On Allard's interior regularity theorem for varifolds with bounded generalized mean curvature

Julio César Correa Hoyos

Dissertao o de Mestrado apresentada ao Programa de Ps-graduao do Instituto de Matemtica, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessrios obteno do ttulo de Mestre em Matemtica.

Orientador: Stefano Nardulli.

11 de Dezembro de 2015

On the

On Allard's interior regularity theorem for varifolds with bounded generalized mean curvature

Julio César Correa Hoyos

Orientador: Stefano Nardulli.

Dissertao de Mestrado submetida ao Programa de Ps-graduao do Instituto de Matemtica da Universidade Federal do Rio de Janeiro - UFRJ como parte dos requisitos necessrios obteno do ttulo de Mestre em Matemtica.

Aprovada por:

(Presidente) Prof. Stefano Nardulli - IM/UFRJ

Prof. Wladimir Augusto das Neves - IM/UFRJ

Prof. Sergio Almaraz - UFF

Rio de Janeiro

11 de Dezembro de 2015

To my family back in Colombia, my Brazilian family, my local and non local friends.

Acknowledgements

The author is greatly indebted to his advisor Stefano Nardulli for introducing him to Geometric Measure Theory and calculus of variations, furthermore for his encouragement and support during the preparation of this dissertation.

The author wish also to express his thanks to Professors Wladimir Augusto das Neves and Sergio Almaraz for reading this work and accepting to participate of the examining board. Finally, the author gratefully acknowledges professor Camillo De Lellis for the many helpful suggestions and explanations concerning several technical points treated in what follows.

Resumo

Motivados pela crescente importância que as varifolds têm assumido na análise geométrica, nesta tese de mestrado estudamos os detalhes da prova agora clássica do teorema de regularidade de Allard. Seguiremos de perto a abordagem deste tópico desenvolvido nas notas de Camillo De Lellis [Lel12]. Por esta razão uma versão mais fraca do Teorema 8.1 de [All72], que concerne as varifolds com curvatura media generalizada é demonstrada. Isto permite de obter demonstraões simplificadas apesar de manter incluídas todas as técnicas e as dificuldades do caso geral.

Palavras chaves: Varifolds, Curvatura média geralizada, Teorema interior de Allard,Teoria Geométrica da Medida.

Abstract

Motivated by the fast growing importance that varifolds are assuming in geometric analysis, in this master thesis we study the details of the proof of the by now classical Allard's regularity theory for integral varifolds. We follow very closely the approach to this topic developed in the notes of Camillo De Lellis [Lel12]. For this reason a weaker version of Theorem 8.1 of [All72], which concerns varifolds with bounded generalized mean curvature is proved. This allows simplified proofs even if all the principal technics and difficulties of the general case appears already here.

Key words: Varifolds, First variation of a Varifold, Generalized mean curvature, Allard's interior regularity Theorem, Geometric Measure Theory.

Contents

Introduction

In the 1920, Besicovitch studied linearly measurable sets in the plane, that is sets with locally finite lenght. This marks the beginning of the study of the geometry of measures and the associated field known as Geometric Measure Theory. When considering geometric variational problem like the Plateau's problem or the isoperimetric problem, in general we minimize some geometric functional like the area functional over an admissible family of hypersurfaces subjected to various topological and/or analytical family of conditions, as, fixed volume, fixed boundary, etc. Even if the family of admissible hypersurfaces could be chosen as nice (smooth) as possible, it could happens that a minimizing sequence could converge to an object which lives outside the world of smooth objects, allowing for the raising of singularities. These new objects could be interpreted as the real mathematical model for the physical problem of Plateau or of interfaces. The advantage of using varifolds or other objects of the Geometric Measure Theory, like finite perimeter sets, is related to compactness an regularity properties which guarantee existence and regularity results once impossible to obtain. Furthermore those results make abstraction of the complete list of singularities that we have to include in the definition of an object which models our physical soap-film like hypersurfaces. Of course it is useful to know all possibles types of singularities that could appear, but the current state of the art still does not permits to achieve this goal. However varifolds offer geometrically significant solutions to a wide number of variational problems without having to know what all the possible singularities can be. In this thesis we are concerned with the proof of Theorem 8.1 of [All72], but actually we will give just a proof of a weaker result following the treatment of the proof of Theorem 3.2 of [Lel12]. One of the big differences between the result treated here and [All72] is that we restrict the theory to varifolds with bounded generalized mean curvature, whereas a suitable integrability assumption is usually sufficient. A second drawback is that hypothesis $(H2)$ in Allards interior ε-regularity Theorem 4.1 is redundant. Still, the statement given here suffices to draw the two major conclusions of Allards theory. A third disadvantage is that a few estimates coming into the proof of Theorem 4.1 are stated in a fairly suboptimal form. This work could be intended as a preparatory text to the reading and understanding of the more complete and sophisticated sources [Sim83] and [All72].

Chapter 1

Measures and outer measures

Definition 1.1. *Let $\mathcal{P}(X)$ be the set of all subsets of an arbitrary set X. An **outer measure** μ on X is a set function on X with values in $[0, \infty]$, i.e.,*

$$\mu : \mathcal{P}(X) \to [0, \infty],$$

such that the following properties are satisfied

1. $\mu(\emptyset) = 0$,
2. *if $E \subset \bigcup_{n\in\mathbb{N}} E_n$, then $\mu(E) \leq \sum_{n\in\mathbb{N}} \mu(E_n)$.*

Definition 1.2 (σ-algebra)**.** *Given a family $\mathcal{F} \subset \mathcal{P}(X)$ such that*

1. $\emptyset \in \mathcal{F}$,
2. *if $E \in \mathcal{F}$ then $X \setminus E \in \mathcal{F}$,*
3. *if $\{E_k\}_{k\in\mathbb{N}}$ is a sequence of elements of $\mathcal{F}$, then the countable union $\bigcup_{k\in\mathbb{N}} E_k$ belongs to $\mathcal{F}$.*

*The family $\mathcal{F}$ is said a **σ-algebra**.*

Definition 1.3 ($\sigma -$ *aditivity*)**.** *Given a family $\mathcal{F}$ of subsets of X, closed with respect to countable unions, we say that a set function μ on X is **σ-additive** on $\mathcal{F}$, provided that*

$$\mu\left(\mathring{\bigcup}_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \mu(E_n),$$

for every disjoint sequence $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$.

Definition 1.4. *The pair $(X, \mathcal{F})$, where X is a set and $\mathcal{F}$ a $\sigma-algebra$ on X, is said a* ***measure space****. Let $(X, \mathcal{F})$ be a measure space, we say that a function $\mu : \mathcal{F} \to [0, +\infty]$, is a (abstract)* ***measure****, if μ is an outer measure and satisfy the σ-additivity property on $\mathcal{F}$.*

There exist various examples of (concrete) measures, but among them we will consider two that are of particular interest in the calculus of variations and geometric measure theory.

Example 1.1 (Lebesgue Measure)**.** *Let $E \subset \mathbb{R}^n$, the n-**dimensional Lebesgue measure** of E is defined as*

$$\mathcal{L}^n(E) = \inf_{\mathcal{F}} \sum_{Q \in \mathcal{F}} r(Q)^n,$$

where $\mathcal{F}$ is a countable covering of E by cubes with sides parallel to the coordinates axis, and $r(Q)$ denote the length of a side of Q.
$\mathcal{L}^n$ is an outer measure (by construction). Moreover it is translation-invariant, that is

$$\mathcal{L}^n(x + E) = \mathcal{L}^n(E),$$

and satisfy the scaling law

$$\mathcal{L}^n(\lambda E) = \lambda^n \mathcal{L}^n, \quad \forall \lambda > 0.$$

If $B = \{x \in \mathbb{R}^n : |x| \leq 1\}$ is the ***Euclidean unit ball of*** *$\mathbb{R}^n$, we set $\omega_n = \mathcal{L}^n(B)$.*

Example 1.2 (Hausdorff Measure)**.** *Given $n, k \in \mathbb{N}$, $\delta > 0$, the k-dimensional* ***Hausdorff measure of step*** *δ, of $E \subset \mathbb{R}^n$ is defined as*

$$\mathcal{H}^k_\delta(E) := \inf_{\mathcal{F}} \sum_{f \in \mathcal{F}} \omega_k \left(\frac{diam(F)}{2} \right)^k, \tag{1.1}$$

where $\mathcal{F}$ is a countable covering of E by sets $F \subset \mathbb{R}^n$ such that $diam(F) < \delta$.
The k-dimensional ***Hausdorff measure*** *of E is*

$$\mathcal{H}^k(E) := \sup_{\delta > 0} \mathcal{H}^k_\delta = \lim_{\delta \downarrow 0} \mathcal{H}^k_\delta(E). \tag{1.2}$$

$\mathcal{H}^k_\delta$ is an outer measure, hence $\mathcal{H}^k$ is an outer measure too. $\mathcal{H}^k$ is translation-invariant, and satisfy a scaling law. The Hausdorff measure is a quite suitable generalization of the notion of area on a k-dimensional parametrized surface.

The following are basic results in measure theory and the only purpose is to illustrate the properties of Hausdorff and Lebesgue's measures, the proof of those can be found in any good textbook on measure theory as for example [Fed14] or [Mag12].

Theorem 1.1 (Caratheodory's theorem)**.** *If μ is an outer measure on $\mathbb{R}^n$ and define $\mathcal{M}(\mu)$ as the family of those $E \subset \mathbb{R}^n$ such that*

$$\mu(F) = \mu(E \cap F) + \mu(F \setminus E) \quad \forall F \subset \mathbb{R}^n,$$

then $\mathcal{M}(\mu)$ is a $\sigma - algebra$, and μ is a measure on $\mathcal{M}(\mu)$.

Remark 1.1. *When we define a measure μ' starting from an outer measure μ we always consider μ' defined as the restriction of μ on $\mathcal{M}(\mu)$, so μ' is always a complete measure. This is what we do with the Hausdorff measure $\mathcal{H}^k$.*

1.1 Measure and Integration

Definition 1.5. *A function $u : E \subset \mathbb{R}^n \to [-\infty, +\infty]$ is said μ-**mesurable** on $\mathbb{R}^n$ if $\mu(\mathbb{R}^n \setminus E) = 0$, and if for all $t \in \mathbb{R}$*

$$\{u > t\} = \{x \in E : u(x) > t\}$$

*belongs to $\mathcal{M}(\mu)$. A function u is a μ-**simple function** on $\mathbb{R}^n$, if is μ-mesurable and the image of u is countable.*

Definition 1.6. *Given a non-negative, μ-simple function u, the **integral of** u **with respect to** μ is defined in $[0, \infty]$ as the series*

$$\int_{\mathbb{R}^n} u d\mu = \sum_{t \in u(\mathbb{R}^n)} t\mu(\{u = t\}),$$

with the convention that $0 \cdot \infty = 0$.
When u is a μ-simple function, and either

$$\int_{\mathbb{R}^n} u^- d\mu, \text{ or}, \int_{\mathbb{R}^n} u^+ d\mu,$$

*is finite, we say that u is μ-**integrable simple function**, and set*

$$\int_{\mathbb{R}^n} u d\mu = \int_{\mathbb{R}^n} u^+ d\mu - \int_{\mathbb{R}^n} u^- d\mu.$$

Let $u : E \subset \mathbb{R}^n \to [-\infty, +\infty]$ be a function such that $\mu(\mathbb{R}^n \setminus E) = 0$; we define the ***upper and lower integrals with respect to*** *μ as*

$$\int_{\mathbb{R}^n}^{*} u d\mu = \inf_{v \in \mathcal{S}} \left\{ \int_{\mathbb{R}^n} v d\mu : v \geq u \ \mu - a.e. \ on \ \mathbb{R}^n \right\},$$
$$\int_{*\mathbb{R}^n} u d\mu = \sup_{v \in \mathcal{S}} \left\{ \int_{\mathbb{R}^n} v d\mu : v \leq u \ \mu - a.e. \ on \ \mathbb{R}^n \right\},$$

where $\mathcal{S}$ is the family of μ-integrable simple functions of $\mathbb{R}^n$.
If u is μ-mesurable and

$$\int_{\mathbb{R}^n}^{*} u d\mu = \int_{*\mathbb{R}^n} u d\mu,$$

we say that u is a ***μ-integrable function****, and this common value is called the* ***integral of*** *u* ***with respect to*** *μ, denoted by*

$$\int_{\mathbb{R}^n} u d\mu.$$

Theorem 1.2 (Monotone Convergence Theorem). *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of μ-mesurable functions $u_n : \mathbb{R}^n \to [0, \infty]$ such that $u_n \leq u_{n+1}$, μ-a.e. on $\mathbb{R}^n$, then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^n} u_n d\mu = \int_{\mathbb{R}^n} \sup_{n\in\mathbb{N}} u_n d\mu.$$

If, instead, $u_n \geq u_{n+1}$ and $u_1 \in L^1(\mathbb{R}^n; \mu)$, then

$$\lim_{n\to\infty} \int_{\mathbb{R}^n} u_n d\mu = \int_{\mathbb{R}^n} \inf_{n\in\mathbb{N}} u_n d\mu.$$

Theorem 1.3 (Fatou's Lemma). *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of μ-mesurable functions $u_n : \mathbb{R}^n \to [0, \infty]$, then*

$$\int_{\mathbb{R}^n} \liminf_{n\to\infty} u_n d\mu \leq \liminf_{n\to\infty} \int_{\mathbb{R}^n} u_n d\mu.$$

Theorem 1.4 (Dominated Convergence Theorem). *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of μ-mesurable functions, such that converges pointwise to some limit function u, μ-a.e. on $\mathbb{R}^n$ and if exist $v \in L^1(\mathbb{R}^n; \mu)$ such that*

$$|u_n| \leq v \quad \mu - a.e. \ on \ \mathbb{R}^n,$$

then

$$\int_{\mathbb{R}^n} u d\mu = \lim_{n\to\infty} \int_{\mathbb{R}^n} u_n d\mu.$$

Theorem 1.5 (Egoroff's Theorem). *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of μ-mesurable functions, such that converges pointwise to some limit function u, μ-a.e. on $\mathbb{R}^n$, then for every $\varepsilon > 0$ and for every $E \in \mathcal{M}(\mu)$ with $\mu(E) < \infty$, there exists $F \in \mathcal{M}(\mu)$ such that*

$$\mu(E \setminus F) \leq \varepsilon \text{ and } u_n \to u \text{ uniformly on } F.$$

Definition 1.7 (Product Measure). *Let μ be an outer measure on $\mathbb{R}^n$, and let ν be an outer measure on $\mathbb{R}^m$, an outer measure*

$$\mu \times \nu : \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^m) \to [0, \infty],$$

is defined at $G \subset \mathbb{R}^n \times \mathbb{R}^m$ by setting

$$\mu \times \nu(G) := \inf_{\mathcal{F}} \sum_{E\times F\subset\mathcal{F}} \mu(E)\nu(F),$$

*where $\mathcal{F}$ is a covering of G by sets of the form $E \times F$, where $E \in \mathcal{M}(\mu)$ and $F \in \mathcal{M}(\nu)$; $\mu \times \nu$ is called the **product measure**.*
To every $x \in \mathbb{R}^n$ there corresponds a vertical section $G_x \subset \mathbb{R}^n \times \mathbb{R}^m$

$$G_x := \{y \in \mathbb{R}^m : (x, y) \in G\}.$$

Theorem 1.6 (Fubini's Theorem). *Let μ be an outer measure on $\mathbb{R}^n$, and let ν be an outer measure on $\mathbb{R}^m$*

1. *If $E \in \mathcal{M}(\mu)$ and $F \in \mathcal{M}(\nu)$, then $E \times F \in \mathcal{M}(\mu \times \nu)$ and*

$$\mu \times \nu(E \times F) = \mu(E)\nu(F)$$

2. *For every $G \subset \mathbb{R}^n \times R^m$, there exists $H \in \mathcal{M}(\mu \times \nu)$ such that*

$$G \subset H, \text{ and } \mu \times \nu(G) = \mu \times \nu(H).$$

3. *If $G \subset \mathbb{R}^n \times \mathbb{R}^m$ is σ-finite with respect to $\mu \times \nu$, then $G_x \in \mathcal{M}(\nu)$ for μ-a.e. $x \in \mathbb{R}^n$. Moreover*

$$x \in \mathbb{R}^n \mapsto \nu(G_x) \text{ is } \mu - \text{mesurable on } \mathbb{R}^n,$$
$$\mu \times \nu(G) = \int_{\mathbb{R}^n} \nu(G_x) d\mu(x).$$

4. *If $u \in L^1(\mathbb{R}^n \times \mathbb{R}^m; \mu \times \nu)$ then*

$$x \in \mathbb{R}^n \mapsto \int_{\mathbb{R}^m} u(x, y) d\nu(y) \in L^1(\mathbb{R}^n; \mu),$$
$$\int_{\mathbb{R}^n\times\mathbb{R}^m} ud(\mu \times \nu) = \int_{\mathbb{R}^n} d\mu(x) \int_{\mathbb{R}^m} u(x, y) d\nu(y).$$

1.2 Borel and Regular Measures

Definition 1.8. *Let μ be a measure on $\mathbb{R}^n$, we say that μ is a* ***Borel measure*** *on $\mathbb{R}^n$, if $\mathcal{M}(\mu) = \mathcal{B}(\mathbb{R}^n)$, where $\mathcal{B}(\mathbb{R}^n)$ is the σ-algebra generated by the open sets of $\mathbb{R}^n$, i.e., the smallest σ-algebra containing the topology of $\mathbb{R}^n$.*

Theorem 1.7 (Caratheodory's Criterium)**.** *If μ is an outer measure on $\mathbb{R}^n$, then μ is a Borel measure on $\mathbb{R}^n$ if and only if*

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2),$$

for every $E_1, E_2 \subset \mathbb{R}^n$, such that $dist(E_1, E_2) > 0$.

Example 1.3. *The first example of a Borel measure, is the Lebesgues's measure restricted to $\mathcal{B}(\mathbb{R}^n)$. to prove this is a Borel measure, we will use Theorem 1.7, so it is enough to prove*

$$\mathcal{L}^n(E_1 \cup E_2) \geq \mathcal{L}^n(E_1) + \mathcal{L}^n(E_2),$$

for every $E_1, E_2 \subset \mathbb{R}^n$ such that $dist(E_1, E_2) > 0$. Let $\mathcal{F}$ be a countable family of disjoint cubes with sides parallel to the axis such that

$$E_1 \cup E_2 \subset \bigcup_{Q \in \mathcal{F}} Q.$$

Since $\mathcal{L}^n$ is additive on finite disjoint cubes, up to further divisions of each $Q \in \mathcal{F}$ into finite sub-cubes, we may assume $diam(Q) < d = dist(E_1, E_2)$ for every $Q \in \mathcal{F}$. Set

$$\mathcal{F}_j = \{Q \in \mathcal{F} : Q \cap E_j \neq \emptyset\},$$

then $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ and $E_j \subset \bigcap_{Q \in \mathcal{F}_j} Q$, $j = 1, 2$, so that

$$\begin{aligned} \sum_{Q \in \mathcal{F}} r(Q)^n &\geq \sum_{Q \in \mathcal{F}_1} r(Q)^n + \sum_{Q \in \mathcal{F}_2} r(Q)^n \\ &\geq \mathcal{L}^n(E_1) + \mathcal{L}^n(E_2). \end{aligned}$$

Finally, by the arbitrariness of $\mathcal{F}$, we conclude

$$\mathcal{L}^n(E_1 \cup E_2) \geq \mathcal{L}^n(E_1) + \mathcal{L}^n(E_2).$$

Thus by Theorem 1.7 $\mathcal{L}^n$ is a Borel measure.

Example 1.4. *Let $E_1, E_2 \subset \mathbb{R}^n$ be such that $dist(E_1, E_2) = d > 0$, choose $0 < \delta < d/4$, and let $\mathcal{F}$ a family of disjoint subsets of $\mathbb{R}^n$ such that $diam(F) < \delta$ for all $F \in \mathcal{F}$. Set*

$$\mathcal{F}_j = \{F \in \mathcal{F} : F \cap E_j\} \neq \emptyset \quad j = 1, 2,$$

then $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ and $E_j \subset \bigcap_{Q \in \mathcal{F}_j} Q$, $j = 1, 2$, so that

$$\begin{aligned}\sum_{F \in \mathcal{F}} \omega_k \left(\frac{diam(F)}{2}\right)^k &\geq \sum_{F \in \mathcal{F}_1} \omega_k \left(\frac{diam(F)}{2}\right)^k + \sum_{F \in \mathcal{F}_2} \omega_k \left(\frac{diam(F)}{2}\right)^k \\ &\geq \mathcal{H}^k_\delta(E_1) + \mathcal{H}^k_\delta(E_2).\end{aligned}$$

Taking the supremum over all coverings $\mathcal{F}$, we find

$$\mathcal{H}^k_\delta(E_1 \cup E_2) \geq \mathcal{H}^k_\delta(E_1) + \mathcal{H}^k_\delta(E_2),$$

provided $0 < 4\delta < d$. Letting $\delta \to 0$ we get

$$\mathcal{H}^k_\delta(E_1 \cup E_2) \geq \mathcal{H}^k(E_1) + \mathcal{H}^k(E_2).$$

Hence, by Theorem 1.7, $\mathcal{H}^k$ is a Borel measure.

Definition 1.9. *A Borel measure μ is said **regular**, if for every $F \subset \mathbb{R}^n$ there exists a Borel set E such that $F \subset E$, satistying*

$$\mu(E) = \mu(F).$$

Example 1.5. *Let us prove that $\mathcal{H}^k$ is regular. Let $m \in \mathbb{N}$ and let $E \subset \mathbb{R}^n$. Since*

$$\mathcal{H}^k_{1/m}(E) = \inf_{\mathcal{F}} \omega_k \sum_{F \in \mathcal{F}} \left(\frac{diam(F)}{2}\right)^k,$$

where $\mathcal{F}$ is a covering of E such that for every $F \in \mathcal{F}$, $diam(F) < \delta = \frac{1}{m}$, there exist a covering $\mathcal{F}_m$ of closed sets such that

$$\omega_k \sum_{F \in \mathcal{F}_m} \left(\frac{diam(F)}{2}\right)^k \leq \mathcal{H}^k_{1/m}(E) + \frac{1}{m}.$$

Set

$$G = \bigcap_{m \in \mathbb{N}} \bigcup_{F \in \mathcal{F}_m} F.$$

Notice that G is a Borel set, $E \subset G$ and

$$\mathcal{H}^k_{1/m}(G) \leq \omega_k \sum_{f \in \mathcal{F}_m} \left(\frac{diam(F)}{2}\right)^k \leq \mathcal{H}^k_{1/m}(E) + \frac{1}{m}.$$

Then, making $m \to \infty$ we get

$$\mathcal{H}^k(G) \leq \mathcal{H}^k(E),$$

by monotonicity of $\mathcal{H}^k$ we get

$$\mathcal{H}^k(E) \leq \mathcal{H}^k(G),$$

Then $\mathcal{H}^k$ is a regular measure on $\mathbb{R}^n$.

1.3 Approximation Theorems for Borel Measures

Theorem 1.8 (Inner approximation by compact sets). *If μ is a Borel measure on $\mathbb{R}^n$, $E \in \mathcal{B}(\mathbb{R}^n)$ with $\mu(E) < \infty$, then for every $\varepsilon > 0$ there exists $K \subset E$ compact set such that $\mu(E \setminus K) \leq \varepsilon$, in particular*

$$\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}.$$

Definition 1.10. *An outer measure μ on $\mathbb{R}^n$ is* ***locally finite*** *if*

$$\mu(K) < \infty, \quad \forall K \subset \mathbb{R}^n \text{ compact}.$$

Theorem 1.9 (Outer Approximation by Open Sets). *If μ is a locally finite measure on $\mathbb{R}^n$ and $E \in \mathcal{B}(\mathbb{R}^n)$, then*

$$\begin{aligned} \mu(E) &= \inf\{\mu(A) : E \subset A,\ A \text{ open}\} \\ &= \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}. \end{aligned}$$

1.4 Radon Measures. Restriction, Support and Push-forward

Definition 1.11. *An outer measure μ is a* ***Radon measure*** *in $\mathbb{R}^n$, if it is a locally finite, Borel regular measure on $\mathbb{R}^n$.*

By the previous theorem if μ is a Radon measure then

$$\begin{aligned} \mu(E) &= \inf\{\mu(A) : E \subset A,\ A \text{ open set}\} \\ &= \sup\{\mu(K) : K \subset E,\ K \text{ compact set}\}, \end{aligned}$$

for every $E \in \mathcal{B}(\mathbb{R}^n)$. Thus by Borel regularity a Radon measure μ is characterized on $\mathcal{M}(\mu)$ by its behaviour on compact (or on open) sets.

Example 1.6. *Since $\mathcal{L}^n$ is trivially locally finite, and by the previous comments it is a Borel regular measure, then $\mathcal{L}^n$ is a Radon measure. On the other hand if $0 \leq k < n$, $\mathcal{H}^k$ is not locally finite because*

$$\mathcal{H}^k(A) = \infty, \quad \forall A, \text{ open set.}$$

*To prove this let us introduce the notion of **Hausdorff dimension** of a set, to do this it is needed a generalization of our classical Hausdorff measure. Given $s \in [0, \infty)$, the s-**dimensional Hausdorff measures** $\mathcal{H}^s_\delta$ and $\mathcal{H}^s$ are defined by simply replacing k with s in (1.1) and (1.2) and replacing the normalization constant ω_k by,*

$$\omega_s = \frac{\pi^{s/2}}{\Gamma(1 + s/2)}, \quad s \geq 2,$$

*where $\Gamma :]0, \infty[\to [1, \infty[$ is the **Euler Gamma function***

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \quad s > 0.$$

*Then we define the **Hausdorff dimension** of E as*

$$\dim_{\mathcal{H}}(E) := \inf\{s \in [0, \infty[: \mathcal{H}^s(E) = 0\}.$$

Now, it is easy to check that if $E \in \mathbb{R}^n$, then $\dim(E) \in [0, n]$, so let us prove that, if $\mathcal{H}^s(E) < \infty$, for some $s \in [0, n[$, then $\mathcal{H}^t(E) = 0$ for every $t > s$. Indeed, if $\mathcal{F}$ is a countable covering of E by sets of diameter less than δ, then

$$\mathcal{H}^t_\delta(E) \leq \omega_t \sum_{F \in \mathcal{F}} \left(\frac{diam(F)}{2} \right)^t \leq \left(\frac{\delta}{2} \right)^{t-s} \frac{\omega_t}{\omega_s} \omega_s \sum_{F \in \mathcal{F}} \left(\frac{diam(F)}{2} \right)^s .$$

That is $\mathcal{H}^t_\delta(E) \leq C(t, s) \delta^{t-s} \mathcal{H}^s(E)$. Letting $\delta \downarrow 0$ we have that $\mathcal{H}^t(E) = 0$. By definition of Hausdorff dimension, we have that, if $A \in \mathbb{R}^n$ is an open set $\mathcal{H}^n(A) > 0$ and $k < n$ then $\mathcal{H}^k(A) = +\infty$. However, if $E \in \mathcal{B}(\mathbb{R}^n)$ with $\mu(E) < \infty$, then $\mathcal{H}^k \llcorner E$ is a Radon measure in $\mathbb{R}^n$, as we can see in the next theorem.

Definition 1.12. *Given an outer measure μ on $\mathbb{R}^n$, and $E \in \mathbb{R}^n$, the **restriction of** μ **to** E is defined as*

$$\mu \llcorner E(F) = \mu(E \cap F), \quad F \subset \mathbb{R}^n.$$

We have $\mathcal{M}(\mu) \subset \mathcal{M}(\mu \llcorner E)$.

Proposition 1.1 (Restriction of Borel Regular Measures). *If μ is a Borel regular measure on $\mathbb{R}^n$, and $E \in \mathcal{M}(\mu)$ is such that $\mu \llcorner E$ is locally finite, then $\mu \llcorner E$ is a Radon measure on $\mathbb{R}^n$.*

Definition 1.13. *An outer measure μ on $\mathbb{R}^n$ is* ***concentrated on*** *$E \subset \mathbb{R}^n$ if $\mu(\mathbb{R}^n \setminus E) = 0$. The intersection of the closed sets E such that μ is concentrated on E is denoted $Spt(\mu)$ and called the* ***support of*** *μ. In particular*

$$\mathbb{R}^n \setminus Spt(\mu) = \{x \in \mathbb{R}^n : \mu(B(x,r)) = 0 \ for \ some \ r > 0\}.$$

Definition 1.14. *Given a funcion $f : \mathbb{R}^n \to \mathbb{R}^m$ and an outer measure μ on $\mathbb{R}^n$, the* ***push-forward of*** *μ* ***trough*** *f is the outer measure $f_\sharp\mu$ on $\mathbb{R}^m$ defined by the formula*

$$f_\sharp\mu(E) = \mu(f^{-1}(E)), \quad E \subset \mathbb{R}^n.$$

Proposition 1.2 (Push-forward of a Radon Measure). *If μ is a Radon measure on $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous proper function, then $f_\sharp\mu$ is a Radon measure on $\mathbb{R}^n$, $Spt(f_\sharp\mu) = f(Spt(\mu))$, and for every Borel mesurable function $u : \mathbb{R}^m \to [0, \infty]$ it holds*

$$\int_{\mathbb{R}^n} u d(f_\sharp\mu) = \int_{\mathbb{R}^n} (u \circ f) d\mu.$$

1.5 Radon Measures and Continuous Functions

Theorem 1.10 (Lusin's Theorem). *If μ is a borel measure on $\mathbb{R}^n$, $u : \mathbb{R}^n \to \mathbb{R}$ is a continuous function , and $E \in \mathcal{B}(\mathbb{R}^n)$ with $\mu(E) < \infty$, then for every $\varepsilon > 0$ there exists a compact set $K \subset E$, such that u is continuous on K and*

$$\mu(E \setminus K) < \varepsilon.$$

Definition 1.15. *Let $L : C_c^0(\mathbb{R}^n; \mathbb{R}^m) \to \mathbb{R}$ be a linear functional, we define the* ***total variation of*** *L as the set function*

$$|L| : \mathcal{P}(\mathbb{R}^n) \to [0, \infty],$$

such that, for any $A \subset \mathbb{R}^n$ open set

$$|L|(A) := \sup\{\langle L, \varphi \rangle : \varphi \in C_c^0(A; \mathbb{R}^m), \ |\varphi| \leq 1\},$$

and for any arbitrary $E \subset \mathbb{R}^n$

$$|L|(E) := \inf\{|L|(A) : E \subset A, \ A \ open \ set\}.$$

Theorem 1.11 (Riez Representation Theorem). *If $L : C_c^0(\mathbb{R}^n;\mathbb{R}^m) \to \mathbb{R}$ is a bounded linear functional, then its total variation $|L|$ is a positive Radon measure on $\mathbb{R}^n$ and there exists a $|L|$-mesurable function $g : \mathbb{R}^n \to \mathbb{R}^m$, with $|g| = 1$, $|L|$-a.e. on $\mathbb{R}^n$ and*

$$\langle L, \varphi\rangle = \int_{\mathbb{R}^n} \langle \varphi, g\rangle d|L|, \quad \forall \varphi \in C_c^0(\mathbb{R}^n, \mathbb{R}^m),$$

that is, $L = g|L|$. Moreover, for every open set $A \subset \mathbb{R}^n$ we have

$$|L|(A) = \sup\left\{\int_{\mathbb{R}^n} \langle \varphi, g\rangle d|L| : \varphi \in C_c^0(A, \mathbb{R}^m),\ |\varphi| \leq 1\right\},$$

and for each arbitrary subset $E \subset \mathbb{R}^n$ the following formula holds

$$|L|(E) := \inf\{|L|(A) : E \subset A \textit{ and } A \textit{ is open}\}.$$

1.6 Weak* convergence of Radon measures

Remark 1.2. *If L is a **monotone linear functional on** $C_c^0(\mathbb{R}^n)$, i.e., $L(\varphi_1) \leq L(\varphi_2)$ whenever $\varphi_1 \leq \varphi_2$, then L is bounded on $C_c^0(\mathbb{R}^n)$, and by Theorem 1.11 it holds*

$$\langle L, \varphi\rangle = \int_{\mathbb{R}^n} \varphi g d|L|,$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is $|L|$-mesurable, with $|g| = 1$ $|L|$-a.e. on $\mathbb{R}^n$, and

$$\langle L, \varphi\rangle = \int_{\mathbb{R}^n} \varphi d|L|, \quad \forall \varphi \in C_c^0(\mathbb{R}^n).$$

Note also that if two Radon measures μ_1, μ_2 on $\mathbb{R}^n$ coincides as linear functionals, that is

$$\int_{\mathbb{R}^n} \varphi d\mu_1 = \int_{\mathbb{R}^n} \varphi d\mu_2, \quad \forall \varphi \in C_c^0(\mathbb{R}^n),$$

then we have $\mu_1 = \mu_2$, so Radon measures can be unambiguously identified with monotone linear functionals on $C_c^0(\mathbb{R}^n)$.

Remark 1.3. *Let $\mathcal{B}_b(\mathbb{R}^n)$ denote the family of bounded Borel sets of $\mathbb{R}^n$ and $\mathcal{B}(E)$ the family of bounded sets contained in $E \subset \mathbb{R}^n$. If L is a bounded linear functional on $C_c^0(\mathbb{R}^n;\mathbb{R}^m)$, then L induces a $\mathbb{R}^m$-valued set function*

$$\nu : \mathcal{B}_b(\mathbb{R}^n) \to \mathbb{R}^m$$
$$E \mapsto \nu(E) := \int_E g d|L|,$$

that enjoys the σ-additivity property

$$\nu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \nu(E_n),$$

on every disjoint sequence $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(K)$ for some K compact set in $\mathbb{R}^n$.

Thus bounded linear functionals on $C_c^0(\mathbb{R}^n;\mathbb{R}^m)$ naturally induces a $\mathbb{R}^m$-valued set function on $\mathbb{R}^n$ that are σ-additive on bounded Borel sets.

Definition 1.16. *Taking into account Remark 1.2 and the Remark 1.3, we define a* $\mathbb{R}^m$***-valued measure Radon measure on*** $\mathbb{R}^n$ *as the bounded linear functionals on $C_c^0(\mathbb{R}^n;\mathbb{R}^m)$, when $m = 1$ we speak of* ***signed Radon measures on*** $\mathbb{R}^m$*, we shall always adopt the Greek symbols μ, ν, etc. in place of L to denote the vector valued Radon measures and also set*

$$\langle \mu, \varphi \rangle := \int_{\mathbb{R}^n} \varphi \cdot d\mu,$$

to denote the value of the $\mathbb{R}^m$-valued Radon measure μ on $\mathbb{R}^n$ at $\varphi \in C_c^0(\mathbb{R}^n;\mathbb{R}^m)$.

Definition 1.17. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ and μ Radon measures with values in $\mathbb{R}^n$. We say that μ_n* ***weak* (pronounced weak star) converge to*** μ*, and we write $\mu_n \overset{\star}{\rightharpoonup} \mu$, if*

$$\int_{\mathbb{R}^n} \varphi d\mu = \lim_{n\to\infty} \int_{\mathbb{R}^n} \varphi d\mu_n, \quad \forall \varphi \in C_c^0(\mathbb{R}^n;\mathbb{R}^m).$$

The following Theorem, is an important property of weak* convergence of Radon measures, and will be used along the text repeatedly, the full proof of that can be found in [Mag12] Proposition 4.26.

Proposition 1.3. *If $\{\mu_n\}_{n\in\mathbb{N}}$ and μ are Radon measures on $\mathbb{R}^n$, then the following are equivalents*

1. *$\mu_n \overset{\star}{\rightharpoonup} \mu$.*

2. *If K is a compact set and A is an open set, then*

$$\mu(K) \geq \limsup_{n\to\infty} \mu_n(K),$$

$$\mu(A) \leq \liminf_{n\to\infty} \mu_n(A).$$

3. *If E is a bounded Borel set with $\mu(\partial E) = 0$, then*

$$\mu(E) = \lim_{n\to\infty} \mu_n(E).$$

Moreover if $\mu_n \overset{\star}{\rightharpoonup} \mu$ the for every $x \in Spt\mu$ there exists $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^n$ with

$$\lim_{n\to\infty} x_n = x, \; x_n \in Spt\mu_n, \; \forall n \in \mathbb{N}.$$

An important result about weak-$*$ convergence of Radon measures is the following compactness criteria, and again the best general reference here is Proposition 4.33 of [Mag12].

Theorem 1.12 (Compactness Criteria for Radon Measures). *If $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of Radon measures on $\mathbb{R}^n$ such that, for all compact set K in $\mathbb{R}^n$*

$$\sup_{n\in\mathbb{N}} \mu_n(K) < \infty,$$

then there exists a Radon measure μ on $\mathbb{R}^n$ and a sequence $n(k) \to \infty$, as $k \to \infty$ such that

$$\mu_{n(k)} \overset{\star}{\rightharpoonup} \mu.$$

1.7 Besicovitch's Covering Theorem

We discuss here Besicovitch's covering theorem, one of the most frequently used technical tool in geometric measure theory. We omit the proof that could be found in [Mag12], compare Theorem 5.1 therein.

Theorem 1.13 (Besicovitch's Covering Theorem). *If $n \geq 1$ then there exists a positive constant $\xi(n)$ with the following property. If $\mathcal{F}$ is a family os closed non-degenerated balls of $\mathbb{R}^n$, and either the set C of centers of the balls in $\mathcal{F}$ is bounded or*

$$\sup_{\overline{B}\in\mathcal{F}} diam(\overline{B}) < \infty,$$

then there exists $\mathcal{F}_1, \ldots, \mathcal{F}_{\xi(n)}$ (possibly empty) subfamilies of $\mathcal{F}$ such that

1. *Each family $\mathcal{F}_i$ is disjoint and at most countable,*

2. $C \subseteq \bigcup_{i=1}^{\xi(n)} \bigcup_{\overline{B}\in\mathcal{F}_i}$.

Corollary 1.1. *If μ is an outer measure on $\mathbb{R}^n$ and $\mathcal{F}$ and C are as in Theorem 1.13, then there exists a countably disjoint subfamily $\mathcal{F}'$ of $\mathcal{F}$ with*

$$\mu(C) \leq \xi(n) \sum_{\overline{B}\in\mathcal{F}'} \mu(C \cap \overline{B}).$$

If, moreover, μ is a Borel measure and C is μ-measurable, then

$$\mu(C) \leq \xi(n)\mu\left(C \cap \bigcup_{\overline{B} \in \mathcal{F}'} \overline{B}\right).$$

Chapter 2

Rectifiable Sets and Geometric Measure Theory

The main purpose of this chapter is to introduce the notation and main theorems of Geometric Measure Theory, required to understand the proof of Allard's Regularity Theorem 8.1 [All72].

2.1 Lebesgue-Besicovitch Differentiation Theorem

The first of a series of results is the **Lebesgue-Besicovitch Theorem** which asserts that given μ and ν Radon measures on $\mathbb{R}^n$, under suitable assumptions, we can decompose ν in "terms" of μ. With this aim in mind, let us begin with some definitions.

Definition 2.1. *Let μ, ν Radon measures on $\mathbb{R}^n$, we say that ν* ***is absolutely continuous with respect to*** *μ, and we write $\nu \ll \mu$, if for every $E \in B(\mathbb{R}^n)$, $\mu(E) = 0$, implies $\nu(E) = 0$. We say that μ and ν are* ***mutually singular****, and we write $\mu \perp \nu$, if for any given Borel set $E \in \mathbb{R}^n$, it holds*

$$\mu(\mathbb{R}^n \setminus E) = \nu(E) = 0.$$

Definition 2.2 (Upper and lower μ densities)**.** *Let μ and ν be Radon measures on $\mathbb{R}^n$, the* ***upper*** *μ* ***density*** *and the* ***lower*** *μ* ***density*** *of ν are functions*

$$D^+_\mu \nu : spt\mu \to [0, \infty],$$

$$D^-_\mu \nu : spt\mu \to [0, \infty],$$

defined respectively as

$$D^+_\mu\nu(x) := \limsup_{r\downarrow 0} \frac{\nu(\overline{B}(x,r))}{\mu(\overline{B}(x,r))}, \quad x \in spt\mu,$$
$$D^-_\mu\nu(x) := \liminf_{r\downarrow 0} \frac{\nu(\overline{B}(x,r))}{\mu(\overline{B}(x,r))}, \quad x \in spt\mu.$$

If the two limits exists and are finite, then we denote by $D_\mu\nu(x)$ their common value and call it the μ-density on ν at x. Thus we have defined a function

$$D_\mu\nu : \{x \in spt\mu : D^-_\mu\nu = D^+_\mu\nu\} \to [0,\infty].$$

We have thus defined a special function on $spt\mu$, but as we already noticed, we are interested in Borel functions, and this make the object of the next proposition.

Proposition 2.1. *Let μ be a Radon measure on $\mathbb{R}^n$, let $r > 0$, let us define $u(x) := \mu(\overline{B}(x,r))$. Then u is an upper semicontinuous function.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^n$ such that $x_n \to x_0$ for some $x_0 \in \mathbb{R}^n$, and define

$$\mu_{x_n} := (\tau_{x_n})_\sharp\mu,$$

where $\tau_{x_n}(y) = y - x_n$, clearly μ_{x_n} is a Radon measure, then applying Theorem 1.4 and Theorem 1.11 we obtain for every $\varphi \in C^0_c(\mathbb{R}^n)$

$$\begin{aligned}
\lim_{n\to\infty} \mu_{x_n}(\varphi) &= \lim_{n\to\infty} (\tau_{x_n})_\sharp\mu(\varphi) \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^n} \varphi \cdot d(\tau_{x_n})_\sharp\mu \\
&= \int_{\mathbb{R}^n} \varphi \circ \tau_{x_n} d\mu(y) \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^n} \varphi(y - x_n) d\mu(y) \\
&= \int_{\mathbb{R}^n} \lim_{n\to\infty} \varphi(y - x_n) d\mu(y) \\
&= \int_{\mathbb{R}^n} \varphi(y - x_0) d\mu(y) \\
&= \int_{\mathbb{R}^n} \varphi \circ \tau_{x_0} d\mu(y) \\
&= \int_{\mathbb{R}^n} \varphi \cdot d(\tau_{x_0})_\sharp\mu(y),
\end{aligned}$$

hence $\mu_{x_n} \overset{\star}{\rightharpoonup} \mu_{x_0}$. Now by Proposition 1.3 and the fact that $\overline{B}(0,r)$ is compact for all $r > 0$ we get easily

$$\begin{aligned}
\limsup_{n\to\infty} \mu_{x_n}(\overline{B}(0,r)) &\le \mu_{x_0}(\overline{B}(0,r)) \\
\limsup_{n\to\infty} \mu(\tau_{x_n}^{-1}(\overline{B}(0,r))) &\le \mu(\tau_{x_0}^{-1}(\overline{B}(0,r))) \\
\limsup_{n\to\infty} \mu(\overline{B}(x_n,r)) &\le \mu(\overline{B}(x_0,r)) \\
\limsup_{n\to\infty} u(x_n) &\le u(x_0),
\end{aligned}$$

proving that u is upper semicontinuous. $\square$

Remark 2.1. *If we define $v(x) := \mu(B(x,r))$ we can prove that v is a lower semicontinuous function, and the proof goes along the same lines.*

Remark 2.2. *By Proposition 2.1, $D_\mu^+\nu$ and $D_\mu^-\nu$ are Borel functions, which can be extended to the whole $\mathbb{R}^n$ and so to $D_\mu\nu$. Now, since there exist many countably $r > 0$ such that either $\mu(\partial B(x,r)) = 0$ or $\nu(\partial B(x,r)) = 0$ (the full proof of this fact can be found in [Mag12] prop 2.16), if $D_\mu\nu$ is defined at x, then*

$$D_\mu\nu(x) = \lim_{r\downarrow 0} \frac{\nu(\overline{B}(x,r))}{\mu(\overline{B}(x,r))}, \quad \forall x \in spt\mu.$$

Theorem 2.1 (Lebesgue-Besicovitch differentiation Theorem)**.** *If μ and ν are Radon measures on $\mathbb{R}^n$, then $D_\mu\nu$ is defined μ-a.e. on $\mathbb{R}^n$, $D_\mu\nu \in L^1_{loc}(\mathbb{R}^n,\mu)$, and, in fact $D_\mu\nu$ is a Borel measure on $\mathbb{R}^n$. Furthermore*

$$\nu = (D_\mu\nu)\mu + \nu_\mu^s, \quad on\ \mathcal{M}(\mu), \tag{2.1}$$

where the Radon measure ν_μ^s is concentrated on the Borel set

$$\begin{aligned}
Y &= \mathbb{R}^n \setminus \{x \in spt\mu : D_\mu^+\nu(x) < \infty\} \\
&= (\mathbb{R}^n \setminus spt\mu) \cup \{x \in spt\mu : D_\mu^+\nu(x) = \infty\},
\end{aligned}$$

in particular; $\nu_\mu^s \perp \mu$.

Proof. The proof will be divided into four steps

Step one: First we show that we can reduce (2.1) on $\mathcal{B}_b(\mathbb{R}^n)$. Clearly $(D_\mu\nu)\mu$ is a measure on $\mathcal{M}(\mu)$, so that, by intersecting balls with increasing radii, we see that it suffice to prove (2.1) on bounded μ-mesurable sets.

Now, if $E \in \mathcal{M}(\mu)$ is bounded, then by Borel regularity of μ, there exists $F \in \mathcal{B}_b(\mathbb{R}^n)$ with $E \subset F$ and $\mu(E) = \mu(F)$; moreover, $(\nu - \nu_\mu^s)$ is a Radon measure on $\mathbb{R}^n$, again by Borel regularity there exists a bounded Borel set G with $E \subset G$ and $(\nu - \nu_\mu^s)(E) = (\nu - \nu_\mu^s)(G)$; combining those facts with the validity of (2.1) on F and G, we thus conclude that

$$\begin{aligned}(\nu - \nu_\mu^s)(E) &= (\nu - \nu_\mu^s)(G)\\ &= \int_G D_\mu \nu d\mu\\ &\geq \int_E D_\mu \nu d\mu\\ &= \int_F D_\mu \nu d\mu\\ &= (\nu - \nu_\mu^s)(F)\\ &\geq (\nu - \nu_\mu^s)(E),\end{aligned}$$

i.e., $(\nu - \nu_\mu^s)(E) = \int_E D_\mu \nu d\mu$.

Step two:

We show that, if $t \in]0, +\infty]$ and $E \in \mathcal{B}_b(\mathbb{R}^n)$ then

$$\begin{aligned}&if\ E \subset \{D_\mu^- \nu \leq t\},\ then\ \nu(E) \leq t\mu(E),\\ &if\ E \subset \{D_\mu^+ \nu \geq t\},\ then\ \nu(E) \geq t\mu(E).\end{aligned} \tag{2.2}$$

It suffices to prove (2.2). Fix $\varepsilon > 0$ and let A be an open bounded set such that $E \subset A$ and $\mu(A) \leq \varepsilon + \mu(E)$, as $E \subset \{D_\mu^- \nu \leq t\}$ the family

$$\mathcal{F} = \{\overline{B}(x,r) : x \in E,\ \overline{B}(x,r) \subset A,\ \nu(\overline{B}(x,r)) \leq (t+\varepsilon)\mu(\overline{B}(x,r))\},$$

satisfies the assumption of Corollary 1.1. Hence there exists a countably disjoint subfamily $\{\overline{B}(x_n, r_n)\}_{n\in\mathbb{N}} \subset \mathcal{F}$ such that

$$\nu\left(E \setminus \bigcup_{n\in\mathbb{N}} \overline{B}(x_n, r_n)\right) = 0,$$

and

$$\begin{aligned}\nu(E) &= \sum_{n\in\mathbb{N}} \nu(\overline{B}(x_n, r_n)) \\ &\le (t+\varepsilon) \sum_{n\in\mathbb{N}} \mu(\overline{B}(x_n, r_n)) \\ &\le (t+\varepsilon)\mu(A) \\ &\le (t+\varepsilon)(\mu(E)+\varepsilon).\end{aligned}$$

Step three:

We prove that $D_\mu \nu$ exists and is finite for μ-a.e. $x \in \mathbb{R}^n$. It is enough to prove that the two sets

$$Z = \{D^+_\mu \nu = \infty\}, \quad Z_{p,q} = \{D^-_\mu \nu < q < p < D^+_\mu \nu\} \ p, q \in \mathbb{Q},$$

have μ-measure zero. Indeed, $Z \subset \{D^+_\mu \nu \ge t\}$ for every $t \in (0, \infty)$, and thus, by step two

$$\mu(Z \cap B_R) \le \frac{\nu(Z \cap B_R)}{t} \le \frac{\nu(B_R)}{t}.$$

Since $\nu(B_R)$ is finite, by letting $t \to \infty$ and $R \to \infty$, we find $\mu(Z) = 0$.

Now again by the step two , we have that for all $R > 0$

$$\nu(Z_{p,q} \cap B_R) \le q\mu(Z_{p,q} \cap B_R) \le \frac{q}{p}\nu(Z_{p,q} \cap B_R).$$

Since $q/p < 1$, we have $\mu(Z_{p,q} \cap B_R) = 0$, and thus $\mu(Z_{p,q}) = 0$.

Step four:

Let us set $\nu = \nu_1 + \nu_2$, where

$$\nu_1 = \nu \llcorner (\mathbb{R}^n \setminus Y), \quad \nu_2 = \nu \llcorner Y, \quad Y = (\mathbb{R}^n \setminus Spt\mu) \cup \{D^+_\mu \nu = \infty\}.$$

By step three $\mu(Y) = 0$, thus $\nu_2 \perp \mu$. We are left to prove that

$$\nu\left(E \cap \{D^+_\mu \nu < \infty\}\right) = \int_E D_\mu \nu d\mu,$$

for all $E \in \mathcal{B}_b(\mathbb{R}^n)$. By step two

$$\begin{aligned}\nu\left(\{D^-_\mu = 0\} \cap B_R\right) &\le \nu\left(\{D^-_\mu \le \varepsilon\} \cap B_R\right) \\ &\le \varepsilon\mu(B_R),\end{aligned}$$

therefore $\nu\left(\{D^-_\mu = 0\}\right) = 0$. As $D_\mu \nu$ exists and is finite μ-a.e. on $spt\mu$, we are thus reduced to prove

$$\nu(W \cap E) = \int_E D_\mu \nu d\mu, \tag{2.3}$$

for all Borel set E, where

$$W = \{x \in spt\mu : D_\mu\nu \ exists, \ 0 < D_\mu\nu < \infty\}.$$

To prove (2.3), fix $t \in (1, \infty)$ and let

$$E_k := E \cap \{w \in W : t^k < D_\mu\nu(w) < t^{k+1}\}, \quad \forall k \in \mathbb{Z}.$$

As $\{E_k\}_{k\in\mathbb{N}}$ is a sequence of disjoint Borel sets, with $E \cap W = \bigcup_{k\in\mathbb{N}} E_k$, we find

$$\int_E D_\mu\nu d\mu = \int_{E\cap W} D_\mu\nu d\mu = \sum_{k\in\mathbb{Z}} \int_{E_k} D_\mu\nu d\mu,$$

$$\nu(E \cap W) = \sum_{k\in\mathbb{Z}} \nu(E_k).$$

By step two, we have $\nu(E_k) \leq t^{k+1}\mu(E_k)$, and thus

$$\begin{aligned}
\nu(E \cap W) &\leq \sum_{k\in\mathbb{Z}} \nu(E_k) \\
&\leq \sum_{k\in\mathbb{Z}} t^{k+1}\mu(E_k) \\
&= t\sum_{k\in\mathbb{Z}} t^k\nu(E_k) \\
&\leq t\sum_{k\in\mathbb{Z}} \int_{E_k} D_\mu\nu d\mu \\
&\leq t\int_{E\cap W} D_\mu\nu d\mu.
\end{aligned}$$

Again by step two, we have $\nu(E_k) \geq t^k\mu(E_k)$, and so

$$\begin{aligned}
\nu(E \cap W) &\geq \sum_{k\in\mathbb{Z}} \nu(E_k) \\
&\geq \sum_{k\in\mathbb{Z}} t^k\mu(E_k) \\
&= \frac{1}{t}\sum_{k\in\mathbb{Z}} t^{k+1}\nu(E_k) \\
&\geq \frac{1}{t}\sum_{k\in\mathbb{Z}} \int_{E_k} D_\mu\nu d\mu \\
&\geq \frac{1}{t}\int_{E\cap W} D_\mu\nu d\mu.
\end{aligned}$$

Letting $t \to 1^+$, in the preceding inequality we have

$$\nu(E \cap W) = \int_{E\cap W} D_\mu\nu d\mu.$$

$\square$

The following Corollary is the natural extension of Theorem 2.1 to vector-valued Radon measures, and the proof is an easy exercise of decomposition of measures.

Corollary 2.1. *If ν is an $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$, and μ is a Radon measure on R^n, then for μ-a.e. $x \in \mathbb{R}^n$ there exists the limit*

$$D_\mu \nu(x) = \lim_{r \downarrow 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} \in \mathbb{R}^m,$$

which defines a Borel vector field $D_\mu \nu(x) \in L^1_{loc}(\mathbb{R}^n, \mu, \mathbb{R}^m)$, with the property that

$$\nu = (D_\mu \nu)\mu + \nu^s_\mu, \quad \text{on } \mathcal{M}(\mu)$$

where $\nu^s_\mu \perp \mu$.

To end this section we show an important and direct consequence of Theorem 2.1, known as *Lebesgue's points Theorem.*

Theorem 2.2 (Lebesgue points). *If μ is a Radon measure on $\mathbb{R}^n$, $p \in [1, \infty[$ and $u \in L^1_{loc}(\mathbb{R}^n, \mu)$, then for μ-a.e. $x \in \mathbb{R}^n$*

$$\lim_{r \downarrow 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u(x) - u(y)|^p d\mu(y) = 0.$$

In this case, we say that x is a ***Lebesgue point*** *of u with respect to μ.*

Proof. First we claim that for μ-a.e $x \in \mathbb{R}^n$

$$\lim_{r \downarrow 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u d\mu = u(x). \tag{2.4}$$

First we observe that the signed measure $\nu = u\mu$ is clearly absolutely continuous with respect to μ, and thus, by Theorem 2.1, for μ-a.e. $x \in \mathbb{R}^n$ the limit

$$D_\mu \nu(x) = \lim_{r \downarrow 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} = \lim_{r \downarrow 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u d\mu,$$

exists, and for every Borel set E

$$\int_E u d\mu = \nu(E) = \int_E D_\mu \nu d\mu,$$

in particular, $u = D_\mu \nu$, μ-a.e. on $\mathbb{R}^n$, so (2.4) is proved.

Now let $\mathbb{Q} = \{q_k\}_{k \in \mathbb{N}}$. For every $k \in \mathbb{N}$ there exists a μ-null set E_k such that

$$\lim_{r \downarrow 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u - q_k|^p d\mu = |u(x) - q_k|^p, \quad \forall x \in \mathbb{R}^n \setminus E_k.$$

If $E = \bigcup_{k\in\mathbb{N}} E_k$, then $\mu(E) = 0$ and for al $x \in \mathbb{R}^n \setminus E$ and $k \in \mathbb{N}$

$$\begin{aligned}\int_{B(x,r)} |u(x) - u|^p d\mu &= \left(\int_{B(x,r)} |u(x) - q_k + q_k - u|^p d\mu\right) \\ &\leq 2^{p-1}\left(|u(x) - q_k|^p \mu(B(x,r)) + \int_{B(x,r)} |q_k - u|^p d\mu\right),\end{aligned}$$

then, dividing by $\mu(B(x,r))$ and taking limits we have

$$\lim_{r\downarrow 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u(x) - u|^p d\mu \leq 2^{p-1}|u(x) - q_k|^p,$$

for all $k \in \mathbb{N}$. So taking a subsequence of $\{q_k\}_{k\in\mathbb{N}}$, such that $q_{k_m} \to u(x)$, we conclude the proof. □

2.2 Area Formula

The purpose of this section is to prove an important result about Lipschitz injective functions, which roughly speaking, relates the Hausdorff measure of the image of a measurable set with the Jacobian of a function f, which by Rademacher's Theorem exists $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$ (The full proof of the Rademacher's Theorem can be found in [Mag12] Theorem 7.8, or [EG15] Theorem 3.1.2).

Before continuing, let us establish a few important facts about linear transformations and define the Jacobian of a function f.

Definition 2.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $1 < n < m$. We define the **Jacobian of** f as the Borel bounded function $Jf : \mathbb{R}^n \to [0, \infty]$, by*

$$Jf(x) = \begin{cases} \sqrt{\det\left((\nabla f(x))^\star \nabla f\right)}, & \text{if } f \text{ is differentiable at } x; \\ \infty, & \text{if } f \text{ is not differentiable at } x. \end{cases}$$

Notice that $\{x \in \mathbb{R}^n : Jf(x) < \infty\}$ coincides with the set of points $x \in \mathbb{R}^n$ at which f is differentiable, then by the Rademacher's Theorem, $\{Jf(x) < \infty\}$ has full Lebesgue measure on $\mathbb{R}^n$.

As already noticed, we will use a lot of terminology from linear algebra, then before to continue let us establish a few facts about linear functions, $T : \mathbb{R}^n \to \mathbb{R}^m$, with $n \leq m$. In first, it is useful to identify the vector space of all $\mathbb{R}$-linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, $Hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$, with $\mathbb{R}^m \otimes \mathbb{R}^n$. This identification is done via the isomorphism of vector spaces that depends on the choice of a fixed orthonormal basis $\mathcal{W} := (v_i)_{1\leq i\leq n}$ of $\mathbb{R}^n$, completed to an orthonormal basis of $\mathcal{V} := (v_i)_{1\leq i\leq m}$ of $\mathbb{R}^m$, that associates to the matrix $A_{ij} = [\delta_{ij}]$, the element $v_i \otimes v_j$. In this way every element $T = a_{ij}A_{ij} \in Hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$ is identified with an element $T = a_{ij}v_i \otimes v_j$. In general, given $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, $w \otimes v : x \mapsto \langle v, x\rangle w$.

Definition 2.4. *Let $T \in \mathbb{R}^m \otimes \mathbb{R}^n$, T is said **orthonormal**, and we write $T \in O(n,m)$, if $\langle Tu, Tv\rangle = \langle u, v\rangle$ for all $u, v \in \mathbb{R}^n$, it is easy to check that, if $T \in O(n,m)$, then* $\ker T = \{0\}$ *and*

$$||T|| := sup\{|Tu| : |u| = 1\} = Lip(T) = 1, \quad \forall T \in O(n,m).$$

*If $T \in O(n,m)$, where $Lip(T)$ denotes the Lipschitz constant of T, then $T^\star T = id_{\mathbb{R}^n}$, where $T^\star$ denotes the adjoint of T. If $T^\star \in O^\star(n,m) := \{T^\star : T \in O(n,m)\}$ we say that $T^\star$ is an **orthogonal projection**. If $T^\star$ is an orthogonal projection we have* $\ker T^\star = T(\mathbb{R}^n)^\perp$ *and*

$$||T^\star|| = Lip(T^\star) = 1, \quad \forall T^\star \in O^\star(n,m),$$

moreover

$$|T^\star u - T^\star v| = |u - v|, \quad \forall u, v \in T(\mathbb{R}^n).$$

*Let $Sym(n) := \{T \in \mathbb{R}^n \otimes \mathbb{R}^n : T^\star = T\}$, if $T \in Sym(n)$ we say that T is **symmetric**, and by the spectral theorem, if $T \in Sym(n)$ then there exists an orthonormal basis $\{v\}_{i=1}^n$ of $\mathbb{R}^n$ of eigenvalues of T, such that*

$$T = \sum_{i=1}^n \lambda_i v_i \otimes v_i,$$

where $\lambda_i = \langle v_i, Tv_i\rangle$.

The next proposition asserts that given $T \in \mathbb{R}^m \otimes \mathbb{R}^n$ we can decompose it in the following way.

Proposition 2.2 (Polar decomposition)**.** *Given $T \in \mathbb{R}^m \otimes \mathbb{R}^n$, then there exist $P \in O(n,m)$ and $S \in Sym(n)$ such that*

$$T = PS.$$

Proof. Since $T^\star T \in Sym(n)$, we have

$$\langle T^\star Tv, v\rangle = \langle Tv, Tv\rangle = ||Tv||^2 \geq 0, \quad \forall v \in \mathbb{R}^n. \tag{2.5}$$

By spectral theorem, there exist $\{\lambda_i\}_{i=1}^n$ non zero real numbers and $\{v_i\}_{i=1}^n$ an orthogonal basis of $\mathbb{R}^n$ such that

$$T^\star T = \sum_{i=1}^n \lambda_i v_i \otimes v_i,$$

and $\lambda_i \geq 0$, for $i = 1, \dots n$, then if we set

$$\sqrt{T^\star T} = \sum_{i=1}^n \sqrt{\lambda_i} v_i \otimes v_i,$$

$I := \{i : \lambda_i > 0\}$, and define

$$w_i := \frac{T v_i}{\sqrt{\lambda_i}} \in \mathbb{R}^m, \quad i \in I,$$

then, by construction, $\{w_i\}_{i \in I}$ is an orthonormal set of vector of $\mathbb{R}^m$, then completing to a basis $\{w_i\}_{i=1}^m$ of $\mathbb{R}^m$, and defining $P \in O(n, m)$ by

$$P v_i = w_i,$$

we get the desired polar decomposition

$$T = P\sqrt{T^\star T}, \tag{2.6}$$

as it is easy to check evaluating (2.6) on the basis $\{v_i\}_{i=1}^n$, and remembering that by (2.5) $\lambda_i = 0$, implies $T v_i = 0$. □

Before continuing with the discussion of the Area formula, let us prove an important lemma, which guarantees under suitable assumption the measurability of the image of a Lebesgue measurable set by a Lipschitz function.

Lemma 2.1. *If E is $\mathcal{L}^n$-measurable set in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ ($1 \leq n \leq m$) is a Lipschitz function, then $f(E)$ is $\mathcal{H}^n$-measurable in $\mathbb{R}^m$*

Proof. We can assume that E is bounded, so $\mathcal{L}^n(E) < \infty$. As E is $\mathcal{L}^n$-measurable, there exists a sequence $\{K_h\}_{h\in\mathbb{N}}$ of compact sets, such that $K_h \subseteq E$ and $\mathcal{L}^n(E \setminus K_h) \to 0$ as $h \to 0$, since $f(K_h)$ is compact, the set $\bigcup_{h\in\mathbb{N}} f(K_h)$ is a Borel set. The $\mathcal{H}^n$-measurability of $f(E)$ follows, since

$$\begin{aligned}
\mathcal{H}^n\left(f(E) \setminus \bigcup_{h\in\mathbb{N}} f(K_h)\right) &\leq \mathcal{H}^n\left(f(E \setminus \bigcup_{h\in\mathbb{N}} K_h)\right) \\
&\leq Lip(f)^n \mathcal{L}^n\left(E \setminus \bigcup_{h\in\mathbb{N}} K_h\right) \\
&= 0.
\end{aligned}$$

which allow us to write $f(E)$ like the union of a Borel set and a set of measure 0. □

Now we introduce the area formula which plays a central role in all the entire theory.

Theorem 2.3 (Area formula for linear maps)**.** *If $T \in \mathbb{R}^m \otimes \mathbb{R}^n$ $(1 \leq n \leq m)$, then for all $E \in \mathbb{R}^n$, it is true that*

$$\mathcal{H}^n(T(E)) = JT(\mathcal{L}^n(E)).$$

Proof. We shall prove the theorem by showing that

$$\mathcal{H}^n(T(E)) = \frac{\mathcal{H}^n(T(B))}{\mathcal{L}^n(B)}\mathcal{L}^n(E), \quad \forall E \subset \mathbb{R}^n, \tag{2.7}$$

$$JT = \frac{\mathcal{H}^n(T(B))}{\mathcal{L}^n(B)}, \tag{2.8}$$

where B is the unit ball. For brevity's sake, set

$$\kappa := \frac{\mathcal{H}^n(T(B))}{\mathcal{L}^n(B)}.$$

We start proving (2.7). First consider the case $\kappa = 0$, then by definition of κ and linearity

$$\mathcal{H}^n(T(B_r)) = 0, \quad \forall r > 0,$$

hence

$$\mathcal{H}^n(T(\mathbb{R}^n)) = 0,$$

thus $\mathcal{H}^n(T(E)) = 0$ for all $E \in \mathbb{R}^n$, and

$$\mathcal{H}^n(T(E)) = \kappa\mathcal{L}^n(E) \quad \forall E \in \mathbb{R}^n.$$

Now, let $\kappa > 0$, so that T is injective and define an outer measure ν on $\mathbb{R}^n$ as

$$\nu(E) = \mathcal{H}^n(T(E)), \quad E \subset \mathbb{R}^n.$$

By the previous Lemma and Proposition 1.1, $\mathcal{H}^n \llcorner (T(\mathbb{R}^n))$ is a Radon measure on $\mathbb{R}^n$. Since T maps compact sets into compact sets, and

$$\nu = (T^{-1})_\sharp(\mathcal{H}^n \llcorner (T(\mathbb{R}^n))),$$

then ν is a Radon measure on $\mathbb{R}^n$, and by linearity of T and the definition of κ

$$\begin{aligned}
\nu(B(x,r)) &= \mathcal{H}^n(T(B(x,r))) \\
&= \mathcal{H}^n(Tx + rT(B)) \\
&= \mathcal{H}^n(rT(B)) \\
&= r^n\mathcal{H}^n(T(B)) \\
&= r^n\mathcal{L}^n(B)\kappa \quad \forall x \in \mathbb{R}^n, r > 0,
\end{aligned}$$

hence $\nu \ll \mathcal{L}^n$, with $D_{\mathcal{L}^n}\nu = \kappa$, on $\mathbb{R}^n$, then by (2.1) we have

$$\nu = \kappa\mathcal{L}^n, \quad on\ \mathcal{M}(\mathcal{L}^n).$$

We deduce finally that $\nu = \kappa\mathcal{L}^n$ on $\mathcal{P}(\mathbb{R}^n)$.

Now, we prove (2.8). Let $T = PS$, the polar decomposition of T, then

$$\mathcal{L}^n(E) \leq \mathcal{H}^n(P(E)) \leq \mathcal{L}^n(E),$$

that is $\mathcal{L}^n(E) = \mathcal{H}^n(E)$, in particular, if $Q \subset \mathbb{R}^n$ and we set $E = S(Q)$ then

$$\begin{aligned}\kappa &= \frac{\mathcal{H}^n(T(Q))}{\mathcal{L}^n(Q)}\\ &= \frac{\mathcal{H}^n(PS(Q))}{\mathcal{L}^n(E)}\frac{\mathcal{L}^n(E)}{\mathcal{L}^n(Q)}\\ &= \frac{\mathcal{L}^n(S(Q))}{\mathcal{L}^n(Q)}.\end{aligned}$$

If $S = \sum_{i=1}^n \lambda_i v_i \otimes v_i$, then the cube

$$Q = \{x \in \mathbb{R}^n : |\langle x, v_i\rangle| \leq 1/2\},$$

with unit side lengths and faces perpendicular to v_i, is mapped by S into

$$S(Q) := \{x \in \mathbb{R}^n : |\langle x, v_i\rangle| \leq |\lambda_i|/2\},$$

a parallel cube, possibly degenerated, with sides length given by $|\lambda_i|$, hence

$$\mathcal{L}^n(S(Q)) = \prod_{i=1}^n |\lambda_i| = |\det S| = |\det S|\mathcal{L}^n(Q),$$

and $\kappa = |\det S|$. Finally we note that $T^\star = S^\star P^\star$, $P^\star P = id_{\mathbb{R}^n}$, and

$$S^\star S = \sum_{i=1}^n \lambda^2 v_i \otimes v_i,$$

then

$$\begin{aligned}JT &= \sqrt{\det(T^\star T)}\\ &= \sqrt{\det(S^\star S)}\\ &= \left(\prod_{i=1}^n \lambda_i^2\right)^{1/2}\\ &= \prod_{i=1}^n |\lambda_i|\\ &= |\det S|\\ &= \kappa.\end{aligned}$$

□

The following result is a suitable formulation of Sard's Lemma for Lipschitz functions instead that for the classical C^1 functions.

Theorem 2.4. *If $f : \mathbb{R}^n \to \mathbb{R}^m$, with $1 \leq n \leq m$, is a Lipschitz function, then*

$$\mathcal{H}^n(f(E)) = 0,$$

where $E = \{x \in \mathbb{R}^n : Jf(x) = 0\}$.

Before continuing, and for notational convenience, let us denote $B_r^k = \{z \in \mathbb{R}^k : |z| < r\}$, the ball of radius r centred at the origin of $\mathbb{R}^k$. If $F \subset \mathbb{R}^m$, set $I_\varepsilon(F) = \{x \in \mathbb{R}^m : dist(x, F) < \varepsilon\}$, for the ε-neighbourhood of F in $\mathbb{R}^m$. Let $D_s = \{(z, y) \in \mathbb{R}^k \times \mathbb{R}^{m-k} : |z| < s, y = 0\}$ be the k-dimensional disk in $\mathbb{R}^m$ of radius $s > 0$.

Proof. The prof of this Theorem 2.4 will be divided into three steps.
Step one:
We claim that, if $1 \leq n \leq m$, then

$$\mathcal{H}^n_\infty(I_\delta(D_s)) \leq C(n, s)\delta, \quad \forall \delta \in]0, 1[.$$

Indeed, if we set

$$K := \{(z, y) : |z| < \delta s, |y| < \delta\} = B^k_{\delta s} \times B^{m-k}_\delta,$$

then there exists a finite covering $\mathcal{F}$ of $I_\delta(D_s)$ such that each $F \in \mathcal{F}$ is a translation of K, and the cardinality of $\mathcal{F}$ is bounded from above by $C\delta^{-1}$, for some $C > 0$. Moreover, if $F \in \mathcal{F}$ then

$$\begin{aligned} diam(F)^2 &= diam(K)^2 \\ &= diam(B^k_{\delta s})^2 + diam(B^{m-k}_\delta)^2 \\ &= 4\delta^2 s^2 + \delta^2 \\ &= 4\delta^2(1 + s^2), \end{aligned}$$

since $\delta \in]0, 1[$ and $n - k \geq 1$, we conclude that

$$\begin{aligned} \mathcal{H}^n_\infty(I_\delta(D_s)) &\leq \omega_n \sum_{F \in \mathcal{F}} \left(\frac{diam F}{2}\right)^n \\ &\leq C(n)(1 + s^2)^{n/2}\delta^{n-k} \\ \mathcal{H}^n_\infty(I_\delta(D_s)) &\leq C(n, s)\delta. \end{aligned} \tag{2.9}$$

Step two:

If $x \in E$, so that $Jf(x) = 0$, then $L_x = \nabla f(x)(\mathbb{R}^n)$ is a linear subspace of $\mathbb{R}^m$, with

$$k = dim(L_x) \leq n - 1 < m.$$

If $k \geq 1$ then $\nabla f(x)(B_r^n)$ is contained into a k-dimensional disk of radius $Lip(f)$ in $\mathbb{R}^m$ for all $r > 0$, i.e.

$$\nabla f(x)(B_r^n) \subset B^m_{Li(f)r} \cap L_x, \quad \forall r > 0.$$

Hence by (2.9), for all $\varepsilon \in]0,1[$ and $r > 0$ we find

$$\begin{aligned}
\mathcal{H}^n_\infty(I_{\varepsilon r}(\nabla f(x)(B_r^n)) &\leq \mathcal{H}^n_\infty(I_{\varepsilon r}(B^m_{Li(f)r} \cap L_x)) \\
&\leq r^n \mathcal{H}^n_\infty(I_\varepsilon(B^m_{Lip(f)} \cap L_x)) \\
&\leq C(n, Lip(f)) r^n \varepsilon.
\end{aligned}$$

If $k = 0$, then $\nabla f(x)(\mathbb{R}^n) = \{0\}$, and for all $\varepsilon \in]0,1[$ and $r > 0$

$$\begin{aligned}
\mathcal{H}^n_\infty(I_{\varepsilon r}(\nabla f(x)(B_r^n)) &= \mathcal{H}^n(B^n_{r\varepsilon}) \\
&\leq \omega_n r^n \varepsilon^n \\
&\leq \omega_n r^n \varepsilon.
\end{aligned}$$

Steep three:

If $x \in E$, and $\varepsilon \in]0,1[$, then as f is differentiable on E, there exists $r(\varepsilon, x) \in]0,1[$ such that

$$|f(x+v) - f(x) - \nabla f(x)v| \leq \varepsilon |v|,$$

whenever $|v| < r(\varepsilon, x)$. In particular, for all $r < r(\varepsilon, x)$ we have that

$$f(B^n(x,r)) \subset f(x) + I_\varepsilon(\nabla f(x)(B_r^n)).$$

Since $Jf(x) = 0$, by the step two we find that $r < r(\varepsilon, x)$ implies

$$\mathcal{H}^n(f(B^n_{r\varepsilon})) \leq C(n, Lip(f)) \varepsilon r^n. \tag{2.10}$$

Given the family of open balls

$$\mathcal{F} = \{B^n(x,r) : x \in E \cap B^n_R, \ 0 < r < r(\varepsilon, x)\},$$

has the bounded set $E \cap B^n_R$ as the set of its centers; let $\{\mathcal{F}_i\}_{i=1}^{\xi(n)}$ the subfamily of $\mathcal{F}$ given by Theorem 1.13. Since $E \cap B_R \subset \bigcup_{i=1}^{\xi(n)} \mathcal{F}_i$, with $\mathcal{F}_i$ countably and disjoint, by the inequality (2.10), easily follows

$$
\begin{aligned}
\mathcal{H}^n(f(B^n(x,r) \cap E) &\leq \sum_{i=1}^{\xi(n)} \sum_{B^n(x,r) \in \mathcal{F}_i} \mathcal{H}^n(f(B^n(x,r))) \\
&\leq C\varepsilon \sum_{i=1}^{\xi(n)} \sum_{B^n(x,r) \in \mathcal{F}_i} r^n \\
&= \frac{C\varepsilon}{\omega_n} \sum_{i=1}^{\xi(n)} \sum_{B^n(x,r) \in \mathcal{F}_i} r^n \omega_n \\
&= \frac{C\varepsilon}{\omega_n} \sum_{i=1}^{\xi(n)} \sum_{B^n(x,r) \in \mathcal{F}_i} \mathcal{L}^n(B^n(x,r)) \\
&= \frac{C\varepsilon}{\omega_n} \sum_{i=1}^{\xi(n)} \mathcal{L}^n(I_1(E \cap B_r)),
\end{aligned}
$$

where in the last inequality we have use the fact that $r(\varepsilon, x) \in]0,1[$, and $C = C(n, Lip(f))$, for $\varepsilon \downarrow 0$ we find

$$\mathcal{H}^n_\infty(f(E \cap B_R)) = 0.$$

Hence

$$\mathcal{H}^n(f(E \cap B_R)) = 0,$$

and letting $R \to \infty$, we finally get

$$\mathcal{H}^n(f(E)) = 0.$$

□

The following theorem is an important result concerning Lipschitz immersions, which we apply in the proof of the area formula, and will also play an important role in the theory of rectifiable sets, which is the keystone of this dissertation.
The main idea, due to Federer, is to reformulate the classical approximation by linear functions of C^1 functions, to Lipschitz functions.

Notice that by Proposition 2.2, if $Jf(x) > 0$ then there exists $S_x \in Sym(n)$ and $P_x \in O(n,m)$ such that $\nabla f(x) = P_x S_x$.

Theorem 2.5 (Lipschitz linearization)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ $(1 \leq n \leq m)$ be a Lipschitz function, and*

$$F := \{x \in \mathbb{R}^n : 0 < Jf(x) < \infty\},$$

then there exists a partition of F into Borel sets $\{F_n\}_{n\in\mathbb{N}}$ such that f is injective on each F_n. Moreover for all $t > 1$, such a partition can be formed with the property that, for all $n \in \mathbb{N}$ there exists an invertible linear function $S_n \in GL(n)$ such that $f|_{F_n} \circ S_n$ is almost an isometry of $\mathbb{R}^n$ into $\mathbb{R}^m$. Precisely, for all $x, y \in F_n$ and $v \in \mathbb{R}^n$ we have

$$t^{-1}|S_n x - S_n y| \leq |f(x) - f(y)| \leq t|S_n x - S_n y|, \tag{2.11}$$

$$t^{-1}|S_n v| \leq |\nabla f(x)| \leq t|S_n v|, \tag{2.12}$$

$$t^{-n} JS_n \leq Jf(x) \leq t^n JS_n. \tag{2.13}$$

Proof. It is suffices to show that F can be covered by sets F_n having the desired properties; indeed, once this has been done, we can replace each F_n with $\overline{F}_n = F_n \setminus \bigcup_{k=1}^{n-1} F_k$, in order to define the desired partition of F.

Recall that if $T, S \in GL(n)$ and $||T - S|| \leq \delta$, then

$$\begin{aligned}
||T - S||||S^{-1}|| &\leq ||S^{-1}||\delta \\
||TS^{-1} - id_{\mathbb{R}^n}|| &\leq \delta||S^{-1}|| \\
||TS^{-1}|| &\leq 1 + \delta||S^{-1}||.
\end{aligned} \tag{2.14}$$

Similarly we prove that

$$||ST^{-1}|| \leq 1 + \delta||T^{-1}||.$$

Now choose $\varepsilon > 0$ so that $t^{-1} + \varepsilon < 1 < t - \varepsilon$, and a dense set (in the operator norm) $\mathcal{G}$ in $GL(n)$. Correspondingly to every $n \in \mathbb{N}$ and $S \in \mathcal{G}$, define

$$F(n, S),$$

as the set of those $x \in F$ such that

$$(t^{-1} + \varepsilon)|Sv| \leq |\nabla f(x)| \leq (t - \varepsilon)|Sv|, \quad \forall v \in \mathbb{R}^n, \tag{2.15}$$

$$|f(x + v) - f(x) - \nabla f(x)v| \leq \varepsilon|Sv|, \quad \forall v \in \mathbb{R}^n, |v| \leq \frac{1}{n}. \tag{2.16}$$

Note that inequalities as in (2.15) imply

$$(t^{-1} + \varepsilon)^n JS \leq Jf(x) \leq (t - \varepsilon)^n JS, \quad \forall x \in \bigcup_{n\in\mathbb{N}} F(n, S). \tag{2.17}$$

Indeed , as $S \in GL(n)$, for every such x we have that

$$B_{t^{-1}+\varepsilon} \subset \nabla f(x)(S^{-1}(B)) \subset B_{t-\varepsilon},$$

and thus, as required, $(t^{-1}+\varepsilon)^n JS \leq Jf(x) \leq (t-\varepsilon)^n JS$. Another relevant property of the sets $F(n,S)$ is that, if $x, y \in F(n,s)$ and $|x-y| < 1/n$, then

$$|f(x) - f(y)| \leq |\nabla f(x-y)| + \varepsilon|Sx - Sy| \leq t|Sx - Sy|, \tag{2.18}$$

$$|f(x) - f(y)| \geq |\nabla f(x-y)| + \varepsilon|Sx - Sy| \geq t|Sx - Sy|. \tag{2.19}$$

If now $\{x_j\}_{j\in\mathbb{N}}$, is a dense subset of F, and we relabel the sequence of sets

$$F(n,S) \cap B\left(x_j, \frac{1}{2n}\right), \quad S \in \mathcal{G},\ n, j \in \mathbb{N},$$

as $\{F_k\}_{k\in\mathbb{N}}$, then by (2.17), (2.18) and (2.19) we see that (2.11) and (2.13) hold true on each F_k.

We also observe that (2.12) holds trivially on each F_k by (2.15).

So,we are left to prove that $F = \bigcup_{S\in\mathcal{G}} F(n,S)$. Let $x \in F$ and consider the polar decomposition

$$\nabla f(x) = P_x S_x.$$

As $Jf(x) > 0$, we have $S_x \in GL(n)$, in particular by (2.14) we can find $S \in \mathcal{G}$ with

$$||S_x S^{-1}|| \leq t - \varepsilon, \quad ||S(S_x)^{-1}|| \leq (t^{-1}+\varepsilon)^{-1}.$$

In that way we ensures that

$$|S_x v| \leq (t-\varepsilon)|Sv|, \quad (t^{-1}+\varepsilon)|Sv| \leq |S_x v|, \quad \forall v \in \mathbb{R}^n,$$

that is (2.15), since $|\nabla f(x)v| = |P_x S_x v| = |S_x v|$.

Concerning (2.16), the differentiability of f at x, implies the existence of a modulus of continuity ω_x such that whenever $|v| \leq n^{-1}$

$$|f(x+v) - f(x) - \nabla f(x)v| \leq \frac{\omega_x}{n}|v| \leq \frac{\omega_x}{n}||S^{-1}|||Sv|.$$

Choosing $n = n(x,S)$ so that $\omega_x(n^{-1})||S^{-1}|| \leq \varepsilon$, proves (2.16). □

Finally we are ready to state and prove the area formula for injective Lipschitz maps. We restrict to prove this version, but there is a version of the area formula for general Lipschitz maps that are not necessarily injective; the reader can find it in Theorem 8.9 of [Mag12].

Theorem 2.6 (Area formula for injective Lipschitz functions)**.** *If $f:\mathbb{R}^n\to\mathbb{R}^m$, $1\le n\le m$, is an injective Lipschitz function and $E\subset\mathbb{R}^n$ is Lebesgue measurable, then*

$$\mathcal{H}^n\llcorner(f(E))=\int_E Jf(x)dx, \tag{2.20}$$

and $\mathcal{H}^n\llcorner(f(\mathbb{R}^n))$ is a Radon measure on $\mathbb{R}^m$.

Proof. Because $\mathcal{H}^n(f(E))\le Lip(f)^n\mathcal{L}^n(E)$ for all $E\subset\mathbb{R}^n$, then both sides of 2.20 are zero if $\mathcal{H}^n(E)=0$.

Therefore by Rademarcher's theorem we can reduce to prove 2.20 on a set E over which f is differentiable. Moreover by Theorem 2.4 we directly can assume

$$E\subset F=\{x\in\mathbb{R}^n:0<Jf(x)<\infty\}.$$

We now fix $t>1$ and consider the partition $\{F_k\}_{k\in\mathbb{N}}$ given by 2.5. we see E as the union of disjoint sets $F_k\cap E$, $k\in\mathbb{N}$, so that by the global injectivity of f, we have that

$$f(E)=\bigcup_{k\in\mathbb{N}}f(F_k\cap E),\quad k\in\mathbb{N},$$

where $f(F_k\cap E)$ is $\mathcal{H}^n$-measurable by Lemma 2.1. Therefore by Theorem 2.5 we find that

$$\begin{aligned}\mathcal{H}^n(f(E))&=\sum_{k\in\mathbb{N}}\mathcal{H}^n(f(F_k\cap E))\\&=\sum_{k\in\mathbb{N}}\mathcal{H}^n(f\mid_{F_k}\circ S_k^{-1})(S(F_k\cap E))\\&\le\sum_{k\in\mathbb{N}}Lip(f\mid_{F_k}\circ S_k^{-1})^n\mathcal{L}^n(S(F_k\cap E))\\&\le t^n\sum_{k\in\mathbb{N}}JS_k\mathcal{L}^n(S(F_k\cap E))\\&\le t^{2n}\sum_{k\in\mathbb{N}}\int_{F_k\cap E}Jf(x)dx\\&=t^{2n}\int_E Jf(x)dx, \end{aligned}\tag{2.21}$$

where we have also applied the fact that thanks to the upper bound in (2.11) the Lipschitz norm of $f\mid_{F_k}\circ S_k^{-1}$ over $S_k(F_k)$ is controlled by t. In a similar way, the lower bound in (2.11) implies that the Lipschitz norm of $S_k\circ(f\mid_{F_k})^{-1}$ over $f(F_k)$ is controlled by t, so that by an analogous argument,

$$
\begin{aligned}
\int_E Jf(x)dx &= \sum_{k\in\mathbb{N}} \int_{F_k\cap E} Jf(x)dx \\
&\leq t^n \sum_{k\in\mathbb{N}} JS_k \mathcal{L}^n(E\cap F_k) \\
&= t^n \sum_{k\in\mathbb{N}} \mathcal{L}^n \left((S_k \circ (f\mid_{F_k})^{-1})(f(F_k\cap E)\right) \\
&\leq t^{2n} \sum_{k\in\mathbb{N}} \mathcal{H}^n(f(E\cap F_k)) \\
&= t^{2n}\mathcal{H}^n(f(E\cap F_k)). \qquad (2.22)
\end{aligned}
$$

Letting $t\downarrow 1$ on (2.21) and (2.23) we prove (2.20).

By (2.1) $f(\mathbb{R}^n)$ is $\mathcal{H}^n$-measurable, while (2.20) implies $\mathcal{H}^n\llcorner(f(\mathbb{R}^n))$ to be locally finite, then $\mathcal{H}^n\llcorner(f(\mathbb{R}^n))$is a Radon measure on $\mathbb{R}^m$. $\square$

2.3 Rectifiable sets

We shall now introduce the notion of rectifiable set, which provides a generalization of the notion of surface and it is, of course, of primary importance in the study of geometric variational problems; in particular **rectifiable sets** are the sets on which rectifiable varifolds lives, so they are in some way the main subject of our study.

Definition 2.5. *Let $M \subset \mathbb{R}^n$, M is said a* ***countably k-rectifiable set*** *if there exists many countably $f_j : \mathbb{R}^k \to \mathbb{R}^n$ Lipschitz functions and $M_0 \subset \mathbb{R}^k$ with $\mathcal{H}^k(M_0) = 0$ such that*

$$M \subset M_0 \cup \left(\bigcup_{j\in\mathbb{N}} f_j(\mathbb{R}^k)\right). \qquad (2.23)$$

Remark 2.3. *By Kirszbraun's Theorem ([Mag12] Theorem 7.2) (2.23) is equivalent to say*

$$M = M_0 \cup \left(\bigcup_{j\in\mathbb{N}} f_j(A_j)\right),$$

where $\mathcal{H}^k(M_0) = 0$ and $f_j : A_j \to \mathbb{R}^n$ are countably many Lipschitz functions, and $A_j \subset \mathbb{R}^k$ are countably many Borel sets.

Remark 2.4. *The decomposition given in (2.23) is clearly not unique, and several properties can be imposed on the f_j's, by decreasing the sets A_j while increasing the $\mathcal{H}^k$-null* *set M_0. Indeed if f_j satisfies a global property on a set $E_j \subset A_j$, and $\mathcal{L}^k(A_j \setminus E_j) = 0$, then we can replace A_j with E_j in (2.23) and set $M_0' = M_0 \cup f_j(A_j \setminus E_j)$, then we have another representation for M; this is why we choose a "special" decomposition of M.*

Definition 2.6. *Given $f : \mathbb{R}^k \to \mathbb{R}^n$ a Lipschitz function, and a bounded Borel set $E \subset \mathbb{R}^k$, we say that the pair (f, E) defines a **regular Lipschitz image** $f(E)$ in $\mathbb{R}^n$ if*

1. *f is injective and differentiable on E, with $Jf(x) > 0$ for all $x \in E$.*

2. *Every $x \in E$ is a point of density 1 for E.*

3. *Every $x \in E$ is a Lebesgue point of ∇f.*

The following Theorem guarantees that always exist a "good decomposition" for a countably k-rectifiable set.

Theorem 2.7 (Decomposition of k-rectifiable sets). *If M is a countably k-rectifiable set in $\mathbb{R}^n$, and $t > 1$, then there exists Borel sets $M_0 \subset \mathbb{R}^k$, and $A_j \subset \mathbb{R}^k$, and many countably Lipschitz maps $f_j : A_j \subset \mathbb{R}^k \to \mathbb{R}^n$ such that*

$$M = M_0 \cup \left(\bigcup_{j \in \mathbb{N}} f_j(A_j) \right), \quad \mathcal{H}^k(M_0) = 0.$$

And each pair (f_j, A_j) defines a regular Lipschitz image, with Lip$(f_j) \leq t$, $j \in \mathbb{N}$, and

$$t^{-1}|x - y| \leq |f_j(x) - f_j(y)| \leq t|x - y|,$$

$$t^{-1}|v| \leq |\nabla f_j(x)v| \leq t|v|,$$

$$t^{-k} \leq Jf_j(x) \leq t^k,$$

for all $x, y \in A_j$ and for all $v \in \mathbb{R}^k$.

Proof. Since M is a countably k-rectifiable set, there exists Borel sets $N_0 \subset \mathbb{R}^k$, and $E_j \subset \mathbb{R}^k$, and many countably Lipschitz maps $g_j : E_j \subset \mathbb{R}^k \to \mathbb{R}^n$ such that

$$M = N_0 \cup \left(\bigcup_{j \in \mathbb{N}} g_j(E_j) \right), \quad \mathcal{H}^k(M_0) = 0.$$

Now, by Rademarcher's theorem and Theorem 2.4

$$E_j = G_j \cup B_j$$

where

$$B_j = \{x \in \mathbb{R}^k : Jg_j = 0\} \cup \{x \in \mathbb{R}^k : f \text{ is not differentiable}\},$$

and $\mathcal{H}^k(B_j) = 0$, and $G_j = \{x \in \mathbb{R}^k : 0 < Jg_j < \infty\}$, then arguing like 2.4, we have

$$M = M_0 \cup \left(\bigcup_{j \in \mathbb{N}} g_j(G_j) \right),$$

where $M_0 = N_0 \cup \left(\bigcup_{j \in \mathbb{N}} g_j(B_j) \right)$, moreover by Theorem 2.5 each of g_j's is injective, and G_j is a Borel set, therefore, every $x \in G_j$ is a point of density 1 in G_j.

Now by Theorem 2.2,

$$\frac{1}{r^k} \int_{B(x,r)} |Jf(z) - Jf(x)| dz \to 0, \quad as\ r \downarrow 0.$$

Then for all $j \in \mathbb{N}$ (g_j, G_j) defines a regular Lipschitz image $g_J(G_j)$. Moreover (again by Theorem 2.5), there exists $\{S_j\}_{j \in \mathbb{N}} \subset GL(k)$ such that for every $x, y \in G_j$ and $v \in \mathbb{R}^k$

$$t^{-1}|S_j x - S_j y| \leq |g_j(x) - g_j(y)| \leq t|S_j x - S_j y|,$$

$$t^{-1}|S_j v| \leq |\nabla g_j(x) v| \leq t|S_j v|.$$

Let us now define $E_j \subset \mathbb{R}^k$ and $f_j : E_j \to \mathbb{R}^n$ setting

$$E_j = S_j(G_j), \quad f_j = g_j \circ S_j^{-1}.$$

Then the f_j's are Lipschitz on E_j, with

$$t^{-1}|x - y| \leq |f_j(x) - f_j(y)| \leq t|x - y|,$$

for all $x, y \in E_j$. By Kirszbraun's Theorem, we can extend $f_j : \mathbb{R}^k \to \mathbb{R}^n$ with $Lip(f_j) \leq t$. Since the g_j's was differentiable on G_j, we have f_j is differentiable on E_j with $\nabla f_j(x) = \nabla g_j(S_j x) \circ S_j^{-1}$, so that

$$t^{-1}|v| \leq |\nabla f_j(x) v| \leq t|v|,$$

for all $x \in E_j$ and $v \in \mathbb{R}^k$, in particular $t^{-k} \leq Jf_j(x) \leq t^k$ on E_j.

Since the pair (f_j, E_j) defines a regular Lipschitz image $f_j(E_j) = g_j(G_j)$ we have established the Theorem. □

Chapter 3

First Variation of a Varifold, Radial Deformations and Monotonicity Formula

Introduced by J. F. Almgren in [Alm65], the theory of varifolds culminated in the works of Allard and in the big regularity paper of Almgren. In this text we are interested in **Allard's Regularity Theorem** which roughly speaking asserts that under suitable assumptions a varifold is a $C^{1,\alpha}$ submanifold. The mean references for the material covered in this thesis are [All72], [Sim83], and [Lel12]. Our exposition follows closely that of [Lel12].

3.1 Varifolds

In this section we will define the concept of k-dimensional varifold which is a substitute for ordinary k-dimensional submanifolds in a n-dimensional space ($0 \leq k \leq n$) suitable to tackle geometric variational problems.

Definition 3.1 (Abstract Varifold). *Let Σ be a m-dimensional submanifold of $\mathbb{R}^n$, V is said a k-**dimensional Varifold**, if V is a Radon measure over*

$$G_k(\Sigma) = (\Sigma \times G(n,k)) \cap \{(x,S) : S \subset Tan(\Sigma, x)\}.$$

To see the formal definition of $G_k(\Sigma)$, and $G(n,k)$ see the Appendix A.

Let

$$\mathbf{V}_k(\Sigma),$$

be the weakly topologized space of k-dimensional varifolds in Σ, i.e., $V_k(\Sigma)$ is endowed with the weak topology introduced in Definition 1.17. Whenever $V \in \mathbf{V}_k(\Sigma)$, we define the **weight of** V, and write $||V||$, as the Radon measure*

$$||V||(A) := V\left(G_k(\Sigma) \cap \{(x, S) : x \in A\}\right), \quad \forall A \subset \Sigma,$$

i.e., $||V|| = \pi_\sharp V$, where π is the canonical projection onto Σ.

As the reader has noticed, an abstract varifold can be a quite strange object, because it is hard to work with Borel sets on $G_k(\Sigma)$ in an operative way (here operative way has to be understate in a sense to be specified later in this section); but we also define the weight of V which is a Radon measure on Σ obtained from V by ignoring the fiber variable. The next theorem illustrates how to "simplify" a varifold in an operative way. This result is a direct application of a well known **disintegration Theorem**, which can be found in [AFP00] Theorem 2.28. However our proof is an adaptation of it to the context of varifolds.

Theorem 3.1 (Disintegration Theorem for Varifolds)**.** *Let $V \in \mathbf{V}_k(\Sigma)$, $\pi : G_k(\Sigma) \to \Sigma$ the canonical projection onto Σ. Then there exists a family of Radon measures $\{\pi_x\}_{x\in\Sigma}$ such that, the map $x \mapsto \pi_x$ is $||V||$-measurable and the following relations are satisfied*

$$\pi_x(B) := \lim_{r\downarrow 0} \frac{V(B(x,0) \times B)}{||V||(B(x,r))}, \ \forall B \in \mathcal{B}(G_x(n,k)), \tag{3.1}$$

$$\pi_x\left(G_x(n,k) \setminus \{S : S \subset Tan(\Sigma, x)\}\right) = 0, and\ \pi_x(G_x(n,k)) = 1, \tag{3.2}$$

$$f(x,\cdot) \in L^1(G(n,k), \pi_x),\ \ for\ ||V|| - a.e.\ x \in \Sigma, \tag{3.3}$$

$$x \mapsto \int_{G_x(n,k)} f(x,S) d\pi_x(S) \in L^1(\Sigma, ||V||), is\ ||V|| - measurable, \tag{3.4}$$

$$\int_{G_k(\Sigma)} f(x,S) dV(x,S) = \int_\Sigma \left(\int_{G_x(n,k)} f(x,S) d\pi_x(S)\right) d||V||(x), \tag{3.5}$$

for any $f \in L^1(G_k(\Sigma), V)$. Moreover, if π'_x is any other $||V||$-measurable map satisfying (3.4) and (3.5) for every bounded Borel function with compact support and such that $x \mapsto \pi'_x(G(n,k)) \in L^1(\Sigma, ||V||)$, then $\pi_x = \pi'_x$ for $||V||$-a.e. $x \in \Sigma$.

Proof. The idea of the proof is to construct the measures π_x by application of the Theorem 1.11 to a family of linear functionals defined on $C^0_c(G(n,k))$, and show the validity of the disintegration formula (3.5), then we will prove (3.2) and finally uniqueness.

Step 1:

First see that given any $g \in C_c^0(G(n,k))$ it is possible to construct a Radon measure $||V||_g$ such that $||V||_g \ll ||V||$. Indeed, let $||V||_g = \pi_\sharp(gV)$, i.e.

$$||V||_g(B) = \int_{B\times G(n,k)} g(S)dV(x,S), \quad \forall B \in \mathcal{B}(\Sigma),$$

then, for all $B \in \mathcal{B}(\Sigma)$, and $g \in C_c^0(G(n,k))$

$$\begin{aligned} ||V||_g(B) &= \int_{B\times G(n,k)} g(S)dV(x,S) \\ &\le ||g||_\infty V(B \times G(n,k)) \\ &= ||g||_\infty V(\pi^{-1}(B)) \\ &= ||g||_\infty \pi_\sharp(V)(B) \\ &= ||g||_\infty ||V||(B). \end{aligned}$$

So $||V||_g$ satisfy

$$||V||_g \le ||g||_\infty ||V||,$$

then $||V||_g \ll ||V||$, and by Theorem 2.1, there exists $h_g = D_{||V||}||V||_g \in L^1_{loc}(\Sigma, ||V||)$ for $||V||$-a.e. $x \in \Sigma$, such that $||h_g||_\infty \le ||g||_\infty$ and

$$||V||_g = h_g||V||.$$

Furthermore, by the construction of h_g

$$\begin{aligned} ||V||_{g+g'} &= ||V||_g + ||V||_{g'} \\ &= h_g||V|| + h_{g'}||V|| \\ &= (h_g + h_g')||V||. \end{aligned}$$

Now since $C_c^0(G(n,k))$ is separable, there exists $\mathcal{D} \subset C_c^0(G(n,k))$ a dense countable set, moreover for each $g \in \mathcal{D}$ there exists a $||V||$-measurable set N_g with $||V||(N_g) = 0$, such that $h_g(x)$ exists for all $x \in \Sigma \setminus N_g$. Let $N = \bigcup_{g \in \mathcal{D}} N_g$, then by Theorem 2.1

$$\begin{aligned}
h_{g+g'}(x) &= \lim_{r \downarrow 0} \frac{1}{||V||(B(x,r))} \int_{B(x,r)\times G(n,k)} (g+g')(S) dV(x,S) \\
&= \lim_{r \downarrow 0} \frac{1}{||V||(B(x,r))} \left(\int_{B(x,r)\times G(n,k)} g(S) dV(x,S) + \right. \\
&+ \left. \int_{B(x,r)\times G(n,k)} g'(S) dV(x,S) \right) \\
&= \lim_{r \downarrow 0} \frac{1}{||V||(B(x,r))} \int_{B(x,r)\times G(n,k)} g(S) dV(x,S) + \\
&+ \lim_{r \downarrow 0} \frac{1}{||V||(B(x,r))} \int_{B(x,r)\times G(n,k)} g'(S) dV(x,S) \\
&= h_g(x) + h_{g'}(x), \quad \forall x \in \Sigma \setminus N,\ g, g' \in \mathcal{D}.
\end{aligned}$$

Now, define the map

$$\begin{aligned}
T_x &: \mathcal{D} \to \mathbb{R} \\
&g \mapsto T_x(g) = h_g(x),
\end{aligned}$$

for $x \in \Sigma \setminus N$, which is additive, and since

$$|T_x(g)| = |h_g(x)| \leq ||h_g||_\infty \leq ||g||_\infty,$$

T_x is continuous, therefore we can extend it to an additive, bounded $\mathbb{R}$-valued operator defined on the whole $C^0_c(G(n,k))$. Then by Theorem 1.11 there exists π_x a Radon measure such that

$$T_x(g) = \int_{G_x(n,k)} g d\pi_x, \quad \forall g \in C^0_c(G(n,k)),$$

and

$$\pi_x(G(n,k)) = \int_{G_x(n,k)} d\pi_x = T_x(\chi_{G_x(n,k)}) \leq ||\chi_{G_x(n,k)}||_\infty = 1.$$

Then for all $B \in \mathcal{B}(\Sigma)$ and $g \in \mathcal{D}$

$$\begin{aligned}
\int_{G_k(\Sigma)} \chi_B(x) g(S) dV(x,S) = ||V||_g(B) \\
&= \int_B h_g(x) d||V||(x) \\
&= \int_B \left(\int_{G_x(n,k)} g(S) d\pi_x(S) \right) d||V||(x) \\
&= \int_\Sigma \left(\int_{G_x(n,k)} \chi_B(x) g(S) d\pi_x(S) \right) d||V||(x).
\end{aligned}$$

By an approximation argument the same identity is true for $g \in C_c^0(G(n,k))$ and $g = \chi_A$, with A open, so (3.5) holds for $f(x,S) = \chi_B(x)\chi_A(S)$, moreover it still holds for characteristic functions of any Borel set $B \subset G_k(\Sigma)$. In particular if $B \in \mathcal{B}(\Sigma)$ is such that $V(B) = 0$ then $\chi_B(x,\cdot) \in L^1(G(k,n),\pi_x)$ and

$$\int_{G_x(k,n)} \chi_B(x,S)d\pi_x(S) = 0, \quad for\ ||V|| - a.e.\ x \in \Sigma.$$

Thus (3.1), (3.3), (3.4) and (3.5) holds for $f = \chi_B$ where B is any set in $\mathcal{B}(G_k(\Sigma))$, and the general case follows from an argument of approximation by simple functions.

Step 2:

In order to prove (3.2), we recall that

$$V(G_k(\Sigma)) = V(\pi^{-1}(\Sigma)) = (\pi_\sharp V)(\Sigma) = ||V||(\Sigma) = \int_\Sigma 1d||V||(x),$$

and by (3.5)

$$V(G_k(\Sigma)) = \int_\Sigma \left(\int_{G_x(n,k)} 1d\pi_x(S)\right) d||V||(x) = \int_\Sigma \pi_x(G_x(n,k))d||V||(x).$$

Since $||V||$ is finite, comparing with the preceding equality, yields

$$\pi_x(G_x(n,k)) = 1.$$

Step 3: Finally, let π_x be as in the statement of the theorem. For any $g \in \mathcal{D}$ and any $B \in \mathcal{B}(\Sigma)$ relatively compact we have

$$\begin{aligned}\int_B \left(\int_{G_x(n,k)} g(S)d\pi_x(S)\right) d||V||(x) &= \int_{B\times G_x(n,k)} g(S)dV(x,S)\\ &= \int_B \left(\int_{G_x(n,k)} g(S)d\pi'_x(S)\right) d||V||(x),\end{aligned}$$

and therefore there is a $||V||$-negligible set N' such that $\pi_x(g) = \pi'_x(g)$ for any $g \in \mathcal{D}$ and any $x \in E \setminus N'$. Since $\mathcal{D}$ is dense in $C_c^0(G(n,k))$, $\pi_x = \pi'_x$ for any $x \in E \setminus N'$ and the proof is complete. $\square$

Corollary 3.1. *Let $V \in \mathbf{V}_k(\Sigma)$, with the same notation of the Theorem 3.1, the equality*

$$V = ||V|| \otimes \pi_x,$$

holds.

Now we are ready to introduce some important sub-families of $\mathbf{V}_k(\Sigma)$, which are relevant for our work and motivates it.

Definition 3.2. *Whenever E is an $\mathcal{H}^k$-measurable subset of Σ which meets every compact subset of Σ in a k-rectifiable subset of Σ, we let*

$$\mathbf{v}(E)(A) := \mathcal{H}^k\left(\{(x, Tan^k(\mathcal{H}^k \llcorner E, x)) \in A\}\right), \quad A \subset G_k(\Sigma), \tag{3.6}$$

where $Tan^k(\mathcal{H}^k \llcorner E, x) := \bigcap \left\{Tan(S,x) : \Theta^k(\mathcal{H}^k \llcorner (E \setminus S), x) = 0\right\}$ is the ***k-dimensional approximate tangent space*** *of the measure $\mathcal{H}^k \llcorner E$ and*

$$Tan(S,x) := [0, +\infty[\bigcap_{\varepsilon>0} Closure\left\{\frac{y-x}{|y-x|} : y \in S \cap B(x,\varepsilon)\right\}.$$

Remark 3.1. *It is clear that $\mathbf{v}(E) \in V_k(\Sigma)$.*

Definition 3.3 (Integer-Rectifiable Varifold)**.** *$V \in \mathbf{V}_k(\Sigma)$ is said a* ***rectifiable varifold*** *if there exists $k_i \in \mathbb{R}$ and $\mathcal{H}^k$-measurable sets $E_1, E_2, \ldots$ of Σ which meets every compact subset of U in a k-rectifiable subset of U such that*

$$V = \sum_{i=0}^{\infty} k_i \mathbf{v}(E_i). \tag{3.7}$$

If the $k_i \in \mathbb{N}$ we say that V is an ***integral varifold****.*
We define

$$\mathbf{RV}_k(\Sigma) \text{ and } \mathbf{IV}_k(M),$$

as the space of ***k-dimensional rectifiable varifolds in*** *Σ and* ***k-dimensional integral varifolds in*** *Σ, respectively.*

Remark 3.2. *Notice that if $E \subset \Sigma$ is a k-rectifiable set, by Theorem 3.1 and (3.6) we have that $||\mathbf{v}(E)|| = \mathcal{H}^k \llcorner E$.*

3.2 First Variation of a Varifold

Now we begin the study of the first variation of a varifold, which is a key notion in the theory of varifolds. Roughly speaking, for every varifold in a smooth Rimannian manifold Σ corresponds a vector valued distribution on Σ. This first variation relates the initial rate of change of the weight of a varifold under a deformation of Σ into itself with the initial velocity vector field of the deformation, which in turn gives a measure theoretic notion of boundary of a varifold. To define our variation, it is needed to have a way to map varifolds measure theoretically and compatible with the intuition of what a k-varifold has to be, i.e., a generalized k-submanifold of Σ.

Definition 3.4. *Let Σ_1^n and Σ_2^m be smooth manifolds of dimension n and m respectively, $V \in \mathbf{V}_k(\Sigma_1^n)$, and $F : \Sigma_1^n \to \Sigma_2^m$ a smooth map, we define the* ***push-forward of** V **by** F, as the Borel regular measure*

$$F_\sharp V,$$

on $G_k(\Sigma_2)$, characterized by

$$F_\sharp V(B) = \int_{\{(x,S):(F(x),D_xF(S))\in B\}} |JF(x) \circ S| dV(x,S),$$

Before introducing our so expected first variation, let us first fix some notations. Let $G \subset \Sigma$ be an open set, $\varepsilon > 0$ and $X \in \mathcal{X}(\Sigma)$ be a vector field, the one parameter family of diffeomorphisms generated by X is

$$\begin{aligned} \Phi : (-\varepsilon, \varepsilon) \times \Sigma &\to \Sigma \\ (t,x) &\mapsto \Phi(t,x) = \Phi_t(x), \end{aligned}$$

where Φ is the unique solution of

$$\begin{cases} \frac{\partial \Phi}{\partial t} = X(\Phi), \\ \Phi_0(x) = x, \end{cases}$$

and $\{x : \Phi_t(x) \neq x,\ t \in (-\varepsilon, \varepsilon)\}$ has compact closure on G.

Definition 3.5 (Variational Formula)**.** *Whenever $V \in \mathbf{V}_k(\Sigma)$ and $||V||(G) < \infty$, we define the* ***first variation of a varifold*** *V in G, as*

$$\delta V(X) = \frac{d}{dt} \left(||\Phi_{t\sharp}(V)||(G) \right) |_{t=0} \,.$$

Remark 3.3. *By the definition of push-forward of varifolds, and a derivation under the integral sign, we have*

$$\begin{aligned} \delta V(X) &= \frac{d}{dt} \left(||\Phi_{t\sharp}(V)||(G) \right) |_{t=0} \\ &= \int \frac{\partial}{\partial t} |J\Phi_t(x) \circ S| \,|_{t=0}\, dV(x,S) \\ &= \int \left(J \left(\frac{\partial \Phi_0}{\partial t}(x) \right) \circ S \right) \cdot S dV(x,S). \end{aligned}$$

Then we notice that the initial rate of change of $||\Phi_{t\sharp}||(G)$ depends linearly on $\frac{\partial \Phi}{\partial t}$.

Motivated by the preceding computations, we introduce in the functional theoretic language the first variation of a varifold. Later on we are going to prove the equality with the variational formulation.

Definition 3.6 (First Variation of a Varifold)**.** *Suppose* $V \in \mathbf{V}_k(\Sigma)$*, define the linear functional over the* $\mathbb{R}$ *vector space* $\mathcal{X}(\Sigma)$

$$\delta V : \mathcal{X}(\Sigma) \to \mathbb{R}$$
$$X \mapsto \delta V(X) = \int (\nabla X \circ S) \cdot S dV(x, S),$$

and we define the ***first variation of*** V ***on*** X *as* $\delta V(X)$*.*
Given $V \in \mathbf{V}_k(\Sigma)$*,* $||\delta V||$ *is the* ***total variation of*** δV*, if* $||\delta V||$ *is the largest Borel regular measure on* Σ *satisfying*

$$||\delta V||(G) := sup\{\delta V(g) : g \in \mathcal{X}(\Sigma), spt(g) \subseteq G, |g| \leq 1\},$$

for every open set G*. If* $\delta V = 0$ *we say that* V *is* ***stationary****, if* $G \subset \Sigma$ *is open, and* $||\delta V||(G) = 0$ *we say that* V *is* ***stationary on*** G*.*

Remark 3.4. *For each* $S \in G(n, k)$*, let* $\{e_1, \ldots e_k\}$ *be an orthonormal base of* S*, then*

$$\begin{aligned}
\delta V(X) &= \int (\nabla X \circ S) \cdot S dV(x, S) \\
&= \int tr(S^*(\nabla X \circ S)) dV(x, S) \\
&= \int \langle S^*(\nabla X \circ S)(e_i), e_i \rangle dV(x, S) \\
&= \int \langle (\nabla X \circ S)(e_i), S(e_i) \rangle dV(x, S) \\
&= \int \langle \nabla_{e_i} X, e_i \rangle dV(x, S) \\
&= \int div_S X dV(x, S),
\end{aligned}$$

where we have used in the preceding equalities the Einstein's summation convention, and the bracket $\langle \cdot, \cdot \rangle$ *corresponds to the inner product due to a Riemannian metric.*

Proposition 3.1. *Let* $V \in \mathbf{V}_k(\Sigma)$*, such that* $||\delta V||$ *is a Radon measure,* $X \in \mathcal{X}(\Sigma)$*, then there exist* $||V||$*-measurable functions* $H \in \mathbb{R}^n$ *and* $\eta \in \mathbb{S}^{n-1}$*,* $||\delta V||$*-measurable functions, and a measure* $||\delta V||_{sing} \perp ||V||$ *such that*

$$\delta V(X) = -\int X \cdot H d||V|| + \int X \cdot \eta d||\delta V||_{sing}.$$

Proof. Since $||\delta V||$ is the total variation of the linear functional δV, by Theorem 1.11 for all $X \in \mathcal{X}(\Sigma)$, there exists a vector valued, $||\delta V||$-measurable function η, with $|\eta| = 1$, $||\delta V||$-a.e. and

$$\delta V(X) = \int X \cdot \eta d||\delta V||. \tag{3.8}$$

Since $||\delta V||$ is by hypothesis a Radon measure and $||V|| \in \mathbf{V}_k(\Sigma)$, by Theorem 2.1 there exists the $||V||$-measurable function $h = D_{||V||}||\delta V||$, and

$$h(x) = \lim_{r \downarrow 0} \frac{||\delta V||(B(x,r))}{||V||(B(x,r))},$$

and there also exists a measure $||\delta V||_{sing} \perp ||V||$ such that

$$||\delta V|| = h||V|| + ||\delta V||_{sing}.$$

Replacing in (3.8), we have

$$\begin{aligned}\delta V(X) &= \int X \cdot \eta d(h||V||) + \int X \cdot \eta d||\delta V||_{sing} \\ &= \int X \cdot h\eta d||V|| + \int X \cdot \eta d||\delta V||_{sing}.\end{aligned}$$

Letting

$$-H := h\eta,$$

we have

$$\delta V(X) = -\int X \cdot H d||V|| + \int X \cdot \eta d||\delta V||_{sing}.$$

□

Definition 3.7 (Generalized Mean Curvature Vector)**.** *The vector field H as in Proposition 3.1 is called the* ***generalized mean curvature vector of the varifold*** *V.*

Definition 3.8. *Let $V \in \mathbf{V}_k(\Sigma)$. If there exists $H \in L^\infty(||V||) \cap \mathcal{X}^1(\Sigma)$ such that*

$$\delta V(X) = -\int H \cdot X d||V||, \ \forall X \in \mathcal{X}^1_c(\Sigma), \tag{3.9}$$

we say that V has ***bounded generalized mean curvature*** *H. In particular by the decomposition given by Theorem 3.1 we have that $||\delta V||_{sing} = 0$.*

Now we have almost all the tools to start with our regularity theorem, before that let us analyze the special case of integer-rectifiable varifolds.

3.3 First Variation of Integer Rectifiable Varifolds

As mentioned before, we are interested in a succinct formula for the first variation of a varifold, the Proposition 3.3 proves that in the integer-rectifiable case, the first variation is well defined and has a useful representation.

Proposition 3.2. *Let $\Phi_t : \Sigma \to \Sigma$ the family of diffeomorphisms as defined before, Let $E \subset \Sigma$ an $\mathcal{H}^k$-rectifiable set, then*

$$(\Phi_{t\sharp})\mathbf{v}(E) = \mathbf{v}(\Phi_t(E)).$$

Moreover, if $V \in \mathbf{IV}(\Sigma)$

$$(\Phi_{t\sharp})V = \sum_{i=1}^{\infty} k_i \mathbf{v}(\Phi_t(E)) = \sum_{i=1}^{\infty} k_i \mathcal{H}^k \llcorner (\Phi_y(E)),$$

and then

$$\delta V(X) = \frac{d}{dt}\left(\sum_{i=1}^{\infty} k_i \mathcal{H}^k \llcorner (\Phi_t(E)) \right) |_{t=0}$$

Proof. Let $B \in \mathcal{B}(G_k(\Sigma))$, then by definition

$$\begin{aligned}(\Phi_{t\sharp})\mathbf{v}(E)(B) &= \mathbf{v}(E)(\Phi^{-1}(B)) \\ &= \mathcal{H}^k\left(\{x \in E : (x, Tan^k(\mathcal{H}^k \llcorner E, x)) \in \Phi_t^{-1}(B)\}\right),\end{aligned}$$

then, if $(x, Tan^k(\mathcal{H}^k \llcorner E, x)) \in \Phi_t^{-1}(B)$, $(\Phi_t(x), Tan^k(\mathcal{H}^k \llcorner \Phi_t(E), \Phi_t(x))) \in B$, so

$$\begin{aligned}&\mathcal{H}^k\left(\{x \in E : (x, Tan^k(\mathcal{H}^k \llcorner E, x)) \in \Phi_t^{-1}(B)\}\right) = \\ &= \mathcal{H}^k\left(\{x \in \Phi_t(E) : (\Phi_t(x), Tan^k(\mathcal{H}^k \llcorner \Phi_t(E), \Phi_t(x))) \in B\}\right) \\ &= \mathbf{v}(\Phi_t(E))(B),\end{aligned}$$

thus

$$\mathbf{v}(\Phi_t(E)) = (\Phi_{t\sharp})\mathbf{v}(E). \tag{3.10}$$

The part that remains to prove is a direct consequence of the definitions and (3.10). □

Proposition 3.3. *Let $\Sigma \subset \mathbb{R}^n$ and $V \in \mathbf{IV}_k(\Sigma)$. Then $\delta V(X)$ is well defined, it holds*

$$\delta V(X) = \sum_{i=0}^{\infty} k_i \int_{E_i} div_{T_x\Gamma} X dV, \quad \forall X \in C_c^1(U, \mathbb{R}^n), \tag{3.11}$$

where

$$div_\pi X = \sum_{i=1}^{k} \langle \nabla_{e_i} X, e_i \rangle,$$

with $\{e_1, \ldots, e_k\}$ being an orthonormal basis over π, and $\Gamma := spt||V||$.

Proof. Let $\{E_i\}_{i\in\mathbb{N}}$ a sequence of sets such that for every $i \in \mathbb{N}$, E_i meets every compact subset of Σ in a k-rectifiable set and $\Sigma \subset \bigcup E_i$. Then given $X \in C^1_c(U, \mathbb{R}^n)$

$$\delta V(X) = \sum_i k_i \mathcal{H}^k \llcorner (\Phi_t(E_i)).$$

Hence, by the area formula

$$\mathcal{H}^k \llcorner (\Phi_t(E_i)) = \int_{\Phi_t(E_i)} d\mathcal{H}^k = \int_{E_i} J\Phi_t(x) d\mathcal{H}^k(x). \tag{3.12}$$

Without loss of generality we can assume E_i being k-rectifiable, then there exists many countable C^1 embeddings $F_j : \mathbb{R}^k \to \mathbb{R}^n$ and compact sets K_j such that

1. $F_j(K_j) \cap F_l(K_l) = \emptyset$
2. $F_j(K_j) \subset E_i$ for all j
3. $\{F_j(K_j)\}_{i\in\mathbb{N}}$ covers $\mathcal{H}^k$ a.e. E_i

So, by 3.12, and appliying the area formula, again, we conclude

$$\begin{aligned}
\mathcal{H}^k \llcorner (\Phi_t(E_i)) &= \int_{E_i} J\Phi_t(x) d\mathcal{H}^k(x) \\
&= \sum_j \int_{F_j(K_j)} J\Phi_t(x) d\mathcal{H}^k(x) \\
&= \sum_j \int_{K_j} J(\Phi_t \circ F)(y) d\mathcal{H}^k(y) \\
&= \sum_j \int_{K_j} \mid d(\Phi \circ F_i) \mid_y e_1 \wedge \ldots d(\Phi \circ F_i) \mid_y e_k \mid dy,
\end{aligned}$$

where $\{e_1, \ldots, e_k\}$ is an orthonormal basis of $\mathbb{R}^k$.

Fix $y \in K_j$, set $x = F_j(y)$, and recalling that $\{dF_j \mid_y e_1, \ldots, dF_j \mid_y e_k\}$ is a basis for $T_x\Gamma$, $\mathcal{H}^k$-a.e., we have that $x \in F_j(K_j)$, so if $\{v_1, \ldots, v_k\}$ is an orthonormal basis for $T_x\Gamma$ we can deduce

$$\mid d(\Phi \circ F_j) \mid_y e_1 \wedge \cdots \wedge d(\Phi \circ F_j) \mid_y e_k \mid = \mid d\Phi_t \mid_y v_1 \wedge \cdots \wedge d\Phi_t \mid_y v_k \mid\mid dF_j \mid_y e_1 \wedge \cdots \wedge dF_j \mid_y e_k \mid .$$

Setting $h_x(t) = \mid d\Phi_t \mid_y v_1 \wedge \cdots \wedge d\Phi_t \mid_y v_k \mid$, we have

$$\int_{\Phi_t(E_i)} d\mathcal{H}^k = \int_{E_i} h_x(t) dV(x).$$

Recalling that $\Phi_0 = id$ we have that

$$\frac{1}{t}\left(\mathcal{H}^k \llcorner (\Phi_t(E_i)) - \mathcal{H}^k \llcorner (\Phi_0(E_i))\right) = \frac{1}{t} \int_{E_i} (h_x(t) - h_x(0)) dV(x).$$

Since $h_x(t) = det\sqrt{M_x(t)}$ where $(M_x(t))_{ij} = \langle d\Phi_t \mid_x v_i, d\Phi_t \mid_x v_j \rangle$; differentiating $(M_x(t))_{ij}$ with respect to t yields to

$$(M_x'(t))_{ij} = \left\langle \frac{\partial}{\partial v_i}(X \circ \Phi_t) \mid_x v_i, d\Phi_t \mid_x v_j \right\rangle + \left\langle , d\Phi_t \mid_x v_i, \frac{\partial}{\partial v_j}(X \circ \Phi_t) \mid_x v_j \right\rangle .$$

Since $X(\Phi_t(x))$ is a tangent vector, there exists C a constant, independent of x, such that

$$\mid M_x'(t) \mid \leq C. \tag{3.13}$$

Hence, h_x is differentiable and there exist $\delta > 0$ and C such that

$$\mid h_x(t) - h_x(0) \mid \leq Ct, \tag{3.14}$$

for all $t \in [-\delta, \delta]$, and $x \in U$. Therefore

$$\lim_{t \downarrow 0} \frac{1}{t} \left(\mathcal{H}^k \llcorner (\Phi_t(E_i)) - \mathcal{H}^k \llcorner (\Phi_0(E_i)) \right) = \int_{E_i} h_x'(t) dV(x). \tag{3.15}$$

Finally

$$\begin{aligned} (M_x'(0))_{ij} &= \delta_{ij} h_x'(0) \\ &= \frac{1}{2} Tr M_x'(0) \\ &= \sum_i \langle \nabla_{v_i} X(x), v_i \rangle \\ &= div_{T_x\Sigma} X(x). \end{aligned}$$

The final result follows summing over every E_i. □

The following proposition is a straightforward consequence of Riez's Representation Theorem and Radon-Nikodym's theorem.

Proposition 3.4. *If there exists a constant $C \geq 0$ such that*

$$\mid \delta V(X) \mid \leq C \int \mid X \mid d||V||, \tag{3.16}$$

then there exists a bounded Borel map $H : U \to \mathbb{R}^n$ such that

$$\delta V(X) = - \int H \cdot X d||V||.$$

3.4 The Monotonicity Formula

The following result is a monotonicity formula, which ensures the monotone behaviour of the density ratio of the weight of a varifold, modulo a correction factor depending on the generalized mean curvature. This kind of results can be considered as a step zero for any regularity theory, the proof of which is achieved via a calculation of the first variation of a varifold by a radial deformation.

Theorem 3.2 (Monotonicity Formula)**.** *Let $V \in \mathbf{IV}_k(\Sigma)$ with bounded mean curvature H and fix $\xi \in U$. Then for $0 < \sigma < \rho < dist(\xi, \partial U)$ we have*

$$\frac{||V||(B_\rho(\xi))}{\rho^k} - \frac{||V||(B_\sigma(\xi))}{\sigma^k} = \int_{B_\rho(\xi)} \frac{H}{k} \cdot (x-\xi) \left(\frac{1}{m(r)^k} - \frac{1}{\rho^k} \right) d||V|| + \\ + \int_{B_\rho(\xi) \setminus B_\sigma(\xi)} \frac{| \nabla^\perp r |^2}{r^k} d||V||, \quad (3.17)$$

where $r(x) = | x - \xi |$, $\nabla^\perp r = P_{T_x \Gamma^\perp} \left(\frac{x-\xi}{|x-\xi|} \right)$, *and* $m(r) = \max\{|x|, \sigma\}$.

Proof. By a translation we can assume that $\xi = 0$. Let $\gamma \in C^1_c(]-1,1[)$, such that $\gamma \equiv 1$ in a neighbourhood of 0, and define the vector field $X_s(x) = \gamma\left(\frac{|x|}{s}\right) x$.
By Proposition 3.3 it is easily seen that

$$-\int X_s(x) \cdot H dV = \int div_{T_x \Gamma} X_s(x) d||V||(x). \quad (3.18)$$

Putting $\pi = T_x V$, fixing $\{e_1, \ldots, e_k\}$ an orthonormal basis of π, and the natural completion $\{e_1, \ldots, e_k, e_{k+1}, \ldots, e_n\}$ to an orthonormal basis of $\mathbb{R}^n$ leads to

$$\begin{aligned} div_{T_x V} X_s(x) &= \sum_i e_i \cdot \nabla_{e_j} X(s) \\ &= k\gamma\left(\frac{r}{s}\right) + \sum_{j=1}^k e_j \cdot x \gamma'\left(\frac{r}{s}\right) \frac{x \cdot e_j}{|x| s} \\ &= k\gamma\left(\frac{r}{s}\right) + \frac{r}{s}\gamma'\left(\frac{r}{s}\right) \sum_{j=1}^k \left(\frac{x \cdot e_j}{|x|}\right)^2 \\ &= k\gamma\left(\frac{r}{s}\right) + \frac{r}{s}\gamma'\left(\frac{r}{s}\right) \left(1 - \sum_{j=k+1}^n \left(\frac{x \cdot e_j}{|x|}\right)^2\right) \\ &= k\gamma\left(\frac{r}{s}\right) + \frac{r}{s}\gamma'\left(\frac{r}{s}\right) \left(1 - | \nabla^\perp r |^2\right). \end{aligned}$$

Now, inserting in (3.18) and dividing by s^{k+1} it follows

$$-\int_{\mathbb{R}^n}\gamma\left(\frac{|x|}{s}\right)\frac{H\cdot x}{s^{k+1}}dV(x)=\int_{\mathbb{R}^n}\frac{k}{s^{k+1}}\gamma\left(\frac{|x|}{s}\right)dV(x)+$$
$$+\int_{\mathbb{R}^n}\frac{r}{s^{k+2}}\gamma'\left(\frac{|x|}{s}\right)\left(1-\mid\nabla^{\perp}r\mid\right)^2 dV(x).$$

Integrating in s, between σ and ρ

$$-\int_{\sigma}^{\rho}\int_{\mathbb{R}^n}\gamma\left(\frac{|x|}{s}\right)\frac{H\cdot x}{s^{k+1}}d||V||(x)=\int_{\sigma}^{\rho}\int_{\mathbb{R}^n}\frac{k}{s^{k+1}}\gamma\left(\frac{|x|}{s}\right)d||V||(x)+$$
$$+\int_{\sigma}^{\rho}\int_{\mathbb{R}^n}\frac{|x|}{s^{k+2}}\gamma'\left(\frac{r}{s}\right)\left(1-\mid\nabla^{\perp}r\mid\right)^2 d||V||(x).$$

Applying the Fubini's Thorem at all the terms of the preceding equality we get

$$-\int_{\mathbb{R}^n}\int_{\sigma}^{\rho}\gamma\left(\frac{|x|}{s}\right)\frac{H\cdot x}{s^{k+1}}d||V||(x)=\int_{\mathbb{R}^n}\int_{\sigma}^{\rho}\frac{k}{s^{k+1}}\gamma\left(\frac{|x|}{s}\right)d||V||(x)+$$
$$+\int_{\mathbb{R}^n}\int_{\sigma}^{\rho}\frac{|x|}{s^{k+2}}\gamma'\left(\frac{|x|}{s}\right)\left(1-\mid\nabla^{\perp}r\mid\right)^2 d||V||(x).$$

Notice that

$$\int_{\sigma}^{\rho}\frac{k}{s^{k+1}}\gamma\left(\frac{|x|}{s}\right)+\frac{|x|}{s^{k+2}}\gamma'\left(\frac{|x|}{s}\right)ds=-\int_{\sigma}^{\rho}\frac{d}{ds}\left(\frac{1}{s^k}\gamma\left(\frac{|x|}{s}\right)\right)ds.$$

Applying the Fundamental Theorem of Calculus we get

$$\begin{aligned}-\left(\frac{1}{\rho^k}\int_{\mathbb{R}^n}\gamma\left(\frac{|x|}{\rho}\right)d||V||(x)-\frac{1}{\sigma^k}\int_{\mathbb{R}^n}\gamma\left(\frac{|x|}{\sigma}\right)d||V||(x)\right) &= \\ \int_{\mathbb{R}^n}|\nabla^{\perp}r|^2\int_{\sigma}^{\rho}\frac{|x|}{s^{k+2}}\gamma'\left(\frac{|x|}{s}\right)dsd||V||(x) &+ \\ \int_{\mathbb{R}^n}H\cdot x\int_{\sigma}^{\rho}\frac{1}{s^{k+1}}\gamma\left(\frac{|x|}{s}\right)dsd||V||(x). &\end{aligned} \tag{3.19}$$

Integrating by parts

$$\begin{aligned}\int_{\mathbb{R}^n}|\nabla^{\perp}r|^2\int_{\sigma}^{\rho}\frac{|x|}{s^{k+2}}\gamma'\left(\frac{|x|}{s}\right)dsd||V||(x) &= \int_{\mathbb{R}^n}|\nabla^{\perp}r|^2\int_{\sigma}^{\rho}-\frac{1}{s^k}\left[-\frac{|x|}{s^2}\gamma'\left(\frac{|x|}{s}\right)\right)dsd||V||(x)\\ &= \int_{\mathbb{R}^n}|\nabla^{\perp}r|^2\int_{\sigma}^{\rho}-\frac{1}{s^k}\frac{d}{ds}\left(\gamma\left(\frac{|x|}{s}\right)\right)dsd||V||(x)\\ &= \int_{\mathbb{R}^n}|\nabla^{\perp}r|^2\left[-\frac{1}{s^k}\gamma\left(\frac{|x|}{s}\right)\Big|_{\sigma}^{\rho}-\right.\\ &\quad\left.-\int_{\sigma}^{\rho}\frac{k}{s^{k-1}}\gamma\left(\frac{|x|}{s}\right)ds\right]d||V||(x)\\ &= \int_{\mathbb{R}^n}|\nabla^{\perp}r|^2\left[\frac{1}{\sigma^k}\gamma\left(\frac{|x|}{\sigma}\right)-\frac{1}{\rho^k}\gamma\left(\frac{|x|}{\rho}\right)-\right.\\ &\quad\left.-\int_{\sigma}^{\rho}\frac{k}{s^{k-1}}\gamma\left(\frac{|x|}{s}\right)ds\right]d||V||(x).\end{aligned}$$

Replacing in (3.19)

$$\left(\frac{1}{\rho^k}\int_{\mathbb{R}^n}\gamma\left(\frac{|x|}{\rho}\right)d||V||(x)-\frac{1}{\sigma^k}\int_{\mathbb{R}^n}\gamma\left(\frac{|x|}{\sigma}\right)dV(x)\right) =$$

$$\int_{\mathbb{R}^n}|\nabla^\perp r|^2\left[\frac{1}{\rho^k}\gamma\left(\frac{|x|}{\rho}\right)-\frac{1}{\sigma^k}\gamma\left(\frac{|x|}{\sigma}\right)\int_\sigma^\rho\frac{k}{s^{k-1}}\gamma\left(\frac{|x|}{s}\right)ds\right]d||V||(x) +$$

$$\int_{\mathbb{R}^n}H\cdot x\int_\sigma^\rho\frac{k}{s^{k+1}}\gamma\left(\frac{|x|}{s}\right)dsd||V||(x). \tag{3.20}$$

Now testing (3.20) with a sequence of cut-off functions $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $\gamma_n\to\chi_{]-1,1[}$ from below, by the dominated convergence theorem

$$\frac{1}{\rho^k}||V||(B_\rho(0))-\frac{1}{\sigma^k}||V||(B_\sigma(0)) =$$

$$\int_{\mathbb{R}^n}|\nabla^\perp r|^2\left[\frac{1}{\rho^k}\chi_{]0,1[}\left(\frac{|x|}{\rho}\right)-\frac{1}{\sigma^k}\chi_{]0,1[}\left(\frac{|x|}{\sigma}\right)+\int_\sigma^\rho\frac{k}{s^{k-1}}\chi_{]0,1[}\left(\frac{|x|}{s}\right)ds\right]d||V||(x)$$

$$+\int_{\mathbb{R}^n}H\cdot x\int_\sigma^\rho\frac{k}{s^{k+1}}\chi_{]0,1[}\left(\frac{|x|}{s}\right)dsd||V||(x).$$

Since

$$\int_\sigma^\rho\frac{k}{s^{k+1}}\chi_{]0,1[}\left(\frac{|x|}{s}\right)ds=\int_{m(r)}^\rho\frac{k}{s^{k+1}}ds=\left[\frac{1}{m(r)^k}-\frac{1}{\rho^k}\right]\chi_{]0,\rho[}(|x|),$$

the monotonicity formula (3.17) readily follows. □

The following results are simple but very important consequences of the monotonicity formula.

Corollary 3.2. *The function*

$$\rho\mapsto e^{||H||_\infty\rho}\rho^{-k}||V||(B_\rho(\xi)), \tag{3.21}$$

is monotone increasing.

Proof. Without loss of generality we can assume $\xi=0$. Let $f(\rho)=\rho^{-k}||V||(B_\rho(0))$, then

$$\frac{f(\rho)-f(\sigma)}{\rho-\sigma}=\frac{\rho^{-k}||V||(B_\rho(0))-\sigma^{-k}||V||(B_\sigma(0))}{\rho-\sigma}.$$

By the monotonicity formula (3.17) of Theorem 3.2 it is easily checked that

$$\frac{f(\rho)-f(\sigma)}{\rho-\sigma}=\frac{1}{\rho-\sigma}\left[\int_{B_\rho(0)}\frac{H\cdot x}{k}\left(\frac{1}{m(r)^k}-\frac{1}{\sigma^k}\right)dV+\int_{B_\rho(0)\setminus B_\sigma(0)}\frac{|\nabla^\perp r|^2}{|x|^k}dV\right].$$

Hence

$$\begin{aligned}\frac{f(\rho)-f(\sigma)}{\rho-\sigma}&\geq\frac{1}{\rho-\sigma}\int_{B_\rho(0)}\frac{H\cdot x}{k}\left(\frac{1}{m(r)^k}-\frac{1}{\rho^k}\right)dV\\&\geq-\frac{||H||_\infty}{k}\int_{B_\rho(0)}|x|\frac{m(r)^{-k}-\rho^{-k}}{\rho-\sigma}dV\\&\geq-\frac{||H||_\infty}{k}V(B_\rho(0)\rho\frac{\sigma^{-k}-\rho^{-k}}{\rho-\sigma}\end{aligned}$$

Since $g : \rho \mapsto \rho^{-k}$ is convex, setting $\rho = \sigma + \varepsilon$ we get

$$\begin{aligned}
\frac{f(\rho)-f(\sigma)}{\rho-\sigma} &\geq -\frac{||H||_\infty}{k} V(B_\rho(0))(\sigma+\varepsilon)\frac{\sigma^{-k}-(\sigma+\varepsilon)^{-k}}{\varepsilon} \\
&= \frac{||H||_\infty}{k} V(B_\rho(0))(\sigma+\varepsilon) g'(\eta) \\
&\geq -||H||_\infty V(B_\rho(0))(\sigma+\varepsilon)\sigma^{-(k+1)},
\end{aligned}$$

where $\eta \in]\sigma, \rho[$. Therefore

$$\frac{f(\sigma+\varepsilon)-f(\sigma)}{\varepsilon} \geq -||H||_\infty f(\sigma+\varepsilon)\frac{(\sigma+\varepsilon)^{k+1}}{\sigma^{k+1}}. \tag{3.22}$$

If ψ_δ is a standard non-negative mollifier, we can first take the convolution with ψ_δ integrating with respect to the variable σ, in both sides of (3.22) yields

$$\frac{f(\sigma+\varepsilon)-f(\sigma)}{\varepsilon} * \psi_\delta \geq -||H||_\infty \left(f(\sigma+\varepsilon)\frac{(\sigma+\varepsilon)^{k+1}}{\sigma^{k+1}} * \psi_\delta \right),$$

and only after letting $\varepsilon \downarrow 0$. We obtain in this way

$$(f * \psi_\delta)' \geq -||H||_\infty (f * \psi_\delta).$$

Hence, multiplying by $e^{||H||_\infty \rho}$

$$e^{||H||_\infty \rho}(f * \psi_\delta)' + ||H||_\infty e^{||H||_\infty \rho}(f * \psi_\delta) \geq 0,$$

or equivalently

$$\frac{d}{d\rho}\left(e^{||H||_\infty \rho}(f * \psi_\delta)\right) \geq 0.$$

Finally taking the limit when $\delta \to 0$ in the preceding inequality the result follows easily.

□

The following proposition is an interesting application of Theorem 3.2.

Proposition 3.5. *Let $V \in \mathbf{IV}_k(U)$, where $U \subset \mathbb{R}^n$ is open and with bounded mean curvature. Then*

(i) the limit

$$\Theta(||V||, x) = \lim_{\rho \downarrow 0} \frac{||V||(B_\rho(x))}{\omega_k \rho^k}, \tag{3.23}$$

exists at every $x \in U$,

(ii) $\Theta(||V||, x)$ is upper semicontinuous,

(iii) $\Theta(||V||, x) \geq 1$, *for all* $x \in Spt||V|| \cap U$,

(iv)

$$||V||(B_\rho(x)) \geq \omega_k e^{-||H||_\infty \rho} \rho^k, \tag{3.24}$$

for all $x \in spt(||V||)$ *and for all* $\rho < dist(x, \partial U)$,

(v) $\mathcal{H}^k \left(Spt||V|| \setminus \bigcup_{i=0}^{\infty} E_i\right) = 0$.

Proof. (i) The existence of the limit is guaranteed by the monotonicity of

$$\rho \to e^{||H||_\infty \rho} \rho^{-k} ||V||(B_\rho(x)).$$

(ii) Fix $x \in U$ and $\varepsilon > 0$, Let $0 < 2\rho < dist(x, \partial U)$ such that

$$e^{||H||_\infty r} \frac{||V||(B_\rho(x))}{r^{-k}\omega_k} \leq \Theta(||V||, x) + \frac{\varepsilon}{2}, \quad \forall r < 2\rho. \tag{3.25}$$

If $\delta < \rho$ and $|x - y| < \delta$, then

$$\begin{aligned}
\Theta(||V||, y) := \lim_{\rho \downarrow 0} \frac{||V||(B_\rho(y))}{\omega_k \rho^k} &\leq \lim_{\rho \downarrow 0} e^{||H||_\infty \rho} \frac{||V||(B_\rho(y))}{\omega_k \rho^k} \\
&\leq \lim_{\rho \downarrow 0} e^{||H||_\infty (\rho+\delta)} \frac{||V||(B_{(\rho+\delta)}(x))}{\omega_k \rho^k} \\
&= \lim_{\rho \downarrow 0} e^{||H||_\infty (\rho+\delta)} \frac{||V||(B_{(\rho+\delta)}(x))}{\omega_k (\rho+\delta)^k} \left(\frac{\rho+\delta}{\rho}\right)^k \\
&\leq \left(1 + \frac{\delta}{\rho}\right)^k \left[\Theta(||V||, x) + \frac{\varepsilon}{2}\right],
\end{aligned}$$

where the last inequality is true because of (3.25). If δ is small enough

$$\Theta(||V||, y) \leq \Theta(||V||, x) + \varepsilon,$$

which proves the upper semicontinuity.

(iii) Since $\Theta(||V||, \cdot) \equiv k_i$ $||V||$-a.e. the set $\{\Theta(||V||, \cdot) \geq 1\}$ has full measure. Thus $\{\Theta(||V||, \cdot) \geq 1\}$ must be dense in $Spt(||V||)$ and so, for every $x \in Spt(||V||) \cap V$ the inequality $\Theta(||V||, x) \geq 1$ follows from the upper semi continuity.

(iv) The remaining follows trivially from Theorem 3.2 and the corollary.

(v) Finally by the classical Density theorems $\Theta(||V||, \cdot) = 0$ $\mathcal{H}^k$-a.e. On $U \setminus E_i$ for each $i \in \mathbb{N}$. Hence the result follows from (iii).

□

Chapter 4

Allard's Interior Regularity theorem

In this chapter we introduce, and prove the **Allard's Interior Regularity Theorem** (Theorem 4.1), which roughly speaking asserts that under suitable conditions a Integer-Rectifiable varifold V is a $C^{1,\alpha}$ submanifold of a certain ball. The proof of Theorem 4.1 is achieved by an *Excess-Decay* argument, joint with a suitable *Lipschitz approximation* of the varifold $V \in \mathbf{IV}_k(\Sigma)$.

To do this is necessary to introduce the **tilt excess** and **height excess** of a varifold.

4.1 Excess and Allard's Theorem

Definition 4.1 (Excess). *Let $V \in \mathbf{IV}_k(U)$ with $U \subset \mathbb{R}^n$, $B_r(x) \subset U$ an open ball and π a k-dimensional plane. The **tilt excess of** V **in** $B_r(x)$ **with respect to** π is the nonnegative real number $E(V, \pi, x, r) \in [0, +\infty[$ defined as follows*

$$E(V, \pi, x, r) := r^{-k} \int_{B_r(x)} ||T_y\Gamma - \pi||^2 d||V||(y). \tag{4.1}$$

The tilt excess of a rectifiable varifold measures the square local deviation of the tangent planes of the support of the varifold to a fixed plane π.

Definition 4.2. *Let $V \in \mathbf{IV}_k(U)$ with $U \subset \mathbb{R}^n$, $B_r(x) \subset U$ an open ball and π a k-dimensional plane. The L^2**-height-excess of** V **in** $B_r(x)$ **with respect to** π is the nonnegative real number $Hexc(V, \pi, x, r) \in [0, +\infty[$ defined as follows*

$$Hexc(V, \pi, x, r) := r^{-k-2} \int_{B_r(x)} d(y - x_0, \pi)^2 d||V||(y). \tag{4.2}$$

The height excess of a rectifiable varifold gives a measure of the distance of the support of the varifold to the fixed plane π.

Proposition 4.1 (Rescaling properties of the tilt excess)**.** *For every* $V \in \mathbf{IV}_k(\Sigma)$ *holds*

(i): $E(\mu_{r\sharp}V, x_0, \pi, r) = E(V, x_0, \pi, 1)$.

Proposition 4.2 (Rescaling properties of the height excess)**.** *For every* $V \in \mathbf{IV}_k(\Sigma)$ *holds*

(i): $Hexc(\mu_{r\sharp}V, x_0, \pi, r) = Hexc(V, x_0, \pi, 1)$.

Theorem 4.1 (Allard's ε-regularity Theorem)**.** *For every* $k < n$*,* $k \in \mathbb{N}$ *there exist positive constants* $\alpha, \varepsilon, \gamma$ *such that, if* $V \in \mathbf{IV}_k(\Sigma)$ *and* V *is with bounded mean curvature* H *in* $B_r(x_0) \subset \mathbb{R}^n$ *satisfying:*

$(H1)$ $||V||(B_r(x_0)) < (\omega_k + \varepsilon)r^k$*, and* $||H||_\infty \leq \frac{\varepsilon}{r}$.

$(H2)$ *There exists* $\pi \in G(n,k)$ *such that* $E(V, \pi, x, r) < \varepsilon$.

Then $spt||V|| \cap B_{\gamma r}(x_0)$ *is a* $C^{1,\alpha}$ *submanifold (without boundary) in* $B_r(x_0)$.

The remaining of the text is devoted to prove the Theorem 4.1, and to do this we follow the scheme below:

- In Section 4.2 we prove an inequality for the excess which is a direct analogue of the Caccioppoli's inequality for solutions of elliptic partial differential equations.
- In Section 4.3 we show that, under the assumptions of Theorem 4.1, the varifold can be well approximated by a Lipschitz graph.
- In Section 4.4 we use the previous sections to prove an excess-decay theorem.
- In Section 4.5 we use Theorem 4.4 and the Lipschitz approximation and a excess-decay argument (which is the an instance of the well known De Giorgi-Nash-Moser iteration scheme) to conclude the proof of Theorem 4.1

4.2 Tilt Excess Inequality

The first step to prove Theorem 4.4 is an analogue of the *Caccioppoli's inequality* for elliptic PDE's.

Theorem 4.2 (Tilt-Excess Inequality). *Let $k < n$ be a positive integer. Then there is a constant $C > 0$, such that the following inequality holds for every varifold $V \in \mathbf{IV}_k(\Sigma)$ with bounded mean curvature H in $B_r(x_0) \subset \mathbb{R}^n$, satisfying $(H1), (H2)$ of Theorem 4.1, and every k-plane $\pi \in G(n,k)$*

$$E(V,\pi,x_0,r/2) \leq \frac{C}{r^{k+2}} \int_{B_r(x_0)} dist(y-x_0,\pi)^2 dV(y) + \frac{2^{k+1}}{r^{k-2}} \int_{B_r(x_0)} |H|^2 dV. \tag{4.3}$$

To get an intuition of what happens in Theorem 4.2, suppose that $V = \sum \mathbf{v}(E_i)$ and each E_i is the graph of a function f with small Lipschitz constant the boundedness of H translates in a suitable elliptic system of partial differential equations, then $E(V,\pi,x_0,r/2)$ approximates the *Dirichlet Energy*, in fact when π is the hyperplane $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^{k+1}$, $E(V,\mathbb{R}^k \times \{0\},0,r/2) = \frac{2^k}{r^k}\int_{B(0,r/2)} |\nabla f|^2$ and the height excess in the right hand of inequality (4.3) is compared with the L^2 norm of f. Then the inequality can be translated into

$$\frac{1}{r^k}\int_{B_{r/2}(x_0)} |\nabla f|^2 \leq \frac{C}{r^{k+2}} \int_{B_r(x_0)} |f|^2 + \frac{2^{k+1}}{r^{k-2}} \int_{B_r(x_0)} |div(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}})|^2,$$

which is the *Caccioppoli's inequality* for elliptic partial differential equations, in the case in which the elliptic operator considered is the mean curvature operator of a graph. About this point the reader can compare Theorem 4.4 of [GM12].

Before to start the proof of Theorem 4.2 we need a technical computation, which will be useful all along the text. First of all we introduce some notation. Given a k-dimensional plane π we denote P_π and $P_\pi^\perp$ respectively the orthonormal projection onto π and $\pi^\perp$, similarly, for $f \in C^1(\mathbb{R}^n)$, $\nabla_\pi f$ and $\nabla_\pi^\perp f$ will denote, respectively $P_\pi \circ \nabla f$ and $P_\pi^\perp \circ \nabla f$. Finally, if $\Phi \in C^1(\mathbb{R}^n,\mathbb{R}^k)$, $J_\pi\Phi$ will denote the absolute value of the Jacobian determinant of $\Phi\mid_\pi$.

Lemma 4.1. *Consider two k-dimensional planes π, T in $\mathbb{R}^n$. Let $X : \mathbb{R}^n \to \mathbb{R}^n$ be the vector field $X(x) := P_\pi^\perp(x)$ and fix an orthonormal base $\{\nu_{k+1},\ldots,\nu_n\}$ of $\pi^\perp$, consider the function $f_j(x) := \langle x,\nu_j\rangle_{\mathbb{R}^n}$. Then*

$$\frac{1}{2}||T-\pi||^2 = div_T X = \sum_{i=k+1}^{n} |\nabla_T f_i|^2. \tag{4.4}$$

Moreover there is a positive constant C_0, depending only on N and k such that

$$|J_T P_\pi - 1| \leq C_0 ||T-\pi||^2. \tag{4.5}$$

Proof. Let $\{\xi_1, \ldots, \xi_k\}$ an orthonormal base of T and $\{e_{k+1}, \ldots, e_n\}$ an orthonormal base of $T^{\perp}$. Notice that

$$P_\pi = id_{\mathbb{R}^n} - \sum_{j=k+1}^{n} \nu_j \otimes \nu_j, \;\; P_T = id_{\mathbb{R}^n} - \sum_{j=k+1}^{n} e_i \otimes e_i,$$

with $id_{\mathbb{R}^n}$ denoting the identity map of $\mathbb{R}^n$. Thus

$$\begin{aligned}
\frac{1}{2}||\pi - T||^2 &= \frac{1}{2} \langle P_\pi - P_T, P_\pi - P_T \rangle_{\mathbb{R}^n \otimes \mathbb{R}^n} \\
&= \frac{1}{2} \left\langle - \sum_{j=k+1}^{n} \nu_j \otimes \nu_j + \sum_{i=k+1}^{n} e_i \otimes e_i, - \sum_{j=k+1}^{n} \nu_j \otimes \nu_j + \sum_{i=k+1}^{n} e_i \otimes e_i \right\rangle_{\mathbb{R}^n \otimes \mathbb{R}^n} \\
&= \frac{1}{2} \left\langle \sum_{j=k+1}^{n} \nu_j \otimes \nu_j - \sum_{i=k+1}^{n} e_i \otimes e_i, \sum_{j=k+1}^{n} \nu_j \otimes \nu_j - \sum_{i=k+1}^{n} e_i \otimes e_i \right\rangle_{\mathbb{R}^n \otimes \mathbb{R}^n} \\
&= \frac{1}{2} \left\langle \sum_{j=k+1}^{n} \nu_j \otimes \nu_j, \sum_{j=k+1}^{n} \nu_j \otimes \nu_j \right\rangle_{\mathbb{R}^n \otimes \mathbb{R}^n} - \left\langle \sum_{j=k+1}^{n} \nu_j \otimes \nu_j, \sum_{i=k+1}^{n} e_i \otimes e_i \right\rangle_{\mathbb{R}^n \otimes \mathbb{R}^n} \\
&+ \frac{1}{2} \left\langle \sum_{i=k+1}^{n} e_i \otimes e_i, \sum_{i=k+1}^{n} e_i \otimes e_i \right\rangle_{\mathbb{R}^n \otimes \mathbb{R}^n} \\
&= \frac{1}{2}(n-k) + \frac{1}{2}(n-k) - \sum_{j=k+1}^{n} \sum_{i=k+1}^{n} \langle \nu_j \otimes \nu_j, e_i \otimes e_i \rangle_{\mathbb{R}^n \otimes \mathbb{R}^n} \\
&= (n-k) - \sum_{j=k+1}^{n} \sum_{i=k+1}^{n} (\nu_j \cdot e_i)^2 \\
&= \sum_{j=k+1}^{n} \left(1 - \sum_{i=k+1}^{n} (\nu_j \cdot e_i)^2 \right) \\
&= \sum_{j=k+1}^{n} \sum_{i=1}^{k} (\nu_j, \xi_i)^2 = \sum_{j=k+1}^{N} |\nabla_T f_j|^2.
\end{aligned}$$

The last equality is due to the fact that in general $\nabla_{\xi_i} f_j = \langle \nabla f_j, \xi_i \rangle$ and a simple calculation using the very definition of f_j, gives also $D_{\xi_i} f_j = \langle \xi_i, \nu_j \rangle$. So

$$\nabla_T f_j = P_T \circ \nabla f_j = \sum_{i=1}^{k} (\xi_i \cdot \nu_j) \xi_i,$$

and we conclude from this that the following equality holds

$$\frac{1}{2}||\pi - T||^2 = \sum_{j=k+1}^{N} |\nabla_T f_j|^2. \tag{4.6}$$

On the other hand

$$
\begin{aligned}
\sum_{i=k+1}^{N} |\nabla_T f_j|^2 &= \sum_{i=k+1}^{N} \sum_{j=1}^{k} (\xi_j \cdot \nu_i)^2 \\
&= \sum_{i=k+1}^{N} \sum_{j=1}^{k} (\xi_j \cdot \nu_i)(\xi_j \cdot \nu_i) \\
&= \sum_{i=k+1}^{N} \sum_{j=1}^{k} (\xi_j \cdot (\nu_i(\xi_j \cdot \nu_i)) \\
&= \sum_{j=1}^{k} \left(\xi_j \cdot \sum_{i=k+1}^{N} (\nu_i(\xi_j \cdot \nu_i)) \right) \\
&= \sum_{j=1}^{k} \sum_{i=k+1}^{N} \left(\xi_j \cdot \frac{\partial f_i}{\partial \xi_j} \nu_i \right) \\
&= \sum_{j=1}^{k} \left(\xi_j \cdot \frac{\partial}{\partial \xi_j} \sum_{i=k+1}^{N} f_i \nu_i \right) \\
&= \sum_{j=1}^{k} \left(\xi_j \cdot \frac{\partial}{\partial \xi_i} X \right) \\
&= div_T X.
\end{aligned}
$$

Next, recall that $J_T P_\pi = \sqrt{detM}$, where

$$
\begin{aligned}
M_{ij} &= P_\pi(\xi_j) \cdot P_\pi(\xi_i) \\
&= \delta_{ij} - (P_\pi^\perp(\xi_j) \cdot \xi_i) - (\xi_j \cdot P_\pi^\perp(\xi_i)) + (P_\pi^\perp(\xi_j) \cdot P_\pi^\perp(\xi_i)) \\
&:= \delta_{ij} + A_{ij} + A_{ji} + B_{ij}.
\end{aligned}
$$

Observe that there exists a positive constant $C = C(n,k)$, such that

$$
\begin{aligned}
|P_\pi^\perp(\xi_j)| = |P_\pi^\perp(\xi_j) - P_T^\perp(\xi_j)| &\leq ||T^\perp - \pi^\perp||^{op} |\xi_j| \\
&\leq ||T^\perp - \pi^\perp|| = ||T - \pi||,
\end{aligned} \tag{4.7}
$$

where by $||\cdot||^{op}$ we denote the operator norm that as it is very well known it is equivalent to the Hilbert-Schmidt norm $||\cdot||$, since we are dealing with finite dimensional normed vector spaces. Thus taking the supremum over the set of unit vectors of T, in (4.7) and again considering the equivalence of the norms we obtain

$$
||A|| \leq ||T - \pi|| \ and \ ||B|| \leq ||T - \pi||^2. \tag{4.8}
$$

Moreover

$$
\begin{aligned}
A_{ij} &:= \xi_i \cdot P_\pi^\perp(\xi_j) \\
&= \sum_{i=1}^{k}\left(\xi_i \cdot \sum_{l=k+1}^{N}(\xi_j \cdot \nu_l)\nu_l\right) \\
&= \xi_i \cdot \frac{\partial}{\partial \xi_j} X.
\end{aligned}
$$

Hence, the usual Taylor expansion of the determinant gives

$$
\begin{aligned}
detM &= 1 - 2TrA + O(||T-\pi||^2) \\
&= 1 - 2div_T X + O(||T-\pi||^2).
\end{aligned}
$$

Since $J_T P_\pi \geq 0$, it is straightforward to check that $|J_T P_\pi - 1| \leq |J_T P_\pi - 1||J_T P_\pi + 1| = |detM - 1|$. As a consequence of this we get the following estimates

$$
\begin{aligned}
|J_T P_\pi - 1| &\leq |2div_T X - O(||T-\pi||^2)| \\
&\leq 2|div_T X| + C||T-\pi||^2.
\end{aligned}
$$

Using (4.4) combined with the preceding estimates we finally are lead to show the existence of a positive constant $C_0 = C_0(N,k)$, such that

$$
|J_T P_\pi - 1| \leq C_0 ||T-\pi||^2,
$$

and (4.5) is proved. □

Now we are ready to prove Theorem 4.2.

Proof of the Tilt-Excess Inequality (4.3). Consider a smooth cut-off function $\phi(\frac{x}{r}) =: \zeta_r \in C_c^\infty(B_r(x_0))$ (that for simplicity of notation, we will denote just with ζ forgetting the dependence on r) such that $\zeta \equiv 1$ on $B_{r/2}(x_0)$, $\zeta \equiv 0$ outside $B_r(x_0)$, and $|\nabla\zeta| \leq \frac{||\nabla\phi||_\infty}{r} = \frac{C}{r}$, where C can be chosen as a numerical constant not less than 2, for example $C = 3$. Let $X(x) = P_\pi^\perp(x)$, $\Gamma := spt||V||$, we test the first variation of $V \in \mathbf{IV}_k(\Sigma)$ with the vector field $\zeta^2 X$, this gives

$$
\delta V(\zeta^2 X) = \int div_{T_y\Gamma}(\zeta^2 X) d||V||(y) = -\int H \cdot (\zeta^2 X) d||V||. \tag{4.9}
$$

Hence

$$
\int \zeta^2 div_{T_y\Gamma} X d||V||(y) = -\int \zeta^2 H \cdot X d||V|| - \int 2\zeta X \cdot \nabla_{T_y\Gamma}\zeta d||V||. \tag{4.10}
$$

By the previous lemma it is straightforward to verify that

$$\begin{aligned} \frac{2^{k+1}}{r^k}\int_{B_r(x_0)} \zeta^2 div_{T_y\Gamma} X d||V||(y) &\geq \frac{2^k}{r^k}\int_{B_{r/2}(x_0)} ||T_y\Gamma - \pi||^2 d||V||(y) \\ &= E(V,\pi,x_0,r/2). \end{aligned} \tag{4.11}$$

On the other hand applying the Young's inequality to the first term of the right hand side of (4.10) we have

$$\begin{aligned} \int \zeta^2 div_{T_y\Gamma} X d||V|| &\leq \frac{1}{2}\int \zeta^4 |X|^2 d||V|| + \frac{1}{2}\int_{B_r(x_0)} |H|^2 d||V|| \\ &+ 2\int \zeta |X\cdot \nabla_{T_yV}\zeta| d||V||. \end{aligned} \tag{4.12}$$

Now to obtain L^2 estimates we need to estimate $\zeta|X\cdot\nabla_{T_y\Gamma}\zeta|$, set $T = T_y\Gamma$, recall that $X(x) = \sum_{i=k+1}^n \langle x,\nu_i\rangle \nu_i$, and use the notation of the previous lemma, to check the following

$$\begin{aligned} 2\zeta|X\cdot\nabla_T\zeta| &= \zeta\left|\sum_{j=1}^k (\nabla\zeta\cdot\xi_j)(\xi_j\cdot X)\right| \\ &\leq \zeta\sum_{j=1}^k |(\nabla\zeta\cdot\xi_j)| \sum_{i=k+1}^n |f_j||(\nu_j\cdot\xi_i)| \\ &\leq C\zeta|\nabla\zeta||X| \sum_{i=k+1}^n |\nabla_T f_i|. \end{aligned}$$

Applying the Young's inequality again we have

$$C\zeta|\nabla\zeta||X|\sum_{i=k+1}^n |\nabla_T f_i| \leq \frac{(n-k)C}{2}\zeta^2|\nabla\zeta|^2|X|^2 + \frac{1}{2}\zeta^2\sum_{k+1}^n |\nabla_T f_i|^2.$$

Applying again Lemma 4.1 we deduce

$$2\zeta|X\cdot\nabla_T\zeta| \leq C(N,k)\zeta^2|\nabla\zeta|^2|X|^2 + \frac{1}{2}\zeta^2 div_T X. \tag{4.13}$$

Inserting (4.13) in (4.12) we obtain

$$\begin{aligned} \frac{1}{2}\int \zeta^2 div_{T_yV} X d||V||(y) &\leq \frac{1}{2}\int \zeta^4|X|^2 d||V|| + \frac{1}{2}\int_{B_r(x_0)} |H|^2 d||V|| \\ &+ C\int \zeta^2|\nabla\zeta|^2|X|^2 d||V||. \end{aligned} \tag{4.14}$$

Since $|\nabla\zeta| \leq \frac{C}{r}$, $|\zeta| \leq C$ and $|X(x)| = dist(x,\pi)$ we conclude

$$\frac{1}{2}\int_{B_{r/2}(x_0)} \zeta^2 div_{T_y||V||} \leq \frac{C}{r^2}\int_{B_r(x_0)} dist(x,\pi)^2 d||V|| + \frac{1}{2}\int_{B_r(x_0)} r^2|H|^2 d||V||. \tag{4.15}$$

Combining all this with (4.11) leads to prove (4.3) and establish the proof of Theorem 4.2. $\square$

4.3 The Lipschitz approximation

Theorem 4.3 (Lipschitz Approximation)**.** *For any positive integer $k < N$, there is a constant C with the following property. For any $l, \beta \in]0,1[$ there are $\lambda_l > 0$ (depending on l, but not on β) and $\varepsilon_L = \varepsilon_L(l,\beta)$ (depending on l and β) such that the following holds. If $V \in \mathbf{IV}_k(\Sigma)$ and $\pi \in G(n,k)$ satisfy the assumptions of the Theorem 4.1 with $\varepsilon = \varepsilon_L$, then there exists a Lipschitz map*

$$f : (\pi + x_0) \cap B_{r/8}(x_0) \to x_0 + \pi^\perp,$$

such that:

(i): the Lipschitz constant of f is less than l and the graph of f (denoted in the sequel by $Gr(f)$) is contained in the $\mathcal{U}_{\beta r}(\pi + x_0)$.

(ii): $\Theta \equiv 1$ $\mathcal{H}^k$-a.e. on $spt(||V||) \cap B_{r/8}(x_0)$, and $spt(||V||) \cap B_{r/8}(x_0) \subset \mathcal{U}_{\beta r}(\pi + x_0)$.

(iii): $Gr(f) \supset G := \left\{x \in \Gamma \cap B_{r/8}(x_0) | E(V,\pi,x,\rho) \le \lambda_l,\ \forall \rho \in]0, r/2[\right\}$.

(iv): The following estimate holds

$$\mathcal{H}^k\left[(\Gamma \setminus G) \cup (Gr(f) \setminus G)\right] \le C\lambda_l^{-1} E(V, x_0, \pi, r) r^k + C||H||_\infty r^{k+1}.$$

The proof of this theorem is based on the two next lemmas.

Lemma 4.2. *Let V_i be a sequence in $\mathbf{IV}_k(B_1(0))$, $B_1(0) \subset \mathbb{R}^N$ such that each V_i satisfies the assumptions of Theorem 4.1 with $\varepsilon_i = \varepsilon(V_i) \downarrow 0$ for the same fixed plane π, then*

$$||V_i|| \overset{*}{\rightharpoonup} \mathcal{H}^k \llcorner \pi \ in\ B_1(0).$$

Proof. Let $\rho \in]0,1[$ be fixed, and H_i be the generalized mean curvature of V_i, according with the monotonicity formula we have

$$\begin{aligned}
\int_{B_1 \setminus B_\rho} \frac{|\nabla^\perp r|^2}{r^k} d||V_i|| &= ||V_i||(B_1) - \frac{||V_i||(B_\rho)}{\rho^k} - \int_{B_1} \frac{H_i}{k} \cdot |x| \left(\frac{1}{m(r)^k} - 1\right) d||V_i|| \\
&\le ||V_i||(B_1) - \frac{||V_i||(B_\rho)}{\rho^k} + \int_{B_1} \left|\frac{H_i}{k} \cdot |x| \left(\frac{1}{\rho^k} - 1\right)\right| d||V_i|| \\
&\le ||V_i||(B_1) - \frac{||V_i||(B_\rho)}{\rho^k} + C_\rho(\omega_k + 1)||H_i||_\infty \qquad (4.16) \\
&\le ||V_i||(B_1) - e^{-||H_i||_\infty \rho} \omega_k + C_{\rho,k}||H_i||_\infty \\
&\le (\omega_k + \varepsilon(V_i)) - e^{-||H_i||_\infty \rho}\omega_k + C_{\rho,k}\varepsilon(V_i).
\end{aligned}$$

The first inequality is trivial, the second is a consequence of $(H1)$, the third is a straightforward application of (ii) of Proposition 3.5, the forth is again an application of $(H1)$. Hence, if $i \to 0$, then the right hand side of the last inequality goes to zero provided $(H1)$ and $(H2)$ are satisfied. As it is easy to check for every fixed $\rho \in]0,1[$ we have

$$\begin{aligned}
\int_{B_1} |P_\pi^\perp(y)|^2 d||V_i|| &\leq \int_{B_1} |P_\pi^\perp(y) - P_{T_y\Gamma_i}^\perp(y)|^2 + \int_{B_1} |P_{T_y\Gamma_i}^\perp(y)|^2 d||V_i|| \\
&\leq C\int_{B_1} ||\pi - T_y\Gamma_i||^2 d||V_i|| \\
&+ C\rho^2||V_i||(B_\rho) + C\int_{B_1\setminus B_\delta} |r\nabla^\perp r|^2 d||V_i|| \\
&\leq C\int_{B_1} ||\pi - T_y\Gamma_i||^2 d||V_i|| \\
&+ C\int_{B_1\setminus B_\rho} \frac{|\nabla^\perp r|}{r^k} d||V_i|| + C\rho^2||V_i||(B_\delta) \\
&\leq C\int_{B_1} ||\pi - T_y\Gamma_i||^2 d||V_i|| \\
&+ C\int_{B_1\setminus B_\rho} \frac{|\nabla^\perp r|}{r^k} d||V_i|| + C\rho^2(\omega_k + \varepsilon(V_i)). \qquad (4.17)
\end{aligned}$$

Letting $i \to +\infty$ in (4.17) we conclude that

$$\lim_{i\to+\infty} \int_{B_1} |P_\pi^\perp(y)|^2 d||V_i|| \leq C\rho^2\omega_k, \qquad (4.18)$$

letting now $\rho \to 0$, it follows

$$\lim_{i\to+\infty} \int_{B_1} |P_\pi^\perp(y)|^2 d||V_i|| = 0. \qquad (4.19)$$

Now suppose that exists a Radon measure μ such that a subsequence of $\{||V_i||\}_{i\in\mathbb{N}}$ converge to μ, and fix $\varphi \in C_0(B_1)$, by the definition of weak* convergence and (4.19) we argue

$$\begin{aligned}
\left|\int_{B_1} |P_\pi^\perp(y)|^2 \varphi(y) d\mu(y)\right| &= \left|\lim_{i\to\infty} \int_{B_1} |P_\pi^\perp(y)|^2 \varphi(y) d||V_i||(y)\right| \\
&\leq ||\varphi||_\infty \lim_{i\to\infty} \int_{B_1} |P_\pi^\perp(y)|^2 d||V_i||(y) = 0. \qquad (4.20)
\end{aligned}$$

This last assertion readily permits to see that $spt\mu \subseteq \pi$. On the other hand, for any $x \in B_1$ and $\rho < 1 - |x|$, we have trivially

$$\frac{\mu(B_\rho(x))}{\rho^k} \leq \liminf_{i\to 0} \frac{||V_i||(B_\rho(x))}{\rho^k} \leq \liminf_{i\to 0} \frac{e^{||H_i||_\infty \rho}||V_i||(B_\rho(x))}{\rho^k}. \qquad (4.21)$$

Then, applying the monotonicity formula (3.21) we get

$$\liminf_{i\to 0} \frac{e^{||H_i||_\infty \rho}||V_i||(B_\rho(x))}{\rho^k} \leq \liminf_{i\to 0} \frac{e^{||H_i||_\infty (1-|x|)}||V_i||(B_{1-|x|}(x))}{(1-|x|)^k}. \qquad (4.22)$$

Now by the hypothesis $(H1)$ of Theorem 4.1

$$||V_i||(B_{1-|x|}(x)) \leq ||V_i||(B_1(0)) \leq (\omega_k + \varepsilon(V_i)), \tag{4.23}$$

which have as a consequence that

$$\frac{\mu(B_\rho(x))}{\rho^k} \leq \frac{\omega_k}{(1-|x|)^k}. \tag{4.24}$$

From the preceding equation, we argue immediately that

$$\Theta^{*k}(\mu, x) = \limsup_{\rho \downarrow 0} \frac{\mu(B_\rho(x))}{\omega_k \rho^k} \leq \frac{\omega_k}{(1-|x|)^k},$$

for all $x \in B_\rho$, and this joint with (3.23) guarantees the existence of a nonnegative Borel map θ such that

$$\mu = \theta \mathcal{H}^k \llcorner \pi.$$

Given $X \in C_c^1(B_1(0))$ be a fixed vector field, $\pi, T \in G(n,k)$, the following estimates follows

$$\begin{aligned} |div_\pi X - div_T X| &\leq \sum_{i=1}^k |\langle \nabla_{e_i} X, e_i \rangle - \langle \nabla_{\tilde{e}_i} X, \tilde{e}_i \rangle| && (4.25) \\ &\leq \sum_{i=1}^k 2||\nabla_{(\cdot)} X||_\infty |e_i - \tilde{e}_i| && (4.26) \\ &\leq 2k||\nabla X||_\infty ||\pi - T||, && (4.27) \end{aligned}$$

since a simple direct, but cumbersome, computation shows that in general $|e_i - \tilde{e}_i| \leq ||\pi - T||$ for all $1 \leq i \leq k$, if $(e_i), (\tilde{e}_i)$ are orthonormal basis of π and T respectively. Therefore

$$\begin{aligned} \left| \int_\pi div_\pi X \theta d\mathcal{H}^k \right| &= \lim_{i\to\infty} \left| \int div_\pi X d||V_i|| \right| && (4.28) \\ &\leq \liminf_{i\to\infty} \left| \int_{TV_i} div_{T\Gamma_i} X d||V_i|| \right| && (4.29) \\ &+ \liminf_{i\to\infty} C(X) \int ||T\Gamma_i - \pi|| d||V_i|| && (4.30) \\ &\leq \liminf_{i\to\infty} (||H_i||_\infty ||X||_\infty ||V_i||(B_1(0)) && (4.31) \\ &+ \liminf_{i\to\infty} C(X)(||V_i||(B_1(0)))^{1/2} E(V_i, \pi, 0, 1)))^{\frac{1}{2}} && (4.32) \\ &= 0. && (4.33) \end{aligned}$$

Now let $z_1, \ldots, z_k, y_1, \ldots, y_{N-k}$ be a system of coordinates such that $\pi = \{y = 0\}$, then by (4.28)

$$\int \theta(z) div_z Y(z) dz = 0, \quad \forall Y \in C_c^1(B_{\mathbb{R}^k}(0,1), \mathbb{R}^k). \tag{4.34}$$

A way to prove this fact is to take for instance a standard mollifier φ_δ and test (4.34) with the smooth vector fields $Y * \varphi_\delta$ to conclude, via the divergence theorem, that the derivative of $\theta * \varphi_\delta$ vanishes on $B_{1-\delta}$; letting $\delta \downarrow 0$, we then conclude that θ is a constant θ_0. On the other hand, since $\mu(\partial B_\rho) = 0$, we have $\theta_0 \omega_k \rho^k = \mu(B_\rho) = \lim_{i\to\infty} ||V_i||(B_\rho)$. However, as already observed, by the monotonicity formula and $(H1)$,

$$\omega_k e^{-||H_i||_\infty \rho} \rho^k \leq ||V_i||(B_\rho) \leq (\omega_k + \varepsilon_i)\rho^k.$$

Thus $\theta_0 = \lim_{i\to\infty} \frac{||V_i||(B_\rho)}{\omega_k \rho^k} = 1$. Summarizing, any convergent subsequence of $\{||V_i||\}$ converges *weak** to $\mathcal{H}^k \llcorner \pi$. By the *weak** compactness of closed bounded convex sets in the space of measures, we conclude the proof. $\square$

Lemma 4.3. *Let $k < N$ be positive integers. For any $\delta \in]0, 1/2[$ there is a positive number $\varepsilon_H(\delta)$ such that. If V satisfies the assumptions of Theorem 4.1 with $\varepsilon = \varepsilon_H$ then:*

(i): spt$(||V||) \cap B_{r/2}(x_0) \subset (\delta)r$-neighborhood of $\pi + x_0$.

(ii): $||V||(B_\rho(x)) \leq (\omega_k + \delta)\rho^k$, $\forall x \in B_{r/4}$ and $\rho \leq r/2$.

The proof of this lemma is based on the blow up argument explained in Lemma 4.2.

Proof. Scaling an translating we can assume $x_0 = 0$ and $r = 1$. Arguing by contradiction, suppose that the proposition is false, then there would be a positive constant $\delta < 1/2$, a plane π and a sequence of varifolds $V_i \in \mathbf{IV}_k(\Sigma)$ satisfying the assumptions of the previous lemma, and for each i, one of the following alternatives holds:

1. There is a point $x_i \in spt(\mu) \cap B_{1/2}(0)$ such that $|P_\pi^\perp(x_i)| \geq \delta$
2. there is a point $x_i \in B_{1/4}(0)$ and a radius $\rho_i \in]0, 1/2[$ such that $||V_i||(B_{\rho_i}(x_i)) \geq (\omega_k + \delta)\rho_i^k$.

Remark 4.1. *Observe here that the plane π is fixed and is the same for all V_i. If we take the negation of the statement of the theorem this ensures just the existence of a sequence η_i and a sequence of planes π_i such that the remaining it is true with η_i in place of ε_i. But it is also true that from the sequence π_i we can extract a subsequence that we call again (π_i) that converges to a plane $\pi \in G(n,k)$, by the compactness of $G(n,k)$. Furthermore as it is easy to check, we have $E(V_i, x_0, \pi, r) \leq E(V_i, x_0, \pi_i, r) + \frac{1}{2}||\pi - \pi_i||^2 ||V_i||(B_r(x_0)) \leq E(V_i, x_0, \pi_i, r) + \frac{1}{2}||\pi - \pi_i||^2(\omega_k + \eta_i)r^k$. So for i large enough* *we have* $E(V_i, x_0, \pi, r) \le \eta_i \left(1 + \frac{1}{2}(\omega_k + \eta_i) r^k\right)$. *Taking* $\varepsilon_i := \eta_i \left(1 + \frac{1}{2}(\omega_k + \eta_i) r^k\right) > \eta_i$, *we obtain the desired sequence of* ε_i *and* V_i *satisfying the hypotheses of Lemma 4.2.*

Since Lemma 4.2 guarantees that $V_i \overset{*}{\rightharpoonup} \mathcal{H}^k \llcorner \pi$ *in* $B_1(0)$, without loss of generality we can assume that one of the two alternatives holds for every i.

Suppose that 1 holds, we can also assume $x_i \to x$. Then $x \in \overline{B_{1/2}}(0)$ and $|P^\perp(x)| \ge \delta$. Thus $B_\delta(x) \subset B_1(0)$ and $B_\delta(x) \cap \pi = \emptyset$. On the other hand, if i is large enough, $B_{\delta/2}(x_i) \subset B_\delta(x)$. Since $\mathcal{H}^k(\partial B_\delta(x) \cap \pi) = 0$, using (iv) of Proposition 3.5 we get

$$\begin{aligned} 0 = \mathcal{H}^k(B_\delta(x) \cap \pi) = \lim_{i\to\infty} ||V_i||(B_\delta(x)) \;&\ge\; \limsup_{i\to\infty} ||V_i||(B_{\delta/2}(x_i)) \\ &\ge\; \limsup_{i\to\infty} e^{-||H_i||_\infty \frac{\delta}{2}} \omega_k \left(\frac{\delta}{2}\right)^k \\ &\ge\; \omega_k \left(\frac{\delta}{2}\right)^k > 0, \end{aligned}$$

which is manifestly a contradiction.

Now assume that 2 holds. By (iv) of Proposition 3.5

$$||V_i||(B_{1/2}(x_i)) \ge e^{-||H_i||_\infty 1/2}(\omega_k + \delta) 2^{-k}.$$

Without loss of generality we can assume $x_i \to x \in \overline{B}_{1/4}(0)$. Fix $r > 1/2$, and notice that for i large enough $B_r(x) \supset B_{1/2}(x_i)$. Since $\mathcal{H}^k(\partial B_r \cap \pi) = 0$, then

$$\mathcal{H}^k(B_r(x) \cap \pi) = \lim_{i\to\infty} ||V_i||(B_r(x)) \ge \lim_{i\to\infty} ||V_i||(B_{1/2}(x_i)) \ge (\omega_k + \delta) 2^{-k}.$$

Letting $r \downarrow 1/2$ we then conclude $\mathcal{H}^k(B_{1/2}(x) \cap \pi) \ge (\omega_k + \delta) 2^{-k}$ which is a contradiction, because $\mathcal{H}^k(B_{1/2}(x) \cap \pi)$ can be at most $\omega_k 2^{-k}$ which correspond to the case $x \in \pi$. □

Lipschitz Approximation: Proof of Theorem 4.3. Without loss of generality we can assume $x_0 = 0$ and $r = 1$. To simplify the notation set $\mathbf{E} := E(V, \pi, 0, 1)$.

C1 First choose $\lambda < \varepsilon_H(\delta_1)$ (given by Lemma 4.3) with $\delta_1 := \frac{(N-k)^{-1/2} l}{3}$.

C2 Then choose $\varepsilon_L < \min\{\lambda, \varepsilon_H(\delta_2)\}$ (given by Lemma 4.3) with $\delta_2 := \min\{\lambda, (N-k)^{-1/2}\beta\}$.

Observe here that as a consequence of the estimate (4.36) below and (iv) of Proposition 3.5, G is not empty and have k-dimensional Hausdorff measure that goes to $\omega_k \left(\frac{1}{8}\right)^k$ when

$\varepsilon_L \to 0$. This allow us to make the following construction. Suppose $x \in G$ and pick $y \in G$. Observe that $|y - x| < 1/4$. Therefore choose $r > |x - y|$ so that

$$2r < \min\{1/2, 3|x - y|\}.$$

Since $2r < 1/2$, by the choice C2 of ε_L we have

$$||V||(B_{2r}(x)) \leq (\omega_k + \lambda)(2r)^k,$$

and since $x \in G$ we also have $E(V, \pi, x, 2r) < \lambda$. So applying Lemma 4.3 again

$$spt(||V||) \cap B_r(x) \subset \mathcal{U}_{\delta_2 r}(\pi).$$

Since $y \in B_r(x)$

$$\begin{aligned} |P_\pi^\perp(x) - P_\pi^\perp(y)| &\leq \delta_1 r = 3^{-1} l (N-k)^{-1/2} r \\ |P_\pi^\perp(x) - P_\pi^\perp(y)| &\leq \frac{1}{2}|x - y|. \end{aligned} \tag{4.35}$$

On the other hand we have that $|P_\pi^\perp(x) - P_\pi^\perp(y)| + |P_\pi(x) - P_\pi(y)| \geq |x - y|$, then subtracting (4.35), we have

$$|P_\pi(x) - P_\pi(y)| \geq \frac{1}{2}|x - y|,$$

which implies that $P_\pi : G \to \pi$ is an injective map. So if we set $D = P_\pi(G)$ we can define

$$\begin{aligned} f : D \subset \pi \to \pi^\perp \\ v \mapsto f(v) = P_\pi^\perp(z), \end{aligned}$$

where z is such that $P_\pi(z) = v$.

Since $P_\pi : G \to \pi$ is injective, P_π^{-1} is well defined, and since $P_\pi^\perp : G \to \pi^\perp$ is already a function, f is well defined, and $Gr(f) = G$.

Notice that by construction

$$||f||_\infty = \sup_{x \in D} |f(x)| \leq (N-k)^{-1/2}\beta,$$

and that

$$\begin{aligned} |f(v) - f(w)| &= |P_\pi^\perp(v, f(v)) - P_\pi^\perp(w, f(w))| \\ &\leq \frac{2}{3}(N-k)^{-1/2} l |(v, f(v)) - (w, f(w))| \\ &\leq (N-k)^{-1/2} l |P_\pi(v, f(v)) - P_\pi(v, f(v))| \\ &= (N-k)^{1/2} l |v - w| \end{aligned}$$

Thus $f : D \to \pi^{\perp}$ has Lipschitz constant $(N-k)^{1/2}l$. Now fixing a system of orthonormal coordinates on $\pi^{\perp}$ and let $f_1 \dots, f_{N-k}$ be the corresponding coordinate functions of f. We can extend each f_i to $B_{1/8} \cap \pi$ preserving Lipschitz constant and L^{∞} norm. Thus the resulting extended function (abusing of notation) f will have Lipschitz constant at most l and L^{∞} norm at most β. Thus (i) is satisfied, and also (iii) by construction.

Now consider any point $x \in Spt(V) \cap B_{1/8}(x)$ by Lemma 4.3 and our choice C2 we have

$$\begin{aligned} V(B_r(x)) &\leq (\omega_k + \lambda) r^k \quad \forall r < 1/2 \\ \frac{V(B_r(x))}{\omega_k \rho^k} &\leq \left(1 + \frac{\lambda}{\omega_k}\right). \end{aligned}$$

So letting $r \downarrow 0$ we have

$$\Theta^k(V, x) = \lim_{r \downarrow 0} \frac{V(B_r(x))}{\omega_k \rho^k} \leq \left(1 + \frac{\lambda}{\omega_k}\right) < 2.$$

Since $\Theta \in \mathbb{N}$ for $\mathcal{H}^k$- a.e. and $x \in Spt(V)$, then $\Theta \equiv 1$ $\mathcal{H}^k$-a.e. on $Spt(V) \cap B_{1/8}(x)$.

Also notice that, by our choice of ε_L on C2, $Spt(V) \cap B_{1/8}(0)$ is contained in a $(N-k)^{-1/2}\beta$-neighbourhood of π. So (ii) is satisfied.

Finally, for each $x \in F := (spt||V|| \setminus G) \cap B_{1/8}(0)$ choose a radius $\rho_x < 1/2$ such that $E(V, \pi, x, \rho_x) \geq \lambda$.

Then by Besicovich's Theorem 1.13 we can find countably many pairwise disjoint balls $B_{\rho_i}(x_i)$ such that

$$\{B_{\rho_i}(x_i)\}_i \textit{ covers } F \textit{ and } E(V, \pi, x_i, \rho_i) \geq \lambda,$$

and

$$\begin{aligned} \mathcal{H}^k(F) &\leq 5^k \omega_k \sum_i \rho_i \\ &= \frac{5^k \omega_k}{\lambda} \sum_i \lambda \rho_i \\ &\leq 5^k \omega_k \lambda^{-1} \sum_i E(V, \pi, x_i, \rho_i) \\ &\leq 5^k \omega_k \lambda^{-1} \sum_i \int_{B_{\rho_i}(x_i)} ||T_{x_i} V - \pi||^2 dV \\ &\leq C \lambda^{-1} \int_{B_1(0)} ||T_0 V - \pi||^2 dV \\ \mathcal{H}^k(F) &\leq C \lambda^{-1} \mathbf{E} \end{aligned} \tag{4.36}$$

To estimate $F' := \Gamma \setminus G$ we have

$$
\begin{aligned}
\mathcal{H}^k(F') &\leq C(\omega_k 8^{-k} - \mathcal{H}^k(D)) \\
&\leq C\left(\frac{\omega_k}{8^k} - \int_G J_{TV} P_\pi d\mathcal{H}^k\right) \\
&\leq C\left(\frac{\omega_k}{8^k} - \int_G |J_{TV} P_\pi - 1| d\mathcal{H}^k - \int_G d\mathcal{H}^k\right) \\
&\leq C\left(\frac{\omega_k}{8^k} + C_0 \int_G ||TV - \pi||^2 - \mathcal{H}^k(G)\right) \\
&\leq C\left(\frac{\omega_k}{8^k} + C_0 \mathbf{E} - \mathcal{H}^k(G)\right) \\
&\leq C\left(\frac{\omega_k}{8^k} + C_0 \mathbf{E} - V(B_{1/8}(0)) + \mathcal{H}^k(F)\right) \\
&\leq C(C_0 \mathbf{E} + C\lambda^{-1}\mathbf{E}) + (\omega_k 8^{-k} - V(B_{1/8}(0))) \\
&\leq \frac{C_1}{\lambda}\mathbf{E} + C\omega_k 8^k (1 - e^{-||H||_\infty 1/8}) \\
\mathcal{H}^k(F') &\leq \frac{C_1}{\lambda}\mathbf{E} + C||H||_\infty.
\end{aligned} \tag{4.37}
$$

The result follows from combining (4.36) and (4.37). $\square$

4.4 Allard's decay theorem

The aim of this chapter is to prove Theorem 4.4, the idea of the proof is divided in four steeps, making use of all the tools developed until now. Before to start with the proof of Theorem 4.4, we will establish two useful lemmas about harmonic functions.

Lemma 4.4. *Let $k \in \mathbb{N} \setminus \{0\}$, for every $\rho \in]0, +\infty[$ there exists $\varepsilon_{Har,\rho} > 0$ such that, if $f \in W^{1,2}(B_r(x))$, $||\nabla f||^2_{2,B_r(x)} \leq r^k$, with $B_r(x) \subseteq \mathbb{R}^k$, and*

$$\left| \int \langle \nabla \varphi, \nabla f \rangle \right| \leq \varepsilon_{Har,\rho} r^k ||\nabla \varphi||_\infty,$$

for all $\varphi \in C^1_c(B_r(x))$, then there exists $u \in \mathcal{H}(B_r(x))$ where $\mathcal{H}(B_r(x)) := \{u : B_r(x) \to \mathbb{R} : \Delta u = 0\}$ satisfying the following properties

1. *$||\nabla u||^2_{2,B_r(x)} \leq r^k$,*

2. *$||f - u||^2_{2,B_r(x)} \leq \rho r^{2+k}$.*

Proof. We argue the lemma by contradiction. To this aim, suppose that there exists $\rho > 0$ such that $\forall \varepsilon > 0$ there exist a function $f_\varepsilon \in W^{1,2}(B_r(x))$, $||\nabla f_\varepsilon||^2_{2,B_r(x)} \leq r^k$, and

$|\int\langle\nabla\varphi,\nabla f_\varepsilon\rangle|\leq\varepsilon r^k||\nabla\varphi||_\infty$, $\forall\varphi\in C^1_c(B_r(x))$ such that there not exists $u\in\mathcal{H}(B_r(x))$, with $||\nabla u||^2_{2,B_r(x)}\leq r^k$, and $||f-u||^2_{2,B_r(x)}\leq\varepsilon\rho r^{2+k}$. Put $L_\varphi[\psi]:=\int\nabla\varphi\cdot\nabla\psi$, whenever $\varphi,\psi\in W^{1,2}(B_r(x))$. Taking a sequence $\varepsilon_j\to 0$, we can find a sequence of functions $f_j\in W^{1,2}(B_r(x))$, such that

$$\lim_{j\to+\infty}\sup_{\varphi\in C^1_c(B_r(x)),||\nabla\varphi||_\infty\leq 1}|L_{f_j}[\varphi]|=0, \tag{4.38}$$

but

$$\int(u-f_j)^2\geq\rho r^{k+2}>0. \tag{4.39}$$

Observe now that the two preceding conditions remains invariant by adding a constant c_j to f_j. In particular we can assume without loss of generality that $\int f_j=0$. The Poincare's inequality then applies, which in turn allows us to conclude that $||f_j||_{W^{1,2}(B_r(x))}\leq 2r^k$. We know that $W^{1,2}(B_r(x))$ is reflexive so we can extract a subsequence weakly convergent to some $u\in W^{1,2}(B_r(x))$, furthermore Rellich-Kondrakov's theorem ensures that this sequence is strongly convergent to the same u in $L^2(B_r(x))$. The weakly lower semicontinuity of the Dirichlet energy guarantees that $||\nabla u||^2_2\leq r^k$. Since $f_j\rightharpoonup^* u$, implies $\int\nabla\varphi\cdot\nabla u=\lim_{j\to+\infty}\int\nabla\varphi\cdot\nabla f_j$, and the right hand side of the last equation is zero by (4.38), u satisfies $\int\nabla\varphi\cdot\nabla u=0$ that is a weak formulation of the equation $\Delta u=0$, which by classical elliptic regularity theory means that u is harmonic on $B_r(x)$. To finish the proof we just have to observe that $f_j\to u$ in $L^2(B_r(x))$, which is in contradiction with (4.39). □

Lemma 4.5. *Let $u\in\mathcal{H}(B_r(x_0))$. Then there exists a constant $C=C(k)>0$ such that*

$$\sup_{x\in B_\rho}|u(x)-u(x_0)-\nabla u(x_0)\cdot x|\leq C\rho^2 r^{-\frac{k}{2}-1}||\nabla u||_{2,B_r},\ \forall\rho\leq\frac{r}{2}. \tag{4.40}$$

Proof. As it is known, by standard elliptic regularity theory, an harmonic function is an analytic function and so, in particular is a $C^2(B_r(x))$ function. Thus by a classical Taylor's expansion argument we immediately get

$$\sup_{x\in B_\rho}|u(x)-u(x_0)-\nabla u(x_0)\cdot x|\leq C\rho^2 r^{-\frac{k}{2}-1}||\nabla^2 u||_{\infty,B_{r/2}(x_0)},\ \forall\rho\leq\frac{r}{2}. \tag{4.41}$$

Indeed ∇u and $\nabla^2 u$ are also harmonic so by the mean value property, that is a special property satisfied only by harmonic functions we get

$$|\nabla^2 u(x)|=\left|\frac{1}{|B_{1/2}|}\int_{B_{r/2}(x_0)}\nabla^2 u(x)dx\right|\leq C_k||\nabla^2 u||_{1,B_{r/2}(x_0)}\overset{\text{Hölder}}{\leq}C||\nabla^2 u||_{2,B_{r/2}(x_0)}.$$

Taking the supremum with respect to x, in the preceding inequality we obtain

$$||\nabla^2 u||_{\infty,B_{r/2}(x_0)} \leq C||\nabla^2 u||_{2,B_{r/2}(x_0)}.$$

Now, from a straightforward application of Caccioppoli's inequality (see for example Theorem 4.1 of [GM12]) easily follows (4.40). □

Theorem 4.4. *There exist two positive mutually independent constants* $\eta \in]0, \frac{1}{2}[$ *and* $\varepsilon_E > 0$ *which depends only on the dimension and on the geometry of the ambient space* Σ^n*, such that whenever* $V \in \mathbf{IV}_k(\Sigma)$ *with bounded generalized mean curvature satisfies for a given point* $x_0 \in \Sigma$ *and radius* $r > 0$*, the following three assumptions*

(i): $||V||(B(x_0, r)) \leq (\omega_k + \varepsilon_E)r^k$*,*

(ii): there exists $\pi \in G(n, k)$ *such that* $\mathbf{E} := E(V, x_0, \pi, r) \leq \varepsilon_E$*,*

(iii): $||H||_\infty r \leq \mathbf{E}$*,*

then there exists $\tilde{\pi} \in G(n, k)$ *such that*

$$E(V, x_0, \tilde{\pi}, \eta r) \leq \frac{1}{2} E(V, x_0, \pi, r). \tag{4.42}$$

The idea underlying the proof is to take the Lipschitz approximation f given by Theorem 4.3 and to show that f could be approximated by an harmonic function u applying Lemma 4.4. After taking as $\tilde{\pi}$ the tangent plane to the graph of u in $u(x_0)$ we show that thanks to the mean value property of harmonic functions we find an upper bound for the L^2 height excess which inserted in the tilt excess inequality (4.3) gives the desired decay estimate (4.42). We sometimes refer to Inequality (4.42) as the tilt excess decay inequality (or estimate) to distinguish it from the tilt excess inequality (4.3). The principal reason to be forced to use an harmonic approximation is that for harmonic functions the mean value properties leading to (4.40) holds. This ensures the control of the height of a function in a point by the L^2 norm of the function in a neighborhood, which is in general, not possible for an arbitrary Lipschitz function

Proof. Without loss of generality, assume $r = 1$, $x_0 = 0$ and $\mathbf{E} := E(V, \pi, 0, 1)$.

Lipschitz approximation:

Assume $\varepsilon_0 < \varepsilon_L$ given by Theorem 4.3 for some choice of l and β, and consider the Lipschitz approximation

$$f : B(0, 1/8) \cap \pi \to \pi^\perp,$$

and λ, again given by Theorem 4.3.

Now assume $(y_1, \ldots, y_k, z_1, \ldots, z_{n-k})$ an orthonormal system of coordinates in $\mathbb{R}^n$ such that $\pi = \{z = 0\}$, and denote $f = (f_1, \ldots, f_{n-k})$, and $\{e_1, \ldots, e_{n-k}\}$ the canonical orthonormal base of $\mathbb{R}^{n-k}$ (i.e. for fixed $j \in \{1, \ldots, n-k\}$, $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is in the $j-th$ place), let $\varphi \in C_c^1(B(0, 1/16) \cap \pi)$, and consider the vector field

$$X(y, z) = \varphi(y)e_j.$$

Notice that $spt(X)$ is not compact in $B(0, 1/8)$, by the mere definition of X. However recall that $\Gamma \cap B(0, 1/8) \subset \mathcal{U}_{\beta r}(\pi)$ (remember that we have denoted $\Gamma := spt(||V||)$), then assume $\beta < 1/16$, so we can multiply X ba a cut-off function in the z variables to make it compactly supported in $B(0, 1/18)$ without afecting its values on $\Gamma \cap B(0, 1/16)$. Since, by Theorem 4.3 $\Theta \equiv 1$ $||V||$-a.e. on $\Gamma \cap B(0, 1/8)$, testing the first variation formula with the field X, we get by the estimates in Theorem 4.3 that

$$\begin{aligned}
\left| \int_{Gr(f)} \langle \nabla_{T_x Gr(f)} \varphi, e_j \rangle d\mathcal{H}^k(x) \right| &\leq \left| \int_{Gr(f)} \langle \nabla_{T_x Gr(f)} \varphi, e_j \rangle d\mathcal{H}^k(x) - \delta V(X) \right| + |\delta V(X)| \\
&= \left| \int_{Gr(f)} \langle \nabla_{T_x Gr(f)} \varphi, e_j \rangle d\mathcal{H}^k(x) - \int div_{T_x \Gamma} X \right| + |\delta V(X)| \\
&= \left| \int_{Gr(f)} \langle \nabla_{T_x Gr(f)} \varphi, e_j \rangle d\mathcal{H}^k(x) - \int_{\Gamma} \langle \nabla_{T_x \Gamma} \varphi, e_j \rangle \right| + |\delta V(X)|,
\end{aligned}$$

since

$$\begin{aligned}
\int_{Gr(f)} \langle \nabla_{T_x Gr(f)} \varphi, e_j \rangle d\mathcal{H}^k(x) &= \int_{Gr(f) \cap \Gamma \cap B(0,1/16)} \langle \nabla_{T_x Gr(f)} \varphi, e_j \rangle d\mathcal{H}^k(x) \\
&\quad + \int_{(Gr(f) \setminus \Gamma) \cap B(0,1/16)} \langle \nabla_{T_x Gr(f)} \varphi, e_j \rangle d\mathcal{H}^k(x), \\
\int_{\Gamma} \langle \nabla_{T_x \Gamma} \varphi, e_j \rangle d\mathcal{H}^k(x) &= \int_{\Gamma \cap Gr(f) \cap B(0,1/16)} \langle \nabla_{T_x \Gamma} \varphi, e_j \rangle d\mathcal{H}^k(x) \\
&\quad + \int_{(\Gamma \setminus Gr(f)) \cap B(0,1/16)} \langle \nabla_{T_x Gr(f)} \varphi, e_j \rangle d\mathcal{H}^k(x),
\end{aligned}$$

and by Theorem 4.3, we know that f is Lipschitz and by definition Γ is rectifiable thus

$$\mathcal{H}^k\left(\left\{x \in \mathbb{R}^n : \langle \nabla_{T_x Gr(f)} \varphi, e_j \rangle \neq \langle \nabla_{T_x \Gamma} \varphi, e_j \rangle \right\}\right) = 0,$$

and then

$$\int_{Gr(f) \cap \Gamma \cap B(0,1/16)} \langle \nabla_{T_x Gr(f)} \varphi, e_j \rangle d\mathcal{H}^k(x) = \int_{Gr(f) \cap \Gamma \cap B(0,1/16)} \langle \nabla_{T_x \Gamma} \varphi, e_j \rangle d\mathcal{H}^k(x).$$

This shows that

$$
\begin{aligned}
\left|\int_{Gr(f)} \langle \nabla_{T_xGr(f)}\varphi, e_j\rangle d\mathcal{H}^k(x)\right| &\leq \left|\int_{Gr(f)\setminus\Gamma\cap B(0,1/16)} \langle \nabla_{T_xGr(f)}\varphi, e_j\rangle d\mathcal{H}^k(x)\right| \\
&+ \left|\int_{\Gamma\setminus Gr(f)\cap B(0,1/16)} \langle \nabla_{T_x\Gamma}\varphi, e_j\rangle d\mathcal{H}^k(x)\right| + |\delta V(X)| \\
&\leq ||\nabla\varphi||_\infty \left[\mathcal{H}^k(Gr(f)\setminus\Gamma\cap B(0,1/16))\right. \\
&+ \left.\mathcal{H}^k(\Gamma\setminus Gr(f)\cap B(0,1/16))\right] + \left|\int \langle H, X\rangle d||V||\right| \\
&\leq ||\nabla\varphi||_\infty \left(\frac{C}{\lambda}\mathbf{E} + C||H||_\infty\right) \\
&+ ||\varphi||_\infty ||H||_\infty \mathcal{H}^k(\Gamma\cap B(0,1/16)) \\
&\leq ||\nabla\varphi||_\infty \left(\frac{C}{\lambda}\mathbf{E} + C||H||_\infty\right) + C||\nabla\varphi||_\infty ||H||_\infty \\
&\leq ||\nabla\varphi||_\infty \left(\frac{C}{\lambda}\mathbf{E} + C||H||_\infty\right),
\end{aligned}
$$

by hypothesis (iii) we have $||H||_\infty \leq \mathbf{E}$, then

$$\left|\int_{Gr(f)} \langle \nabla_{T_xGr(f)}\varphi, e_j\rangle d\mathcal{H}^k(x)\right| \leq ||\nabla\varphi||_\infty \frac{C}{\lambda}\mathbf{E}. \tag{4.43}$$

Next, let $\{\xi_1, \ldots, \xi_k\}$ be an orthonormal basis for π, and consider the first fundamental form in $T_xGr(f)$, i.e., the $n\times n$ matrix

$$g_{ij} = \langle \xi_i + \sum_{l=1}^{n-k} \partial_{y_i} f_l e_l, \xi_j + \sum_{m=1}^{n-k} \partial_{y_j} f_m e_m\rangle := \langle v_i, v_j\rangle,$$

where $v_i := d\Phi_y(\xi_i)$, and $\Phi : y \mapsto (y, f(y))$. Then, since $\xi_i \perp e_j$

$$
\begin{aligned}
|g_{ij} - \delta_{ij}| &= |\langle v_i, v_j\rangle - \delta_{ij}| \\
&= \left|\langle \xi_i, \xi_j\rangle + \sum_l \partial_{y_i} f_l \langle e_l, \xi_j\rangle + \sum_m \partial_{y_j} f_m \langle e_m, \xi_i\rangle\right. \\
&\qquad \left. + \sum_l \sum_m \partial_{y_i} f_l \partial_{y_j} f_m \langle e_l, e_m\rangle - \delta_{ij}\right| \\
&= \left|\delta_{ij} + \sum_l \sum_m \partial_{y_i} f_l \partial_{y_j} f_m \langle e_l, e_m\rangle - \delta_{ij}\right| \\
&\leq |Df|^2.
\end{aligned}
$$

Thus, if $l < 1$ then $|g^{ij} - \delta^{ij}| \leq C|Df|^2 O(l)$, where g^{ij} are the entries of the inverse matrix of $G := [g_{ij}]$, i.e., $G^{-1} := [g^{ij}]$. This in turn implies, that

$$|g^{ij} - \delta_{ij}| \leq C|Df|^2,$$

provided l is less than a geometric constant ensuring that $O(l) < 1$ holds. Now let us calculate the projection $P_{T_xGr(f)} : \mathbb{R}^n \to T_xGr(f)$,

$$P_{T_xGr(f)}(w) = \sum_{i,j} \langle w, v_i \rangle g^{ij}(y) v_j,$$

since

$$\begin{aligned}
\langle e_j, v_l \rangle &= \langle e_j, \xi_l + \sum_{m=1}^{n-k} \partial_{y_l} f_m e_m \rangle \\
&= \langle e_j, \xi_l \rangle + \sum_{m=1}^{n-k} \partial_{y_l} f_m \langle e_j, e_m \rangle \\
&= \partial y_l f_j,
\end{aligned}$$

and

$$\begin{aligned}
\langle \nabla\varphi, v_m \rangle &= \langle \nabla\varphi, \xi_m + \sum_{l=1}^{n-k} \partial_{y_m} f_l e_l \rangle \\
&= \langle \nabla\varphi, \xi_m \rangle + \sum_{l=1}^{n-k} \langle \partial_{y_m} f_l e_l, \nabla\varphi \rangle \\
&= \partial_{y_m} \varphi.
\end{aligned}$$

Therefore, if we fix the point $x = (w, f(w))$

$$\begin{aligned}
\langle P_{T_xGr(f)}(\nabla\varphi(w)), e_j \rangle &= \langle \sum_{l,i} \langle \nabla\varphi(w) \rangle g^{li}(w) v_l, e_j \rangle \\
&= \sum_{i,j} \langle \nabla\varphi(w), v_l \rangle g^{ij}(w) \langle v_l, e_j \rangle \\
&= \sum_{l,i} \partial_{y_i} f_j(w) g^{ij}(w) \partial_{y_l} \varphi(w) \\
&= \sum_{l} \sum_{i} \partial_{y_i} f_j(w) g^{ij}(w) \partial_{y_l} \varphi(w) \\
&= \sum_{l} \partial_{y_l} f_j(w) \partial_{y_l} \varphi(w) + O(|Df|^3(w) |\nabla\varphi(w)|). \qquad (4.44)
\end{aligned}$$

Now applying the Area Formula we have

$$\int_{Gr(f)} \langle \nabla_{T_xGr(f)} \varphi, e_j \rangle d\mathcal{H}^k(x) = \int_{B(0,1/16)\cap\pi} \langle P_{T_xGr(f)} \varphi(w), e_j \rangle Jf(w) dw.$$

Setting $\overline{\nabla}\varphi = (\partial_{y_i}\varphi, \ldots, \partial_{y_k}\varphi)$, we have by (4.44)

$$\begin{aligned}
\int_{B(0,1/16)\cap\pi} \langle P_{T_xGr(f)} \varphi(w), e_j \rangle Jf(w) dw \leq& \int_{B(0,1/16)\cap\pi} \langle \overline{\nabla}\varphi(w), \overline{\nabla} f(w) \rangle Jf(w) dw + \\
&+ C \int_{B(0,1/16)\cap\pi} |Df|^3 |\nabla\varphi(w)| Jf(w) dw. \quad (4.45)
\end{aligned}$$

On the other hand, by a simple Taylor expansion we get

$$|Jf(w)-1| \leq C|Df|^2, \tag{4.46}$$

then combining (4.43), (4.45) and (4.46), and notice that $spt(\varphi) \subset B(0,1/16)\cap\pi$

$$\left|\int_{B(0,1/16)\cap\pi} \langle\overline{\nabla}\varphi(w), \overline{\nabla}f_j(w)\rangle dw\right| \leq \frac{C}{\lambda}\mathbf{E}||\overline{\nabla}\varphi||_\infty + C||\overline{\nabla}\varphi||_\infty \int_{B(0,1/16)\cap\pi} |Df|^2. \tag{4.47}$$

On the other hand, notice

$$\begin{aligned}
||\pi - T_x\Gamma||^2 &\geq |P_\pi(e_j) - P_{T_x\Gamma}(e_j)|^2 \\
&\geq |P_{T_x\Gamma}(e_j)|^2 \\
&= \left|\sum_{l,m} \partial_{y_l} f_j(w) g^{lm}(w) v_m(w)\right|^2 \\
&\geq \sum_l |\partial_{y_l} f_j(w) e_j|^2 - 2\sum_l |Df(w)||g^{ll}-1|^2 - 2\sum_{l\neq m} |Df(w)| \left(g^{lm}(w)\right)^2 \\
&\geq |\nabla f_j(w)|^2 - C|Df(w)|^3.
\end{aligned}$$

Summing over j and using the fact that $Lip(f) < l$, we conclude

$$||\pi - T_x\Gamma||^2 \geq |Df(w)|^2 - Cl|Df(w)|^2 = (1-Cl)\,|Df(w)|^2.$$

Then if l is less than a geometric constant, i.e., $l \leq \frac{1}{2C}$ we have

$$2||\pi - T_x\Gamma||^2 \geq |Df|^2.$$

Inserting this in (4.47) we have

$$\begin{aligned}
\left|\int_{B(0,1/16)\cap\pi} \langle\overline{\nabla}\varphi(w), \overline{\nabla}f_j(w)\rangle dw\right| &\leq \frac{C}{\lambda}\mathbf{E}||\overline{\nabla}\varphi||_\infty + 2C||\overline{\nabla}\varphi||_\infty \int_G ||\pi - T_x\Gamma||^2 d||V||(x) \\
&\leq \frac{C}{\lambda}\mathbf{E}||\overline{\nabla}\varphi||_\infty.
\end{aligned}$$

Finally, since l has been chosen smaller than a geometric constant, by Theorem 4.3 $\lambda = \lambda(l)$ (i.e. λ depends only on l), then

$$\left|\int_{B(0,1/16)\cap\pi} \langle\overline{\nabla}\varphi(w), \overline{\nabla}f_j(w)\rangle dw\right| \leq C\mathbf{E}||\overline{\nabla}\varphi||_\infty \quad \forall\varphi \in C_c^1(B(0,1/16)\cap\pi). \tag{4.48}$$

Moreover

$$\int_{B(0,1/16)\cap\pi} |\overline{\nabla}f_j(y)|dy \leq \frac{Cl^2}{\lambda}\mathbf{E} + \int_{\pi(G)} |\overline{\nabla}f_j(y)|^2 dy \leq C\mathbf{E}.$$

Harmonic Approximation:

Let ϑ (to be specified soon) and consider $\varepsilon_{Har,\rho}$ given by Lemma 4.4. Choosing $\vartheta = \rho$, define

$$\overline{f}_j := c_o \mathbf{E}^{-1/2} f_j,$$

where c_0 has been chosen so that $\int_{B(0,1/16)\cap\pi} |\overline{f}_j|^2 \leq 1$. Then

$$\begin{aligned}
\left| \int_{B(0,1/16)\cap\pi} \langle \overline{\nabla}\varphi(y), \overline{\nabla}, \overline{f}_j(y) \rangle dy \right| &= \left| \int_{B(0,1/16)\cap\pi} \langle \overline{\nabla}\varphi(y), c_0 \mathbf{E}^{-1/2} \overline{\nabla} f_j(y) \rangle dy \right| \\
&= \left| \int_{B(0,1/16)\cap\pi} c_0 \mathbf{E}^{-1/2} \langle \overline{\nabla}\varphi(y), \overline{\nabla} f_j(y) \rangle dy \right| \\
&\leq c_0 \mathbf{E}^{-1/2} (C\mathbf{E} ||\overline{\nabla}\varphi||_\infty) \\
&\leq C\mathbf{E} ||\overline{\nabla}\varphi||_\infty.
\end{aligned}$$

Assuming $\varepsilon_0 \leq (\varepsilon_{Harm,\rho}/C)^2$, we can apply the Lemma 4.4, to conclude the existence of an harmonic function

$$\overline{u}_j : B(0,1/16) \cap \pi \to \mathbb{R},$$

with $\int |\nabla \overline{u}_j|^2 \leq 1$ and

$$\int_{B(0,1/16)\cap\pi} (\overline{f}_j - \overline{u}_j)^2 \leq \vartheta.$$

Setting

$$u_j := \frac{1}{c_0} \mathbf{E}^{-1/2} \overline{u}_j,$$

we have

$$\int_{B(0,1/16)\cap\pi} (f_j - u_j)^2 \leq C\vartheta\mathbf{E}. \tag{4.49}$$

Notice, in particular, that if we define $u := (u_1, \ldots, u_{n-k})$, we have

$$||Du||^2_{L^2(B(0,1/16)\cap\pi))} \leq C\mathbf{E}. \tag{4.50}$$

Height excess estimates:

Denote by

$$\begin{aligned}
L : B(0,1/16) \cap \pi &\to \pi^\perp \\
y \mapsto L(y) &= \sum_j \langle \nabla u_j(0), y \rangle e_j,
\end{aligned}$$

by $x_0 = u(0)$, and by $\bar{\pi}$, the plane

$$\bar{\pi} := \{(y, L(y)) : y \in B(0,1/16) \cap \pi\}.$$

We are interested in estimating

$$\frac{1}{\eta^{k+2}}\int_{B(x_0,4\eta)} dist(x-x_0,\overline{\pi})^2 d||V||(x),$$

for $\eta\in]0,1/2[$.

We start by observing, that by the mean value property for harmonic functions

$$\begin{aligned} dist(x-x_0,\overline{\pi}) &= |u(0)| \\ &\leq C||u||_{L^1} \\ &\leq C||u||_{L^2} \\ &\leq C||u-f||_{L^2}+C||u||_{L^2} \\ &\leq C\vartheta^{1/2}\mathbf{E}^{1/2}+C\beta, \end{aligned}$$

where in the last inequality we use the fact that by Theorem 4.3 $||f||_\infty<\beta$, on the other hand

$$\begin{aligned} ||P_\pi^\perp - P_{\overline{\pi}}^\perp|| &\leq C\sum_j |\overline{\nabla} u_j(0)| \\ &\overset{B.3}{\leq} ||\nabla u||_{L^1} \\ &\overset{\text{Hölder}}{\leq} C||\nabla u||_{L^2} \\ &\overset{(4.50)}{\leq} C\vartheta^{1/2}\mathbf{E}^{1/2} \\ &\leq C\vartheta^{1/2}\mathbf{E}^{1/2}+C\beta. \end{aligned}$$

Since

$$||P_\pi^\perp - P_{\overline{\pi}}^\perp|| \geq |P_\pi^\perp(x-x_0) - P_{\overline{\pi}}^\perp(x-x_0)| \geq |P_\pi^\perp(x-x_0)| - |P_{\overline{\pi}}^\perp(x-x_0)|,$$

then, for $x\in\Gamma\cap B(0,1/16)$, we have

$$\begin{aligned} dist(x-x_0,\overline{\pi}) &= |P_{\overline{\pi}}^\perp(x-x_0)| \\ &\leq C\vartheta^{1/2}\mathbf{E}^{1/2}+C\beta+|P_\pi^\perp(x-x_0)| \\ &\leq C\vartheta^{1/2}\mathbf{E}^{1/2}+C\beta. \end{aligned}$$

Then, we conclude

$$\begin{aligned} \int_{B(x_0,4\eta)\setminus Gr(f)} dist(x-x_0,\bar{\pi})^2 d||V||(x) &= \int_{(\Gamma\setminus Gr(f))\cap B(x_0,4\eta)} dist(x-x_0,\bar{\pi})^2 d\mathcal{H}^k(x) \\ &\leq C(\vartheta^{1/2}\mathbf{E}^{1/2}+\beta)^2\mathbf{E}. \qquad (4.51) \end{aligned}$$

Finally, observe, that, if $x = (y, f(y)) \in Gr(f)$, then

$$dist(x - x_0, \bar{\pi}) \leq |f(y) - u(0) - L(y)|.$$

Recalling that $L(y) = \langle Du(0), y\rangle$, from Lemma 4.5 we have

$$\sup_{y \in B(x_0, 4\eta)\cap\pi} |u(y) - u(0) - L(y)|^2 \leq C\eta^4 ||\overline{D}u||^2_{L^2(B(x_0,4\eta))} \leq C\eta^4 \mathbf{E}.$$

Summarizing:

$$\int_{Gr(f)\cap B(x_0,4\eta)} dist(x - x_0, \overline{\pi})^2 d||V||(x) \leq C\vartheta\mathbf{E} + C\eta^{k+4}\mathbf{E}. \tag{4.52}$$

Then to conclude

$$\begin{aligned}
\frac{1}{\eta^{k+2}} \int_{B(x_0,4\eta)} dist(x - x_0, \overline{\pi})^2 d||V||(x) &= \frac{1}{\eta^{k+2}} \bigg(\int_{B(x_0,4\eta)\setminus Gr(f)} dist(x - x_0, \overline{\pi})^2 d||V||(x) \\
&+ \int_{B(x_0,4\eta)\cap Gr(f)} dist(x - x_0, \overline{\pi})^2 d||V||(x) \bigg) \\
&\leq \frac{1}{\eta^{k+2}} \left(C(\vartheta^{1/2}\mathbf{E}^{1/2})^2 \mathbf{E} + C\vartheta\mathbf{E} + \eta^{k+4}\mathbf{E} \right) \\
&\leq \frac{1}{\eta^{k+2}} \left(C\vartheta\mathbf{E}^2 + 2C\vartheta^{1/2}\mathbf{E}^{1/2}\beta + C\beta^2\mathbf{E} + C\vartheta\mathbf{E} + C\eta^{k+4}\mathbf{E} \right) \\
&\leq \frac{1}{\eta^{k+2}} \left(C\vartheta\mathbf{E} + C\beta^2\mathbf{E} + \eta^{k+4}\mathbf{E} + 2C\vartheta^{1/2}\mathbf{E}^{1/2}\beta \right) \\
&\leq \frac{C\vartheta}{\eta^{k+2}}\mathbf{E} + \frac{C\beta^2}{\eta^{k+2}}\mathbf{E} + C\eta^2\mathbf{E}.
\end{aligned}$$

Then

$$\frac{1}{\eta^{k+2}} \int_{B(x_0,4\eta)} dist(x - x_0, \overline{\pi})^2 d||V||(x) \leq \frac{C\vartheta}{\eta^{k+2}}\mathbf{E} + \frac{C\beta^2}{\eta^{k+2}}\mathbf{E} + C\eta^2\mathbf{E}. \tag{4.53}$$

Tilt decay inequality:

c Now, impose that ϑ and β satisfies

$$C\vartheta^{1/2} \leq \frac{\eta}{2} \quad and \quad C\beta \leq \frac{\eta}{2}. \tag{4.54}$$

Then $B(0, \eta) \subset B(x_0, 2\eta)$, so

$$\frac{1}{\eta^k} \int_{B(0,\eta)} ||\bar{\pi} - T_x\Gamma||^2 d||V||(x) \leq 2^k \frac{1}{(2\eta)^k} \int_{B(x_0,2\eta)} ||\bar{\pi} - T_x\Gamma||^2 d||V||(x),$$

therefore

$$E(V, \bar{\pi}, 0, \eta) \leq 2^k E(V, \bar{\pi}, x_0, 2\eta).$$

Applying Theorem 4.2

$$2^k E(V,\bar{\pi},x_0,2\eta) \leq \frac{C}{\eta^{k+2}}\int_{B(x_0,4\eta)} dist(x-x_0,\bar{\pi})^2 d||V||(x) + \frac{C}{\eta\eta^{k-2}}\int_{B(x_0,4\eta)} |H|^2 dV, \quad (4.55)$$

then applying previous bound on first term, and the fact that $||H||_\infty \leq \mathbf{E}$, we have

$$E(V,\bar{\pi},0,\eta) \leq \frac{C\vartheta}{\eta^{k+2}}\mathbf{E} + \frac{C\beta^2}{\eta^{k+2}}\mathbf{E} + C\eta^2\mathbf{E} + \eta^2\mathbf{E}^2. \quad (4.56)$$

Finally, notice that C never depends on η, β, ϑ or ε_0, then we can choose

$$C\eta^2 = \frac{1}{8},$$

and, β, and ϑ, such that

$$\frac{C\vartheta}{\eta^{k+2}} \leq \frac{1}{8}, \quad and \quad \frac{C\beta^2}{\eta^{k+2}} \leq \frac{1}{8},$$

which are compatibles with (4.54), then, replacing in (4.56) we have

$$E(V,\bar{\pi},0,\eta) \leq \frac{1}{8}\mathbf{E} + \frac{1}{8}\mathbf{E} + \frac{1}{8}\mathbf{E} + \frac{1}{8}\mathbf{E} = \frac{1}{2}\mathbf{E}.$$

□

4.5 Allards interior regularity theorem

In this section we finally prove Theorem 4.1, using all the tools developed until here. The proof is divided into four steps, first we apply Theorem 4.4 to conclude a power law decay for the excess due to the very nature of this, then we show that we can include $spt||V||$ into the graph of a Lipschitz function f, and prove the "absence of holes". Finally we prove a Morrey's type estimate for the derivative of the Lipschitz function f, which by classical arguments implies that f is actually a $C^{1,\alpha}$ function.

Proof of Allard's ε-regularity theorem 4.1. Without loss of generality we can assume $x_0 = 0$ and $r = 1$.

Power-law decay of the excess: Let $\varepsilon_E > 0$ be as in Theorem 4.4 and choose ε so small that Lemma 4.3 can be applied with $\delta = \varepsilon_E$, thus

$$||V||(B(x,r)) \leq (\omega_k + \varepsilon_0)r^k, \ \forall r < \frac{1}{2}, \ \forall x \in \Gamma \cap B(0,1/4).$$

Fix $x \in \Gamma \cap B(0,1/4)$, and define

$$F(r) := E(r) + \frac{4}{\eta^k}||H||_\infty r := \min_\tau E(V,\tau,x,r) + \frac{4}{\eta^k}||H||_\infty r.$$

If $F(r) \leq \epsilon_0$, then either

$$||H||_\infty r \leq E(r) \quad or \quad E(r) \leq ||H||_\infty r.$$

If $||H||_\infty \leq E(r)$, by Theorem 4.4, for $0 < \eta < 1/2$

$$\begin{aligned}
F(\eta r) &= \min_\tau E(V,\tau,x,\eta r) + \frac{4}{\eta^k}||H||_\infty(\eta r)\\
&\leq \frac{1}{2}E(V,\tau,x,r) + \frac{4}{\eta^k}(\eta r)\\
&\leq \frac{1}{2}E(r) + \frac{4}{\eta^k}||H||_\infty \frac{r}{2}\\
&= \frac{1}{2}F(r),
\end{aligned}$$

and, if $E(r) \leq ||H||_\infty r$, again by Theorem 4.4

$$\begin{aligned}
F(\eta r) &= E(\eta r) + \frac{4}{\eta^k}||H||_\infty(\eta r)\\
&\leq \frac{1}{2}E(r) + \frac{4}{\eta^k}||H||_\infty(\eta r)\\
&\leq \frac{1}{2}||H||_\infty r + \frac{4}{\eta^k}||H||_\infty r\\
&= \frac{4}{\eta}||H||_\infty r\left(\frac{1}{2}\frac{\eta^k}{4} + \eta\right)\\
&\leq \frac{4}{\eta^k}||H||_\infty r\left(\frac{1}{4} + \eta\right)\\
&\leq \frac{3}{4}\left(\frac{4}{\eta^k}||H||_\infty r\right)\\
&= \frac{3}{4}F(r).
\end{aligned}$$

Thus, we can conclude that, if $F(r) < \varepsilon_E$, then $F(\eta r) \leq \frac{3}{4}F(r)$. In particular $F(\eta r) < \varepsilon_E$, for $0 < \eta < 1/2$, and we can iterate again with ηr in place of r.

Notice that

$$F(\frac{1}{2}) \leq \frac{1}{2^k}E(V,\pi,0,1) + \frac{1}{2}\frac{4}{\eta}||H||_\infty \leq \left(\frac{1}{2} + \frac{1}{2}\frac{4}{\eta^k}\right)\varepsilon.$$

Thus, if $\varepsilon > 0$ is sufficiently small, we can start from $r = 1/2$, and iterate the argument to infer

$$F\left(\eta^n\frac{1}{2}\right) \leq C\left(\frac{3}{4}\right)^n \varepsilon \quad \forall n \in \mathbb{N}.$$

Then, given any $r < 1/2$, let $n = \lfloor \log_\eta(2r) \rfloor$, we conclude

$$\begin{aligned} E(r) &\leq \frac{1}{\eta^k} E\left(\eta^n \frac{1}{2}\right) \\ &\leq \frac{1}{\eta^n} F\left(\eta^n \frac{1}{2}\right) \\ &\leq C\left(\frac{3}{4}\right)^{\lfloor \log_\eta 2r \rfloor - 1} \varepsilon \\ E(r) &\leq C^{2\alpha}\varepsilon, \end{aligned} \tag{4.57}$$

where C and $\alpha > 0$ depends only of the dimension of the varifold and the ambient Euclidian space.

<u>**Inclusion in a Lipschitz graph:**</u>

Fix $x \in B(0, 1/4)$. Set $\pi_0 = \pi$, and for $n \geq 1$ let π_n be a plane such that

$$E(V, \pi_n, x, \frac{1}{2^n}) = \min_\tau E(V, \tau, x, \frac{1}{2^n}) = E(\frac{1}{2^n}).$$

Recalling that by Theorem (3.24)

$$||V||(B(x,r)) \geq \omega_k e^{-||H||_\infty r} r^k \geq C^{-1} r^k, \quad for\ any\ r < 1 - |x|,$$

then by (4.57) we have

$$\begin{aligned} ||\pi_n - \pi_{n+1}|| &\leq \frac{1}{||V||(B(x,r))} \int_{B(x,\frac{1}{2^{n+1}})} (||\pi_n - T_y\Gamma|| + \pi_{n+1} - T_y\Gamma||) d||V||(y) \\ &\leq \frac{C}{r^k} \left(\int_{B(x,\frac{1}{2^{n+1}})} ||\pi_n - T_y\Gamma|| d||V||(y) \right) + \frac{C}{r^k} (||\pi_{n+1} - T_y\Gamma|| d||V||(y)) \\ &\leq C \left(E(V\pi_n, x, \frac{1}{2^{n+1}})^{1/2} + E(V\pi_{n+1}, x, \frac{1}{2^{n+1}})^{1/2} \right) \\ &\leq C \left(\left(\frac{1}{2^n}\right)^{2\alpha} \varepsilon^{1/2} + \left(\frac{1}{2^{n+1}}\right)^{2\alpha} \varepsilon^{1/2} \right) \\ ||\pi_n - \pi_{n+1}|| &\leq C \left(\frac{1}{2^n}\right)^{2\alpha} \varepsilon^{1/2}. \end{aligned} \tag{4.58}$$

Now, summing over n from 0 to $j-1$

$$\begin{aligned} ||\pi_j - \pi|| &= ||\sum_{n=0}^{j-1} \pi_n - \pi_{n+1}| \\ &\leq \sum_{n+0}^{j-1} ||\pi_n - \pi_{n+1}|| \\ &\leq \sum_{n=0}^{j-1} C \left(\frac{1}{4^\alpha}\right)^n \epsilon^{1/2} \\ ||\pi_j - \pi|| &\leq C\epsilon^{1/2}, \end{aligned}$$

where C is a dimensional constant.

Thus

$$E(V, x, \pi, r) \leq C\epsilon \quad \forall x \in \Gamma \cap B(0, 1/4),\ and\ r < \frac{1}{2}.$$

Next, fix a constant $l \leq 1/2$ and let $\lambda(l)$ and $\varepsilon_L(\lambda)$ the corresponding constants given by Theorem 4.3, and assume $\varepsilon < \varepsilon_L$ and also small that the set G of the same Theorem contains in $\Gamma \cap B(0, 1/4)$. We then conclude that there exists a Lipschitz function

$$f : B(0, 1/4) \cap \pi \to \pi^{\perp},$$

such that $\Gamma \cap B(0, 1/4) \subset Gr(f)$ and $Lip(f) \leq l$.

<u>Absence of "holes":</u>

Let $B^k(0,1)$ the k-ball, $x \in \partial B^k(0,1)$ and let

$$\mathcal{L}^k(B^k(0,1) \setminus B^k(x,1)) = 2\vartheta.$$

Assume that $B(0, 1/16) \cap \pi \not\subset D = P_\pi(\Gamma \cap B(0, 1/4))$ (D as defined in Theorem 4.3), and let $w \in (B(0, 1/16) \cap \pi)$. Define

$$r := \inf_{z \in D} |w - z|.$$

It is clear that $r < 1/16$, because $0 \in \Gamma$, therefore any infimizing sequence $\{z_n\}_{n \in \mathbb{N}}$ must be contained in $B(0, 1/8) \cap \pi$. Then modulo a subsequence, we can suppose

$$z_n \to z \in \overline{B(0, 1/8) \cap \pi},$$

and recalling that $0 \in \Gamma$ we conclude $||f||_\infty \leq l$.

If l is sufficiently small we conclude that $x_n = (z_n, f(z_n)) \in B(0, 3/16)$ and thus x_n converges to $x = (z, f(z)) \in B(0, 3/16)$, where $z \in \overline{B(0, 1/8) \cap \pi}$.

Observe that $\Gamma \cap B(x, r) \subset Gr(f)$ because $r, 1/16$. In particular, considering that $B^k(w, r) \cap D = \emptyset$, using the area formula we can estimate

$$\begin{aligned} ||V||(B(x,r)) &= \mathcal{H}^k(B(x,r) \cap \Gamma) \\ &= \int_{B^k(z,r) \setminus B^k(w,r)} Jf(u) du \\ &\leq (\omega_k - 2\vartheta)(1 + Cl^2) r^k. \end{aligned}$$

Now we choose l such that

$$(\omega_k - 2\vartheta)(1 + Cl^2) r^k = \omega_k - \vartheta.$$

On the other hand, by Theorem 3.2 and the hypothesis, we have

$$||V||(B(x,r)) \geq \omega_k r^k e^{||H||_\infty r} \geq \omega_k r^k e^{-\varepsilon}.$$

then choosing ε such that

$$\omega_k - \vartheta < \omega_k r^k e^{-\varepsilon},$$

we reach a contradiction (in fact it is enough to have ε being smaller than $k\log(r)$).

Morrey Estimate for Df

So far we have conclude that $spt||V||$ coincides with the graph of a Lipschitz function on the intersection of $(B(0,1/8)\cap\pi)\times\pi^\perp$ and $B(0,1/4)$.

Now, for every $z \in B(0,1/4)$ and every $r < 1/16$ let $\pi_{z,r}$, the k-dimensional plane such that

$$E(V,\pi_{z,r},(z,f(z))) = \min_\tau E(V,\tau,(z,f(z)),r) \leq C\epsilon r^{2\alpha}.$$

Recalling that $E(V,\pi,(z,f(z)),r) \leq C\epsilon$, we conclude

$$||\pi - \pi_{z,r}|| \leq C\epsilon^{1/2}.$$

If ε is sufficiently small, $\pi_{z,r}$ is the graph of a linear function

$$T_{z,r} : \pi \to \pi^\perp,$$

with

$$||T_{z,r}||_{HS} \leq 1.$$

Consider two linear maps $T,\overline{T} : \pi \to \pi^\perp$, with $||T||_{HS}, ||\overline{T}||_{HS} \leq Cl$, the k-dimensional planes $\tau_T, \tau_{\overline{T}}$, given by the corresponding graphs and, P_T and $P_{\overline{T}}$ the orthogonal projection onto τ_T and $\tau_{\overline{T}}$ respectively. Observe that, if l is smaller than a geometric constant

$$|P_T(v)| \leq \frac{1}{2}|v|, \quad for\ any\ v \in \pi^\perp.$$

Fix an orthonormal basis $\{e_1,\ldots,e_k\}$ for π, then

$$\begin{aligned}
|T(e_i) - \overline{T}(e_i)| &= |(e_i + T(e_i)) - (e_i + \overline{T}(e_i))| \\
&= |P_T(e_i + T(e_i)) - P_{\overline{T}}(e_i - \overline{T}(e_i))| \\
&\leq |P_T(e_i) - P_{\overline{T}}(e_i)| + |P_T(T(e_i)) - P_{\overline{T}}(T(e_i))| \\
&+ |P_{\overline{T}}(T(e_i) - \overline{T}(e_i))|,
\end{aligned}$$

since $T(e_i) - \overline{T}(e_i) \in \pi^\perp$, then

$$|T(e_i) - \overline{T}(e_i)| \leq C||\tau_T \tau_{\overline{T}}|| + \frac{1}{2}|T(e_i) - \overline{T}(e_i)|,$$

so

$$|T(e_i) - \overline{T}(e_i)| \leq C||\tau_T - \tau_{\overline{T}}||.$$

Finally, from the previous discussion, for $r < 1/16$,

$$\begin{aligned}\int_{B(z,r/2)\cap\pi} |Df(y) - T_{z,r}|^2 dy &\leq \int_{B(z,r/2)\cap\pi} |Df(y) - T_{z,r}|^2 Jf(y)dy \\ &\leq Cr^k E(V, \pi_{z,r}, (z, f(z)), 2r) \\ &\leq Cr^{k+2\alpha}.\end{aligned}$$

Then, using the notation on C.1, we have

$$\int_{B(z,r)\cap\pi} \left|Df(y) - (Df)_{B(z,r)\cap\pi}\right|^2 dy = \min_T \int_{B(z,r)\cap\cap\pi} |Df(y) - T|^2 dy \leq Cr^{k+2\alpha}. \quad (4.59)$$

<u>**Conclusion:**</u> The conclusion of this theorem is a simple application of the **Campanato's criterion**, wich can be found in [Mag12] Theorem 6.1, and Proposition 16.23 on [CDK11]. However, we give a sketch of this, as a direct consequence of our previous estimates.

In first, arguing as in the proof of (4.58), we conclude

$$|(Df)_{B(x,2^{-k})} - (Df)_{B(x,2^{-k-1})}| \leq C\frac{1}{2^{k\alpha}}, \quad \forall k > 5, \ and, x \in B(0, 1/32) \cap \pi. \quad (4.60)$$

Hence the sequence of continuous functions $x \mapsto (Df)_{B(x,2^k)}$ is a Cauchy sequence, with the supremum norm, then there exists a continuous function g, such that

$$\{(Df)_{B(x,2^{-k})}\}_{k\in\mathbb{N}} \to g,$$

uniformly, and $g = Df$ for all the Lebesgue points on $B(x, 1/32) \cap \pi$. Summing (4.60) over different scales, we have

$$|(Df)_{B(x,r)} - (Df)_{B(x,\rho)}| \leq C(\max\{r, \rho\})^\alpha, \quad \forall x \in B(0, 1/32) \cap \pi, \ and \ all \ r, \rho < \frac{1}{32}. \quad (4.61)$$

Observe that, if $r = |x - y|$ and $x, y \in B(0, 1/64) \cap \pi$, applying the triangular inequality

$$\begin{aligned}|(Df)_{x,r} - (Df)_{x,\rho}|^2 \leq Cr^{-k} &\int_{B(x,r)\cap\pi} |Df - (Df)_{B(x,r)}|^2 + \\ &+ Cr^{-k} \int_{B(y,r)\cap\pi} |Df - (Df)_{B(y,r)}|^2 \leq Cr^{2\alpha}.\end{aligned}$$

Combining (4.60) and (4.61) and (4.5) we conclude the existence of a dimensional constant such that

$$|(Df)_{B(x,2^{-k})} - (Df)_{B(y,2^{-k})}| \leq C \left(\max\{\frac{1}{2^k}, |x-y|\} \right)^{\alpha}. \tag{4.62}$$

Thus, fixing x and y and letting $k \to \infty$ we obtain

$$|g(x) - g(y)| \leq C|x-y|^{\alpha}.$$

Finally, mollifying f with a standard kernel φ_δ to get $f * \varphi_\delta$. Then we have

$$D(f * \varphi_\delta) = g * \varphi_\delta,$$

and therefore $||f * \varphi_\delta||_{C^{1,\alpha}(B(0,1/64)\cap\pi)}$ is bounded, and independently of δ. Letting $\delta \downarrow 0$ by [CDK11] Proposition 16.23, which shows that

$$f \in C^{1,\alpha}(B(0,1/64) \cap \pi),$$

as we claimed. □

Appendices

Appendix A

Grassmanian Manifold

In this section we will study the geometry of the set of all k-dimensional subspaces of a Euclidean space. Let n, k be fixed integers, with $n \geq 0$ and $0 \leq k \leq n$; we will denote by $G(n, k)$ the set of all k-dimensional vector subspaces of $\mathbb{R}^n$; $G(n, k)$ is called the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$.

Our goal is to describe a differentiable atlas for $G(n, k)$, and the main idea is to view the points of $G(n, k)$ as graphs of linear maps defined on a fixed k-dimensional subspace of $\mathbb{R}^n$ and taking values in another fixed (nk)-dimensional subspace of $\mathbb{R}^n$, where these two fixed subspaces are transversal. To this aim, we consider a direct sum decomposition $\mathbb{R}^n = W_0 \oplus W_1$, where $dim(W_0) = k$ (and obviously $dim(W_1) = nk$). For every linear map $T : W_0 \to W_1$, the graph of T given by:

$$Gr(T) = \{v + T(v) : v \in W_0\}$$

is an element in $G(n, k)$. Moreover, an element $W \in G(n, k)$ is of the form $Gr(T)$ if and only if it is transversal to W_1 , i.e., iff it belongs to the set:

$$G^0_{W_1}(n, k) = \{W \in G(n, k) : W \cap W_1 = \{0\}\} \subset G(n, k).$$

In this situation, the linear map T is uniquely determined by W . We can therefore define a bijection:

$$\varphi_{W_0,W_1} : G^0_{W_1}(n, k) \to Lin(W_0, W_1), \tag{A.1}$$

by setting $\varphi_{W_0,W_1}(W) = T$ when $W = Gr(T)$.

More concretely, if π_0 and π_1 denote respectively the projections onto W_0 and W_1 in the decomposition $\mathbb{R}^n = W_0 \otimes W_1$, then the linear map $T = \varphi_{W_0,W_1}(W)$ is given by:

$$T = (\pi_1 \mid_W) \circ (\pi_1 \mid_W) .$$

Observe that the condition that W be transversal to W_1 is equivalent to the condition that the restriction $\pi_0 \mid_W$ be an isomorphism onto W_0.

We will now show that the collection of the charts ${}_{W_0,W_1}$, when (W_0, W_1) run over the set of all direct sum decomposition of $\mathbb{R}^n$ with $dim(W_0) = k$, is a differentiable atlas for $G(n, k)$. To this aim, we need to study the transition functions between these charts. Let us give the following definition.

Definition A.1. *Given subspaces $W_0, W_0' \subset \mathbb{R}^n$ and given a common complementary subspace $W_1 \subset \mathbb{R}^n$ of theirs, i.e., $R^n = W_0 \otimes W_1 = W_0' \otimes W_1$, then we have an isomorphism:*

$$\eta = \eta^{W_1}_{W_0,W_0'} : W_0 \to W_0',$$

obtained by the restriction to W_0 of the projection onto W_0' relative to the decomposition $\mathbb{R}^n = W_0' \otimes W_1$. We say that $\eta^{W_1}_{W_0,W_0'}$ is the isomorphism of W_0 and W_0 determined by the common complementary subspace W_1.

The inverse of $\eta^{W_1}_{W_0,W_0'}$ is simply $\eta^{W_1}_{W_0',W_0}$.

Let us consider charts φ_{W_0,W_1} and φ_{W_0,W_1} in $G(n, k)$, with $k = dim(W_0) = dim(W_0')$; observe that they have the same domain. In this case it is easy to obtain the following formula for the transition function:

$$\varphi_{W_0',W_1} \circ \varphi^{-1}_{W_0,W_1} = (\pi_1' \mid_{W_0} + T) \circ \eta^{W_1}_{W_0',W_0}, \tag{A.2}$$

where π_1' denotes the projection onto W_1 relative to the decomposition $R^n = W_0' \otimes W_1$.

Let us now consider decompositions $R^n = W_0 \otimes W_1' = W_0 \otimes W_1$, with $dim(W_0) = k$, and let us look at the transition function $\varphi_{W_0,W_1'} \circ \varphi^{-1}_{W_0,W_1}$. First, we observe that its domain consists of those linear maps $T \in Lin(W_0, W_1)$ such that $Gr(T) \in G^0_{W_1'}(n, k)$; it is easy to see that this condition is equivalent to the invertibility of the map

$$Id + (\pi_0' \mid_{W_1}) \circ T,$$

where π_0' denotes the projection onto W_0 relative to the decomposition $\mathbb{R}_n = W_0 \otimes W_1'$ and Id is the identity map on W_0 . We have the following formula for $\varphi_{W_0',W_1} \circ \varphi^{-1}_{W_0,W_1}$

$$\varphi_{W_0',W_1} \circ \varphi^{-1}_{W_0,W_1}(T) = \eta^{W_0}_{W_1,W_1'} \circ T \circ (Id + (\pi_0' \mid_{W_1}) \circ T)^{-1}. \tag{A.3}$$

We have therefore proven the following proposition.

Proposition A.1. *The set of all charts φ_{W_0,W_1} in $G(n,k)$, where the pair (W_0, W_1) run over the set of all direct sum decompositions of $\mathbb{R}^n$ with $dim(W_0) = k$, is a differentiable atlas for $G(n,k)$.*

Proof. Since every subspace of $\mathbb{R}_n$ admits one complementary subspace, it follows that the domains of the charts φ_{W_0,W_1} cover $G(n,k)$. The transition functions A.2 and A.3 are differentiable maps defined in open subsets of the vector space $Lin(W_0, W_1)$. The general case of compatibility between charts φ_{W_0,W_1} and $\varphi_{W_0',W_1'}$ follows from transitivity. $\square$

Theorem A.1. *The differentiable atlas in Proposition A.1 makes $G(n,k)$ into a differentiable manifold of dimension $k(n-k)$.*

Proof. If $dim(W_0) = k$ and $dim(W_1) = n-k$, then $dim(Lin(W_0, W_1)) = k(n-k)$. It remains to prove that the topology defined by the atlas is Hausdorffand second countable. The Hausdorff property follows from the fact that every pair of points of $G(n,k)$ belongs to the domain of a chart. The second countability property follows from the fact that, if we consider the finite set of chart φ_{W_0,W_1}, where both W_0 and W_1 are generated by elements of the canonical basis of $\mathbb{R}^n$, we obtain a finite differentiable atlas for $G(n,k)$. $\square$

Finally we introduce the Grassmannian bundle of a given manifold.

Definition A.2. *Let Σ^n a n-dimensional manifold, we define the k-**Grassmannian bundle of** Σ as the topological space*

$$G_k(\Sigma) = \bigsqcup_{x\in\Sigma} G_x(n,k),$$

where $G_x(n,k)$ is the k-Grassmannian manifold of $T_x\Sigma$, which is isomorphic to $\mathbb{R}^n$.

Proposition A.2. *$G_k(\Sigma^n)$ as defined before, is a fiber bundle with fiber $G_x(n,k)$.*

Proof. Let

$$\pi : G_k(\Sigma) \to \Sigma$$
$$(x,S) \mapsto x,$$

the canonical projection onto Σ, which is a continuous function, and since $\pi^{-1}(x) = \{x\} \times G_x(n,k) \approx G_x(n,k)$, which is a manifold.

To prove the local trivialization, let $x \in U$ an open neighbourhood in Σ, and define

$$\begin{aligned} \psi : U \times Lin(W_0, W_1) &\to \pi^{-1}(U) \\ (x, T) &\mapsto (x, \varphi^{-1}_{W_0,W_1}(T)), \end{aligned}$$

where $W_0 \oplus W_1 = \mathbb{R}^n$, $dim(W_0) = k$, $dim(W_1) = n - k$, and φ_{W_0,W_1} is a chart of $G_x(n, k)$. Notice that the fact that $T_x\Sigma \approx \mathbb{R}^n$ guarantees that φ_{W_0,W_1} is a chart of $G_x(n, k)$ also. Finally since $Lin(W_0, W_1) \approx \mathbb{R}^{k(n-k)}$ we can define

$$\overline{\psi} : U \times \mathbb{R}^{k(n-k)} \to \pi^{-1}(U),$$

as the corresponding identification. Then $\overline{\psi}$ is clearly an isomorphism. $\square$

Appendix B

Harmonic Functions

The purpose of this appendix is to introduce the **harmonic functions**, and some of their main proprieties, illustrating some aspects of the classical model problem in the theory of elliptic regularity: the Dirichlet problem for the Laplace operator.

Definition B.1. *Given a function $u \in C^2(\Omega)$, where Ω is an open, connected and bounded subset of $\mathbb{R}^n$, we say that u is:*

- ***harmonic**, if $\Delta u = 0$,*
- ***subharmonic**, if $\Delta u \geq 0$,*
- ***superharmonic**, if $\Delta u \leq 0$,*

where

$$\Delta u(x) := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x),$$

is the Laplacian operator.

We shall be concerned with the problem of the existence of harmonic functions with prescribed boundary value, i.e., the solution of the following Dirichlet problem:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

in $C^2(\Omega) \cap C^0(\overline{\Omega})$, for a given function $g \in C^0(\partial\Omega)$. The problem of finding a harmonic function with prescribed boundary value $g \in C^0(\partial\Omega)$ is tied, but not equivalent, to the following one: find a minimizer function to the functional

$$\mathcal{D}(u) = \frac{1}{2} \int_{\Omega} ||D(u)||^2 dx,$$

on the set

$$\mathcal{A} := \{u \in C^2(\Omega) \cap C^0(\overline{\Omega}) : u = g \text{ } on \text{ } \partial\Omega\}.$$

The functional $\mathcal{D}$ is called the **Dirichlet integral**.

In fact, if such minimizer, call it u, exists, then the first variation of Dirichlet integral, vanishes, i.e.

$$\frac{d}{dt}(D(u + t\varphi)) \mid_{t=0} = 0,$$

for all $\varphi \in C_c^0(\Omega)$, then by integrating by parts

$$\begin{aligned}
0 &= \frac{d}{dt}(D(u + t\varphi)) \mid_{t=0} \\
&= \frac{1}{2}\frac{d}{dt}\left(\int_\Omega ||D(u + t\varphi)||^2 dx\right) \\
&= \frac{1}{2}\int_\Omega \frac{d}{dt}\langle \nabla u + t\nabla\varphi, \nabla u + t\nabla\varphi\rangle \mid_{t=0} dx \\
&= \int_\Omega \langle \nabla u, \nabla\varphi\rangle dx \\
&= -\int_\Omega \Delta u\varphi dx \quad \forall \varphi \in C_c^0(\Omega),
\end{aligned}$$

then $\Delta u = 0$, i.e. minimizers of Dirichlet integral are harmonic functions.

Then we have proved that to minimize the Dirichlet integral is equivalent to find harmonic functions with prescribed boundary value, then we can state the next principle.

Theorem B.1 (Dirichlet's Principle)**.** *A minimizer u of the Dirichlet integral in Ω with prescribed boundary value g always exists, it is unique and it is a harmonic function; moreover, solves the problem*

$$\begin{cases} \Delta u = 0, & in \ \Omega \\ u = g, & on \ \partial\Omega. \end{cases} \tag{B.1}$$

Conversely, any solution of B.1 is a minimizer of the Dirichlet integral in the class of functions with boundary value g.

But this principle is not always true, because we faces two problems, first, a minimizer of a functional always exists?, and second, since we know that if the minimizer is a harmonic function, is possible to assert the converse? To answer the first question, we exhibit an example where this is not true, to answer the second question the reader can consult [GM12] 1.2.2.

Example B.1. *Consider the area functional defined on the unit ball* $B(0,1) \subset \mathbb{R}^2$

$$\mathcal{F}(u) = \int_{B(0,1)} \sqrt{1 + ||D(u)||^2},$$

defined on

$$A = \{u \in Lip(B(0,1)) : u = 0 \text{ on } \partial B(0,1),\ u(0) = 1\}.$$

As $\mathcal{F}(u) \geq \pi$ *for every* $u \in \mathcal{A}$*, the sequence of functions*

$$u_n(x) = \begin{cases} 1 - n||x||, & for\ ||x|| \in [0, \frac{1}{n}] \\ 0, & for\ ||x|| \in [\frac{1}{n}, 1] \end{cases}$$

shows that $\inf_{\mathcal{A}} \mathcal{F} = \pi$*. On the other hand if* $\mathcal{F}(u) = \pi$ *for some* $u \in \mathcal{A}$*, then* u *is constant, thus cannot belong to* $\mathcal{A}$*.*

Before to prove the solvability of the Dirichlet problem, we introduce some properties of harmonic functions, the full proof of those can be found in [GM12] section 1.3.

Proposition B.1 (Weak maximum principle)**.** *If* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *is subharmonic, then*

$$\sup_{\Omega} u = \max_{\partial\Omega} u;$$

If u *is superharmonic, then*

$$\inf_{\Omega} u = \min_{\partial\Omega} u.$$

Proposition B.2 (Comparison principle)**.** *Let* $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *be such that* u *is subharmonic,* v *is superharmonic and* $u \leq v$ *on* $\partial\Omega$*, then* $u \leq v$ *in* Ω*.*

Corollary B.1 (Maximum estimate)**.** *Let* u *and* v *be two harmonic functions in* Ω*, then*

$$\sup_{\Omega} |u - v| \leq \max_{\partial\Omega} |u - v|.$$

Corollary B.2 (Uniqueness)**.** *Two harmonic functions on* Ω *that agree on* $\partial\Omega$ *are equal.*

Proposition B.3 (Mean value inequalities)**.** *Suppose that* $u \in C^2(\Omega)$ *is subharmonic, then for every ball* $B(x,r) \subsetneq \Omega$

$$u(x) \leq \frac{1}{\mathcal{H}^{n-1}(B(x,r))} \int_{\partial B(x,r)} u(y) d\mathcal{H}^{n-1}(y)$$
$$u(x) \leq \frac{1}{\mathcal{L}^{n-1}(B(x,r))} \int_{B(x,r)} u(y) dy.$$

If u *is superharmonic, the reverse inequalities hold; consequently for* u *harmonic equalities are true.*

Corollary B.3 (Strong maximum principle). *If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is subharmonic (resp. superharmonic), then it cannot attain its maximum (resp. minimum) in Ω unless it is constant.*

Lemma B.1 (Weyl). *A function $u \in L^1_{loc}(\Omega)$ is harmonic if and only if*

$$\int_\Omega u \Delta \varphi dx = 0, \quad \forall \varphi \in C_c^\infty(\Omega).$$

Proposition B.4. *Given $u \in C^0(\Omega)$, the following facts are equivalent:*

1. *For every ball $B(x,R) \subsetneqq \Omega$ we have*

$$u(x) \leq \frac{1}{\mathcal{H}^{n-1}(B(x,r))} \int_{\partial B(x,r)} u(y) d\mathcal{H}^{n-1}(y);$$

2. *for every ball $B(x,R)$*

$$u(x) \leq \frac{1}{\mathcal{L}^{n-1}(B(x,r))} \int_{B(x,r)} u(y) dy;$$

3. *for every $x \in \Omega$, $R_0 > 0$, there exist $R \in]0, R_0[$ such that $B(x,R) \subsetneqq \Omega$ and*

$$u(x) \leq \frac{1}{\mathcal{L}^{n-1}(B(x,r))} \int_{B(x,r)} u(y) dy;$$

4. *for each $h \in C^0(\Omega)$ harmonic in $\Omega' \subsetneqq \Omega$ whit $u \leq h$ in $\partial\Omega'$, we have $u \leq h$ in Ω';*

5. *For all $\varphi \in C_c^\infty(\Omega)$ and $\varphi \geq 0$*

$$\int_\Omega u(x) \Delta\varphi(x) dx \geq 0.$$

Appendix C

Morrey and Campanato spaces

The aim of this appendix is to introduce the reader to certain sub-spaces of the well known L^p-spaces with a finer structure and which describe the scaling of the L^p-norm in small balls in terms of powers of the radii of these balls.
Before to begin let us introduce some notation, based on the definition of L^p-spaces and their norms.

Definition C.1. *Let $\Omega \subset \mathbb{R}^n$ be an bounded, open set, $p \in [1, \infty[$ and $\lambda \geq 0$.*

1. *We denote by $L^{p,\lambda}(\Omega, \mathbb{R}^m)$ the* ***Morrey space*** *of all functions $f \in L^p(\Omega, \mathbb{R}^m)$ such that*

$$||f||^p_{L^{p,\lambda}} := \sup_{x_0 \in \overline{\Omega}, 0<\rho<diam(\Omega)} \frac{1}{\rho^\lambda} \int_{B(x_0)\cap\Omega} |f|^p dx,$$

is finite.

2. *We denote by $\mathcal{L}^{p,\lambda}(\Omega, \mathbb{R}^m)$ the* ***Campanato space*** *of all functions $f \in L^p(\Omega, \mathbb{R}^m)$ such that*

$$[f]_{L^{p,\lambda}(\Omega,\mathbb{R}^m)} := \sup_{x_0 \in \overline{\Omega}, 0<\rho<diam(\Omega)} \frac{1}{\rho^\lambda} \int_{B(x_0)\cap\Omega} |f - (f)_{B(x_0,\rho)\cap\Omega}|^p dx,$$

is finite, where

$$(f)_{B(x_0,\rho)\cap\Omega} := \frac{1}{\mathcal{L}^n(B(x_0,\rho)\cap\Omega)} \int_{B(x_0,rho)\cap\Omega} f dx,$$

i.e. the average of f over $B(x_0,\rho)\cap\Omega$.

The following remarks are direct consequences of the definitions above, and their proof are left to the reader, or may also be found in [GM12] Chapter 5.

Remark C.1. *Endowed with the norm* $||\cdot||_{L^{p,\lambda}(\Omega,R^m)}$ *the Morrey spaces* $L^{p,\lambda}(\Omega,\mathbb{R}^m)$ *are Banach spaces for all* $p\in[1,\infty[$ *and* $\lambda\geq 0$.
The Campanato spaces $\mathcal{L}^{p,\lambda}(\Omega,\mathbb{R}^m)$ *are also Banach spaces,endowed with the norm*

$$||\cdot||_{\mathcal{L}^{p,\lambda}(\Omega,\mathbb{R}^m)} := [\cdot]_{\mathcal{L}^{p,\lambda}(\Omega,\mathbb{R}^m)} + ||\cdot||_{L^p(\Omega,\mathbb{R}^m)}.$$

Remark C.2. *From the definition is it clear that both conditions only depend on the behaviour of the functions* f *for small radii. Therefore, it is sufficient so check that the supremum remains bounded for all* $\rho<\rho_0$ *for some fixed, positive* ρ_0.

Remark C.3. *For the Morrey spaces we have* $L^{p,0}(\Omega,\mathbb{R}^m)=L^p(\Omega,\mathbb{R}^m)$ *and* $L^{p,n}(\Omega,\mathbb{R}^m)=L^\infty(\Omega,\mathbb{R}^m)$. *Moreover,* $L^{p,\lambda}(\Omega,\mathbb{R}^m)\approx\{0\}$ *for* $\lambda>n$ *in view of Theorem 2.2.*
We further have $\mathcal{L}^{p,\lambda}(\Omega,\mathbb{R}^m)=L^{p,\lambda}(\Omega,\mathbb{R}^m)$ *for the intermediate parameters* $\lambda\in[0,n[$.

Remark C.4. *The Campanato space* $\mathcal{L}^{1,n}(\Omega,\mathbb{R}^m)$ *has a special role and is usually known as the bounded mean oscillation space. It is smaller than any Lebesgue space* $L^p(\Omega,\mathbb{R}^m)$ *with* $p<\infty$ *but still containing* $L^\infty(\Omega,\mathbb{R}^m)$ *as a strict subspace.*

To end this appendix we enunciate the next useful theorem which we use in crucial way in the last part of the proof of Theorem 4.1, and whose proof can be found in [GM12] Theorem 5.5.

Theorem C.1. *Let* Ω *be a bounded, open set in* $\mathbb{R}^n$ *which satisfies*

$$\mathcal{L}^n(B(x_0,\rho)\cap\Omega)\geq A\rho^n, \quad \forall x_0\in\overline{\Omega},\ \text{and}\ \forall\rho\leq diam(\Omega)$$

for some $A>0$. *Then we have* $\mathcal{L}^{p,n+\alpha}(\Omega,\mathbb{R}^m)\approx C^{0,\alpha}(\Omega,\mathbb{R}^m)$ *for all* $\alpha\in]0,1]$.

Appendix D

Caccioppoli's inequality

As we mentioned before, the Theorem 4.3, is a version of the **Caccioppili's inequality**, which enables us to give a priori estimates of the L^2-norm of the derivatives of a weak solution u, of a linear elliptic of PDE's system, in terms of the L^2-norm of u.
We start our discussion with the simpler case, the harmonic case.

Theorem D.1 (Caccioppoli's inequality for harmonic functions)**.** *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $\Delta u = 0$, that is*

$$\int_\Omega D_\alpha u D_\alpha \varphi dx = 0, \quad \forall \varphi W_0^{1,2}(\Omega). \tag{D.1}$$

Then for each $x_0 \in \Omega$, $0 < \rho < R \leq dist(x_0, \partial\Omega)$ we have

$$\int_{B(x_0,\rho)} |Du|^2 dx \leq \frac{c}{(R-\rho)^2} \int_{B(x_0,R)\setminus B(x_0,\rho)} |u-\lambda|^2 dx, \quad \forall \lambda \in \mathbb{R}$$

for some universal constant c.

Proof. Define a cut-off function $\eta \in C_c^\infty(\Omega)$ such that

- $0 \leq \eta \leq 1$;
- $\eta \equiv 1$ on $B(x_0, \rho)$ and $\eta \equiv 0$ on $B(x_0, R) \setminus B(x_0, \rho)$;
- $|D\eta| \leq \frac{2}{R-\rho}$.

Choosing as test function $\varphi := (u - \lambda)\eta^2$ we get

$$\begin{aligned}
\int_\Omega D_\alpha u D_\alpha \varphi dx &= \int_\Omega D_\alpha u D_\alpha \left((u-\lambda)\eta^2\right) dx \\
&= \int_\Omega D_\alpha u \left(\eta^2 D_\alpha u + 2\eta u D_\alpha \eta - 2\eta\lambda D_\alpha \eta\right) dx \\
&= \int_\Omega |Du|^2 \eta^2 + \int_\Omega 2\eta D_\alpha u \, (u-\lambda)\, D_\alpha \eta dx,
\end{aligned}$$

then, replacing in in D.1

$$\int_{\Omega}|Du|^2\eta^2 + \int_{\Omega} 2\eta D_\alpha u\,(u-\lambda)\,D_\alpha\eta dx = 0,$$

therefore

$$\int_{B(x_0,R)}|Du|^2|\eta|^2 \leq \int_{B(x_0,R)}(|\eta||Du|)\,(2|u-\lambda||D\eta|)\,dx.$$

Now applying the Holder inequality, we have

$$\int_{B(x_0,R)}|Du|^2|\eta|^2 \leq \left(\int_{B(x_0,R)}|\eta|^2|Du|^2dx\right)^{1/2}\left(4\int_{B(x_0,R)}|u-\lambda|^2|D\eta|^2dx\right)^{1/2}.$$

Now dividing by

$$\left(\int_{B(x_0,R)}|\eta|^2|Du|^2dx\right)^{1/2},$$

and squaring in both sides, we obtain

$$\int_{B(x_0,R)}|Du|^2|\eta|^2 \leq 4\int_{B(x_0,R)}|u-\lambda|^2|D\eta|^2dx,$$

or equivalently

$$\int_{B(x_0,R)}|Du|^2|\eta|^2 \leq 4\left(\int_{B(x_0,R)}|u-\lambda|^2|D\eta|^2dx + \int_{B(x_0,R)\setminus B(x_0,\rho)}|u-\lambda|^2|D\eta|^2dx\right),$$

finally taking account the properties of η, we have that

$$\begin{aligned}
\int_{B(x_0,\rho)}|Du|^2dx &\leq \int_{B(x_0,R)}|Du|^2|\eta|^2dx,\\
\int_{B(x_0,R)}|u-\lambda|^2|D\eta|dx &= 0,\\
\int_{B(x_0,R)\setminus B(x_0,\rho)}|u-\lambda|^2|D\eta|^2dx &\leq \frac{4}{(R-\rho)^2}\int_{B(x_0,R)\setminus B(x_0,\rho)}|u-\lambda|^2.
\end{aligned}$$

Then

$$\int_{B(x_0,\rho)}|Du|^2dx \leq \frac{c}{(R-\rho)^2}\int_{B(x_0,R)\setminus B(x_0,\rho)}|u-\lambda|^2dx, \quad \forall\lambda\in\mathbb{R}.$$

□

The following result is the general case of the Caccioppoli's inequality for elliptic system of PDE's

Theorem D.2 (Caccioppoli's inequality for elliptic systems)**.** *Let $u\in W^{1,2}(\Omega,\mathbb{R}^m)$ be a weak solution of*

$$-D_\alpha(A_{ij}^{\alpha\beta}D_\beta u^j) = f_i - D_\alpha F_i^\alpha,$$

with $f_i, F_i^\alpha \in L^2(\Omega)$ and assume one of the following conditions holds:

1. $A_{ij}^{\alpha\beta} \in L^\infty(\Omega)$ *and there exists* $\lambda \geq 0$ *such that*

$$A_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j \geq \lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^m \times \mathbb{R}^n.$$

This condition is known as the ***Legendre ellipticity condition****.*

2. $A_{ij}^{\alpha\beta} \equiv$ *constant and there exists* $\lambda \geq 0$ *such that*

$$A_{ij}^{\alpha\beta}\xi_\alpha\xi_\beta\eta^i\eta^j \geq \lambda|\xi|^2|\eta|^2, \quad \forall \xi \in \mathbb{R}^n, \forall \eta \in \mathbb{R}^m.$$

This condition is known as the ***Legendre-Hadamard condition****.*

3. $A_{ij}^{\alpha\beta} \in C^0(\overline{\Omega})$ *satisfying the* ***Legendre-Hadamard condition****.*

Then for any ball $B(x_0, R) \subset \Omega$ *(*$R \leq R_0$ *small enough under condition* 3.*) and* $0 \leq \rho \leq R$*, the following Caccioppoli's inequality holds:*

$$\int_{B(x_0,\rho)} |Du|^2 dx \leq c\left(\frac{1}{(R-\rho)^2}\int_{B(x_0,R)\setminus B(x_0,\rho)} |u-\xi|^2 dx + R^2\int_{B(x_0,R)} |f|^2 dx + \int_{B(x_0,R)} |F|^2 dx\right).$$

For all $\xi \in \mathbb{R}^m$*, where under conditions* 1. *or* 2.

$$c = c(\lambda, \Lambda), \quad \Lambda := \sup|A|,$$

under condition 3.*, the constant* c *also depends on the modulus of continuity of* $A_{ij}^{\alpha\beta}$ *and* R_0*.*

The proof of the Theorem D.2 under the assumption 1. is very similar to the case of harmonic functions when $f_i = 0$. In the other case we get the desired bounds applying the Jensen inequality; under the assumptions 2. and 3. after applying the Garding's inequality. The details can be found in [GM12] 4.2.

Bibliography

[AFP00] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*, volume 254. Clarendon Press Oxford, 2000.

[All72] William K Allard. On the first variation of a varifold. *Annals of mathematics*, pages 417–491, 1972.

[Alm65] Frederick J Almgren. The theory of varifolds. *Mimeographed notes, Princeton*, page l2, 1965.

[CDK11] Gyula Csató, Bernard Dacorogna, and Olivier Kneuss. *The pullback equation for differential forms*, volume 83. Springer Science & Business Media, 2011.

[EG15] Lawrence Craig Evans and Ronald F Gariepy. *Measure theory and fine properties of functions*. CRC press, 2015.

[Fed14] Herbert Federer. *Geometric measure theory*. Springer, 2014.

[GM12] Mariano Giaquinta and Luca Martinazzi. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, volume 11 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, second edition, 2012.

[Lel12] Camillo De Lellis. Allard's interior regularity theorem: an invitation to stationary varifolds. http://www.math.uzh.ch/fileadmin/user/delellis/publikation/allard_35.pdf, 2012.

[Mag12] Francesco Maggi. *Sets of finite perimeter and geometric variational problems: an introduction to Geometric Measure Theory*, volume 135. Cambridge University Press, 2012.

[Sim83] Leon Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.