Uniform Limits and Pointwise Dynamics

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Dissertação de Mestrado apresentada ao Programa de pós-graduação do Instituto de Matemática, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Mestre em Matemática.

Orientador: Alexander Eduardo Arbieto Mendoza

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Resumo

Nesta dissertação abordamos dois problemas relacionados à dinâmica topológica. O primeiro é enunciado na seguinte forma. "Se uma sequencia f_n de aplicações contínuas possuem uma propriedade dinâmica (P) e convergem uniformemente para uma aplicação f, é verdade que f também satisfaz (P)? Caso contrário, como caracterizar as sequências de aplicações para as quais a pergunta acima tem resposta satisfatória?

O segundo problema versa sobre a possibilidade de redefinir propriedades dinâmicas globais em termos pontuais. Além do mais é possível recuperar a propriedade global em termos das locais? Finalmente, quais são as consequências dinâmicas dadas por estas novas definições?

Neste trabalho, discutiremos resultados existentes sobre ambas as questões e também apresentamos novos resultados relacionados a elas. De fato, mostramos que se uma sequência possui sombreamento uniforme então o limite possui sombreamento. Além disso mostramos que a existência de um ponto não-errante, não periódico, expansivo e sombreável garante que a entropia topológica é positiva. Mais resultados relacionados serão apresentados.

Uniform Limits and Pointwise Dynamics

Elias Ferraz Rego

Advisor: Alexander Eduardo Arbieto Mendoza

Abstract

In this thesis we deal with two questions related to topological dynamics. The first one can be stated as follows. "Let f_n be a sequence of continuous maps possessing a dynamical property (P) and which converges uniformly to a map f, is it true that f possess property (P)? If not, is it possible to characterize the sequences for which the answer of the previous question is affirmative?

The second one deals with the possibility of to redefine global dynamical properties in a pointwise manner. Moreover, is it possible to recover the global property from the local ones? Finally, what are the consequences of these new definitions?

In this work, we will discuss the existent results concerning the two above questions and we will present new results about them. Actually, we show that if a sequence posses uniform shadowing then the limit has the shadowing property. Moreover, we show that the existence of a non-wandering, non-periodic, expansive and shadowable point implies the the positiveness of the topological entropy. More related results will be presented.

Contents

Introduction

In this work we deal with the topological theory of dynamical systems. By topological dynamics we mean to study dynamical systems in the framework of general topology. Our initial wondering is the following question:

"If a sequence f_n of continuous systems possessing some property (P) converges uniformly to a system f, is it true that the limit system f inherits the property (P)?"

Since our very first undergraduate courses in real analysis, we have faced this question. Indeed, it is well know that the uniform limit of continuous maps is a continuous map. The same is true when (P) is integrability, or even differentiability (assuming that the derivatives converges too). However, we are interested in this question when (P) is a dynamical property.

Is that true that the limit map of a sequence of maps inherits their dynamical properties?

As we will see, this question is too general and its answer is no.

Actually, if (P) is topological transitivity, it is not difficult to see that the answer is no. Just take irrational rotations converging to the identity. We were very surprised which the fact that some authors wrote articles claiming that the answer is yes. Likewise, in 2005 Abu-Saris an Al-Hami claimed this ([1]). Later, in 2006 Roman-Flores noticed that Abu-Saris and Al-Hami were wrong, giving a counter example to their claiming. Furthermore, he impose extra conditions to the sequence of maps, in order to assure the transitivity of the limit map.

The proof was correct. However, we can notice that the hypothesis are so strong such that it does not use the transitivity of the sequence maps in the proof. That is, he gave conditions to assure the transitivity of the limit, independent of the sequence maps be transitive or not. One of the goals of this work is to present the proof of these results in their original forms and discuss those issues about the proofs.

Pointwise dynamics deals with the study of dynamical properties generated by points. Likewise, it is well known that the existence of a point with dense orbit is enough to guarantees the transitivity of the system. Not too well known, is the fact that a point such that the closure of its orbit is minimal is syndetically recurrent (for instance, see chapter 3).

Since then many articles deals with such ideas. Like, X. Ye in 2007 ([13]), where entropy points and full entropy points were defined and used to understand the topological entropy of a system, for instance they used it to show upper semicontinuity of the entropy map. Also, Morales in 2016 ([7]), the notion of shadowable points is given and the author discuss its consequences for distal and equicontinuous maps. Moothathu in 2011 ([11]), use the notion of entropy points and minimal points to give results on the positiveness of the topological entropy. Actually, some of new results are based in some of his ideas.

The second goal of this work is to present some of the results mentioned before and to give new results, some of them generalizes some of the previous results (see chapter 4 for more details).

This thesis is organized as follows. In chapter one we discuss all the above facts in detail, as well as, we discuss some properties of chaotic systems.

In chapter two we gave the main tools of topological dynamical systems, in order to furnishes a theory background to the reader understand the chapters three and four.

In chapter three we begin to study pointwise dynamics. We aim to give pointwise versions of the tools defined in chapter two. We begin defining minimal points and in the sequence we define shadowable points. Next we define the sensitive points and the entropy points. Next we show some consequences of these definitions(e.g. we give conditions to a map possesses positive entropy).

In the last chapter we pose our results concerning the initial question and the advances we obtained about pointwise dynamics. Our first main theorem is about the shadowing property for uniform limits. This result generalizes proposition 9 in [5].

Theorem A. *Let X be a compact metric space and $f_n : X \to X$ be a sequence of continuous functions which converges uniformly to a function f. Suppose that f_n has the uniform shadowing property. Then f has the shadowing property.*

As a consequence of the previous theorem we can give positive answers to our initial questions for topologically transitive, topologically mixing and now-wandering maps.

Our second main theorem is about pointwise dynamics and it gives conditions for a continuous maps possesses positive topological entropy.

Theorem I. *Let X be a compact metric space and let (X, f) be a TDS. If there exists a non-periodic point x which is non-isolated, shadowable, expansive and non-wandering then f has positive topological entropy.*

We end this work with some consequences of theorem I. The first one is a generalization of it. The second one is a consequence of its proof which states that any positively expansive map with the shadowing property and a infinite number of periodic points, must to have positive topological entropy.

The reading of this work requires a previous knowledge about general topology and the topology of metric spaces. We recommend to the reader without these requisites, the reading of [8] and [6]

Chapter 1

Dynamical Behavior of Uniform Limits

In this chapter we investigate the works concerning uniform limits dynamics, as well, as we give the basic tools to understand them.

1.1 Some Basic Definitions

We call a topological dynamical system (TDS) a pair (X, f) where X is a topological space and $f : X \to X$ is a continuous function. Let f^n denote the n-fold composition of f with itself, f^0 denote the identity map on X and if f is invertible, let f^{-n} denote $(f^{-1})^n$.

Intuitively fixing some point x in X after applying the transformation f on x, $f(x)$ represents the state of x after one unity of time. Similarly $f^2(x)$ represents the state of x after two units of time and so forth. Following this idea we have the following definition.

Definition 1.1.1. *Let $f : X \to X$ be a topological dynamical system.*

- *For each $x \in X$ the set $\mathcal{O}^+(x) = \{f^n(x); n \in \mathbb{N}\}$ is called the forward orbit of the point x.*
- *If f is invertible, we define the backward orbit of x by $\mathcal{O}^-(x) = \{f^{-n}(x); n \in \mathbb{N}\}$ and the orbit of x by $\mathcal{O}(x) = \{f^{-n}(x); n \in \mathbb{Z}\}$*

One of the main purposes of the study of dynamical systems is to understand how the systems evolves over time. We can do this trying to understand the asymptotic behavior of the orbits of the system. In the following examples this behavior is quite simple to describe.

Example 1.1.2. *Let X be a topological space and consider the TDS $f : X \to X$ defined by $f(x) = x$. It is clear that $\mathcal{O}(x) = \{x\}$ for every $x \in X$.*

The next example displays a more complicated behavior, but it is easy give a complete asymptotic description of its orbits anyway.

Example 1.1.3. *Consider the dynamical system $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x(1-x)$.*

First we notice that if we solve the equation

$$x = f(x) = 2x(1-x)$$

we will find that $f(0) = 0$ and $f(\frac{1}{2}) = \frac{1}{2}$. Furthermore $f(1) = 0$.

Let $x < 0$. Thus $f(x) = 2x(1-x) < x$ and therefore $f^n(x) < f^{n-1} < \ldots < f(x) < x$. This implies that the sequence $(f^n(x))_{n\in\mathbb{N}}$ is strictly decreasing. If there exists p such that $f^n(x) \to p$, then $p = \lim f^{n+1}(x) = f(p) < p$ a contradiction. Therefore $f^n(x) \to -\infty$. If $x > 1$ then $f(x) = 2x(1-x) < 0$ and therefore $f^n(x) \to -\infty$ by an analogous argument.

Next we will use the mean value theorem from calculus to study the orbits of points in $(0,1)$. We begin computing the derivative of f. Since $f'(x) = -4x+2$ and f is a C^1 map there are $\varepsilon > 0$ and $C < 1$ such that we have $|f'(x)| < C$ if $x \in (\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon)$. Now take a point $x \in (\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon)$ then

$$|f(x) - f(\frac{1}{2})| = |f(x) - \frac{1}{2}| \leq \sup_{y\in(\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon)} |f'(y)||x - \frac{1}{2}| < |x - \frac{1}{2}|$$

Inductively, we obtain $|f^n(x) - f(\frac{1}{2})| < C^n|x - \frac{1}{2}|$ and therefore $f^n(x) \to \frac{1}{2}$.

Finally, if $x \in (0,1) \setminus (\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon)$, there exists k such that $f^k(x) \in (\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon)$ and therefore $f^n(x) \to \frac{1}{2}$.

The map of the previous example is often called logistic map.

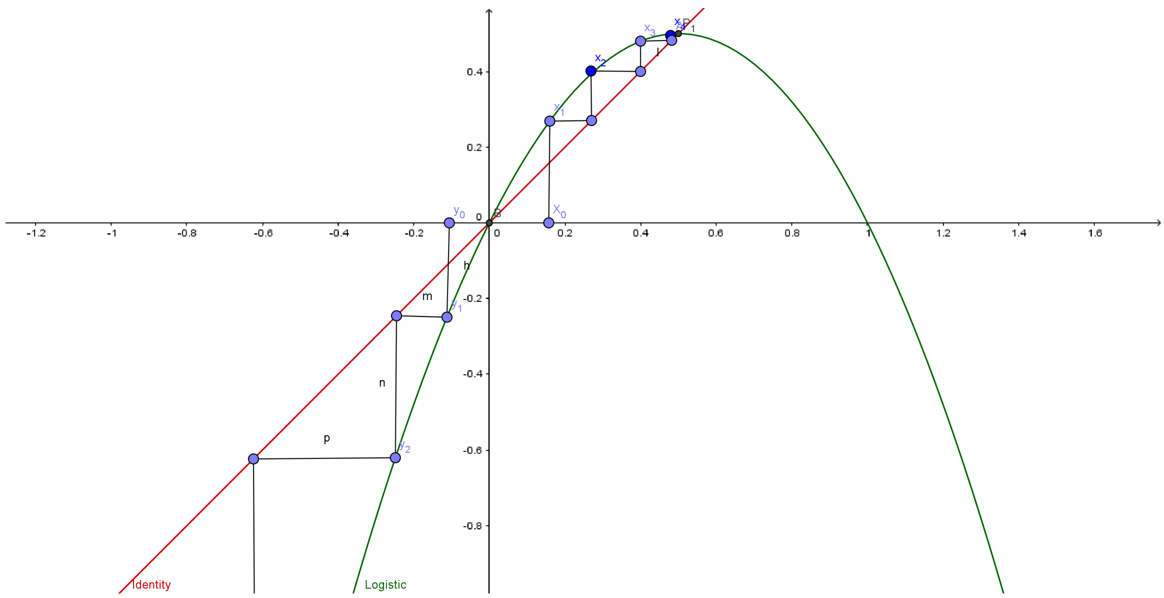

Figure 1.1: The graphics of the logistic map. The sequence y_n represents the iterates of a negative point y_0 and the sequence x_n represents the iterates of the point x_0 which belongs to the interval $[0,1]$. The points p_0 and p_1 represents the repelling and the attracting fixed point respectively.

In figure 1.1 we illustrate an useful technique to get some intuition on the asymptotic behavior of a orbit. We begin choosing a point x_0, then we project vertically this point on the graphics of f to obtain the point x_1. Next we project this point horizontally on the graphics of identity map. We continue projecting vertically on the graphics of f to obtain the point x_2 and so forth. Thus $x_n = f^n(x_0)$. Even this method can get us some intuition, we alert that it is not a rigorous mathematical proof as the one wrote in the example.

The previous examples evidence the existence of special types of orbits and motivate the following definitions.

Definition 1.1.4. *Let $f : X \to X$ be a topological dynamical system.*

- *A point $x \in X$ is called a periodic point for f if there exists a natural number n such that $f^k(x) = x$. The period of x is the lowest natural n satisfying this condition. Let $Per(f)$ denote the the sets of periodic points of $Per(f)$*
- *A point $x \in X$ is called a fixed point for f if it is periodic and has period 1. Let $Per(f)$ denote the the sets of periodic points of f.*
- *A point $x \in X$ is called eventually periodic for f if there exists a natural number k such that $f^k(x)$ is a periodic point.*

In example 1.1.2 all the points are fixed. On the other hand, on example 1.1.3 only the points 0 and $\frac{1}{2}$ are fixed, while 1 is a eventually periodic point.

It would be amazing if all dynamical systems exhibit behaviors as simple as the ones in previous examples. Indeed, a lot of physical, chemical, climate phenomena, can be modeled by dynamical systems. Thus it would be easy work on such models. However a great number of this models displays complicated dynamical behavior.

1.2 Circle Rotations

Our next aiming is to give a example of map which is quite simple, but can exhibit a more complicate dynamical behavior, namely, the circle rotations.

Define on $\mathbb{R}$ the following equivalence relation:

$$x \sim y \text{ if and only if } (x - y) \in \mathbb{Z}$$

Let $\mathbb{R}/\mathbb{Z}$ denote the quotient of $\mathbb{R}$ by the above relation. We endow it with the quotient topology. We can also identify $\mathbb{R}/\mathbb{Z}$ with the circle $S^1 = \{x \in \mathbb{C}; |x| = 1\}$ by the application

$$\Phi : \mathbb{R}/\mathbb{Z} \to S^1, \Phi(x) = e^{2\pi i x}$$

We first remark that Φ is well defined. Indeed, let x_1, x_2 be two members of an equivalence class $[x]$. Then $e^{2\pi i x_1} = e^{2\pi i x_2}$, since $(x_1 - x_2) \in \mathbb{Z}$. Furthermore, it is easily checked that Φ is a homeomorphism.

We can turn $\mathbb{R}/\mathbb{Z}$ an abelian group with the operation $[x]+[y]=[x+y]$. It folows then that Φ is a group isomorphism. Indeed, $\Phi([x]+[y]) = e^{(x+y)i} = e^{xi}e^{yi} = \Phi([x])\cdot\Phi([x])$.

Actually, the topology can be recovered by a metric d in $\mathbb{R}/\mathbb{Z}$ defined as follows:

$$d([x],[y]) = |x-y| \mod \mathbb{Z}$$

One can easily check that d is a metric on S^1, turning it a compact metric space. Geometrically, in S^1, the induced metric between two points in the circle is the geometric arc length (up to multiple by 2π).

Now, we are able to define the circle rotations.

Definition 1.2.1. *Let $\theta \in \mathbb{R}$, the application $R_\theta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ defined by $R_\theta([x]) = [x+\theta]$ is called the θ-rotation of S^1.*

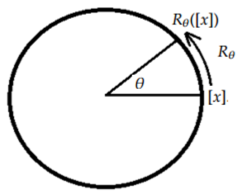

Figure 1.2: The Circle Rotation

In order to understand the dynamical behavior of the circle rotations, we shall proceed by separating them in to particular cases depending on the rationality of θ.

First suppose θ is rational. We claim that every orbit of R_θ is periodic. Indeed, we can write $\theta = \frac{p}{q}$ with p and q integers. Then for any $x \in S^1$, $R_\theta^q([x]) = [x]+[\frac{qp}{p}] = [x]+[p] = [x]$.

On the other hand, if θ is a irrational number, the behavior of the orbits is quite different.

Proposition 1.2.2. *If θ is irrational there exists $x \in S^1$ such that $\mathcal{O}^+(x)$ is dense in S^1.*

Proof. Fix $[x] \in \mathbb{R}/\mathbb{Z}$ and $m, n \in \mathbb{N}$. Then $R_\theta^m([x]) = R_\theta^n([x])$ if and only if $(m-n)\theta \in \mathbb{Z}$. In this case we must to have $m = n$. Thus, $\mathcal{O}^+(x)$ cannot be a finite set. Hence, since S^1 is compact, there exist a convergent subsequence $R_\theta^{k_n}(x)$.

Now, fix $\varepsilon > 0$ and let $m < n$ such that $d(R_\theta^{k_m}([x]), R_\theta^{k_n}([x])) < \varepsilon$. Thus

$$\begin{aligned}\varepsilon > d(R_\theta^m([x]), R_\theta^n([x])) &= |x + m\theta - x - n\theta| \mod \mathbb{Z} = \\ &= |x + (m-n)\theta - x| \mod (\mathbb{Z}) = d(R_\theta^{n-m}([x]), x).\end{aligned}$$

Let $k = m - n$, then the arcs connecting $x, R_\theta^l([x]), R_\theta^{2l}([x]), \ldots$ form a cover to S^1. Since ε is arbitrary, $\mathcal{O}^+(x)$ is dense in S^1. $\square$

1.3 Transitivity and Minimality

Irrational rotations motivate us to define an important dynamical property called topological transitivity.

Definition 1.3.1. *Let (X, f) be discrete dynamical system and let $U, V \subset X$ be any two non-empty sets. We define $\mathcal{N}(U, V) = \{n \in \mathbb{N}; f^n(U) \cap V \neq \emptyset\}$.A system (X, f) is called topologically transitive if for every pair of open subsets U and V of X, $\mathcal{N}(U, V) \neq \emptyset$.*

Next we shall show that in presence of separability and completeness, topological transitivity is equivalent to the existence of a dense orbit. In particular this holds for compact spaces.

Theorem 1.3.2. *Let X be a complete separable topological space without isolated points and (X, f) be a TDS. Then f is transitive if and only if the set of points in X whose orbit is dense is non-empty.*

Proof. Suppose f has a point x with dense orbit and let U and V be non-empty subsets of X. Then there exist $m', n' \in \mathbb{N}$ such that $f^{m'}(x) \in U$ and $f^{n'}(x) \in V$. If $m' < n'$ then $f^{n'-m'}(U) \cap V \neq \emptyset$. Suppose $n' < m'$. Since X has no isolated points, the set $U \setminus \{x, f(x), f^2(x), ..., f^{m'}(x)\}$ is open. Then there exists $n > m'$ such that $f^n(x) \in V$ and therefore $f^{n-m'}(U) \cap V \neq \emptyset$.

Conversely, suppose that f is transitive. Since X is separable, let $\{U_k\}_{k\in\mathbb{N}}$ be a countable basis for the topology of X. Consider the set $A = \cap_{k\in\mathbb{N}} \bigcup_{i=0}^{\infty} f^{-i}(U_k)$ and suppose that exists a point $x \in A$. Thus for each k there exists n such that $f^n(x) \in U_k$. Hence x must to have dense orbit.

Now, since f is transitive, for each k, $\bigcup_{i=0}^{\infty} f^{-i}(U_k)$ is dense in X. By continuity this set is also open. By completeness of X, we can use Baire's theorem and conclude that A is a dense set, since it is a countable intersection of dense open sets. In particular A is non-empty and therefore there exists a point x whose orbit is dense. □

We call x a transitive point of f if the orbit of x under f is dense on the space X. Let $Tr(f)$ denote the set of transitive points of f. As a remark on the proof of the previous theorem, one can notice that $Tr(f)$ is more than an non-empty set. Indeed, $Tr(f)$ is a residual subset of X.

The previous result shows that irrational rotations are topologically transitive since every orbit is dense. On the other hand, there are dynamical systems which are topologically transitive but $tr(f) \neq X$.

Example 1.3.3. *Consider the TDS $f : [0, 1] \to [0, 1]$ defined by*

$$f(x) = \begin{cases} 2x & \text{if } x \in [0, -\frac{1}{2}] \\ -2x + 2 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

This map is frequently called the tent map.

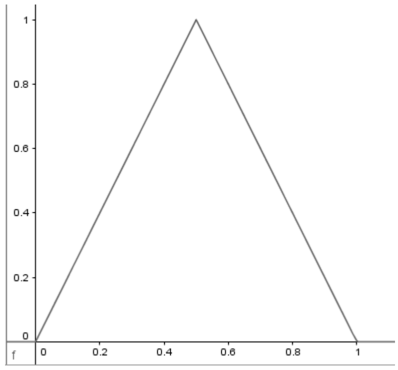

Figure 1.3: The tent map

First we will show that f is transitive. To do this, consider a natural n and the intervals $[\frac{k}{2^n}, \frac{k+1}{2^n}]$, $k = 0, 1, 2, ..., 2^{n-1}$. Then f^n maps each interval in $[0, 1]$. Now, let U, V be open sets and choose $n, k \in \mathbb{N}$ such that $[\frac{k}{2^n}, \frac{k+1}{2^n}] \in U$. Then $f^n(U) = [0, 1]$. Thus $f^n(U) \cap V \neq \emptyset$ and f is transitive.

On the other hand, the same argument shows that the graph of $f^n|_{[\frac{k}{2^n}, \frac{k+1}{2^n}]}$ needs to intersect the graph of the identity map on $[0, 1]$. Therefore $Perf(f)$ is dense in $[0, 1]$.

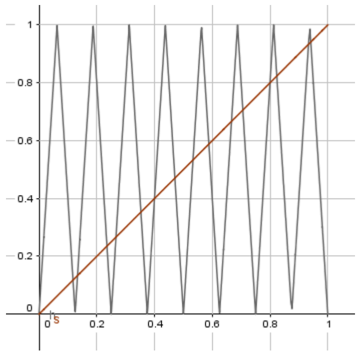

Figure 1.4: This is the graph of the 4-th iterate of f to illustrate the claim made before.

The fact of $Per(f) \neq \emptyset$ for the tent map shows the existence of a topologically transitive map such that not all orbits are dense, in contrast with irrational rotations. However, systems which all orbits are dense have important properties and it motivates the definition of minimal systems.

Definition 1.3.4. *A topological dynamical system (X, f) is called minimal if the orbit of every point is dense in X.*

Next we will give other characterization of minimal dynamical systems based on invariant sets.

Definition 1.3.5. *Let (X, f) be a TDS.*

- *A set $A \subset X$ is called f-positively invariant if $f(A) \subset A$*

- *A set $A \subset X$ is called f-negatively invariant if $f^{-1}(A) \subset A$*
- *A set $A \subset X$ is called f-invariant if $f(A) = A$*

One can notice that if a set A is f-invariant, then $f^n(A) = A$ for every $n \in \mathbb{N}$. Also The respective results for positively and negatively invariant are valid.

Proposition 1.3.6. *Let X be a separable topological space, and let $f : X \to X$ be a continuous map. If A is f-positively (respectively negative) invariant, then $\overline{A}$ is f-positively (respectively negative) invariant.*

Proof. Let A be a f-positively invariant set and take $y \in f(\overline{A})$. Then there exists a point $x \in \overline{A}$ such that $f(x) = y$. Since X is separable, there exists a sequence $(x_n) \in A$ such that $x_n \to x$. Since A is positively invariant, then $f(x_n) \in A$, and since f is continuous, then $f(x_n) \to y$. Thus $y \in \overline{A}$ and $\bar{A}$ is f-positively invariant. □

As a direct consequence of the above proposition we obtain that if A is an invariant set, then $\overline{A}$ is invariant.

The next theorem shows the connection between minimal systems and invariant subsets.

Theorem 1.3.7. *f is a minimal system if and only if the only f-positively-invariant non-empty closed subset of X is X.*

Proof. If there exists a non-empty closed f-positively invariant proper subset U o X we can choose $x \in U$. Now $f(\overline{O^+(x)}) \subset U \neq X$. Then the orbit of x can not be dense in X and therefore $Tr(f) \neq X$.

Conversely,if we prove $\overline{O^+(x)}$ is positively invariant we are done. Let $y \in \overline{O^+(x)}$. Then there exists a sequence n_k such that $f^{n_k}(x) \to y$. By continuity $f^{n_k+1}(x) \to f(y)$ and therefore $\overline{O^+(x)}$ is invariant. Then it must to be X. □

The previous theorem gives a justification to the term minimal. Roughly speaking, if f is a minimal system we cannot divide the map f in subsystems and study them individually. In other words, X is the smallest set we can consider to try understand the system evolution.

1.4 Devaney Chaotic Systems

We have seen that there exists examples of dynamical systems displaying orbits with complicate behavior. Devaney in [4] gave a definition of chaotic maps. He realized that a chaotic map should exhibit some kind of unpredictability

Definition 1.4.1. *A TDS (X,f) is said to have sensitive dependence on initial condition if there exists $\delta > 0$ such that for every $x \in X$ and every open neighborhood U of x, there exists $y \in V$ such that $d(f^n(x), f^n(y)) > \delta$ for some $n \in \mathbb{N}$*

The constant δ on previous definition is called the sensitive constant of f.

Definition 1.4.2. *A TDS (X,f) is called a chaotic dynamical system in the sense of Devaney if :*

- *f is topologically transitive*
- *$Per(f)$ is dense in X*
- *f has sensitive dependence on the initial conditions.*

Example 1.4.3. *The tent map defined on the example 1.3.3 is chaotic in the sense of Devaney. Indeed, we just need to prove that f has sensitive dependence on initial conditions.*

To do this let $\delta = \frac{1}{4}$, $x \in [0,1]$ and let U be an open neighborhood of x. Then there exists $n, k \in \mathbb{N}$ such that $[\frac{k}{2^n}, \frac{k+1}{2^n}] \subset U$. Since $f^n|_{[\frac{k}{2^n}, \frac{k+1}{2^n}]} = [0,1]$, there exists $y \in U$ such that $d(x,y) > \delta$. Then f has sensitive dependence and therefore is chaotic on Devaney's sense.

Unfortunately, in [3] the authors noticed a redundancy on Devaney's definition. More precisely we have the following theorem

Theorem 1.4.4. *Let X be an infinite metric space and (X, f) be a TDS. If f is topologically transitive and $Per(f)$ is dense in X, then f is chaotic in the Devaney's sense.*

Proof. Since X is infinite we can take p_1 and p_2 be two distinct periodic points with disjoint orbits. Let ρ be the distance between the orbits of p_1 and p_2. Let x be any point of X. Then

$$\rho = d(O(p_1), O(p_2)) \leq d(O(p_1), x) + d(x, O(p_2))$$

Then x is at least a distance $\frac{\rho}{2}$ from p_1 or p_2

Let $\delta = \frac{\rho}{8}$. Let x be a point in X. Since the periodic points of f are dense on X, there exists a periodic point $p \in B_\delta(x)$. Let n denote the period of p. Now let q be a periodic point with orbit at least a distance 4δ from x. Define $V = \bigcap_{i=0}^{n} f^{-i}(B_\delta(f^i(q)))$.

Since f is transitive and V is a non-empty open set, there exists $y \in B_\delta(x)$ such that $f^k(y) \in V$. Let j be the integer part of $\frac{k}{n} + 1$. Then we have

$$f^{nj}(y) = f^{nj-k}(f^k(y)) \in f^{nj-k}(V) \subseteq B_\delta f^{nj-k}(q)$$

Since $f^{nj}(p) = p$, we have

$$d(f^{nj}(p), f^{nj}(y)) = d(p, f^{nj}(y)) \geq d(x, f^{nj-k}(q)) - d(f^{nj-k}(q), f^{nj}(y)) - d(p, x)$$

Note that since $p \in B_\delta(p))$ and $f^{nj}(y) \in B_\delta(f^{nj-k}(q))$ then $d(f^{nj}(p), f^{nj}(y)) > 2\delta$. Finally, using the triangle inequality we obtain that $d(f^{nj}(x), f^{nj}(y)) > \delta$ or $d(f^{nj}(p), f^{nj}(x)) > \delta$. Therefore, f has sensitive dependence on initial condition with sensitive constant δ. $\square$

Later in [12] the authors noticed that for interval maps, topologically transitivity implies the denseness of periodic points. Thus in order to prove that an interval map is chaotic on the sense of Devaney, one can just prove its transitivity .

Theorem 1.4.5. *Let* $[a, b]$ *be a compact interval and* $([a, b], f)$ *be a TDS. If* f *is topologically transitive, then* $\overline{Per(f)} = X$.

Proof. Otherwise, there would exist an interval $J \subset [a, b]$ without periodic points. Since f is transitive there exists $x \in J$ such that $x, f^m(x), f^n(x) \in J$ with $m < n$. In this scenario, we asserts that $x < f^m(x) < f^n(x)$ or $x > f^m(x) > f^n(x)$. Indeed, if not, we could have $x < f^m(x)$ and $f^m(x) > f^n(x)$ or $x > f^m(x)$ and $f^m(x) < f^n(x)$. We will proceed in the first case, the another one is quite analogous.

Define $g = f^m$. Then $x < g(x)$. We claim that for ever $k \geq 1$ we have $x < g(x) < g^{k+1}(x)$. To prove the claiming we proceed by induction.

Consider $k = 1$. If $g^2(x) < g(x)$, then setting $h(z) = g(z) - z$ we have $h(z) =< 0$ when $z = x$ and $h(z) > 0$ when $z = g(x)$. This intermediate value theorem gives a fixed point $x_0 \in [x, g(x)]$ for h and therefore a periodic point for f. Therefore our claiming holds for $k = 1$.

Now suppose the claiming holds for $k = n$. If $g^{n+1}(x) < g^n(x)$, then setting $h(z) = g^n(z) - z$ we have $h(z) > 0$ when $z = x$ and $h(z) < 0$ when $z = g(x)$. Again the intermediate value theorem implies the existence of a periodic point for f in $[x, g(x)]$ and it is a contradiction. Thus our claiming holds for every k.

Thus we have $x < g^k(x)$ for every k. Set $k = n - m$ and define $h = f^{n-m}$. Then we have $x < h^m(x)$ and by assumption $h(f^m(x)) > f^n(x)$. Now we claiming that $h^k(f^m(x)) < h(f^m(x)) < f^m(x)$. In order to prove that we use an induction argument which is totally analogous to one used in the previous claiming. We just need to replace x by $f^m(x)$ and g by h.

Since $x < h^m(x)$ and $h^m(f^m(x)) < f^m(x)$, if we define $s(z) = h^m(z) - z$, then the intermediate value theorem gives us a point $x_1 \in [x, f^m(x)]$ which is fixed for s and therefore is periodic for f. This is a contradiction, since $x_1 \in J$ and J has not periodic points. This proves our assertion.

Since f is transitive there exists a point x whose orbit is dense in I. Thus there exists m such that $f^m(x) \in J$. Let $J' = J \cap (f^m(x), b]$. Since $O^+(x)$ is dense in I, $O^+(x) \setminus \{x, f(x), ..., f^m(x)\}$ is also. Thus there exists n such that $f^n(x) \in J'$. Now consider $J'' = (f^m(x), f^n(x))$, the denseness of $O^+(x)$ again implies that there exists $o > n$ such that $f^o(x) \in J''$. This contradicts our claiming if we set $z = f^m(x)$. Therefore $\overline{Per(f)} = I$.

□

1.5 Uniform Limits

In this section we ask if the uniform limit of a sequence of topological dynamical systems possessing a property (P), also posses (P). With the theory developed in the previous sections, we can easily see that it is not true when (P) is topological transitivity. Indeed, the next example shows that.

Example 1.5.1. *Let θ_n be a sequence of positive irrationals converging to* 0*. We claim that* $\{R_{\theta_n}\}$ *converges uniformly to the identity on S^1. Indeed, fix $0 < \varepsilon < 1$ and choose $n \in \mathbb{N}$ such that $|\theta_n - 0| < \varepsilon$. Then $d(R_{\theta_n}(x), x) = |x + \theta_n - x| mod(\mathbb{Z}) = 2|\theta_n| mod(\mathbb{Z}) = |\theta_n| < \varepsilon$ for every $x \in S^1$. Now each R_{θ_n} is topologically transitive, but the identity trivially is not topologically transitive.*

As we just see, only uniform convergence is not enough to ensure the transitivity of limit map. We were very surprised which the fact that some authors wrote articles claiming that it is possible to assure the transitivity of the limit map without extra conditions. Likewise, in 2005, Abu-Saris an Al-Hami claimed this ([1]). Before to continue, we will reproduce ipsis-litteris the proof given by Abu-Saris. In the sequence, we will analyze their proof in order to discover why it does not work.

Ipsis-litteris proof

Let U, V be two nonempty open subsets of X. Since f^n is topologically transitive on X, there exists a positive integer l_n such that $f_n^{l_n}(U) \cap V \neq \emptyset$. Choose a point $y_0 \in f_n^{l_n}(U) \cap V$ and let us have x_0 such that $y_0 = f_n^{l_n}(x_0)$. Since V is open there is an $\varepsilon > 0$ such that $B_\varepsilon(y_0) = \{y \in X : d(y, y_0) < \varepsilon\} \subseteq V$. But, by Lemma 3.1, one can take n sufficiently large that $f^{l_n}(x_0) \in B_\varepsilon(y_0) \in V$. Hence,$f^{l_n}(x_0) \in f^{l_n}(U) \cap V$. This completes the proof.

We remark that lemma 3.1 cited in the proof states that if $f_n \to f$, then $f_n^l \to f^l$, for any positive integer i.

If we do a careful analysis on the previous proof, we can notice that the missed point is the fact that the point x_0 obtained above depends on the map f_n as well as l_n and ε. Therefore we can not make n grow without changes in x, l_n and ε and that is the reason that their attempt of proof fail.

We will now discuss Roman-Flores work in [10]. He assumed stronger hypothesis concerning the convergence in order to obtain an positive answer to Abu-Saris question.

Define the metric $d_\infty(f, g) = \sup_{x \in X}\{d(f(x), g(x))\}$, where d denotes the metric on X. Clearly, d_∞ is a metric on the space $C^0(X)$ of the continuous self-transformations of X.

The first attempt to correct the wrong points of Abu-Saris and Al-Hami proof consisted of two steps. The first was assume that x and the ε obtained on their proof were uniform and the second one was try to control the speed of convergence of the iterates f_n^n. More precisely, they assumed $\lim\limits_{n\to\infty} d_\infty(f_n^n, f^n) = 0$.

Unfortunately, this hypothesis alone is not enough as we can see exploring more carefully the example of irrational rotations.

Example 1.5.2. *Let $\lambda \in [0,1]$ be a irrational number and define the sequence $\lambda_n = \frac{\lambda}{n^2}$. Then $d(f_n^n(x), f^n(x)) = |n\lambda_n| mod(\mathbb{Z}) = |\frac{\lambda}{n}| mod(\mathbb{Z})$ wich converges to 0. Furthermore, $\lambda_n \to 0$, and therefore $R_{\lambda_n} \to Id$ uniformly.*

The previous example shows us that we need more conditions to answer our question. The next lemma, two theorems and proofs are ipsis-litteris writing of the ones found in [10]. They give sufficient conditions to guarantee the transitivity of the limit map.

Lemma 1. *Let X be a perfect metric space and consider $U \subset X$ a nonempty open set. If (x_n) is a dense sequence in X and $x_{n_0} \in U$, then there exists $n_1 > n_0$ such that $x_{n_1} \in U$.*

Proof. It is sufficient to observe that $U \setminus \{x_1; x_2; ...; x_{n_0}\}$ is a nonempty open set.

Theorem 2. *Let (X, d) be a perfect metric space, and let $f_n : X \to X$ be a sequence of continuous and topologically transitive functions such that (f_n) converges uniformly to a function f. Additionally, suppose that*

(T1) $d_\infty(f_n^n, f^n) \to 0$ as $n \to \infty$,

(T2) $\{f_n^n(x)\}$ is dense in X, for some $x \in X$.

Then f is topologically transitive.

Proof. Let U, V be two nonempty open subsets of X. Then, due to (T2), there exists $x_0 \in X$ such that $\{f_n^n(x_0)\}$ is dense in X. Thus, by Lemma 1 and condition (T1), we obtain that the sequence

$\{f^n(x_0)\}$ is also dense in X

Thus, there exists $p \in \mathbb{N}$ such that

$z = f^p(x_0) \in U.$

Now, consider the set $G = \backslash\{f(x_0), ..., f^p(x0)\}$. Then, because X is a perfect metric space, G is a nonempty open set. Thus, due to denseness of $\{f^n(x0)\}$, there exists $q > p$ such that $f^q(x0) \in G \subset V$, which implies that

$f^q(x_0) = f^{q-p}(f^p(x_0)) = f^{q-p}(z) \in f^{q-p}(U) \cap V$

And consequently $f^{q-p}(U) \cap V$ is nonempty and f is topologically transitive. This completes the proof.

With a very similar argument the author prove that if we suppose conditions (T1), then (T2) is equivalent to the limit possesses transitivity.

However, exploring the previous proof one can notice that in any time the authors used the fact that the maps f_n are topologically. Then we conclude that they did not answered the original question and what they proved can be correctly expressed by the next theorem.

Theorem 1.5.3. *Let $f_n \in C^0(X)$ a sequence of maps converging uniformly to f. In addition,* *suppose that $d_\infty(f_n^n, f^n) \to 0$. Then f is transitive if and only if $\{f_n^n(x)\}$ is dense in X, for some x.*

Corollary 1.5.4. *Let I be an interval and Let $f_n \in C^0(I)$ be a sequence of maps converging uniformly to f. In addition, suppose that $d_\infty(f_n^n, f) \to 0$. The f is chaotic on Devaney's sense if and only if $\{f_n^n(x)\}$ is dense in I, for some x.*

As a corollary of the proof of the previous theorem we can obtain

Corollary 1.5.5. *Let $\{f_n\}$ be a sequence of continuous maps converging to on $C^0(X)$ to f. In Addition, suppose:*

$$\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0$$

Then f is minimal if and only if $\{f_n^n(x)\}$ is dense in X, for every $x \in X$.

Proof. The same argument used in the proof of the theorem works here. We just use it for every point in X. □

In the definition of topological transitivity we have that for any pair of open sets U and V, U meets V at some time. But one could ask how many times U meets V. Since topological transitivity is equivalent to the existence of a dense orbit we can imagine the following situation. Take U and V non-empty open sets and let x be a point with dense orbit. Then there are $m < n$ such that $f^m(x) \in U$ and $f^n(x) \in V$. Since $O^+(x) \setminus \{x, f(x), ..., f^n(x)\}$ in X there exists $o > n$ such that $f^o(x) \in V$. Now we can keep proceeding in this way and therefore we conclude that U meets V in an infinite number of times.

Example 1.5.6. *Let $I = [-1, 1]$ and consider the TDS (I, f) defined by*

$$f(x) = \begin{cases} 2x+2 & \text{if } x \in [-1, -\frac{1}{2}] \\ -2x & \text{if } x \in [-\frac{1}{2}, 0] \\ -x & \text{if } x \in [0, 1] \end{cases}$$

We notice that f fixes the point 0, $f([-1, 0]) = [0, 1]$ and $f([0, 1]) = [-1, 0]$. moreover

$$f^2(x) = \begin{cases} -2x-2 & \text{if } x \in [-1, -\frac{1}{2}] \\ 2x & \text{if } x \in [-\frac{1}{2}, \frac{1}{2}] \\ -2x-2 & \text{if } x \in [0, 1] \end{cases}$$

We have that f^2 is the tent map on $[0, 1]$ and it is the product of the tent map with the constant map $g(x) = -1$ on $[-1, 0]$.

Now, let U be an open subset of I. If $U \cap [-1, 0] \neq \emptyset$, there is k such that $f^{2k}(U) \supset [-1, 0]$ and $f^{2k+1}(U) \supset [0, 1]$. If we use a similar argument, we can deduce the same if $U \cap [0, 1] \neq \emptyset$. Therefore $I = f^k(U) \cup f^{2k+1}(U)$ and f must to be transitive. We remark that for every $n > k$ is an open set then $f^n(U) \supset [-1, 0]$ if n is even or $f^n(U) \supset [0, 1]$ if n is odd. Then $f^{k+2j}(U) \cap V \neq \emptyset$ or $f^{k+1+2j}(U) \cap V \neq \emptyset$, for every $j \in \mathbb{N}$ and any non-empty open set V.

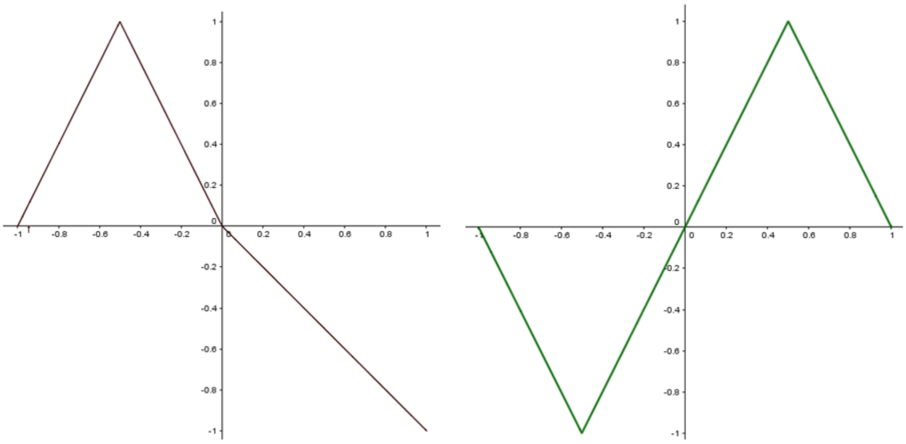

Figure 1.5: The graphics of f (left) and f^2 (right).

In the last example we had a transitive map such that for every pair of open sets there exists a bounded amount of time that the sets need to wait until they meet again, once they have already met. This motivates the following definition.

Definition 1.5.7. *A subset $F \in \mathbb{N}$ is syndetic when it has bounded gaps. That is, there exists an integer N such that the maximum length of any sequence of consecutive integers in $\mathbb{N} \setminus F$ is N.*

Definition 1.5.8. *A map is called syndetically transitive if for every open subsets U, V of X, $\mathcal{N}(U, V)$ is syndetic.*

Following the ideas of Roman-Flores, Risong-Li proved in [9] the following results.

Theorem 1.5.9. *Let X be a metric space with metric d, and let $f_n : X \to X$ be a sequence of continuous functions such that (f_n) converges uniformly to a function f. Additionally, suppose that*

$$\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0$$

Then f is syndetically transitive if and only if for every non-empty $U, V \subset X$ the set $\{n | f_n^n(U) \cap V \neq \emptyset\}$ is syndetic.

Proof. Let U and v be non-empty open sets. Let $x \in V$ and $\varepsilon > 0$ Such that $B_{2\varepsilon}(x) \subset V$. Suppose f is syndetically transitive, thus $\mathcal{N}(U, B_\varepsilon(x))$ is syndetic. On the other hand, since $d_\infty(f_n^n(x), f^n(x)) \to 0$ there is N such that $d(f_n^n(x), f^n(x)) < \varepsilon$ for ever $n \geq N$. Since $\mathcal{N}(U, B_\varepsilon(x))$ is syndetic then $A = \mathcal{N}(U, B_\varepsilon(x)) \cap \{N, N+1, N+2, ...\}$ is also. Thus for each natural $j \in A$ there is $x_j \in U$ such that $f^j(x_j) \in B_\varepsilon(x)$. Then $d(f_j^j(x_j), x) \leq d(f_j^j(x_j), f^j(x_j)) + d(f^j(x_j), x) < 2\varepsilon$ and therefore $f_j^j(x) \in V$ for every $j \in A$, i.e. $\{n | f_n^n(U) \cap V \neq \emptyset\}$ is syndetic.

Conversely suppose $\{n | f_n^n(U) \cap V \neq \emptyset\}$ is syndetic for every open setes U and V. Since $d_\infty(f_n^n(x), f^n(x)) \to 0$ there is N such that $d(f_n^n(x), f^n(x)) < \varepsilon$ for ever $n \geq N$. Since $\{n | f_n^n(U) \cap B_\varepsilon(x) \neq \emptyset\}$ is syndetic then $A = \{n | f_n^n(U) \cap B_\varepsilon(x) \neq \emptyset\} \cap \{N, N+1, N+2, ...\}$ is also. Thus for each natural $j \in A$ there is $x_j \in U$ such that $f_j^J(x_j) \in B_\varepsilon(x)$. Then $d(f^j(x_j), x) \leq d(f^j(x_j), f_j^j(x_j)) + d(f_j^j(x_j), x) < 2\varepsilon$ and therefore $f^j(x) \in V$ for every $j \in A$, i.e. $\mathcal{N}(U, V)$ is syndetic. □

Let us look again to the example 1.3.3. When we proved that the tent is transitive, we have gone further. We have shown that for every open set, there is a natural number k such that $f^k(U) = I$. Thus $f^i(U) = I$ for every $i \geq K$ and therefore there exists a time k such that U meets any open set V for a time greater than k. This motivates us the following definition

Definition 1.5.10. *A subset K of $\mathbb{N}$ is co-finite when $\mathbb{N} \setminus K$ is finite .*

Definition 1.5.11. *A map is called topologically mixing if for every open subsets U, V of X, $\mathcal{N}(U, V)$ is co-finite.*

Evidently every topologically mixing map is syndetically transitive and every syndetically transitive map is topologically transitive. However, example 1.5.6 shows a map which is syndetically transitive but not topologically mixing. In chapter two we will investigate some connections between transitive and topologically mixing maps.

Theorem 1.5.12. *Let X be a metric space with metric d, and let $f_n : X \to X$ be a sequence of continuous functions such that (f_n) converges uniformly to a function f. Additionally, suppose that*

$$\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0$$

Then f is topologically mixing if and only if for every non-empty $U, V \subset X$ the set $\{n | f_n^n(U) \cap V \neq \emptyset\}$ is co-finite.

Proof. Let U and V be non-empty open sets. Let $x \in V$ and $\varepsilon > 0$ Such that $B_{2\varepsilon}(x) \subset V$. Suppose f is topologically mixing, thus $\mathcal{N}(U, B_\varepsilon(x))$ is co-finite. On the other hand, since $d_\infty(f_n^n(x), f^n(x)) \to 0$ there is N such that $d(f_n^n(x), f^n(x)) < \varepsilon$ for ever $n \geq N$. Since $\mathcal{N}(U, B_\varepsilon(x))$ is co-finite then $A = \mathcal{N}(U, B_\varepsilon(x)) \cap \{N, N+1, N+2, ...\}$ is also. Thus for each natural $j \in A$ there is $x_j \in U$ such that $f^j(x_j) \in B_\varepsilon(x)$. Then $d(f_j^j(x_j), x) \leq d(f_j^j(x_j), f^j(x_j)) + d(f^j(x_j), x) < 2\varepsilon$ and therefore $f_j^j(x) \in V$ for every $j \in A$, i.e. $\{n | f_n^n(U) \cap V \neq \emptyset\}$ is co-finite.

Conversely suppose $\{n | f_n^n(U) \cap V \neq \emptyset\}$ is co-finite for every open sets U and V. Since $d_\infty(f_n^n(x), f^n(x)) \to 0$ there is N such that $d(f_n^n(x), f^n(x)) < \varepsilon$ for ever $n \geq N$. Since $\{n | f_n^n(U) \cap B_\varepsilon(x) \neq \emptyset\}$ is co-finite then $A = \{n | f_n^n(U) \cap B_\varepsilon(x) \neq \emptyset\} \cap \{N, N+1, N+2, ...\}$ is also. Thus for each natural $j \in A$ there is $x_j \in U$ such that $f_j^J(x_j) \in B_\varepsilon(x)$. Then $d(f^j(x_j), x) \leq d(f^j(x_j), f_j^j(x_j)) + d(f_j^j(x_j), x) < 2\varepsilon$ and therefore $f^j(x) \in V$ for every $j \in A$, i.e. $\mathcal{N}(U, V)$ is co-finite. □

Chapter 2

Topological Dynamics

Our objective in this chapter is to establish some important results concerning topological dynamical systems that will be used in our main results. We begin defining properties which very often arises on topological dynamical systems. Next we establish conditions to guarantee when a systems has such properties. Finally, we establish some relations between them. For the remainder of this chapter, X will denote a compact metric space and f will denote a continuous self-map of X.

2.1 Topological Conjugacy

In this section we are going to spend some time discussing what we mean by a dynamical property. Suppose we have two systems that in some sense are equivalent. Then one can expect that their dynamical properties are the same. Next we will to turn this notion of equivalence more precise.

Definition 2.1.1. *Let (X, f) and (Y, g) be TDS. We say that f and g are topologically conjugated if there is $h : X \to Y$ a homeomorphism such that $h \circ f = g \circ h$. In this case we denote $f \cong g$ and the map h is called a conjugacy homeomorphism*

Topological conjugation is an equivalence relation. Indeed, f is always conjugated to itself if we take $Id : X \to X$ as a topological conjugacy. If f and g are topological conjugated by h, then $h^{-1} \circ g = f \circ h^{-1}$. Thus g is conjugated to f. In order to prove the transitivity of $\cong$, suppose $f : X \to X$ is conjugated to $g : Y \to Y$ by h_1 and g is conjugated to $t : Z \to Z$ by h_2. Therefore $h_2 \circ h_1 \circ f = h_2 \circ g \circ h_1 = t \circ h_2 \circ h_1$. Therefore f is conjugated to t.

Remark: Let $x \in X$. If h is an conjugacy between f and g, then $h(f(x)) = g(h(x))$. Thus $h(f^2(x)) = h(f(f(x))) = g(h(f(x))) = g^2(h(x))$. If we keep proceeding this way, we will find $h(f^n(x)) = g^n(h(x))$ for every n. Therefore we conclude that h carries orbits of f into orbits of g.

Since we have an natural identification between the orbit of f and g, we expect that their dynamical behavior are the same. This is why we call properties which are preserved by topological conjugacy topological dynamical properties.

The first one we shall analyze is the persistence of periodic orbits by topological conjugacy. Namely, we have

Proposition 2.1.2. *Let (X, f) and (Y, g) be TDS and let h be a conjugacy between f and g. Then*

1. $h(Fix(f)) = Fix(g)$
2. $h(Per(f)) = Per(g)$.

Proof. 1. If $y \in h(Fix(f))$, then there exists $x \in Fix(f)$ such that $h(x) = y$. Thus $h(x) = h(f(x)) = g(h(x))$ and therefore $y \in Fix(g)$.

If $y \in Fix(g)$, since h is a homeomorphism, there exists an unique x such that $h^{-1}(y) = x$. Thus

$$x = h^{-1}(y) = h^{-1}(g(y)) = h^{-1}(g(h(x)) = h^{-1}h(f(x)) = f(x)$$

and thus $x \in Fix(f)$. Therefore $y \in h(Fix(f))$.

2. If $x \in Per(f)$, then there exists n such that $f^n(x) = (x)$. Thus $h(x) = h(f^n(x)) = g^n(h(x))$ and therefore $h(x) \in Per(g)$.

If $y \in Per(g)$, there exists n such that $g^n(y) = y$. Since there exists a unique x such that $h^{-1}(y) = x$, we have

$$x = h^{-1}(y) = h^{-1}(g^n(y)) = h^{-1}(g^n(h(x)) = h^{-1}h(f^n(x)) = f^n(x)$$

and thus $x \in Per(f)$. Therefore $y \in h(Per(f))$. Notice that m must to be smaller or equal than n.

□

Remark: If $f \cong g$ and x is a periodic point of period n for f, then $h(x)$ is a periodic point of period n for g. Indeed, suppose that the periods of x and $h(x)$ are n and m respectively. Thus $m \leq n$. If $m < n$, $h(x) = g^m(h((x)) = h(f^m(x))$. But this implies $f^m(x) = x$, since h is an homeomorphism. Therefore the period of x must to be m and that is a contradiction.

I view of previous result we have that the density of periodic orbits is a dynamical property.

Theorem 2.1.3. *Let (X, f) and (Y, g) be TDS and let h be a conjugacy between f and g. If $\overline{Per(f)} = X$ then $\overline{Per(g)} = Y$.*

Proof. Suppose that $\overline{Per(f)} = X$ and let $U \subset Y$ be a non-empty open set. Since h is a homeomorphism $h^{-1}(U) \subset X$ is a non-empty open set. Thus we can take a periodic point x in $h^{-1}(U)$. By the previous proposition, $h(x)$ must to be a periodic point which lies in U. That is, $\overline{Per(g)} = Y$. □

Example 2.1.4. *Let R_1 be a rational rotation and R_2 be an irrational rotation. Since every point is periodic R_1 and R_2 does not admits periodic points, then R_1 and R_2 cannot be topologically conjugated.*

In chapter one we defined sensitive dependence on initial conditions. Now we will prove that it is a dynamical property.

Theorem 2.1.5. *Let (X, f) and (Y, g) be TDS and let h be a conjugacy between f and g. If f has sensitive dependence on initial conditions then g has it also.*

Proof. Let C be the sensitive constant of f. Let d^X and d^Y denote the metrics on X and Y respectively. Since Y is compact h^{-1} is uniformly continuous. Thus there exists $\delta > 0$ such that $d^Y(x, y) < \delta$ implies $d^X(h^{-1}(x), h^{-1}(y)) < C$.

Now suppose g has not sensitive dependence on initial conditions. Thus if for every $K > 0$ there is an y and an open neighborhood U of y such that $d^Y(g^n(x), g^n(y)) < K$ for every n. In particular, we can take $K < \delta$. Set $U' = U \cap B^Y_\delta(y)$. Thus $h^{-1}(U') \subset B^X_C(h^{-1}(y))$ and it is an open neighborhood of $h^{-1}(y)$. If $x \in U'$ then $d^Y(g^n(x), g^n(y)) < K$ for every n and therefore

$$d^X(h^{-1}(g^n(x)), (h^{-1}(g^n(y))) = d^X((f^n(h^{-1}(x)), (f^n(h^{-1}(y)))) < C \text{ for every n}$$

But this implies that f has not sensitive dependence on initial conditions which is a contradiction. □

In topological conjugation we require the conjugacy map be a homemorphism. But sometimes it can be hard to find such homeomorphism. So, in the next definition we present a weaker form of topological conjugacy.

Definition 2.1.6. *Let (X, f) and (Y, g) be TDS. We say that f is semi-conjugated to g if there is a continuous surjection $h : X \to Y$ such that $h \circ f = g \circ h$. In this case we say that f and g are semi-conjugated or sometimes that g is a factor of f. The map h is called the semi-conjugation map or the factor map.*

Semi-conjugacy is a weaker form of topological conjugation. One of the consequences of admit h be only a continuous surjection is that, in contrast of conjugacy, semi-conjugacy is not a equivalence relation. Moreover, properties which are preserved by conjugacy can be preserved by semi-conjugacy, but not with the same "precision" as in conjugacy. To be more explicit let us analyze the changes in the proposition 2.1.2 if we accept h be a semi-conjugacy.

One can reproduce the argument of proposition 2.1.2 step by step with no failure, but the remark of end of proposition cannot be reproduced. Indeed, if h is only a surjection we can only guarantee that is x is a periodic point of f then $h(x)$ is a periodic point of g whose period is not necessarily equal to the period of x.

2.2 Transitive, Weakly Mixing and Mixing Systems

In this section we continue the discussion started in chapter one about transitivity and topologically mixing. We begin defining an intermediate property between topological transitivity and topological mixing.

Definition 2.2.1. *Let (x, f) be a TDS. We say that f is a topologically weakly mixing map if $f \times f$ is transitive in X^2.*

Let us explain what we mean by "intermediate property". We first notice that every topologically mixing map is a weakly mixing map. Indeed, let $U, V \subset X^2$ be open sets and consider $U_1 \times U_2 \subset U$ and $V_1 \times V_2 \subset V$ be basic open sets. Since f is topologically mixing there are n_1 and n_2 such that $f^i(U_1) \cap V_1 \neq \emptyset$ and $f^j(U_2) \cap V_2 \neq \emptyset$ for every $i \geq n_1$ and $j \geq n_2$. If we take $N = \max\{n_1, n_2\}$, we have $(f \times f)^N(U) \cap V \neq \emptyset$. That is, f weakly mixing.

On the other hand, weakly mixing implies transitivity. Indeed, let U, V be non-empty open subsets of X. Since the f is weakly mixing there is a natural n such that $(f \times f)^n(U \times U) \cap V \times V \neq \emptyset$. Thus $\mathcal{N}(U, V) \neq \emptyset$ and f is transitive.

In our previous discussion we proved that weakly mixing implies transitivity. But it is not the only consequence of weakly mixing. Indeed, weakly mixing implies a property stronger than transitivity, namely, total transitivity.

Definition 2.2.2. *A TDS (X, f) is called totally transitive if f^n is topologically transitive for every n.*

Example 2.2.3. *Let θ be a irrational number. Then the irrational rotation R_θ is totally transitive. Indeed, if n is any natural number, then $R_\theta^n(x) = x + n\theta$, for any $x \in S^1$. Thus $R_\theta^n = R_{n\theta}$. Since $n\theta$ is a irrational number, R_θ is totally transitive.*

Remark: The previous example evidences the existence of transitive map which is not topologically mixing. Indeed, let U, V and W be open intervals of S^1 with length $\frac{1}{8}$ which are equidistribuited on S^1. Since R_θ is transitive, then U must to meets V and W, but since R_θ is an isometry, it cannot happen simultaneously. Therefore R_θ cannot be topologically mixing.

Actually, the previous example displays a map which is not weakly mixing. To see that, we need to keep in mind that $S^1 \times S^1$ is the torus $\mathbb{T}^2$ and that every orbit of $R_\theta \times R_\theta$ lies on line which rational slope. Thus $R_\theta \times R_\theta$ cannot be transitive, since lines with rational slope are not dense on $\mathbb{T}^2$

Proposition 2.2.4. *Let (X, f) be a TDS. If f is weakly mixing, then n-product $f \times .... \times f$ is topologically transitive, for every n.*

Proof. Let U_1, U_2, V_1, V_2 be non empty open subsets of X. Since $f \times f$ is transitive, there exists $n \in \mathcal{N}(U_1, U_2) \cap \mathcal{N}(V_1, V_2)$. Define $U := U_1 \cap f^{-n}(U_2)$ and $V := V_1 \cap f^{-n}(V_2)$. The sets U and V are non-empty open sets. Now, since f is also transitive, there exists k

such that $U \cap f^{-k}(V) \neq \emptyset$. That is $U_1 \cap f^{-n}(U_2) \cap f^{-k}(V_1) \cap f^{-n-k}(V_2) \neq \emptyset$. But this implies $\mathcal{N}(U, V) \subset \mathcal{N}(U_1, V_1) \subset \mathcal{N}(U_1, V_2)$.

Using a induction method we can prove that for every n, if $U_1, ..., U_n, V_1, ..., V_n$ are non-empty open sets, there exists U, V such that $\mathcal{N}(U, V) \subset \mathcal{N}(U_1, V_1) \subset ... \subset \mathcal{N}(U_n, V_n)$. This is enough to prove that the n product $f \times ... \times f$ is topologically transitive, since $\mathcal{N}(U, V)$ is always non-empty. □

Corollary 2.2.5. *A weakly mixing TDS is totally transitive.*

Proof. Let f be a weakly mixing map. Let U_1, U_2, V_1, V_2 be non empty subsets of X and fix $n > 0$. Consider the sets

$$U = U_1 \times f^{-1}(U_1) \times ... \times f^{-n+1}(U_1) \times V_1 \times f^{-1}(V_1) \times ... \times f^{-n+1}(V_1)$$

and

$$V = U_2 \times ... \times U_2 \times V_2 \times ... \times V_2$$

where we have n products of U_2 and n products of V_2.

Since f is weakly mixing then f is transitive and this implies that for every point open set U, $f^{-1}(U) \neq \emptyset$. Indeed, this is guaranteed by the existence of a point with dense orbit. Thus U is a non-empty subset of X^{2n}.

The previous proposition implies that the $2n$ product of f is transitive. Then there is k such that $f^{k+i}(V_1) \cap V_2 \neq set$ $f^{k+i}(U_1) \cap U_2 \neq set$ for every $i = 0, ..., n$. Choose i such that $k + i = np$ for some natural p. Then $(f \times f)^{np}(U_1, U_2) \cap (V_1, V_2) \neq \emptyset$. Thus f^n is weakly mixing and therefore f^n is transitive. □

The next proposition will be useful in future to prove one of the mains theorems of chapter three. Therefore we shall give a proof for it now.

Proposition 2.2.6. *Let (X, f) be a TDS. If for every U, V non-empty sets we have $\mathcal{N}(U, U) \cap \mathcal{N}(U, V) \neq \emptyset$ then f is topologically weakly mixing.*

Proof. Let U_1, U_2, V_1, V_2 be non-empty subsets of X. Since $\mathcal{N}(U, V) \neq \emptyset$ for every U, V then f is transitive. Thus, choose n, k such that $U := U_1 \cap f^{-n}(U_2) \cap f^{-k}(V_1) \neq \emptyset$ and define $V = f^{-n-k}(V_2)$.

Then if $m \in \mathcal{N}(U, U) \cap \mathcal{N}(U, V)$ we have that $m + k \in \mathcal{N}(U_1, V_1) \cap \mathcal{N}(U_2, V_2)$ and this implies f is weakly mixing. □

We end this section proving that transitivity, weakly mixing, mixing, and minimality are dynamical properties.

Theorem 2.2.7. *Let (X, f) and (Y, g) be TDS and let h be a conjugacy between f and g. If f is transitive, then g is transitive.*

Proof. Let x be a transitive point for f. We claim that $h(x)$ is a transitive point of g. Indeed, let U be an non-empty subset of Y. Since h is a homeomorphism, then $h^{-1}(U)$ is a non-empty open set. Since x has dense orbit, there exists n such that $f^n(x) \in h^{-1}(U)$. Thus $h(f^n(x)) = g^n(h(x)) \in U$. Therefore g is transitive. $\square$

Theorem 2.2.8. *Let (X, f) and (Y, g) be TDS and let h be a conjugacy between f and g. If f is weakly mixing, then g is weakly mixing.*

Proof. In order to prove this this theorem we consider $h' = h\times h$, $f' = f\times f$ and $g' = g\times g$. Since h is a homeomorphism, the h' is also. Furthermore, if $(x_1, x_2) \in X \times X$ then

$$h'(f'(x_1, x_2)) = (h(f(x_1)), h(f(x_2))) = (g(h(x_1)), g(h(x_2))) = g'(h'(x_1, x_2))$$

This shows that h' is a topological conjugacy between f' and g'. Thus we just need to apply the previous result to f' and g'. $\square$

In the proof of theorem 2.2.7, we saw that topological conjugacy carries the dense orbits of f into dense orbits of g. This implies the following result

Theorem 2.2.9. *Let (X, f) and (Y, g) be TDS and let h be a conjugacy between f and g. If f is minimal, then g is minimal.*

Theorem 2.2.10. *Let (X, f) and (Y, g) be TDS and let h be a conjugacy between f and g. If f is topologically mixing, then g is topologically mixing.*

Proof. Let U and V be non-empty subset of Y. Chose $y \in U$ and $y' \in V$ and $\varepsilon > 0$ such that $B^Y_\varepsilon(y) \in U$ and $B^Y_\varepsilon(y') \in V$. Since h is uniformly continuous chose $\delta > 0$ such that $h(B^X_\delta(h^{-1}(y)) \subset B^Y_\varepsilon(y)$ and $h(B^X_\delta(h^{-1}(y')) \subset B^Y_\varepsilon(y')$

Since f is topologically mixing there exists N such that for every $n \geq N$ there is a point $x_n \in B^X_\delta(h^{-1}(y))$ such that $f^n(x_n) \in B^X_\delta(h^{-1}(y'))$. But this implies $h(x_n) \in B^Y_\varepsilon(y)$ and $h(f^n(x_n) = g^n(h(x_n)) \in B^Y_\varepsilon((y'))$ for every $N \geq n$. Thus g is topologically mixing. $\square$

2.3 Expansive Systems

In this section we will define expansive systems. Roughly speaking an expansive system is a system that every pair of different points must to move away some time.

Definition 2.3.1. *Let f be a TDS.*

- *We say that f is positively expansive if there exists a positive constant e such that for every different points x and y we have $d(f^i(x), f^i(y)) > e$ for some positive integer i.*
- *If f is a homeomorphism, We say that f is negatively expansive if there exists a positive constant e such that for every different points x and y we have $d(f^i(x), f^i(y)) > e$ for some negative integer i.*

- *If f is a homeomorphism, We say that f is expansive if there exists a positive constant e such that for every different points x and y we have $d(f^i(x), f^i(y)) > e$ for some integer i.*

The constant e is called expansive constant of f.

Example 2.3.2. *Let $C > 1$. The map $f : S^1 \to S^1$ defined by $f(x) = Cx$ is an positively expansive map. Indeed, take $e = \frac{1}{8}$. If $d(x, y) > 0$ then $d(f^n(x), f^n(y)) = C^n d(x, y) > e$ for some n. Then f is expansive.*

Notice that expansiveness looks very similar to sensitive dependence. But they are not the same property.

Theorem 2.3.3. *Let X be a compact metric space without isolated points and let (X, f) be a TDS. If f is positively expansive, then it has sensitive dependence on initial conditions.*

Proof. Let e be the expansive constant of f. We claim that it is a suitable sensitive constant for f. Since f has no isolated points every neighborhood of x has a point y different from x. Then the positive expansiveness of f implies that there is a natural i such that $d(f^i(x), f^i(y)) > e$ and that is the condition for sensitivity. □

On the other hand, the next example shows a map which is sensitive, but it is not positively expansive.

Example 2.3.4. *Let $X = [0, 1]$ and let $\{x_i\}_{i\in\mathbb{Z}}$ be a sequence of points such that $\lim_{i\to\infty} x_i = 1$ and $\lim_{i\to-\infty} x_i = 0$. Let I_i be the interval $[x_i, x_{i+1}]$. We define a map f from X to X as follows. For each I_i define $f_i : I_i \to I_{i-1} \cup I_i \cup I_{i+1}$ such that*

$$f_i(x_i) = x_i,\ f_i(x_{i+1}) = x_{i+1},\ f(\frac{2x_i + x_{i+1}}{3}) = x_{i+2} \text{ and } f(\frac{x_i + 2x_{i+1}}{3}) = x_{i-1}$$

Let f_i be linear between the whose it is already defined. Now define f by parts, setting $f|I_i = f_i$.

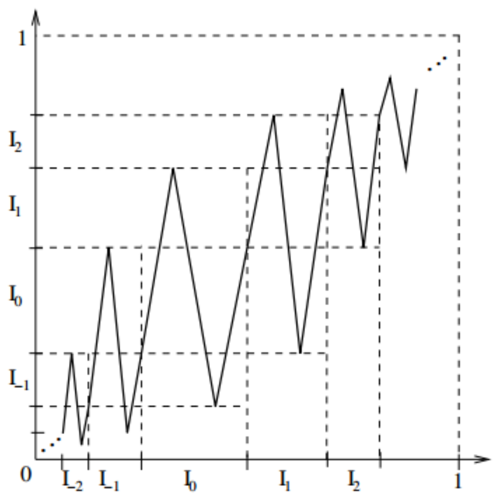

Figure 2.1: The Graphics of f

Since f is piecewise linear, it must to be continuous and therefore $f(0) = 0$. Now, if we take any $e > 0$, there exists a fixed point $x_k \in (0, e)$ and therefore f cannot be expansive.

To see that f is sensitive, we notice that the length of any open sub-interval of I_i grows to the length I_i at some time. This implies the sensitivity of f.

The next theorem is a property of positively expansive maps whose we will enunciate without proof. In chapter four we will prove a pointwise version, which trivially implies it. Their proofs can be found in [2] anyway. We remark that there are similar results and proofs if f is negatively expansive or expansive homeomorphism.

Theorem 2.3.5. *f is positively (resp. negatively) expansive if, and only if, for every f^k is positively (resp. negatively) expansive for every k.*

Proof. [2] Theorem 2.2.4, pag.38 □

Expansiveness is a dynamical property as we can see in next theorem.

Theorem 2.3.6. *Let (X, f) and (Y, g) be TDS. Let h be a topological conjugacy between f and g. If f is positively (reps. negatively) expansive, then g is positively (reps. negatively) expansive.*

Proof. We shall prove for the case of f be a positively expansive map. The proof for the negatively expansive case is quite analogous. Let e be the constant of expansiveness of f. Let d^X and d^Y denote the metrics on X and Y respectively. Since Y is compact, h^{-1} is uniformly continuous. Thus there exists $\delta > 0$ such that $d^Y(x, y) < \delta$ implies $d^X(h^{-1}(x), h^{-1}(y)) < e$.

Now suppose g is not positively expansive. Thus if $0 < e' < \delta$ there are different point $y, y' \in Y$ such that $d^Y(g^n(y), g^n(y')) < e'$ for every $n \geq 0$. Thus

$$d^X(f^n(h^{-1}(y)), f^n(h^{-1}(y'))) = d^X(h^{-1}(g^n(y)), h^{-1}(g^n(y'))) < e, \text{ for every } n \in \mathbb{N}$$

But this implies that f is not positively expansive, since $h^{-1}(y) \neq h^{-1}(y')$ and this is a contradiction. □

Example 2.3.7. *If f is the expansive map of example 2.3.2, then it is not conjugated to a rotation R_θ. Indeed, otherwise the rotation could be expansive systems. But this cannot be true, because for every $e > 0$ and $x, y \in S^1$ different point such that $d(x, y) = c < e$, we have $d(R_\theta(x), R_\theta(y)) = c < e$.*

The next theorem shows that if a system is expansive, then any of its subsystems is also expansive.

Theorem 2.3.8. *Let (X, f) be a TDS. If f is positively (resp. negatively) expansive and $Y \subset X$ is a closed invariant subset, then $(Y, f|_Y)$ is positively (resp. negatively) expansive.*

Proof. Suppose f is positively expansive and let e be the constant of expansiveness of f. If x and y are different points of Y, then they are different points of X. Since Y is invariant then every iterated of x and y belongs to Y. Now since f is expansive there exists an integer i such that $d(f^i(x), f^i(y)) > e$. Therefore $(Y, f|_Y)$ is expansive. The proof for the negatively expansive case is analogous. □

The product of positively expansive maps is positively expansive too.

Theorem 2.3.9. *Let (X, f) and (Y, g) be TDS. If f and g are positive expansive, then the product $f \times g$ is positive expansive in the space $X \times Y$ endowed with the product metric $d((x_1, y_1), (x_2, y_2)) = \max\{d^X(x_1, x_2), d^Y(y_1, y_2)\}$.*

Proof. Let e_f and e_g be the constants of expansiveness of f and g respectively. Set $e = \min\{e_f, e_g\}$. If (x_1, y_1) and (x_2, y_2) are different points of $X \times Y$, then $x_1 \neq x_2$ or $y_1 \neq y_2$. In the first case there is i such that $d^X(f^i(x_1), f^i(x_2)) > e_1 > e$ then

$$d((f^i(x_1), g^i(y_1)), (f^i(x_2), g^i(y_2)) > e$$

In the second case there is j such that $d^Y(g^j(y_1), g^j(y_2)) > e_1 > e$. Thus

$$d((f^j(x_1), g^j(y_1)), (f^j(x_2), g^j(y_2)) > e$$

and this concludes the proof. □

2.4 Shadowing Property

In this section we are aiming to study the shadowing property for topological dynamical systems. For a moment, suppose we have a topological dynamical system. In addition, suppose we need to compute some orbits with a specific behavior, but the complexity of the system does not allow us do it directly. It could be a tragic scene. In despite of it, suppose that we can compute a set which is somewhat similar to the orbits that we need to compute. Would it be reasonable expect the existence of real orbits approximating this set? For some kind of systems we can get a positive answer to the above question. The property which allow us to do this approximations is called shadowing property.

To define the systems whose dynamical behavior includes this property, we begin defining a kind sets which is not a real orbit, but has a similar behavior. If $x_n = f^n(x)_{n \in N}$ is an orbit of some point x, we have $f(x_{n-1}) = x_n$, which is equivalent to $d(f(x_{n-1}), x_n) = 0$. Instead of it, we shall admit a weaker condition on $\{x_n\}_{n \in \mathbb{N}}$.

Definition 2.4.1. *A sequence of points $(x_n) \in X$ is called a δ-pseudo orbit of a map f if $d(f(x_{n-1}), x_n) < \delta$ for every index n.*

δ-pseudo orbits are sets which are somehow similar to orbits. Next we shall give the meaning of an orbit of f "approximate" a pseudo-orbit.

Definition 2.4.2. *We say that a point $x \in X$ ε-shadows the δ-pseudo orbit (x_n) if $d(f^n(x), x_n) < \varepsilon$ for every n.*

Finally, we define the property that we are looking for in the above discussion.

Definition 2.4.3. *A TDS (X, f) is said to have the shadowing property if for every $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo orbit of f is ε-shadowed by some point in X.*

Remark: In literature shadowing property is often called pseudo-orbit trancing property, or abbreviating, P.O.T.P.

Remark: If f is a homeomorphism, then in order to define the shadowing property for f we just need change to $\mathbb{Z}$ the index set of the pseudo-orbits and imitate the above definitions.

Remark: Suppose for a moment that f is a homeomorphism. We claim that in order to assure that f the shadowing property, we just need to proof that for every $\epsilon > 0$ there exists $\delta > 0$ such that every $\{x_n; n \geq 0\}$ δ-pseudo orbit is ϵ shadowed by some point in X. Indeed, let $\varepsilon > 0$ and $\delta >$ such that every δ-pseudo-orbit is $\frac{\varepsilon}{2}$ -shadowed. Let $\{x_i\}_{i\in\mathbb{Z}}$ be a δ-pseudo-orbit. For each n, define $y_n^i = x_{n-i}$ for every $i \geq 0$. Let y_n be a point that ϵ shadows $\{y_n^i\}$. Since X is compact, we can suppose $y_n \to y$, up to a sub-sequence. now, $d(f^j(y), f^j(x_i)) < d(f^j(y), f^j(y_n^i)) + d(f^j(y_n^i), f^j(x_i))$ for every j. Then for any j we can take n large enough, to conclude $d(f^j(y), f^j(x_i)) < \varepsilon$. This proves our claiming.

One can wonder if there exists maps with the shadowing property. The next theorems shows us that the simplest map ever can have the shadowing property depending on the space where it is defined.

Theorem 2.4.4. *Let X be a compact metric space with more that one point. Then the identity map on X has the shadowing property if, and only if, X is totally disconnected.*

Proof. Suppose X is totally disconnected. Let $\varepsilon > 0$, then there exists a finite cover $U_1, ..., U_n$ of X such that $U_i \cap U_j = \emptyset$ if $i \neq j$ and every U_I has diameter smaller than ε. Now chose $\delta < \min\{d(U_i), (U_j); i \neq j\}$. Then every δ-pseudo orbit is entirely contained in some U_i. and therefore any point of U_i ε-shadows it.

Conversely if X is not totally disconnected there exists $x_0 \in X$ such that its connected component $C(x_0)$ has more that one point. Let ε' be the diameter of $C(x_0)$. Since X is compact, there exist points x and y in $C(X)$ such that $d(x, y) = \varepsilon'$. Let $\varepsilon = \frac{\varepsilon'}{4}$.Now, for every $\delta > 0$ we can cover $C(x_0)$ with a finite number of open sets $U_1, ..., U_{n_\delta}$ with diameter δ. Since $C(x_0)$ is connected then we can assume $x \in U_1$, $y \in U_{n_\delta}$ and $C(x) \cap U_i \cap U_{i+1} \neq \emptyset$ for every i, reordering the $U_i's$ if necessary.

Let $\delta < \varepsilon$. The set $\{x, x_1, ..., x_{n_\delta - 1}, y\}$ where x_i is any point of $C(x) \cap U_i \cap U_{i+1}$ is clearly a δ-pseudo-orbit. But it cannot be ε-shadowed for any point in X, by the choice of δ. □

Now let us explore some properties of maps with the shadowing property. We remark that the next theorems are trivial consequences of the pointwise versions of them. These will be stated and proved in chapter three. Thus in this chapter we will not give a proof for the following theorems. However, we point out that the proofs can be found in [2].

Lemma 2.4.5. *Let (X, f) be a TDS and let k be a positive integer. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that every $\{x_n\}_{n=0}^k$ finite δ-pseudo orbit for f is ε-traced by x_0.*

Proof. In chapter three we will present a proof for a pointwise version of this theorem which trivially will imply it. □

Theorem 2.4.6. *Let (X, f) be a TDS and let k be a positive integer. Then f has the shadowing property if and only if f^k has the shadowing property.*

Proof. [2]Theorem 2.3.3 pag. 80. □

Theorem 2.4.7. *Let (X, f) be a TDS. If f is a homeomorphism and has the shadowing property, then f^1 has the shadowing property.*

Proof. [2]Theorem 2.3.4 pag. 80. □

We now will prove that f is dynamical invariant.

Theorem 2.4.8. *Let (X, f) and (Y, g) be TDS. If f and g are conjugated and f has the shadowing property, then g has the shadowing property.*

Proof. Let $\varepsilon > 0$ and let $h : X \to Y$ be the conjugation between f and g. Since X is compact, there exists $\eta > 0$ such that $h(x_1, x_2) < \delta$ if $d(x_1, x_2) < \eta$. let $\eta' > 0$ be given by the η-shadowing of f. Since Y is compact, there exists $\delta > 0$ such that $h^{-1}(y_1, y_2) < \eta'$ if $d(y_1, y_2) < \delta$.

If $\{y_n\}_{n\in\mathbb{N}}$ is an δ-pseudo-orbit of g then $d(g(y_n), y_{n+1}) < \delta$ for every n and therefore

$$d(h^{-1}(g(y_n)), h^{-1}(y_{n+1})) = d(f(h^{-1}(y_n)), h^{-1}(y_{n+1})) < \eta'.$$

Thus $\{h(y_n)\}_{n\in\mathbb{N}}$ is a η'-pseudo-orbit of f. Then there exists x such that $d(f^n(x), h^{-1}(y_n)) < \eta$ for every n. Thus $d(h(f^n(x)), y_n) = d(g^n(h(x)), y_n) < \varepsilon$ for every n. Therefore g has the shadowing property. □

We end this section proving two facts that will reveal to be useful to us in the next chapters.

Definition 2.4.9. *Let (X, f) be a TDS.*

- *f is called a chain-transitive map if for every $\varepsilon > 0$ and any pair x, y of points of X, there exists $\{x_i\}_{i=0}^n$ an ε-pseudo-orbit of f such that $x_0 = x$ and $x_n = y$*
- *f is called a chain-mixing map if for every $\varepsilon > 0$ and any pair x, y of points of X, there exists K_0 such the for every $k \geq K - 0$ the exists $\{x_i\}_{i=0}^k$ an ε-pseudo-orbit of f such that $x_0 = x$ and $x_k = y$.*

Proposition 2.4.10. *Let (X, f) be a TDS.*

1. *If f is topologically transitive, then f is chain-transitive*
2. *If f is topologically mixing, then f is chain-mixing*

Proof. 1. Suppose f is transitive. Let $\varepsilon > 0$ and take $x, y \in X$. By uniform continuity let $0 < \delta < eps$ such that $d(z, z') < \delta$ then $d(f(z), f(z')) < \varepsilon$. Since f is transitive, there exists n and a point $p \in B_\delta(x)$ such that $f^n(x_n) \in B_\delta(y)$. Now, $d(f(x_n), f(x_n)) < \varepsilon$ and therefore the set $\{x, f(x_n), f^2(x_n), ..., f^{n-1}(x_n), y\}$ is ε-pseudo-orbit from x to y. Thus f is chain-transitive.

2. To prove 2, we repeat the previous argument noticing that since f is topologically mixing, there exists K such that for every $n \geq K$ there exists a x_n as in the proof of 1.

□

Next we will show that the converse for the previous proposition holds if we assume that f has the shadowing property.

Theorem 2.4.11. *Let (X, f) be a TDS with the shadowing property.*

1. *If f is chain-transitive, then f is topologically transitive*
2. *If f is chain-mixing, then f is topologically mixing.*

Proof. 1. Let U and V be non-empty open subsets of X. Let $x \in U$, $x \in V$ and $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$ and $B_\varepsilon(y) \subset V$. Let $0 < \delta < \varepsilon$ be given by shadowableness of f. Since f is chain-transitive, there exists $\{x_0 = x, x_1, ..., x_n = y$ a δ-pseudo-orbit of f. Since f has the shadowing property, there exists p such that $d(f^i(p), x_i), \varepsilon$ for $i = 0, ..., n$. Thus $p \in U$ and $f^n(p) \in V$ and therefore f is topologically mixing.

2. To prove 2, we proceed as in the previous proof noticing that since f is chain-mixing there exists k such that if $n \geq K$ there exists $\{x_0, ..., x_n\}$ a δ-pseudo orbit of f.

□

2.5 Topological Entropy

In this section we will deal with another dynamical property which is called topological entropy. Roughly speaking, topological entropy is a number which measures in average how the number of orbits that move away grows exponentially. In order to turn the its meaning more natural, we shall define topological entropy in steps.

First we define the dynamical balls. Let f be a TDS and fix n. Define

$$d_{n,f}(x, y) = \max_{0 \leq i \leq n}\{d(f^i(x), f^i(y))\}$$

When there is no confusion about the map f, we simplify the notation writing only d_n. We first remark that d_n is a metric. Indeed $d_n(x, y) = 0$ if, and only if, $d(x, y) = 0$ and if, and only if, $x = y$. The symmetry of d_n is obvious from the symmetry of d.

To prove the triangular inequality we notice that for every $x, y, z \in X$ and every i

$$d(f^i(x), f^i(z)) \leq d(f^i(x), f^i(y)) + d(f^i(y), f^i(z))$$

therefore

$$\max_{0\leq i\leq n} d(f^i(x), f^i(z)) \leq \max_{0\leq i\leq n} d(f^i(x), f^i(y)) + \max_{0\leq i\leq n} d(f^i(y), f^i(z))$$

and this completes the proof.

The metrics d_n and d are equivalents. Indeed, Let $B^n_\varepsilon(x)$ denote the open ball of radius ε in the metric d_n. Since $y \in B^n_\varepsilon(x)$ implies $d(x, y) < \varepsilon$ then $B^n_\varepsilon(x) \subset B_\varepsilon(x)$. On the other hand, since f is uniformly continuous, f^i is also, for every i. Thus there exists $\delta > 0$ such that if $d(x, y)$ then $d(f^i(x), f^i(y)) < \varepsilon$ for every $0 \leq i \leq n$. Then $B_\delta(x) \subset B^n_\varepsilon(x)$. This proves that the d and d_n are equivalents.

The balls $B^n_\varepsilon(x)$ is called the n-dynamical ball centered in x with radius ε.

Definition 2.5.1. *Let (X, f) be a TDS.*

- *We say that a subset F is a (n, ε)-generator for X, if $\{B^n_\varepsilon(x); x \in F\}$ form a covering for X.*
- *We say that a subset of E is (n, ε)-separated if $B^n_\varepsilon(x) \cap (F \setminus \{x\}) = \emptyset$ for every $x \in F$.*

Let $r_n(\varepsilon)$ denote the minimal cardinality of the (n, ε)-generators of X and let $s_n(\varepsilon)$ denote maximal cardinality of the (n, ε)-separated subsets of X. Since X is a compact space with the metric d_n these numbers are aways finite.

Now we define $r(\varepsilon) := \limsup_{n\to\infty} \frac{1}{n} \log r_n(\varepsilon)$ and $s(\varepsilon) := \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon)$.

Let us say a few words about the last definition. Roughly speaking $r(\varepsilon)$ computes in exponential average the minimum amount of orbits needed to go along with any orbit of the system within a distance ε. On the other hand, $s(\varepsilon)$ computes in exponential average the maximum amount of orbits which are at least ε alway from each other.

Proposition 2.5.2. *Let (X, f) be a TDS. Then $r(\varepsilon) \leq s(\varepsilon) \leq r(\frac{\varepsilon}{2})$ for every $\varepsilon > 0$.*

Proof. Fix $\varepsilon > 0$ and n. Let E be a (n, ε)-separated of maximal cardinality. If E is not a (n, ε)-generator of X then there is a point $x \in X$ such that $d_n(x, y) > \varepsilon$, contradicting the maximality of E. Then $r(\varepsilon) \leq s(\varepsilon)$.

On the other hand, let F be a $(n, \frac{\varepsilon}{2})$-generator of X and E be a (n, ε)-separated subset of X. For each $x \in E$ chose a point $y \in F$ such that $d_n(x, y) < \frac{\varepsilon}{2}$. For each x, we can find y such as all y are different. Therefore $\#E \leq \#F$ and this implies $s_n(\varepsilon) \leq r_n(\frac{\varepsilon}{2})$.

Thus we have

$$r_n(\varepsilon) \leq s_n(\varepsilon) \leq r_n(\frac{\varepsilon}{2})$$

Therefore if we take the superior limits in the previous inequality we conclude the proposition. □

Finally we are able to define the topological entropy of a map f. We only need to remove the error ε in the previous amounts.

Definition 2.5.3. *Let (X, f) be a TDS. The amount $h(f)$ defined by*

$$h(f) = \lim_{\varepsilon \to 0} r(\varepsilon) = \lim_{\varepsilon \to 0} s(\varepsilon)$$

is called the topological entropy of f.

A remark on the definition is that the previous limit is increasing, then it always exists and it can be infinity.

Example 2.5.4. *Let X be a compact metric space and consider the identity map $Id : X \to X$. Let us calculate the topological entropy of Id. Fix $\varepsilon > 0$ and a natural n. Since $f^n(x) = x$ for every $x \in X$ then $d_n(x, y) = d(x, y)$ for every $x, y \in X$. Thus $s_n(\varepsilon)$ has the same value for every n. Therefore $s(\varepsilon) = \limsup_{n\to\infty} \frac{1}{n} s_n(\varepsilon) = 0$ for every $\varepsilon > 0$. Therefore $h(id) = 0$.*

More generally, we have the following result.

Theorem 2.5.5. *Let (X, f) be a TDS. If f is a isometry, that is, $d(x, y) = d(f(x), f(y))$ for every $x, y \in X$, then $h(f) = 0$.*

Proof. In order to proof this theorem we only notice that $d_n(x, y) = d(x, y)$ for every $x, y \in X$ since f is an isometry. Therefore we just need to repeat the argument in the previous example. □

In the same way, we can prove that the entropy of f is independent of the choice of equivalent metrics for X.

Theorem 2.5.6. *Let (X, f) be a TDS. If d' is a metric equivalent to the metric d. Then the entropy of (X', f) is equal to the entropy of (X, f). Here X' denotes the space (X, d').*

The previous proof is quite analogous to the proof of the theorem 2.5

Next we will discuss some properties of topological entropy which will be very useful to us in the next chapters.

Theorem 2.5.7. *Let (X, f) be a TDS. If $Y \in X$ is a closed invariant subset, then $h(f|Y) \leq h(f)$.*

Proof. Fix $\varepsilon > 0$ and a natural n. Let E be a (n, ε)-separated subset of Y with maximum cardinality. Then F is a (n, ε)-separated subset of X. Thus $s_n(\varepsilon) \leq s'_n(\varepsilon)$ where $s'_n(\varepsilon)$ is the maximal cardinality (n, ε)-separated subsets o X. Then $r(\varepsilon) \leq r'(\varepsilon)$ for every $\varepsilon > 0$ and therefore $h(f|Y) \leq h(f)$. □

The next theorems gives a useful tool to calculate the topological entropy of some systems.

Theorem 2.5.8. *Let (X, f) be a TDS. Then $h(f^n) = nh(f)$, for any $n > 0$.*

Proof. before to start the proof let us fix some notation. Here, the prefix (n, ε, g) will denote the (n, ε)-generator (or separated) subsets of X with respect to g.

Let $\varepsilon > 0$ and let F be a (nm, ε, f)-generator set for X with minimal cardinality. Then for every $x \in X$ there exists $x' \in F$ such that $d(f^i(x), f^i(x') < \varepsilon$ for every $i = 1, ..., nm$. Thus E is (m, ε, f^n)-generator for X and this implies $r_m(f^n, \varepsilon) \le r_n m(f, \varepsilon)$. Therefore

$$r(\varepsilon, f^n) = \limsup_{m\to\infty} \frac{1}{n} \log r_n(\varepsilon) \le n \limsup_{m\to\infty} \frac{1}{mn} \log \le n \limsup_{m\to\infty} \frac{1}{m} \log r_m(\varepsilon) = nr(f, \varepsilon)$$

Therefore $h(f^n) \le nh(f)$.

Now since f is uniformly continuous there exists $0 < \delta \le \varepsilon$ such that for every $d(f^i(x), f^i(y)) < \varepsilon$ for every $i = 0, ..., n$. Thus a (m, δ, f^n)-generator set for X with minimal cardinality must to be (nm, ε, f)-generator for X. Thus $mr_n(f, \delta) \le r_m(f^n, \varepsilon)$ and therefore $nr(\delta, f) \le r(\varepsilon, f^n)$. This implies the contrary inequality. □

Next we will prove that topological entropy is a dynamical property.

Theorem 2.5.9. *Let (X, f) and (Y, g) be TDS. If f and g are topologically conjugated then $h(f) = h(g)$.*

Proof. Let h be the conjugacy map between f and g and fix $\varepsilon > 0$. Since X is compact then h is uniformly continuous. Thus there exists $0 < \delta < \varepsilon$ such that if $d^Y(h(x), h(x')) < \varepsilon$ when $d^X(x, x') < \delta$. Let F be a $(n, \frac{\delta}{2})$-generator of X with minimal cardinality. Since h is bijective for every $y \in Y$ there is $x \in X$ such that $h(x) = y$. Since F is a generator for X there is $x' \in F$ such that $d(f^i(x), f^i(x')) < \frac{\delta}{2}$ for every $0 \le i \le n$. Thus $d^Y(h(f^i(x)), h(f^i(x'))) = d^Y(g^i(h(x)), g^i(h(x'))) < \varepsilon$ for every $0 \le i \le n$. In other words $h(F)$ is an (n, ε)-generator for Y. Therefore $r_n(\varepsilon, g) \le r_n(\frac{\delta}{2}, f)$ for every n. Since $\frac{\delta}{2} \to 0$ when $\varepsilon \to 0$ we conclude that $h(g) \le h(f)$.

The contrary inequality is proved in a totally analogous way. □

Notice that in the previous proof if the conjugacy is only a surjection we can prove only one of the inclusions. Indeed, the proof of the previous theorem gives the following corollary.

Corollary 2.5.10. *Let (X, f) and (Y, g) be TDS. If f is semi-conjugated to g then $h(g) \le h(f)$.*

2.6 The Shift Map

We finalize this chapter with an example of map with positive topological entropy which will be used in one of the main theorems of this work. However this maps presents a very rich dynamical behavior. For that reason we will pays special attention to it.

Let Σ be set of all functions from the non-negative integers to the set $\{0,1\}$. We can visualize this set as the set of the unilateral infinite sequences of zeros and ones.

$$\Sigma = \{(s_i)_{i=0}^{\infty}; s_i \in \{0,1\}\}$$

Consider the following map $d : \Sigma \times \Sigma \to [0,\infty]$ defined by $d(s,t) = \Sigma_{i=0}^{\infty} \frac{1}{2^{i+1}}|s_i - t_i|$. Notice that for every s,t we have that $d(s,t)$ is majored by the geometric series $\Sigma_{i=0}^{\infty} \frac{[1]}{2^i i+1}$ whose sum is 1 and therefore $d(s,t)$ is always well defined.

We claim that d is a metric in Σ. Indeed, the symmetry of d is obvious. If $d(t,s) = 0$ then all the coordinates of s and t must to be equal and then $t = s$. If $s = t$ then $s_i = t_i$ for every i and therefore $d(t,s) = 0$. In order to prove the triangle inequality we fix n and notice that is $r,s,t \in \Sigma$ then

$$\Sigma_{i=0}^{n} \frac{1}{2^{i+1}}|r_i - t_i| \le \Sigma_{i=0}^{n} \frac{1}{2^{i+1}}|r_i - s_i| + \Sigma_{i=0}^{n} \frac{1}{2^{i+1}}|s_i - t_i|$$

and this implies the triangle inequality when n grows to infinity.

Let us make some remarks on (Σ, d). If we use the classical Cantor's diagonal argument, we can prove that Σ is a non-enumerable space. Σ is a compact metric space since it is a product of compact metric spaces. If $d(t,s) < \frac{1}{2^n}$ therefore the n first coordinates must to be equal. This implies that Σ has not isolated point. Indeed, let $s \in \Sigma$ and $\varepsilon > 0$. Then there is a natural n such that $\frac{1}{2^n} < \varepsilon$. So if t is a sequence such that $t_i = s_i$ for every $0 \le i \le n$ and $t_{n+1} \ne s_{n+1}$ we have that t is ε-close to s and $t \ne s$.

Definition 2.6.1. *The map $\sigma : \Sigma \to \Sigma$ defined by $\sigma((s_i)) = (s_{i+1})$ is called the shift map.*

As we have said the shift map has a very complex dynamical behavior.

Theorem 2.6.2. $\overline{Per(f)} = \Sigma$

Proof. A periodic point of σ with period n must to satisfies $\sigma^n(s) = s$. This implies $(s_{n+i} = s_i)$ for every $i \ge 0$. Thus let $s \in \Sigma$ and $\varepsilon > \frac{1}{2^n}$. If $t = (t_i)$ where $t_{i+n} = t_i$ and $t_i = s_i$ for $i = 0, ..., n$ then t is periodic point ϵ-close to s. This proves that the periodic points of σ are dense in Σ. □

Next we will prove that σ is transitive.

Theorem 2.6.3. *σ is a topologically transitive map.*

Proof. Let t be the point of Σ defined as follows. The first coordinates of t are 0 and 1. The next are all the possibles combinations of 0's and 1" with two digits. The next are are all the possibles combinations of 0's and 1" with three digits and so forth. Thus for any $s \in \Sigma$ we can make $\sigma^n(t)$ be so close as we want making n great. Therefore the orbit of t is dense in Σ anf σ is transitive. □

We end this chapter proving that the shift map has positive entropy.

Theorem 2.6.4. *The entropy of the shift map is* $\log 2$

Proof. Fix $\varepsilon < 1$ and $n > 0$. Let $k > 0$ such that $\frac{1}{2^{k+1}} \leq \varepsilon \leq \frac{1}{2^k}$. Let us compute the maximal cardinality of (n, ε)-generator sets for Σ. Fix a point $x \in \Sigma$. If $y \in B^n_\varepsilon(x)$ then the k first coordinates of $\sigma^i(x)$ and $\sigma^i(y)$ must to coincide for every $i = 1, ..., n$. This implies that the $n+k$ first coordinates of x and y must to coincide. Now, with combinatorial analysis we can conclude that there are 2^{n+k} different possible combinations of the $n+k$ coordinates for a point in $x \in \Sigma$.

We have that there exists 2^{n+k} balls $B^n_\varepsilon(x)$ covering Σ where each x has one of the 2^{n+k} possibilities of such combinations. Moreover it is the minimum number of such balls. Indeed, if there are a smaller number, one of the possibilities would be not contemplated and therefore these ball cannot form a cover for Σ.

Then

$$r(\varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log r_n(\varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log 2^{n+k} = \limsup_{n\to\infty} \frac{1}{n}(n+k)\log 2 = \log 2$$

Thus we conclude that $h(\sigma) = \lim_{\varepsilon\to 0} r(\varepsilon) = \log 2$. □

Our previous discussion was entirely focused on the shift map of two symbols. But we remark that we can make modifications on Σ admitting that it is the set of sequences of any finite collection of symbols instead of zeros and ones. The previous properties are still valid with very similar proofs, except the entropy which now assume the value $\log p$, where p is the number of symbols.

On the other, hand we can make modifications on Σ to define the two-sided shift map. To do this we need define Σ as the sequence of bilateral sequences of zeros and ones and make a little adaptation on the metric on Σ. Thus the shift now become a homeomorphism and again all the properties are still valid.

Chapter 3

Pointwise Dynamics

In this chapter we begin to deal with pointwise dynamics. We are aiming to rewrite some properties that we defined globally into local ones. To clarify ideas, let us recall an example. In chapter one, when we defined what is a topologically transitive map, we gave the definitions in terms of a global property. Namely, we said that a map is transitive if for any pair of open sets the iterates of one set meets the other set in some time. On the other hand, we proved an equivalence for that definition in therms of a pointwise property, namely, the existence of a dense orbit. We called transitive point, a point whose orbit is a dense set. Thus, the definition of transitivity cold be rephrased as "A map is topologically transitive if its set of transitive points is non-empty".

We are going to follow this direction. We will define what are minimal points, sensitive points, shadowable points, entropy points and others and give the relations between their sets and topological dynamical properties.

We notice that the definitions and results here can be found in [11] and [13]. For the remainder of this chapter by a topological dynamical system we mean a continuous self-map of a compact metric space.

3.1 Minimal Points

In this section we spend some time discussing the meaning of minimal point. We begin with its definition.

Definition 3.1.1. *Let (X, f) be a TDS. A point $x \in X$ is called minimal if $\overline{O^+(x)}$ is a minimal set. Let $M(f)$ denote the set of minimal points of f.*

It follows immediately from the definition that x if is a minimal point, then $f|_{\overline{O^+(X)}}$ is a minimal subsystem of f. This makes clear why the use of the term minimal point.

If f is a minimal system, then for every $x \in X$ we have $\overline{O^+(x)} = X$ and therefore every point is a minimal point. The converse does not hold as we can see in the next example.

Example 3.1.2. *Let f be any compact metric space and let f be the identity map on X. Since every point in X is a fixed point then it must to be a minimal point for f*

As in the case of fixed points, we have that periodic points are always minimal points. Indeed, if x is a periodic point, then $O^+(x)$ is a finite f-invariant subset of X. Thus $\overline{O^+(x)} = O^+(x)$ and for every point $y \in O^+(x)$ we have $O^+(y) = O^+(x)$. This implies that x is a minimal point.

Proposition 3.1.3. *Let (X, f) be a TDS. Then $x \in M(f)$ if, and only if, $x \in \overline{O^+(y)}$ for every $y \in \overline{O^+(x)}$.*

Proof. If x is minimal point of f, then $f|_{\overline{O^+(x)}}$ is a minimal system. Thus the orbit of y is dense in $\overline{O^+(x)}$, for every $y \in \overline{O^+(x)}$. Therefore $x \in \overline{O^+(y)} = \overline{O^+(x)}$, for every $y \in \overline{O^+(x)}$

Conversely, suppose $x \in \overline{O^+(y)}$. If $y \in \overline{O^+(x)}$, then $\overline{O^+(x)} \subset \overline{O^+(y)}$ since the orbit of y is closed invariant set. On the other hand, we have trivially the contrary inclusion. Thus the orbit of y is dense on $\overline{O^+(x)}$ an this proves the minimality of x. □

Since the restriction of f to the closure of the orbit of a minimal point is a minimal system, in particular, this point needs to meet each of its neighborhoods in some time. A point satisfying this condition is called a recurrent point.

Definition 3.1.4. *Let (X, f) be a TDS. A point x is called a recurrent point of f if*

$$\mathcal{N}(x, U) := \{n \in \mathbb{N}; f^n(x) \in U\} \neq \emptyset$$

for every open neighborhood U of x. Let $R(f)$ denote the recurrent points of f.

As examples of recurrent points we have periodic points and transitive points. Minimal point are also recurrent points. Indeed, the next theorem shows that minimal points are syndetically recurrent points.

Theorem 3.1.5. *Let (X, f) be a TDS. Then $x \in M(f)$ if, and only if, $\mathcal{N}(x, U)$ if is syndetic for every neighborhood U of x.*

Proof. Suppose x is a minimal point and let U be a neighborhood of x. If $\mathcal{N}(x, U)$ is not syndetic, then for every k there exists an integer n_k such that $f^i(x) \notin U$ for every $i = n_k, ..., n_k + k$. Since X is compact we can suppose that the sequence $f^{n_K}(x)$ converges to a point y, when k tends to infinity. Since U is open then $y \notin U$. Fix $i > 0$, $f^i(y) = \lim_{k\to\infty} f^{n_k+i}(x)$ for every i. Thus $f^i(y) \notin U$. Then $x \notin \overline{O^+(y)}$ and this contradicts the previous proposition.

Conversely, suppose $\mathcal{N}(x, U)$ is syndetic for every U. Fix $\varepsilon > 0$ and let k be a limitation for gaps of $\mathcal{N}(x, B_\varepsilon(x))$. Take $y \in \overline{O^+(x)}$. Since f is uniformly continuous there exists $\delta > 0$ such that $f^i(B_\delta(z)) \subset B_\varepsilon(f(z))$ for every $z \in X$ and $j = 1, ..k$. Since $y \in \overline{O^+(x)}$ there exists N such that $d(f^N(x), y) < \delta$. Thus $d(f^{N+i}(x), f^i(y)) < \varepsilon$ for every $i = 1, ..., k$. But the triangle inequality implies $d(x, f^i(y)) \leq d(x, f^{N+i}(x)) + d(f^{N+i}(x), f^i(y)) < 2\varepsilon$ for some i between 1 and k. This implies $x \in \overline{O^+(y)}$ and by the previous proposition $x \in M(f)$. □

Corollary 3.1.6. *Every minimal systems is syndetically transitive*

Proof. Let U and V be non-empty open sets. Since every orbit is is dense, for any $x \in U$ there exist N such that $f^N(x) \in V$. Now, Since V is an open neighborhood of $f^N(x)$ and $f^N(x)$ is a minimal point, then $\mathcal{N}(f^N(x), V)$ is a syndetic set. But this implies that $\mathcal{N}(U, V)$ is a syndetic set. Therefore f is syndetically transitive. □

In chapter one we gave an example of map which if transitive but not minimal. In this case, $\mathcal{N}(U, U) \neq \emptyset$, for every open set U, since f is transitive, but we cannot assume that every point in U is recurrent. Indeed, there exists a larger class of points, namely, non-wandering points.

Definition 3.1.7. *Let (X, f) be a TDS. A point x is called a non-wandering point if $\mathcal{N}(U, U) \neq \emptyset$ for every open neighborhood U of x. Let $\Omega(f)$ denote the set of non-wandering points of f.*

We have an trivial hierarchy between the classes of points we have studied until now, namely, $Fix(f) \subset Per(f) \subset M(f) \subset R(f) \subset \Omega(f)$.

The next theorem tells us that the set of non-wandering is well-behaved topologically and dynamically.

Theorem 3.1.8. *$\Omega(f)$ is closed positively f-invariant set.*

Proof. To show that $\Omega(f)$ is closed, let $x_0 \in \overline{\Omega(f)}$. Let $\varepsilon > 0$, then there exists $y \in B_\varepsilon(x_0) \cap \Omega(f)$. Thus let $\delta > 0$ such that $B_\delta(y) \subset B_\varepsilon(x_0)$. Then there exists $x \in B_\delta(y)$ and k such that $f^k(x) \in B_\delta(y)$. Therefore $x_0 \in \Omega(f)$.

Now, let $x \in \Omega(f)$ and fix $\varepsilon > 0$. Since f is uniformly continuous there is a $\delta > 0$ such that $f(B_\delta(x)) \subset B_{\varepsilon(f(x))}$. Since x is non-wandering, there exists $y \in B_\delta(x)$ and k such that $f^K(y) \in B_\delta(x)$. Thus $f(y) \in B_\varepsilon(f(x))$ and $f^{k+1}(y) \in B_\varepsilon(f(x))$. Therefore $f(x)$ is a non-wandering point. □

We end this section with two results similar to the lasts theorems of the previous chapter.

Definition 3.1.9. *A map $f : X \to X$ is called a non-wandering map if $\Omega(f) = X$*

Definition 3.1.10. *Let (X, f) be a TDS. A point $x \in X$ is called a chain-recurrent point of f if for every $\varepsilon > 0$, there exists a ε-pseudo-orbit $\{x_0, ..., x_n\}$ such that $x_0 = x_n = x$. Let $CR(f)$ denote the set of chain-recurrent points of f. The map f is called a chain-recurrent map if $CR(f) = X$.*

Proposition 3.1.11. *Let (X, f) be a TDS. If f is non-wandering, then f is chain-recurrent*

Proof. The proof is analogous to the one for transitive systems, the only difference is that here we need to obtain ε-pseudo-orbits from x to x, for every $x \in X$ □

Theorem 3.1.12. *Let (X, f) be a TDS with the shadowing property. If f is chain-recurrent, then f is non-wandering.*

Proof. Again, the proof is analogous to the proof for chain-transitive maps. □

3.2 Shadowable Points

In this section we will deal with the pointwise definition of shadowing given by C.A. Morales in [7]. In his work, the author have considered homeomorphisms of compact metric spaces and defined the notion of shadowable point. Next he proved some properties that maps possessing such points must to have. We will start defining shadowable points for homeomorphisms. For the remainder of this section, let f denote a self-homeomorphism from a compact metric space.

Definition 3.2.1. *Let X be a compact metric space and $f : X \to X$ be a homeomorphism. Let $A \subset X$. We say that a sequence $\{x_n\}_{n\in\mathbb{N}}$ is a δ-pseudo-orbit of f through A if it is an δ-pseudo-orbit of f and $x_0 \in A$.*

Definition 3.2.2. *Let X be a compact metric space and $f : X \to X$ be a homeomorphism. We say that a point $x \in X$ is a shadowable point for f if for $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit through x is ϵ-shadowed by some point in X.*

The idea behind a shadowable point x is that if we take a pseudo-orbit starting on x there exists a real orbit that goes along the pseudo-orbit within an error. In [7] the author proved that for a compact metric space, a homeomorphism possesses the shadowing property is equivalent to all point of X be shadowable.

Theorem 3.2.3. *Let X be a compact metric space and $f : X \to X$ be a homeomorphism. Then has the shadowing property if, and only if, $Sh(f) = X$.*

Proof. [7] □

In chapter four, we will give a direct proof for the previous theorem requiring only that f be a continuous map.

Next we shall investigate the structure of the set of shadowable points. Since we are considering homeomorphism we can think in the inverse map. Indeed, the next result states that if a point is shadowable for a homeomorphism, then it must to be shadowable for its inverse.

Theorem 3.2.4. *If $x \in Sh(f) = Sh(f^{-1})$.*

Proof. Let $x \in sh(f)$ and fix $\varepsilon > 0$. Let $\delta > 0$ such that every δ-pseudo-orbit of f through $\{x\}$, if ε shadowed. Since f is uniformly continuous let δ_0 such that $d(f(x), f(y)) < \varepsilon$ is $d(x, y)) < \delta$.

Let $\{x_i\}$ be a δ-pseudo-orbit of f^{-1} through $\{x\}$. Since $d(f^{-1}(x_i), x_{i-1}) =< \delta$ for every i, then $d(x_i, f(x_{i-1})) < \delta_0$ for every $i \in \mathbb{Z}$. Thus $\{x_{-i}\}$ is a δ_0 pseudo orbit of f through $\{x\}$. Then there exists $y \in X$ such that $d(f^i(y), x_{x_{-i}}) < \varepsilon$ and therefore $d(f^{-i}(y), x_i) < \varepsilon$. Thus $x \in Sh(f^{-1})$.

The proof of contrary inclusion is analogous. □

As a consequence of the previous theorem we have the following

Theorem 3.2.5. *Let X be a compact metric space and $f : X \to X$ be a homeomorphism. Then $Sh(f)$ is an invariant set.*

Proof. Let $x \in Sh(f)$ and fix $\varepsilon_1 > 0$. Since f uniformly continuous, then there exists $\delta_1 > 0$ Such that $d(f(x), f(y)) < \varepsilon_1$, when $d(x, y) < \delta_1$. Let $0 < \varepsilon_2 < \delta_1$ be given by the δ_1-shadowableness through $\{x\}$ by f^{-1} . Since $f^{-1}(x)$ is uniformly continuous, then there exists $0 < \delta_2 < \varepsilon_2$ such that $d(f^{-1}(x), f^{-1}(y)) < \varepsilon_2$ when $d(x, y) < \delta_2$.

Now let $\{x_n\}_{n\in\mathbb{Z}}$ a δ_2-pseudo orbit of f trough $f(x)$. Since $d(f(x_i), x_{i+1}) < \varepsilon_2$ for every $i \in \mathbb{Z}$, then $d(f^{-1}(x_{i+1}), x_i) < \delta_2$, for every $i \in \mathbb{Z}$. Thus $\{f^{-1}(x_{-n})\}$ is a δ_2-pseudo-orbit of f^{-1} through $\{x\}$. Then there exists $y \in X$ such that $d(f^{-n}(y), f^{-1}(x_{-n})) < \delta_1$. Thus $d(f^n(f(y)), x_n) < \varepsilon_1$ for every n and therefore $f(x) \in Sh(f)$.

Analogously, one can proof that $f(Sh(f)) \subset Sh(f)$. □

Theorem 3.2.6. *$CR(f) \cap Sh(f) \subset \Omega(f)$. Therefore $CR(f) = \Omega(f)$, if $CR(f) \subset Sh(f)$*

Proof. Let $x \in CR(f) \cap Sh(f)$ and let U be an open neighborhood of x. Let $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$ and let $0 < \delta < \frac{\varepsilon}{2}$ be given by the $\frac{\varepsilon}{2}$-shadowing trough $\{x\}$.

Since $x \in CR(f)$, then there exists $\{x_0, ...x_n\}$ a δ-pseudo-orbit from x to x, which is obviously through $\{x\}$. Then there exists $y \in B_{\frac{\varepsilon}{2}}(x) \subset U$. Such that $d(f^n(y), x) < \frac{\varepsilon}{2}$. Then $x \in \Omega(f)$.

For the second part of the statement we have $CR(f) \supset \Omega(f)$ is quite obvious. The other inclusion is a trivial consequence of the first part. □

Lemma 3.2.7. *Let k be an integer number and let $x \in X$. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\{x_i\}_{i=0}^k$ is a δ finite pseudo orbit of f through $\{x\}$, then $\{x_i\}$ is ε-shadowed by x_0.*

Proof. We will proceed by induction. Suppose $k = 1$ and Let $\varepsilon > 0$. If we make $\delta = \varepsilon$ and if $\{x, x_1\}$ is an $\varepsilon > 0$ pseudo orbit through $\{x\}$, then $d(x, x) = 0 < \varepsilon$ and $d(f(x), x_1) < \varepsilon$.

Suppose the statement be true for k and let us prove for $k + 1$. Fix $\varepsilon > 0$. Since f is uniformly continuous, there exists $0 < \delta' < \frac{\varepsilon}{2}$ such that if $d(x, y) < \delta$ then $d(f(x), f(y)) < \frac{\varepsilon}{2}$.

On the other hand, the induction hypothesis implies that there exists $0 < \delta < \delta'$ such that if $\{x_i\}_{i=0}^{k+1}$ if a δ-pseudo orbit through x, then $d(f^i(x), x_i) < \delta'$ for $i = 0, 1, ..., k$.

Thus, let $\{x_i\}_{i=0}^{k+1}$ be a δ-pseudo orbit through x. Therefore

$$d(f^{k+1}(x), x_{k+1}) < d(f^{k+1}(x), f(x_k)) + d(f(x_k), x_{k+1})$$

But

$$d(f^{k+1}(x), f(x_k)) = d(f(f^k(x), f(x_k) < \frac{\varepsilon}{2}$$

Since $d(f^k(x), x_k) < \delta < \delta'$. Then $d(f^i(x), x_i) < \varepsilon$ for every $i = 0, ..., k + 1$ and this proves the theorem. □

As in case of shadowing maps in chapter two, we have a pointwise version of the statement for the powers of f.

Theorem 3.2.8. *For every $k \in \mathbb{N}$ we have $Sh(f) = Sh(f^k)$*

Proof. Let $x \in Sh(f)$. If $\{x_n\}_{n\mathbb{Z}}$ is a δ pseudo orbit for f^k trough $\{x\}$, then $d(f^k(x_n), x_{n+1}) < \delta$, for every n.

Consider the set

$$A = \{x_0 = x, f(x_0), ..., f^{k-1}(x_0), x_1, f(x_1), ..., f^{k-1}(x_1), x_2, ...\}.$$

Then A is clearly a δ-pseudo-orbit of f through $\{x\}$. Now, the shadowableness through $\{x\}$ by f implies the shadowableness through $\{x\}$ for f^k.

To prove the other inclusion let $x \in Sh(f^k)$ and fix $\varepsilon > 0$. Since f is uniformly continuous, there exists $\varepsilon_1 > 0$ such that $d(f^i(x), f^i(y)) < \frac{\varepsilon}{2}$ for every $i = 0, 1, ..., k$.

On the other hand, by lemma 3.2.7, there exists $0 < \delta_0 < \varepsilon_1$ such that for every δ_0-pseudo-orbit $\{x_i\}_{i=0}^{i=k}$ of f through $\{x\}$ is ε_1-shadowed by x.

Since $x \in Sh(f^k)$, take $0 < \delta_1 < \delta_0$ such that every δ_1-pseudo-orbit of f^k through $\{x\}$ is ε_1-shadowed. Let $0 < \delta < \delta_1$ such that every finite δ_1-pseudo-orbit $\{x_i\}_{i=0}^{k}$ of f through $\{x\}$ is δ_1-shadowed by x.

Let $\{z_i\}_{i\in\mathbb{N}}$ be a δ-pseudo-orbit of f through x. Define $x_i = z_{ki}$ for every $i \in \mathbb{Z}$ and fix i. Since $\{z_{ki}, z_{ki+1}, ..., z_{(k+1)i}\}$ is a finite δ-pseudo-orbit of f. Thus, $d(f^j(z_{ki}), z_{ki+j}) < \delta_1$ for every $j = 0, 1, ..., k$. In particular $d(f^k(z_{ki}), z_{(k+1)i}) < \delta_1$. Then $\{x_i\}_{\imath\in\mathbb{N}}$ is a δ_1-pseudo-orbit through $\{x\}$ for f^k. Thus, there exists $y \in X$ such that $d(f^{ik}(y), x_i) < \frac{\varepsilon}{2}$, for every $i \in \mathbb{N}$.

Finally, since $d(y, z_{ki}) < \delta_0$, we have $d(f^{ki+j}(y)), f^{ki+j}(z_{ki})) < \frac{\varepsilon}{2}$, for ever $j = 0, 1, ..., k$.

Thus

$$d(f^{ki+j}(y)), z_{jk+i}) < d(f^{ki+j}(y)), f^{ki+j}(z_{ki})) + d(f^{ki+j}(z_{ki}), z_{ki+j}) < \varepsilon,$$

for every integer i. Then $x \in Sh(f)$. □

Notice that the previous results trivially imply their global versions stated in chapter two. Indeed, this is a consequence of theorem 3.2.3.

3.3 Sensitive Points

We begin with the definition of a sensitive point.

Definition 3.3.1. *Let (X, f) be a TDS. A point $x \in X$ is called a sensitive point for f if there exists $\varepsilon_x > 0$ such that, for every open neighborhood V of x, there exists $y \in V$ such that $d(f^n(x), f^n(y)) > \varepsilon_x$, for some natural n. We call ε_x the sensitiveness constant of f on x. Let $Sen(f)$ denote the set of the sensitive points of f.*

The idea here is derived from the definition of sensitivity on initial conditions. But here there is not an uniform constant of sensitivity which works for every point in X. Notice that if $Sen(f) = X$ and $\inf_{x \in X}\{\varepsilon_x\} > 0$ then f is sensitive.

Example 3.3.2. *Consider the TDS $f : I \to I$ where I is the unit interval and f is defined by*
$f(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}] \\ 1 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$

Clearly the orbit of every positive point converges to 1 *then the fixed point* 0 *is a sensitive point. On the other hand if* $x = 1$ *there exists a neighborhood U of x such that $f^n(y) = 1$ for every n. Then the fixed point* 1 *is not a sensitive point.*

The previous example exhibits the existence of maps which the sensitive set is not the entire space. In the other hand, there are maps such that the sensitive sets are empty. The last ones are characterized in the following. The points which are not sensitive points are called equicontinuity points.

Definition 3.3.3. *A map $f : X \to X$ is called an equicontinuous map if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(f^n(x), f^n(y)) < \varepsilon$, for every n, when $d(x, y) < \delta$.*

Proposition 3.3.4. *Let X be a compact metric space and (X, f) a TDS. Then $Sen(f) = \emptyset$ if and only if f is equicontinuous.*

Proof. Suppose $Sen(f) = \emptyset$ and chose $\varepsilon > 0$. Then for every $x \in X$ we can choose a $\delta_x > 0$ such that $d(f^n(x), f^n(y)) < \varepsilon$ for every n. Since X is compact and $\{B_{\delta_x(x)}\}$ form an open cover for X, there exists an open finite sub-cover $\mathcal{B} = \{B_{\delta_{x_i}}\}_{i=1}^n$. Now let $\delta = \min\{\eta, \delta_{x_1}, \delta_{x_2}, ..., \delta_{x_n}\}$, here η is the Lebesgue number of $\mathcal{B}$. Thus, if $d(x, y) < \delta$ then x and y are in some element of the cover $\mathcal{B}$ which implies $d(f^n(x), f^n(x)) < \varepsilon$ for every n. Thus f is equicontinuous.

Conversely, suppose $Sen(f) \neq \emptyset$. Then there exist $x \in X$ and $\varepsilon > 0$ such that if we chose any $\delta > 0$, there exists y satisfying $d(x, y) < \delta$, such that $d(f^n(x), f^n(y)) > \varepsilon$, for some natural n. Thus f is not equicontinuous. □

Isometries are important examples of equicontinuous maps. Heuristically an isometry is a map which does not distorts distances. More specifically, we have the following definition

Definition 3.3.5. *A TDS is called an ismetry if $d(x, y) = d(f(x), f(y))$ holds for every $x, y \in X$.*

Isometries are trivially equicontinuous maps. Circle rotations are our familiar examples of maps which are isometries. In chapter two we defined them and proved that they are minimal system whenever the rotation angle theta is irrational. Now we will use the above concepts to prove a more general fact, namely, every transitive map must to be a minimal map in presence of equicontinuity.

Theorem 3.3.6. *Let (X, f) be a transitive TDS. If $Sen(f) = \emptyset$, then f is a minimal system.*

Proof. Let y be a point of X and let $\varepsilon > 0$. Let x be a point in X whose orbit is a dense set and Let x' be any point of X. Since x' is an equicontinuity point, there exists $\delta > 0$ such that if $d(x', z) < \delta$ then, $d(f^n(x'), f^n(z)) < \frac{\varepsilon}{2}$, for every n.

Since the orbit of x is dense, there exists i such that $d(f^i(x), x') < \delta$. Therefore $d(f^n(f^i(x)), f^n(x')) < \frac{\varepsilon}{2}$, for every n.

Since the orbit o x is dense, there exists j such that $d(f^j(f^i(x)), y) < \frac{\varepsilon}{2}$. Then

$$d(f^j(x'), y) \leq d(f^j(x'), f^j(f^i(x))) + d(f^j(f^i(x)), y) < \varepsilon.$$

Since y and x' are arbitrary, the orbit of x' is dense and this implies the minimality of f. □

3.4 Entropy Points

Next we will give a pointwise definition for topological entropy. The notion of entropy points was first introduced by Xiandong Ye and Guohua Zhang in [13]. Now we present this notion and later we will use it on order to establish when a system having the shadowing property has positive entropy.

Definition 3.4.1. *Let (X, f) be a TDS.*

- *We say that $x \in X$ is an entropy point if $h(f, \overline{U}) > 0$ for every neighborhood U of x. Let $E_p(X, f)$ denote the set of entropy points for f in X. When there is no confusion, we will write E_p instead of $E_p(X, f)$*
- *If $h(f, \overline{U}) = h(f)$ for every open neighborhood U of x, we say that x is a full entropy point for f. Let $E_p^F(X, f)$ denote the set of full entropy points for f in X. When there is no confusion, we will write E_p^F instead of $E_p^F(X, f)$*

It's an immediate fact from the definition that if $E_p \neq \emptyset$, then $h(f) > 0$. Next we will prove the converse. But, first we need to prove the following lemma.

Lemma 3.4.2. *Let (X, f) be TDS and let $U_1, U_2, ..., U_k$ be closed subsets of x. Then $r(\varepsilon, \bigcup_{i=1}^k U_i) = \max_{1\leq i\leq k} r(\varepsilon, U)$ for every $\varepsilon > 0$. Therefore $h(f, \bigcup_{i=1}^k U_i) = \max_{1\leq i\leq k} h(f, U_i)$*

Proof. Since $U_i \subset X$ for each i, therefore an (n, ε)-generator for the $\bigcup_{i=1}^k U_i$ with minimal cardinality is an (n, ε)generator for any U_i. Therefore $r_n(\varepsilon, \bigcup_{i=1}^k U_i) \geq \max_{1\leq i\leq k} r_n(\varepsilon, U_i)$ for every i. Then

$$\limsup_{n\to\infty} \frac{1}{n} \log r_n(\varepsilon, \bigcup_{i=1}^k U_i) \geq \max_{1\leq i\leq k} \limsup_{n\to\infty} \frac{1}{n} \log r_n(\varepsilon, U_i)$$

To prove the other inequality, we first fix n. Now we notice that $k(\max_{1\leq i\leq k} r_n(\varepsilon, U_i) \geq r_n(\varepsilon, \bigcup_{i=1}^k U_i))$. Thus

$$\limsup_{n\to\infty} \frac{1}{n} \log r_n(\varepsilon, \bigcup_{i=1}^{k} U_i) \leq \limsup_{n\to\infty} k(\max_{1\leq i\leq k} \frac{1}{n} \log r_n(\varepsilon, U_i)) = \limsup \frac{1}{n}k + \limsup_{n\to\infty} \max_{1\leq i\leq k} \frac{1}{n} \log$$

This completes the proof of the lemma. □

Theorem 3.4.3. *Let f be a TDS. Suppose that $h(f) > 0$. Then $E_p^F(f) \neq \emptyset$.*

Proof. Fix $\gamma > 0$ and consider the open cover $\mathcal{B} = \{B_\gamma(x)\}_{x\in X}$ for X. Since X is compact we can cover X with a finite number of elements of $\mathcal{B}$. Let $\mathcal{B}_0$ be such finite sub-cover. The previous lemma tells us that there exists a member B_0 of $\mathcal{B}_0$ such that $h(f) = h(f, \overline{B_0})$.

Now, cover $\overline{B_0}$ with an finite open cover $\mathcal{B}_1$ whose elements are balls of radius $\frac{\gamma}{2}$. Then there exists an element B_1 of $\mathcal{B}_1$ such that $h(f) = h(f, \overline{B_1})$. If we continue proceeding in the same way, we will find a sequence of balls $B_0 \supset B_1 \supset .... \supset B_n \supset ....$ Such that $h(f) = h(f, B_i)$ for every natural i.

Since the radium of these balls converge to 0, then $\bigcap_{i=0}^{\infty} \overline{B_i} = \{y\}$. But our construction implies that $y \in E_n^F(f)$. □

Next we show that $E_p(f)$ and $E_p^F(f)$ are closed invariant sets.

Theorem 3.4.4. *$E_p(f)$ and $E_p^F(f)$ are closed sets.*

Proof. Let $x \in \overline{E_p(f)}$. Thus there exists a point $y \in U \cap E_P(y)$, for every open neighborhood U of x. Since U is a neighborhood of y, then $h(f, \overline{U}) > 0$. Thus $x \in E_p(f)$. The proof for the set of full entropy point is analogous. □

Theorem 3.4.5. *Let (X, f) be a TDS. Then $f(E_p(f)) \subset E_p(f)$ and $f(E_p^F(f) \subset E_p^F(f)$.*

Proof. Take $y \in f(E_p(f))$. Thus there is $x \in E_p(f)$ such that $f(x) = y$. Fix $\varepsilon > 0$ and $n > 0$. Since f is uniformly continuous there exists $0 < \delta < \varepsilon$ such that $d(f^i(x), f^i(y)) < \varepsilon$ for every $i = 1, ...n$ if $d(x, y) < \delta$. Let U be an open neighborhood of y, then $f^{-1}(U)$ is an open neighborhood for x. If E is an (n, δ)-separated subset of $\overline{f^{-1}(U)}$ with maximal cardinality, then $f(E)$ is a (n, ε)-separated subset of $\overline{U}$. Then $s_n(\varepsilon, \overline{U}) \geq s_n(\delta, \overline{f^{-1}(U)})$ and this implies $h(f, \overline{U}) \geq h(\overline{f^{-1}(U)}) > 0$. Thus $y \in E_p(f)$.

The proof for $E_p^F(f)$ is quite analogous. □

3.5 Some Consequences

In the following we will turn back to systems with the shadowing property and use the pointwise properties to obtain global properties for these systems. The following results can be found in [11]

Theorem 3.5.1. *Let (X, f) be a TDS. If f has the shadowing property and $x \in \Omega(f)$, then for every neighborhood U of x, there exists k such that $k\mathbb{N} \subset \mathcal{N}(U, U)$.*

Proof. Suppose $x \in \Omega(f)$ and let U be an open neighborhood of x. Let $\varepsilon > 0$ such that $B_{\frac{\varepsilon}{2}}(x) \subset U$. Since f has the shadowing property, there exists $0 < \delta < \frac{\varepsilon}{2}$ such that every δ pseudo orbit of f is $\frac{\varepsilon}{2}$-shadowed by some point in X.

Since $x \in \Omega(f)$, there are $y \in B_{\frac{\delta}{2}}(x)$ and $k \in \mathbb{N}$ such that $f^k(y) \in B_{\frac{\delta}{2}}(x)$. Thus, the sequence $(y, f(y), ..., f^k(y), y, f(y), ..., f^k(y), ...)$ is a δ-pseudo orbit of f. By shadowing, there exists a point $z \in X$ Such that $d(f^{i+kj}(z), f^i(y)) < \frac{\varepsilon}{2}$ for every $0 \leq i \leq k$ and every $j \in \mathbb{N}$. Particularly, $d(f^{kj}(z), f^k(y)) < \frac{\varepsilon}{2}$ for every j. Therefore $d(x, f^k(z)) \leq d(x, f^k(y)) + d(f^{kj}(z), f^k(y)) < \varepsilon$. But it implies that $k\mathbb{N} \subset \mathcal{N}(U, U)$.

□

Corollary 3.5.2. *Let (X, f) be a TDS having the shadowing property. Then*

1. $\Omega(f) = \overline{M(f)}$
2. $\overline{M(f^{\times n})} = \Omega(f)^n$.

Proof. 1)-Since $\Omega(f)$ is closed and $M(f) \subset \Omega(f)$ therefore $\overline{M(f)} \subset \Omega(f)$. Now if $x \in \Omega(f)$ the previous shows that every open neighborhood U of x has a point y such that $f^nk(y) \in U$ for every n and some k. Consider f^k and let $b \in \overline{O^+_{f^k}(y)}$ be a minimal point. Therefore b is a minimal point which belongs to U and it concludes the proof.

2)- Fix $n > 0$ and let $x = (x_1, ..., x_n) \in \Omega(f)^n$. Let U be an open neighborhood of x. Then for each $i - 1, ..., n$, there is $U_i \subset X$ an open neighborhood of x_i, such that $U_1 \times ... \times U_n \subset U$. The previous theorem gives us points $p_1 \in U_i, ..., p_n \in U_n$ and naturals $k_1, ..., k_n$ such that $f^{jk_i}(p_i) \in U_i$ for every $i = 1, ...n$ and every $j \in \mathbb{N}$. If we set $k = k_1k_2...K_n$, then $f^{jk}(p_1, ..., p_n) \in U$ for every $j \in \mathbb{N}$. Finally, If we proceed in the same way as in the proof of part 1) to obtain 2). □

Theorem 3.5.3. *Let (X, f) be a TDS having shadowing property. If f is totally transitive, then f is weakly mixing.*

Proof. Let U and V be non-empty open sets. Since f is totally transitive, we have $\Omega(f) = X$. Then by theorem 3.5.1 there exists k such that $k\mathbb{N} \subset \mathcal{N}(U, U)$. On the other hand, f^k is transitive, then there exists n such that $f^kn(U) \cap V \neq \emptyset$ and therefore $\mathcal{N}(U, U) \cap \mathcal{N}(U, V) \neq \emptyset$ and this implies f is weakly mixing by proposition 2.2.6. □

Theorem 3.5.4. *Let (X, f) be a TDS having the shadowing property. If f is totally transitive, then f is topologically mixing.*

Proof. Let $x, y \in X$ and $\varepsilon > 0$. Let $0 < \delta \leq \varepsilon$ such that every δ-pseudo orbit of f is ε-shadowed for some point in X.

Write $X = \bigcup_{i=1}^{n} B_{\frac{\delta}{2}}(x_i)$ (Remember it is aways possible, since X is compact). Reordering if it is necessary, we can assume that $x \in B_{\frac{\delta}{2}}(x_1)$ and $y \in B_{\frac{\delta}{2}}(x_n)$.

Since f is topologically weakly mixing, $f^{\times n}$ is transitive. Then there exists $p_i \in B_{\frac{\delta}{2}}(x_i)$ and k such that $p_i \in f^k(B_{\frac{\delta}{2}}(x_i)) \cap B_{\frac{\delta}{2}}(x_n))$ for every $1 \leq i \leq n$.

Fix $j \in \mathbb{N}$. Then $f^j(x) \in B_{\frac{\delta}{2}}(x_i))$ for some $1 \leq i \leq n$. Thus, the sequence

$$(x, f(x), ...f^{j-1}(x), p_i, f(p_i), ..., f^k(p_i))$$

is a δ-pseudo-orbit from x to y, since $d(f^j(x), p_i) < \delta$.

Since j is arbitrary, we can construct in a similar way a δ-pseudo-orbit from x to y with any length grater than p. Therefore, f is chain mixing. Finally, since f has the shadowing property, it is topologically mixing. □

Theorem 3.5.5. *Let (X, f) be a TDS having the shadowing property. If X is connected and f is non-wandering, then f is topologically mixing.*

Proof. Since f is non-wandering and by the part 2) of corollary 3.5.2, we have that $\overline{M(f \times f)} = X^2$. Furthermore, since f has the shadowing property, $f \times f$ has also. Then define $Y := X^2$ and $g := f \times f$. Let $x, y \in Y$ and $\varepsilon > 0$.

Since X is connected, then Y is also and therefore there are points $p_0 = x, p_1, p_2, ..., p_n = y$ such that $d(p_i, p_{i+1}) < \frac{\varepsilon}{2}$ for every $0 \leq i \leq n$. Let $0 < \delta < \frac{\varepsilon}{2}$.

Since g is non-wandering, there are $\frac{\varepsilon}{3}$-pseudo orbits of length k_i from p_i to p_i. If we set

$$A = p_0, ...f^{K_1-1}(p_0), ..., p_n, ..., f^{k_n-1}(p_n)\}$$

then A is a ε-pseudo-orbit of g from x to y and therefore g is chain transitive. Since g has the shadowing property, g is transitive and therefore f is weakly mixing. Then in order to conclude, we just need to apply the previous theorem. □

The next result gives conditions to a map possesses positive topological entropy.

Theorem 3.5.6. *Let (X, f) be a TDS having the shadowing property. Let $Y \subset X$ be a f-invariant closed set, let $g = f|_Y$, and consider (Y, g). Then $z \in E_p(f)$, if there is $z \in Sen(g)$ with $(z, z) \in int[\overline{R(g \times g)}]$.*

Proof. Let U be an open neighborhood of z and let $\varepsilon > 0$ such that $B_\varepsilon(z) \subset U$. Since $z \in Sen(g)$, there exists $\eta > 0$ such that for every open neighborhood V in Y of z, there exists k such that the diameter of $g^k(V)$ is bigger than η. If necessary, one can reduce ε in order to assure that $3\varepsilon < \eta$. Choose $0 < \delta < \varepsilon$ given by the ε-shadowing of f.

Let V be an open neighborhood of z in Y such that $V \times V \subset int(\overline{R(g \times g)})$ such that $diam(V) < \frac{\delta}{2}$. Thus there are $(a, b) \in (V \times V) \cap R(g \times g)$ and $k < p$ such that $d(f^k(a), f^k(b)) > 3\varepsilon$, $d(g^p(a), a) < \frac{\delta}{2}$ and $d(g^p(b), b) < \frac{\delta}{2}$

We will show that $h(f,\overline{U}) \geq \frac{\log 2}{p}$. To do this, consider $A = \{a, f(a), ..., f^{p-1}(a)\}$ and $B = \{b, f(b), ..., f^{p-1}(b)\}$. Since the distances between a and b, a and $f^{p-1}(a)$, and between b and $f^{p-1}(b)$ are smaller than $\frac{\delta}{2}$, we have that for any $n \in \mathbb{N}$ any $C = \{C_1, ..., C_n\}$ where $C_i = A$ or $C_i = B$ for any i, C is a δ-pseudo-orbit of f consisting of np Elements.

For each of such C, let $x_C \in X$ be a point that ε-shadows C. If $y \in \{A, B\}$ is the first element of C, then $d(z, x_C) \leq d(z, y) + d(y, x_c) < \delta + \varepsilon < 2\varepsilon$ and therefore $x_C \in U$. If $C \neq D$, then for some $j \in \{0, 1, 2, ..., np-1\}$, the j-th elements of the pseudo-orbits C and D are at least 3ε apart. Therefore, by triangle inequality $d(f^j(x_C), f^j(x_D) > \frac{\varepsilon}{2}$ for some $j \in \{0, 1, 2, ..., np-1\}$. This means that the set $\{x_C : C \in \{A, B\}^n\}$ is a (U, np, ε)-separated subset of $\overline{U}$ and hence (U, np, β)-separated for f for any $\beta \in (0, \varepsilon)$.

On the other hand, $\#\{x_C : C \in \{A, B\}^n\} = 2^n$. Thus, we notice that the cardinality of $s_{np}(f, \overline{U}, \beta)$ is bigger than 2^n for every $\beta < \varepsilon$ and this implies $h(f, \overline{U}) \geq \frac{\log 2}{p} > 0$. Thus x is an entropy point and $h(f) > 0$

□

One can notice that if (X, f) has a minimal non-equicontinuous subsystem (Y, g) then $R(f \times f) = Y^2$, therefore the previous theorem guarantees the positiveness of the entropy of f. Then proceeding in a similar way, one can derive the following result.

Theorem 3.5.7. *Let (X, f) a TDS having the shadowing property. If $R(f) \setminus M(f) \neq \emptyset$ then $h(f) > 0$.*

Proof. Suppose that $x \in R(f) \setminus M(f)$. Let Y be the closure of the orbit of x under f. Then $g = |_Y$ is a minimal subsystem of f. Furthermore $Sen(g) \neq \emptyset$. Then applying theorem 3.5.6 we obtain than $h(f) > 0$. □

As a corollary we have the following result

Corollary 3.5.8. *Let (X, f) a TDS having the shadowing property. Suppose that f is non-wandering. Then $Sen(f) \subset E_p(f)$.*

Proof. Since f is non-wandering and has the shadowing property , then the previous theorem gives $X^2 = \overline{M(f \times f)} \subset \overline{R(f \times f)}$. Thus if $x \in Sen(f)$ theorem 3.5.6 implies that $x \in E_p(f)$. □

Chapter 4

Contributions to the theory

In this chapter we present some of our results about the theory we presented in the previous chapters.

4.1 Dynamics of Uniform Limits

In this section we will turn back to discuss the main question of chapter one. We were asking when the uniform limit of a sequences of maps with some dynamical property (P) inherits (P). In chapter one, we saw that the conditions imposed by Flores and Risong-Li are too strong. Indeed, these properties forces the limit map to possesses property (P), even if the sequence maps do not. Next we will investigate this question when (P) is the shadowing property.

First we remark that without extra conditions the answer to question is negative. Indeed, the next example shows that.

Example 4.1.1. *For any $n > 0$, let $f_n : [0,1] \to [0,1]$ defined by* $f_n(x) = \begin{cases} (\frac{n+1}{n})x & \text{if } x \in [0, 1 - \frac{n}{n+1}] \\ 1 & \text{if } x \in (1 - \frac{n}{n+1}, 1] \end{cases}$

Fix $\varepsilon < 1 - \frac{n}{n+1}$. If we chose $\delta' = \frac{\varepsilon}{n}$ then there exists $k \in \mathbb{N}$ and such that any δ'-pseudo-orbit is such that $x_k > 1 - \frac{\varepsilon}{2}$. Thus for this k and ε, we can take $\delta < \delta'$ given by lemma 2.4.5 such that x_0 $\frac{\varepsilon}{2}$-shadows $\{x_i\}_{i=0}^{k}$. Now, by the choose of δ and ε then $f^i(x_0) = 1$ and $x_i = 1$ for every $i \geq k$. But this implies that x_0 ε-shadows $\{x_i\}$. However, the sequence f_n converges uniformly to the identity map, which has not the shadowing property, since $[0,1]$ is connected.

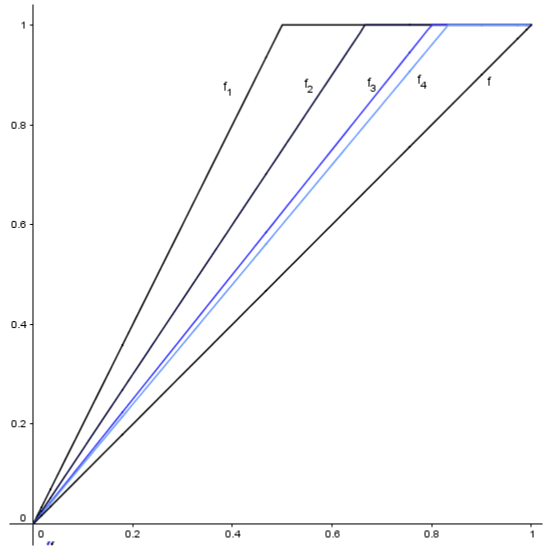

Figure 4.1: Some maps of the sequence f_n and the limit map f.

In the previous example, one can notice that the δ's of f_n converges to 0, when n tends to infinity. Keeping it in mind, we can try to turn the δ' uniform in order to obtain the shadowableness of the limit map.

Definition 4.1.2. *We say that a sequence $f_n : X \to X$ has the uniform shadowing property if every f_n has the shadowing property and the functions δ_n are the same.*

The following theorem slightly improves the result in [5] where it is assumed that $\delta_n(\varepsilon) = \varepsilon$ for every n.

Theorem A. *Let X be a compact metric space and $f_n : X \to X$ be a sequence of continuous functions which converges uniformly to a function f. Suppose that f_n has the uniform shadowing property. Then f has the shadowing property.*

Proof. Fix $\varepsilon > 0$. Since f_n has the uniform shadowing property, chose $\delta > 0$ such that for every n, we have that every δ-pseudo-orbit of f_n is $\frac{\varepsilon}{3}$-shadowed by some point y_n. We will show that every $\frac{\delta}{2}$-pseudo orbit for f is ε shadowed.

Let $\{x_i\}$ be a $\frac{\delta}{2}$-pseudo-orbit of f. We claim that $\{x_i\}$ is a δ-pseudo-orbit for f_n if n is sufficiently large. Indeed, since f_n converges uniformly to f, there exists N_0 such that $d(f_n(x), f(x)) < \frac{\delta}{2}$ for every $x \in X$ and $n \geq N_0$. Then

$$d(f_n(x_i), x_{i+1}) \leq d(f_n(x_i), f(x_i)) + d(f(x_i), x_{i+1}) < \delta,$$

if $n \geq N_0$.

Uniform shadowing implies that for every $n \geq N_0$ there exists a point y_n such that $d(f_n^i(y_n), x_i) \leq \frac{\varepsilon}{3}$ for every i. Let $(y_n)_{n \geq N_0}$ be the sequence of such points. Since X is compact we can assume that y_n converges for some point $y \in X$. We claim that y ε-shadows $\{x_i\}$. Indeed, we have

$$d(f^i(y), x_i) \leq d(f^i(y), f^i(y_n)) + d(f^i(y_n), f_n^i(y_n)) + d(f_n^i(y_n), x_i)$$

Fix i. Since f_n converges to f, then f_n^i converges to f^i. On the other hand, f^i is continuous. Thus we can make $d(f^i(y), f^i(y_n)) \leq \frac{\varepsilon}{3}$ and $d(f^i(y_n), f_n^i(y_n)) \leq \frac{\varepsilon}{3}$ if we chose n sufficiently large. Therefore $d(f^i(y), x_i) \leq \varepsilon$, for every $i \geq 0$. Therefore f has the shadowing property.

$\square$

Example 4.1.3. *For any $n > 0$, let $f_n : [0,1] \to [0,1]$ defined by* $f_n(x) = \begin{cases} 4(\frac{n+1}{n+3})x & \text{if } x \in [0, \frac{1}{4}\frac{n+3}{n+1}] \\ 1 & \text{if } x \in (\frac{1}{4}\frac{n+3}{n+1}, 1] \end{cases}$

Here, the uniform limit of f_n is the map defined by $f(x) = \begin{cases} 4x & \text{if } x \in [0, \frac{1}{4}] \\ 1 & \text{if } x \in (\frac{1}{4}, 1] \end{cases}$

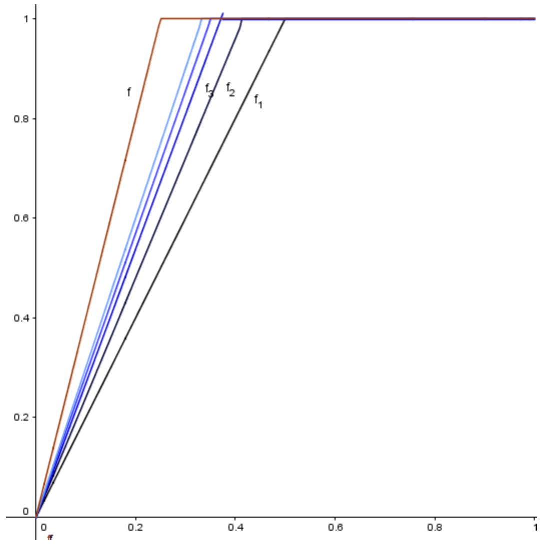

Figure 4.2: Some maps of the sequence f_n and the limit map f

Which has the shadowing property as we have already seen. Moreover, if we proceed as in the previous example we conclude that given any $\varepsilon > 0$, we can chose an uniform $\delta < \frac{\varepsilon}{4}$ for the shadowing of f_n, for every n. Thus, f_n has the uniform shadowing property.

Remark: Looking the proof of the previous theorem again, one can notice that we proved that any $\frac{\varepsilon}{2}$-pseudo-orbit of f_n is an ε-pseudo-orbit of f if we take n sufficiently large. keeping this in mind we can prove the following theorem. The first item of it was proved by Fedeli and Le Donne in [5].

Theorem 4.1.4. *Let X be a compact metric space and let $f_n : X \to X$ be a sequence of continuous maps converging uniformly to f.*

1. *If f_n is chain-transitive for every n, then f is chain-transitive.*
2. *If f_n is chain-mixing for every n, then f is chain-mixing.*
3. *If f_n is chain-recurrent for every n, then f is chain-recurrent.*

Proof. 1) Let $x, y \in X$ and fix $\varepsilon > 0$. Since f_n is chain-transitive for every n there exists $\{x_i^n\}$ finite $\frac{\varepsilon}{2}$-pseudo orbits of f_n starting on x and ending on y. Therefore the previous remark tells us that $\{x_i^{n_0}\}$ is an ε-pseudo orbit of f if we take n_0 sufficiently large. Therefore f is chain-transitive

2) Let $x, y \in X$ and fix $\varepsilon > 0$. Since f_n is chain-transitive for every n there exists K_n and a $\{x_i^n\}_{i=0}^k$ finite $\frac{\varepsilon}{2}$-pseudo orbits of f_n starting on x and ending on y for every $k > k_n$. Therefore the previous remark tells us that $\{x_i^{n_0}\}_{i=0}^k$ is an ε-pseudo orbit of f for every $k > k_0$ if we take n_0 sufficiently large. Therefore f is chain-mixing.

3) Let $x, y \in X$ and fix $\varepsilon > 0$. Since f_n is recurrent for every n there exists $\{x_i^n\}$ finite $\frac{\varepsilon}{2}$-pseudo orbits of f_n starting on x and ending on x. Therefore the previous remark tells us that $\{x_i^{n_0}\}$ is an ε-pseudo orbit of f if we take n_0 sufficiently large. Therefore f is chain-recurrent. □

Joining the two last theorems, we can give a positive answer to our question when the property (P) is topological transitivity, topological mixing, or non-wandering.

Theorem B. *Let X be a compact metric space and $f_n : X \to X$ be a sequence o topologically transitive maps that converges uniformly to a maps f. Suppose that f_n has the uniform shadowing property. Then f is topologically transitive.*

Proof. Since each f_n is transitive, then it is chain-transitive. Thus theorem 4.1.4 implies f is chain-transitive. Since theorem A implies f has the shadowing property, then f is transitive.

□

Theorem C. *Let X be a compact metric space and $f_n : X \to X$ be a sequence o topologically mixing maps that converges uniformly to a maps f. Suppose that f_n has the uniform shadowing property. The f is topologically mixing.*

Proof. Since each f_n is topologically, then it is chain-mixing. Thus theorem 4.1.4 implies f is chain-mixing. Since theorem A implies f has the shadowing property, then f is topologically mixing.

□

Theorem D. *Let X be a compact metric space and $f_n : X \to X$ be a sequence of non-wandering maps that converges uniformly to a maps f. Suppose that f_n has the uniform shadowing property. Then f is non-wandering.*

Proof. Since each f_n is non-wandering, then it is chain-recurrent. Thus theorem 4.1.4 implies f is chain-recurrent. Since theorem A implies f has the shadowing property, then f is recurrent. □

We end this section giving conditions to assure the positiveness of the entropy of a uniform limit map of a sequence of maps (with positive entropy or not).

Corollary E. *Let X be a connected compact metric space and $f_n : X \to X$ be a sequence of non-wandering maps that converges uniformly to a maps f. Suppose that f_n has the uniform shadowing property. Then f has positive entropy.*

Proof. By the previous theorem, f is non-wandering. Since X is connected f is topologically mixing and therefore is sensitive to initial conditions. Thus f is a non-wandering map, with shadowing property and a sensitive point. Therefore theorem 3.5.6 implies that f has positive topological entropy. □

4.2 Pointwise Dynamics

In this section we will present some results concerning pointwise dynamics. We begin giving a proof for the theorem 3.2.3 of chapter three. We remark that this theorem was discovered by Morales in [7]. Here, we present a new and shorter proof for it, which works when the map f is an homeomorphism or not. First we need the following lemma.

Lemma 4.2.1. *A point $x \in X$ is shadowable if, and only if, for every $\epsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit through $B_\delta[x]$ is ϵ shadowed for some point in X.*

Proof. ($\Leftarrow$) Fix $\epsilon > 0$ and let $\delta > 0$ be as in the hypothesis. Obviously every δ-pseudo-orbit through x is a δ-pseudo-orbit through $B_\delta[x]$ and therefore is ϵ-shadowed by some point in x.

($\Rightarrow$) Now suppose x is shadowable and fix $\epsilon > 0$. Let δ' be given by the $\frac{\epsilon}{2}$-shadowing through x. Since f is uniformly continuous there exists $\delta > 0$ such that if $d(y,z) < \delta$ then $d(f(y), f(z)) < \frac{\delta'}{2}$. We can suppose $\delta < \frac{\epsilon}{2}$ and $\delta < \frac{\delta'}{2}$. Let $\{x_n\}$ be a δ-pseudo-orbit through $B_\delta[x]$. And define the following sequence:

$$y_n = \begin{cases} x_n, & n < 0 \\ x, & n = 0 \end{cases}$$

We claim that y_n is a δ'-pseudo-orbit through x. In fact, we only need to show that $d(f(x), x_1) < \delta'$. This follow immediately by the triangle inequality since $d(f(x), x_1) \le d(f(x), f(x_0)) + d(f(x_0), x_1) \le \frac{\delta'}{2} + \frac{\delta'}{2} = \delta'$. Thus there exists $y \in X$ such that $d(f^n(y), y_n) < \frac{\epsilon}{2}$ for every $n \ge 0$. Finally observe that $d(y, x_0) \le d(y, x) + d(x, x_0) \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This means that y ϵ-shadows $\{x_n\}$. □

Theorem 4.2.2. *A continuous map on a X compact metric space has the shadowing property if, and only if, every point in X is shadowable.*

Proof. ($\Rightarrow$) This direction is obvious.

($\Leftarrow$) Let us consider an arbitrary $\epsilon > 0$. Then for every $x \in X$ there exists $\delta_x > 0$ such that every δ_x-pseudo-orbit through $B_{\delta_x}[x]$ is ϵ-shadowed by some point in X. Now

$\{B_{\delta_x}(x)\}_{x\in X}$ is an open cover for X and by compacity we can extract a finite subcover $\{B_{\delta_{x_i}}(x_i)\}$. Let $\delta = \min_i\{\delta_{x_i}\}$ and consider $\{y_n\}$ any δ-pseudo-orbit of f. Clearly $y_0 \in B_{\delta_{x_i}}[x_i]$ for some i and since $\delta \leq \delta_{x_i}$ $\{y_n\}$ is ϵ-shadowed by some point in X. □

In the following, we extend the previous theorem for others forms of shadowing property. The first one is the average shadowing property. The idea behind average shadowing property is similar to the shadowing property, but it is slightly different. Indeed, for average shadowing we admits that the pseudo-orbit are similar to real orbits in average.

Definition 4.2.3. *A subset $\{x_i\}_{i\in\mathbb{N}}$ is an δ-average-pseudo-orbit if there exist k such that*

$$\frac{1}{n}\Sigma_{i=0}^{n} d(f(x_{i+k}), x_{i+k+1}) < \delta$$

for every $n \geq k$.

Definition 4.2.4. *We say that a δ-average-pseudo-orbit $\{x_i\}_{i\in\mathbb{N}}$ is ε-shadowed in average by x if*

$$\lim_{n\to\infty} \frac{1}{n}\Sigma_{i=0}^{n} d(f^i(x), x_i) < \varepsilon$$

.

Next we prove a version of previous result for average shadowing.

Definition 4.2.5. *Let $A \subset X$. We say that a sequence $\{x_n\}_{n\geq 0}$ is a δ-average-pseudo-orbit through x if it is a δ-average-pseudo-orbit and $x_0 \in A$.*

Definition 4.2.6. *We say that a point $x \in X$ is a average-shadowable point if for $\epsilon > 0$ there is $\delta > 0$ such that every δ-average-pseudo-orbit through x is ϵ-average-shadowed by some point in X.*

Lemma 4.2.7. *A point $x \in X$ is average-shadowable if, and only if, for every $\epsilon > 0$ there exists $\delta > 0$ such that every δ-average-pseudo-orbit through $B_\delta[x]$ is ϵ average-shadowed for some point in X.*

Proof. ($\Leftarrow$) Fix $\epsilon > 0$ e let δ be as in the hypothesis. Obviously every δ-average-pseudo-orbit through x is a δ-average-pseudo-orbit through $B_\delta[x]$ and therefore is ϵ-average-shadowed by some point in x.

($\Rightarrow$) Now suppose x is average-shadowable and fix $\epsilon > 0$. Let δ' be given by the $\frac{\epsilon}{2}$-average-shadowing through x. Proceeding as in the lemma 4.2 chose an $\delta > 0$ satisfying the same conditions . Now let $\{x_n\}$ be a δ-average-pseudo-orbit through $B_\delta[x]$ and define the following sequence:

$$y_n = \begin{cases} x_n, & n < 0 \\ x, & n = 0 \end{cases}$$

We clain that y_n is a δ'-average-pseudo-orbit through x. Indeed, since x_n is an avererage pseudo orbit there exists an integer N such that

$$\frac{1}{n}\Sigma_{i=0}^{n} d(f(x_{k+i}), x_{k+i+1}) < \delta$$

for every k and $n \geq N$. Then $\frac{1}{n}\Sigma_{i=0}^{n} d(f(y_{k+i}), y_{k+i+1}) < \frac{\delta}{n} + \delta = \delta'$ for every integer k and $n \geq 1$. Therefore there exists $y \in X$ such that y $\frac{\epsilon}{2}$-shadows y_n. Finally we observe that

$$\limsup \frac{1}{n}\Sigma_{n=0}^{\infty} d(f^n(y), x_n) < \limsup \frac{1}{n} d(y, x_0) + \limsup \frac{1}{n}\Sigma_{n=0}^{\infty} d(f^n(y), x_n) < 0 + \frac{\epsilon}{2} < \epsilon$$

Thus x_n is ϵ-shadowable.

□

Theorem F. *A continuous map on a X compact metric space has the average-shadowing property if, and only if, every point in X is average-shadowable.*

Proof. ($\Rightarrow$) This direction is obvious.

($\Leftarrow$) Let us consider an arbitrary $\epsilon > 0$. Then for every $x \in X$ there exists $\delta_x > 0$ such that every δ_x-average-pseudo-orbit through $B_{\delta_x}[x]$ is ϵ-average-shadowed by some point in X. Now $\{B_{\delta_x}(x)\}_{x\in X}$ is an open cover for X and by compacity we can extract a finite subcover $\{B_{\delta_{x_i}}(x_i)\}$. Let $\delta = \min_i\{\delta_{x_i}\}$ and consider $\{y_n\}$ any δ-average-pseudo-orbit of f. Clearly $y_0 \in B_{\delta_{x_i}}[x_i]$ for some i and since $\delta \leq \delta_{x_i}$ $\{y_n\}$ is ϵaverage-shadowed by some point in X. □

We end our discussion about shadowing points noticing that sometimes it makes no sense try to extend some characterization which works on a specific case, for all other possible cases. indeed, the next theorem does not add extra information about the previous knowledge concerning the limit shadowing property.

Definition 4.2.8. *We say that a sequence $\{x_i\}_{i\in\mathbb{N}}$ is a limit-pseudo-orbit of f if $d(f(x_i), x_{i+1}) \to 0$ when $i \to \infty$.*

Definition 4.2.9. *We say that a limit-pseudo-orbit $\{x_i\}_{i\in\mathbb{N}}$ is limit-shadowed by a point x if $d^i(f^i(x), x_i) \to 0$ when $i \to \infty$*

The idea behind limit-shadowing property is when i increases the pseudo-orbit looks more similar to an real orbit. Furthermore, the pseudo orbit is followed by a real orbit which get closer to it as i grows.

Next we try to obtain a similar theorem for the limit-shadowing property.

Definition 4.2.10. *Let $x \in X$. We say that a sequence $\{x_n\}_{n\geq 0}$ is a δ-limit-pseudo-orbit through x if it is a δ-limit-pseudo-orbit and $x_0 = x$.*

Definition 4.2.11. *We say that a point $x \in X$ is a limit-shadowable point if every limit-pseudo-orbit through x is limit shadowable for some point in x.*

Proposition 4.2.12. *A continuous map on a X compact metric space has limit-shadowing property if, and only if, every point in X is limit-shadowable.*

Proof. The proof quite obvious, since the limit shadowing property does not depends on the point x_0. □

As we have seen, the previous theorems does not say anything new to improve our understanding about limit-shadowing property.

Now we will give a result in the same spirit of 3.5.6 but with a weaker hypothesis about the shadowing property. Indeed, we will give a pointwise version of it. Before to start let us define a pointwise version of expansiveness.

Definition 4.2.13. *Let (X, F) be a TDS. We say that a point is x is an positively-expansive point if there are $e > 0$ and V a neighborhood of x such that for any $y, z \in V$ distinct points, there exists $i \in \mathbb{N}$ such that $d(f^i(y), f^i(z)) > e$. Let $Ex(f)$ denote the set of positively-expansive points of f.*

Now we shall discuss some properties of expansive points.

Proposition G. *Let x be a compact metric space and let $f : x \to X$ be a continuous map. Then f is expansive if, and only, $Ex(f) = X$*

Proof. If f is positively-expansive then every x is obviously an expansive point.

Conversely, if every point of X is positively-expansive, then for every x there are U_x an open neighborhood of x and e_x an expansiveness constant for $f|_{U_x}$. Since the sets U_x form an open covering for X and X is compact, there exists an finite sub-covering $U_{x_1}, ..., U_{x_n}$. set $e = \min\{e_{x_1}, ..., e_{x_n}, \eta\}$, where is the Lebesgue number of the covering. Therefore if $d(f^j(y), f^j(z)) < e$ for every $j \geq 0$, then $y, z \in U_(x_i)$ for some $i = 1, ..., n$ and this implies $y = z$. Therefore f is positively-expansive. □

Proposition H. *Let x be a compact metric space and let $f : X \to X$ be a continuous map. Then $f(Ex(f)) \subset Ex(f)$.*

Proof. Let $x \in f(Ex(f))$. Then there exists a point $y \in Ex(f)$ such that $f(y) = x$. Since y is an expansive point, there exists $e > 0$ and an open neighborhood V of y, such that for each pair of distinct point $z_1, z_2 \in V$ we have $d(f^i(z_1), f^i(z_2)) > e$ for some i. Since f is continuous, then $U = f^{-1}(B_e(x))$ is an open set containing y. Set $W = U \cap V$. Now, W is an open set containing x. Thus every pair of distinct points in W must to be e-apart for some time greater than 1. Therefore $y \in Ex(f)$. □

Now we prove our main theorems on pointwise dynamics.

Theorem I. *Let X be a compact metric space and let (X, f) be a TDS. If there exists a non-periodic point x which is non-isolated, shadowable, positively-expansive and non-wandering then f has positive topological entropy.*

Proof. Let x be a non-periodic, non-wandering, positiveli-expansive and shadowable point. Let U be a neighborhood of x. Since x is expansive, there are $e > 0$ and an open neighborhood V of x such that for any pair of points $y, z \in V$ there exists l such that $d(f^l(x), f^l(y)) > e$.

Take $U' = U \cap V$. Now let $0 < \varepsilon < \frac{e}{10}$ such that $B_\varepsilon(x) \subset U'$ Let $0 < \delta < \frac{\varepsilon}{2}$ be given by the $\frac{\varepsilon}{2}$-shadowing through $\{x\}$.

Since X is compact, f is uniformly continuous and therefore there is $0 < \eta < \delta$ such that $d(f(y), f(z)) < \delta$ when $d(y, z) < \eta$.

Claim 4.2.14. *There are k and two distinct points $a, b \in U$ such that $f^{nk}(a), f^{nk}(b) \in U$ for every n.*

Since x is a non-periodic non-wandering point, we shall divide the proof of the claiming in two cases.

1) If x is a recurrent point, then there exists m_1 such that $f^{m_1}(x) \in B_\eta(x)$. Since X has not isolated points the set $W = B_\eta(X) \setminus \{f(x), ..., f^{m_1}(x)\}$ is an open neighborhood of x and therefore there exists m_2 such that $f^{m_2}(x) \in W$. Let $x_a = f^{m_1}(x)$ and $x_b = f^{m_2}(x)$. Clearly $x_a \neq x_b$ since $x_a \notin W$. Since $B_\eta \subset V$, there exists l such that $d(f^l(x_a), f^l(x_b)) > e$.

Now, $W' = B_\eta(x) \setminus \{f(x), f^2(x), ..., f^{m_2+l}\}$ is an open neighborhood of x, then there exists $m_3 > m_2 + l$ such that $f^{m_3}(x) \in W'$. From a similar argument we can find $m_4 > m_3$ such that $f^{m_4}(x) \neq f^{m_3}(x)$ and $f^{m_4}(x) \in W'$. Set $k_1 = m_3 - m_1$ and $k_2 = m_4 - m_2$.

Consider the sets

$$A = \{x, f(x_a), ..., f^{k_1-1}(x_a), f^{k_1}(x_a)\} \text{ and } B = \{x, f(x_b), ..., f^{k_2-1}(x_b), f^{k_2}(x_b)\}$$

We have $d(x, x_a), d(x, x_b) < \eta$ and therefore $d(f(x), f(x_a)), d(f(x)f(x_b)) < \delta$. Thus the above sets are δ pseudo orbits through $\{x\}$. The shadowableness of x implies that there exist a and b that $\frac{\varepsilon}{2}$-shadows A and B respectively. Thus $d(f^{nk_1}(a), x) < \frac{\varepsilon}{2}$ and $d(f^{nk_1}(a), x) < \frac{\varepsilon}{2}$ for every n. Then if $k = k_1k_2$ we have $f^{nk}(a), f^{nk}(b) \in U$ for every n. To prove that $a \neq b$ we notice that $d(f^l(x_b), f^l(b)), d(f^l(x_a), f^l(a)), < \varepsilon < \frac{e}{10}$ and $d(f^l(x_a), f^l(x_b)) > e$. Thus $f^l(a) \neq f^l(b)$ and therefore $a \neq b$.

2) If x is not a recurrent point there are $x_a \in B_\eta(x)$ and k_1 such that $x_a \neq x$ and $f^{k_1}(x_a) \in B_\eta(x)$. Since X has not isolated points the set $W = B_\eta(x) \setminus \{x_a, f(x_a), ..., f^{k_1}(x_a)\}$ is an open neighborhood of x. Thus there are x_b and k_2 such that $f^{k_2}(x_b) \in W$. Now we proceed in the same way as in the case 1) to obtain the points a and b and this proves our claiming.

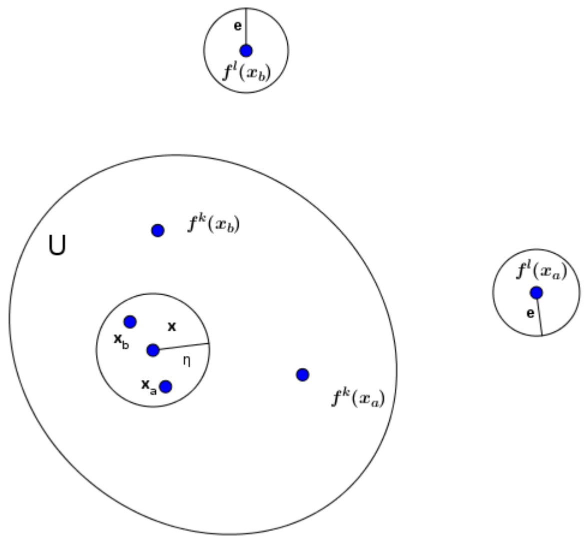

Figure 4.3: An illustration of the main idea of the proof of the claiming.

Let a, b given by the claiming and consider the sets $B_0 = \overline{B_\varepsilon(f^l(a))}$ and $B_1 = \overline{B_\varepsilon(f^l(b))}$. Since $\varepsilon < \frac{e}{10}$ and $d(f^l(a), f^l(b)) > e$, then $B_0 \cap B_1 = \emptyset$. Let (Σ, ρ) denote the space of the sequences of zeros and ones endowed with the standard metric. For each $s = (s_i)_{i\in\mathbb{N}} \in \Sigma$ define the set $Y_s := \{y \in \overline{B_\varepsilon(x)}; f^{ki} \in B_\varepsilon(x) \wedge f^{ki+l}(y) \in B_{s_i}, \forall i \in \mathbb{N}\}$.

We claim that Y_s is non empty for every s. indeed, fix $s \in \Sigma$. Define the following δ-pseudo-orbit through $\{x\}$

$$A_s = (Z_{s_1} Z_{s_2} .. Z_{s_i} ...)$$

Where $Z_{s_i} = (x, f(a), ..., f^{k-1}(a))$ if $s_i = 0$ and $Z_{s_i} = (x, f(a), ..., f^{k-1}(b))$ if $s_1 = 1$. Then the shadowing through $\{x\}$ gives a point y_s that ε-shadows A_s and therefore Y_s is non-empty.

Fix $s \in \Sigma$ and let $(y_j)_{j\in\mathbb{N}} \in Y_s$ be a sequence of points converging to y_s. Then for every i we have $f^{ki}(y_j) \in \overline{B_\eta(x)}$ and $f^{ki+l}(y_j) \in B_{s_i}$ for every j. The continuity of f^{ki} and f^{ki+l} implies that $f^{ki}(y_j) \to f^{ki}(y)$ and $f^{ki+l}(y_j) \to f^{ki+l}(y)$. Therefore $f^{ki}(y) \in \overline{B_\eta(x)}$ and $f^{ki+l}(y) \in B_{s_i}$. Thus Y_s is closed for every $s \in \Sigma$.

Set $Y = \bigcup_{s\in\Sigma} Y_s$. Let $(y_n)_{n\in\mathbb{N}} \in Y$ be a sequence of points in Y converging to a point y. Thus for each y_n there is an s_n such that $y_n \in Y_{s_n}$. By compacity of Σ we can assume that $s_n \to s$, up to take a subsequence of (s_n). Thus y must to belong to Y_s and therefore Y is closed.

Next we are going to show that Y is f^k-positively invariant. Indeed, take $y \in Y$. Then there exists s such that $y \in Y_s$. Thus $f^{ki}(y) \in \overline{B_\eta(x)}$ and $f^{ki+l}(y) \in B_{s_i}$ for every i. If we apply f^K on y, we obtain that $f^{k(i+i)}(y) \in \overline{B_\eta(x)}$ and $f^{k(i+1)+l}(y) \in B_{s_{(i+1)}}$, for every i. Then $f^k(y) \in Y_{s'}$ where $s'_i = s_{i+1}$ and therefore Y is f^k-positively invariant.

Now we will construct a semi-conjugation between the subsystem (Y, f^k) and the two symbol shift (Σ, σ). indeed, consider the map $\Pi : Y \to \Sigma$ defined by $\Pi(y) = s$ if

$y \in Y_s$. Since Y_s is non-empty for every s, Π is a surjection. Furthermore, by the above discussion $\Pi(f^k(y)) = s' = \sigma(\Pi(y))$. Thus Π satisfies the conjugation equation. The continuity of Π follows by an argument analogous to the one used to prove that Y is closed.

Thus (Y, f^k) is semi-conjugated to the two symbol shift and therefore $kh(f|_Y) = h(f^k|_Y) \geq h(\sigma) > 0$ and therefore $h(f) \geq h(f|_Y) > 0$. □

Next we will argue that we can weaken the hypothesis about the expansiveness of x.

Definition 4.2.15. *Let (X, F) be a TDS. We say that a point is x is an positively n-expansive point if there are $e > 0$ and V a neighborhood of x such that for any $y \in V$ the cardinality of the set $D(y, e) := \{z \in X; d(f^n(y), f^n(z)) < e, \forall n \in \mathbb{N}\}$ is smaller or equal than n.*

Clearly if x is an expansive point, then it is an n-expansive point for any n. Thus we can generalize the previous theorem to obtain the following

Theorem J. *Let X be a compact metric space without isolated points and let (X, f) be a TDS. If there exists a non-periodic point x which is shadowable, positively n-expansive and non-wandering then f has positive topological entropy.*

Proof. The kernel of the proof of theorem I is to find two distinct points points in U which in sometime be apart, but after this, they return to U. Then if we can do this in our case, we can reproduce the previous proof straightforward. To obtain these points we notice that since in x is not an isolated point, we can find any finite amount of points which behaves in a way similar to the a and b on the previous proof. Now suppose f is n-expansive and let $a_1, ..., a_{n+2} \in U$ and $k \in \mathbb{N}$ such that $f^{nk}(a_i) \in U$ for every n. Since f is n-expansive, if we fix a_i then at least one a_j is such that $d(f^m(a_i), f^m(a_j)) > e$ for some m. Now, we proceed in the same manner as in the proof of J, to prove the theorem. □

We end this work with a corollary of the proof of theorem I.

Corollary K. *Let X be a compact metric space and let (X, f) be a TDS. If f is expansive, f has the shadowing property and $\#Per(f) = \infty$, then $h(f) > 0$.*

Proof. Since X is compact, then the periodic points of f must to accumulate in some point x. Since f is a expansive map with the shadowing property, then x is shadowable and expansive. Now f is a non-wandering point since it is accumulated by periodic points. Here, x can be a periodic point, but it does not matter. Indeed, by the proof of theorem I we just need to find two distinct points with returns to U, then we can take two distinct periodic points in U. Hereafter, the proof is analogous to the the proof of I. □

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