $f_n \rightharpoonup f$ (as was said in the theorem). All that needs to be shown is that

$$Z_n(t,x,v) = \frac{Q(f_n, f_n)}{1 + f_n} \qquad (\forall n \ge 1)$$

is weakly compact. We can again prove the entropy inequality (2.1.21) in this case, again reducing the weak compactness proof to the $Q^-(f,f)$ part. Taking the exact same steps as before, one can say that

$$Q^-(f_n, f_n)\frac{1}{1+f_n} f_n(A * (f_n)) \le A * (f_n)$$

And using the same truncation argument for A as in step (2) (Defining for $K > 0$ $A_K = A(z)\chi_{\|z\|<K}(z)$), we finish the proof. The only difference in in the proof this case and theorem 2.1.1 will appear in Steps (3) and (4). Firstly, in any argument where we used the chain rule of differentiation, we instead use lemma B.1.2. In step (3), since we are using Q instead of Q_n, what we prove is that upon passing to a subsequence, for each $\delta > 0$,

$$\int_{\mathbb{R}^d} Q^\pm(f_n, f_n) \frac{1}{1 + \delta \int_{\mathbb{R}^d} f_n(\cdot,\cdot,\xi)\, d\xi} \phi\, dv \to \int_{\mathbb{R}^d} Q^\pm(f,f) \frac{1}{1 + \delta \int_{\mathbb{R}^d} f(\cdot,\cdot,\xi)\, d\xi} \phi\, dv \text{ in } L^1((0,T)\times\mathbb{R}^d). \tag{2.2.2}$$

For the minus case, again we define ψ_n^-, this time as

$$\psi_n^-(t,x,v) = \frac{A * f_n(t,x,v)}{1 + \delta \int_{\mathbb{R}^d} f_n(t,x,\xi)\, d\xi}\phi,$$

so that our integral can be written as

$$\int_{\mathbb{R}^d} Q^\pm(f_n,f_n) \frac{1}{1 + \delta \int_{\mathbb{R}^d} f_n(\cdot,\cdot,\xi)\, d\xi}\phi\, dv = \int_{\mathbb{R}^d} f_n \psi_n^-\, dv.$$

By Young's convolution inequality we have that, for the v variable

$$\|A * f_n\|_{L^\infty(B_R)} \leq \|A\|_{L^\infty(B_R)} \|f_n\|_{L^1(B_R)} \leq C_R \int_{B_R} f_n(\cdot,\cdot,\xi)\, d\xi,$$

so that

$$\begin{aligned}\|\psi_n^-(t,x,v)\|_{L^\infty((0,T)\times\mathbb{R}_x^d\times\mathbb{R}_v^d)} &\leq C_R \|\phi\|_{L^\infty((0,T)\times\mathbb{R}_x^d\times\mathbb{R}_v^d)} \frac{\int_{B_R} f_n(\cdot,\cdot,\xi)\, d\xi}{1 + \delta \int_{\mathbb{R}^d} f_n(t,x,\xi)\, d\xi} \\ &\leq C_{T,R}.\end{aligned}$$

where $C_R, C_{T,R}$ are postive constants depending only on the subscripted variables. For the positive part, we define as before

$$\tilde{A}(t,x,v,v_*) = \int_{\mathbb{S}^{d-1}} B(v - v_*, \alpha)\varphi'\, d\mathcal{H}^{d-1}(\alpha),$$

$$L'(f)(t,x,v) = \int_{\mathbb{R}^d} \tilde{A}(t,x,v,v_*) f(t,x,v_*)\, dv_*,$$

$$\psi_n^+ = \frac{L'(f_n)}{1 + \delta \int_{\mathbb{R}^d} f_n\, dv}\phi.$$

In both cases we have a.e. convergence of a subsequence, but this time, since we are considering a fixed $\delta > 0$, the sequences converge to $\frac{L'(f_n)}{1+\delta \int_{\mathbb{R}^d} f_n\, dv}\phi$ and $\frac{A*f(t,x,v)}{1+\delta \int_{\mathbb{R}^d} f(t,x,\xi)\, d\xi}\phi$, respectively. So proceeding as in step (3), we demonstrate that (2.2.2) holds. Finally for step (4), the only change in the previous reasoning we have to make is in the proof of proposition 2.1.6: for any nonnegative $\psi \in L^\infty((0,T\times)\mathbb{R}_x^d)$ and $\varphi \in L^\infty(\mathbb{R}^d)$, we can say that

$$\int_{\mathbb{R}^{2d}} \frac{Q^+(f_n,f_n)}{1+\delta f_n} \frac{\varphi\psi}{1 + \int_{\mathbb{R}^d} f_n(\cdot,\cdot,\xi)\, d\xi}\, dxdv \leq \int_{\mathbb{R}^d} \psi \left[\int_{\mathbb{R}^d} \varphi \frac{Q^+(f_n,f_n)}{1 + \int_{\mathbb{R}^d} f_n(\cdot,\cdot,\xi)\, d\xi}\, dv \right] dx \quad \forall \delta > 0.$$

from our modified averaging lemmas, we can say that the term in square brackets on the right hand side converges strongly after passing to a subsequence and taking the limit, so with all the more reason it converges weakly, and the right hand side will converge. The term $\frac{Q^+(f_n,f_n)}{1+\delta f_n}$ on the left hand side is weakly convergent in L^1 and $1 + \int_{\mathbb{R}^d} f_n(\cdot,\cdot,\xi)\, d\xi$ converges strongly from the averaging lemmas modulo passing to a subsequence, so we have

$$\int_{\mathbb{R}^{2d}} Q_\delta^+ \frac{\varphi\psi}{1 + \int_{\mathbb{R}^d} f(\cdot,\cdot,\xi)\, d\xi}\, dxdv \leq \int_{\mathbb{R}^{2d}} Q^+(f,f) \frac{\varphi\psi}{1 + \int_{\mathbb{R}^d} f(\cdot,\cdot,\xi)\, d\xi}\, dxdv \quad \forall \delta > 0.$$

So we still find our desired result that $Q_\delta^+ \leq Q^+(f,f)$ a.e. Doing the necessary replacements, one can follow the same steps in the proof of Step (4), and conclude that f is indeed a renormalized solution. □

The second item of this theorem is interesting, for one of the main reasons why weakly converging sequences do not converge strongly is that they may develop oscillations, as seen from the sequence $u_n(x) = \sin(nx)$, $x \in \mathbb{R}$, $u_n \rightharpoonup 0$. This theorem then shows that, *if oscillations are not already present in the initial data, they will not be generated at later times* [1].

3. The first item of theorem 2.2.1, together with remark 2.1.8 allows us to conclude that *the long time behavior of a renormalized solution is a local Maxwellian.* To see this, take $t_n \geq 0$, $t_n \to \infty$, f a renormalized solution in the sense of theorem 2.1.1, and define $f_n(t,x,v) = f(t+t_n,x,v)$. f_n defined this way is a sequence of renormalized solutions, satisfying the estimates (2.1.47), (2.1.50),(2.1.38), (2.1.45), and (2.1.51). Therefore, upon passing to a subsequence they converge weakly to some renormalized solution $\overline{f}$. Also, notice that, by making the change of variables $u = t + t_n$

$$\begin{aligned}\int_0^T \iint_{\mathbb{R}^{2d}} D(f_n)\, dxdvdt &= \frac{-1}{4}\int_0^T \iiiint_{\mathbb{R}^d_x\times\mathbb{R}^d_v\times\mathbb{R}^d_{v_*}\times\mathbb{S}^{d-1}} (f_n' f_{n,*}' - f_n f_{n,*}) \log\left(\frac{f_n' f_{n,*}'}{f_n f_{n,*}}\right) B(w,\alpha) d\mathcal{H}^{d-1}(\alpha) dvdxdv_*dt \\ &= \frac{-1}{4}\int_{t_n}^T \iiiint_{\mathbb{R}^d_x\times\mathbb{R}^d_v\times\mathbb{R}^d_{v_*}\times\mathbb{S}^{d-1}} (f' f_*' - f_n f_*) \log\left(\frac{f' f_*'}{f f_*}\right) B(w,\alpha) d\mathcal{H}^{d-1}(\alpha) dvdxdv_*du \\ &\underset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

since t_n can be arbitrarily large, and we have estimate (2.1.51). Therefore, from the convexity of $f(x,v) : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $f(x,y) = (x-y)\log(x/y)$, we find again by the same argument used in remark 2.1.8,

$$0 \leq -\int_0^T \int_{\mathbb{R}^{2d}} D[\overline{f}](s,x,v)\, dxdv\, ds \leq \varliminf_{k\to\infty} -\int_0^t \int_{\mathbb{R}^{2d}} D[f_{n_k}](s,x,v)\, dxdv\, ds = 0.$$

Since the integrand is non-negative, we find that $D[\overline{f}] = 0$ for a.e. $(t,x,v) \in (0,T)\times\mathbb{R}^{2d}$, which implies $\overline{f}'\overline{f}_*' = \overline{f}\overline{f}_*$ for a.e. $(t,x,v,\alpha) \in (0,T)\times\mathbb{R}^{2d}\times\mathbb{S}^{d-1}$. Finally, from theorem A.2.1, $\overline{f}$ is of the form

$$\overline{f} = a(t,x)e^{b(t,x)\|v-u(t,x)\|^2} \quad \text{for a.e. } (t,x,v) \in (0,T)\times\mathbb{R}^{2d}$$

(see Appendix A.2.1, as well as reference [10]). Furthermore, if we impose periodic boundary conditions [25] (replacing $\mathbb{R}^d$ by $\mathbb{T}^d$ as the domain for the x variable),or consider renormalized solutions for the Boltzmann equation with bounded domains [32], then it's possible to show that the limit $\overline{f}$ is a global maxwellian (equilibrium distribution).

4. Uniqueness and propagation of regularity is still an open problem for this class of solutions. An improvement that can be easily made to theorem 2.1.1 is to place an extra restriction on the Collision kernel, which makes the matrix m in equation (2.1.44) null. The requirement is that

$$\int_{\mathbb{S}^{d-1}} (1+\cos\theta)B(v_*-v)\, d\mathcal{H}(\alpha) \geq r \int_{\mathbb{S}^{d-1}} B(v_*-v)\, d\mathcal{H}(\alpha),$$

for some $r > 0$ [32].

5. Renormalized solutions are not classical, and so may fail to have all physical properties of the true physical system (e.g. it may have decreasing total energy even if the gas is isolated). However, they exist for all times and for all initial data obeying only the minimal restrictions the system must have (limited energy, momentum and entropy). If we allow ourselves to treat special cases - in which for instance the initial data satisfies additional restrictions, or the solution holds only for short times, or for selected kinds of interaction potential - then we can find stronger solutions, indeed even smooth (C^∞) solutions, as in [20]. For an extensive list of results in this direction, we recommend the reading of [36], [8] and [39].

Renormalized solutions for cross-sections without cutoff

Perhaps the most important generalization of the concept of renormalized solutions in the context of the Boltzmann equation, was that of renormalized solutions with defect measure ,proposed by [1]. The need for a generalized notion of renormalized solution first appeared when trying to use the theory of renormalized solutions as defined in 2.1.3 in other in other so called kinetic equations (e.g. in the Cauchy problem for the Vlasov-Poisson equation, and in the hydrodynamic limit problem). The definition goes as follows

Definition 2.2.1. Let $\beta \in C^2(\mathbb{R}^+, \mathbb{R}^+)$, $\beta(0) = 0$ be a function satisfying, for any $C > 0$,

$$0 < \beta'(x) \leq \frac{C}{1+x}, \quad \beta''(x) < 0.$$

$f \in L^1_{loc}([0,\infty) \times \mathbb{R}^{2d})$ is a *renormalized solution with defect measure* to equation 2.1.1 with initial condition $f_0 \in L^1(\mathbb{R}^{2d})$ if it solves the following inequality in the weak sense

$$\frac{\partial \beta(f)}{\partial t} + v \cdot \nabla_x \beta(f) \geq \beta'(f) Q(f,f),$$

or explicitly, $\forall \phi \in C_0^\infty([0,T) \times \mathbb{R}^d)$,

$$\int_0^T \int_{\mathbb{R}^{2d}} -\beta(f) \left[\frac{\partial \phi}{\partial t} + v \cdot \nabla_x \phi \right] dxdvdt - \int_{\mathbb{R}^{2d}} \beta(f_0(x,v)) \phi(0,x,v)\, dxdv \tag{2.2.3}$$

$$\geq \int_0^T \int_{\mathbb{R}^{2d}} \phi \beta'(f) Q(f,f)\, dxdvdt, \tag{2.2.4}$$

along with the following conditions

$$\int_{\mathbb{R}^{2d}} f(t,x,v)\, dxdv = \int_{\mathbb{R}^{2d}} f_0(x,v)\, dxdv; \quad \frac{Q(f,f)}{1+f} \in L^1_{loc}([0,\infty) \times \mathbb{R}^{2d}).$$

Remark 2.2.1. Notice that if we assume that the distribution

$$\langle T, \phi \rangle = \left\langle \frac{\partial \beta(f)}{\partial t} + v \cdot \nabla_x \beta(f) - \beta'(f) Q(f,f), \phi \right\rangle$$

has a definite sign, then it must be a Radon measure [5]. This measure can be formally constructed by considering the standard parabollic perturbation of the Boltzmann equation: mollify the collision operator and the initial data by making a convolution with a standard mollifier

$$\rho^\epsilon = \frac{1}{\epsilon^{2d}} \rho \left(\frac{x}{\epsilon}, \frac{v}{\epsilon} \right),$$

where ρ is a non-negative element of $C_0^\infty(\mathbb{R}^{2d})$. Consider then the following problem

$$\begin{cases} \frac{\partial f_\epsilon}{\partial t} + v \cdot \nabla_x f_\epsilon = Q^\epsilon(f_\epsilon, f_\epsilon) + \epsilon \Delta f_\epsilon, & (t,x,v) \in (0,T) \times \mathbb{R}^{2d}. \\ f_\epsilon(0,x,v) = f_0^\epsilon(x,v) & (x,v) \in \mathbb{R}^{2d} \end{cases}$$

Formally, by multiplying both sides of the equation by $\beta'(f_\epsilon)$ and using the chain rule, gives us

$$\frac{\partial \beta(f_\epsilon)}{\partial t} + v \cdot \nabla_x \beta(f_\epsilon) = \beta'(f_\epsilon) Q^\epsilon(f_\epsilon, f_\epsilon) + \epsilon \Delta \beta(f_\epsilon) - \epsilon \beta''(f_\epsilon) \|\nabla f_\epsilon\|^2.$$

From the concavity of $\beta(x)$ we see that the term $-\epsilon\beta''(f_\epsilon)\|\nabla f_\epsilon\|^2 \geq 0$, so we have

$$\frac{\partial\beta(f_\epsilon)}{\partial t} + v\cdot\nabla_x\beta(f_\epsilon) \geq \beta'(f_\epsilon)Q^\epsilon(f_\epsilon, f_\epsilon) + \epsilon\Delta\beta(f_\epsilon).$$

and formally as we let $\epsilon \to 0$, we would reach equation (2.2.3). For a more in depth discussion of the vanishing viscosity method with the necessary steps to make this argument rigorous, we recommend the reading of [29].

The theorem 2.1.1, despite its many merits, is still not completely general due to its restrictions on the form of the collision kernel $B(z,\alpha)$. Throughout the theorem we applied Grad's cutoff assumption, which is what guarantees that the collision operator can be separated into "gain" and "loss" terms, both well defined [39].To tackle the problem of the Boltzmann equation withoutGrad's cutoff assumption, one has to analyze the cancellations between the positive and negative parts of the Boltzmann equation. It was believed on physical grounds that the cutoff assumption was not detrimental to the analysis of the Boltzmann equation, since the interaction between particles far apart would be almost negligible [1]. However, as was remarked in [7], if we take the limit of the parameter $\sigma \to \infty$, where σ describes the radius of the shock cross-section, formally it is expected that the Boltzmann equation can be approximated by a non-linear Fokker-Planck-type equation, of the form

$$\frac{\partial}{\partial t}f + v\cdot\nabla_x f = \Delta_v(D(v)f(v)) + \nabla_v\cdot(R(v)f),$$

where we use the following definitions

$$R(\cdot,\cdot,v) = \int_{\mathbb{R}^3\times\mathbb{S}^2}(v'-v)f(\cdot,\cdot,v_*)B(v-v_*,\alpha)\,dv_*d\mathcal{H}^2(\alpha),$$
$$D(\cdot,\cdot,v) = \int_{\mathbb{R}^3\times\mathbb{S}^2}\|v'-v\|^2 f(\cdot,\cdot,v_*)B(v-v_*,\alpha)\,dv_*d\mathcal{H}^2(\alpha).$$

The laplacian on the right hand-side of the equation suggests that this equation, and also perhaps the full Boltzmann equation without cutoff, may exhibit better regularity properties than the Boltzmann equation in the cutoff case. This conjecture was proven true by [2], whose main result was an estimate of the type

$$\|\sqrt{f}\|_{H^s(\{\|v\|<R\})} \leq (D(f) + \|f\|^2_{L^1_2(\mathbb{R}^d_v)}),$$

for each $R > 0$, where the exponent $0 < s < 2$ depends on the particular type of interaction potential assumed between the particles. Finally, [1] used this estimate to prove a generalized version of theorem 2.1.1, which we state below.

Definition 2.2.2. We say that a collision kernel is *at most borderline singular* if it satisfies the following assumptions

1. For $p \in [0,2]$, $k = \frac{z}{|z|}$, define

$$M^p(|z|) = \int_{\mathbb{S}^{d-1}} B(z,\alpha)(1-k\cdot\alpha)^p d\mathcal{H}^{d-1}.$$

   Then, for each $R > 0$,

$$\lim_{|z_0|\to\infty}\frac{1}{|z_0|^{2-p}}\int_{|z-z_0|\leq R} M^p(|z|)dz = 0$$

2. $B(z,\alpha) \geq \Phi(|z|)b(k\cdot\alpha)$, where $\Phi \in C(\mathbb{R}_+;\mathbb{R}_+)$ is a nonnegative function, and

$$\int_{\mathbb{S}^{d-1}} b(k\cdot\alpha)\,d\mathcal{H}^{d-1}(\alpha) = \infty.$$

3. Assume that
$$B = \frac{\beta_0(\alpha \cdot k)}{|z|^d} + B_1(z, \alpha),$$
where β_0, B_1 are measurable functions, and define
$$\mu_0 = \int_{\mathbb{S}^{d-1}} \beta_0(k \cdot \alpha)(1 - k \cdot \alpha)\, d\mathcal{H}^{d-1}, \tag{2.2.5}$$
$$M_1(|z|) = \int_{\mathbb{S}^{d-1}} B_1(z, \alpha)(1 - k \cdot \alpha)\, d\mathcal{H}^{d-1}, \tag{2.2.6}$$
$$M'(|z|) = \int_{\mathbb{S}^{d-1}} B'(z, \alpha)(1 - k \cdot \alpha)\, d\mathcal{H}^{d-1}, \tag{2.2.7}$$
where
$$B'(z, \alpha) = \sup_{\lambda \in (1, \sqrt{2}]} \frac{|B_1(\lambda z, \alpha) - B_1(z, \alpha)|}{(\lambda - 1)|z|}. \tag{2.2.8}$$
We require that $\mu_0 < \infty$, and $M_1(|z|), M'(|z|) \in L^1_{loc}(\mathbb{R}^d)$.

Theorem 2.2.2. *Assume the collision operator is at most borderline singular, as in definition 2.2.2, and that f_0 satisfies.*
$$f_0 \geq 0; \quad \int_{\mathbb{R}^d} f_0(1 + |v|^2 + |x|^2 + |\ln f_0|)\, dv < \infty. \tag{2.2.9}$$
Then, there exists a solution f to 2.1.1 with initial condition f_0, valid for all $t > 0$ in the sense of definition 2.2.1, which satisfies
$$\int_{\mathbb{R}^{2d}} f(t, x, v) v\, dxdv = \int_{\mathbb{R}^{2d}} f(t, x, v) v\, dxdv,$$
$$\int_{\mathbb{R}^d} \frac{\|v\|^2}{2} f_0\, dv \geq \int_{\mathbb{R}^d} \frac{\|v\|^2}{2} f\, dv.$$

Theorem 2.2.3. *Let (f_0^n) and B be a sequence of initial datum and a collision kernel, respectively, such that they satisfy hypotheses* (2.1.10) *and 2.2.2 as in theorem 2.1.1. Let f_n be the sequence of renormalized solutions associated with the initial data f_0^n, and assume without loss of generality that $(f^n) \rightharpoonup f \in L^p((0, T) \times L^1(\mathbb{R}^{2d}))$, $(1 \leq p < \infty)$. Then, the following holds*

1. *f is a renormalized solution.*
2. *$f_n \to f$ strongly in $L^p((0, T) \times L^1(\mathbb{R}^{2d}))$.*

Notice the difference between this theorem and theorem 2.2.1. This theorem implies that any oscillations in the initial data are smoothed out in time, a behavior remarkably different from the cutoff case [39]. This smoothing behavior of the non-cutoff collision operator is what allows for existence theorems of C^∞ solutions in this case, as in [20].

Chapter 3

Conclusion

In this thesis, we presented the basic theory of the Boltzmann equation, covering both its physical origins with its connection to classical mechanics and thermodynamics, as well as introducing the theory of renormalized solutions for the Boltzmann equations in reasonable detail. The theorem 2.1.1 was the first existence theorem for the Boltzmann equation capable of handling initial data satisfying only the natural a priori bounds of limited energy, mass and entropy [19], and the techniques developed in its construction were instrumental to providing existence theorems for the Vlasov-Poisson system, as well as the Fokker-Planck-Boltzmann and the Landau equations [39].

For the Boltzmann equation, theorem 2.1.1 guarantees the existence of a solution $f \in L^1(\mathbb{R}^{2d}_{x,v})$ that

- is valid for all times ($t \in (0,T)$ for each $T > 0$), is continuous with respect to the time variable ($f \in C\left((0,T); L^1_+(\mathbb{R}^{2d}_{x,v})\right)$) and for all initial data satisfying the natural bounds

$$f_0 \geq 0; \quad \int_{\mathbb{R}^d} f_0(1 + |v|^2 + |x|^2 + |\ln f_0|)\, dv < \infty,$$

- for some fixed $C > 0$, obeys the estimates

$$\begin{cases} \int_{\mathbb{R}^{2d}} f(1 + \|x\|^2 + \|v\|^2)\, dxdv \leq \int_{\mathbb{R}^{2d}} f_0(1 + \|x\|^2 + \|v\|^2)\, dxdv < C(1 + t^2), \\ \int_{\mathbb{R}^{2d}} f \log(f)\, dxdv - \int_0^t \int_{\mathbb{R}^{2d}} D[f](s,x,v)\, dxdv\, ds \leq \int_{\mathbb{R}^{2d}} \log(f_0) f_0(x,v)\, dxdv, \end{cases}$$

- has the local and global conservation laws

$$\begin{cases} \int_{\mathbb{R}^{2d}} f(t,x,v)v\, dxdv = \int_{\mathbb{R}^{2d}} f(t,x,v)v\, dxdv, \\ \int_{\mathbb{R}^d} \frac{\|v\|^2}{2} f_0\, dv = \frac{Tr(m)}{2} + \int_{\mathbb{R}^d} \frac{\|v\|^2}{2} f\, dv, \\ \int_0^T \int_{\mathbb{R}^d} \left[-\frac{\partial \phi}{\partial t} \left(\int_{\mathbb{R}^d} f v\, dv \right) - \left(\int_{\mathbb{R}^d} v f\, dv \right) \nabla_x \phi \right] dtdx = 0, \\ \int_0^T \int_{\mathbb{R}^d} \left[-\frac{\partial \phi}{\partial t} \left(\int_{\mathbb{R}^d} f v\, dv \right) - \left(\int_{\mathbb{R}^d} v \otimes v f\, dv \right) \nabla_x \phi \right] dtdx = \int_0^T \int_{\mathbb{R}^d} \nabla_x \phi\, dm(t,x), \end{cases}$$

  where $m \in L^\infty((0,T) \times \mathbb{M}(\mathbb{R}^{3\times 3})$ is a symmetric matrix, and $\phi \in C^1_c((0,T) \times \mathbb{R}^d_x)$ is arbitrary.

- converges weakly in L^1 for large times to a local maxwellian.

There are many applications of the Boltzmann equation that may be of interest to a reader of this text. Here, we provide a short description of some of these applications.

One directions in which renormalized solutions can be generalized and applied would be: in the semi-classical treatment of electrons and other particles usually described by a quantum mechanical formalism, in which a 'quantum Boltzmann equation' can be derived [35]. In fact, many quantum mechanical versions of the Boltzmann equation can be derived, starting for instance from the Von Neumann equation of evolution for mixed quantum states:

$$\frac{\partial}{\partial t}|\psi\rangle\langle\psi| = \left[|\psi\rangle\langle\psi|, \hat{H}\right] \tag{3.0.1}$$

This equation is the quantum mechanical analogue of the Liouville equation, so a similar procedure to the one applied in section 1.5.2 can be used to find a hierarchy of equations alogous to the BBGKY hierarchy, and from it a quantum mechanical Boltzmann equation could be obtained as a limit. Another possible application would be to consider the Linear Boltzmann equation, in which the Collision operator is linearized around a global Maxwellian, acquiring the form

$$L(\phi) = \int\limits_{\mathbb{R}^3} \iint\limits_{\mathbb{S}^2} P_M P_{M*} \left[\phi' + \phi'_* - \phi - \phi_*\right] B(\eta, ||w||)\, d\mathcal{H}^2(\alpha)\, dv_*. \tag{3.0.2}$$

This equation shares some similar features with the Boltzmann equation, and is actually a standard model in neutron transport problems arising in the design and analysis of nuclear reactors [28].

One of the most important problems where renormalized solutions were applied to is the problem of hydrodynamic limits, which as defined before, consists of finding what are the macroscopic constitutive relations a solution of the Boltzmann equation must obey.

The first attempt to solve this problem was made by David Hilbert (1862-1943), as part of his goal of making the foundations of physical theories mathematically rigorous (which later would become known as Hilbert's 6th problem). The starting point for his attempt is the nondimensionalized form of the Boltzmann equation.

Let l_0 be a characteristic macroscopic length scale, and u_0 a macroscopic velocity scale. From them, it is possible to define a macroscopic time scale $t_0 = l_0/u_0$. It is posible to derive as well the sound speed of the gas as a microscopic velocity scale, as well as a microscopic lenght scale from the mean free path λ.

Using the sound speed as a microscopic scale is not a problem, since according to the Maxwell-Boltzmann distribution for gases in equilibrium, it is related to the average thermal speed by the following equation $\left(c = \sqrt{\frac{5\pi}{24}}||\bar{u}||_{MB}\right)$, and so we can define a microscopic time scale by c/λ, which will be related to the mean free time by the the same ratio.

The collision kernel has units of $[=]\,1/t$, so it can be made adimensional using the microscopic scales, by defining

$$B(\eta, ||w||) = \frac{c}{\lambda}\tilde{B}(\eta, ||w||).$$

Finally, introduce new non-dimensional variables through the following relationships: $t = t_0 t'$,$x = l_0 x'$, $v = cv'$.

From the chain rule, inserting the above expressions on the original Boltzmann equation, we find

$$\frac{u_0}{l_0}\frac{\partial f}{\partial t'} + \frac{c}{l_0}v' \cdot \nabla_{x'} f + \frac{u_0 W'(t,x,v)}{ct_0} \cdot \nabla_{v'} f = \frac{c}{\lambda}\mathbb{C}'(f)(t,x,v)$$

Multiplying both sides by l_0 and dividing by c, we have:

$$\frac{u_0}{c}\frac{\partial f}{\partial t'} + v' \cdot \nabla_{x'} f + \left(\frac{u_0}{c}\right)^2 W'(t,x,v) \cdot \nabla_{v'} f = \frac{l_0}{\lambda}\mathbb{C}'(f)(t,x,v)$$

Define the Mach numbaer as $Ma = u_0/c$, and the Knudsen number as $Kn = \frac{\lambda}{l_0}$. Then, our equation can be rewritten as:

$$Kn\left(Ma\frac{\partial f}{\partial t'} + v' \cdot \nabla_{x'} f + Ma^2 W'(t,x,v) \cdot \nabla_{v'} f\right) = \mathbb{C}'(f)(t,x,v) \tag{3.0.3}$$

The Knudsen number encodes how separated are the microscopic and macroscopic length scales. In general, for liquids or dense gases, the Knudsen number is very small (for air at room temperature at sea level, the Knudsen number is of the order of 10^{-8}). Therefore, if the Mach number is not too large so that all terms on the left hand side can be considered to have the same order of magnitude (subsonic flows), one can treat Kn as a perturbation, and 'expand' our solution in powers of Kn: this method is known as Hilbert's expansion. A more modern version of the method, known as Chapman-Enskog expansion is used below. In the above equation, if we set $Kn = 0$, we will find that

$$\mathbb{C}'(f)(t,x,v) = 0 \Rightarrow f = M_{\rho,u,\epsilon}(t,x,v)$$

Therefore, in our 'zeroth order' perturbation, the solution of the Boltzmann equation should be a local maxwellian if the Knudsen number is identically 0, and if we insert $f = M_{\rho,u,\epsilon}(t,x,v)$ in the Boltzmann equation, we recover the fact that the macroscopic fields ρ, u, ϵ obey the following system of equations

$$\begin{cases} \frac{\partial}{\partial t}\rho(x,t) + \nabla\cdot(\rho(x,t)u(x,t)) = 0 \\ \frac{\partial}{\partial t}\rho(t,x)u(x,t) + \nabla\cdot(\rho(t,x)u\otimes u + (RT)Id) = 0 \\ \frac{\partial}{\partial t}\left[\rho(x,t)\left(\frac{3}{2}RT + \frac{||u||^2}{2}\right)\right] + \nabla\cdot\rho(x,t)u\left[\frac{5}{2}RT + \frac{||u||^2}{2}\right] = 0 \end{cases}$$

which describe the gas as a perfect fluid obeying the ideal gas law, with a fixed ratio $\gamma = \frac{Cp}{Cv} = \frac{5}{3}$ between specific heats, which characterizes the gas as monatomic (for a derivation of this system of equations, see section 1.3.3). Indeed, one can relate the Mach and Knudsen numbers by the following expression:

$$Kn = \frac{Ma}{Re}\sqrt{\frac{\gamma\pi}{2}} = \frac{Ma}{Re}\sqrt{\frac{5\pi}{6}}$$

where Re is the usual Reynolds number. Therefore, assuming $Ma \approx 1$ and $Kn = 0$ implies assuming that $Re \to \infty$, and it is known that in turbulent flows, the effects of viscosity can be neglected away from the walls. From the above analysis, the CHapman-Enskog method consists in using the ansatz

$$P = P_M\left(1 + \sum_{n\geqslant 1}(Kn)^n f_n\right) \tag{3.0.4}$$

where f_n is the n-th order perturbation correction. using a first order perturbation, it is possible to show (see [14]) that for small Knudsen numbers, the system's macroscopic fields (ρ, u, ϵ) obey the following system of equations

$$\begin{cases} \frac{\partial}{\partial t}\rho(x,t) + \nabla\cdot(\rho(x,t)u(x,t)) = 0, \\ \frac{\partial}{\partial t}\rho(t,x)u(x,t) + \nabla\cdot(\rho(t,x)u\otimes u + (RT)Id) = Kn\nabla\cdot(\mu(\rho,T)Du) + O(Kn^2), \\ \frac{\partial}{\partial t}\left[\rho(x,t)\left(\frac{3}{2}RT + \frac{||u||^2}{2}\right)\right] + \nabla\cdot\rho(x,t)u\left[\frac{5}{2}RT + \frac{||u||^2}{2}\right] = Kn\nabla\cdot(\kappa(\rho,T)\nabla T + \mu(\rho,T)\left[Du\right]u) + O(Kn^2), \end{cases}$$

which again corresponds to models a monatomic ideal gas, with fixed $\gamma = \frac{Cp}{Cv} = \frac{5}{3}$. However, this time the momentum and energy balances obtained show that the fluid obeys the Navier-Stokes-Fourier constitutive relationships for strain and heat flow. Despite this success, higher order perturbations give us the Burnett equations and super-Burnett equations, which are not necessarily more precise than the Navier-Stokes-Fourier description.

Although this precise ansatz was not amenable to rigorous mathematical analysis, the heuristic idea of considering Kn as a perturbative parameter resulted in an extensive program, driven by Bardos, Golse and Levermore [4], and later by many other authors [19], [32], [27], to rigorously derive macroscopic balance laws for energy and momentum from the Boltzmann equation, in the limit as $\epsilon \to 0$. Not only this allows us to have a better justification for the usage of these balance laws, but the procedure allows us to associate different conditions on the Boltzmann equation with different macroscopic behaviors.

All of these results are built upon generalizations of the concept of renormalized solution for the Boltzmann equation, since as shown before, this type of solution exists for all times and with very mild restrictions on the initial data.

Finally, the fact that the fluid described by the Boltzmann equation satisfies all the usual balance laws from continuum mechanics suggests that the Boltzmann equation can be used for the purpose of computational fluid dynamics simulations, which is indeed the case. There is a growing class of numerical methods known as Lattice-Boltzmann methods [14],[34], which provide an alternative method of simulating fluids, without trying to simiulate directly the Navier-Stokes-Fourier system of equations for a liquid, for instance. Lattice Boltzmann methods are easily parallelizable, and may be simpler to implement than traditional methods for certain situations, such as flows on porous media.

Appendix A

Introduction Theorems

A.1 Existence of the total collision operator

We wish to prove theorem 1.3.2:

Theorem A.1.1. *Let $\mathcal{A}$ be a disk, and $g(v) : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a function satisfying the condition*

$$|g| \leq C(1 + \|v\|)^n$$

for some $n \in \mathbb{N}$ and $C > 0$. If $(1 + \|v\|)^{n+1} f$, $(1 + \|v\|)^{n+1} h \in L^1(\Omega \times \mathbb{R}^d)$, then $\overline{Q_{f,h}} g(t, x)$ is well defined, and equations 1.3.17 and 1.3.18 are valid.

Proof. we follow a proof by [36]. If $\mathcal{A}$, it means our cross section is finite, and our potentials have a cutoff. Assume that the Hausdorff measure of A is M. Notice that the following simple inequalities holds

$$\begin{cases} \|w\| = \|v - v_*\| \leq \|v\| + \|v_*\| \leq (1 + \|v\|)(1 + \|v_*\|) \\ 1 + \|v'\| = 1 + \|v + (w \cdot \alpha)\alpha\| \leq 1 + \|v\| + \|w\| \leq 2(1 + \|v\|)(1 + \|v_*\|) \end{cases}$$

and therefore, we can establish the following estimates

$$\|w\| f f_* g \leq (1 + \|v\|)(1 + \|v_*\|) f f_* g \leq C \left[(1 + \|v\|)^{n+1} f \right] \left[(1 + \|v_*\|) f_* \right],$$

and

$$\begin{aligned} \|w\| f f_* g' &\leq (1 + \|v\|)(1 + \|v_*\|) f f_* g' \leq C(1 + \|v\|)(1 + \|v_*\|) f f_* (1 + \|v'\|)^n \\ &\leq 2^n C \left[(1 + \|v\|)^{n+1} f \right] \left[(1 + \|v_*\|)^{n+1} f_* \right]. \end{aligned}$$

Finally, since the potential has a cutoff, both the 'gain' and 'loss' parts of the collision operator converge and can be analyzed separately (see theorem 1.3.1), which gives us

$$\begin{aligned} \int_{\mathbb{R}^3} Q^-(f, f) g \, dv &= \iiint\limits_{\mathcal{A} \times \mathbb{R}^3 \times \mathbb{R}^3} f(t, x, v) f(t, x, v_*) g \, \|w\| dS_* dv_* dv \\ &\leq C \iiint\limits_{\mathcal{A} \times \mathbb{R}^3 \times \mathbb{R}^3} \left[(1 + \|v\|)^{n+1} f \right] \left[(1 + \|v_*\|) f_* \right] dS_* dv_* dv \\ &\leq CM \|(1 + \|v\|)^{n+1} f\|_{L^1_v}^2 < \infty \end{aligned}$$

and for Q^+ we have

$$Q^-(f,f)g\,dv = \iiint\limits_{\mathcal{A}\times\mathbb{R}^3\times\mathbb{R}^3} f(t,x,v')f(t,x,v'_*)g\,\|w\|dS_*dv_*dv$$

performing a change of variables $(v, v_*, \alpha) \to (v', v'_*, \alpha')$, which has unit jacobian, this can be rewritten as

$$Q^-(f,f)g\,dv = \iiint\limits_{\mathcal{A}\times\mathbb{R}^3\times\mathbb{R}^3} ff_*g'\,\|w\|dS_*dv_*dv$$

$$\leq 2^nC \iiint\limits_{\mathcal{A}\times\mathbb{R}^3\times\mathbb{R}^3} \left[(1+\|v\|)^{n+1}f\right]\left[(1+\|v_*\|)^{n+1}f_*\right]dS_*dv_*dv$$

$$\leq 2^nCM\|(1+\|v\|)^{n+1}f\|^2_{L^1_v} < \infty$$

Therefore, $\overline{Q_f g}$ is well defined, and a slight modification of the above argument allows us to prove that $\overline{Q_{f,h}}g$ is as well. □

A.2 Boltzmann-Gronwall Theorem

Theorem A.2.1 ((Boltzmann-Gronwall)). *Let $g : \Gamma \longrightarrow \mathbb{R}^n$ be a measurable function with respect to $\mathcal{L}$. g satisfies condition V, namely $g(x,v') + g(x,v'_*) = g(x,v) + g(x,v_*)$, where v, v_*, v', v'_* are solutions to the following system*

$$\begin{cases} v + v_* = v' + v'_* \\ \|v\|^2 + \|v_*\|^2 = \|v'\|^2 + \|v'_*\|^2 \end{cases}$$

if and only if g is a second order polynomial in v:$g(x,v) = a(x)\|v\|^2 + \langle b(x), v\rangle + c(x)$ for a.e. (x,v).

Proof. To prove this we'll need the following lemma:

Lemma A.2.1. *Let $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ be an odd function additive on orthogonal pairs, i.e., a function which satisfies*

$$\begin{cases} f(x^1) + f(x^2) = f(x^1 + x^2) & \text{whenever } x^1 \cdot x^2 = 0 \\ f(-x) = -f(x) & \forall x \in \mathbb{R}^N \end{cases}$$

Then, if g is measurable w.r.t. the Lebesgue measure, g is also continuous and linear.

Corollary A.2.1. *The same result is valid if f is additive in the whole space.*

Proof. In this proof, we use a mix of strategies from the proofs of [36] and [23]. The first step will be proving that if f is additive on orthogonal pairs and odd, then it is additive everywhere. To achieve that, we'll consider the case in which the vectors x^1 and x^2 are parallel, i.e. $x^2 = \lambda x^1$. If $\lambda > 0$, then, take a vector u orthogonal to x^1, x^2 and satisfying $\|u\|^2 = x^1 \cdot x^2 = \lambda\|x^1\|^2$. Then, we have that $(x^1 + u)\cdot(x^2 - u) = x^1 \cdot x^2 - \|u\|^2 + u\cdot(x^2 - x^1) = 0$. Therefore

$$f(x^1 + x^2) = f(x^1 + u + x^2 - u) = f(x^1 + u) + f(x^2 - u)$$

$$= f(x^1) + f(x^2) + f(u) + f(-u) = f(x^1) + f(x^2).$$

For the case when $\lambda < 0$, we can say at least that the above result is valid for the pair $-x^2$ and x^1:

$$f(x^1 - x^2) = f(x^1) + f(-x^2) = f(x^1) - f(x^2)$$

$$\Rightarrow f(x^1) = f(x^1 - x^2) + f(x^2) = f((1-\lambda)x^1) + f(\lambda x^1)$$

Define $w_1 = (1-\lambda)x^1, w_2 = \lambda x^1$; we have then $w_1 = \frac{(1-\lambda)}{\lambda} w_2$, $\frac{(1-\lambda)}{\lambda} < 0$, and

$$f(w_1 + w_2) = f(w_2) + f(w_1),$$

So additivity is established in the parallel case. Finally, consider the case of 2 arbitrary vectors x^1, x^2. Decomposing them in an orthonormal basis $\{e_1, e_2\}$:

$$\begin{aligned} f(x^1 + x^2) &= f(a_{11}e_1 + a_{21}e_2 + a_{12}e_1 + a_{22}e_2) \\ &= (a_{11} + a_{12})f(e_1) + (a_{21} + a_{22})f(e_2) \\ &= f(x^1) + f(x^2), \end{aligned} \tag{A.2.1}$$

So we have additivity, as claimed. It is clear that if we prove the theorem's main conclusion using additivity and measurability, then the corollary will be true as well.

Since f is additive, $f(0) + f(0) = f(0) \Rightarrow f(0) = 0$. Consider now the set $A = [0,1]^N$. A is a Borel set, with $\mathcal{L}(A) = 1$. By Lusin's theorem, for each $\epsilon > 0$, we can always find a compact set $K \subseteq A$ such that g restricted to K is continuous, and $\mathcal{L}(A - K) < \epsilon$. Moreover, since K is compact, f is actually uniformly continuous on K, so for every $\eta > 0$ there is $\delta > 0$ such that

$$\|x - y\| < \delta \Rightarrow |f(x) - f(y)| < \eta, \quad \forall x, y \in K.$$

since we can choose an arbitrary ϵ, this means that not all points of K are isolated, otherwise $\mathcal{L}(A-K) = \mathcal{L}(A) = 1$. Take an accumulation point $x_0 \in K$, and a sequence $\{\eta\}_{n\in\mathbb{N}}$ such that $\eta_n \to 0$. Then, for each η_n we will find a corresponding δ_n and a sequence $y_n \to x_0$ such that

$$0 < \|x_0 - y_n\| < \delta_n \Rightarrow |f(x_0) - f(y_n)| = |f(y_n - x_0)| < \eta_n$$

defining $h_n = y_n - x_0$,

$$0 < \|h_n\| < \delta_n \Rightarrow |f(h_n)| < \eta_n$$

and this would entail that f is continuous at 0, sending $n \to \infty$. However, it must then be continuous everywhere, since choosing any point $z \in \mathbb{R}^N$ and $\eta > 0$, by the continuity at 0 there is $\delta > 0$ such that

$$0 < \|h - 0\| < \delta \Rightarrow |f(h) - f(0)| = |f(h)| < \eta$$

Substituting $h = z - x$ means that

$$0 < \|z - x\| < \delta \Rightarrow |f(z - x)| = |f(z) - f(x)| < \eta,$$

and so we have continuity at z. For linearity, the proof is rather standard: by induction, one can prove that for any $p \in \mathbb{N}$

$$f(px) = g\left(\sum_{i=1}^{p} x\right) = \sum_{i=1}^{p} g(x) = pf(x).$$

Replacing x by x/p:

$$f(x) = pg\left(\frac{x}{p}\right); \; g\left(\frac{x}{p}\right) = \frac{1}{p} f(x)$$

So we can choose any rational number q and have: $f(qx) = qf(x)$; and by density of $\mathbb{Q}$, that will apply to irrational numbers as well. Therefore, our function is linear (being homogeneous of degree 1 and additive). □

Returning to the proof of A.2.1: fix a given position x. $g + g_*$ is constant whenever $v + v_*$and $\|v\|^2 + \|v_*\|^2$ are. So we can define a function:

$$U(v+v_*, \|v\|^2+\|v_*\|^2) = g+g_*.$$

If we set $v_*=0$, we find from the above equation that $U(v,\|v\|^2)=g(v)+g(0)$, and so the above equation can be cast as

$$U(v+v_*, \|v\|^2+\|v_*\|^2) = U(v,\|v\|^2)+U(v_*,\|v_*\|^2)-2g(0)$$

if v, v_* are orthogonal, then the left hand side becomes $U(v+v_*, \|v+v_*\|^2)$, and the function U becomes additive for orthogonal pairs! Sadly it's not odd, so we can't yet apply our lemma. Setting $v_*=-v$ in the above equation, we get that

$$U(0,2\|v\|^2) = U(v,\|v\|^2)+U(-v,\|v\|^2)-2g(0)$$

Then, consider a pair of orthogonal vectors v_*, v. We can say that

$$\begin{aligned}
U(0,2\|v+v_*\|^2) &= U(v+v_*,\|v+v_*\|^2)+U(-v-v_*,\|v+v_*\|^2)-2g(0)\\
&= U(v+v_*,\|v\|^2+\|v_*\|^2)+U(-v-v_*,\|v\|^2+\|v_*\|^2)-2g(0)\\
&= U(v,\|v\|^2)+U(-v,\|v\|^2)+U(v_*,\|v_*\|^2)+U(-v_*,\|v_*\|^2)-6g(0)\\
&= U(0,2\|v\|^2)+U(0,2\|v_*\|^2)-2g(0)\\
\Rightarrow \left[U(0,2\|v+v_*\|^2)-2g(0)\right] &= \left[U(0,2\|v\|^2)-2g(0)\right]+\left[U(0,2\|v_*\|^2)-2g(0)\right]
\end{aligned}$$

Therefore, the function $f_1(x)=U(0,x)-2g(0);\ x>0$ is an additive measurable function, and so by our lemma: $U(0,2\|v\|^2)=2a\|v\|^2+2g(0)$. This implies that

$$U(0,\|v\|^2)+U(0,\|v\|^2) = U(0,2\|v\|^2) = U(v,\|v\|^2)+U(-v,\|v\|^2)$$

$$\Rightarrow 0 = \left[U(v,\|v\|^2)-U(0,\|v\|^2)\right]+\left[U(-v,\|v\|^2)-U(0,\|v\|^2)\right]$$

Therefore, defining $f_2(v)=U(v,\|v\|^2)-U(0,\|v\|^2)$, the equation above states that

$$f_2(v)+f_2(-v)=0,$$

so f is odd. Furthermore, notice that, for any pair of orthogonal vectors v, v_*:

$$\begin{aligned}
&\left[U(v+v_*,\|v+v_*\|^2)-U(0,\|v+v_*\|^2)\right]\\
&= \left[U(v+v_*,\|v\|^2+\|v_*\|^2)-a\|v\|^2-a\|v_*\|^2\right]\\
&= \left[U(v,\|v\|^2)-U(0,\|v\|^2)\right]+\left[U(v_*,\|v_*\|^2)-U(0,\|v_*\|^2)\right].
\end{aligned}$$

Or, in terms of f_2:

$$f_2(v+v_*)=f_2(v)+f_2(v_*), \qquad v\cdot v_*=0.$$

Therefore $f_2(v)$ is additive for orthogonal pairs, measurable and odd, so by the previous lemma, it is linear, with $f_2(v)=b\cdot v$. Finally

$$\begin{aligned}
f_2(v) = U(v,\|v\|^2)-U(0,\|v\|^2) &= U(v,\|v\|^2)-f_1(\|v\|^2)-2g(0)\\
&= g(v)+g(0)-a\|v\|^2-2g(0)\\
\Rightarrow b\cdot v = f_2(v) &= g(v)-a\|v\|^2-g(0)\\
g(v) &= a\|v\|^2+b\cdot v+g(0).
\end{aligned}$$

Defining $c=g(0)$ finishes the proof. □

In the above proof, any dependence of g on the position variable was omitted, for the sake of simplifying notation, but in the end result, if g is a function of (t, x, v), then our result will become

$$g(t, x, v) = a(t, x)\|v\|^2 + b(t, x) \cdot v + c(t, x).$$

Notice that the functions a, b, c will inherit the regularity of g. At present, they are only measurable functions, but if one assumes that g is in fact continuous with respect to (t, x) then so are a, b and c. If g is a function of v alone, then a, b, c are constants.

A.3 The encounter problem

In this section of the Appendix, we solve the problem of a binary collision in classical mechanics, obtaining a few properties of the collision dynamics that are used throughout the text.

Since the heuristic used to derive Boltzmann's equation is grounded on 4 basic assumptions, we will concern ourselves only with collisions that match those assumptions. Namely, we will assume that **external forces are negligible** in the moment of a collision. We define now some notation to be used in this Appendix:

$$\begin{cases} x_1, x_2 \in \mathbb{R}^N & \text{positions of particles 1 and 2} \\ p_1, p_2 \in \mathbb{R}^N & \text{momenta of particles 1 and 2} \\ P = p_1 + p_2 & \text{Total momentum of the system} \\ \tilde{\phi}(R) : \mathbb{R}^N \longrightarrow \mathbb{R} & \text{potential energy for particle interaction} \\ R = x_2 - x_1 & \text{relative position of particle 2} \\ \|R\| = r & \end{cases}$$

From Newton's second law, we have that:

$$(i)\ \frac{d}{dt}P = \sum F_{external} \approx 0 \ \therefore P = p_1 + p_2 = constant \quad \text{(Momentum conservation)}$$

We can also write Newton's second law for both particles:

$$\begin{cases} m_1 \dfrac{d}{dt} x_1 = -\nabla \tilde{\phi}(R) & \text{(A.3.1a)} \\ m_2 \dfrac{d}{dt} x_2 = -\nabla \tilde{\phi}(-R) & \text{(A.3.1b)} \end{cases}$$

We make the following assumptions on the potential $\tilde{\phi}$:

1. Interactions are local, such that $\lim_{\|x\| \longrightarrow \infty} \tilde{\phi}(x) = \lim_{\|x\| \longrightarrow \infty} \|\nabla \tilde{\phi}(x)\| = 0$; and they have a singularity at the origin: $\lim_{\|x\| \longrightarrow 0} \tilde{\phi}(x) = \infty$ (particles repel each other at close range)

2. Since our gas is assumed in principle to be *monoatomic,* we can assume that $\tilde{\phi}$ is spherically symmetric (in the case of non-monoatomic gases, this can be thought as a mean-field approximation).

Therefore, expanding $-\nabla \tilde{\phi}(R)$ in spherical coordinates, we have:

$$-\nabla \tilde{\phi}(R) = -\left[\frac{\partial}{\partial r} \tilde{\phi}(R)\right] \hat{R};\ where\ \hat{R} = \frac{R}{r}$$

Again from spherical symmetry, we can define uniquely$\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}$ as $\phi(r) = \tilde{\phi}(r\alpha)$; for any unit vector α. So we can rewrite our equations as:

$$\begin{cases} m_1 \dfrac{d^2}{dt^2} x_1 = \left[\dfrac{\partial}{\partial r}\phi(r)\right] \dfrac{R}{r}; & \text{(A.3.2a)} \\ m_2 \dfrac{d^2}{dt^2} x_2 = -\left[\dfrac{\partial}{\partial r}\phi(r)\right] \dfrac{R}{r}, & \text{(A.3.2b)} \\ & \text{(A.3.2c)} \end{cases}$$

where $t \in \mathbb{R}; x_1, x_2 \in C_t^3(\mathbb{R};\mathbb{R}^n)$, and with the boundary conditions: $\lim_{t \longrightarrow \pm\infty} \|x_2 - x_1\| = \infty$. Adding both equations one recovers conservation of momentum, as stated in (i). Diving (A.3.2a) by m_1 and (A.3.2b) by m_2 and subtracting both equations, one recovers:

$$\frac{d^2}{dt^2}(x_2 - x_1) = -\left(\frac{1}{m_2} + \frac{1}{m_1}\right)\left[\frac{\partial}{\partial r}\phi(r)\right]\frac{R}{r}$$

Defining the reduced massμ as twice the harmonic mean of the particle masses: $\frac{1}{\mu} = \left(\frac{1}{m_2} + \frac{1}{m_1}\right)$; one finds the equation for the relative motion between particles:

$$\mu \frac{d^2}{dt^2} R = -\left[\frac{\partial}{\partial r}\phi(r)\right]\frac{R}{r} \tag{A.3.3}$$

So that the problem can be analyzed as a single particle with mass μ moving according to the central potential ϕ. From this we can use the standard procedure to define energy equations:

$$\begin{cases} m_1 \left\langle \dfrac{d}{dt}x_1, \dfrac{d^2}{dt^2}x_1 \right\rangle = \left[\dfrac{\partial}{\partial r}\phi(r)\right] \left\langle \dfrac{d}{dt}x_1, \dfrac{R}{r} \right\rangle & \text{(A.3.4a)} \\ m_2 \left\langle \dfrac{d}{dt}x_2, \dfrac{d^2}{dt^2}x_2 \right\rangle = -\left[\dfrac{\partial}{\partial r}\phi(r)\right] \left\langle \dfrac{d}{dt}x_2, \dfrac{R}{r} \right\rangle & \text{(A.3.4b)} \\ \mu \left\langle \dfrac{d}{dt}R, \dfrac{d^2}{dt^2}R \right\rangle = \left[\dfrac{\partial}{\partial r}\phi(r)\right] \left\langle \dfrac{d}{dt}R, \dfrac{R}{r} \right\rangle & \text{(A.3.4c)} \end{cases}$$

Adding the first 2 equations; and noting that

$$\left\langle \frac{d}{dt}x_i, \frac{d^2}{dt^2}x_i \right\rangle = \frac{1}{2}\frac{d}{dt}\left\|\frac{d}{dt}x_i\right\|^2; \; i = \{1, 2\}$$

we get

$$m_1 \frac{1}{2}\frac{d}{dt}\left\|\frac{d}{dt}x_i\right\|^2 + m_2 \frac{1}{2}\frac{d}{dt}\left\|\frac{d}{dt}x_2\right\|^2 = -\left[\frac{\partial}{\partial r}\phi(r)\right]\left\langle \frac{d}{dt}(x_2 - x_1), \frac{R}{r} \right\rangle$$

So summing up, we have

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{m_1}{2}\left\|\dfrac{d}{dt}x_i\right\|^2 + \dfrac{m_2}{2}\left\|\dfrac{d}{dt}x_2\right\|^2\right) = -\left[\dfrac{\partial}{\partial r}\phi(r)\right]\left\langle \dfrac{d}{dt}(x_2 - x_1), \dfrac{R}{r} \right\rangle & \text{(A.3.5a)} \\ \dfrac{d}{dt}\dfrac{\mu}{2}\left\|\dfrac{d}{dt}R\right\|^2 = -\left[\dfrac{\partial}{\partial r}\phi(r)\right]\left\langle \dfrac{d}{dt}R, \dfrac{R}{r} \right\rangle & \text{(A.3.5b)} \end{cases}$$

We note that the right hand sides of both equations are indeed the same, recalling that $(x_2 - x_1) \equiv R$. Examining it further, we find:

$$\left[\frac{\partial}{\partial r}\phi(r)\right]\left\langle\frac{d}{dt}R,\frac{R}{r}\right\rangle = \frac{1}{r}\left[\frac{\partial}{\partial r}\phi(r)\right]\left\langle\frac{d}{dt}R,R\right\rangle$$
$$= \frac{1}{2r}\left[\frac{\partial}{\partial r}\phi(r)\right]\frac{d}{dt}r^2$$
$$= \left[\frac{\partial}{\partial r}\phi(r)\right]\frac{dr}{dt} = \frac{d}{dt}\phi\left(r(t)\right)$$

Finally, eliminating the time derivatives, we get:

$$\begin{cases} \left(\frac{m_1}{2}\|\frac{d}{dt}x_i\|^2 + \frac{m_2}{2}\|\frac{d}{dt}x_2\|^2 + \phi\left(r(t)\right)\right) = E & \text{(A.3.6a)} \\ \frac{\mu}{2}\|\frac{d}{dt}R\|^2 + \phi\left(r(t)\right) = E & \text{(A.3.6b)} \end{cases}$$

Fix a value $E \geq 0$. Then, taking advantage that $\lim_{x\longrightarrow\infty}\phi(x) = 0$, one finds from (A.3.6b) that $\lim_{t\longrightarrow\pm\infty}\|\frac{d}{dt}R\| = \sqrt{\frac{2E}{\mu}}$. In fact, reversing this one can say:

$$E = \frac{\mu}{2}\|w\|^2,\ where\ \|w\| = \lim_{t\longrightarrow\pm\infty}\|\frac{d}{dt}R\|$$

Consider now the Frénét Trihedron centered in the trajectory $R(t)$. One can then find the following decompositions:

$$\begin{cases} \frac{d}{dt}R = uT & \text{(A.3.7a)} \\ \frac{d^2}{dt^2}R = \frac{du}{dt}T + ku^2N & \text{(A.3.7b)} \\ B = T\times N = \frac{\frac{d}{dt}R\times\frac{d^2}{dt^2}R}{\|\frac{d}{dt}R\times\frac{d^2}{dt^2}R\|} & \text{(A.3.7c)} \end{cases}$$

Where $k(t)$ is the scalar curvature of $R(t)$, T is the unit tangent vector, N the principal normal vector and B the binormal vector (defined wherever $k \neq 0$), and $u = \|\frac{d}{dt}R\|$.

Taking now equation (A.3.3), we see that:

$$\lim_{t\longrightarrow\pm\infty}\mu\left\|\frac{d^2}{dt^2}R\right\| = \lim_{t\longrightarrow\pm\infty}\left|\frac{\partial}{\partial r}\phi(r(t))\right| = 0$$

Using the following

Lemma A.3.1. *Let $f \in C^1(\mathbb{R},\mathbb{R}^n)$. Then, $\left|\frac{d}{dx}\|f\|\right| \leq \left\|\frac{d}{dx}f\right\|$;*

Inserting $f = \frac{d}{dt}R$, we can deduce that:

$$\lim_{t\longrightarrow\pm\infty}\left|\frac{dw}{dt}\right| \leq \lim_{t\longrightarrow\pm\infty}\left\|\frac{d^2}{dt^2}R\right\| = 0.$$

Therefore, we can use decomposition (A.3.7b) in the left-hand side of equation (A.3.3), and obtain the expression:

$$kw^2N = \frac{dw}{dt}T - \left[\frac{\partial}{\partial r}\phi(r)\right]\frac{R}{r}$$

Finally:

$$\begin{aligned}\lim_{t\longrightarrow\pm\infty}\left\|kw^2N\right\| &= \lim_{t\longrightarrow\pm\infty} kw^2\\ &\leq \lim_{t\longrightarrow\pm\infty}\left\|\frac{dw}{dt}T\right\| + \left\|\left[\frac{\partial}{\partial r}\phi(r)\right]\frac{R}{r}\right\|\\ &= \lim_{t\longrightarrow\pm\infty}\left|\frac{dw}{dt}\right| + \left|\frac{\partial}{\partial r}\phi(r(t))\right| = 0\end{aligned}$$

Proof. To prove said lemma, notice that :

$$\left|\frac{\|f(t+x)\|-\|f(x)\|}{h}\right| = \frac{1}{|h|}\left|\|f(h+x)\|-\|f(x)\|\right| \leq \frac{1}{|h|}\|f(h+x)-f(x)\| = \left\|\frac{f(h+x)-f(x)}{h}\right\|$$

And take the limit as $h \to 0$ □

We can conlude then that $\lim_{t\longrightarrow\pm\infty} k(t) = 0$; so that trajectories behave approximately as straight lines in the limit. From this, we can deduce that

$$\exists \lim_{t\longrightarrow\pm\infty} T, \text{ thus: } \exists \lim_{t\longrightarrow\pm\infty}\tfrac{d}{dt}R = \lim_{t\longrightarrow\pm\infty} wT.$$

So,

Remark A.3.1.

$$\exists \lim_{t\longrightarrow\pm\infty}\frac{dx_i}{dt} = \lim_{t\longrightarrow\pm\infty}\frac{P+(-1)^i\,\frac{d}{dt}R}{2} \quad i=\{1,2\}$$

(Recalling the definitions of P and R). We can then introduce the following

Definition A.3.1. The following limits:

$$\lim_{t\longrightarrow\infty}\frac{dx_1}{dt} := v',\ \lim_{t\longrightarrow-\infty}\frac{dx_1}{dt} := v,\ \lim_{t\longrightarrow\infty}\frac{dx_2}{dt} := v'_*\ \lim_{t\longrightarrow-\infty}\frac{dx_2}{dt} := v_*$$

Are said to be the **asymptotic velocities** of particles 1 and 2.

Since P is constant, we must have that $m_1v + m_2v_* = P = m_1v' + m_2v'_*$

Taking the limit as $t \to \pm\infty$ in equation (A.3.6a) then produces the following system of equations for the asymptotic velocities:

$$(B)\ \begin{cases} m_1v + m_2v_* = m_1v' + m_2v'_* & \text{Momentum conservation}\\ m_1\|v\|^2 + m_2\|v_*\|^2 = m_1\|v'\|^2 + m_2\|v'_*\|^2 & \text{Energy conservation}\end{cases}$$

Assuming (v, v_*) as known.

This system captures the essential features of a collision. Indeed, since we used that $\tau_{collision} \ll \tau_{streaming}$ among our assumptions when deriving the Boltzmann equation, all that matters to us is the asymptotic behavior of a binary collision, since collisions were assumed to be instantaneous. Any solution to the system (B) is said to be a **solution to the Encounter problem**, which is a system of algebraic equations.

Recalling the reasoning given in section 1.3, the knowledge of v, v_* does not determine v', v'_* uniquely. Rather, there are $N-1$ degrees of freedom in the system (B), so we expect that there is a family of parameters k such that:

$$\begin{cases}\exists F : \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{M} \longrightarrow \mathbb{R}^N \times \mathbb{R}^N\\ (v, v_*, k) \longmapsto (v', v'_*)\end{cases}$$

Where k takes values in a N-1 dimensional Manifold $\mathcal{M}$ in $\mathbb{R}^n$; each choice of k indexes a unique possible solution. From the first equation of (B) , we can define:

$$m_1(v' - v) = -m_2(v'_* - v_*) \equiv g$$

Such that

$$\begin{cases} v' = v + \frac{1}{m_1} g \\ v'_* = v_* - \frac{1}{m_2} g \end{cases}$$

Inserting in the left hand side of the conservation of energy equation:

$$m_1\|v'\|^2 + m_2\|v'_*\|^2 = m_1 \left\langle v + \frac{1}{m_1}g,\, v + \frac{1}{m_1}g \right\rangle + m_2 \left\langle v_* - \frac{1}{m_2}g,\, v'_* - \frac{1}{m_2}g \right\rangle$$

$$= m_1\|v\|^2 + m_2\|v_*\|^2 + 2\langle g, v\rangle - 2\langle g, v_*\rangle + \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\|g\|^2 = m_1\|v\|^2 + m_2\|v_*\|^2$$

$$\therefore \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\|g\|^2 = 2\langle g, v_* - v\rangle$$

Recalling the definition of μ, and defining $w = v_* - v$; $w' = v'_* - v'$ $\alpha = \frac{g}{\|g\|}$, this reads

$$\frac{1}{\mu}\|g\| = 2\langle \alpha, w\rangle$$

And inserting back, we have:

$$\begin{cases} v' = v + \frac{1}{m_1} 2\mu \langle \alpha, w\rangle \alpha & \text{(A.3.8a)} \\ v'_* = v_* - \frac{1}{m_2} 2\mu \langle \alpha, w\rangle \alpha & \text{(A.3.8b)} \end{cases}$$

Subtracting (A.3.8a) from (A.3.8b) yields:

$$w' = w - 2\langle \alpha, w\rangle \alpha \tag{A.3.9}$$

Looking at equations (A.3.8a) and (A.3.8b), we see that if α is taken as a parameter this corresponds exactly to a solution to the encounter problem. Being a unit vector, the parameter α belongs to a $N-1$ dimensional sphere, which fulfills our requirements. We will now focus in the more classical case where particles move in 3-D space.

The spherical parametrization for α would be in terms of polar angles (spherical coordinates), and that is the form most used in this text. Another parametrization common in physics textbooks is given in terms of the impact parameter, which will be defined below.

From this point onward, we consider only the relative motion of the particles, so the only equations that will concern us will be (A.3.4c), (A.3.6b) and (A.3.9).

We make the following remarks about (A.3.9):

1. $\|w'\| = \|w\|$
2. The motion of particle 2 in this frame (relative to 1) is constrained to a plane
3. α bissects the angle between $-w$ and w'

Proof. First, $\|w'\|^2 = \langle w - 2\langle \alpha, w\rangle \alpha,\, w - 2\langle \alpha, w\rangle \alpha\rangle = \|w\|^2 + 4\langle \alpha, w\rangle^2 - 4\langle \alpha, w\rangle^2 = \|w\|^2$.

Now, to prove the second assertion, multiply vectorially equation (A.3.3) by $R(t)$, one finds that:

$$\mu\left(\frac{d^2}{dt^2}R\right) \times R = \frac{d}{dt}\left[\mu\left(\frac{d}{dt}R\right) \times R\right] = -\left[\frac{\partial}{\partial r}\phi(r)\right]\frac{R \times R}{r} = 0$$

Therefore, comparing the second and third terms, we see that a derivative vanishes, such that:

$$\mu\left(\frac{d}{dt}R\right) \times R = \vec{C_1}\ (Constant\, angular\, momentum!)$$

This is sufficient to guarantee the motion is restricted to a plane Π_1 orthogonal to $\vec{C_1}$. Indeed, multiplying vectorially equation III this time by $\frac{d}{dt}R$, and recalling the definition of the Binormal vector for this trajectory:

$$\mu\left(\frac{d^2}{dt^2}R\right) \times \frac{d}{dt}R = \mu\left\|\left(\frac{d^2}{dt^2}R\right) \times \frac{d}{dt}R\right\| B = \frac{1}{r}\left[\frac{\partial}{\partial r}\phi(r)\right]\frac{d}{dt}R \times R = \frac{1}{\mu r}\left[\frac{\partial}{\partial r}\phi(r)\right]\vec{C_1}$$

Which means the Binormal vector always has the direction $\vec{C_1}$. By definition the other vectors from the Frénét Frame T, N lie in the plane Π_1, and so do the first 2 derivatives of $R(t)$, being linear combinations of T and N .

To prove the third, multiply scalarly (A.3.9) by α. One finds:

$\langle \alpha, w'\rangle = \langle \alpha, w\rangle - 2\langle \alpha, w\rangle = \langle \alpha, -w\rangle$. Defining: $\langle \alpha, -w\rangle = \cos(\theta)$, $\langle \alpha, w'\rangle = \cos(\theta')$, $\theta', \theta \in [0, \pi]$; the equation means that $\theta' = \theta$. □

So, a typical trajectory should have the aspect described below, with the parameter α accounting for the amount of angular deflection in the trajectory through the angle θ: (Imagem).

Now, take equation (A.3.3) along with a Frénet Frame associated to its trajectory, and decompose $R(t)$ in this frame. We find that:

$$R(t) = a(t)T(t) + b(t)N(t)$$

Where T and N belong to the plane Π_1. Taking the cross product with $\frac{d}{dt}R$ yields:

$$\begin{aligned}\left(\frac{d}{dt}R\right) \times R = \frac{\vec{C_1}}{\mu} &= \left(\frac{d}{dt}R\right) \times [a(t)T(t) + b(t)N(t)] \\ &= b(t)\left(\frac{d}{dt}R\right) \times N(t) \\ &= b(t)u(t)\left[T(t) \times N(t)\right].\end{aligned}$$

Since The binormal vector has a fixed direction and unit norm, it is in fact constant. Taking norms on both sides, we can say:

$$\frac{\left\|\vec{C_1}\right\|}{\mu} = b(t)u(t)$$

Since the left-hand side of this equation is constant, we can say that:

$$\lim_{t \longrightarrow \infty} |b(t)|u(t) \equiv b\|w'\| = \lim_{t \longrightarrow -\infty} |b(t)|u(t) \equiv \tilde{b}\|w\|$$

Since $\|w'\| = \|w\|$, we find that : $\tilde{b} = b$. This allows us to find a closed expression for the angular momentum:

$$\vec{C_1} = \lim_{t \longrightarrow \infty} \mu\left(\frac{d}{dt}R\right) \times R = \mu b\|w\|B$$

where B is a constant unit vector.

So b is a quantity conserved across a collision, if we consider only the asymptotic behavior. Since α determines uniquely the asymptotic outcome of a collision, we guess that there may be a way to parametrize α using b. b determines how aligned the collision is: b=0 means R(t) is always in the direction of its tangent vector; and therefore collision is frontal. b is thus named the **impact parameter**.

The Orbital equation

Consider again the orthonormal basis set $\{\xi, \eta, \hat{w}\}$, where $\hat{w}$ is the unit vector in the direction of $w = v_* - v$. Since the angle between our vector α and w is θ, and since $\|\alpha\| = 1$, we can parametrize α by saying:

$$\alpha = e(\theta, \varphi) = (\cos\varphi \sin\theta)\, \xi + (\sin\varphi \sin\theta)\, \eta + (\cos\theta)\, \hat{w}$$

where $\theta \in [0, \pi]; \varphi \in [0, 2\pi]$

However, since θ measures the angular deflection due to collisions, intuitively it must be related to b. Namely, we want to find an expression of the form: $\theta = g(b, \|w\|)$. Consider again the equations of relative motion for our original (time dependent) problem.

$$\begin{cases} \mu \frac{d^2}{dt^2} R = -\left[\frac{\partial}{\partial r}\phi(r)\right] \frac{R}{r} & , \ t \in \mathbb{R} \\ \lim\limits_{t \longrightarrow \pm\infty} \|R\| = \infty & \text{Boundary conditions} \\ E = \frac{\mu}{2} \|\frac{d}{dt} R\|^2 + \phi\,(r(t)) = \frac{\mu}{2}\|w\|^2 & \text{Conservation of energy} \\ \mu \left(\frac{d}{dt} R\right) \times R = \mu b \|w\| B & \text{Constant Angular momentum} \end{cases}$$

Recalling that our motion is constrained to a plane Π_1, we can write it in polar coordinates, defining:

$$\begin{cases} \hat{\mathbf{r}} = (\cos\theta, \sin\theta, 0) & \hat{\theta} = (-\sin\theta, \cos\theta, 0) \\ \frac{d\hat{\mathbf{r}}}{d\theta} = \hat{\theta} & \frac{d\hat{\theta}}{d\theta} = -\hat{\mathbf{r}} \end{cases}$$

Writing our solution $R(t) = r(t)\hat{\mathbf{r}}(\theta(t))$, we see that:

$$\frac{d}{dt} R(t) = r'(t)\hat{\mathbf{r}}(\theta(t)) + r(t)\frac{d\hat{\mathbf{r}}}{d\theta}\theta'(t)$$

Inserting this expression back on the last 2 equations yields

- Energy

$$\frac{\mu}{2}\left[(r'(t))^2 + (r(t)\theta'(t))^2\right] + \phi\,(r(t)) = \frac{\mu}{2}\|w\|^2 \tag{A.3.10}$$

- Angular momentum
  Since $B \perp \Pi_1$; $\|B\| = 1$, we can take $B = \hat{\mathbf{r}} \times \hat{\theta}$. Then:

$$\mu\left(r'(t)\hat{\mathbf{r}}(\theta(t)) + r(t)\theta'(t)\hat{\theta}\right) \times r(t)\hat{\mathbf{r}}(\theta(t)) = \mu b \|w\| \hat{\mathbf{r}} \times \hat{\theta}$$

$$(r(t))^2\, \theta'(t) = b\|w\| \ = J/\mu \tag{A.3.11}$$

  Or alternatively

$$\theta'(t) = \frac{J}{\mu r(t)^2} \tag{A.3.12}$$

where J is our notation for the total angular momentum (in absolute value). If we compare equations A.3.10 and A.3.11, we can see that total energy and angular momentum are related by

$$b^2 E = \frac{J^2}{2\mu},$$

and using equation A.3.11, equation A.3.10 can be cast as

$$E = \frac{\mu}{2}\left[\left(r'(t)\right)^2 + \frac{J^2}{\mu^2 r(t)^2}\right] + \phi\left(r(t)\right)$$
$$r'(t) = \frac{1}{\sqrt{\frac{2}{\mu}}\sqrt{E - \left(\phi\left(r(t)\right) + \frac{J^2}{2\mu(r(t))^2}\right)}} \tag{A.3.13}$$

Here we can see a physical interpretation for the second term in brackets in equation A.3.10: when writing the equations of motion in the reference frame $\{\hat{\mathbf{r}}, \hat{\theta}\}$ there appears a fictitious force term ($\frac{J^2}{2\mu r^2}$), due to the fact that this frame is not inertial (it is rotating as a function of $\theta(t)$). Therefore we can define

$$V_{\text{eff}} = \phi\left(r(t)\right) + \frac{J^2}{2\mu(r(t))^2}$$

as an effective potential for the particle interaction in this frame. The term ($\frac{J^2}{2\mu r^2}$) is defined as the **centrifugal barrier**, and acts as a repulsive potential. In order to obtain the total change in θ, one could integrate directly equation $A.3.12$, however this would require a direct solution of the equations of motion. Instead, we can do the following: notice that

$$\frac{d\theta}{dr} = \frac{\theta'(t)}{r'(t)} \Rightarrow \theta = \int_{\infty}^{r_{min}} \frac{d\theta}{dr}\, dr$$

If we integrate the following equation from $x = \infty(pre - collision)$ to to the minimal value it achieves (particle 2 closest to particle 1), the change in the angle θ will correspond to the actual value of the angle θ between α and w. And then, in order to find the total deflection angle, one can simply do $p = \pi - 2\theta$. Calculating the above expression we find

$$\frac{\theta'(t)}{r'(t)} = \frac{\frac{J}{\sqrt{2\mu}}}{r^2\sqrt{E - \left(\phi\left(r(t)\right) + \frac{J^2}{2\mu(r(t))^2}\right)}}$$

Or, inserting our relationship between J and E,

$$= \frac{b}{r^2}\frac{\sqrt{E}}{\sqrt{E - \left(\phi\left(r\right) + \frac{b^2 E}{r^2}\right)}} = \frac{b}{r^2}\frac{1}{\sqrt{1 - \left(\frac{\phi(r)}{E} + \frac{b^2}{r^2}\right)}}.$$

Integrating, we find

$$\theta = \int_{\infty}^{r_{min}} \frac{b}{r^2}\frac{1}{\sqrt{1 - \left(\frac{\phi(r)}{E} + \frac{b^2}{r^2}\right)}}\, dr = \int_{\infty}^{r_{min}} \frac{b}{r^2}\frac{1}{\sqrt{1 - \left(\frac{2\phi(r)}{\mu\|w\|^2} + \frac{b^2}{r^2}\right)}}\, dr$$

Now, define $r = \frac{b}{s}$; $\frac{dr}{ds} = -\frac{b}{s^2}$. When r is minimal, s reaches its maximum, such that: $\frac{ds}{d\theta}(s_{max}) = 0$. Therefore

$$s_{max}^2 + \frac{2\phi\left(b/s_{max}\right)}{\mu\|w\|^2} = 1$$

and, if ϕ is a potential with cutoff at $r = \sigma$,

$$\theta = \int_{\frac{b}{\sigma}}^{smax} \frac{1}{\sqrt{1 - s^2 - \frac{2\phi(b/s)}{\mu\|w\|^2}}} ds + \arcsin\left(\frac{b}{\sigma}\right) \tag{A.3.14}$$

Equation A.3.12 is called the **Orbital equation,** considering a finite range potential ϕ. Should ϕ have an unbounded support, it suffices to pass through the limit when $\sigma \to \infty$.

A.4 Derivation of the BBGKY hierarchy

Throughout this section of the Appendix, Ω denotes a bounded open set in $\mathbb{R}^3$ with a sufficiently regular (e.g. Lipschitz) orientable boundary. The derivation here is for the physical case with 3 spatial dimensions, but the derivation is the same, mutatis mutandis, when $x \in \mathbb{R}^d$. We also assume that P^N (or simply P) is a classical solution to the Liouville equation, in order to use the divergence theorem as done here, but the spirit of the derivation remains the same if we take P^N to be only a weak solution to the Liouville equation.

We start by applying the operator $E_s : L^1(\Gamma_N) \longrightarrow L^1(\Gamma_s)$ as defined in section 1.5.1 to both sides of the Liouville equation. We start by considering the case when particles interact only through hard collisions. The result would then be:

$$\frac{\partial P^{(s)}}{\partial t} + \iint_{\Gamma_{N-s}^{\sigma\neq}} \{P, H\} \prod_{i=s+1,j=1}^{i=N,j=3} dx_{3i+j} dp_{3i+j} = 0$$

Expanding the Poisson bracket, we get:

$$\frac{\partial P^{(s)}}{\partial t} + \iint_{\Gamma_{N-s}^{\sigma\neq}} \sum_{i\leqslant s} \sum_{j=1}^{3} \frac{\partial P}{\partial x_{3i+j}} \frac{\partial H}{\partial p_{3i+j}} - \frac{\partial P}{\partial p_{3i+j}} \frac{\partial H}{\partial x_{3i+j}} \prod_{i=s+1,j=1}^{i=N,j=3} dx_{3i+j} dp_{3i+j}$$

$$+ \iint_{\Gamma_{N-s}^{\sigma\neq}} \sum_{i\geqslant s}^{N} \sum_{j=1}^{3} \frac{\partial P}{\partial x_{3i+j}} \frac{\partial H}{\partial p_{3i+j}} - \frac{\partial P}{\partial p_{3i+j}} \frac{\partial H}{\partial x_{3i+j}} \prod_{i=s+1,j=1}^{i=N,j=3} dx_{3i+j} dp_{3i+j} = 0$$

Since our summations are finite, and in this case there is no interparticle potential, we can switch the order of integration and sum operators, and find

$$\frac{\partial P^{(s)}}{\partial t} + \sum_{i\leqslant s} \iint_{\Gamma_{N-s}^{\sigma\neq}} \sum_{j=1}^{3} \frac{\partial P}{\partial x_{3i+j}} \frac{\partial H}{\partial p_{3i+j}} \prod_{i=s+1,j=1}^{i=N,j=3} dx_{3i+j} dp_{3i+j}$$

$$= -\sum_{i\geqslant s}^{N} \iint_{\Gamma_{N-s}^{\sigma\neq}} \sum_{j=1}^{3} \frac{\partial P}{\partial x_{3i+j}} \frac{\partial H}{\partial p_{3i+j}} \prod_{i=s+1,j=1}^{i=N,j=3} dx_{3i+j} dp_{3i+j}$$

Then, changing variables from momentum to velocity, in vector notation, we find

$$\frac{\partial P^{(s)}}{\partial t} + \sum_{i\leqslant s} \iint_{\Gamma_{N-s}^{\sigma\neq}} v_i \cdot \nabla_{x_i} P \prod_{j=s+1}^{j=N} dx_j dv_j + \sum_{k\geqslant s} \iint_{\Gamma_{N-s}^{\sigma\neq}} v_k \cdot \nabla_{x_k} P \prod_{j=s+1}^{j=N} dx_j dv_j = 0 \tag{A.4.1}$$

Where for each fixed $j \leq N$, $v_j \in \mathbb{R}^3$ is a vector, and dv_j implies integration with respect to all coordinates of v_j. We consider each term on the left hand side of equation (A.4.1) separately. For $i \leq s$,

$$\iint_{\Gamma^{\sigma\neq}_{N-s}} v_i \cdot \nabla_{x_i} P \prod_{j=s+1}^{j=N} dx_j dv_j = \iint_{\Gamma^{\sigma\neq}_{N-s}} \nabla_{x_i} \cdot (v_i P) \prod_{j=s+1}^{j=N} dx_j dv_j$$

Notice that the domain of integration is dependent on x_i, so we find that we cannot simply take the divergence "outside" of the integral. rather, define the change of variables: $x_j = \sigma z_j + x_i$ for $j \neq i$. we get then

$$\iint_{\Gamma^{\sigma\neq}_{N-s}} \nabla_{x_i} \cdot (v_i P) \prod_{j=s+1}^{j=N} dx_j dv_j = \sigma^{3(N-s)} \iint_{\{\|z_j\| \geq 1\}} \nabla_{x_i} \cdot (v_i P)(\sigma z_j + x_i, v_i) \prod_{j=s+1}^{j=N} dz_j dv_j,$$

and from the chain rule

$$\nabla_{x_i} \cdot (v_i P)(\sigma z_j + x_i, v_i) = \nabla_{x_i} \cdot (v_i P(\sigma z_j + x_i, v_i)) - \sum_{k \geqslant s}^{N} \nabla_{x_k} \cdot (v_i P)(\sigma z_j + x_i, v_i)$$

Inserting this onto the last integral, since now the domain of integration is independent from the variable x_i, we find that

$$\iint_{\Gamma^{\sigma\neq}_{N-s}} \nabla_{x_i} \cdot (v_i P) \prod_{j=s+1}^{j=N} dx_j dv_j = \nabla_{x_i} \cdot \iint_{\{\|z_j\| \geq 1\}} \sigma^{3(N-s)} (v_i P)(\sigma z_j + x_i, v_i) \prod_{j=s+1}^{j=N} dz_j dv_j$$

$$- \sum_{k \geqslant s}^{N} \sigma^{3(N-s)} \iint_{\{\|z_j\| \geq 1\}} \nabla_{x_k} \cdot (v_i P)(\sigma z_j + x_i, v_i) \prod_{j=s+1}^{j=N} dz_j dv_j,$$

Undoing the change of variables, and using the divergence theorem on the second integral on the right hand side, , we arrive at the following expression;

$$\iint_{\Gamma^{\sigma\neq}_{N-s}} \nabla_{x_i} \cdot (v_i P) \prod_{j=s+1}^{j=N} dx_j dv_j = \nabla_{x_i} \cdot \iint_{\{\|z_j\| \geq 1\}} (v_i P) \prod_{j=s+1}^{j=N} dx_j dv_j$$

$$- \sum_{k \geqslant s}^{N} \iint_{\{\|z_k\| = 1\}} (v_i \cdot n_{ik}) P d\sigma_{ik} \prod_{j=s+1, j \neq k}^{j=N} dx_j \prod_{j=s+1}^{j=N} dv_j,$$

$$= \nabla_{x_i} \cdot (v_i P^{(s)}) - \sum_{k \geqslant s}^{N} \iint_{\{\|z_k\| = 1\}} (v_i \cdot n_{ik}) P^{(s+1)} d\sigma_{ik} dv_k, \tag{A.4.2}$$

where n_{ik} is the unit normal vector to the sphere $\|x_i - x_k\| = \sigma$ (equivalently, $\|z_k\| = 1$), and $d\sigma_{ik}$ is the area element of that sphere.

Proceeding similarly, we now turn to the second term on the left hand side of equation (A.4.1). For $s + 1 \leq k \leq N$, we say that

$$\iint_{\Gamma^{\sigma\neq}_{N-s}} v_k \cdot \nabla_{x_k} P \prod_{j=s+1}^{j=N} dx_j dv_j = \iint_{\Gamma^{\sigma\neq}_{N-s}} \nabla_{x_k} \cdot (v_k P) \prod_{j=s+1}^{j=N} dx_j dv_j$$

Now, since x_k is one of the integration variables, we can start by using the Divergence theorem, and turn the right hand side into an integration over the boundary. The boundary has 2 terms: an integration over the physical boundary $\partial\Omega$ (absent in the case when our domain is unbounded), and a term corresponding to integrations over the spheres given by the condition $\|x_j - x_i\| = \sigma$. This gives us

$$\iint\limits_{\Gamma^{\sigma\neq}_{N-s}} \nabla_{x_k} \cdot (v_k P) \prod_{j=s+1}^{j=N} dx_j dv_j = \int\limits_{\partial\Omega\times\mathbb{R}^3} \iint\limits_{\Gamma^{\sigma\neq}_{N-s-1}} (v_k \cdot n) P \prod_{j=s+1, j\neq k}^{j=N} dx_j dv_j \, dx_k dv_k$$

$$+ \sum_{i=1, i\neq k}^{N} \int\limits_{\{\|z_k\|=1\}\times\mathbb{R}^3} \left[\iint\limits_{\Gamma^{\sigma\neq}_{N-s-1}} (v_k \cdot n_{ik}) P \prod_{j=s+1, j\neq k}^{j=N} dx_j dv_j \right] d\sigma_{ik} dv_k.$$

The first integral on the right hand side is 0 if we restrict ourselves to the case of closed boundaries. For the second term on the right hand side, since $k > s$, we can separate the summation in 2 parts; one for $i \leq s$, and one for $s < i \leq N$:

$$= \sum_{i=1}^{s} \int\limits_{\{\|z_k\|=1\}\times\mathbb{R}^3} \left[\iint\limits_{\Gamma^{\sigma\neq}_{N-s-1}} (v_k \cdot n_{ik}) P \prod_{j=s+1, j\neq k}^{j=N} dx_j dv_j \right] d\sigma_{ik} dv_k$$

$$+ \sum_{i=s+1, i\neq k}^{N} \int\limits_{\{\|z_k\|=1\}\times\mathbb{R}^3} \left[\iint\limits_{\Gamma^{\sigma\neq}_{N-s-1}} (v_k \cdot n_{ik}) P \prod_{j=s+1, j\neq k}^{j=N} dx_j dv_j \right] d\sigma_{ik} dv_k.$$

Since in the first term of the above expression there is no integration with respect to $dx_i dv_i$, if we solve all integrals (in both terms) with respect to variables not containing the indices i or k, we are left with

$$\iint\limits_{\Gamma^{\sigma\neq}_{N-s}} \nabla_{x_k} \cdot (v_k P) \prod_{j=s+1}^{j=N} dx_j dv_j = \sum_{i=1}^{s} \int\limits_{\{\|z_k\|=1\}\times\mathbb{R}^3} (v_k \cdot n_{ik}) P^{(s+1)} \, d\sigma_{ik} dv_k$$

$$+ \sum_{i=s+1, i\neq k}^{N} \int\limits_{\{\|z_k\|=1\}\times\mathbb{R}^3} (v_k \cdot n_{ik}) P^{(s+2)} dx_i dv_i \, d\sigma_{ik} dv_k. \tag{A.4.3}$$

Next, we wish to combine the results from equations (A.4.2),(A.4.3). We find the expression

$$\sum_{i=1}^{s} \iint\limits_{\Gamma^{\sigma\neq}_{N-s}} v_i \cdot \nabla_{x_i} P \prod_{j=s+1}^{j=N} dx_j dv_j + \sum_{k=s+1}^{N} \iint\limits_{\Gamma^{\sigma\neq}_{N-s}} \nabla_{x_k} \cdot (v_k P) \prod_{j=s+1}^{j=N} dx_j dv_j$$

$$= \sum_{i=1}^{s} \nabla_{x_i} \cdot (v_i P^{(s)}) - \sum_{i=1}^{s} \sum_{k=s+1}^{N} \iint\limits_{\{\|z_k\|=1\}} (v_i \cdot n_{ik}) P^{(s+1)} d\sigma_{ik} dv_k + \sum_{i=1}^{s} \sum_{k=s+1}^{N} \int\limits_{\{\|z_k\|=1\}\times\mathbb{R}^3} (v_k \cdot n_{ik}) P^{(s+1)} dx_i dv_i \, d\sigma_{ik} dv_k$$

$$+ \sum_{k=s+1}^{N} \sum_{i=s+1, i\neq k}^{N} \int\limits_{\{\|z_k\|=1\}\times\mathbb{R}^3} (v_k \cdot n_{ik}) P^{(s+2)} dx_i dv_i \, d\sigma_{ik} dv_k.$$

First, we notice that the second and third terms on the right hand side are very similar, so they can be combined. Also, in the last term of the right hand side, since $n_{ik} = -n_{ki}$, most terms of the summation will disappear. After these simplifications, defining the variable $V_{ik} = v_i - v_k$, we arrive at the the expression:

$$\sum_{i=1}^{s}\iint_{\Gamma_{N-s}^{\sigma\neq}} v_i\cdot\nabla_{x_i}P\prod_{j=s+1}^{j=N}dx_jdv_j+\sum_{k=s+1}^{N}\iint_{\Gamma_{N-s}^{\sigma\neq}}\nabla_{x_k}\cdot(v_kP)\prod_{j=s+1}^{j=N}dx_jdv_j$$

$$=\sum_{i=1}^{s}\nabla_{x_i}\cdot(v_iP^{(s)})-\sum_{i=1}^{s}\sum_{k=s+1}^{N}\int_{\{\|z_k\|=1\}\times\mathbb{R}^3}(V_{ik}\cdot n_{ik})P^{(s+1)}\,d\sigma_{ik}dv_k$$

$$+\frac{1}{2}\sum_{i=s+1,i\neq k}^{N}\int_{\{\|z_k\|=1\}\times\mathbb{R}^3}(V_{ki}\cdot n_{ik})P^{(s+2)}dx_idv_i\,d\sigma_{ik}dv_k. \tag{A.4.4}$$

In order to proceed further, we have to make a continuity assumption for the probability distribution P^N:

- we assume that P^N is continuous over the boundary, which implies that

$$P^N(x_1,v_1,...,x_i,v_i,...,x_j,v_j,...,x_N,v_N)=P^N(x_1,v_1,...,x_i,v_i',...,x_j,v_j',...,x_N,v_N) \tag{A.4.5}$$

  whenever $\|x_i-x_j\|=\sigma$,where v_i',v_j' denote velocities after a collision.

We argue that using (A.4.5), the last term on the right hand side is identically 0. Introduce the change of variables $(v_i,v_k,n_{ik})\to(v_i',v_k',-n_{ik})$. Since we are integrating over a sphere, separating it into the hemispheres $\{V_{ki}\cdot n_{ik}>0\}$ and $\{V_{ki}\cdot n_{ik}<0\}$, this change of variables takes points from one hemisphere to the other, and thanks to (A.4.5), $P^{(s+2)}$ has the same value over both hemispheres. Therefore the integrals over the hemispheres cancel each other. Lastly, using the fact that particles are indistinguishable, we can disregard one of the summations on the second term on the right hand side of (A.4.4). So we find

$$\sum_{i=1}^{s}\iint_{\Gamma_{N-s}^{\sigma\neq}} v_i\cdot\nabla_{x_i}P\prod_{j=s+1}^{j=N}dx_jdv_j+\sum_{k=s+1}^{N}\iint_{\Gamma_{N-s}^{\sigma\neq}}\nabla_{x_k}\cdot(v_kP)\prod_{j=s+1}^{j=N}dx_jdv_j$$

$$=\sum_{i=1}^{s}\nabla_{x_i}\cdot(v_iP^{(s)})-(N-s)\sum_{i=1}^{s}\int_{\{\|z_k\|=1\}\times\mathbb{R}^3}(V_{ik}\cdot n_{ik})P^{(s+1)}\,d\sigma_{ik}dv_k. \tag{A.4.6}$$

Thanks to the indistinguishability, the index k is now fixed, and we can take $k=s+1$. Finally, we can insert this back onto our first expression (A.4.1), yielding

$$\frac{\partial P^{(s)}}{\partial t}+\sum_{i\leqslant s}\iint_{\Gamma_{N-s}^{\sigma\neq}} v_i\cdot\nabla_{x_i}P\prod_{j=s+1}^{j=N}dx_jdv_j+\sum_{k\geqslant s}\iint_{\Gamma_{N-s}^{\sigma\neq}} v_k\cdot\nabla_{x_k}P\prod_{j=s+1}^{j=N}dx_jdv_j$$

$$=\frac{\partial P^{(s)}}{\partial t}+\sum_{i=1}^{s}\nabla_{x_i}\cdot(v_iP^{(s)})-(N-s)\sum_{i=1}^{s}\int_{\{\|z_k\|=1\}\times\mathbb{R}^3}(V_{ik}\cdot n_{ik})P^{(s+1)}\,d\sigma_{ik}dv_k=0 \tag{A.4.7}$$

Again, we separate the last term on the left hand side as integrals over two hemispheres, and recall that the change of variables $(v_i,v_k,n_{ik})\to(v_i',v_k',-n_{ik})$ takes points from one hemisphere to the other. So we can convert this into 2 integrals over the same hemisphere:

$$\int\limits_{\{\|z_k\|=1\}\times\mathbb{R}^3} (V_{ik}\cdot n_{ik})P^{(s+1)}\, d\sigma_{ik}dv_k = \int\limits_{\{V_{ki}\cdot n_{ik}>0\}\times\mathbb{R}^3} |V_{ik}\cdot n_{ik}|P^{(s+1)}\, d\sigma_{ik}dv_k - \int\limits_{\{V_{ki}\cdot n_{ik}<0\}\times\mathbb{R}^3} |V_{ik}\cdot n_{ik}|P^{(s+1)}\, d\sigma_{ik}dv_k$$

$$= \int\limits_{\{V_{ki}\cdot n_{ik}>0\}\times\mathbb{R}^3} |V_{ik}\cdot n_{ik}|\left(P^{(s+1)'} - P^{(s+1)}\right) d\sigma_{ik}dv_k = \sigma^2 \int\limits_{\mathbb{S}^2_+\times\mathbb{R}^3} |V_{ik}\cdot n_{ik}|\left(P^{(s+1)'} - P^{(s+1)}\right) d\mathcal{H}(\alpha)dv_k$$

where $\mathbb{S}^2_+$ denotes the hemisphere of the unit sphere where $V_{ki}\cdot n_{ik} > 0$. As claimed, we reach the expression

$$\frac{\partial P^{(s)}}{\partial t} + \sum_{i=1}^{s} v_i\cdot\nabla_{x_i}P^{(s)} = (N-s)\sigma^2\sum_{i=1}^{s}\int\limits_{\mathbb{S}^2_+\times\mathbb{R}^3} |V_{ik}\cdot n_{ik}|\left(P^{(s+1)'} - P^{(s+1)}\right) d\mathcal{H}(\alpha)dv_k, \qquad \text{(A.4.8)}$$

which defined the BBGKY hierarchy for hard spheres, for each $s \leqslant N$.

Next, we consider the case when particles interact through a short range potential. Starting in the same way as in the previous case, applying E_s to both sides of the Liouville equation, we find

$$\frac{\partial P^{(s)}}{\partial t} + \iint\limits_{(\Omega\times\mathbb{R}^3)^{N-s}} \{P, H\} \prod_{i=s+1,j=1}^{i=N,j=3} dx_{3i+j}dp_{3i+j} = 0$$

Expanding the Poisson bracket we get:

$$\frac{\partial P^{(s)}}{\partial t} + \iint\limits_{(\Omega\times\mathbb{R}^3)^{N-s}} \sum_{i\leqslant s}\sum_{j=1}^{3} \frac{\partial P}{\partial x_{3i+j}}\frac{\partial H}{\partial p_{3i+j}} - \frac{\partial P}{\partial p_{3i+j}}\frac{\partial H}{\partial x_{3i+j}} \prod_{i=s+1,j=1}^{i=N,j=3} dx_{3i+j}dp_{3i+j}$$

$$+ \iint\limits_{(\Omega\times\mathbb{R}^3)^{N-s}} \sum_{i\geqslant s}^{N}\sum_{j=1}^{3} \frac{\partial P}{\partial x_{3i+j}}\frac{\partial H}{\partial p_{3i+j}} - \frac{\partial P}{\partial p_{3i+j}}\frac{\partial H}{\partial x_{3i+j}} \prod_{i=s+1,j=1}^{i=N,j=3} dx_{3i+j}dp_{3i+j} = 0$$

Since our summations are finite, we can switch the order of integration and sum operators

$$\frac{\partial P^{(s)}}{\partial t} + \sum_{i\leqslant s}\sum_{j=1}^{3} \iint\limits_{(\Omega\times\mathbb{R}^3)^{N-s}} \frac{\partial P}{\partial x_{3i+j}}\frac{\partial H}{\partial p_{3i+j}} - \frac{\partial P}{\partial p_{3i+j}}\frac{\partial H}{\partial x_{3i+j}} \prod_{i=s+1,j=1}^{i=N,j=3} dx_{3i+j}dp_{3i+j}$$

$$= -\sum_{i\geqslant s}^{N}\sum_{j=1}^{3} \iint\limits_{(\Omega\times\mathbb{R}^3)^{N-s}} \frac{\partial P}{\partial x_{3i+j}}\frac{\partial H}{\partial p_{3i+j}} - \frac{\partial P}{\partial p_{3i+j}}\frac{\partial H}{\partial x_{3i+j}} \prod_{i=s+1,j=1}^{i=N,j=3} dx_{3i+j}dp_{3i+j}$$

$$\frac{\partial P^{(s)}}{\partial t} + \sum_{i\leqslant s}\sum_{j=1}^{3} p_{3i+j} \iint\limits_{(\Omega\times\mathbb{R}^3)^{N-s}} \frac{\partial P}{\partial x_{3i+j}} \prod_{i=s+1,j=1}^{i=N,j=3} dx_{3i+j}dp_{3i+j} - \iint\limits_{(\Omega\times\mathbb{R}^3)^{N-s}} \frac{\partial H}{\partial x_{3i+j}}\frac{\partial P}{\partial p_{3i+j}} \prod_{i=s+1,j=1}^{i=N,j=3} dx_{3i+j}dp_{3i+j}$$

$$= -\sum_{i\geqslant s}^{N}\sum_{j=1}^{3} \iint\limits_{(\Omega\times\mathbb{R}^3)^{N-s}} \frac{\partial P}{\partial x_{3i+j}}p_{3i+j} - \frac{\partial P}{\partial p_{3i+j}}\frac{\partial H}{\partial x_{3i+j}} \prod_{i=s+1,j=1}^{i=N,j=3} dx_{3i+j}dp_{3i+j}$$

Where in the last step, we used the fact that on the left hand side, p_{3i+j} is independent of the integration coordinates. On the Right hand side, one can use the divergence theorem (assuming $\Omega$to be simply connected) to conclude that:

$$\sum_{i\geqslant s}^{N}\iint_{(\Omega\times\mathbb{R}^3)^{N-s}}\sum_{j=1}^{3}\frac{\partial P}{\partial x_{3i+j}}p_{3i+j}\prod_{i=s+1,j=1}^{i=N,j=3}dx_{3i+j}dp_{3i+j}=\sum_{i\geqslant s}^{N}\iint_{(\Omega\times\mathbb{R}^3)^{N-s}}\nabla_{\mathbf{x}_i}\cdot(P\mathbf{p}_{3i})\prod_{i=s+1,j=1}^{i=N,j=3}dx_{3i+j}dp_{3i+j}$$

$$=\sum_{i\geqslant s}^{N}\int_{(\mathbb{R}^3)^{N-s}}\int_{\Omega^{N-s-1}}\left[\int_{\partial\Omega}P\mathbf{p}_{3i}\,dS_x\right]\prod_{l=s+1,l\neq i}^{l=N}dx_{3l+j}\prod_{i=s+1,j=1}^{i=N,j=3}dx_{3i+j}dp_{3i+j}$$

Assuming our vessel is impermeable to the gas, then $P|_{\partial\Omega}=0$, so this term vanishes.

Also on the left hand side, one can express $\frac{\partial H}{\partial x_{3i+j}}$ as

$$\frac{\partial H}{\partial x_{3i+j}}=\frac{\partial V}{\partial x_j}(x_{3i})+\sum_{k\neq i,k\leqslant s}F_j^{ik}(\mathbf{x}_i,\mathbf{x}_k)+\sum_{k\neq i,k\leqslant s}F_j^{ik}(\mathbf{x}_i,\mathbf{x}_k)$$

Therefore, the first 2 terms of this sum will not depend on the integration variables for the integral transformation. Assuming sufficient regularity conditions, one can take $\frac{\partial}{\partial x_{3i+j}}$ out of the integral in the left hand side, ending with:

$$\frac{\partial P^{(s)}}{\partial t}+\sum_{i\leqslant s}\sum_{j=1}^{3}p_{3i+j}\frac{\partial P^{(s)}}{\partial x_{3i+j}}-\left[\frac{\partial V}{\partial x_j}(x_{3i})+\sum_{k\neq i,k\leqslant s}F_j^{ik}(\mathbf{x}_i,\mathbf{x}_k)\right]\iint_{(\Omega\times\mathbb{R}^3)^{N-s}}\frac{\partial P}{\partial p_{3i+j}}\prod_{i=s+1,j=1}^{i=N,j=3}dx_{3i+j}dp_{3i+j}$$

$$=-\sum_{i\leqslant s}\sum_{j=1}^{3}\iint_{(\Omega\times\mathbb{R}^3)^{N-s}}\sum_{k>s}F_j^{ik}(\mathbf{x}_i,\mathbf{x}_k)\frac{\partial P}{\partial p_{3i+j}}\prod_{i=s+1,j=1}^{i=N,j=3}dx_{3i+j}dp_{3i+j}$$

$$-\sum_{i\geqslant s}^{N}\sum_{j=1}^{3}\iint_{(\Omega\times\mathbb{R}^3)^{N-s}}\frac{\partial P}{\partial x_{3i+j}}p_{3i+j}-\frac{\partial P}{\partial p_{3i+j}}\frac{\partial H}{\partial x_{3i+j}}\prod_{i=s+1,j=1}^{i=N,j=3}dx_{3i+j}dp_{3i+j}$$

Which can be rewritten as:

$$\frac{\partial P^{(s)}}{\partial t}+\{P^{(s)},H\}_s=-\sum_{i\leqslant s}\sum_{j=1}^{3}\frac{\partial}{\partial p_{3i+j}}\sum_{k>s}\iint_{(\Omega\times\mathbb{R}^3)^{N-s}}F_j^{ik}(\mathbf{x}_i,\mathbf{x}_k)P\prod_{i=s+1,j=1}^{i=N,j=3}dx_{3i+j}dp_{3i+j}$$

$$+\sum_{i\geqslant s}^{N}\sum_{j=1}^{3}\iint_{(\Omega\times\mathbb{R}^3)^{N-s}}\frac{\partial P}{\partial p_{3i+j}}\frac{\partial H}{\partial x_{3i+j}}\prod_{i=s+1,j=1}^{i=N,j=3}dx_{3i+j}dp_{3i+j}$$

Assuming indistinguishable particles, the value of $\iint_{(\Omega\times\mathbb{R}^3)^{N-s}}F_j^{ik}(\mathbf{x}_i,\mathbf{x}_k)P\prod_{i=s+1,j=1}^{i=N,j=3}dx_{3i+j}dp_{3i+j}$ on the right hand side will not be dependent on k. Therefore, choosing $k=s+1$, the expression simplifies to:

$$\frac{\partial P^{(s)}}{\partial t}+\{P^{(s)},H\}_s=$$

$$-(N-s)\sum_{j=1}^{3}\frac{\partial}{\partial p_{3(s+1)+j}}\iint_{\Omega\times\mathbb{R}^3}F_j^{i\,s+1}(\mathbf{x}_i,\mathbf{x}_{s+1})P\,dx_{3(s+1)+j}dp_{3(s+1)+j}\prod_{i>s+1,j=1}^{i=N,j=3}dx_{3i+j}dp_{3i+j}$$

$$+\sum_{i\geqslant s}^{N}\sum_{j=1}^{3}\iint\limits_{(\Omega\times\mathbb{R}^3)^{N-s}}\frac{\partial P}{\partial p_{3i+j}}\frac{\partial H}{\partial x_{3i+j}}\prod_{i=s+1,j=1}^{i=N,j=3}dx_{3i+j}dp_{3i+j}$$

On the first term on the right hand side, performing the integrals for $i > s+1$, one arrives in:

$$\frac{\partial P^{(s)}}{\partial t}+\{P^{(s)},H\}_s=-(N-s)\sum_{j=1}^{3}\frac{\partial}{\partial p_{3(s+1)+j}}\iint\limits_{\Omega\times\mathbb{R}^3}F_j^{i\,s+1}(\mathbf{x}_i,\mathbf{x}_{s+1})P^{(s+1)}\,dx_{3(s+1)+j}dp_{3(s+1)+j}$$

$$+\sum_{i\geqslant s}^{N}\sum_{j=1}^{3}\iint\limits_{(\Omega\times\mathbb{R}^3)^{N-s}}\frac{\partial P}{\partial p_{3i+j}}\frac{\partial H}{\partial x_{3i+j}}\prod_{i=s+1,j=1}^{i=N,j=3}dx_{3i+j}dp_{3i+j}$$

Under sufficient decay conditions for P, the term $\sum_{i\geqslant s}^{N}\sum_{j=1}^{3}\iint_{(\Omega\times\mathbb{R}^3)^{N-s}}\frac{\partial P}{\partial p_{3i+j}}\frac{\partial H}{\partial x_{3i+j}}\prod_{i=s+1,j=1}^{i=N,j=3}dx_{3i+j}dp_{3i+j}$ vanishes. To prove this claim, we can rearrange the last term as

$$\sum_{i\geqslant s}^{N}\sum_{j=1}^{3}\iint\limits_{(\Omega\times\mathbb{R}^3)^{N-s}}\frac{\partial P}{\partial p_{3i+j}}\frac{\partial H}{\partial x_{3i+j}}\prod_{i=s+1,j=1}^{i=N,j=3}dx_{3i+j}dp_{3i+j}$$

$$=\sum_{i\geqslant s}^{N}\sum_{j=1}^{3}\int\limits_{\Omega^{N-s}}\frac{\partial H}{\partial x_{3i+j}}\left[\int\limits_{(\mathbb{R}^3)^{N-s}}\frac{\partial P}{\partial p_{3i+j}}\prod_{i=s+1,j=1}^{i=N,j=3}dp_{3i+j}\right]\prod_{i=s+1,j=1}^{i=N,j=3}dx_{3i+j}$$

Where we used the fact that the forces in this systems (both internal and external) are independent on the particle's momenta. The following expression equals:

$$\int\limits_{\Omega^{N-s}}\nabla_x H\cdot\left[\int\limits_{(\mathbb{R}^3)^{N-s}}\nabla_p P\prod_{i=s+1,j=1}^{i=N,j=3}dp_{3i+j}\right]\prod_{i=s+1,j=1}^{i=N,j=3}dx_{3i+j}$$

Taking the innermost integral, $\mathbf{I}=\int_{(\mathbb{R}^3)^{N-s}}\nabla_p\prod_{i=s+1,j=1}^{i=N,j=3}dp_{3i+j}$, one can see that $\mathbf{I}=\lim_{R\to\infty}I(R)$, where:

$$I(R)=\int\limits_{(\mathbb{R}^3)^{N-s}}\nabla_p P\prod_{i=s+1,j=1}^{i=N,j=3}dx_{3i+j}dp_{3i+j}$$

Using the Gauss's Theorem:

$$I(R)=\int\limits_{\partial B_R(0)}P\hat{\mathbf{n}}\,dS_{\mathbf{p}}$$

Where $B_R(0)=\left\{x\in\left(\mathbb{R}^3\right)^{N-s}\mid\|x\|\leqslant R\right\}$. Thus we have:

$$\|I(R)\|\leqslant\omega_{3(N-s)}R^{3(N-s)-1}\sup_{\|\mathbf{p}\|=R}P$$

Since $\int_{(\mathbb{R}^3)^{N-s}}P\prod_{i=s+1,j=1}^{i=N,j=3}dp_{3i+j}$ is finite for every $\mathbf{x}$, in spherical coordinates:

$$\int_{(\mathbb{R}^3)^{N-s}} P \prod_{i=s+1,j=1}^{i=N,j=3} dp_{3i+j} = \int_0^\infty r^{3(N-s)-1} \int_{\partial B_R(0)} P\, dS_{\mathbf{p}}\, dr \leqslant \omega_{3(N-s)} \int_0^\infty r^{3(N-s)-1} \sup_{\|\mathbf{p}\|=r} P\, dr \leqslant C(x)$$

Therefore, we can say that:

$$\sup_{\|\mathbf{p}\|=r} P \leqslant r^{-3(N-s)-\epsilon}$$

Finally:

$$\|I(R)\| \leqslant \omega_{3(N-s)} R^{-1-\epsilon} \therefore \mathbf{I} = \lim_{R\to\infty} I(R) = \mathbf{0}$$

Our expression final expression then reads:

$$\frac{\partial P^{(s)}}{\partial t} + \{P^{(s)}, H\}_s = -(N-s) \sum_{j=1}^{3} \frac{\partial}{\partial p_{3(s+1)+j}} \iint_{\Omega\times\mathbb{R}^3} F_j^{i\,s+1}(\mathbf{x}_i, \mathbf{x}_{s+1}) P^{(s+1)}\, dx_{3(s+1)+j} dp_{3(s+1)+j}, \quad \text{(A.4.9)}$$

for $s \leqslant N$, as claimed.

A.5 Factorization property of equilibrium probability distributions in the limit of large N

We follow in this part the demonstration given by [7]

Any time independent solution to equation (1.5.2) (Liouville equation) in 3 dimensions will be a function of the 6N-1 constants of motion of our system. If we know their equilibrium values, we could say:

$$P(x, v) = A \prod_{j}^{6N-1} \delta(K_j(x, v) - K_j^{(observed)})$$

If we can assume the dynamics of our system to be ergodic, then the $K_j(x, v)$ can become arbitratily close to the total energy of our system, so that our expression reduces to $P(x, v) = A\delta(E(x, v) - E^{(observed)})$. Up to this point, the expressions considered are general for systems of classical particles (apart from the simplification ergodicity assumption). Assuming now that the potential energy is small in most regions of the phase space (as particles in a gas are usually very separated), we can approximate the expression for E(x,y), saying:

$$P(x, v) = A\delta\left(\sum_{j=1}^{N} \frac{m_j v_j^2}{2} - Ne\right)$$

Where e is the energy per particle. We now use this expression to obtain a value for A, our normalization constant:

$$A = \frac{1}{\int \delta\left(\sum_{j=1}^{N} \frac{m_j v_j^2}{2} - Ne\right) \prod dx_i \prod dv_i}$$

Since our expression doesn't depend on x (no potential energy), and since we're assuming an ensemble at constant volume and energy, integrations in the x coordinate can be factored out. We can also change variables, setting: $w_j = \sqrt{(m_j/2)}v_j$, which then gives:

$$A = \frac{\prod_j^N m_j^{\frac{3}{2}}}{2^{3N/2} V_N \int \delta\left(\sum_{j=1}^N w_j^2 - Ne\right) \prod dw_j}$$

Where V_N is the volume occupied by the system in phase space. Passing to polar coordinates: $\sum_{j=1}^N w_j^2 = r^2$, thus

$$A = \frac{\prod_j^N m_j^{\frac{3}{2}}}{2^{3N/2} V_N W_N \int \delta\left(r^2 - Ne\right) r^{3N-1} dr} = \frac{\prod_j^N m_j^{\frac{3}{2}}}{2^{3N/2} V_N W_{3N} (Ne)^{\frac{3N-2}{2}}}$$

where we integrated out the angular part, yielding the factor W_N (area of the unit sphere in N dimensions).

This has the following remarkable consequence: suppose we want to have a probability density for a single particle in the phase space. We must then define:

$$P^1(x_1, v_1) = \int P \prod_{i \neq 1} dx_i dv_i$$

which gives us

$$P_N^1(x_1, v_1) = A \int \delta\left(\sum_{j>1}^N \frac{m_j v_j^2}{2} - \left(Ne - \frac{m_1 v_1^2}{2}\right)\right) \prod_{j>1} dx_j dv_j = A W_{3N-3} \frac{V_{N-1}}{2} (2Ne - m_1 v_1^2)^{(3N-5)/2} \prod_{j>1}^N m_j^{\frac{3}{2}}$$

$$= \frac{\prod_j^N m_j^{\frac{3}{2}}}{2^{3N/2} V_N W_{3N} (Ne)^{\frac{3N-2}{2}}} W_{3N-3} \frac{V_{N-1}}{2} (2Ne - m_1 v_1^2)^{(3N-5)/2} \prod_{j>1}^N m_j^{\frac{-3}{2}}$$

(The W_{3N-3} term appears since by subtracting a particle we in fact take away 3 dimensions from our

integral). This becomes after cancellation:

$$= \frac{\left(\frac{m_1}{2Ne}\right)^{\frac{3}{2}}}{V} \left(1 - \frac{m_1 v_1^2}{2Ne}\right)^{(3N-5)/2} \left(\frac{W_{3N-3}}{W_{3N}}\right) = \frac{\left(\frac{m_1}{2Ne}\right)^{\frac{3}{2}}}{V} \left(\left(1 - \frac{(m_1 v_1^2/2e)}{N}\right)^N\right)^{3/2} \left(\frac{W_{3N-3}}{W_{3N}}\right) \left(1 - \frac{m_1 v_1^2}{2Ne}\right)^{-5/2}$$

Taking the limit as $N \to \infty$, we see that the last term in brackets approaches 0, while for the second term we have:

$$\lim_{N \to \infty} \left(\left(1 - \frac{(m_1 v_1^2/2e)}{N}\right)^N\right)^{3/2} = exp\left(\frac{-3 m_1 v_1^2}{4e}\right)$$

One can also show that:

$$\lim_{N\to\infty} \frac{\left(\frac{m_1}{2Ne}\right)^{\frac{3}{2}}}{V} \frac{W_{3N-3}}{W_{3N}} = \frac{1}{V}\left(\frac{4\pi e}{3m_1}\right)^{-3/2},$$

Therefore:

$$\lim_{N\to\infty} P_N^1(x_1,v_1) = \frac{1}{V}\left(\frac{4\pi e}{3m_1}\right)^{-3/2} exp\left(\frac{-3m_1v_1^2}{4e}\right) \equiv P^{(1)}(x1,v1)$$

So in the thermodynamic limit (infinite number of particles), the probability density approaches a gaussian distribution! Moreover, if we calculate the 2-particle distribution (probability density of finding a pair of particles in a given region of the phase space) by integrating out the other particle's coordinates, we find that:

$$P^2(x_1,x_2,v_1,v_2) = \int P\prod_{i>2} dx_i dv_i = A\int \delta\left(\sum_{j>2}^{N} \frac{m_j v_j^2}{2} - \left(Ne - \frac{m_1v_1^2}{2} - \frac{m_2v_2^2}{2}\right)\right) \prod_{j>2} dx_j dv_j$$

$$= \frac{\left(\frac{1}{2Ne}\right)^3}{V^2} (m_1m_2)^{\frac{3}{2}} \left(1 - \frac{m_1v_1^2 + m_2v_2^2}{2Ne}\right)^{(3N-8)/2} \left(\frac{W_{3N-6}}{W_{3N}}\right)$$

$$= \frac{\left(\frac{1}{2Ne}\right)^3}{V^2} (m_1m_2)^{\frac{3}{2}} \left(\left(1 - \frac{((m_1v_1^2 + m_2v_2^2)/2e)}{N}\right)^N\right)^{3/2} \left(\frac{W_{3N-6}}{W_{3N}}\right) \left(1 - \frac{m_1v_1^2 + m_2v_2^2}{2Ne}\right)^{-8/2}$$

Again from the same limiting procedure, we find that:

$$\lim_{N\to\infty} \left(\frac{1}{2eN}\right)^3 \frac{W_{3N-6}}{W_{3N}} = \left(\frac{3}{4\pi e}\right)^3$$

Therefore:

$$\lim_{N\to\infty} P_N^2(x_1,x_2,v_1,v_2) = \frac{1}{V^2}\left(\frac{4\pi e}{3m_1m_2}\right)^{-3} exp\left(\frac{-3m_1v_1^2}{4e}\frac{-3m_2v_2^2}{4e}\right)$$

$$= \left[\frac{1}{V}\left(\frac{4\pi e}{3m_1}\right)^{-3/2} exp\left(\frac{-3m_1v_1^2}{4e}\right)\right]\left[\frac{1}{V}\left(\frac{4\pi e}{3m_1}\right)^{-3/2} exp\left(\frac{-3m_2v_2^2}{4e}\right)\right]$$

$$= P^{(1)}(x_1,v_1)P^{(1)}(x_2,v_2)$$

Therefore, in the thermodynamic limit, particles become statistically uncorrelated (probability of finding a pair of particles in a given state becomes simply the product of the probabilities of each individual state).

Appendix B

Section 1.1 Theorems

B.1 Equivalence lemma

The lemma we wish to prove is

Lemma B.1.1. *Let $f \in L^1_{loc}([0,T] \times \mathbb{R}^{2d})$ be an essentially non-negative function. Then the following conditions hold*
(i) If $Q^{\pm}(f,f), \in L^1_{loc}((0,T) \times \mathbb{R}^{2d})$, then f is a distributional solution to the Boltzmann equation if and only if f is a mild solution.
(ii) If f satisfies the definition 2.2.4 at least for the particular choice of $\beta(x) = \log(1+x)$, $x > 0$, then it is a mild solution.
(iii) If f is a mild solution and $Q^{\pm}(f,f)\beta'(f)$ are both in $L^1_{loc}((0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d)$, then f is a renormalized solution.
(iv) If $Q^-(f,f) \in L^1_t(0,T)$ for a.e. x, v, then, if f is an exponentially mild solution, f is also a mild solution.

Proof. (i): take any $\varphi \in C_0^\infty(\mathbb{R}^{2d}), \zeta(t) \in C_0^\infty([0,T])$. By definition, f is a distributional solution if $-\langle f, \mathfrak{D}\psi\rangle - \langle Q(f,f), \psi\rangle = 0 \ \ \forall \psi \in C_0^\infty([0,T] \times \mathbb{R}^{2d})$. The pairing $\langle \cdot, \cdot \rangle$ becomes an integral over $[0,T] \times \mathbb{R}^{2d}$ in case both arguments are functions in L^1_{loc}, and reduces to $\langle \mathfrak{D} f - Q(f,f), \psi\rangle = 0 \ \ \forall \psi \in C_0^\infty([0,T] \times \mathbb{R}^{2d})$ if f is regular enough. In particular

$$-\left\langle f, \left[\frac{\partial}{\partial t} + v \cdot \nabla\right] \varphi(x - vt, v)\zeta(t)(t) \right\rangle - \langle Q(f,f), \varphi(x - vt, v)\zeta(t)(t)\rangle = 0$$

(choosing $\psi = \varphi(x - vt, v)\zeta(t)$). But

$$\begin{aligned} -\left\langle f, \left[\frac{\partial}{\partial t} + v \cdot \nabla\right] \varphi(x - vt, v)\zeta(t) \right\rangle &= -\left\langle f, \zeta(t) \left[\frac{\partial}{\partial t} + v \cdot \nabla\right] \varphi(x - vt, v) \right\rangle \\ &\quad - \left\langle f, \varphi(x - vt, v) \left[\frac{\partial}{\partial t} + v \cdot \nabla\right] \zeta(t) \right\rangle \\ &= -\langle f, \varphi(x - vt, v)\zeta'(t)\rangle \end{aligned}$$

Therefore

$$-\langle f, \varphi(x - vt, v)\zeta'(t)\rangle = \langle Q(f,f), \varphi(x - vt, v)\zeta(t)\rangle$$

After performing a change of variables ($x' = x - vt$)

$$-\langle f(x' + vt, v), \varphi(x', v)\zeta'(t)\rangle = \langle Q(f,f)(x' + vt, v), \varphi(x', v)\zeta(t)\rangle$$

Since φ is arbitrary and $Q^{\pm}(f,f), f \in L^1_{loc}((0,T)\times\mathbb{R}^{2d})$, we can write down our pairing as an integral, and for almost every x', v

$$-\int_0^T f(x'+vt)\zeta'(t)\,dt = \int_0^T Q(f,f)(x'+vt)\zeta(t)\,dt \tag{B.1.1}$$

Thus $Q(f,f)$ is a weak derivative of f in this case, and we can use the following theorem, taken from [6]:

Theorem B.1.1. *Let $f \in W^{1,1}_{loc}(I)$, $I \in \mathbb{R}$ a bounded open set. Then f has a continuous representative $\tilde{f}$ which is absolutely continuous in every compact set $K \subseteq I$, meaning that*

$$\tilde{f} = f \;\; a.e., \;\; \tilde{f} \in C(I)$$

$$\tilde{f}(t_2) - \tilde{f}(t_1) = \int_{t_1}^{t_2} \partial^w f\,dt; \quad [t_1,t_2] \subseteq I$$

Where $\partial^w f$ denotes the weak derivative of f.

□

We pick $I = [0,T]$, and the above theorem tells us that there is a continuous representative, almost everywhere equal to f, which satisfies

$$f^{\#}(t,x,v) - f^{\#}(t,x,v) = \int_s^t Q^{\#}(f,f)(r,x,v)\,dr \quad \text{for a.e.}(x,v) \in \mathbb{R}^{2d}$$

To prove the converse, we simply follow the argument backwards: Starting from the definition of mild solution, we have that $f^{\#}$ has na ordinary derivative almost everywhere which is equal to $Q^{\#}(f,f)$. this means that $\partial^w f^{\#} = Q(f,f)^{\#}$, which brings us back to equation B.1.1. Then, we multiply by an arbitrary test function φ, and return to our definition of distribution, the only thing we have to check is that any function $\psi \in C_0^\infty([0,T]\times\mathbb{R}^{2d})$ can be approximated by a product of the form $\varphi\zeta$ where $\varphi(x,v) \in C_0^\infty(\mathbb{R}^{2d}), \zeta(t) \in C_0^\infty([0,T])$.

(ii) Suppose that f is a renormalized solution, but only for $\beta(x) = \log(1+x)$. By definition of a renormalized solution, we have that

$$-\left\langle \beta(f), \left[\frac{\partial}{\partial t} + v\cdot\nabla\right]\psi \right\rangle - \langle Q(f,f)\beta'(f), \psi\rangle = 0.$$

For any test function ψ. We wish to show that if this is the case, then f is also a renormalized solution with another choice of renormalization, namely $\beta_\delta(x) = \frac{1}{\delta}\log(1+\delta x)$. For this, notice that $\beta_\delta(x) = \frac{1}{\delta}\beta(\delta(e^{\beta(x)}-1)) = g_\delta(\beta(x))$, with $g_\delta(x) = \frac{1}{\delta}\beta(\delta(e^x-1))$. For each fixed $\delta > 0$, the function $g_\delta(x)$ is Lipschitz, since $g'_\delta(x) = \frac{e^x}{(1-\delta)+\delta e^x} < \frac{1}{\delta}$, and we have the following

Lemma B.1.2. *let f, $\mathfrak{D}f \in L^1_{loc}(\mathbb{R}^{N+1};\mathbb{R})$, and $g:\mathbb{R}\longrightarrow\mathbb{R}$ any lipschitz function. Then*

$$\mathfrak{D}g(f) = g'(f)\mathfrak{D}f \quad a.e. \Rightarrow \;\; \mathfrak{D}g(f) \in L^1_{loc}$$

Therefore, applying the above lemma we have

$$-\left\langle g_\delta(\beta(f)), \left[\frac{\partial}{\partial t} + v\cdot\nabla\right]\psi \right\rangle - \langle Q(f,f)g'_\delta(\beta(f))\beta'(f), \psi\rangle = 0$$

Choosing again $\psi = \varphi(x-vt,v)\zeta(t)$ and noticing that $g'_\delta(\beta(f))\beta'(f) = \frac{1+f}{1+\delta f} \in L^\infty$, we can perform the exact same steps as before, and arrive that

$$-\left\langle \beta_\delta(f)^{\#}, \varphi(x,v)\zeta'(t)\right\rangle = \left\langle \frac{Q(f,f)}{1+\delta f}^{\#}, \varphi(x,v)\zeta(t)\right\rangle \tag{B.1.2}$$

Since $\frac{Q(f,f)}{1+f} \in L^1_{loc}$, then so is $\frac{Q(f,f)}{1+\delta f} = \frac{1+f}{1+\delta f}\frac{Q(f,f)}{1+f}$. Also, since $\beta_\delta(x) \leq x$, we have that $\beta_\delta(f) \leq f \therefore \beta_\delta(f) \in L^1_{loc}$.

$$-\int_0^T \beta_\delta(f)^{\#}\zeta'(t)\,dt = \int_0^T \frac{Q(f,f)}{1+\delta f}^{\#}\zeta(t)\,dt \qquad a.e.$$

$$\therefore\ \ \beta_\delta(f)^{\#}(t,x,v) - \beta_\delta(f)^{\#}(s,x,v) = \int_s^t \frac{Q(f,f)}{1+\delta f}^{\#}(r,x,v)\,dr \quad \text{for a.e.}(x,v) \in \mathbb{R}^{2d} \tag{B.1.3}$$

again using theorem B.1.1. Taking the limit when $\delta \to 0$, the left hand side converges to $f^{\#}(t,x,v) - f^{\#}(s,x,v)$. Since $\frac{Q(f,f)}{1+\delta f}$ is decreasing as a function of the parameter $\delta > 0$, we can use the monotone convergence theorem, so we can conlude that the limit equation is

$$f^{\#}(t,x,v) - f^{\#}(s,x,v) = \int_s^t Q(f,f)^{\#}(r,x,v)\,dr \quad \text{for a.e.}(x,v) \in \mathbb{R}^{2d}$$

which proves that f is indeed a mild solution.

(iii) Let f be a mild solution. Then, $\beta_1(f)$ is also absolutely continuous, and by lemma B.1.2 we can claim that

$$\beta_1(f)^{\#}(t,x,v) - \beta_1(f)^{\#}(s,x,v) = \int_s^t \frac{Q(f,f)}{1+f}^{\#}(r,x,v)\,dr \quad \text{for a.e.}(x,v) \in \mathbb{R}^{2d}$$

Finally, from item (i) we know that $\beta_1(f)$ is a distributional solution, which proves that f is a renormalized solution.

(iv) Let f be an exponentially mild solution. Then, this implies $Q_e^+(f,f), A * f \in L^1_t(0,T)$ for a.e. x, v, and also by assumption $Q^-(f,f) \in L^1_t(0,T)$ for a.e. x, v. By definition of an exponentially mild solution, we have that

$$\partial^w f_e = Q_e^+(f,f)$$

$$(\partial^w f^{\#})e^{\int_0^t (A*f)^{\#}(r,w,v)\,dr} + f^{\#}(A*f)^{\#}e^{\int_0^t (A*f)^{\#}(r,w,v)\,dr} = Q^+(f,f)^{\#}e^{\int_0^t (A*f)^{\#}(r,w,v)\,dr}$$

$$\therefore \partial^w f^{\#} + Q^-(f,f)^{\#} = Q^+(f,f)^{\#}$$

Since $Q_e^+(f,f) = Q^+(f,f)^{\#}e^{F^{\#}}$ and $e^{-F^{\#}} \leq C$, $Q_e^+(f,f) \in L^1_t(0,T) \Rightarrow Q^+(f,f) \in L^1_t(0,T)$. Also, since f_e was absolutely continuous with respect to t, $f^{\#}$ will be as well. Therefore,

$$\partial^w f^{\#} = Q^+(f,f)^{\#} - Q^-(f,f)^{\#} = Q(f,f)^{\#} \in L^1_t(0,T)$$

Finally then, this means

$$f(t,x,v) - f_0(x,v) = \int_0^t Q(f,f)^{\#}\,dr$$

And f is a mild solution.

B.2 Renormalized collision lemma

What we wish to prove is that

$$\|Q_n(f,f) - Q_n(g,g)\|_{L^1_v(\mathbb{R}^d)} \leq C(n,d)\|f-g\|_{L^1_v(\mathbb{R}^d)}$$

Proof. As is often the case in when dealing with the Boltzmann equation, we will deal with the 'gain' and 'loss' terms on the collision operator separately. Consider the case of the 'loss' term. We have that

$$\|Q_n^-(f,f) - Q_n^-(g,g)\|_{L^1_v} =$$

$$\left| \frac{\iiint_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} \left[\left(1+\delta_n \int_{\mathbb{R}^d} g\, d\xi\right) ff_* - \left(1+\delta_n \int_{\mathbb{R}^d} f\, d\xi\right) gg_*\right] B_n(v-v_*,\alpha)\, d\mathcal{H}^{d-1}(\alpha) dv_* dv}{\left(1+\delta_n \int_{\mathbb{R}^d} f\, d\xi\right)\left(1+\delta_n \int_{\mathbb{R}^d} g\, d\xi\right)} \right|$$

$$= \left| \frac{\iiint_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} \left[ff_* - gg_* + \delta_n \left(ff_* \int_{\mathbb{R}^d} g\, d\xi - gg_* \int_{\mathbb{R}^d} f\, d\xi\right)\right] B_n(v-v_*,\alpha)\, d\mathcal{H}^{d-1}(\alpha) dv_* dv}{\left(1+\delta_n \int_{\mathbb{R}^d} f\, d\xi\right)\left(1+\delta_n \int_{\mathbb{R}^d} g\, d\xi\right)} \right| \tag{B.2.1}$$

where for shorthand we ommited the arguments of f, g in the nominator, which can be written explicitly as

$$\left(1+\delta_n \int_{\mathbb{R}^d} g(t,x,\xi)\, d\xi\right)\left(1+\delta_n \int_{\mathbb{R}^d} f(t,x,\xi)\, d\xi\right)$$

and in the numerator, $ff_* = f(t,x,v)f(t,x,v_*)$, $gg_* = g(t,x,v)g(t,x,v_*)$. Separate the integral in the numerator into two terms, namely

$$[\mathrm{I}] = \iiint\limits_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} [ff_* - gg_*]\, B_n(v-v_*,\alpha)\, d\mathcal{H}^{d-1}(\alpha) dv_* dv$$

$$[\mathrm{II}] = \iiint\limits_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} \left(ff_* \int_{\mathbb{R}^d} g\, d\xi - gg_* \int_{\mathbb{R}^d} f\, d\xi\right) B_n(v-v_*,\alpha)\, d\mathcal{H}^{d-1}(\alpha) dv_* dv$$

$$= \iiiint_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} [f(v)f(v_*)g(\xi) - g(v)g(v_*)f(\xi)]\, B_n(v-v_*,\alpha)\, d\mathcal{H}^{d-1}(\alpha) d\xi dv dv_*$$

ommiting the dependence on (x,t). By construction, $B_n(v-v_*,\alpha)$ is C_0^∞ for each n. Therefore, let $C'(n,d) = \|B_n\|_{L^\infty_\alpha} \mathfrak{L}(\mathbb{S}^{d-1})$, where $\mathfrak{L}$ is the lebesgue measure. Then, insert $\pm fg_*$ in the first integral and $\pm g(v_*)f(v)g(\xi) \pm f(v)g(v_*)f(\xi)$ on the second. Using the triangle inequality one then arrives at

$$|\mathrm{I}| \le C'(n,d) \iint\limits_{\mathbb{R}^{2d}} |ff_* - fg_* + fg_* - gg_*|\, dv_* dv \le C'(n,d)\|f-g\|_{L^1_v}\left[\|f\|_{L^1_v} + \|g\|_{L^1_v}\right]$$

$$|\mathrm{II}| \le C'(n,d) \iiint\limits_{\mathbb{R}^{3d}} |f(v)f(v_*)g(\xi) - g(v_*)f(v)g(\xi)|\, d\xi dv dv_* +$$

$$+C'(n,d)\left[\iiint\limits_{\mathbb{R}^{3d}} |g(v_*)f(v)g(\xi) - f(v)g(v_*)f(\xi)|\, d\xi dv dv_* + \iiint\limits_{\mathbb{R}^{3d}} |f(v)g(v_*)f(\xi) - g(v)g(v_*)f(\xi)|\, d\xi dv dv_*\right]$$

$$\le 3C'(n,d)\left[\|f\|_{L^1_v}\|g\|_{L^1_v}\|f-g\|_{L^1_v}\right]$$

Finally, returning to equation B.2.1 yields

$$\|Q_n^-(f,f) - Q_n^-(g,g)\|_{L^1_v} \le \frac{|\mathrm{I}| + \delta_n|\mathrm{II}|}{\left(1+\delta_n\|f\|_{L^1_v}\right)\left(1+\delta_n\|g\|_{L^1_v}\right)}$$

$$\le \frac{C'(n,d)\|f-g\|_{L^1_v}\left[\|f\|_{L^1_v} + \|g\|_{L^1_v} + 3\delta_n\|f\|_{L^1_v}\|g\|_{L^1_v}\right]}{\left[1+\delta_n\left(\|f\|_{L^1_v} + \|g\|_{L^1_v}\right) + \delta_n^2\|f\|_{L^1_v}\|g\|_{L^1_v}\right]}$$

$$\le \frac{3C'(n,d)\|f-g\|_{L^1_v}\left[\delta_n\left(\|f\|_{L^1_v} + \|g\|_{L^1_v}\right) + \delta_n^2\|f\|_{L^1_v}\|g\|_{L^1_v}\right]}{\delta_n\left[1+\delta_n\left(\|f\|_{L^1_v} + \|g\|_{L^1_v}\right) + \delta_n^2\|f\|_{L^1_v}\|g\|_{L^1_v}\right]} \le C_1(n,d)\|f-g\|_{L^1_v}$$

where $C_1(n,d) = 3C'(n,d)/\delta_n$, proving the assertion. For the 'gain' term on the collision operator, we perform the exact same calculations as in equation B.2.1, the only difference is that all functions now appear primed (recall that $f' = f(t,x,v'), f'_* = f(t,x,v'_*)$). We define then [I] and [II] in the same way as before, splitting the numerator. The only difference is that in order to establish bounds for [I],[II], we now need to perform a change of variables. Defining $\sigma = \langle v - v_*, \alpha\rangle\alpha$ and ommiting the dependence on x, t:

$$\iiint_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} f' g'_* B_n(v - v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha) dv_* dv$$

$$= \iiint_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} f(v - \langle v - v_*, \alpha\rangle\alpha) g(v_* + \langle v - v_*, \alpha\rangle\alpha) B_n(v - v_*, \alpha)\, d\mathcal{H}^{d-1}(\alpha) dv_* dv$$

$$= \iiint_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} \frac{1}{|\langle v - v_*, \alpha(\sigma)\rangle|^d} f(v - \sigma) g(v_* + \sigma) B_n(v - v_*, \alpha(\sigma))\, d\sigma dv_* dv.$$

Recall now that $B_n = 0$ if $\langle (v - v_*), \alpha\rangle < \delta_n$, so we can consider only the the case when $\langle (v - v_*), \alpha\rangle \geq \delta_n$, and claim then that the above expression satisfies

$$\leq \frac{1}{\delta_n^d} \iiint_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} f(v - \sigma) g(v_* + \sigma) B_n(v - v_*, \alpha(\sigma))\, d\sigma dv_* dv$$

$$\leq \frac{\|B_n\|_{L^\infty}}{\delta_n^d} \iiint_{\mathbb{R}^{2d}\times\mathrm{supp}(B_n)} f(v - \sigma) g(v_* + \sigma)\, d\sigma dv_* dv = C_*(n,d) \|f\|_{L^1_v} \|g\|_{L^1_v}$$

Where in the above, we used the fact that the B_n are compactly supported and smooth, therefore bounded. We also used Fubini's theorem to swap the order of integration, performing first the integrals with respect to v, v_*, which gives us the L^1 norms. Setting $\mathfrak{L}(A)$ as the lebesgue measure, the constant that appears in this bound is given by $C_*(n,d) = \frac{\|B_n\|_{L^\infty}}{\delta_n^d} \mathfrak{L}(\mathrm{supp}(B_n))$. Performing the exact same splitting as before, we get

$$|\mathrm{I}| \leq C_*(n,d) \|f - g\|_{L^1_v} \left[\|f\|_{L^1_v} + \|g\|_{L^1_v} \right]$$
$$|\mathrm{II}| \leq 3C_*(n,d) \left[\|f\|_{L^1_v} \|g\|_{L^1_v} \|f - g\|_{L^1_v} \right]$$

$$\|Q_n^+(f,f) - Q_n^+(g,g)\|_{L^1_v} \leq \frac{|\mathrm{I}| + \delta_n |\mathrm{II}|}{\left(1 + \delta_n \|f\|_{L^1_v}\right)\left(1 + \delta_n \|g\|_{L^1_v}\right)}$$

$$\leq \frac{3C_*(n,d)\|f - g\|_{L^1_v} \left[\delta_n \left(\|f\|_{L^1_v} + \|g\|_{L^1_v}\right) + \delta_n^2 \|f\|_{L^1_v} \|g\|_{L^1_v}\right]}{\delta_n \left[1 + \delta_n \left(\|f\|_{L^1_v} + \|g\|_{L^1_v}\right) + \delta_n^2 \|f\|_{L^1_v} \|g\|_{L^1_v}\right]} \leq C_2(n,d) \|f - g\|_{L^1_v}, \quad C_2(n,d) = 3C_*(n,d)/\delta_n$$

Finally, defining $C(n,d) = C_1(n,d) + C_2(n,d)$, we get the desired result from the triangle inequality. □

B.3 Dunford-Pettis and De-la-Valéé-Poussin's Lemmas

The key ingredients in the theorem of Diperna and Lions for the existence of renormalized solutions are the averaging lemmas (which we treat in detail in the next subsection) and the lemmas of Dunford-Pettis and De-la-Valéé-Poussin, which allows us to extract weakly converging subsequences from a bounded sequence in L^1, something which is not possible in general. Although they are not new results, we use arguments based on them so frequently in the main proof that they deserve a demonstration of their own. The arguments used here come mainly from [15],[5] and [11].

Theorem B.3.1. *Let $\mathcal{F} \subseteq L^1(\Omega, d\mu)$be a uniformly bounded family. Then the following are equivalent*

- $$\lim_{R\to\infty} \sup_{f\in\mathcal{F}} \int_{f\geq R} f\, d\mu = 0$$

- *$\forall \epsilon > 0$ exists $\delta > 0$ such that $\mu(A) < \delta$ implies*

$$\sup_{f \in \mathcal{F}} \int_A f \, d\mu < \epsilon$$

- *(De-la-Valée-Poussin) there is a continuous, nonnegative, nondecreasing convex function $G : [0, \infty) \longrightarrow [0, \infty)$ such that:*

$$\lim_{t \to \infty} \frac{G(t)}{t} = \infty \qquad \sup_{f \in \mathcal{F}} \int G(|f|(y)) \, d\mu(y) < \infty$$

Proof. We will prove that $(iii) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iii)$. For the first step, define

$$M = \sup_{f \in \mathcal{F}} \int G(|f|) \, d\mu(y)$$

Then, since $\lim_{t \to \infty} \frac{G(t)}{t} = \infty$, let t_0 be such that $\frac{G(t)}{t} > \frac{2M}{\epsilon} \Leftrightarrow t < \frac{\epsilon}{2M} G(t)$ if $t > t_0$. Let A be a set with $\mathfrak{L}(A) < \delta$. Then

$$\int_A |f(y)| \, dy = \int_{A \cap \{|f| \le t_0\}} |f(y)| \, dy + \int_{A \cap \{|f| > t_0\}} |f(y)| \, dy \le t_0 \int_A dy + \int_{A \cap \{|f| > t_0\}} |f(y)| \, dy$$

For the second integral, since $|f| > t_0$, we have that

$$|f(y)| < \frac{\epsilon}{2M} G(|f|)(y)$$

Therefore

$$t_0 \int_A dy + \int_{A \cap \{|f| > t_0\}} |f(y)| \, dy < \delta t_0 + \frac{\epsilon}{2M} \int G(|f|(y)) \, dy = \delta t_0 + + \frac{\epsilon}{2}$$

If we choose $\delta = \frac{\epsilon}{2 t_0}$, then the conclusion follows.

To prove that $(ii) \Rightarrow (i)$, we use Chebyshev's inequality, and say that

$$\mathfrak{L}(\{|f| > C\}) \le \frac{\|f\|_{L^1}}{C}$$

Then, for every $\epsilon > 0$ there is $C > 0$ such that $\mathfrak{L}(\{|f| > C\}) < \delta \ \ \forall f \in \mathcal{F}$, and therefore

$$\sup_{f \in \mathcal{F}} \int_{\{|f| > C\}} |f(y)| \, dy < \epsilon.$$

The last step ($(i) \Rightarrow (iii)$) is the longest since our proof is contructive: we exhibit a function G satisfying all properties in the theorem, following the proof given by [5]. By our hypothesis, there is a sequence of real numbers C_n such that $C_n \to \infty$ and

$$\sup_{f \in \mathcal{F}} \int_{\{|f| > C_n\}} |f(y)| \, dy < 2^{-n}$$

Define $E_f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $E_f(y) = \mathfrak{L}(\{|f(x)| > y\})$. $E_f(y)$ is trivially decreasing, and is also integrable since

$$\int f(y) \, dy = \int_0^\infty E_f(y) \, dy.$$

Then

$$\int_{\{|f| > C_n\}} |f(y)| \, dy = \sum_{i = C_n}^{\infty} \int_{\{i < |f| \le i+1\}} |f(y)| \, dy > \sum_{i = C_n}^{\infty} i \int_{\{i < |f| \le i+1\}} dy$$

$$= \sum_{i=C_n}^{\infty} i(E_f(i) - E_f(i+1)) = \sum_{i=C_n}^{\infty} iE_f(i) - \sum_{j=C_n+1}^{\infty} (j-1)E_f(j) = \sum_{i=C_n}^{\infty} E_f(i)$$

$$\therefore \sum_{i=C_n}^{\infty} E_f(i) < \int_{\{|f|>C_n\}} |f(y)|\, dy \Rightarrow \sum_{n=1}^{\infty}\sum_{i=C_n}^{\infty} E_f(i) \le \sum_{n=1}^{\infty} \int_{\{|f|>C_n\}} |f(y)|\, dy < \sum_{n=1}^{\infty} 2^{-n} = 1$$

Notice that if we switch the order of summation,

$$\sum_{n=1}^{\infty}\sum_{i=C_n}^{\infty} E_f(i) = \sum_{i=C_1}^{\infty} \sum_{n=1}^{\max\{k:C_k<i\}} E_f(i) = \sum_{i=C_1}^{\infty} E_f(i)\,(\max\{k : C_k \le i\})$$

Then, Define the following functions

$$\theta_i = \begin{cases} 0, & \text{for } i < C_1 \\ \max\{k : C_k < i\}, & \text{for } i \ge C_1 \end{cases}$$

$$g, G : \mathbb{R}^+ \to \mathbb{R}^+,\ g(t) = \sum_{i=1}^{\infty} \theta_i \chi_{[i,i+1)}(t),\ \ G(t) = \int_0^t g(s)\, ds$$

Since $g \ge 0$ and G is defined by taking an integral of g, we have that G is non-negative, increasing and continuous (in fact, it is piecewise C^∞, and can be made smooth in the whole domain by applying a mollifier). To prove that it is convex, notice that since g is also non-decreasing, we have for any $t_1 < t_2 < t_3 \in \mathbb{R}^+$

$$\frac{\int_{t_1}^{t_2} g(t)\, dt}{t_2 - t_1} \le \frac{\int_{t_2}^{t_3} g(t)\, dt}{t_3 - t_2} \therefore \frac{G(t_2) - G(t_1)}{t_2 - t_1} \le \frac{G(t_3) - G(t_2)}{t_3 - t_2}$$

which implies convexity. Finally, we have that

$$\int G(|f(y)|)\, dy = \sum_{i=0}^{\infty} \int_{\{i<|f|\le i+1\}} G(|f(y)|)\, dy \le \sum_{i=0}^{\infty} G(i+1) \int_{\{i<|f|\le i+1\}} dy$$

$$= \sum_{i=0}^{\infty} G(i+1)(E_f(i) - E_f(i+1)) = \sum_{i=0}^{\infty} G(i+1)E_f(i) - \sum_{j=1}^{\infty} G(j)E_f(j) = \sum_{i=0}^{\infty} E_f(i)(G(i+1) - G(i))$$

From the definition of G, we see that $G(i+1) = \sum_{j=0}^{i} \theta_i$, so that

$$\int G(|f(y)|)\, dy = \sum_{i=0}^{\infty} E_f(i)\theta_i = \sum_{n=1}^{\infty}\sum_{i=C_n}^{\infty} E_f(i) < \sum_{n=1}^{\infty} 2^{-n} = 1$$

The final claim is that $\lim_{t\to\infty} \frac{G(t)}{t} = \infty$. To prove this, we construct an auxiliary function h by the following rule: θ_i defines a function $\theta : \mathbb{N} \to \mathbb{N}$. From the well-ordering property of $\mathbb{N}$, define the increasing sequence: $k_1 = \inf(\theta(\mathbb{N}))$, $k_{n+1} = \inf(\theta(\mathbb{N}) - \{k_1, k_2, ..., k_n\})$. Let x_n be a sequence defined by $x_n = \sup(\theta^{-1}(k_n))$. θ defines a step function, so that x_n defines the endpoint of each step. From this notation, one can reexpress the function g as

$$g(t) = \sum_{n=1}^{\infty} k_{n+1} \chi_{[x_n, x_{n+1})}(t)$$

Define the auxiliary functions h, H as

$$h, H : \mathbb{R}^+ \to \mathbb{R}^+, h(t) = \sum_{n=1}^{\infty} \left(\frac{k_{n+1} - k_n}{x_{n+1} - x_n}(t - x_n) + k_n \right) \chi_{[x_n, x_{n+1})}(t),\ \ H(t) = \int_0^t h(s)\, ds$$

In the definition of h, inside the summation sign we have a linear interpolation between k_n and k_{n+1}. Therefore, by the monotonicity of the sequence k_n, we have that $g(t) \geq h(t) \therefore G(t) \geq H(t) \quad \forall t$. The advantage of using the auxiliary function h, is that it is continuous, so that $H(t)$ is everywhere differentiable (Actually, we could take G to be defined as H was, instead of how we treated in this demonstration. G would satisfy all requirements in the theorem and would be everywhere differentiable). Then, we can use l'Hôpital's rule, and say that

$$\lim_{t\to\infty} \frac{G(t)}{t} \geq \lim_{t\to\infty} \frac{H(t)}{t} = \lim_{t\to\infty} h(t) = \infty$$

Since $dH(t)/dt = h(t)$.

□

Therefore, all definitions that we used for uniform integrability are indeed equivalent. All that remains is to show how uniform integrability and weak L^1 compactness are related. We will prove one 'direction' of the following theorem

Lemma B.3.1 (Dunford-Pettis). *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. The bounded subset $\mathcal{F} \subseteq L^1(\Omega)$ is weakly compact if and only if $\mathcal{F}$ is uniformly integrable and tight, the last condition meaning that, for any sequence $R_n \to \infty$,*

$$\sup_{f\in\mathcal{F}} \int_{\{\|x\|>R_n\}\cap\Omega} |f| dx \to 0$$

Proof. we see that if Ω is bounded, the tightness condition is trivially satisfied as the domain $\{\|x\| > R_n\} \cap \Omega$ becomes empty for some n. We will concern ourselves with the unbounded case, as the bounded case is a corollary. Also, we'll take $\Omega = \mathbb{R}^N$ for convenience; if not, then we can extend the domains of functions in $\mathcal{F}$ to $\mathbb{R}^N$, it suffices to define $\tilde{f}(x) = f(x), \quad \text{if } x \in \Omega, 0 \text{ if } x \in \Omega^c$.

Construct a sequence $f_n \in \mathcal{F}$.If $f_n \in L^\infty((0,T); L^1(\mathbb{R}^N))$, then the sequence μ_n of measures induced by f_n, converges weakly to some μ upon passing to a subsequence. f_n uniformly integrable means that $\forall\epsilon > 0$ exists $\delta > 0$ such that $\mu(A) < \delta$ implies

$$\sup_{n>0} \int_A |f_n|\, dx = \sup_{n>0} \mu_n(A) < \epsilon$$

Which means no concentration of the measures μ_n occurs. Therefore, the limit measure μ is absolutely continuous, meaning there is a density function $f \in L^1(\mathbb{R}^N)$ associated with it. What we have to show is that $f_n \rightharpoonup f$ upon passing to a subsequence. We take the following demonstration from [15].

Let $h \in L^\infty(\mathbb{R}^N)$. h can be approximated by continuous and uniformly bounded functions h_j. From the tightness condition, choose $R > 0$ such that

$$\sup_{n>0} \int_{\{\|x\|>R\}} |f_n| dx < \epsilon'$$

Then,

$$\int_{\mathbb{R}^N} (f_n - f)h\, dx = \int_{\{\|x\|>R\}} (f_n - f)h\, dx + \int_{\{\|x\|\leq R\}} (f_n - f)h\, dx$$

Since the set $B_R = \{\|x\| \leq R\}$ is clearly bounded, by Egorov's theorem, one can find a set E of arbitrarily small measure, such that $h_j \to h$ uniformly on $B_R - E$. So we write

$$\int_{\{\|x\|\leq R\}} (f_n - f)h\, dx = \int_{B_R - E} (f_n - f)h_j\, dx + \int_{B_R-E} (f_n - f)(h - h_j)\, dx + \int_E (f_n - f)h\, dx$$

We get then

$$\left|\int_{\mathbb{R}^N} (f_n - f)h\, dx\right| \leq \int_{\{\|x\|>R\}} |f_n - f||h|\, dx + \int_E |f_n - f||h|\, dx$$

$$+\left|\int_{B_R-E}(f_nh_j - fh_j)\,dx\right| + \int_{B_R-E}|f_n - f||h-h_j|\,dx$$
$$\leq \|h\|_{L^\infty}\left(\int_{\{\|x\|>R\}}|f_n|\,dx + \mu_n(E) + \mu(\{\|x\|>R\}) + \mu(E)\right)$$
$$+\left|\int_{B_R-E}(f_nh_j - fh_j)\,dx\right| + \int_{B_R-E}|f_n - f||h-h_j|\,dx$$

Since $|f| \in L^1(\mathbb{R}^N)$, by definition, $\mu(\{\|x\| > R\}) = \int_{\{\|x\|>R\}}|f|\,dx < \epsilon'$ for sufficiently large $R > 0$. Similarly, since μ is absolutely continuous and $\mathfrak{L}(E) < \delta$, then we can choose δ such that $\mu(E) < \epsilon'$. Finally, from uniform integrability, we can choose $\delta, R > 0$ such that the first 2 terms inside brackets are smaller than ϵ' uniformly in n. For the fourth term, take $j > j_0$ so that $|h - h_j| < \epsilon'$ by uniform convergence, and define $R = sup_{n\in\mathbb{N}}\int_{B_R-E}|f_n| + |f|\,dx$. with all this in place, the expression above becomes

$$< 4\epsilon'|h\|_{L^\infty} + R\epsilon' + \left|\int_{B_R-E}(f_nh_j - fh_j)\,dx\right|$$
$$= 4\epsilon'|h\|_{L^\infty} + R\epsilon' + \left|\int_{B_R-E}h_j\,d\mu_n(x) - \int_{B_R-E}h_j\,d\mu(x)\right|$$

in which we used the definition of the measures μ_n, μ. Then, by weak convergence of measures, there is a subsequence μ_{n_k} such that, for large k, the term in absolute value is small for any j. Let $C = \max\{4|h\|_{L^\infty}, R\}$. We can bound the above expression as

$$4\epsilon'|h\|_{L^\infty} + R\epsilon' + \left|\int_{B_R-E}h_j\,d\mu_n(x) - \int_{B_R-E}h_j\,d\mu(x)\right| < (C+1)\epsilon'$$

Choosing $\epsilon' = \epsilon/(C+1)$ gives us that, for every $\epsilon > 0$ there is a subsequence f_{n_k} such that, if $k > k_0$, then

$$\left|\int_{\mathbb{R}^N}(f_{n_k} - f)h\,dx\right| < \epsilon \quad \forall h \in L^\infty(\mathbb{R}^N)$$

And our proof is complete. The converse assertion (that weak compactness in L^1 implies tightness and uniform integrability) is not as important for us as the direction we just proved, so we will end our discussion at this point. For a complete account of the demonstration of the converse, see [3]. □

Informally, the 'punchline' is that 2 phenomena can make a uniformly bounded family $f_n \subseteq L^1(\mathbb{R}^N)$ not have a weakly converging subsequence:

- the functions can concentrate around a value, and in the limit become a combination of Dirac masses (which is not forbidden by weak convergence of measures)
- the sequence can have the behavior of a travelling wave, which has always the same integral and yet doesn't converge to any specific function.

Using the first definition of Uniform integrability given in this Appendix, we see that if a sequence has this property it avoids concentration around any particular values (first item); and tightness ensures the decay at infinity of this sequence is uniform, preventing the travelling wave behavior (second item).

B.4 Averaging lemmas

The averaging lemmas are necessary in order to guarantee that $Q_n(f_n, f_n)$ converges when we pass to the limit $n \to \infty$. Since this is one of the most technical parts of the demonstration, and since these lemmas reappear constantly in the theory of the Boltzmann equation and its hydrodynamic limits, we provide a proof for them below. Some arguments used in this section are taken from [22] and [32].

The simplest of the average lemmas is the following

Lemma B.4.1. *Let $f(x,v) : \mathbb{R}^N \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be a solution in the sense of distributions to the following equation*

$$\alpha(v) \cdot \nabla_x f = g$$

and let μ be a measure in $\mathbb{R}^d$ which satisfies, for a given $s > 0$ and for all $\epsilon > 0$ the following restriction

$$\mu(\{v \in \mathbb{R}^d \text{such that} |\zeta \cdot \alpha(v)| < \epsilon\}) < M\epsilon^s \tag{B.4.1}$$

where ζ is a unit vector in $\mathbb{R}^d$. Define

$$m = \int_{\mathbb{R}^d} f \, d\mu(v)$$

If $f, g \in L^2(\mathbb{R}^N \times B_R)$ for every $R > 0$, $B_R \subseteq \mathbb{R}^d$, then $m \in H^s(\mathbb{R}^N)$ with $s \in (0,1)$.

We follow here the demonstration in [22]

Proof. Recall that

$$\|f\|^2_{\dot{H}^s} = \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{f}|^2 \, d\xi$$

The trick of this demonstration, started by BOUCHUT, is the following: add zf to both sides of the equation, $z \in \mathbb{R}$:

$$zf + \alpha(v) \cdot \nabla_x f = g + zf$$

Now taking the fourier transform with respect to the x variable on both sides, we get that

$$\hat{f}(x,\xi)(z + 2i\pi\xi \cdot \alpha(v)) = z\hat{f} + \hat{g}$$

$$\therefore \hat{f} = \frac{z}{z + 2i\pi\xi \cdot \alpha(v)} \hat{f} + \frac{1}{z + 2i\pi\xi \cdot \alpha(v)} \hat{g}$$

Then, we have that

$$|\hat{m}| = \left| \int_{\mathbb{R}^d} \hat{f} \, d\mu(v) \right| \leq \int_{\mathbb{R}^d} |\hat{f}| \, d\mu(v)$$

$$\leq \int_{\mathbb{R}^d} \left| \frac{z}{z + 2i\pi\xi \cdot \alpha(v)} \right| \left| \hat{f} \right| d\mu(v) + \int_{\mathbb{R}^d} \left| \frac{1}{z + 2i\pi\xi \cdot \alpha(v)} \right| \left| \hat{g} \right| d\mu(v)$$

By the Cauchy-Schwartz inequality, this gives us

$$|\hat{m}| \leq [|z| \|\hat{f}\|_{L^2(d\mu)} + \|\hat{g}\|_{L^2(d\mu)}] \left(\int_{\mathbb{R}^d} \frac{1}{|z + 2i\pi\xi \cdot \alpha(v)|^2} \, d\mu(v) \right)^{1/2}$$

where we see that $|z + 2i\pi\xi \cdot \alpha(v)|^2 = z^2 + 4\pi^2 |\xi \cdot \alpha(v)|^2$. define $\phi(y) = \frac{1}{z^2 + 4\pi^2 y^2}$. Then, the integral above can be written as

$$\int_{\mathbb{R}^d} \frac{1}{|z + 2i\pi\xi \cdot \alpha(v)|^2} \, d\mu(v) = \int_{\mathbb{R}^d} \phi(|\xi \cdot \alpha(v)|) \, d\mu(v)$$

Define $g(v) = |\xi \cdot \alpha(v)|$. We use then the following standard trick

$$\int_{\mathbb{R}^d} \phi(g(v)) \, d\mu(v) = -\int_{\mathbb{R}^d} \int_{g(v)}^{\infty} \phi'(y) \, dy d\mu(v) = -\int_{\mathbb{R}^d} \int_0^{\infty} \phi'(y) \chi_{\{g(v) < y\}} \, dy d\mu(v)$$

since $g(v) > 0$. swapping the order of the integrals by Fubini's theorem, we get that

$$= -\int_0^{\infty} \phi'(y) \int_{\mathbb{R}^d} \chi_{\{g(v) < y\}} \, d\mu(v) dy = -\int_0^{\infty} \phi'(y) \mu(\{|\xi \cdot \alpha(v)| < y\}) \, dy$$

$\mu(\{|\xi \cdot \alpha(v)| < y\}) = \mu(\{|\frac{\xi}{|\xi|} \cdot \alpha(v)| < \frac{y}{\xi}\})$, so using the property B.4.1 we stated at the beginning for the measure μ, we have that $\mu(\{|\frac{\xi}{|\xi|} \cdot \alpha(v)| < \frac{y}{|\xi|}\}) < M\frac{y^s}{|\xi|^s}$. Trivially $\phi'(y) = \frac{-8\pi^2 y}{(z^2+4\pi^2y^2)^2}$. Inserting this onto the expression yields

$$\left(\int_{\mathbb{R}^d} \frac{1}{|z + 2i\pi\xi \cdot \alpha(v)|^2}\, d\mu(v)\right)^{1/2} \leq \frac{2\pi\sqrt{M}}{|\xi|^{s/2}} \left(\int_0^\infty \frac{2y^{1+s}}{(z^2+4\pi^2y^2)^2}\, dy\right)^{1/2}$$

The integral on the right hand side times $\sqrt{A}$ will converge if $1 + s - 4 < -1 \Leftrightarrow s < 2$, to a constant we'll define as $C_{z,s}$. Define for convenience $s = 2k \Rightarrow k \in (0,1)$. Inserting back into our expression

$$|\xi|^k |\hat{m}| \leq 2\pi C_{z,2k}[|z| \|\hat{f}\|_{L^2(d\mu)} + \|\hat{g}\|_{L^2(d\mu)}].$$

Squaring both sides and setting z=1, we get that

$$|\xi|^{2k}|\hat{m}|^2 \leq 8\pi C_{1,2k}\left[\frac{1}{2}\|\hat{f}\|_{L^2(d\mu)} + \frac{1}{2}\|\hat{g}\|_{L^2(d\mu)}\right]^2 \leq 4\pi C_{1,2k}\left(\|\hat{f}\|^2_{L^2(d\mu)} + \|\hat{g}\|^2_{L^2(d\mu)}\right)$$

By Jensen's inequality. Finally, integrating on both sides with respect to the ξ variable, and using Plancherel's formula on the right hand side, we get

$$\|m\|^2_{\dot{H}^k} \leq 4\pi C_{1,2k}\left(\|f\|^2_{L^2_{x,d\mu(v)}} + \|g\|^2_{L^2_{x,d\mu(v)}}\right)$$

and the conclusion follows.

□

Corollary B.4.1. *In the above theorem, let μ be given by*

$$\mu(A) = \int_A \psi\, dv$$

where $\psi \in L^\infty_v(\mathbb{R}^d)$ is a compactly supported function. then define m as above, with $f, g \in L^2(\mathbb{R}\times\mathbb{R}^d\times B_R)$ for every $R, T > 0$, and f a distributional solution to the following equation

$$\partial_t f + v \cdot \nabla_x f = g \tag{B.4.2}$$

Then, $m \in L^2_{v,t}(\mathbb{R}\times\mathbb{R}^d; H^{1/2}_x(\mathbb{R}^d))$

Proof. Notive that we can always rewrite equation B.4.2, by defining new variables as

$$\alpha(v) \cdot \nabla_{x'} f(x', v) = g(x', v)$$

Where $x' = (t, x), \alpha(v) = (1, v) \in \mathbb{R}^{d+1}$. Then, we can follow exactly the same procedure as above until the point we have

$$|\hat{m}|(\xi') \leq [|z| \|\hat{f}\|_{L^2(d\mu)} + \|\hat{g}\|_{L^2(d\mu)}]\left(\int_{\mathbb{R}^d} \frac{\psi^2}{|z + 2i\pi\xi' \cdot \alpha(v)|^2}\, dv\right)^{1/2}$$

Where our transformed variable is given by $\xi' = (\tau, \xi)$.Then we have to show that the measure $\nu(A) = \int_A \psi^2\, dv$ satisfies the restriction B.4.1. For this we have to bound the following integral

$$\nu(\{|\zeta \cdot \alpha(v)| < \epsilon\}) = \nu(\{|\xi' \cdot \alpha(v)| < |\xi'|\epsilon\})$$
$$= \nu(\{|\tau + \xi \cdot v| < |\xi'|\epsilon\}) = \int_{|\tau+\xi\cdot v|<|\xi'|\epsilon} \psi^2\, dv$$

The domain of this integral is a strip contained between 2 hyperplanes, with normal vector ξ and a spacing between them proportional to ϵ. Since ψ is compactly supported we know this integral will converge, and we wish to show that its result is proportional to ϵ. There are many ways to proceed at this point, our choice following [22], [32] is to use a change of variables that realigns our axes, such that one of them coincides with the direction of the normal vector to the hyperplanes. Starting with the vector $\xi/|\xi|$, construct an orthornormal basis set in $\mathbb{R}^d$, and decompose the vector v as

$$v = \left(y - \frac{\tau}{|\xi|}\right)\frac{\xi}{|\xi|} + v_\perp$$

Where $v_\perp$ denotes the components of v in the orthogonal complement of $\frac{\xi}{|\xi|}$. Since this basis set is orthonormal, the change of variables has unit jacobian. with this change of variables

$$v = \left(y - \frac{\tau}{|\xi|}\right)\frac{\xi}{|\xi|} + v_\perp \Leftrightarrow v \cdot \frac{\xi}{|\xi|} = \left(y - \frac{\tau}{|\xi|}\right) \Leftrightarrow y|\xi| = \tau + \xi \cdot v$$

So that our integral can be rewritten as

$$\int_{|\tau+\xi\cdot v|<|\xi'|\epsilon} \psi^2\, dv = \iint_{|y|<\frac{|\xi'|}{|\xi|}\epsilon} \psi^2\, dy dv_\perp \leq \|\psi^2\|_{L^\infty(\mathbb{R}^d)} \iint_{|y|<\frac{|\xi'|}{|\xi|}\epsilon} \chi_{\text{supp}(\psi)}\, dy dv_\perp$$

(abusing notation, the integral with respect to $dv_\perp$ corresponds to an integral over the coordinates associated to the orthogonal complement of $\xi/|\xi|$ in the new basis set). Let $L > 0$ be such that $\text{supp}(\psi) \subseteq B_L \subseteq \mathbb{R}^d$. Notice that by definition

$$B_L = \{v \in \mathbb{R}^d \text{ such that } y^2 + v_\perp^2 \leq L^2\} \subseteq \{v \in \mathbb{R}^d \text{ such that } y^2 \leq L^2, v_\perp^2 \leq L^2\} = A_L$$

The set A_L is a cylinder in $\mathbb{R}^d$, therefore assuming that $\frac{|\xi'|}{|\xi|}\epsilon < L$

$$\iint_{|y|<\frac{|\xi'|}{|\xi|}\epsilon} \chi_{\text{supp}(\psi)}\, dy dv_\perp \leq \iint_{|y|<\frac{|\xi'|}{|\xi|}\epsilon} \chi_{A_L}\, dy dv_\perp = A(d)\frac{|\xi'|}{|\xi|}\epsilon L^{d-1}$$

Where $A(d) = 2\omega(d-1)$, and $\omega(d)$ is the volume of the sphere in dimension d. Therefore, we see that this measure satisfies our constraint with exponent $s = 1$. We get then

$$\left(\int_{\mathbb{R}^d} \frac{1}{|z + 2i\pi\xi' \cdot \alpha(v)|^2}\, d\mu(v)\right)^{1/2} \leq \frac{2\pi}{|\xi'|^{1/2}}\sqrt{A(d)\frac{|\xi'|}{|\xi|}L^{d-1}}\left(\int_0^\infty \frac{2y^2}{(z^2+4\pi^2y^2)^2}\, dy\right)^{1/2}$$

$$= \frac{2}{|\xi|^{1/2}}\sqrt{\omega(d-1)L^{d-1}}\left(\int_0^\infty \frac{4\pi y^2}{(z^2+4\pi^2y^2)^2}\, dy\right)^{1/2}$$

Then the integral on the right hand side can be computed explicitly using the change of variables $y = \frac{z}{2\pi}\tan(\theta)$. After some calculations, we find that

$$= \frac{2}{|\xi|^{1/2}}\sqrt{\omega(d-1)L^{d-1}}\left(\frac{1}{z}\arctan\left(\frac{2\pi y}{z}\right) - \frac{4\pi^2 y}{z^2+4\pi y^2}\Big|_0^\infty\right)^{1/2} = \frac{C(d,L)}{2\sqrt{z|\xi|}}$$

Inserting back into our equation, considering $z > 0$ yields

$$|\hat{m}|(\xi') \leq \frac{C(d,L)}{2\sqrt{z|\xi|}}[|z|\|\hat{f}\|_{L_v^2} + \|\hat{g}\|_{L_v^2}]$$

$$|\xi|^{\frac{1}{2}}|\hat{m}|(\tau,\xi) \le C(d,L)\left[\frac{\sqrt{z}}{2}\|\hat{f}\|_{L^2_v} + \frac{1}{2\sqrt{z}}\|\hat{g}\|_{L^2_v}\right]$$

Again setting $z = 1$, squaring both sides and using Jensen's inequality, we conclude

$$|\xi\||\hat{m}|^2(\tau,\xi) \le \frac{C(d,L)}{2}\left[\|\hat{f}\|^2_{L^2_v} + \|\hat{g}\|^2_{L^2_v}\right]$$

$$\therefore \|m\|^2_{L^2_{v,t}(\mathbb{R}\times\mathbb{R}^d;\dot{H}^{1/2}_x(\mathbb{R}^d))} \le \frac{C(d,L)}{2}\left[\|f\|^2_{L^2_{t,x,v}} + \|g\|^2_{L^2_{t,x,v}}\right]$$

□

Corollary B.4.2. *Let f_n, g_n be uniformly bounded sequences in $L^2(\mathbb{R}\times\mathbb{R}^d\times B_R)$ for every $R, T > 0$, and f_n distributional solutions to the following problem*

$$\partial_t f_n + v\cdot\nabla_x f_n = g_n \tag{B.4.3}$$

And define the average m_n as before

$$m_n = \int_{\mathbb{R}^d} f_n\psi\, dv$$

where $\psi \in L^\infty_v(\mathbb{R}^d)$ is a compactly supported function. Then, there is a subsequence f_{n_k} such that m_{n_k} converges strongly in $L^2(\mathbb{R}\times\mathbb{R}^d)$

Proof. If f_n is uniformly bounded in L^2, then we can always extract a convergent subsequence, and m_n is uniformly bounded in $L^2_{v,t}(\mathbb{R}\times\mathbb{R}^d; H^{1/2}_x(\mathbb{R}^d))$. Then, by Rellich-kondrachov, we know that $H^{\frac{1}{2}} \subseteq H^0 = L^2$, so that m_{n_k} is convergent in $L^2(\mathbb{R}\times\mathbb{R}^d)$. □

From this fact, we deduce the truly important case to us, the case when f_n is weakly convergent in L^1.

Theorem B.4.1. *Let $(f_n) \subseteq L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ be a weakly compact subset.*
Assume that $\mathfrak{D}f_n$ is also weakly compact in $L^1_{loc}((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$. Let (ψ_n) be a uniformly bounded sequence in $L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ converging almost everywhere to ψ. Then the sequence

$$(\Psi_n(t,x)) = \int_{\mathbb{R}^d} f_n(t,x,v)\psi_n(t,x,v)\, dv$$

forms a strongly compact subset of $L^1((0,T)\times\mathbb{R}^d)$

Corollary B.4.3. *If in addition $f_n \rightharpoonup f \in L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$, then*

$$\Psi_n(t,x) = \int_{\mathbb{R}^d} f_n\psi_n\, dv \to \int_{\mathbb{R}^d} f\psi\, dv = \Psi(t,x)$$

strongly in $L^1((0,T)\times\mathbb{R}^d)$

Before starting the proof, it will be important to have a closed expression for the resolvent of the operator $\alpha(v)\cdot\nabla$, $R_z = (zI + \alpha(v)\cdot\nabla)^{-1}$. One can obtain it by adding zf to both sides of the equation

$$zf(x-\alpha(v)t,v) + \alpha(v)\cdot\nabla f(x-\alpha(v)t,v) = g(x-\alpha(v)t,v) + zf(x-\alpha(v)t,v) \quad (e^{-zt})$$

$$zf(x-\alpha(v)t,v)e^{-zt} - e^{-zt}\frac{d}{dt}f(x-\alpha(v)t,v) = -f(x-\alpha(v)t,v)\frac{d}{dt}e^{-zt} - e^{-zt}\frac{d}{dt}f(x-\alpha(v)t,v)$$

$$= e^{-zt}[g(x-\alpha(v)t,v) + zf(x-\alpha(v)t,v)]$$

$$-\frac{d}{dt}[e^{-zt}f(x-\alpha(v)t,v)] = e^{-zt}[g(x-\alpha(v)t,v) + zf(x-\alpha(v)t,v)]$$

Or equivalently

$$f(x,v) = \int_0^\infty e^{-zt} g(x - \alpha(v)t, v)\, dt + z \int_0^\infty e^{-zt} f(x - \alpha(v)t, v)\, dt = R_z(g + zf)$$

Therefore

$$R_z(g) = \int_0^\infty e^{-zt} g(x - \alpha(v)t, v)\, dt, \quad z \in \mathbb{C}$$

This operator has the following important property:

$$\|R_z(g)\|_{L^p_{x,v}} \leq \int_{\mathbb{R}^d} \int_0^\infty e^{-zt} \|g\|_{L^p_{x,v}}\, dt = \frac{1}{z} \|g\|_{L^p_{x,v}}$$

Now we can prove lemma 2.1.2.

Proof. Starting from equation B.4.2, doing the same trick as before, we know that f_n must satisfy

$$f_n = R_z(g_n + z f_n)$$

Define $G_{n,z} = g_n + z f_n$, and for some choice of $M > 0$, split this term as

$$G_{n,z} = G_{n,z}\chi_{|G_{n,z}| \leq M} + G_{n,z}\chi_{|G_{n,z}| > M}$$

Since R_z is a linear operator, this allows us to write

$$f_n = R_z[G_{n,z}\chi_{|G_{n,z}| \leq M}] + R_z[G_{n,z}\chi_{|G_{n,z}| > M}] = f_n^- + f_n^+$$

where $f_n^+ = R_z[G_{n,z}\chi_{|G_{n,z}| > M}]$, $f_n^- = R_z[G_{n,z}\chi_{|G_{n,z}| \leq M}]$. Crucial to this proof is the following lemma:

Lemma B.4.2. *Let $H \subseteq V$, V Banach space. H is relatively compact if for every $\epsilon > 0$ $\exists \mathcal{K}_\epsilon$ compact set such that $H \subseteq \mathcal{K}_\epsilon + B_\epsilon(0)$*

Then, we can perform the following splitting

$$\int_{\mathbb{R}^d} f_n \psi_n\, dv = \int_{|x'|+|v| \leq R} f_n^- \psi_n\, dv + \int_{|x'|+|v| > R} f_n^- \psi_n\, dv + \int_{\mathbb{R}^d} f_n^+ \psi_n\, dv$$

Since the first integral is performed over a compact set, we can use Egorov's theorem, and say that there is a set E such that $\mathfrak{L}(E) < \delta$ and $\psi_n \to \psi$ uniformly in E^c, so we can further split the first term as

$$\int_{|x'|+|v| \leq R} f_n^- \psi_n\, dv = \int_{\{|x'|+|v| \leq R\} \cap E} f_n^- \psi_n\, dv + \int_{\{|x'|+|v| \leq R\} \cap E^c} f_n^- (\psi_n - \psi)\, dv$$
$$+ \int_{\{|x'|+|v| \leq R\} \cap E^c} f_n^- \psi\, dv$$

Then, let $n > n_0$ such that $|\psi_n - \psi| < \epsilon/4A$, the constant A to be defined later. We can safely say that

$$\int_{|x'|+|v| \leq R} f_n^- \psi_n\, dv < \sup_n \|\psi_n\| \int_{\{|x'|+|v| \leq R\} \cap E} f_n^-\, dv + \frac{\epsilon}{4A} \int_{\mathbb{R}^d} f_n^-\, dv + \int_{E^c} f_n^- \psi \chi_{\{|x'|+|v| \leq R\}}\, dv$$

Then, we see that

$$\left\| \int_{\mathbb{R}^d} f_n \psi_n\, dv - \int_{E^c} f_n^- \psi \chi_{\{|x'|+|v| \leq R\}}\, dv \right\|_{L^1_{x',v}} < \sup_n \|\psi_n\| \int \int_{\{|x'|+|v| \leq R\} \cap E} f_n^-\, dv dx'$$

$$+\frac{\epsilon}{4A}\int_{(0,T)\times\mathbb{R}^d}\int_{\mathbb{R}^d} f_n^- \,dvdx' + \int\int_{|x'|+|v|>R} f_n^- \,dvdx' + \int_{(0,T)\times\mathbb{R}^{2d}} f_n^+ \,dvdx'$$

Since the f_n are uniformly integrable, we can choose $\delta > 0$ such that the first integral is less than $\epsilon/(4\sup_n \|\psi_n\|_{L^\infty})$. Since the f_n are uniformly bounded in L^1, choose A to be that bound. We get that

$$\left\|\int_{\mathbb{R}^d} f_n\psi_n \,dv - \int_{E^c} f_n^- \psi\chi_{\{|x'|+|v|\le R\}}\,dv\right\|_{L^1_{x',v}} < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \int\int_{|x'|+|v|>R} f_n^- \,dvdx' + \int_{(0,T)\times\mathbb{R}^{2d}} f_n^+ \,dvdx'$$

Recalling that $f_n^- \le f_n$ and our property for the Resolvent operator, the right hand side then satisfies

$$\le \frac{\epsilon}{2} + \int\int_{|x'|+|v|>R} f_n \,dvdx' + \frac{1}{z}\left[\int_{(0,T)\times\mathbb{R}^{2d}} G_{n,z}\chi_{|G_{n,z}|>M}\,dvdx'\right]$$

Finally, by uniform integrability, we can choose $R, M > 0$ such that the last two terms are each bounded by $\epsilon/4$. placing all of this together yields

$$\left\|\int_{\mathbb{R}^d} f_n\psi_n \,dv - \int_{E^c} f_n^- \psi\chi_{\{|x'|+|v|\le R\}}\,dv\right\|_{L^1_{x',v}} < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon$$

Therefore, if we can say that $\int_{E^c} f_n^- \psi\chi_{\{|x'|+|v|\le R\}}\,dv$ forms a compact set in L^1, we are done by the above lemma. By the property of the resolvent operator,

$$\|f_n^-\|_{L^p_{x',v}} \le \frac{1}{z}\|G_{n,z}\chi_{|G_{n,z}|\le M}\|_{L^p_{x',v}} < \infty \qquad \text{for } p = 2, \infty.$$

Therefore, we can apply corollary B.4.2 and finish the proof, since $\int_{E^c} f_n^- \psi\chi_{\{|x'|+|v|\le R\}}\,dv$ will be compact in $L^2(x', v)$, and also in $L^1(x', v)$ by compactness of the support.

□

B.5 Product limit theorem

Here we provide a proof to a corollary of the well-known Egorov's theorem, which is used in many proofs in the theory of renormalized solutions to the Boltzmann equation [32]. The theorem goes as follows:

Theorem B.5.1. *Let (X, Σ, μ) be a measurable space with finite measure. Let $f_n \in L^1(X)$, $g_n \in L^\infty(X)$ be sequences of functions, such that $f_n \rightharpoonup f$ and $g_n \to g$ almost everywhere, with g_n uniformly bounded. Then, we have that*

$$g_n f_n \rightharpoonup fg$$

in $L^1(X)$.

Proof. without loss of generality, we can assume that $g_n \to 0$ almost everywhere, and from this follows the general case. From the Dunford-Pettis lemma, f_n is uniformly integrable, so that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\mu(A) < \delta$,

$$\sup_{n>0}\int_A |f_n|\,dxdv < \epsilon.$$

Also from Egorov's theorem, we know that for any $\delta > 0$, there exists $E \in \Sigma$ such that $g_n \to g$ uniformly in $X - E$ and $\mu(E) < \delta$. Then, fix $\epsilon > 0$, take δ from the uniformly integrability definition and find the set E from Egorov's theorem. Then, for any $\phi \in L^\infty(X)$ we find that

$$\left|\int_X f_n g_n \phi \,d\mu\right| \le \left|\int_{X-E} f_n g_n \phi\,d\mu\right| + \left|\int_E f_n g_n\phi\,d\mu\right|$$

$$\leq \int_{X-E} |f_n g_n \phi| \, d\mu + \int_E |f_n g_n \phi| \, d\mu$$

$$\leq \|\phi\|_{L^\infty} \left[\sup_{n>0} \|g_n\|_{L^\infty(X)} \int_E |f_n| \, d\mu + \|g_n\|_{L^\infty(X-E)} \int_{X-E} |f_n| \, d\mu \right].$$

Then, on the right hand side, the first integral is small using the uniform integrability of f, whereas $\|g_n\|_{L^\infty(X-E)}$ goes to 0 from the uniform convergence of g_n. Therefore

$$\left| \int_X f_n g_n \phi \, d\mu \right| < \epsilon \phi\|_{L^\infty} \left[\sup_{n>0} \|g_n\|_{L^\infty(X)} + \int_{X-E} |f_n| \, d\mu \right] < C\epsilon,$$

and $f_n g_n \rightharpoonup 0$. In case g_n converges to g a.e., then $g_n - g$ converges to 0, and $f_n g_n \rightharpoonup fg \Leftrightarrow f_n(g_n - g) \rightharpoonup 0$. □

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