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Exotic spheres in dimensions 7 and 15

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Dissertação de mestrado apresentada ao Programa de Pós-graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Mestre em Matemática

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Trabalho aprovado por

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Rio de Janeiro, Brasil
9 de maio de 2022

Dedico este trabalho aos meus
pais & avós,
em sinal de gratidão.

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Resumo

A descoberta de 7-esferas exóticas por J. W. Milnor deixou surpresa a comunidade matemática pois respondia negativamente à *Conjectura de Poincaré Suave* em dimensão 7. Com efeito, uma *n-esfera exótica* é uma n-variedade fechada homeomorfa, mas não difeomorfa, à n-esfera canônica $\mathbb{S}^n$ e, ademais, a Conjectura de Poincaré Suave (em dimensão n) afirma ser difeomorfa à $\mathbb{S}^n$ toda n -variedade suave fechada com o mesmo tipo de homotopia de $\mathbb{S}^n$.

Posteriormente, N. Shimada adapatou as construções de Milnor e foi capaz de mostrar a existência de 15-esferas exóticas. Uma classificação completa foi dada por J. W. Milnor e M. Kervaire para dimensões $n \neq 3, 4$.

Surpreendentemente, J. W. Milnor foi capaz de mostrar que existe ao menos uma $(4k-1)$-esfera exótica para todo $k \geq 2$ e, portanto, a Conjectura de Poincaré Suave é inválida para um número infinito de dimensões. Se a conjectura de G. Wang e Z. Xu estiver correta, a Conjectura de Poincaré Suave é verdadeira apenas em dimensões $n = 1, 2, 3, 5, 6, 12, 56, 61$ e, possivelmente, $n = 4$.

A *Conjectura de Poincaré Generalizada*, no entanto, é válida em todas as dimensões, isto é, toda n-variedade suave fechada com o mesmo tipo de homotopia da n-esfera é homeomorfa à $\mathbb{S}^n$.

O objetivo do presente trabalho é apresentar os invariantes de Milnor e Shimada e usá-los para mostrar a existência de esferas exóticas em dimensões 7 e 15, respectivamente.

Palavras-chave: esferas exóticas, conjectura de poincaré suave, invariante de Milnor, invariante de Shimada.

Abstract

The discovery of exotic 7-spheres by J. W. Milnor was a surprise for the mathematical community for it showed the *Smooth Poincaré Conjecture* to be false in dimension 7. Indeed, an *exotic n-sphere* is a smooth closed n-manifold which is homeomorphic but not diffeomorphic to the standard n-sphere $\mathbb{S}^n$ and the *Smooth Poincaré Conjecture* (in dimension n) asserts that any smooth closed n-manifold with the same homotopy type of the n-sphere is diffeomorphic to $\mathbb{S}^n$

Shortly after that, N. Shimada adapted Milnor's construction and was able to show the existence of exotic 15-spheres. A complete classification was given by J. W. Milnor and M. Kervaire for dimensions $n \neq 3, 4$.

Surprisingly, J. W. Milnor established the existence of $(4k-1)$-exotic spheres for all $k \geq 2$ and, therefore, Smooth Poincaré Conjecture is false for an infinite number of dimensions. If a recent Conjecture by G. Wang and Z. Xu proves to be true, then Smooth Poncaré Conjecture holds only in dimensions $n = 1, 2, 3, 5, 6, 12, 56, 61$ and, possibly, $n = 4$.

On the other hand, the *Generalized Poincaré Conjecture* is true for all dimensions, i.e., every smooth closed n-manifold with the same homotopy type of the n-sphere is homeomorphic to $\mathbb{S}^n$.

The goal of this work is to present Milnor and Shimada invariants and use them to show the existence of exotic spheres in dimensions 7 and 15, respectivelly.

Keywords: exotic spheres, smooth poincaré conjecture, Milnor's invariant, Shimada's invariant.

Contents

1 Introduction

The goal of this introductory chapter is twofold. On the one hand, to set some terminology and notation used throughout. On the other hand, to present the reader some *exoticness* phenomena in the differential topology of smooth manifolds, first by exploring exotic spheres and, finally, in a more general setting.

1.1 Terminological Convetions

Recall that a *differentiable structure* on a locally euclidean, Hausdorff, second countable topological space M is a maximal $\mathscr{C}^\infty$ atlas.

A *differentiable manifold* (also called *smooth manifold*) is a pair $(M, \mathcal{A})$ where M is a locally euclidean, Hausdorff, second countable topological space and $\mathcal{A}$ is a differentiable structure on it. When there is no risk of confusion, we simply say M *is smooth manifold*, omitting any reference to the differentiable structure.

When employing the word *manifold*, we mean a *smooth manifold* that may have a non-empty boundary. In the passages where the underlying manifold is required to have no boundary, it will be explicitly stated. As usual, a *closed manifold* is a compact manifold without boundary.

We say that a manifold is *orientable* if there exists an *oriented atlas* for it. Recall that an oriented atlas is a smooth atlas for which the transition functions are orientation preserving diffeomorphisms between open sets of the euclidean space $\mathbb{R}^n$, where $n = dim(M)$. An *orientation* for a manifold is a maximal oriented atlas. An *oriented manifold* is a manifold together with a orientation. We usually drop any mention to the chosen orientation and simply say *an oriented manifold M*.

If M is a connected orientable manifold, then it has precisely two orientations. When M is oriented with one of them, we write $-M$ to mean the same manifold oriented with the other atlas, i.e., the *inverse orientation* of M. In general, if M is an oriented manifold, the symbol $-M$ means that we are taking the inverse orientation in each connected component of M.

With the terminology introduced, for an oriented manifold M, the map $id : M \to -M$ is an orientation reversing diffeomorphism.

1.2 Historical Note

An *exotic n-sphere* is a manifold M which is homeomorphic but not diffeomorphic to $\mathbb{S}^n$, where the later manifold is given its standard differentiable structure induced by the immersion into $\mathbb{R}^{n+1}$.

The first mathematician to come up with exotic 7-spheres was John W. Milnor in his 1956 paper [Mil56]. In the subsequent year, Nobuo Shimada adapted Milnor's construction and obtained exotic 15-spheres, see [Shi57]. It is the object of this work to present such constructions and to show the existence of exotic spheres in these dimensions.

Once we know that exotic spheres exist, it is natural to ask how many of them are there up to diffeomorphism. This classification problem becomes simpler if we consider instead *oriented exotic spheres*. An *oriented exotic n-sphere* is an oriented n-manifold which is homeomorphic but not diffeomorphic to $\mathbb{S}^n$. We then seek to classify oriented exotic spheres up to orientation preserving diffeomorphism. This is the original approach of Milnor and Kervaire in [KM63]. In this paper, the authors prove that the set Θ_n of *h-cobordism classes*[1] of *homotopy n-spheres*[2] is finite for $n \neq 3$.

It is not immediately clear what Θ_n has to do with the classification problem for oriented exotic sphere. For n at least 5, Smale's *h-cobordism theorem* [Sma62, Theorem 1.1] implies that the map

$$\begin{array}{rccc} \Lambda_n : & \Theta_n & \longrightarrow & \pi_n^{SO} \\ & [M]_{h-cob} & \longmapsto & [M]_{diff^+} \end{array}$$

is a well-defined bijective correspondence between Θ_n and the set

$$\pi_n^{SO} \quad := \quad \{\, [M]_{diff^+} \mid M \text{ is a homotopy } n\text{-sphere} \,\}$$

of orientation-preserving diffeomorphism classes of homotopy n-spheres. By Smale's solution of *Generalized Poincaré Conjecture* [Sma61, Theorem A], for n at least 5, we have

$$\pi_n^{SO} \;=\; \{\, [M]_{diff^+} \mid M \text{ oriented, homeomorphic to } \mathbb{S}^n \,\} \,.$$

In particular, writing $Exot_n^{SO}$ for the set of orientation-preserving diffeomorphism classes of oriented exotic n-spheres and noticing that $[\mathbb{S}^n]_{diff^+} = [-\mathbb{S}^n]_{diff^+}$ for the standard n-sphere, we get

$$\Theta_n \;\longleftrightarrow\; \pi_n^{SO} \;=\; Exot_n^{SO} \sqcup \{[\mathbb{S}^n]_{diff^+}, [-\mathbb{S}^n]_{diff^+}\} \;=\; Exot_n^{SO} \sqcup \{[\mathbb{S}^n]_{diff^+}\} \,.$$

[1] Two closed, oriented n-manifolds M_1 and M_2 are *h-cobordant* if there is a compact, oriented $(n+1)$-manifold W for which $\partial W = M_1 \sqcup -M_2$ and W deformation retracts onto both M_1 and $-M_2$. This is an equivalence relation on the class of closed, oriented n-manifolds. In particular, we can consider $[M]_{h-cob}$ the h-cobordism class of any closed, oriented n-manifold M.

[2] A *homotopy n-sphere* is a closed, oriented n-manifold with the same homotopy type of $\mathbb{S}^n$.

This is the link between Θ_n and the classification of oriented exotic spheres. It follows that there are $\#\, Exot_n^{SO} = \#\,\Theta_n - 1$ oriented exotic n-spheres up to orientation preserving diffeomorphism for $n \geq 5$.

We now outline the proof of the main theorem of [KM63], namely that Θ_n is finite for $n \neq 3$. The connected sum gives the set Θ_n the structure of an abelian group. The neutral element is the class $[\mathbb{S}^n]_{h-cob}$ of the standard n-sphere. The inverse of $[M]_{h-cob}$ is obtained by reversing orientation, i.e., it is $[-M]_{h-cob}$. Milnor and Kervaire introduce the subgroup $bP_{n+1} \leq \Theta_n$ of h-cobordism classes of homotopy n-spheres that bound a compact, oriented, *parallelizable*[3] $(n+1)$-manifold. They show that the quotient group Θ_n/bP_{n+1} is finite and that, in addition, bP_{n+1} is trivial, for n even, and finite cyclic, for $n \neq 3$ odd. Besides, the authors were also able to prove more refined results concerning the order of bP_{n+1} and also computed the order of some Θ_n for small values of n, as one can see by the table

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\#\,\Theta_n$	1	1	?	1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16

herein [KM63].

Remark 1.1. By Perelman's solution of Poincaré's Conjecture, the group Θ_3 is trivial, so $\#\,\Theta_3 = 1$.

From the preceding discussion, it follows that there is a finite number of oriented exotic spheres up to orientation preserving diffeomorphism, for $n \geq 5$. It follows that the number of (unoriented) exotic n-spheres is also finite ($n \geq 5$). Write $Exot_n = \{\, [M]_{diff} \mid M \text{ is an exotic } n\text{-sphere} \,\}$ for the set of diffeomorphism classes of exotic n-spheres and observe that every exotic n-sphere M has precisely two orientations. Denote by $\tilde{M}$ the oriented exotic sphere obtained by choosing an orientation for M. There are only two possibilities: $\tilde{M}$ either admits an orientation reversing diffeomorphism or there is no orientation reversing diffeomorphism of $\tilde{M}$ onto itself. In the first case, $[\tilde{M}]_{diff^+} = [-\tilde{M}]_{diff^+}$ and $[\tilde{M}]_{diff^+} \neq [-\tilde{M}]_{diff^+}$, in the second. Let $Exot_n^=$ be the set of diffeomorphism classes $[M]_{diff}$ of exotic n-spheres for which $\tilde{M}$ admits an orientation reversing diffeomorphism. Similarly, let $Exot_n^{\neq}$ stand for the set of diffeomorphism classes $[M]_{diff}$ of n-spheres for which $\tilde{M}$ does not admit an orientation reversing diffeomorphism. Then, $Exot_n = Exot_n^= \sqcup Exot_n^{\neq}$ and

$$\#\, Exot_n \;=\; \#\, Exot_n^= \;+\; \#\, Exot_n^{\neq} \;\leq\; \#\, Exot_n^= \;+\; 2 \times \#\, Exot_n^{\neq} \;=\; \#\, Exot_n^{SO}\,.$$

[3] A n-manifold X is parallelizable if its tangent bundle is trivial, i.e., X-isomorphic to $(X \times \mathbb{R}^n, \pi_1, X)$.

For n at least 5, we have already seen that $\#\, Exot_n^{SO}$ is finite. Therefore, $\#\, Exot_n$ is finite for $n \geq 5$, as we wanted to show. For $n = 1, 2, 3$, two closed smooth n-manifolds are homeomorphic if ad only if they are diffeomorphic, so there are no exotic n-spheres for $n = 1, 2, 3$. This proves the following result.

Proposition 1.2. *For $n \neq 4$, there is a finite number of exotic n-spheres up to diffeomorphism.*

Remark 1.3. Recall that Poncaré Conjecture asserts that an *unoriented homotopy n-sphere*[4] is homeomorphic to $\mathbb{S}^n$. It is true in all dimensions. In low dimensions ($n \leq 3$), homeomorphism classification and diffeomorphism classification coincide, so Poincaré Conjecture implies that unoriented homotopy n-spheres are diffeomorphic to $\mathbb{S}^n$. This implies that there is neither exotic spheres nor oriented exotic spheres in low dimensions. In high dimensions ($n \geq 5$), the *h-cobordism theorem* allowed us to establish a bijection between Θ_n and $Exot_n^{SO} \sqcup \{[\mathbb{S}^n]_{diff+}\}$, i.e, the set of orientation-preserving diffeomorphism classes of oriented exotic n-spheres together with the orientation-preserving diffeomorphism class of the standard n-sphere. It turns out that the h-cobordism theorem fails in dimension four, as the counter-example found by S. K. Donaldson shows (see [Don87]). Thus, in dimension four, there is no warranty that we have a bijective correspondence between Θ_4 and $Exot_4^{SO} \sqcup \{[\mathbb{S}^4]_{diff+}\}$.

Remark 1.4. The classification of exotic n-spheres up to diffeomorphism is equivalent to the classification of *exotic differentiable structures* of $\mathbb{S}^n$ up to diffeomorphism[5]. Thus, by the proposition above, there is a finite number of diffeomorphism classes of differentiable structures of $\mathbb{S}^n$, for $n \neq 4$.

The following questions remain unanswered. Are there exotic 4-spheres? If so, how many are there up to diffeomorphism: a finite or infinite number? In case the second possibility holds, we have a countable or uncountable number of diffeomorphism classes of exotic 4-spheres?

Coming back to oriented exotic n-spheres ($n \geq 5$), it follows from

$$\#\, Exot_n^{=} \;+\; 2 \times \#\, Exot_n^{\neq} \;=\; \#\, Exot_n^{SO} \;=\; \#\, \Theta_n \;-\; 1\,.$$

that $\#\, Exot_n^{=} \neq 0$ whenever $\#\, \Theta_n$ is even. Thus, we obtain the following result.

[4] In other words, a closed smooth n-manifold with the same homotopy type as $\mathbb{S}^n$.

[5] Two differentiable structures $\mathcal{A}_1$ and $\mathcal{A}_2$ of a manifold X are said *diffeomorphic* if the manifolds $(X, \mathcal{A}_1)$ and $(X, \mathcal{A}_2)$ are diffeomorphic. An *exotic differentiable structure* of $\mathbb{S}^n$ is a differentiable structure which is not diffeomorphic to the standard one.

Corollary 1.5. *Let $n \geq 5$. If $\#\,\Theta_n$ is even, there is an oriented exotic n-sphere which admits an orientation reversing diffeomorphism.*

From the preceding table and the corollary above, it follows that there is an oriented exotic n-sphere admitting an orientation reversing diffeomorphism for $n = 7, 8, 9, 10, 11, 14, 15, 16, 17, 18$.

One can further ask for which values of n there are oriented exotic n-spheres. Milnor proved that, for $n = 4k - 1$ $(k \geq 2)$, there is at least one oriented exotic n-sphere [Mil07, Corollary 4.3]. In particular, there are infinite values of n for which $\mathbb{S}^n$ has a non-standard differentiable structure. In 2017, G. Wang and Z. Xu went onto show that the only odd dimensional spheres with a unique smooth structure are $\mathbb{S}^1$, $\mathbb{S}^3$, $\mathbb{S}^5$ and $\mathbb{S}^{61}$ [WX17, Corollary 1.13]. They further proved that, for $5 \leq n \leq 61$, the only dimensions for which $\mathbb{S}^n$ has a unique smooth structure are $n = 5, 6, 12, 56, 61$ [WX17, Corollary 1.15] and conjecture that these values of n give the only dimensions greater than four for which the n-sphere has a unique smooth structure. If their conjecture proves to be right, then there are only a finite number of n for which $\mathbb{S}^n$ has a unique smooth structure, namely $n = 1, 2, 3, 5, 6, 12, 56, 61$ and, possibly, $n = 4$.

It is also possible to study differential geometric aspects of exotic sphere. We refer the reader to [Wra97], [GZ00], [Boy+05] and [JW08].

1.3 Exotic Smooth Manifolds

Let M be a smooth n-manifold. We say that a smooth n-manifold X is an *exotic* M if it is homeomorphic but not diffeomorphic to M. In this case, we seek to classify exotic M's up to diffeomorphism. If, in addition, M is oriented, an *oriented exotic M* is an oriented n-manifold which homeomophic but not diffeomorphic to M. We then want to classify oriented exotic M's up to orientation preserving diffeomorphism.

For $n \neq 4$, there are no exotic $\mathbb{R}^n$'s. When $n = 4$, there are uncountable many exotic $\mathbb{R}^4$'s [DF92, Corollary 4.1].

We have already mentioned that it is not known whether there are exotic 4-spheres. If they exist after all, one may ask the following questions. Is there an exotic $\mathbb{S}^4$ for which the complement of a point of is diffeomorphic to the standard $\mathbb{R}^4$? Is there an exotic $\mathbb{S}^4$ for which the complement of a point is diffeomorphic to an exotic $\mathbb{R}^4$. These questions also remain unanswered.

In their 1962 paper [EK62], J. Eells and N. H. Kuiper introduced the invariant μ, defined for $(4k - 1)$-manifolds satisfying certain conditions. It turns out that μ is computable for a reasonable class of manifolds.

In [CN21], D. Crowley and J. Nordstöm gave the first examples *exotic G_2-manifolds*, i.e., closed, Riemannian 7-manifolds with holonomy G_2 that are homeomorphic but not diffeomorphic. To distinguish between certain G_2-manifolds, they made use a generalized Eells-Kuiper invariant introduced by them in a previous paper, namely [CN19].

1.4 Structure of this article

The goal of this work is to present Milnor and Shimada invariants and use them to show the existence of oriented exotic spheres in dimensions 7 and 15, respectively.

In Chapter 1, we introduce the necessary prerequisites. The sections on *vector bundles* is of central importance, for our exotic spheres appear as sphere bundles of vector bundles.

In Chapter 2, we study *Pontryagin classes*, which are isomorphism invariants of vector bundles. We also introduce *Pontryagin numbers* and compute them for manifolds of interest.

Finally, in Chapter 3 we define Milnor and Kervaire invariants and, among other results, show the existence of exotic spheres.

2 Background Material

2.1 Quaternions and Octonions

We can define a product operation on $\mathbb{R}^2$ such that it becomes a field, namely the field $\mathbb{C}$ of *complex numbers*. As sets, $\mathbb{C} = \mathbb{R}^2$, but we write $\mathbb{C}$ instead of $\mathbb{R}^2$ to emphasise that we are considering the aforementioned product operation on the vector space $\mathbb{R}^2$. We adopt the same convention for *quaternions* and *octonions*, as we shall se below.

Let us introduce the *associative* real algebra of quaternions. We define a product between the elements of the canonical basis $\{e_1, e_2, e_3, e_4\}$ accordingly to the multiplication table:

$$
\begin{array}{c|cccc}
\cdot & e_1 & e_2 & e_3 & e_4 \\
\hline
e_1 & e_1 & e_2 & e_3 & e_4 \\
e_2 & e_2 & -e_1 & e_4 & -e_3 \\
e_3 & e_3 & -e_4 & -e_1 & e_2 \\
e_4 & e_4 & e_3 & -e_2 & -e_1
\end{array} \quad ,
$$

where it is understood that the element (i-th line, j-th column) is the result of the product $e_i \cdot e_j$. We extend such product to $\mathbb{R}^4$ by putting

$$
x \cdot y \;=\; \left(\sum_{i=1}^{4} x_i e_i \right) \cdot \left(\sum_{j=1}^{4} y_j e_j \right) \;:=\; \sum_{i=1}^{4} \sum_{j=1}^{4} x_i y_j \; e_i \cdot e_j \; .
$$

This product turns $\mathbb{R}^4$ into an associative real algebra with unity $1 = e_1$. We say that $\mathbb{H} := \mathbb{R}^4$ is the set of *quaternions*. Just as in the case of complex numbers, we write $\mathbb{H}$ instead of $\mathbb{R}^4$ to emphasise the product operation on the later set. We can also think of the product $x \cdot y$ as the value that a linear map $M_4(x) : \mathbb{R}^4 \to \mathbb{R}^4$ takes when applied to y. In fact, defining

$$
\begin{array}{cccc}
M_4 : & \mathbb{R}^4 & \longrightarrow & Lin(\mathbb{R}^4; \mathbb{R}^4) \\
 & x & \longmapsto & (M_4(x) : v \mapsto x \cdot v)
\end{array} \quad ,
$$

we automatically have $x \cdot y = M_4(x)(y)$. The matrix representation of $M_4(x)$ with respect to the canonical basis of $\mathbb{R}^4$ is

$$
\mathcal{M}_4(x) \;:=\; \begin{pmatrix}
x_1 & -x_2 & -x_3 & -x_4 \\
x_2 & x_1 & -x_4 & x_3 \\
x_3 & x_4 & x_1 & -x_2 \\
x_4 & -x_3 & x_2 & x_1
\end{pmatrix}
$$

and the map

$$\begin{array}{rccc} \mathcal{M}_4 : & \mathbb{H} & \longmapsto & Mat_{4\times 4}(\mathbb{R}) \\ & x & \longmapsto & \mathcal{M}_4(x) \end{array}$$

is an isomorphism of the algebra $\mathbb{H}$ onto a subalgebra of $Mat_{4\times 4}(\mathbb{R})$.

We will now study the *non-associative* real algebra of octonions. As before, we begin by defining a product between the elements of the canonical basis $\{e_1, \cdots, e_8\}$:

$$\begin{array}{c|cccccccc} \cdot & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ \hline e_1 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ e_2 & e_2 & -e_1 & e_4 & -e_3 & e_6 & -e_5 & -e_8 & e_7 \\ e_3 & e_3 & -e_4 & -e_1 & e_2 & e_7 & e_8 & -e_5 & -e_6 \\ e_4 & e_4 & e_3 & -e_2 & -e_1 & e_8 & -e_7 & e_6 & -e_5 \\ e_5 & e_5 & -e_6 & -e_7 & -e_8 & -e_1 & e_2 & e_3 & e_4 \\ e_6 & e_6 & e_5 & -e_8 & e_7 & -e_2 & -e_1 & -e_4 & e_3 \\ e_7 & e_7 & e_8 & e_5 & -e_6 & -e_3 & e_4 & -e_1 & -e_2 \\ e_8 & e_8 & -e_7 & e_6 & e_5 & -e_4 & -e_3 & e_2 & -e_1 \end{array} \quad ,$$

where it is understood that the element in (i-th line, j-th column) is the result of the product $e_i \cdot e_j$. Extend the product to $\mathbb{R}^8$ by setting

$$x \cdot y \;=\; \left(\sum_{i=1}^{8} x_i e_i\right) \cdot \left(\sum_{j=1}^{8} y_j e_j\right) \;:=\; \sum_{i=1}^{8}\sum_{j=1}^{8} x_i y_j \, e_i \cdot e_j \; .$$

This product turns $\mathbb{R}^8$ into a non-associative real algebra with unity $1 = e_1$. To see that it is not associative, note that the multiplication table gives

$$(e_2 \cdot e_3).e_5 \;=\; e_4 \cdot e_5 \;=\; e_8 \;\neq\; -e_8 \;=\; e_2 \cdot e_7 \;=\; e_2 \cdot (e_3 \cdot e_5) \; .$$

We say that $\mathbb{O} := \mathbb{R}^8$ is the set of *octonions*. Just as in the case of complex numbers, we write $\mathbb{O}$ instead of $\mathbb{R}^8$ to emphasise the product operation on the later set. We can also think of the product $x \cdot y$ as the value that a linear map $M_8(x) : \mathbb{R}^8 \to \mathbb{R}^8$ takes when applied to y. Indeed, defining

$$\begin{array}{rccc} M_8 : & \mathbb{R}^8 & \longrightarrow & Lin(\mathbb{R}^8; \mathbb{R}^8) \\ & x & \longmapsto & (M_8(x) : v \mapsto x \cdot v) \end{array} \quad ,$$

we readily get $x \cdot y = M_8(x)(y)$. The matrix representation of $M_8(x)$ with respect to the canonical basis of $\mathbb{R}^8$ is

$$\mathcal{M}_8(x) \quad := \quad \begin{pmatrix} x_1 & -x_2 & -x_3 & -x_4 & -x_5 & -x_6 & -x_7 & -x_8 \\ x_2 & x_1 & -x_4 & x_3 & -x_6 & x_5 & x_8 & -x_7 \\ x_3 & x_4 & x_1 & -x_2 & -x_7 & -x_8 & x_5 & x_6 \\ x_4 & -x_3 & x_2 & x_1 & -x_8 & x_7 & -x_6 & x_5 \\ x_5 & x_6 & x_7 & x_8 & x_1 & -x_2 & -x_3 & -x_4 \\ x_6 & -x_5 & x_8 & -x_7 & x_2 & x_1 & x_4 & -x_3 \\ x_7 & -x_8 & -x_5 & x_6 & x_3 & -x_4 & x_1 & x_2 \\ x_8 & x_7 & -x_6 & -x_5 & x_4 & x_3 & -x_2 & x_1 \end{pmatrix} .$$

Unlike the case of quaternions, the map

$$\begin{array}{rccc} \mathcal{M}_8 : & \mathbb{O} & \longmapsto & Mat_{8\times 8}(\mathbb{R}) \\ & x & \longmapsto & \mathcal{M}_8(x) \end{array}$$

is *not* an isomorphism of the algebra $\mathbb{O}$ onto a subalgebra of $Mat_{4\times 4}(\mathbb{R})$, for otherwise $\mathbb{O}$ would be an associative algebra. However, it is known that any subalgebra of $\mathbb{O}$ generated by two elements is associative.

Let $\mathbb{A}$ be either quaternionic or octonionic algebra and write a for its dimension as a real vector space. We have linear maps

$$\begin{array}{rccc} Re : & \mathbb{A} & \longrightarrow & \mathbb{R} \\ & x = (x_1, \cdots, x_a) & \longmapsto & x_1 \end{array}$$

and

$$\begin{array}{rccc} \mathcal{C} : & \mathbb{A} & \longrightarrow & \mathbb{A} \\ & x = (x_1, x_2, \cdots, x_a) & \longmapsto & (x_1, -x_2, \cdots, -x_a) \end{array} .$$

We call $Re(x)$ the *real part* of x and $\mathcal{C}(x)$ the *conjugate* of x. We also adopt the classical notation $\overline{x}$ for the conjugate of x. It is usual to omit any reference to the product and simply write xy. Another permitted omission is the one of e_1, i.e., one can simply write

$$x \quad = \quad x_1 + x_2 e_2 + \cdots + x_a e_a \, .$$

We justify this convention by observing that $x \mapsto xe_1$ gives an embedding of the algebra $\mathbb{R}$ into $\mathbb{A}$, so we regard the subalgebra $\{xe_1 | x \in \mathbb{R}\} \subset \mathbb{A}$ just as $\mathbb{R}$. Under this identification, we have $\overline{x} = x$, whenever $x \in \mathbb{R} \subset \mathbb{A}$. We now collect some arithmetic properties of $\mathbb{A}$ that will be important for future work.

Proposition 2.1. *Let $|| \cdot ||$ be the euclidean norm in $\mathbb{A} = \mathbb{R}^a$. The following properties hold:*

1. $||x \cdot y|| \;=\; ||x||\,||y||$, *for all* $x, y \in \mathbb{A}$

2. $x \cdot \overline{x} \;=\; \overline{x} \cdot x \;=\; ||x||^2$, *for all* $x \in \mathbb{A}$

3. $\overline{x \cdot y} \;=\; \overline{y} \cdot \overline{x}$, *for all* $x, y \in \mathbb{A}$

4. $\overline{\overline{x}} \;=\; x$, *for all* $x \in \mathbb{A}$

5. $Re(x \cdot y \cdot \overline{x}) \;=\; ||x||^2 Re(y)$, *for all* $x, y \in \mathbb{A}$[1]

Proof. Item (4) is obvious, for it follows immediately from the definition of the conjugate. Let us prove item (3). Note that the assertion holds for the elements in the canonical basis of $\mathbb{A}$. Indeed, for $i \neq j$ with $i, j \geq 2$, we have $e_i \cdot e_j \neq e_1 = 1$ and, therefore,

$$\overline{e_i \cdot e_j} \;=\; -e_i \cdot e_j \;=\; -(-e_j \cdot e_i) \;=\; e_j \cdot e_i \;=\; (-e_j) \cdot (-e_i) \;=\; \overline{e_j} \cdot \overline{e_i} \;.$$

For $i = 1$, we have

$$\overline{e_i \cdot e_j} \;=\; \overline{e_i \cdot e_j} \;=\; \overline{e_j} \;=\; \overline{e_j} \cdot e_1 \;=\; \overline{e_j} \cdot \overline{e_i} \;.$$

Similarly, when $j = 1$, we get

$$\overline{e_i \cdot e_j} \;=\; \overline{e_i \cdot e_1} \;=\; \overline{e_i} \;=\; e_1 \cdot \overline{e_i} \;=\; \overline{e_j} \cdot \overline{e_i} \;.$$

For $i = j$, the multiplication table give $e_1 \cdot e_1 = e_1$ and $e_i \cdot e_i = -e_1$ ($i \geq 2$). Hence,

$$\overline{e_1 \cdot e_1} \;=\; \overline{e_1} \;=\; e_1 \;=\; e_1 \cdot e_1 \;=\; \overline{e_1} \cdot \overline{e_1}$$

and

$$\overline{e_i \cdot e_i} \;=\; \overline{-e_1} \;=\; -e_1 \;=\; = \; e_i \cdot e_i \;=\; (-e_i) \cdot (-e_i) \;=\; \overline{e_i} \cdot \overline{e_i} \;,$$

for $i \geq 2$. Finally, for $x, y \in \mathbb{A}$, we have

$$\begin{aligned}
\overline{x \cdot y} \;&=\; \overline{\left(\sum_{i=1}^{a} x_i e_i\right) \cdot \left(\sum_{j=1}^{a} y_j e_j\right)} \;=\; \overline{\sum_{i=1}^{a}\sum_{j=1}^{a} x_i y_j e_i \cdot e_j} \;=\; \sum_{i=1}^{a}\sum_{j=1}^{a} x_i y_j \overline{e_i \cdot e_j} \\
&=\; \sum_{i=1}^{a}\sum_{j=1}^{a} x_i y_j \overline{e_j} \cdot \overline{e_i} \;=\; \sum_{j=1}^{a}\sum_{i=1}^{a} y_j x_i \overline{e_j} \cdot \overline{e_i} \;=\; \left(\sum_{j=1}^{a} y_j \overline{e_j}\right) \cdot \left(\sum_{i=1}^{a} x_i \overline{e_i}\right) \;=\; \overline{y} \cdot \overline{x} \;.
\end{aligned}$$

---

[1] Associativity of quaternionic algebra implies that the expression $x \cdot y \cdot \overline{x}$ is well-defined, when quaternionic multiplication is understood. Though octonionic multiplication is not associative, any subalgebra of $\mathbb{O}$ generated by two elements is associative. Thus, for $v \neq 0$, since we have $\overline{v} = ||v||^2 v^{-1} \in \mathbb{O}\langle u, v\rangle$, the associative subalgebra generated by u and v, it follows that the expression $x \cdot y \cdot \overline{x}$ is also well-defined for octonionic multiplication. Of course, the same expression also makes sense for $v = 0$.

Item (2) follows from direct computation. Indeed, with respect to euclidean inner product on $\mathbb{R}^a$, all columns of $\mathcal{M}_a(x)^t$ are orthogonal to $\overline{x}$, except for the first, which is exactly $\overline{x}$. Thus,

$$x \cdot \overline{x} \;=\; \overline{x}\mathcal{M}_a(x)^t \;=\; ||x||^2 e_1 + 0e_2 + \cdots + 0e_a \;=\; ||x||^2 \,.$$

Putting $y := \overline{x}$, we have

$$\overline{x} \cdot x \;=\; y \cdot \overline{y} \;=\; ||y||^2 \;=\; ||\overline{x}||^2 \;=\; ||x||^2 \;.$$

Item (1) follows from the items we have already proved. In fact, note that

$$||x \cdot y||^2 \;=\; (x \cdot y) \cdot \overline{x \cdot y} \;=\; x \cdot (y \cdot \overline{y}) \cdot \overline{x} \;=\; ||y||^2 (x \cdot \overline{x}) \;=\; ||x||^2 \, ||y||^2 \;.$$

Then, taking the square root in both sides of the equation gives $||x \cdot y|| = ||x|| \, ||y||$.

We now prove item (5). In terms of matrices, we have

$$(x \cdot y)^t \;=\; \mathcal{M}_a(x) \, y^t$$

and

$$\mathcal{M}_a(\overline{x}) \;=\; \mathcal{M}_a(x)^t \;,$$

where A^t stands for the transpose of the matrix A. In particular,

$$(x \cdot y \cdot \overline{x})^t \;=\; \mathcal{M}_a(x)(y \cdot \overline{x})^t \;=\; \mathcal{M}_a(x) \, \mathcal{M}_a(y) \, \overline{x}^t \;.$$

If $\mathcal{M}_a(x) \, \mathcal{M}_a(y) = (\alpha_{i,j})_{i,j \in [\![1,a]\!]}$, then

$$Re(x \cdot y \cdot \overline{x}) \;=\; \alpha_{1,1} x_1 \;-\; \sum_{i=2}^{a} \alpha_{1,i} x_i \;.$$

For $a = 4$, we have

$$\begin{cases} \alpha_{1,1} &= x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 \\ \alpha_{1,2} &= -x_1y_2 - x_2y_1 - x_3y_4 + x_4y_3 \\ \alpha_{1,3} &= -x_1y_3 + x_2y_4 - x_3y_1 - x_4y_2 \\ \alpha_{1,4} &= -x_1y_4 - x_2y_3 + x_3y_2 - x_4y_1 \end{cases}$$

and, likewise, for $a = 8$ we get

$$\begin{cases} \alpha_{1,1} &= x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 - x_5y_5 - x_6y_6 - x_7y_7 - x_8y_8 \\ \alpha_{1,2} &= -x_1y_2 - x_2y_1 - x_3y_4 + x_4y_3 - x_5y_6 + x_7y_8 - x_8y_7 \\ \alpha_{1,3} &= -x_1y_3 + x_2y_4 - x_3y_1 - x_4y_2 - x_5y_7 - x_6y_8 + x_7y_5 + x_8y_6 \\ \alpha_{1,4} &= -x_1y_4 - x_2y_3 + x_3y_2 - x_4y_1 - x_5y_8 + x_6y_7 - x_7y_6 + x_8y_5 \\ \alpha_{1,5} &= -x_1y_5 + x_2y_6 + x_3y_7 + x_4y_8 - x_5y_1 - x_6y_2 - x_7y_3 - x_8y_4 \\ \alpha_{1,6} &= -x_1y_6 - x_2y_5 + x_3y_8 - x_4y_7 + x_5y_2 - x_6y_1 + x_7y_4 - x_8y_3 \\ \alpha_{1,7} &= -x_1y_7 - x_2y_8 - x_3y_5 + x_4y_6 + x_5y_3 - x_6y_4 - x_7y_1 + x_8y_2 \\ \alpha_{1,8} &= -x_1y_8 + x_2y_7 - x_3y_6 - x_4y_5 + x_5y_4 + x_6y_3 - x_7y_2 - x_8y_1 \end{cases} .$$

In both cases, a direct computation shows that

$$\alpha_{1,1}x_1 - \sum_{i=2}^{a} \alpha_{1,i}x_i \;=\; ||x||^2 y_1 \;=\; ||x||^2 Re(y) \;.$$

Therefore,

$$Re(x \cdot y \cdot \overline{x}) \;=\; \alpha_{1,1}x_1 - \sum_{i=2}^{a} \alpha_{1,i}x_i \;=\; ||x||^2 Re(y) \;,$$

as we wanted to prove.

□

Corollary 2.2. *We have*

1. $v^{-1} \;=\; \overline{v}$*, for all* $v \in \mathbb{S}^{a-1} \subset \mathbb{A}$
2. $Re(v \cdot x \cdot v^{-1}) \;=\; Re(x)$*, for all* $x, v \in \mathbb{A}$ *with* $v \;\neq\; 0$

Proof. For item (1), observe that $v \cdot \overline{v} \;=\; \overline{v} \cdot v \;=\; ||v||^2 \;=\; 1$ and, consequently, $v^{-1} \;=\; \overline{v}$. Let us prove item (2). It holds for all v with norm 1. Indeed, from $||v|| = 1$, $v^{-1} = \overline{v}$ and the last item of the proposition above we get

$$Re(v \cdot x \cdot v^{-1}) \;=\; Re(v \cdot x \cdot \overline{v}) \;=\; ||v||^2 Re(x) \;=\; Re(x) \;.$$

Now, define $u := v/||v||$ and notice that $v \cdot x \cdot v^{-1} = u \cdot x \cdot u^{-1}$ and $||u|| = 1$. Thus,

$$Re(v \cdot x \cdot v^{-1}) \;=\; Re(u \cdot x \cdot u^{-1}) \;=\; Re(x) \;,$$

as we wanted to prove.

□

Define $\mathbb{A}^* := \mathbb{A} - \{0\}$. The maps

$$\begin{array}{rccc} L: & \mathbb{A}^* & \longrightarrow & GL(a,\mathbb{R}) \\ & u & \longmapsto & (L(u) : v \mapsto u \cdot v) \end{array} \quad \text{and} \quad \begin{array}{rccc} R: & \mathbb{A}^* & \longrightarrow & GL(a,\mathbb{R}) \\ & u & \longmapsto & (R(u) : v \mapsto v \cdot u) \end{array}$$

given by left and right multiplication, respectively, are smooth. Of course, they take values in $GL(a, \mathbb{R})$ and invetibility follows from

$$L(u^{-1}) \circ L(u) \;=\; L(u) \circ L(u^{-1}) \;=\; id_{\mathbb{R}^a} \;=\; R(u^{-1}) \circ R(u) \;=\; R(u) \circ R(u^{-1}) \;,$$

which, in turn, are consequences of associativity[2]. Now, since $\mathbb{A}^*$ is connected, its images $L(\mathbb{A}^*)$ and $R(\mathbb{A}^*)$ lie in a connected component of $GL(a, \mathbb{R})$. Clearly, $L(e_1) = R(e_1) = id_{\mathbb{R}^a}$,

[2] The associativity of either the quaternionic multiplication or any subalgebra of $\mathbb{O}$ generated by two elements.

so the images of $\mathbb{A}^*$ both lie in the set of invertible linear maps with positive determinant. Moreover, if $u \in \mathbb{S}^{a-1} \subset \mathbb{A}^*$, the maps $L(u)$ and $R(u)$ are norm preserving, for

$$||L(u)v|| = ||u \cdot v|| = ||u||||v|| = ||v|| = ||v||||u|| = ||v \cdot u|| = ||R(u)v|| \ .$$

Hence, $L(\mathbb{S}^{a-1})$ and $R(\mathbb{S}^{a-1})$ both lie in $SO(a)$. Therefore, for every pair of integers $(h,j) \in \mathbb{Z} \times \mathbb{Z}$ and every $u \in \mathbb{S}^{a-1}$, the map $f^a_{h,j}(u) := L(u^h) \circ R(u^j)$ lies in $SO(a)$.

2.2 Smooth Homotopy Groups

Let M and N be smooth manifolds. We say that two smooth application $f_0, f_1 : M \to N$ are *smoothly homotopic* if there is a smooth map $H : M \times [0,1] \to N$ such that

$$H(x,0) = f_0(x) \qquad \text{and} \qquad H(x,1) = f_1(x) \ ,$$

for all $x \in M$. Given $f, g \in \mathscr{C}^\infty(M,N)$, we write $f \simeq_\infty g$ to indicate that f and g are smoothly homotopic. This is an equivalence relation in $\mathscr{C}^\infty(M,N)$ [Lee13, Lemma 6.28]. So we can consider $[M,N]_\infty$ the set of equivalence classes $[f]_\infty$.

If N is a manifold without boundary, two smooth maps $f_0, f_1 : M \to N$ are homotopic if and only if they are smoothly homotopic and, furthermore, any continuous map $g_0 : M \to N$ is homotopic to a smooth one [Lee13, Theorems 6.26 and 6.29]. Thus, we get the following

Theorem 2.3. *For N a manifold without boundary, the association*

$$\begin{array}{cccc} \iota_\infty & [M,N]_\infty & \longrightarrow & [M,N] \\ & [f]_\infty & \longmapsto & [f] \end{array}$$

is bijective. In particular, when $M = \mathbb{S}^n$ and N is connected, we have a group isomorphism $\pi_n^\infty(N) \to \pi_n(N)$.

Recall that $f^a_{h,j}(u) := L(u^h) \circ R(u^j)$, where $u \in \mathbb{S}^{a-1}$ and $(h,j) \in \mathbb{Z}^2$. The association

$$\begin{array}{cccc} f^a_{h,j} : & \mathbb{S}^{a-1} & \longrightarrow & SO(a) \\ & u & \longmapsto & \left(f^a_{h,j}(u) : v \mapsto u^h \cdot v \cdot u^j\right) \end{array}$$

so obtained is smooth. It is possible to show that

$$\begin{array}{cccc} \theta : & \mathbb{Z} \times \mathbb{Z} & \longrightarrow & \pi^\infty_{a-1}(SO(a)) \\ & (h,j) & \longmapsto & [f^a_{h,j}]_\infty \end{array}$$

is a group homomorphism.

Remark 2.4. It is a classical result that $\theta_0 : \mathbb{Z}^2 \to \pi_{a-1}(SO(a))$ given by $(h,j) \mapsto [f^a_{h,j}]$ is a group isomorphism. Then, Theorem 2.3 implies that θ is also a group isomorphism.

2.3 Vector Bundles

Vector bundles play a major role in modern Topology and Differential Geometry. Certain geometric concepts, such as curvature, can be formulated in the more general framework of smooth vector bundles. A classical example of a smooth vector bundle is provided by the tangent bundle of a smooth manifold. As we shall be concerned only with smooth vector bundles, we will refer to them only as *vector bundles*, omitting the word smooth.

2.3.1 Terminology and Examples

Definition 2.5. A *vector bundle* is a tuple $\xi = (E, p, B, \mathbb{R}^n, \mathcal{A})$, where E and B are smooth manifolds and $p : E \to B$ is a smooth map. Each one of the *fibers* $E_b := p^{-1}(\{b\})$ $(b \in B)$ must have the structure of an n-dimensional vector space over the reals, i.e., they must be isomorphic to $\mathbb{R}^n$. Finally, $\mathcal{A} = \{(U_i, \varphi_i)\}$ is a collection such that $\{U_i\}$ is an open cover of B and the maps $\varphi_i : p^{-1}(U_i) \to U_i \times \mathbb{R}^n$ are diffeomorphisms satisfying the condition of *local triviality*, namely, that $p = \pi_1^i \circ \varphi_i$ on the open set $p^{-1}(U_i)$, where $\pi_1^i : U_i \times \mathbb{R}^n \to U_i$ stands for the projection map onto the first coordinate, and that the restrictions $E_b \to \{b\} \times \mathbb{R}^n$ are linear.

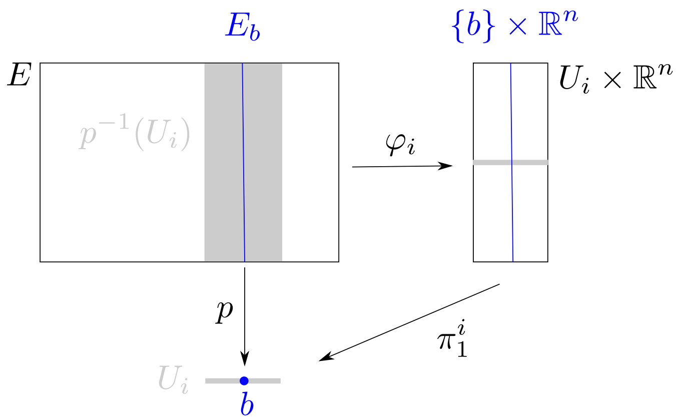

Figure 1 – Illustrating the condition of *local triviality*

We call E and B the *total space* and *base space*, respectively. We say that p is the *projection map* and that ξ has *rank* n, since each fiber is isomorphic to $\mathbb{R}^n$. The collection $\mathcal{A}$ is called *bundle atlas* and a pair (U, φ) in $\mathcal{A}$, a *bundle chart*. For a vector bundle η with base space B, we say η is a vector bundle *over* B.

Remark 2.6. Exchanging smooth manifolds by topological spaces and smooth maps by continuous ones, we obtain the definition of a topological vector bundle.

Remark 2.7. For each $b \in B$, we denote by 0_b the zero element of the vector space E_b. Then, the map $Z : B \to E$ given by $Z(b) = 0_b$ is an embedding and it follows that B and $Z(B)$ are diffeomorphic. We call Z the *zero section* of ξ. We write E_0 for the complement $E - Z(B)$ of the image of the zero section in E.

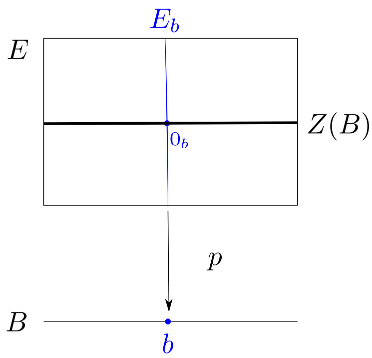

Figure 2 – Illustrating the image of the zero section

Example 2.8. Let B be a smooth manifold. Then, $\theta^n_B = (B \times \mathbb{R}^n, \pi_1, B, \mathbb{R}^n, \{(B, id_{B\times\mathbb{R}^n})\})$ is a vector bundle.

Though a bundle atlas is needed to specify a vector bundle, we shall often omit any reference to it and simply write $\xi = (E, p, B)$ for a vector bundle whenever there is no risk of confusion.

Example 2.9. Let M and N be smooth manifolds with N without boundary, so the cartesian product $M \times N$ can also be regarded as a smooth manifold. From two vector bundles $\xi = (E, p, M)$ and $\nu = (E', p', N)$, we can form a third one, called the *product bundle*. Such bundle is defined by $\xi \times \nu = (E \times E', p \times p', M \times N)$.

Example 2.10. Let $G_n(\mathbb{R}^m)$ be the Grasmannian of n-planes in $\mathbb{R}^m$. An element of $G_n(\mathbb{R}^m)$ is just a n-dimensional vector subspace of $\mathbb{R}^m$. It is possible to give $G_n(\mathbb{R}^m)$ a smooth atlas, making it a closed manifold. We define

$$E^m_n \;:=\; \{\, (V, x) \in G_n(\mathbb{R}^m) \times \mathbb{R}^m \mid x \in V \,\} \;\text{ and }\; p^m_n : (V, x) \in E^m_n \longmapsto V \in G_n(\mathbb{R}^m)\,.$$

We can find a bundle atlas $\mathcal{A}$ for which $\gamma^m_n = (E^m_n, p^m_n, G_n(\mathbb{R}^m))$ is rank n vector bundle. We call γ^m_n the *tautological vector bundle over* $G_n(\mathbb{R}^m)$.

We now describe a process by which we can get a vector bundle over B' from a smooth map $f : B' \to B$ and a vector bundle ξ over B. The vector bundle $f^*(\xi)$ so obtained is called the *pullback bundle* of ξ by f.

Example 2.11. Let $\xi = (E, p, B, \mathbb{R}^n, \mathcal{A})$ be a vector bundle over a manifold without boundary B and let $f : B' \to B$ be a smooth map. Since B has no boundary, so does E and, therefore, we can consider the smooth manifold $B' \times E$. The usual projections $\pi_1 : B' \times E \to B'$ and $\pi_2 : B' \times E \to E$ are smooth. Define

$$f^*(E) := \{(b', v) \in B' \times E \,|\, p(v) = f(b')\} \qquad \text{and} \qquad f^*(p) = \pi_1|_{f^*(E)} \ .$$

We note that $f^*(E)$ is a smooth submanifold of $B' \times E$. Then, the map $f^*(p)$ is smooth because it is the restriction of π_1 to a smooth submanifold. Now, for every $b' \in B'$, the fiber $E'_{b'}$ is just $\{b'\} \times E_{f(b')}$ and, therefore, has the structure of a n-dimensional real vector space because $E_{f(b')}$ does so. Moreover, if $\mathcal{A} = \{(U_i, \varphi_i)\}$ is the bundle atlas of ξ, we can give $f^*(\mathcal{A})$ a bundle atlas in the following way. Recall that f is smooth and $\{U_i\}$ is an open cover of B. Then, putting $V_j := f^{-1}(U_j)$, we get an open cover $\{V_j\}$ of B'. Now, observe that

$$f^*(p)^{-1}(V_j) = \bigcup_{b' \in V_j} f^*(p)^{-1}(b') = \bigcup_{b' \in V_j} \{b'\} \times E_{f(b')} = V_j \times p^{-1}(U_j \cap f(B')) \ .$$

Thus, writing $pr_2^j : f^*(p)^{-1}(V_j) \to p^{-1}(U_j \cap f(B'))$ for the projection onto the second coordinate, the map $\psi_j := \left(id_{V_j}, \varphi_j \circ pr_2^j\right) : f^*(p)^{-1}(V_j) \to V_j \times \mathbb{R}^n$ is a diffeomorphism satisfying local triviliaty: $f^*(p) = \pi_1^j \circ \psi_j$ on $f^*(p)^{-1}(V_j)$. Thus, the collection $f^*(\mathcal{A}) = \{(V_j, \psi_j)\}$ is a bundle atlas for $f^*(\xi) := (f^*(E), f^*(p), B', \mathbb{R}^n, f^*(\mathcal{A}))$.

Pullback bundles play a role in the classification of vector bundles. We will need only a very special case of this theorem, namely Theorem 2.22. We refer the reader to [Hus94], [Ste99], [Hir94], and [AGP02] for detailed accounts on the general classification theorem.

Definition 2.12. Two vector bundles $\xi_1 = (E^1, p_1, B, \mathbb{R}^n)$ and $\xi_2 = (E^2, p_2, B, \mathbb{R}^m)$ over the same base space B are B-*isomorphic*, or simply *isomorphic*, if there is a diffeomorphism $F : E^1 \to E^2$ such that $p_1 = p_2 \circ F$ and the restriction $F|_{E_b^1} : E_b^1 \to E_b^2$ is linear, for every $b \in B$.

Remark 2.13. The commutativity of the diagram below (Figure 3), expressed by the equation $p_1 = p_2 \circ F$, is the algebraic counterpart of the following geometric requirement. For each $b \in B$, the application F maps E_b^1 to E_b^2. Moreover, since F is a fiber-preserving diffeomorphism whose restrictions to the fibers are linear, it follows that each restriction $F : E_b^1 \to E_b^2$ is a *linear isomorphism*. We also notice that B-isomorphic vector bundles have the same rank.

$$\begin{tikzcd} E^1 \arrow[r, "F"] \arrow[dr, "p_1"'] & E^2 \arrow[d, "p_2"] \\ & B \end{tikzcd}$$

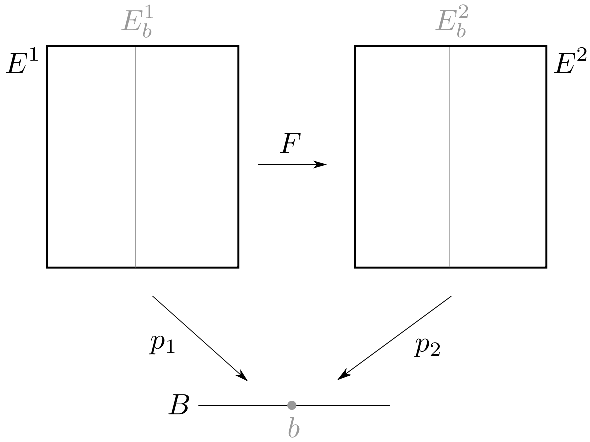

Figure 3 – The commutative diagram and its geometrical interpretation

We declare two rank n vector bundles over the same base space B to be *equivalent* if they are B-isomorphic. This defines an equivalence relation in the class of all rank n vector bundles over B. The collection of equivalence classes $[\xi]_\infty$ for this relation is denoted by $Vec_n^\infty(B)$.

Let ξ be a rank n vector bundle with atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$. By local triviality, whenever the intersection $U_j \cap U_i$ is non-empty, there is a smooth map $g_{j,i} : U_j \cap U_i \to GL(n, \mathbb{R})$ such that

$$\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times \mathbb{R}^n \longrightarrow (U_i \cap U_j) \times \mathbb{R}^n$$

has the form

$$\varphi_j \circ \varphi_i^{-1}(b, v) \quad = \quad (b, g_{j,i}(b)(v)) \ ,$$

for all $(b, v) \in (U_j \cap U_i) \times \mathbb{R}^n$. Such a map is called a *transition function*. The set $\mathcal{T} = \{g_{j,i} : U_j \cap U_i \to GL(n, \mathbb{R})\}$ is called the *collection transition functions of* ξ. Reciprocally, from an open cover $\{U_i\}$ of the smooth manifold B, a collection $\mathcal{T} = \{g_{j,i} : U_i \cap U_j \to GL(n, \mathbb{R})\}$ together with additional data, we can form a vector bundle whose collection of transition functions is $\mathcal{T}$.

Theorem 2.14. *([Lee13, Lemma 10.6]) Let B be a smooth manifold and $\{E_b\}_{b \in B}$ a collection of real vector spaces, all of them with the same dimension n. Let $E := \sqcup_{b \in B} E_b$ and define a map $p : E \to B$ by declaring $p(v) = b$ if and only if $v \in E_b$. Suppose we are given*

1. *an open cover $\{U_i\}$ of B*
2. *a collection $\mathcal{A} = \{\varphi_i : p^{-1}(U_i) \to U_i \times \mathbb{R}^n\}$ of bijective maps such that the restrictions $E_p \to \{p\} \times \mathbb{R}^n$ are linear isomorphisms*

3. *a collection of smooth maps $\mathcal{T} = \{g_{j,i} : U_j \cap U_i \to GL(n, \mathbb{R})\}$ of smooth maps defined for non-empty intersections $U_j \cap U_i$ such that*

$$\varphi_j \circ \varphi_i^{-1} : (U_j \cap U_i) \times \mathbb{R}^n \longrightarrow (U_j \cap U_i) \times \mathbb{R}^n$$

has the form $\varphi_j \circ \varphi_i^{-1}(b, v) = (b, g_{j,i}(b)(v))$, for all $(b, v) \in (U_j \cap U_i) \times \mathbb{R}^n$.

Under these hypothesis, it is possible to give E a unique topology and smooth structure for which $\xi = (E, p, B, \mathbb{R}^n, \mathcal{A})$ is a vector bundle. In particular, the collection of transition functions of ξ is $\mathcal{T}$.

Example 2.15. Let $\{(U_i, \psi_i\}$ be a smooth atlas for the manifold M. Then, for each non-empty overlap $U_j \cap U_i$, the map

$$\begin{array}{rccc} g_{j,i} : & U_j \cap U_i & \longrightarrow & GL(n, \mathbb{R}) \\ & x & \longmapsto & D(\psi_j \circ \psi_i^{-1})_x \end{array}$$

is smooth. Consider the collection $\{T_x M\}_{x \in M}$ and E and p as in Theorem 2.14. If we write $(U_i, \psi_i) = (U_i, x_i^1, \cdots, x_i^n)$, the map

$$\begin{array}{rccc} \varphi_i : & p^{-1}(U_i) & \longrightarrow & U_i \times \mathbb{R}^n \\ & \displaystyle\sum_{k=1}^{n} c_k \left.\frac{\partial}{\partial x_i^k}\right|_x & \longmapsto & (x, c_1, \cdots, c_n) \end{array}$$

is bijective and its restriction to each fiber is a linear isomorphism. Moreover,

$$\varphi_j \circ \varphi_i^{-1}(x, v) \;=\; (x, g_{j,i}(x)v) \;,$$

for all $(x, v) \in (U_i \cap U_j) \times \mathbb{R}^n$. Thus, Theorem 2.14 implies there is a vector bundle $\tau_M = (E, p, M, \mathcal{A})$ such that $\mathcal{A} = \{(U_i, \varphi_i)\}$ is a bundle atlas and $\mathcal{T} = \{g_{j,i}\}$ is its collection of associated transition functions. We call τ_M the *tangent bundle of M*. To stay with classical notation, we rename $TM := E$, so $\tau_M = (TM, p, M)$.

Example 2.16. We proceed as in the previous example. Let $\{(U_i, \psi_i\}$ be a smooth atlas for the manifold B. Then, for each non-empty overlap $U_j \cap U_i$, the map

$$\begin{array}{rccc} g_{j,i} : & U_j \cap U_i & \longrightarrow & GL(n, \mathbb{R}) \\ & x & \longmapsto & D(\psi_j \circ \psi_i^{-1})_x \end{array}$$

is smooth. Consider the collection $\{\bigwedge^k T_x M\}_{x \in M}$ and let E and p as in Theorem 2.14. Writing again $(U_i, \psi_i) = (U_i, x_i^1, \cdots, x_i^n)$, the map

$$\begin{array}{rccc} \varphi_i : & p^{-1}(U_i) & \longrightarrow & U_i \times \mathbb{R}^{\binom{n}{k}} \\ & \displaystyle\sum_{I \in \mathcal{L}} c_I \left.\frac{\partial}{\partial x_i}\right|_{x,I} & \longmapsto & (x, (c_I)_{I \in \mathcal{L}}) \end{array} \quad ,$$

where

$$\mathcal{L} \;=\; \{\,(i_1,\cdots,i_k)\mid 1\le i_1<\cdots<i_k\le n\,\}$$

and

$$\left.\frac{\partial}{\partial x}\right|_{x,I} \;=\; \left.\frac{\partial}{\partial x_i^{i_1}}\right|_x \wedge\cdots\wedge \left.\frac{\partial}{\partial x_i^{i_k}}\right|_x \;,$$

is bijective and its restriction to each fiber is a linear isomorphism. Moreover,

$$\varphi_j\circ\varphi_i^{-1}(x,v) \;=\; (x, g_{j,i}(x)v)\;,$$

for all $(x,v)\in(U_i\cap U_j)\times\mathbb{R}^n$. Thus, Theorem 2.14 implies there is a vector bundle $\wedge^k\tau_M=(E,p,M,\mathcal{A})$ such that $\mathcal{A}=\{(U_i,\varphi_i)\}$ is a bundle atlas and $\mathcal{T}=\{g_{j,i}\}$ is its associated collection of transition functions. We call $\wedge^k\tau_M$ the *k-th exterior product of* τ_M. To stay with classical notation, we rename $\wedge^kTM:=E$, and so $\wedge^k\tau_M=(\wedge^kTM,p,M)$.

Let $\xi_i=(E_i,p_i,B,\mathbb{R}^{n_i},\mathcal{A})$ $(i=1,2)$ be two vector bundles over the same manifold B. Write $\mathcal{A}_1=\{(U_i,\varphi_i)\}$ and $\mathcal{A}_2=\{(V_j,\psi_j)\}$ for their bundle atlases. Define $A=\{(i,j)|U_i\cap V_j\}\neq\emptyset$ and $W_\alpha=U_i\cap V_j$, for $\alpha=(i,j)\in A$. Then, $\{W_\alpha\}$ is an open cover of B and the restrictions $\varphi_\alpha:=\varphi_i:p_1^{-1}(W_\alpha)\to W_\alpha\times\mathbb{R}^{n_1}$ and $\psi_\alpha:=\psi_j:p_2^{-1}(W_\alpha)\to W_\alpha\times\mathbb{R}^{n_2}$, where $\alpha=(i,j)$, form new bundle atlases for ξ_1 and ξ_2, respectively. Moreover, ξ_1 is B-isomorphic to $\xi_1'=(E_1,p_1,B,\mathbb{R}^{n_1},\{(W_\alpha,\varphi_\alpha)\})$. Indeed, $id_{E_1}:E_1\to E_1$ is an isomorphism between them. The same reasoning shows that ξ_2 and $\xi_2'=(E_2,p_2,B,\mathbb{R}^{n_2},\{(W_\alpha,\psi_\alpha)\})$ are B-isomorphic. This proves that, given two vector bundles, we may assume without loss of generality that the open covers of the base space figuring in their bundle atlases are the same. When this occurs, we say that the two bundle atlases are *comparable*.

Example 2.17. Let $\xi_k=(E^k,p_k,B,\mathbb{R}^{n_k},\mathcal{A}_k)$ $(k=1,2)$ be two vector bundles with comparable bundle atlases. The collection $\{E_b^1\oplus E_b^2\}_{b\in B}$ is formed by real vector spaces of rank n_1+n_2. Define $E^1\oplus E^2:=E=\sqcup_{b\in B}(E_b^1\oplus E_b^2)$ and $p:E\to B$ by $p(v)=b\Leftrightarrow v\in E_b^1\oplus E_b^2$. If $\mathcal{A}_k=\{(U_i,\varphi_i^k\}$ is the bundle atlas of ξ_k $(k=1,2)$, then the map

$$\begin{array}{cccc}\varphi_i: & p^{-1}(U_i): & \longrightarrow & U_i\times\mathbb{R}^{n_1+n_2}\\ & v & \longmapsto & (p(v),(\pi_2^i\circ\varphi_i^1(v),\pi_2^i\circ\varphi_i^2(v)))\end{array}$$

is bijective and its restriction to each fiber is a linear isomorphism. The collection of transition functions of ξ_k is of the form $\mathcal{T}_k=\{g_{j,i}^k:U_j\cap U_i\to GL(n_k,\mathbb{R})\}$ $(k=1,2)$. Therefore, the map $g_{j,i}:U_j\cap U_i\to GL(n_1+n_2,\mathbb{R})$ defined by $g_{j,i}(b)=(g_{j,i}^1(b),g_{j,i}^2(b))$ is smooth. Moreover, in the overlapping $U_j\cap U_i$, we have

$$\begin{aligned}\varphi_j\circ\varphi_i^{-1}(b,v) &= \Big(b,(\varphi_j^1\circ(\varphi_i^1)^{-1}(b)v,\varphi_j^2\circ(\varphi_i^2)^{-1}(b)v)\Big)\\ &= \Big(b,(g_{j,i}^1(b)v,g_{j,i}^2(b)v)\Big)\\ &= (b,g_{j,i}(b)v)\;.\end{aligned}$$

Thus, by Theorem 2.14, we may endow $E^1 \oplus E^2$ with a smooth atlas such that the projection map p is smooth and, furthermore, $\xi_1 \oplus \xi_2 = (E^1 \oplus E^2, p, B, \mathbb{R}^{n_1+n_2})$ is a vector bundle. We call $\xi_1 \oplus \xi_2$ the *Whitney sum of* ξ_1 *and* ξ_2.

Remark 2.18. From any vector bundle $\xi = (E, p, B)$, we can form a new vector bundle $SymBil(\xi) = (SymBil(E), p_{Bil}, B)$ whose fiber $SymBil(E)_b$ is the vector space of all symmetric bilinear forms $E_b \times E_b \to \mathbb{R}$.

Corollary 2.19. *Let B be a manifold and $\{U_1, U_2\}$ an open cover such that $U_1 \neq U_2$ and $U_1 \cap U_2 \neq \emptyset$. For any smooth map $g : U_1 \cap U_2 \to GL(n, \mathbb{R})$, there is a vector bundle $\xi_g = (E_g, p_g, B, \mathbb{R}^n, \mathcal{A})$ for which $g_{2,1} = g$, where $\mathcal{T} = \{g_{i,j}\}_{i,j=1,2}$ is the collection of transition functions of ξ_g.*

Proof. For all $b \in B$, let $E_b = \mathbb{R}^n$. Let E and p be as in Theorem 2.14. The maps

$$\begin{array}{cccc} \varphi_1 : & p^{-1}(U_1) & \longrightarrow & U_1 \times \mathbb{R}^n \\ & v & \longmapsto & \begin{cases} (p(v), v) & \text{if } p(v) \in U_1 - U_2 \\ (p(v), g(p(v))^{-1} v) & \text{otherwise} \end{cases} \end{array}$$

and

$$\begin{array}{cccc} \varphi_2 : & p^{-1}(U_2) & \longrightarrow & U_2 \times \mathbb{R}^n \\ & v & \longmapsto & (p(v), v) \end{array}$$

are bijective and their restriction to fiber gives a linear isomorphism. Furthermore, the maps

$$\begin{array}{cccc} g_{1,1} : & U_1 & \longrightarrow & GL(n, \mathbb{R}) \\ & b & \longmapsto & id_{\mathbb{R}^n} \end{array} \quad , \quad \begin{array}{cccc} g_{1,2} : & U_1 \cap U_2 & \longrightarrow & GL(n, \mathbb{R}) \\ & v & \longmapsto & g(b)^{-1} \end{array}$$

$$\begin{array}{cccc} g_{2,1} : & U_2 \cap U_1 & \longrightarrow & GL(n, \mathbb{R}) \\ & b & \longmapsto & g(b) \end{array} \quad \text{and} \quad \begin{array}{cccc} g_{2,2} : & U_2 & \longrightarrow & GL(n, \mathbb{R}) \\ & v & \longmapsto & id_{\mathbb{R}^n} \end{array}$$

are smooth and we have

$$\varphi_j \circ \varphi_i^{-1}(b, v) = (b, g_{j,i}(b) v) ,$$

for all $(b, v) \in (U_i \cap U_j) \times \mathbb{R}^n$ and $i, j = 1, 2$. Therefore, Theorem 2.14 guarantees the existence of the bundle ξ_g. That $g = g_{2,1}$ follows from the equation above.

□

Spheres are smooth manifolds admitting an open cover by two distinct open sets. The preceding corollary yields, for this case, a method by which we can associate a vector bundle of rank n over $\mathbb{S}^m$ from a smooth mapping $f : \mathbb{S}^{m-1} \to GL(n, \mathbb{R})$. We call such a process *clutching construction*.

Example 2.20. Let $U_1 := \mathbb{S}^m - \{-e_{m+1}\}$ and $U_2 := \mathbb{S}^m - \{e_{m+1}\}$. Then, $\{U_1, U_2\}$ is an open cover of $\mathbb{S}^m$ with $U_1 \cap U_2 \neq \emptyset$. Consider the retraction onto the equator given by

$$\begin{array}{cccc} r: & U_1 \cap U_2 & \longrightarrow & \mathbb{S}^{m-1} \\ & (x_1, \cdots, x_m, x_{m+1}) & \longmapsto & \dfrac{1}{\sqrt{1-x_{m+1}^2}}(x_1, \cdots, x_m) \end{array} .$$

Clearly, r is smooth. Thus, for any smooth map $g : \mathbb{S}^{m-1} \to GL(n, \mathbb{R})$, we get a smooth map $g \circ r : U_1 \cap U_2 \to GL(n, \mathbb{R})$. By Corollary 2.19, there is a vector bundle $\xi_{g \circ r} = (E_{g \circ r}, p_{g \circ r}, \mathbb{S}^m, \mathcal{A})$ for which $g_{2,1} = g \circ r$, where $\mathcal{T} = \{g_{j,i} : U_i \cap U_j \to GL(n, \mathbb{R})\}_{i,j=1,2}$ is the collection of transition functions of $\xi_{g \circ r}$.

We now give an important criterion by which one can decide when two vector bundles with comparable atlases are isomorphic.

Theorem 2.21. *Let $\xi_k = (E^k, p_k, B, \mathbb{R}^n, \mathcal{A}_k)$ $(k = 1, 2)$ be two vector bundles with comparable atlases and let $\mathcal{T}_k = \{g^k_{j,i} : U_i \cap U_j \to GL(n, \mathbb{R})\}$ $(k = 1, 2)$ be their associated collection of transition functions. Then, ξ_1 and ξ_2 are isomorphic if and only if there is a collection of smooth maps $\{f_i : U_i \to GL(n, \mathbb{R})\}$ such that $g^1_{j,i}(b) = f_j(b)^{-1} g^2_{j,i}(b) f_i(b)$, for all $b \in U_i \cap U_j$.*

Proof. Let $F : E^1 \to E^2$ be a vector bundle isomorphism. Notice that $F(p_1^{-1}(U_i)) = p_2^{-1}(U_i)$. In fact,

$$v \in p_1^{-1}(U_i) \iff p_1(v) \in U_i \iff (p_2 \circ F)(v) \in U_i \iff F(v) \in p_2^{-1}(U_i) ,$$

where the second equivalence follows from $p_1 = p_2 \circ F$. Then, writing $\mathcal{A}_1 = \{(U_i, \varphi_i)\}$ and $\mathcal{A}_2 = \{(U_i, \psi_i)\}$ for the bundle atlases of ξ_1 and ξ_2, respectively, and since F restricts to linear isomorphisms on the fibers, the map $T_i := \psi_i \circ F \circ \varphi_i^{-1} : U_i \times \mathbb{R}^n \to U_i \times \mathbb{R}^n$ is a diffeomorphism satisfying the following condition. For every $b \in U_i$, there is a linear map $f_i(b) \in GL(n, \mathbb{R})$ such that $T_i(b, v) = (b, f_i(b)v)$, for all $v \in \mathbb{R}^n$. The association $f_i : U_i \to GL(n, \mathbb{R}^n)$ so obtained is smooth and we have

$$\begin{aligned} \left(b, g^1_{j,i}(b)v\right) &= (\varphi_j \circ \varphi_i^{-1})(b, v) \\ &= (T_j^{-1} \circ \psi_j \circ \psi_i^{-1} \circ T_i)(b, v) \\ &= (T_j^{-1} \circ \psi_j \circ \psi_i^{-1})(b, f_i(b)v) \\ &= T_j^{-1}\left(b, g^2_{j,i}(b) f_i(b) v\right) \\ &= \left(b, f_j(b)^{-1} g^2_{j,i}(b) f_i(b) v\right) , \end{aligned}$$

for all $(b, v) \in (U_i \cap U_j) \times \mathbb{R}^n$. Thus, $\{f_i : U_i \to GL(n, \mathbb{R})\}$ is a collection of smooth maps with

$$g^1_{j,i}(b) = f_j(b)^{-1} g^2_{j,i}(b) f_i(b) ,$$

for all $b \in U_i \cap U_j$.

Reciprocally, suppose $\{f_i : U_i \to GL(n,\mathbb{R})\}$ is a collection of smooth maps such that

$$g^1_{j,i}(b) \;=\; f_j(b)^{-1} g^2_{j,i}(b) f_i(b) \;,$$

for all $b \in U_i \cap U_j$. Then, the map $L_i : U_i \times \mathbb{R}^n \to U_i \times \mathbb{R}^n$ given by $(b,v) \mapsto (b, f_i(b))$ is a diffeomorphism such that the restrictions $L_i : \{b\} \times \mathbb{R}^n \to \{b\} \times \mathbb{R}^n$ are linear isomorphisms. Define $F_i := \psi_i^{-1} \circ L_i \circ \varphi_i : p_1^{-1}(U_i) \to p_2^{-1}(U_i)$. Then, F_i is a diffeomorphism whose restriction to the fibers are linear isomorphisms. Now, define $F : E^1 \to E^2$ by the following rule. Given $v \in E^1$, let i be an index such that $p_1(v) \in U_i$. Put $F(v) := F_i(v)$. We claim that the value $F(v)$ does not depend on the particular index chosen. Indeed, if j is another index for which $p_1(v) \in U_j$ and $\varphi_i(v) = (b, v_0)$, then

$$\begin{aligned}
(F_j^{-1} \circ F_i)(v) &= (\varphi_j^{-1} \circ L_j^{-1} \circ \psi_j \circ \psi_i^{-1} \circ L_i)(b, v_0) \\
&= (\varphi_j^{-1} \circ L_j^{-1} \circ \psi_j \circ \psi_i^{-1})(b, f_i(v) v_0) \\
&= (\varphi_j^{-1} \circ L_j^{-1}) \left(b, g^2_{j,i}(b) f_i(b) v_0\right) \\
&= \varphi_j^{-1} \left(b, f_j(b)^{-1} g^2_{j,i}(b) f_i(b) v_0\right) \\
&= \varphi_j^{-1} \left(b, g^1_{j,i}(b) v_0\right) \;=\; \varphi_i^{-1}(b, v_0) \;=\; v \;,
\end{aligned}$$

and we clearly have $F_i(v) = F_j(v)$. Besides well-defined, the map F is a bijective local diffeomorphism, for its restriction to each fiber is a linear isomorphism and its restriction to the open set $p_1^{-1}(U_i)$ gives the diffeomorphism F_i. Thus, F is a diffeomorphism and it follows from the construction that $p_1 = p_2 \circ F$.

□

Theorem 2.22. *There exists a group homomorphism*

$$\begin{array}{rccc}
\psi : & \pi^\infty_{m-1}(SO(n)) & \longrightarrow & \pi^\infty_m(G_n(\mathbb{R}^{2n})) \\
& [f]_\infty & \longmapsto & [\tilde{f}]_\infty
\end{array}$$

such that the vector bundles $\xi_{f \circ r}$ and $\tilde{f}^* \gamma^{2n}_n$ are $\mathbb{S}^m$-isomorphic. Here $\xi_{f \circ r}$ is the rank n vector bundle over $\mathbb{S}^m$ obtained by clutching construction via the map $f \circ r : U_1 \cap U_2 \to SO(n)$, where $r : U_1 \cap U_2 \to \mathbb{S}^{m-1}$ is the retraction onto the equator $\mathbb{S}^{m-1}$. And $\tilde{f} \gamma^{2n}_n$ is obtained by pulling back the tautological vector bundle γ^{2n}_n over the grassmaniann $G_n(\mathbb{R}^{2n})$ via the smooth map $\tilde{f} : \mathbb{S}^m \to G_n(\mathbb{R}^{2n})$.

Proof. For a proof in the topological case, see [Tio09, Proposition 1.3]. The smooth case then follows from the topological case together with the fact that

$$\begin{array}{rccc}
\iota_\infty : & \pi^\infty_k(N) & \longrightarrow & \pi_k(N) \\
& [f]_\infty & \longmapsto & [f]
\end{array}$$

is a group isomorphism, whenever N is a manifold without boundary.

□

We need one more concept from the theory of vector bundles, one which will be of interest mainly in our study of de Rham cohomology.

Definition 2.23. A *section* of a vector bundle $\xi = (E, p, B)$ is a smooth map $s : B \to E$ such that $p \circ s = id_B$.

Remark 2.24. Geometrically, the commutativity $p \circ s = id_B$ means that to each point b in the base space, we associate a vector $s(b)$ in the fiber over it.

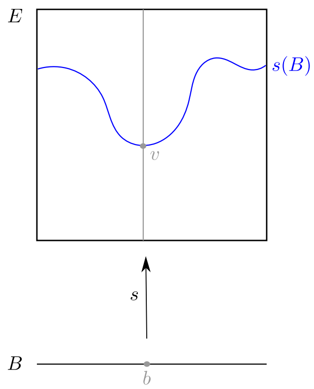

Figure 4 – Illustrating the image of a section

Example 2.25. A *vector field* on a manifold M is a section of its tangent bundle τ_M. A *Riemmanian metric* on M is a section s of $SymBil(\tau_M)$ such that $s(b)$ is positive definite, for all $b \in M$. A *differential form of degree* k on M is a section of $\wedge^k \tau_M$. The zero section $Z : B \to E$ of a vector bundle $\xi = (E, p, B)$ is a section. Compare Figures 2 and 4.

2.3.2 Riemannian Bundles

Definition 2.26. A *Riemannian metric* on a vector bundle $\xi = (E, p, B)$ is a section s of the vector bundle $SymBil(\xi)$ such that $\langle \cdot, \cdot \rangle_b := s(b) : E_b \times E_b \to \mathbb{R}$ is positive-definite, for all $b \in B$.

Remark 2.27. A Riemannian metric on a manifold M is nothing but a Riemannian metric on its tangent bundle τ_M.

Theorem 2.28. *([Tu17, Theorem 10.8]) Every vector bundle can be given a Riemannian metric.*

Proof. Let $\xi = (E, p, B, \mathbb{R}^n, \mathcal{A})$ be a vector bundle. Take a smooth partition of unity $\{f_i : B \to [0,1]\}_{i\in I}$ subordinate to the open cover $\{U_i\}_{i\in I}$ formed by the open sets figuring in the bundle atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$ of ξ. Now, define

$$\langle\cdot,\cdot\rangle_b \quad := \quad \sum_{i\in I} f_i(b)\varphi_i^*(b)\langle\cdot,\cdot\rangle_{eucl} \ ,$$

where $\varphi_i^*\langle\cdot,\cdot\rangle_{eucl}$ is obtained by pulling back the standard euclidean metric $\langle\cdot,\cdot\rangle_{eucl}$ in $\mathbb{R}^n$ to E_b through φ_i. More precisely, for $b \in U_i$, we set

$$\begin{array}{rccc} \varphi_i^*(b)\langle\cdot,\cdot\rangle_{eucl} : & E_b \times E_b & \longrightarrow & \mathbb{R} \\ & (v_1, v_2) & \longmapsto & \langle(\pi_2^i \circ \varphi_i)(v_1), (\pi_2^i \circ \varphi_i)(v_2)\rangle_{eucl} \end{array} ,$$

where $\pi_2^i : U_i \times \mathbb{R}^n \to \mathbb{R}^n$ stands for the projection onto the second coordinate. For $b \notin U_i$, we put $\varphi_i^*(b)\langle\cdot,\cdot\rangle_{eucl}$ to be the everywhere null symmetric bilinear form, so the sum giving $\langle\cdot,\cdot\rangle_b$ above is well-defined. $\square$

If $s : B \to SymBil(E)$ is Riemannian metric for $\xi = (E, p, B)$, then the map $Q : E \to \mathbb{R}$ given by $v \mapsto \langle v, v\rangle_{p(v)}$ is smooth and its restriction to each fiber $Q_b : E_b \to \mathbb{R}$ is a *positive-definite quadratic form* on E_b. Conversely, if $Q : E \to \mathbb{R}$ is a smooth map with this property, for every $b \in B$ we put

$$\begin{array}{rccc} s(b) : & E_b \times E_b & \longrightarrow & \mathbb{R} \\ & (v, w) & \longmapsto & \left((v,w) \mapsto \dfrac{1}{2}\left(Q(v+w) - Q(v) - Q(w)\right)\right) \end{array} .$$

We then get a Riemannian metric $s : B \to SymBil(E)$ on ξ, so the two approaches are equivalent.

We are specially interested in vector bundles for which all the transition functions take values in $O(n)$[3] for, in this case, it is possible to construct a Riemmanian metric such that the bundle charts are fiberwise isometries. In this context, an isometry is a linear map preserving the positive-definite quadratic forms of its domain and codomain. Therefore, we are going to construct a smooth map $Q : E \to \mathbb{R}$ such that the restriction to each fiber $Q : E_b \to \mathbb{R}$ is a positive-definite quadratic form and, moreover,

$$Q(v) \quad = \quad ||(\pi_2^i \circ \varphi_i)(v)||^2 \qquad (v \in E_b) \ ,$$

whenever $b \in U_i$. We recall that $||\cdot||^2 := \langle\cdot,\cdot\rangle_{eucl}$ and note that

$$||v||^2_{(b,i)} \quad := \quad ||(\pi_2^i \circ \varphi_i)(v)||^2 \qquad (b \in U_i\,,\ v \in E_b) \ ,$$

[3] The condition of all transition functions to take values in $O(n)$ is not as restrictive as it may seem at a first glance. In fact, by applying Gram-Schmidt process to every bundle chart, one can reduce the *structure group* $GL(n, \mathbb{R})$ of the fiber bundle to $O(n)$.

gives a positive-definite quadratic form in $\{b\} \times \mathbb{R}^n$, so that it makes sense to speak of the restriction $\varphi_i : E_b \to \{b\} \times \mathbb{R}^n$, to be an isometry between (E_b, Q) and $\Big(\{b\} \times \mathbb{R}^n, ||\cdot||^2_{(b,i)}\Big)$.

Let $\xi = (E, p, B, \mathbb{R}^n, \mathcal{A})$ be a vector bundle whose transition functions take values in $O(n)$. Consider the function $Q : E \to \mathbb{R}$ given by the following rule. For $x \in E$ and $b := p(x)$, if $i \in I$ is such that $b \in U_i$, put $Q(x) := ||\pi_2^i \circ \varphi_i(x)||^2$. Here we write $\pi_2^i : U_i \times \mathbb{R}^n \to \mathbb{R}^n$ for the projection onto the second coordinate. We claim that such definition of $Q(x)$ is independent of the particular index $i \in I$ chosen. Indeed, let $j \in I$ be another index for which $b \in U_j$. Observe that

$$(\pi_2^j \circ \varphi_j)(x) \quad = \quad (\pi_2^j \circ \varphi_j \circ \varphi_i^{-1} \circ \varphi_i)(x) \ .$$

Now, since $\varphi_i(x) = (b, v)$, for some $v \in \mathbb{R}^n$, and

$$\varphi_j \circ \varphi_i^{-1}(b, v) = (b, f(b) \cdot v) \ ,$$

for some $f(b) \in O(n)$, we conclude that

$$(\pi_2^j \circ \varphi_j)(x) \quad = \quad \pi_2^j(b, f(b) \cdot v) \quad = \quad f(b) \cdot v \ .$$

In particular,

$$||\pi_2^i \circ \varphi_i(x)||^2 \quad = \quad ||v||^2 \quad = \quad ||f(b) \cdot v||^2 \quad = \quad ||\pi_2^j \circ \varphi_j(x)||^2 \ .$$

It follows immediately from the definition that Q is a positive-definite quadratic form on each fiber E_b. That Q is smooth follows from the fact that it is smooth on each $p^{-1}(U_i)$.

Lemma 2.29. *The set $crit(Q)$ of critical points of Q is precisely the image of the zero section $Z(B)$ of the vector bundle.*

Proof. Let $x \in E_0 = E - Z(B)$ and put $b := p(x)$. Consider the smooth path $\alpha_x : (0, +\infty) \to E_b$ given by $t \mapsto tx$. Since $Q \circ \alpha_x(t) = t^2 Q(x)$, for all $t > 0$, we have

$$2Q(x) \quad = \quad \frac{d}{dt}(Q \circ \alpha_x)\Big|_{t=1} \quad = \quad dQ(x) \cdot d\alpha_x(1) \ .$$

Now, $Q(x) \neq 0$ since $x \in E_0$ and, therefore, we have $dQ(x) \cdot d\alpha_x(1) \neq 0$. In particular, $dQ(x) \neq 0$, proving that x is not a critical point. This shows that $crit(Q) \subseteq Z(B)$. To prove the reverse inclusion, it suffices to note that all points of $Z(B)$ takes on the minimal value of the function Q. In fact, $Q(x) \geq 0$, for all $x \in E$, with equality holding if and only if $x \in Z(B)$.

□

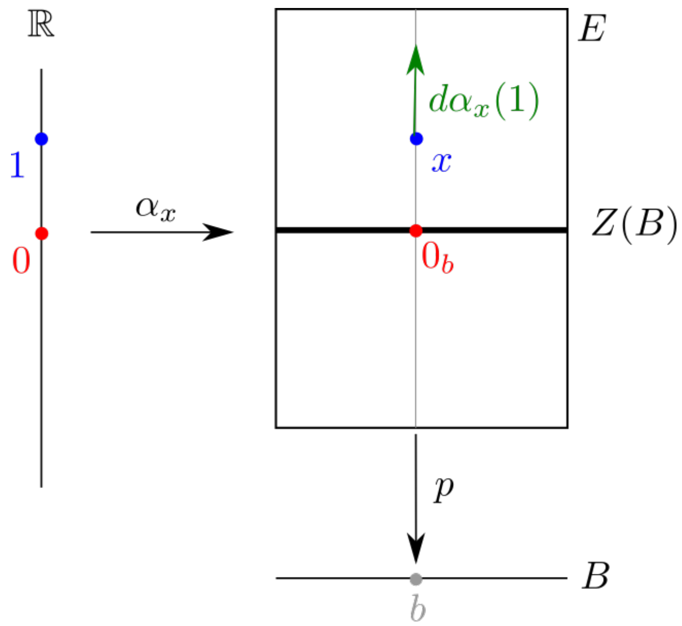

Figure 5 – The path α_x pass trough x at time 1 with velocity vector $d\alpha_x(1)$

Now, since 1 is a regular value of Q, a well-known result from differential topology asserts that the sets

$$D\xi \;:=\; Q^{-1}([0,1]) \;=\; Q^{-1}((-\infty,1])) \qquad \text{and} \qquad S\xi \;:=\; Q^{-1}(\{1\})$$

are smooth submanifolds of E with $S\xi$ bounding $D\xi$.

Remark 2.30. We observe that Q is the square of a norm on each E_b. More precisely, if we write $||\cdot||_b$ for the norm in $\{b\}\times\mathbb{R}^n$ given by $(b,v)\mapsto ||v||$, where $||\cdot||$ is the standard euclidean norm on $\mathbb{R}^n$, then the restrictions

$$\varphi_i|_{E_b} : \left(E_b, \sqrt{Q_b}\right) \longrightarrow (\{b\}\times\mathbb{R}^n, ||\cdot||_b)$$

are isometries of normed vector spaces. In particular, these restrictions map $D\xi_b := D\xi\cap E_b$ and $S\xi_b := S\xi\cap E_b$ to $\{b\}\times\mathbb{D}^n$ and $\{b\}\times\mathbb{S}^{n-1}$, respectively. Therefore, we have *fiber bundles*

$$\mathbb{D}^n \hookrightarrow D\xi \xrightarrow{p} B \qquad\qquad \text{and} \qquad\qquad \mathbb{S}^{n-1} \hookrightarrow S\xi \xrightarrow{p} B$$

with compact fibers. If B is also compact, the total spaces $D\xi$ and $S\xi$ will be compact, for fiber bundles with compact fiber and base space have compact total spaces.

2.4 $\xi^a_{h,j}$: Milnor and Shimada's Bundles

Let $f^a_{h,j} : \mathbb{S}^{a-1} \to SO(a)$ be as in Section 1. We write $\xi^a_{h,j} = (E^a_{h,j}, p^a_{h,j}, \mathbb{S}^a, \mathcal{A}^a_{h,j})$ for the vector bundle over $\mathbb{S}^a$ obtained by clutching construction. Then the transition functions of $\xi^a_{h,j}$ take values on $SO(a) \subset O(a)$ and we can give it the Riemannian metric as in the preceding section. Thus, we can consider $D^a_{h,j} := D\xi^a_{h,j}$ and $S^a_{h,j} := S\xi^a_{h,j}$.

Definition 2.31. For $a = 4$, we call $\xi^a_{h,j}$ a Milnor's vector bundle, and for $a = 8$, a Shimada's vector bundle.

Let $\mathring{D}^a_{h,j} := D^a_{h,j} - S^a_{h,j}$ and notice that $\mathring{D}^a_{h,j}$ is a smooth manifold without boundary. We claim that it is orientable. Therefore, the whole manifold $D^a_{h,j}$ is orientable and, in particular, so is the boundary $S^a_{h,j}$. Indeed, let $\{U_1, U_2\}$ be the open cover of $\mathbb{S}^a$ as in Example 2.20 and let $p_1 : U_1 \to \mathbb{R}^a$ and $p_2 : U_2 \to \mathbb{R}^a$ be the stereographic projections from the south and north poles, respectively. Writing $\mathcal{A}^a_{h,j} = \{(U_i, \varphi_i)\}_{i=1,2}$, we define $V_i := (p^a_{h,j})^{-1}(U_i) \cap \mathring{D}^a_{h,j} = \varphi_i^{-1}(U_i \times \mathring{\mathbb{D}}^a)$ and $\psi_i := (p_i, id_{\mathbb{R}^a}) \circ \varphi_i : V_i \to \mathbb{R}^a \times \mathring{\mathbb{D}}^a$. Then, $\mathcal{B} = \{(V_i, \psi_i)\}_{i=1,2}$ is a smooth atlas for $\mathring{D}^a_{h,j}$ and since

$$(\psi_2 \circ \psi_1^{-1})(u_1, u_1) \;=\; \left(\frac{u_1}{||u_1||^2} \,,\, \left(f^a_{h,j} \circ r\right) (p_1^{-1}(u_1))u_2 \right)$$

holds for all $(u_1, u_2) \in (\mathbb{R}^a - \{0\}) \times \mathring{\mathbb{D}}^a$, we conclude that $\mathcal{B}$ is an oriented atlas for $\mathring{D}^a_{h,j}$. We orient $D^a_{h,j}$ with the orientation whose restriction to the interior $\mathring{D}^a_{h,j}$ coincides with the orientation of $\mathcal{B}$. We give the boundary $S^a_{h,j} = \partial D^a_{h,j}$ the induced orientation.

In the following, we present two results related to the bundles $\xi^a_{h,j}$ that will be important in further developments.

Proposition 2.32. *The vector bundles $\xi^a_{h,j}$ and $\xi^a_{-j,-h}$ are isomorphic.*

Proof. The map $f_i : U_i \to GL(n, \mathbb{R})$ $(i = 1, 2)$ given $b \mapsto \mathcal{C}$ is constant and, consequently, smooth. Moreover, writing $\mathcal{T}^a_{h,j} = \{g^1_{k,l}\}_{k,l=1,2}$ and $\mathcal{T}^a_{-j,-h} = \{g^2_{k,l}\}_{k,l=1,2}$ for the collection of transition functions of the bundles $\xi^a_{h,j}$ and $\xi^a_{-j,-h}$, respectively, we have

$$g^1_{k,l}(b) \;=\; f_k(b)^{-1} g^2_{k,l}(b) f_l(b) \,,$$

for all $b \in U_k \cap U_l$. The conclusion follows from Theorem 2.21.

□

Recall the following result from Morse Theory, known as *Reeb's Theorem*. We will use it to show that some of the $S^a_{h,j}$ are homeomorphic to $\mathbb{S}^{2a-1}$. We say that $f : M \to \mathbb{R}$ is *Morse function* if it is smooth and all of its critical points are non-degenerate.

Theorem 2.33. *Let M be closed n-manifold. If there is a Morse function $f : M \to \mathbb{R}$ with only two critical points, then M is homeomorphic to $\mathbb{S}^n$.*

Corollary 2.34. *If $h+j=1$, then $S^a_{h,j}$ is homeomorphic to $\mathbb{S}^{2a-1}$.*

Proof. Write $\mathcal{A}^a_{h,j} = \{(U_i,\varphi_i)\}_{i=1,2}$ and consider again the stereographic projections $p_i : U_i \to \mathbb{R}^a$ $(i=1,2)$. Define $V_i := (p^a_{h,j})^{-1}(U_i) \cap S^a_{h,j} = \varphi_i^{-1}(U_i \times \mathbb{S}^{a-1})$. Clearly, $\{V_1, V_2\}$ is an open cover of $S^a_{h,j}$ and $\varphi_i|_{V_i} : V_i \to U_i \times \mathbb{S}^{a-1}$ is a diffeomorphism. Thus, the composition

$$\psi_i \;:=\; (p_i, id_{\mathbb{R}^a}) \circ \varphi_i : V_i \longrightarrow \mathbb{R}^a \times \mathbb{S}^{a-1}$$

is a also diffeomorphism. To simplify notation, we write

$$(u,v) \;=\; \left(\pi^1_1 \circ \psi_1, \pi^1_2 \circ \psi_1\right) \;=\; \psi_1 \qquad \text{and} \qquad (u',v') \;=\; \left(\pi^2_1 \circ \psi_2, \pi^2_2 \circ \psi_2\right) \;=\; \psi_2 \;.$$

In particular, $\psi_3 := (u'(v')^{-1}, v') : V_2 \to \mathbb{R}^a \times \mathbb{S}^{a-1}$ is also a diffeomorphism. Again to simplify notation, write $u'' = u'(v')^{-1}$, so $\psi_3 = (u'', v')$. In the intersection $V_1 \cap V_2$, the functional equation

$$(u',v') \;=\; \psi_2 \;=\; (\psi_2 \circ \psi_1^{-1}) \circ \psi_1 \;=\; (\psi_2 \circ \psi_1^{-1})(u,v) \;=\; \left(\frac{u}{||u||^2}, (f^a_{h,j} \circ r)(p_1^{-1}(u))v\right)$$

holds. But since

$$(r \circ p_1^{-1})(u) \;=\; \frac{u}{||u||}\;,$$

we have

$$(f^a_{h,j} \circ r)(p_1^{-1}(u))v \;=\; \left(\frac{u}{||u||}\right)^h v \left(\frac{u}{||u||}\right)^j \;=\; \frac{u^h v u^j}{||u||}\;,$$

where the last equation follows from the hypothesis $h+j=1$. Thus, in the intersection $V_1 \cap V_2$, one has

$$(u',v') \;=\; \left(\frac{u}{||u||^2}, \frac{u^h v u^j}{||u||}\right)\;.$$

In particular, in the same intersection, we have the functional equation

$$u'' \;=\; u'(v')^{-1} \;=\; \frac{u}{||u||^2}||u||u^{-j}v^{-1}u^{-h} \;=\; \frac{u^{1-j}v^{-1}u^{-h}}{||u||} \;=\; \frac{u^h v^{-1} u^{-h}}{||u||}\;,$$

the last equation following again from $h+j=1$.

Now that we have the functional relations between ψ_1 and ψ_3 in the intersection $V_1 \cap V_2$, we are ready to exhibit a Morse function $f : S^a_{h,j} \to \mathbb{R}$ with only two critical points. First, notice that the maps

$$\begin{array}{rccc} f_1 : & \mathbb{R}^a \times \mathbb{S}^{a-1} & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & \dfrac{Re(y)}{\sqrt{1+||x||^2}} \end{array} \qquad \text{and} \qquad \begin{array}{rccc} f_2 : & \mathbb{R}^a \times \mathbb{S}^{a-1} & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & \dfrac{Re(x)}{\sqrt{1+||x||^2}} \end{array}$$

are smooth. Furthermore, f_1 has only two critical points, namely $(0, \pm 1)$, and they both are non-degenerate. Since f_2 always increases in the x-direction, it has no critical points, for in this case the partial derivative with respect to the first coordinate is always positive. Now, define $f : S^a_{h,j} \to \mathbb{R}$ by the following rule. Given $x \in S^a_{h,j}$, we put $f(x) = (f_1 \circ \psi_1)(x)$, if $x \in V_1$. Otherwise, we have $x \in V_2$ and we set $f(x) := (f_2 \circ \psi_3)(x)$. We claim that f is well-defined. Indeed, in the intersection $V_1 \cap V_2$, the functional relations

$$(u'', v') \quad = \quad \left(\frac{u^h v^{-1} u^{-h}}{||u||}, \frac{u^h v u^j}{||u||} \right)$$

hold. Since $u \neq 0$ in the overlapping $V_1 \cap V_2$ and $||v|| = 1$, Corolary 2.2 implies

$$Re(\overline{v}) \quad = \quad Re(v^{-1}) \quad = \quad Re(u^h v^{-1} u^{-h}) \; .$$

But since $Re(v) = Re(\overline{v})$, we see that

$$Re(u'') \quad = \quad Re \left(\frac{u^h v^{-1} u^{-h}}{||u||} \right) \quad = \quad \frac{1}{||u||} Re(u^h v^{-1} u^{-h}) \quad = \quad \frac{Re(v)}{||u||} \; ,$$

in the intersection $V_1 \cap V_2$. Still in this intersection, we get

$$f_2(u'', v') \quad = \quad \frac{Re(u'')}{\sqrt{1 + ||u''||^2}} \quad = \quad \frac{\dfrac{Re(v)}{||u||}}{\sqrt{1 + \dfrac{1}{||u||^2}}} \quad = \quad \frac{Re(v)}{\sqrt{1 + ||u||^2}} \quad = \quad f_1(u, v) \; ,$$

proving that f is well-defined. Finally, since ψ_1 and ψ_3 are diffeomorphisms, it follows that f is a Morse function with only two critical points, namely $\psi_1^{-1}((0, \pm 1))$.

□

2.5 Fundamentals of de Rham Cohomology

2.5.1 Differential Forms

Recall that a *differential form* of degree k, or simply a k*-form*, on a smooth n-manifold M is a section $s : M \to \wedge^k TM$ of the vector bundle $\wedge^k \tau_M$. The collection of all k-forms on M is denoted by $\Omega^k(M)$. For $k = 0$, we define $\Omega^0(M) := \mathscr{C}^\infty(M; \mathbb{R})$. Each $\Omega^k(M)$ is a vector space over the reals and, in particular, so is their direct sum $\Omega(M) := \oplus_{k=0}^n \Omega^k(M)$. For any smooth map $f : M \to N$, there is an induced *linear map* $f^* : \Omega(N) \to \Omega(M)$ taking k-forms on N into k-forms on M. Given $\omega \in \Omega(N)$, we define $f^*\omega$ pointwise by

$$f^*\omega_x(v_1, \cdots, v_k) \quad := \quad \omega_{f(x)} \left(df_x(v_1), \cdots, df_x(v_k) \right) \; ,$$

where $x \in M$ and $v_1, \cdots, v_k \in T_xM$. It can be shown that the section $f^*\omega : M \to \wedge^k TM$ obtained is smooth, i.e., it is a differential k-form. We call $f^*\omega$ the *pullback of* ω *by* f.

We also recall that the *wedge product* $\wedge : \Omega(M) \times \Omega(M) \to \Omega(M)$ is a *bilinear map* such that $\wedge(\Omega^k(M)) \times \Omega^l(M)) \subseteq \Omega^{k+l}(M)$, turning $\Omega(M)$ into a real graded algebra with unity. The unity is given by the constant function $\mathbb{1}_M : M \to \mathbb{R}$ taking all points of M into 1. Furthermore, the following properties are satisfied. For proofs, we refer the reader to [Tu11] or [Lee13].

Proposition 2.35. *The following properties of wedge product hold.*

1. *Graded Commutativity: If ω_1 is a k-form and ω_2 is a l-form, then $\omega_1 \wedge \omega_2$ is a $(k+l)$-form. Moreover,*

$$\omega_1 \wedge \omega_2 \quad = \quad (-1)^{kl} \omega_2 \wedge \omega_1 \ ,$$

i.e, the algebra $\Omega(M)$ is graded commutative. In particular, if ω is differential form of odd degree, then $\omega \wedge \omega = 0$.

2. *Associativity: For any $\omega_1, \omega_2, \omega_3 \in \Omega(M)$, we have*

$$(\omega_1 \wedge \omega_2) \wedge \omega_3 \quad = \quad \omega_1 \wedge (\omega_2 \wedge \omega_3)$$

3. *Algebra homomorphism: For any smooth map $f : M \to N$ and $\omega_1, \omega_2 \in \Omega(N)$, we have*

$$f^*(\omega_1 \wedge \omega_2) \quad = \quad f^*\omega_1 \wedge f^*\omega_2 \ ,$$

or, in other words, $f^* : \Omega(N) \to \Omega(M)$ is an algebra homomorphism. Moreover, we clearly have $f^*(\mathbb{1}_N) = \mathbb{1}_M$.

Let us now introduce the *exterior derivative*. For ω a differential k-form on a smooth manifold M and $(U, \varphi) = (U, x^1 \cdots, x^n)$ a chart for the same manifold, there are smooth functions $a_I : U \to \mathbb{R}$ such that ω can be written as

$$\omega(x) \quad = \quad \sum_{I \in \mathcal{I}_k} a_I(x)\, dx_I|_x \ ,$$

for all $x \in U$. Here,

$$\mathcal{I}_k \quad = \quad \{\, (i_i, \cdots, i_k) \mid 1 \leq i_1 < \cdots < i_k \leq n \,\} \qquad \text{and} \qquad dx_I \quad = \quad dx_{i_1} \wedge \cdots \wedge dx_{i_k} \ .$$

We then define $d\omega$ with respect to the chart $(U, x^1, \cdots, x^n)$ to be

$$d\omega(x) \quad := \quad \sum_{I \in \mathcal{I}_k} \left(\sum_{i=1}^{n} \frac{\partial a_I}{\partial x^i}(x) \right) dx_I \ ,$$

for all $x \in U$. Thus, $d\omega$ gives a $(k+1)$-form on U. It is possible to show that $d\omega$ is pointwise well-defined. More precisely, if $(U, x^1, \cdots, x^n)$ and $(V, y^1, \cdots, y^n)$ are charts of M with $x \in U \cap V$, then the values of $d\omega(x)$ with respect to the charts $(U, x^1, \cdots, x^n)$ and $(V, y^1, \cdots, y^n)$ coincide. Therefore, we have defined a map

$$\Omega^k(M) \xrightarrow{\ d\ } \Omega^{k+1}(M) \ .$$

We further notice that $d^2\omega = 0$. Indeed, for any chart $(U, x^1, \cdots, x^n)$ of M, we have

$$\begin{aligned} d^2\omega(x) &= \sum_{I \in \mathcal{I}_k} \left(\sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 a_I}{\partial x^j \partial x^i}(x)\ dx^i \wedge dx^j \right) dx_I \\ &= \sum_{I \in \mathcal{I}_k} \left(\sum_{1 \le i < j \le n} \left(\frac{\partial^2 a_I}{\partial x^i \partial x^j} - \frac{\partial^2 a_I}{\partial x^j \partial x^i} \right) dx^i \wedge dx^j \right) dx_I \\ &= 0 \ , \end{aligned}$$

the last equation following from *Clairaut-Schwartz's theorem*. The following properties of exterior derivative can also be found in [Tu11] and [Lee13].

Proposition 2.36. *The following properties are true.*

1. *The map $d : \Omega(M) \to \Omega(M)$ is linear.*
2. *For any $\omega_1, \omega_2 \in \Omega(M)$, we have*

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2 \ ,$$

   where k is the degree of ω_1.

3. *Let $f : M \to N$ be a smooth map. The pullback f^* commutes with exterior derivative d, i.e., we have*

$$f^*(d\omega) = d(f^*\omega) \ ,$$

   for all $\omega \in \Omega(N)$.

Definition 2.37. Let $\omega \in \Omega^k(M)$. We say that ω is *closed* if $d\omega = 0$. If there is $\omega_0 \in \Omega^{k-1}(M)$ such that $\omega = d\omega_0$, we say that ω is *exact*.

Since $d^2 = 0$, every exact form is closed. In general, the converse is false. A way by which one can measure how much closed forms are far from being exact is by computing the *de Rham cohomology* of the underlying manifold. Let us introduce such cohomology.

Note that the set of all closed k-forms on a smooth manifold M is given by the vector space

$$Z^k(M) \ := \ ker\{\, d : \Omega^k(M) \to \Omega^{k+1}(M)\,\} \ ,$$

while the set of all exact forms on M is the vector space

$$B^k(M) \ := \ im\{\, d : \Omega^{k-1}(M) \to \Omega^k(M)\,\} \ .$$

The *k-th de Rham cohomology group of M* is defined to be the quotient vector space

$$H^k_{dR}(M) \ := \ \frac{Z^k(M)}{B^k(M)} \ = \ \frac{ker\{\, d : \Omega^k(M) \to \Omega^{k+1}(M)\,\}}{im\{\, d : \Omega^{k-1}(M) \to \Omega^k(M)\,\}}$$

and the de Rham cohomology of M, as being $H_{dR}(M) = \oplus_{k=0}^n H^k_{dR}(M)$. From the properties we have stated so far, it is possible to prove that the pullback f^* and the wedge product descend to cohomology, i.e., the maps

$$\begin{array}{rccc} \wedge : & H_{dR}(M) \times H_{dR}(M) & \longrightarrow & H_{dR}(M) \\ & ([\omega_1], [\omega_2]) & \longmapsto & [\omega_1 \wedge \omega_2] \end{array} \quad \text{and} \quad \begin{array}{rccc} f^* : & H_{dR}(N) & \longrightarrow & H_{dR}(M) \\ & [\omega] & \longmapsto & [f^*\omega] \end{array}$$

are both well-defined. Moreover, the first one is bilinear and the later, linear. Just as for $\Omega(M)$, the wedge product turns the de Rham cohomology $H_{dR}(M)$ into a unitary, associative, and graded commutative algebra.

2.5.2 Two Exact Sequences

Let $\{U_1, U_2\}$ be an open cover of the smooth manifold M such that $U_1 \cap U_2 \neq \emptyset$. Write $i_p : U_p \hookrightarrow M$ and $j_p : U_1 \cap U_2 \hookrightarrow U_p$ $(p = 1, 2)$ for the inclusions. For every k, there is a linear map $\Delta : H^k_{dR}(U_1 \cap U_2) \to H^{k+1}_{dR}(M)$. It is possible to show that

$$\cdots \xrightarrow{\Delta} H^k_{dR}(M) \xrightarrow{i_1^* \oplus i_2^*} H^k_{dR}(U_1) \oplus H^k_{dR}(U_2) \xrightarrow{j_1^* - j_2^*} H^k_{dR}(U_1 \cap U_2) \xrightarrow{\Delta} H^{k+1}_{dR}(M) \xrightarrow{i_1^* \oplus i_2^*} \cdots$$

is a long exact sequence, known as *Mayer-Vietoris long exact sequence*, or *Mayer-Vietoris exact sequence.*

Let M be a smooth n-manifold with non-empty boundary and let $i : \partial M \hookrightarrow M$ be the inclusion. We set

$$\Omega^k(M, \partial M) \ := \ \{\, \omega \in \Omega^k(M) \mid i^*\omega = 0\,\} \ .$$

For $\omega \in \Omega^k(M, \partial M)$, we have $i^*d\omega = di^*\omega = 0$, so $d\omega \in \Omega^{k+1}(M, \partial M)$. Consequently, we can define the *de Rham relative cohomology groups* as the quotient vector spaces

$$H^k_{dR}(M, \partial M) \ := \ \frac{ker(d : \Omega^k(M, \partial M) \longrightarrow \Omega^{k+1}(M, \partial M))}{im(d : \Omega^{k-1}(M, \partial M) \longrightarrow \Omega^k(M, \partial M))} \ .$$

An equivalence class in the relative cohomology group $H^k_{dR}(M, \partial M)$ is denoted by $[\omega]_\partial$. Writing $j := id_M : M \to M$ for the identity map, we have induced linear maps

$$\begin{array}{rccc} i^*: & H^k_{dR}(M) & \longrightarrow & H^k_{dR}(\partial M) \\ & [\omega] & \longmapsto & [i^*\omega] \end{array} \quad \text{and} \quad \begin{array}{rccc} j^*: & H^k_{dR}(M, \partial M) & \longrightarrow & H^k_{dR}(M) \\ & [\omega]_\partial & \longmapsto & [\omega] \end{array} .$$

For each k, there is a linear map $\delta : H^k_{dR}(\partial M) \to H^{k+1}_{dR}(M, \partial M)$. One can prove that

$$\cdots \xrightarrow{\delta} H^k_{dR}(M, \partial M) \xrightarrow{j^*} H^k_{dR}(M) \xrightarrow{i^*} H^k_{dR}(\partial M) \xrightarrow{\delta} H^{k+1}_{dR}(M, \partial M) \xrightarrow{j^*} \cdots$$

is a long exact sequence, called *de Rham relative long exact sequence*, or *relative exact sequence* for short. For a detailed and general account on this topic, we recommend [God71].

2.5.3 Signature

Let V be a finite dimensional real vector space and $B : V \times V \to \mathbb{R}$ a symmetric bilinear form. For any basis $\mathcal{B} = \{v_1, \cdots, v_n\}$ of V, we can consider the square matrix $\mathbb{M} = (B(v_i, v_j))_{i,j}$ giving the representation of B relatively to the basis $\mathcal{B}$. Since B is symmetric, so is the matrix $\mathbb{M}$ and, therefore, the *Spectral Theorem* implies $\mathbb{M}$ is diagonalizable. The elements figuring in a diagonal presentation of $\mathbb{M}$ are either positive, negative or zero. Let e^+ denote the number of positive elements and e^- the number of negative elements in such presentation. We define the index of B as $index(B) = e^+ - e^-$. The fact that the value $index(B)$ is well-defined, i.e., it depends neither on the basis $\mathcal{B}$ choosen nor on the diagonalization of $\mathbb{M}$ found, follows from *Sylvester's Inertial law*.

We recall that a bilinear form $B : V \times V \to \mathbb{R}$ is non-degenerate if and only if the linear map $\tilde{B} : V \longrightarrow V^*$ given by $v \longmapsto (u \mapsto B(v, u))$ is an isomorphism. Equivalently, for every $v \neq 0$ in V, there exists v_0 in V such that $B(v, v_0) \neq 0$.

Remark 2.38. The following properties follow easily from the definitions.

1. Let V be a 1-dimensional real vector space. Then, a symmetric bilinear form $B : V \times V \to \mathbb{R}$ has either index 0 or ± 1. If, in addition, B is non-degenerate, then its index is either 1 or -1.

2. $index(-B) = -index(B)$.

3. Let V_i be a vector space and B_i a symmetric bilinear form on it ($i = 1, 2$). If $V := V_1 \oplus V_2$, then $B_1 \oplus B_2$ is a symmetric bilinear form on V and we have

$$index(B_1 \oplus B_2) \quad = \quad index(B_1) \ + \ index(B_2) \ .$$

In particular,

$$index(B_1 \oplus -B_2) \quad = \quad index(B_1) \ - \ index(B_2) \ .$$

4. Let V and W be isomorphic vector spaces and let $\varphi : W \to V$ be an isomorphism. If $B : V \times V \to \mathbb{R}$ is a symmetric bilinear form on V, then $B \circ (\varphi, \varphi)$ is a symmetric bilinear form on W and

$$index(B \circ (\varphi, \varphi)) \;=\; index(B) \ .$$

Indeed, let $\{v_1, \cdots, v_n\}$ be a basis of B for which the matrix $\mathbb{M} = (B(v_i, v_j))_{i,j}$ is diagonal. Then, putting $w_i := \varphi^{-1}(v_i)$, we get a basis $\{w_1, \cdots, w_n\}$ of W for which the matrix of $B \circ (\varphi, \varphi)$ is also $\mathbb{M}$. Since the index of a bilinear form is given by the number of positive entries minus the negative entries in the diagonal of a diagonal matrix representing it, the result follows.

For M a closed, oriented $4n$-manifold, the map

$$\begin{array}{rccc} B_M : & H^{2n}_{dR}(M) \times H^{2n}_{dR}(M) & \longrightarrow & \mathbb{R} \\ & ([\omega_1], [\omega_2]) & \longmapsto & \displaystyle\int_M \omega_1 \wedge \omega_2 \end{array}$$

is a well-defined bilinear form. Indeed, if

$$(([\omega_1], [\omega_2]) \;=\; (([\omega_1'], [\omega_2']) \ ,$$

there are

$$\alpha_i \in \Omega^{2n-1}(M) \qquad\qquad (i = 1, 2)$$

such that

$$\omega_i - \omega_i' \;=\; d\alpha_i \qquad\qquad (i = 1, 2) \ .$$

Thus,

$$\begin{aligned} \int_M \omega_1 \wedge \omega_2 &= \int_M (\omega_1' + d\alpha_1) \wedge (\omega_2' + d\alpha_2) \\ &= \int_M \omega_1' \wedge \omega_2' + d\alpha_1 \wedge \omega_2' + \omega_1' \wedge d\alpha_2 + d\alpha_1 \wedge d\alpha_2 \\ &= \int_M \omega_1' \wedge \omega_2' + \int_M d(\alpha_1 \wedge \omega_2' + \alpha_2 \wedge \omega_1' + \alpha_1 \wedge d\alpha_2) \\ &= \int_M \omega_1' \wedge \omega_2' \ . \end{aligned}$$

The second equality follows from bilinearity of wedge product. The third one follows from the following remarks. By elementary operational rules of wedge product, we have

$$d(\alpha_1 \wedge \omega_2') \;=\; d\alpha_1 \wedge \omega_2' \;+\; (-1)^{2n-1}\alpha_1 \wedge d\omega_2' \;=\; d\alpha_1 \wedge \omega_2' \ ,$$

where we used that ω_2 is closed in the second equation. Similarly,

$$\omega_1' \wedge d\alpha_2 \;=\; (-1)^{(2n)^2} d\alpha_2 \wedge \omega_1' \;=\; d\alpha_2 \wedge \omega_1' \;+\; (-1)^{2n-1}\alpha_2 \wedge d\omega_1' \;=\; d(\alpha_2 \wedge \omega_1')\;,$$

where we used the closedness of ω_1' in second equation. Moreover,

$$d(\alpha_1 \wedge d\alpha_2) \;= d\alpha_1 \wedge d\alpha_2 \;+\; (-1)^{2n-1}\alpha_1 \wedge d(d\alpha_2) \;=\; d\alpha_1 \wedge d\alpha_2\;,$$

where we used $d^2 = 0$ in second equation. Therefore, by linearity of the exterior derivative, we get

$$d\alpha_1 \wedge \omega_2' + \omega_1' \wedge d\alpha_2 + d\alpha_1 \wedge d\alpha_2 \;=\; d(\alpha_1 \wedge \omega_2' + \alpha_2 \wedge \omega_1' + \alpha_1 \wedge d\alpha_2)\;.$$

Finally, since M has no boundary, the integral of a top dimensional exact form is zero by Stoke's Theorem. That B_M is bilinear and symmetric follow from the properties of the wedge product.

Definition 2.39. The signature of a closed, oriented $4n$-dimensional manifold M is given by $\sigma(M) := index(B_M)$.

Remark 2.40. If we change the orientation of M, the sign of the signature changes, i.e., $\sigma(-M) = -\sigma(M)$.

Example 2.41. The sphere $\mathbb{S}^{4n}$ is closed, oriented with dimension divisible by 4. Therefore, its signature is defined. Since $H^{2n}_{dR}(\mathbb{S}^{4n}) = \{0\}$, we have $\sigma(\mathbb{S}^{4n}) = 0$.

The same formula defines a symmetric bilinear form whenever M is a compact, oriented $4n$-manifold. The only difference is in that we have to use relative cohomology instead of usual cohomology for it to be well-defined. More precisely, the map

$$\begin{array}{rccc} B_{(M,\partial M)}: & H^{2n}_{dR}(M,\partial M) \times H^{2n}_{dR}(M,\partial M) & \longrightarrow & \mathbb{R} \\ & ([\omega_1]_\partial, [\omega_2]_\partial) & \longmapsto & \int_M \omega_1 \wedge \omega_2 \end{array}$$

is well-defined. Indeed, let $([\omega_1],[\omega_2]) = ([\omega_1'],[\omega_2'])$ be in $H^{2n}_{dR}(M,\partial M) \times H^{2n}_{dR}(M,\partial M)$. Then, there are $\alpha_i \in \Omega^{2n-1}(M,\partial M)$ $(i = 1,2)$ such that $\omega_i - \omega_i' = d\alpha_i$. Thus, we have

$$\begin{aligned} \omega_1 \wedge \omega_2 \;&=\; (\omega_1' + d\alpha_1) \wedge (\omega_2' + d\alpha_2) \\ &=\; \omega_1' \wedge \omega_2' + \omega_1' \wedge d\alpha_2 + d\alpha_1 \wedge \omega_2' + d\alpha_1 \wedge d\alpha_2\;. \end{aligned}$$

Now, note that

$$\omega_1' \wedge d\alpha_2 \;=\; d(\alpha_2 \wedge \omega_1')\;,\quad d\alpha_1 \wedge \omega_2' \;=\; d(\alpha_1 \wedge \omega_2') \qquad \text{and} \qquad d\alpha_1 \wedge d\alpha_2 \;=\; d(\alpha_1 \wedge d\alpha_2)\;.$$

Therefore,

$$\int_M \omega_1 \wedge \omega_2 \;=\; \int_M \omega_1' \wedge \omega_2' \;+\; \int_M d(\alpha_2 \wedge \omega_1' + \alpha_1 \wedge \omega_2' + \alpha_1 \wedge d\alpha_2)\ .$$

If we write $\iota : \partial M \hookrightarrow M$ for the inclusion, then Stokes Theorem gives

$$\int_M d(\alpha_2 \wedge \omega_1' + \alpha_1 \wedge \omega_2' + \alpha_1 \wedge d\alpha_2) \;=\; \int_{\partial M} \iota^*(\alpha_2 \wedge \omega_1' + \alpha_1 \wedge \omega_2' + \alpha_1 \wedge d\alpha_2)\ .$$

But since $\alpha_i \in \Omega^{2n-1}(M, \partial M)$, we have $\iota^*(\alpha_i) = 0$ and, consequently, the integral on the right side of the above equation vanishes. This shows that $B_{(M,\partial M)}$ is well-defined. That it is symmetric and bilinear follows again from elementary properties of the wedge product.

Definition 2.42. The signature of a compact, oriented $4n$-dimensional manifold M is defined by $\sigma_\partial(M) := index(B_{(M,\partial M)})$.

Remark 2.43. Changing the orientation of M, we change the sing of its signature, $\sigma_\partial(-M) = -\sigma_\partial(M)$.

Example 2.44. The disk $\mathbb{D}^{4n}$ is compact, oriented manifold with dimension divisible by 4. All of its cohomology groups, except by the 0-th one, are trivial. Thus, the signature of $\mathbb{D}^{4n}$ is well-defined. To compute it, we observe that its boundary $\mathbb{S}^{4n-1}$ has trivial $(2n-1)$-th and $2n$-th cohomology groups. The relative exact sequence then implies $H^{2n}_{dR}(\mathbb{D}^{4n}, \mathbb{S}^{4n-1}) = 0$ and, consequently, $\sigma_\partial(\mathbb{D}^{4n}) = 0$.

Let us take a look on the conditions upon which we can assure that B_M and $B_{(M,\partial M)}$ are non-degenerate.

Theorem 2.45. *Let M be a closed, oriented $4n$-manifold. The symmetric bilinear form B_M is non-degenerate.*

Proof. Give M a Riemannian metric, so we can make use of the powerful results of Hodge Theory. Let $[\omega] \in H^{2n}_{dR}(M)$ with $[\omega] \neq [0]$. We can assume ω is harmonic. Since each cohomology class has a unique harmonic representative and $[\omega] \neq [0]$, we have $\omega \neq 0$. Finally, writing $\star$ for the Hodge star operator, we have $[\star\omega] \in H^{2n}_{dR}(M)$ and

$$B_M([\omega], [\star\omega]) \;=\; \int_M \omega \wedge \star\omega \;=\; ||\omega||^2 \;\neq\; 0\ ,$$

where $||\omega|| \neq 0$ follows from $\omega \neq 0$.

□

Before proving $B_{(M,\partial M)}$ is non-degenerate, we introduce a useful gluing construction for smooth, oriented manifolds. It will also be important for subsequent developments.

Let B_1 and B_2 be two compact, oriented n-manifolds whose boundaries are orientation-preserving diffeomorphic to a closed, oriented $(n-1)-$manifold M, so we have identifications $\partial B_1 = \partial B_2 = M$. Gluing B_1 and $-B_2$ along the boundary, i.e, via the orientation-reversing diffeomorphism $id : M \to -M$, results in a closed, oriented n-manifold, which we will denote by

$$B_1 \diamond B_2 \qquad := \qquad B_1 \bigsqcup_{\partial B_1 = -\partial B_2} -B_2 \ .$$

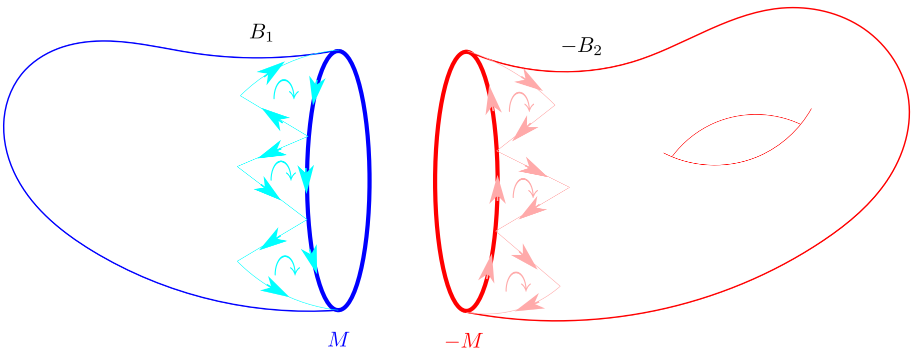

Figure 6 – Gluing two compact oriented manifolds whose boundaries are the same closed, oriented manifold by preserving the orientation of the first and reversing the orientation of the second results in a closed, oriented manifold

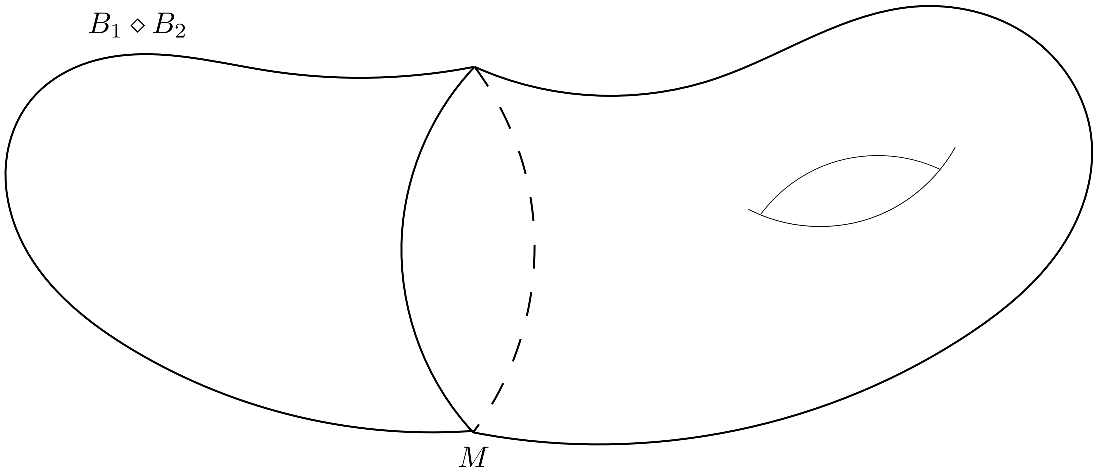

Figure 7 – The closed oriented manifold obtained by gluing construction

The inclusions $i_1 : B_1 \hookrightarrow B_1 \diamond B_2$ and $i_2 : -B_2 \hookrightarrow B_1 \diamond B_2$ are orientation-preserving and, consequently, the images $i_k(B_k)$ $(k = 1, 2)$ have orientations induced by the one of $B_1 \diamond B_2$. Moreover, the intersection $i_1(B_1) \cap i_2(-B_2)$ has measure zero, since it is diffeomorphic to a codimension one submanifold of $B_1 \diamond B_2$. Thus, for any $\omega \in H^n_{dR}(B_1 \diamond B_2)$, the integral splits:

$$\begin{aligned}
\int_{B_1 \diamond B_2} \omega &= \int_{i_1(B_1)\cup i_2(B_2)} \omega \\
&= \int_{i_1(B_1)} \omega \;+\; \int_{i_2(B_2)} \omega \\
&= \int_{B_1} i_1^* \omega \;+\; \int_{-B_2} i_2^* \omega \\
&= \int_{B_1} i_1^* \omega \;-\; \int_{B_2} i_2^* \omega \;.
\end{aligned}$$

Let us state this result for later use.

Proposition 2.46. *Let B_1 and B_2 be compact, oriented n-manifolds having as boundary a closed, oriented $(n-1)$-manifold M. Then, for any $\omega \in H^n_{dR}(B_1 \diamond B_2)$, we have*

$$\int_{B_1 \diamond B_2} \omega \quad = \quad \int_{B_1} \iota_1^* \omega \;-\; \int_{B_2} \iota_2^* \omega \;,$$

where $\iota_1 : B_1 \to B_1 \diamond B_2$ and $\iota_2 : -B_2 \hookrightarrow B_1 \diamond B_2$ stand for the inclusion maps.

Corollary 2.47. *Let M be a compact, oriented $4n$-manifold with $H^{2n-1}_{dR}(\partial M)$ and $H^{2n}_{dR}(\partial M)$ trivial. The symmetric bilinear form $B_{(M,\partial M)}$ is non-degenerate. In particular, if $H^{2n}_{dR}(M, \partial M)$ is 1-dimensional, $\sigma_\partial(M) = \pm 1$.*

Proof. Since the cohomology groups $H^{2n-1}_{dR}(\partial M)$ and $H^{2n}_{dR}(\partial M)$ are trivial, from the relative long exact sequence we conclude that

$$H^{2n}_{dR}(M, \partial M) \xrightarrow{j^*} H^{2n}_{dR}(M)$$

is an isomorphism. Now, let $C := M \diamond -M$ be the closed, oriented $4n$-dimensional manifold obtained by the gluing construction described above. Taking collar neighborhoods of M and $-M$, one sees that there are open sets U_i such that $M_i \subseteq U_i$, U_i deformation retracts onto M_i and $U_1 \cap U_2$ deformation retracts onto ∂M. Therefore, the cohomology groups of U_i are isomorphic to the ones of M and the cohomology groups of $U_1 \cap U_2$ are isomorphic to the ones of ∂M. Replacing these isomorphisms in the Mayer-Vietoris long exact sequence and recalling that $H^{2n-1}_{dR}(\partial M)$ and $H^{2n}_{dR}(\partial M)$ are trivial, we see that

$$H^{2n}_{dR}(C) \xrightarrow{i_1^* \oplus i_2^*} H^{2n}_{dR}(M) \oplus H^{2n}_{dR}(M)$$

is an isomorphism. By composing isomorphisms, we get the isomorphism

$$\varphi \;:\; H^{2n}_{dR}(C) \xrightarrow{((j^*)^{-1} \oplus (j^*)^{-1}) \circ (i_1^* \oplus i_2^*)} H^{2n}_{dR}(M, \partial M) \oplus H^{2n}_{dR}(M, \partial M) \;.$$

If $[\theta]_\partial \neq [0]_\partial \in H^{2n}_{dR}(M, \partial M)$, then, $[\omega] := \varphi^{-1}([\theta]_\partial \oplus [0]_\partial) \neq [0] \in H^{2n}_{dR}(C)$. Since B_C is non-degenerate, there is $[\omega_0] \in H^{2n}_{dR}(C)$ such that

$$\int_C \omega \wedge \omega_0 \; = \; B_C([\omega], [\omega_0]) \; \neq \; 0 \; .$$

Being φ an isomorphism, there is

$$[\theta_0]_\partial \oplus [\theta_1]_\partial \; \in \; H^{2n}_{dR}(M, \partial M) \oplus H^{2n}_{dR}(M, \partial M)$$

such that

$$(i_1^* \oplus i_2^*)[\omega_0] \; = \; (j^* \oplus j^*)([\theta_0]_\partial \oplus [\theta_1]_\partial) \; = \; [\theta_0] \oplus [\theta_1] \; .$$

In particular, $[\theta_0] = i_1^*[\omega_0]$ and $[\theta_1] = i_2^*[\omega_0]$. Observing that

$$(i_1^* \oplus i_2^*)[\omega] \; = \; [\theta] \oplus [0] \; ,$$

it follows that $i_1^*[\omega] = [\theta]$ and $i_2^*[\omega] = [0]$. Thus,

$$\int_M \theta \wedge \theta_0 \quad = \quad \int_M i_1^*\omega \wedge i_1^*\omega_0 \; - \; \int_M i_2^*\omega \wedge i_2^*\omega_0 \quad = \quad \int_C \omega \wedge \omega_0 \quad \neq \quad 0 \; ,$$

where the last equation follows from Proposition 2.46. Hence, $B_{(M,\partial M)}$ is non-degenerate.

Finally, if $H^{2n}_{dR}(M, \partial M)$ is 1-dimensional, for any basis $\{[W]_\partial\}$ we have

$$c \; := \; B_{(M,\partial M)}([W]_\partial, [W]_\partial) \; \neq \; 0 \; ,$$

because $B_{(M,\partial M)}$ is non-degenerate. Then,

$$\left\{ \left[W_0 \; := \; \frac{1}{\sqrt{|c|}} W \right]_\partial \right\}$$

is a basis of $H^{2n}_{dR}(M, \partial M)$ and

$$B_{(M,\partial M)} \left([W_0]_\partial, [W_0]_\partial\right) \; = \; \frac{1}{|c|} B_{(M,\partial M)}([W]_\partial, [W]_\partial) \; = \; \frac{c}{|c|} \; = \; \pm 1 \; .$$

□

There is yet another important consequence of Proposition 2.46, which we will make use to prove that Milnor and Shimada's invariants are well-defined.

Corollary 2.48. *Let B_1 and B_2 be compact, oriented $4n$-manifolds having as boundary a closed, oriented manifold M with $H^{2n-1}_{dR}(M)$ and $H^{2n}_{dR}(M)$ trivial cohomology groups. We have*

$$\sigma(B_1 \diamond B_2) \quad = \quad \sigma_\partial(B_1) \; - \; \sigma_\partial(B_2) \; .$$

Proof. We proceed as in the proof of Corollary 2.47. Since $H^{2n-1}_{dR}(M)$ and $H^{2n}_{dR}(M)$ are trivial, Mayer-Vietoris long exact sequence implies that

$$i_1^* \oplus i_2^* : H^{2n}_{dR}(B_1 \diamond B_2) \longrightarrow H^{2n}_{dR}(B_2) \oplus H^{2n}_{dR}(B_2)$$

is an isomorphism. Moreover, from the relative exact sequence we conclude that

$$j_k^* : H^{2n}_{dR}(B_k, M) \longrightarrow H^{2n}_{dR}(B_k)$$

are isomorphisms ($k = 1, 2$). Thus, the composition

$$\varphi \ := \ ((j_1^*)^{-1} \oplus (j_2^*)^{-1}) \circ (i_1^* \oplus i_2^*) : H^{2n}_{dR}(B_1 \diamond B_2) \to H^{2n}_{dR}(B_1, M) \oplus H^{2n}_{dR}(B_2, M)$$

is an isomorphism. Finally, for all $[\omega_1], [\omega_2] \in H^{2n}_{dR}(B_1 \diamond B_2)$, we have

$$\begin{aligned} B_{B_1 \diamond B_2}([\omega_1], [\omega_2]) \ &= \ \int_{B_1 \diamond B_2} \omega_1 \wedge \omega_2 \\ &= \ \int_{B_1} i_1^*\omega_1 \wedge i_1^*\omega_2 \ - \ \int_{B_2} i_2^*\omega_1 \wedge i_2^*\omega_2 \\ &= \ (B_{(B_1,M)} \oplus -B_{(B_2,M)})(\varphi([\omega_1]), \varphi([\omega_2])) \ . \end{aligned}$$

Therefore,

$$\sigma(B_1 \diamond B_2) \ = \ index(B_{B_1 \diamond B_2}) \ = \ index(B_{(B_1,M)}) - index(B_{(B_2,M)}) \ = \ \sigma_\partial(B_1) - \sigma_\partial(B_2) \ ,$$

by the properties given in Remark 2.38.

□

For more information about signatures and its uses in different branches of Mathematics, we recommend the paper [GR16].

3 Pontryagin Classes

3.1 Connections, Curvature and Invariant Polynomials

Recall that the collection of all vector fields on a smooth manifold M is denoted by $\mathfrak{X}(M)$. Moreover, we will write $\Gamma(\xi)$ for the set of all sections of a vector bundle ξ. We write $Mat_{n\times n}(\mathbb{R})$ for the vector space of $n \times n$ matrices with real coefficients and $Mat^*_{n\times n}(\mathbb{R})$ for the Lie group of all invertible $n \times n$ matrices with real coefficients. The references for this section are [Tu17] and [Lee18].

Definition 3.1. Let $\xi = (E, p, B)$ be a vector bundle. A *Koszul connection* on ξ is a map

$$\begin{array}{cccc} \nabla\colon & \mathfrak{X}(B) \times \Gamma(\xi) & \longrightarrow & \Gamma(\xi) \\ & (X, s) & \longmapsto & \nabla_X s \end{array}$$

which is $\mathscr{C}^\infty(B;\mathbb{R})$-linear in the first entry and $\mathbb{R}$-linear in the second. Moreover, the *Leibniz rule*

$$\nabla_X(fs) \;=\; (Xf)s \;+\; f\nabla_X s$$

must be satisfied for all $f \in \mathscr{C}^\infty(B;\mathbb{R})$, $X \in \mathfrak{X}(B)$ and $s \in \Gamma(\xi)$.

Remark 3.2. Let $\xi = (E, p, B, \mathbb{R}^n, \mathcal{A})$ be a vector bundle and let U be an open subset of B. Write $\mathcal{A} = \{(U_i, \varphi_i)\}$ for the bundle atlas. Setting $E|_U := p^{-1}(U)$ and $\mathcal{A}|_U := \Big\{\Big(U \cap U_i, \varphi_i|_{p^{-1}(U\cap U_i)}\Big)\Big\}$, we obtain a new vector bundle $\xi|U := (E|_U, p|_U, U, \mathbb{R}^n, \mathcal{A}|_U)$. We call $\xi|_U$ the *restricted vector bundle* or the *restriction of* ξ *over* U.

Remark 3.3. Let $\xi = (E, p, B)$ be a vector bundle and let ∇ be a connection on it. For every open subset U, there exists a unique connection ∇^U on the restricted bundle $\xi|_U$ such that

$$\nabla^U_{X|_U}(s|_U) \;=\; (\nabla_X s)\,|_U \;,$$

for all $X \in \mathfrak{X}(B)$ and $s \in \Gamma(\xi)$. We call ∇^U the *restriction of* ∇ *to* U. See [Lee18, Proposition 4.3].

Definition 3.4. Let $\xi = (E, p, B)$ be a vector bundle and let ∇ be a connection on it. The *curvature tensor of* ∇ is the 3-linear map

$$R : \mathfrak{X}(B) \times \mathfrak{X}(B) \times \Gamma(\xi) \longrightarrow \Gamma(\xi)$$

given by $(X, Y, s) \longmapsto \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s$.

Let $\xi = (E, p, B)$ be a vector bundle of rank n and let ∇ be a connection on it. If U is an open subset of B such that $\xi|_U$ is trivial, then there are sections $e_1, \cdots, e_n : U \to E|_U$ such that $\{e_1(b), \cdots, e_n(b)\}$ is a basis of E_b, for all $b \in U$. We call the collection $\mathcal{F} := \{e_1, \cdots, e_n\}$ a *frame over* U. For every section $s : U \to E|_U$ of $\xi|_U$, there are smooth functions $a^1, \cdots, a^n : U \to \mathbb{R}$ such that

$$s(b) \;=\; \sum_{i=1}^{n} a^i(b)\, e_j(b) \;,$$

for all $b \in U$. In particular, there are maps

$$\omega^i_j : \mathfrak{X}(U) \longrightarrow \mathscr{C}^\infty(U;\mathbb{R}) \qquad \text{and} \qquad \Omega^i_j : \mathfrak{X}(U) \times \mathfrak{X}(U) \longrightarrow \mathscr{C}^\infty(U;\mathbb{R})$$

such that

$$\nabla^U_X e_j \;=\; \sum_{i=1}^{n} \omega^i_j(X)\, e_i \qquad \text{and} \qquad R^U(X,Y)e_j \;=\; \sum_{i=1}^{n} \Omega^i_j(X,Y)\, e_i$$

in U, where ∇^U is the restriction of the connection ∇ to U and R^U is the curvature tensor of ∇^U. The matrices

$$\omega^U_{\mathcal{F}}(\cdot) \;:=\; \left[\omega^i_j(\cdot)\right]_{n\times n} \qquad \text{and} \qquad \Omega^U_{\mathcal{F}}(\cdot,\cdot) \;:=\; \left[\Omega^i_j(\cdot,\cdot)\right]_{n\times n}$$

are called *connection matrix* and *curvature matrix*, respectively. The $\mathscr{C}^\infty(U;\mathbb{R})$-linearity of ∇^U with respect to X implies that $\omega^U_{\mathcal{F}} : \mathfrak{X}(U) \to \mathscr{C}^\infty(U;\mathbb{R})$ and $\Omega^U_{\mathcal{F}} : \mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathscr{C}^\infty(U;\mathbb{R})$ are $\mathscr{C}^\infty(U;\mathbb{R})$-linear and $\mathscr{C}^\infty(U;\mathbb{R})$-bilinear, respectively. Now, by the definition of the curvature tensor, one can see that $\Omega^U_{\mathcal{F}}$ is alternating. By the lemma below, we can regard $\omega^U_{\mathcal{F}}$ and $\Omega^U_{\mathcal{F}}$ as matrices of 1-forms and 2-forms, respectively. For a proof, see [Tu17, pg. 59].

Lemma 3.5. *Let M be a smooth manifold. Suppose $\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathscr{C}^\infty(M;\mathbb{R})$ is a map that is $\mathscr{C}^\infty(M;\mathbb{R})$-linear in each of the k entries and, in addition, is alternating, i.e.,*

$$\omega(X_1, \cdots, X_k) \;=\; sgn(\sigma)\omega(X_{\sigma(1)}, \cdots, X_{\sigma(k)}) \;,$$

for all $\sigma \in S_k$. Then, there is a unique differential k-form $\tilde{\omega} : M \to \wedge^k TM$ such that

$$\tilde{\omega}_b(X_1|_b, \cdots, X_k|_b) \;=\; \omega(X_1, \cdots, X_k)(b) \;,$$

for all $b \in M$ and $X_1, \cdots, X_k \in \mathfrak{X}(M)$.

Definition 3.6. Let x^i_j be indeterminates and define $X := [x^i_j]$, the matrix of indeterminates, where $i, j = 1, \cdots, n$. We say that $P(X)$ is a *polynomial in* X if there are constants $c_0, \cdots, c_r \in \mathbb{R}$ and $\varepsilon^k_{i,j} \in \mathbb{N}$, where $i, j = 1, \cdots, n$ and $k = 1, \cdots r$, such that

$$P(X) \;=\; \sum_{k=0}^{r} c_k \left(\prod_{i,j=1}^{n} (x^i_j)^{\varepsilon^k_{i,j}} \right) \;.$$

We say that $P(X)$ is *homogeneous of degree* d, or shortly that $P(X)$ has degree d, if $\sum_{i,j=1}^{n} \varepsilon_{i,j}^{k} \;=\; d$, for all $k = 1, \cdots, r$.

Definition 3.7. Every polynomial $P(X)$ in a $n \times n$ matrix of indeterminates X defines a function $P : Mat_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$ by the formula

$$P(a) \;=\; \sum_{k=0}^{r} c_k \left(\prod_{i,j=1}^{n} (a_j^i)^{\varepsilon_{i,j}^k} \right) \;,$$

where $a = [a_j^i] \in Mat_{n\times n}(\mathbb{R})$. We say that $P(X)$ is an *invariant polynomial* if

$$P(a^{-1}\, b\, a) \;=\; P(b) \;,$$

for all $b \in Mat_{n\times n}(\mathbb{R})$ and $a \in Mat^*_{n\times n}(\mathbb{R})$.

Example 3.8. Let $X = [x_j^i]_{n\times n}$ be a matrix of indeterminates and let λ be another indeterminate. There are polynomials $f_1(X), \cdots, f_n(X)$ such that

$$det\,(\lambda id_{\mathbb{R}^n} + X) \;=\; \lambda^n + f_1(X)\lambda^{n-1} + \cdots + f_{n-1}(X)\lambda + f_n(X) \;.$$

In other words, $f_k(X)$ is the coefficient of λ^{n-k}. In particular, $f_k(X)$ has degree k. Moreover, the polynomials $f_k(X)$ are invariant. See [Tu17, Example 23.1].

Let $\mathcal{G} = \{s_1, \cdots, s_n\}$ be another frame over U and let $\omega_{\mathcal{G}}^U$ and $\Omega_{\mathcal{G}}^U$ be the associated connection and curvature matrices, respectively. There are smooth functions $a_j^i : U \to \mathbb{R}$ such that

$$s_j \;=\; \sum_{i=1}^{n} a_j^i\, e_i$$

in U. Then, the map $a : U \to Mat^*_{n\times n}(\mathbb{R})$ given by $a : b \mapsto a(b) := \left[a_j^i(b)\right]$ is smooth and we have

$$\Omega_{\mathcal{G}}^U(\cdot,\cdot) \;=\; a^{-1}(\cdot)\, \Omega_{\mathcal{F}}^U(\cdot,\cdot)\, a(\cdot)$$

in U. Therefore, every invariant polynomial $P(X)$ of degree d defines a differential $2d$-form on B. Indeed, let $P(\Omega) : B \to \wedge^{2d} T^* B$ be given by the following construction. For $b \in B$, take an open set U such that $b \in U$ and $\xi|_U$ is trivial. Then, take a frame $\mathcal{F}$ over U and define $P(\Omega)_b := P(\Omega_{\mathcal{F}}^U)$. This value depends neither on the open set U nor on the frame $\mathcal{F}$. In fact, let V be another open set with $b \in V$ and $\xi|_V$ trivial, and take $\mathcal{G}$ a frame over V. Then, $W := U \cap V$ is an open set such that $b \in W$ and with two frames $\mathcal{F}|_W$ and $\mathcal{G}|_W$ over it, where $\mathcal{F}|_W$ and $\mathcal{G}|_W$ are obtained by restriction. Thus, there is a smooth map $a : W \to Mat^*_{n\times n}(\mathbb{R})$ such that

$$\Omega_{\mathcal{G}|_W}^W(\cdot,\cdot) \;=\; a^{-1}(\cdot)\, \Omega_{\mathcal{F}|_W}^W(\cdot,\cdot)\, a(\cdot) \;,$$

in W. Thus, since $P(X)$ is invariant,

$$P(\Omega^W_{\mathcal{G}|_W}) \;=\; P(\Omega^W_{\mathcal{F}|_W}) \ ,$$

in W and, in particular,

$$P(\Omega^V_{\mathcal{G}})_b \;=\; P(\Omega^W_{\mathcal{G}|_W})_b \;=\; P(\Omega^W_{\mathcal{F}|_W})_b \;=\; P(\Omega^U_{\mathcal{F}})_b \ ,$$

proving that $P(\Omega)$ is well-defined. Smoothness of $P(\Omega)$ follows from definition, for it implies that $P(\Omega)$ is smooth on each open set U such that $\xi|_U$ is trivial. Finally, $P(\Omega)$ has degree $2d$ because $P(X)$ has degree d and the entries of $\Omega^U_{\mathcal{F}}$ are differential 2-forms.

Remark 3.9. For a real number c, we have $P(c\,\Omega) = c^d\,P(\Omega)$, where d is the degree of $P(X)$. Thus, $P(c\,\Omega) = c^d\,P(\Omega)$. In particular, when d is even and c is a purely imaginary complex number, $P(c\,\Omega)$ is a real form.

Let $\xi = (E, p, B, \mathbb{R}^n)$ be a vector bundle. It is possible to show that we can endow it with a *connection* ∇ and, in particular, we can also consider the form $P(\Omega)$, for any invariant polynomial of degree d. It is possible to show that $P(\Omega)$ is closed and that its de Rham cohomology class $[P(\Omega)]$ does not depend on the connection. See [Tu17] for more information.

Definition 3.10. Let $\xi = (E, p, B)$ be a vector bundle. We define the *k-th Pontryagin class* of ξ as

$$p_k(\xi) \;:=\; \left[f_{2k}\left(\frac{i}{2\pi}\Omega\right)\right] \;\in\; H^{4k}_{dR}(B) \ .$$

The *total Pontryagin class* is defined by

$$p(\xi) \;=\; \sum_{k=0}^{\infty} p_k(\xi) \;=\; \sum_{k=0}^{\lfloor dim(B)/4 \rfloor} p_k(\xi) \ ,$$

where $dim(B)$ is the dimension of the smooth manifold B.

Remark 3.11. Since $f_0(X) = 1$, we have $p_0(\xi) = 1$, for any vector bundle ξ.

Remark 3.12. Since $f_{2k}(X)$ has degree $2k$, the complex number i will disappear. It is included in the definition in order to get sign free formulas. We divide by 2π to obtain the cohomology class of an *integral form*. Recall that a l-form is integral if, whenever it is integrated over a submanifold of dimension l, we obtain an integer number.

Remark 3.13. For k odd, the classes $[f_k(\Omega)]$ are trivial, so we have ignored them and kept only with $[f_{2k}(\Omega)]$. See [Tu17, Theorem 24.3].

Definition 3.14. For M a smooth manifold, we define its Pontryagin classes as the Pontryagin classes $p_k(M) := p_k(\tau_M)$ of its tangent bundle τ_M.

Theorem 3.15. *Pontryagin classes satisfy the following properties:*

1. *Naturality. Let $f : B_0 \to B$ be a smooth map and let ξ be a vector bundle over B. Then,*
$$p_k(f^*\xi) \;=\; f^*(p_k(\xi)) \ ,$$
where $f^*\xi$ is the pullback bundle of ξ and $f^* : H^*_{dR}(B_0) \to H^*_{dR}(B)$ is the induced map in cohomology.

2. *Whitney sum formula. Let ξ_1 and ξ_2 be two vector bundles over B. We have*
$$p(\xi_1 \oplus \xi_2) \;=\; p(\xi_1) \oplus p(\xi_2) \ .$$

3. *If ξ_1 and ξ_2 are B-isomorphic vector bundles, then $p_k(\xi_1) = p_k(\xi_2)$.*

Example 3.16. Regard $\mathbb{S}^n$ embedded in $\mathbb{R}^{n+1}$ and consider its tangent τ_n and normal η_n bundles. The Whitney sum $\tau_n \oplus \eta_n$ is isomorphic to the trivial bundle $\varepsilon_{n+1} = (\mathbb{S}^n \times \mathbb{R}^{n+1}, \pi_1, \mathbb{S}^n, \mathbb{R}^{n+1})$. Therefore, Whitney product formula for Pontryagin classes imply
$$p(\varepsilon_{n+1}) \;=\; p(\tau_n \oplus \eta_n) \;=\; p(\tau_n)p(\eta_n) \ .$$
But since η_n and ε_{n+1} are trivial bundles over $\mathbb{S}^n$, we have $p(\eta_n) = p(\varepsilon_{n+1}) = 1$. Thus, we get $p(\mathbb{S}^n) = 1$. In particular, $p_k(\mathbb{S}^n) = p_k(\tau_n) = 0$, for all $k \geq 1$.

Definition 3.17. Let n be a positive integer number. A *weighted partition* of $4n$ is a n-uple $(i_1, \cdots, i_n)$ of non-negative integer numbers such that
$$4i_1 \;+\; 8i_2 \;+\; \cdots \;+\; 4ni_n \;=\; 4n \ .$$
The collection of all such partitions is denoted by $WP(4n)$.

If M is a closed, oriented, $4n$-manifold and $I = (i_1, \cdots, i_n) \in WP(4n)$, then the cohomology class
$$p_1(\tau_M)^{i_1} \wedge p_2(\tau_M)^{i_2} \wedge \cdots \wedge p_n(\tau_M)^{i_n}$$
has top degree and can be integrated over M. More precisely, if $p_{i_k}(M) = [\omega_k] \in H^{4k}_{dR}(M)$, then
$$p_{i_1,\cdots,i_n}(M) \;:=\; \int_M \omega_1^{i_1} \wedge \omega_2^{i_2} \wedge \cdots \wedge \omega_n^{i_n}$$
is well-defined. The value obtained by the evaluating the above integral is called a *Pontryagin number* of M. From Remark 3.12, Pontryagin numbers are integers.

Though it is possible to define Pontryagin classes for manifolds with boundary, the same can not be said about Pontryagin numbers. In case M is a compact, oriented $4n$-manifold, it may happen that choosing two different representatives for the cohomology class $p_1(M)^{i_1} \wedge \cdots \wedge p_n(M)^{i_n}$, one yields at two different numbers after integration. The problem here is the contribution such different representatives can have on the boundary. However, it is still possible to define relative Pontryagin numbers under certain additional assumptions.

Definition 3.18. Let M be a compact, oriented $4n$-manifold. We say that weighted partition I of $4n$ is *compatible with* M if, for every non-zero entry i_k of I, the homomorphism

$$j^* : H^{4k}_{dR}(M, \partial M) \longrightarrow H^{4k}_{dR}(M)$$

is an isomorphism.

Let M be a compact, oriented $4n$-manifold and let I be a compatible weighted partition of. For every non-zero entry i_k of I, let

$$[\alpha_k]_\partial \quad := \quad (j^*)^{-1}(p_k(M)) \quad \in \quad H^{4k}_{dR}(M, \partial M) \ .$$

Then, if $\{i_{k_1}, \cdots, i_{k_p} | k_1 < \cdots < k_p\}$ is the collection of all non-zero entries of I, the *relative Pontryagin number*

$$p_{i_1,\cdots,i_n}(M) \quad := \quad \int_M \alpha_{k_1}^{i_{k_1}} \wedge \cdots \wedge \alpha_{k_p}^{i_{k_p}}$$

is well-defined and, moreover, belongs to $\mathbb{Z}$.

If B_1 and B_2 are compact, oriented $4n$-manifolds for which we can perform the gluing construction, then the naturality of Pontryagin classes implies that

$$i_l^* \left(p_k(\tau_{B_1 \diamond B_2})\right) \quad = \quad p_k\left(i_l^*(\tau_{B_1 \diamond B_2})\right) \quad = \quad p_k(\tau_{B_l}) \ ,$$

where $i_l : (-1)^{l-1} B_l \hookrightarrow B_1 \diamond B_2$ $(l = 1, 2)$ are the inclusion maps. In particular, if we write

$$[\omega_k] \quad = \quad p_k(B_1 \diamond B_2) \ ,$$

then

$$[i_l^* \omega_k] \quad = \quad p_k(B_l) \ ,$$

for $l = 1, 2$. Let M be the closed, oriented $(4n-1)$-manifold for which $\partial B_1 = \partial B_2 = M$. Suppose I is weighted partition of $4n$ compatible with both B_1 and B_2 and let $\{i_{k_1}, \cdots, i_{k_p} | k_1 < \cdots < k_p\}$ be the collection of all non-zero entries of I. Then, setting

$$[\alpha_{k_s}]_\partial \quad := \quad (j_1^*)^{-1}(p_{k_s}(B_1)) \quad \in \quad H^{4k_s}_{dR}(B_1, M)$$

and

$$[\beta_{k_s}]_\partial \;:=\; (j_2^*)^{-1}(p_{k_s}(B_2)) \;\in\; H^{4k_s}_{dR}(B_2, M)\ ,$$

where $j_l^* : H^k_{dR}(B_l, M) \to H^k_{dR}(B_l)$ $(l = 1, 2)$ is the homomorphism given in the relative long exact sequence, we have

$$\begin{aligned}
p_{i_1,\cdots,i_n}(B_1 \diamond B_2) &= \int_{B_1\diamond B_2} \omega^{i_{k_1}}_{k_1} \wedge \cdots \wedge \omega^{i_{k_p}}_{k_p} \\
&= \int_{B_1} \omega^{i_{k_1}}_{k_1} \wedge \cdots \wedge \omega^{i_{k_p}}_{k_p} \;+\; \int_{-B_2} \omega^{i_{k_1}}_{k_1} \wedge \cdots \wedge \omega^{i_{k_p}}_{k_p} \\
&= \int_{B_1} i_1^*\left(\omega^{i_{k_1}}_{k_1} \wedge \cdots \wedge \omega^{i_{k_p}}_{k_p}\right) \;-\; \int_{B_2} i_2^*\left(\omega^{i_{k_1}}_{k_1} \wedge \cdots \wedge \omega^{i_{k_p}}_{k_p}\right) \\
&= \int_{B_1} \alpha^{i_{k_1}}_{k_1} \wedge \cdots \wedge \alpha^{i_{k_p}}_{k_p} \;-\; \int_{B_2} \beta^{i_{k_1}}_{k_1} \wedge \cdots \wedge \beta^{i_{k_p}}_{k_p} \\
&= p_{i_1,\cdots,i_n}(B_1) \;-\; p_{i_1,\cdots,i_n}(B_2)\ .
\end{aligned}$$

This proves the following result.

Proposition 3.19. *Let B_1 and B_2 be two compact, oriented $4n$-manifold whose boundaries are the same closed, oriented manifold M. If $I = (i_1, \cdots, i_n)$ is a weighted partition of $4n$ compatible with B_1 and B_2, then*

$$p_{i_1,\cdots,i_n}(B_1 \diamond B_2) \;=\; p_{i_1,\cdots,i_n}(B_1) \;-\; p_{i_1,\cdots,i_n}(B_2)\ .$$

Let M be a compact, oriented $4n$-manifold. Suppose I is a weighted partition of $4n$ for which $H^{4k-1}_{dR}(\partial M)$ and $H^{4k}_{dR}(\partial M)$ are trivial whenever $i_k \neq 0$. Then, the relative long exact sequence implies I is compatible with M and we get the following result.

Corollary 3.20. *Let B_1 and B_2 be two compact, oriented $4n$-manifold whose boundaries are the same closed, oriented manifold M. If $I = (i_1, \cdots, i_n)$ is a weighted partition of $4n$ such that $H^{4k-1}_{dR}(M)$ and $H^{4k}_{dR}(M)$ are trivial whenever $i_k \neq 0$, then I is compatible with both B_1 and B_2. In particular, we have*

$$p_{i_1,\cdots,i_n}(B_1 \diamond B_2) \;=\; p_{i_1,\cdots,i_n}(B_1) \;-\; p_{i_1,\cdots,i_n}(B_2)\ .$$

Example 3.21. We now compute the relative Pontryagin numbers of $\mathbb{D}^{4n}$. Let $I = (i_1, \cdots, i_n)$ be a weighted partition of n for which $i_n = 0$. Notice that the cohomology of $\mathbb{S}^{4n-1}$ and the relative long exact sequence imply that $j^* : H^k_{dR}(\mathbb{D}^{4n}, \mathbb{S}^{4n-1}) \to H^k_{dR}(\mathbb{D}^{4n})$ are isomorphisms, for $k = 1, \cdots, 4n-1$. Therefore, I is compatible with $\mathbb{D}^{4n}$. Since $\mathbb{D}^{4n}$ has trivial cohomology in all degrees greater than zero, it follows that the Pontryagin classes $p_k(\mathbb{D}^{4n})$ are exact and, therefore, so are $[\alpha_k]_\partial := (j^*)^{-1}(p_k(\mathbb{D}^n)) \in H^{4k}_{dR}(\mathbb{D}^{4n}, \mathbb{S}^{4n-1})$. Let $p \in \{1, \cdots, 4n-1\}$ be such that $i_p \neq 0$ and take $\beta \in \Omega^{4p-1}(\mathbb{D}^{4n}, \mathbb{S}^{4n-1})$ satisfying $\alpha_p = d\beta$. We have

$$\begin{aligned} d\left(\beta \wedge \alpha_1^{i_1} \wedge \cdots \wedge d\beta^{i_p-1} \wedge \cdots \wedge \alpha_{4n-1}^{i_{4n-1}}\right) &= d\beta \wedge \alpha_1^{i_1} \wedge \cdots \wedge d\beta^{i_p-1} \wedge \cdots \wedge \alpha_{4n-1}^{i_{4n-1}} \\ &- \beta \wedge d\left(\alpha_1^{i_1} \wedge \cdots \wedge d\beta^{i_p-1} \wedge \cdots \wedge \alpha_{4n-1}^{i_{4n-1}}\right) \end{aligned}$$

Since $\alpha_1^{i_1} \wedge \cdots \wedge d\beta^{i_p-1} \wedge \cdots \wedge \alpha_{4n-1}^{i_{4n-1}}$ is the wedge product of closed forms, it is also closed. In particular, its exterior derivative vanishes and we get

$$d\left(\beta \wedge \alpha_1^{i_1} \wedge \cdots \wedge d\beta^{i_p-1} \wedge \cdots \wedge \alpha_{4n-1}^{i_{4n-1}}\right) = \pm\, \alpha_1^{i_1} \wedge \cdots \wedge \alpha_{4n-1}^{i_{4n-1}}$$

after reoganzing the elements in the first term of the above equation. Writing $\iota : \mathbb{S}^{4n-1} \hookrightarrow \mathbb{D}^{4n}$ for the inclusion and observing that $d\beta^{i_p-1} = \alpha_p^{i_p-1}$, Stokes theorem gives

$$\int_{\mathbb{D}^{4n}} \alpha_1^{i_1} \wedge \cdots \wedge \alpha_{4n-1}^{i_{4n-1}} = \pm \int_{\mathbb{S}^{4n-1}} \iota^*\left(\beta \wedge \alpha_1^{i_1} \wedge \cdots \wedge \alpha_p^{i_p-1} \wedge \cdots \wedge \alpha_{4n-1}^{i_{4n-1}}\right)$$

If $i_p > 1$ or $i_l \neq 0$, for some $l \neq p$, then the integral on the right vanishes, because each form α_k is in $\Omega^{4k}(\mathbb{D}^{4n}, \mathbb{S}^{4n-1})$. We claim that it is not possible to have $i_p = 1$ and all the other coefficients i_l equal to zero. Otherwise, we would have $4n = 4pi_p = 4p$. Therefore, $p = n$ and $1 = i_p = i_n$. This is a contradiction, since we have assumed $i_n = 0$. We finally conclude that $p_{i_1,\cdots,i_{n-1},0}(\mathbb{D}^{4n}) = 0$.

3.2 Pontryagin numbers of D_k^{2a}

We recall that $S^a_{h,j}$ is homeomorphic to $\mathbb{S}^{2a-1}$, whenever $h+j = 1$. Then, for $h+j = 1$, the de Rham long exact sequence implies that $j^* : H^a_{dR}(D^a_{h,j}, S^a_{h,j}) \to H^a_{dR}(D^a_{h,j})$ is an isomorphism. But since $D^a_{h,j}$ has the same cohomology of $\mathbb{S}^a$, it follows that $H^a_{dR}(D^a_{h,j}, S^a_{h,j})$ is a 1-dimensional vector space. By Corollary 2.47, $\sigma_\partial(D^a_{h,j}) = \pm 1$.

Definition 3.22. Let k an odd integer and consider the integer numbers

$$h_k := \frac{1+k}{2} \qquad \text{and} \qquad j_k := \frac{1-k}{2} .$$

Since $h_k + j_k = 1$, the preceding discussion shows $\sigma_\partial(D^a_{h_k,j_k}) = \pm 1$. If $\sigma_\partial(D^a_{h_k,j_k}) = 1$, we set $D_k^{2a} := D^a_{h_k,j_k}$. Otherwise, $\sigma_\partial(D^a_{h_k,j_k}) = -1$ and we define $D_k^{2a} := -D^a_{h_k,j_k}$. The point of such convention is that we always have $\sigma_\partial(D_k^{2a}) = 1$.

The goal of this section is to establish the following result, which will play a role in later computations.

Proposition 3.23. *Let k be an odd integer. We have*

1. *D_k^8 is a compact, oriented 8-manifold with*

$$p_{2,0}(D_k^8) \;=\; 4k^2 \qquad \text{and} \qquad \sigma_\partial(D_k^8) \;=\; 1 \ .$$

   We further have that $\partial D_k^8 = S_k^7$ is diffeomorphic to the 7-sphere

2. *D_k^{16} is a compact, oriented 8-manifold satisfying*

$$p_{0,2,0,0}(D_k^{16}) \;=\; 36k^2 \quad , \quad \sigma_\partial(D_k^{16}) \;=\; 1$$

   and

$$p_{1,0,1,0}(D_k^{16}) \;=\; p_{2,1,0,0}(D_k^{16}) \;=\; p_{4,0,0,0}(D_k^{16}) \;=\; 0 \ .$$

   Moreover, $\partial D_k^{16} = S_k^{15}$ is diffeomorphic to the 15-sphere.

We first compute the Pontryagin classes $p_{a/4}(\xi^a_{h,j})$. To this end, let us fix the standard orientation of $\mathbb{S}^a$ and also the generator $[u_a]$ of $H^a_{dR}(\mathbb{S}^a)$ for which

$$\int_{\mathbb{S}^a} u_a \;=\; 1 \ .$$

Being $[u_a]$ a generator of $H^a_{dR}(\mathbb{S}^a)$, for all h, j integers, there is a real number $c^a_{h,j}$ such that $p_{a/4}(\xi^a_{h,j}) = c^a_{h,j}[u_a]$.

Proposition 3.24. *We have $p_{a/4}(\xi^a_{h,j}) = c_a(h-j)[u_a]$, where $c_a := c^a_{1,0}$.*

To prove the above proposition, we need the following result.

Lemma 3.25. *The map $\varphi_a : \mathbb{Z}\times\mathbb{Z} \to H^a(\mathbb{S}^a)$ given by $\varphi_a(h,j) := p_{a/4}(\xi^a_{h,j})$ is a group homomorphism.*

Proof. The idea is to write φ_a as a composition of group homomorphisms. We know from Theorem 2.22 that there is a group homomorphism

$$\begin{array}{rccc} \psi : & \pi^\infty_{a-1}(SO(a)) & \longrightarrow & \pi^\infty_a(G_a(\mathbb{R}^{2a})) \\ & [f]_\infty & \longmapsto & [\tilde{f}]_\infty \end{array}$$

such that the vector bundles $\xi_{f\circ r}$ and $\tilde{f}^*\gamma^{2a}_a$ are $\mathbb{S}^a$-isomorphic. See [Tio09] for the topological case. Moreover, consider the isomorphism

$$\theta : \mathbb{Z}\times\mathbb{Z} \longrightarrow \pi^\infty_{a-1}(SO(a))$$

given in section 2 of the previous chapter.The map

$$\begin{array}{rccc} \lambda_a : & \pi_a^\infty(G_a(\mathbb{R}^{2a})) & \longrightarrow & H^a_{dR}(\mathbb{S}^a) \\ & [\tilde{f}]_\infty & \longmapsto & p_{a/4}(\tilde{f}^*(\gamma_a^{2a})) \end{array} .$$

is also a group homomorphism, for

$$p_{a/4}(\tilde{f}^*(\gamma_a^{2a})) \;=\; \tilde{f}^*(p_{a/4}(\gamma_a^{2a}))$$

and

$$\begin{array}{ccc} \pi_l^\infty(M) & \longrightarrow & Hom(H^k_{dR}(M), H^k_{dR}(\mathbb{S}^l)) \\ [f]_\infty & \longmapsto & f^* \end{array}$$

is a group homomorphism. Finally, we see that

$$\varphi_a(h,j) \;=\; p_{a/4}(\xi^a_{h,j}) \;=\; (\lambda_a \circ \psi \circ \theta)(h,j) \ .$$

$\square$

Proof. (Proposition 3.24) Since φ_a is a group homomorphism, we have

$$\varphi_a(h,j) \;=\; h\varphi_a(1,0) \;+\; j\varphi_a(0,1) \ .$$

Recall from Lemma 2.32 that $\xi^a_{h,j}$ and $\xi^a_{-j,-h}$ are $\mathbb{S}^a$-isomorphic vector bundles. Therefore,

$$\varphi_a(h,j) \;=\; p_{a/4}(\xi^a_{h,j}) \;=\; p_{a/4}(\xi^a_{-j,-h}) \;=\; \varphi_a(-j,-h) \ ,$$

as isomorphic vector bundles over the same base space have equal Pontryagin classes. In particular,

$$\varphi_a(0,1) \;=\; \varphi_a(-1,0) \;=\; -\varphi_a(1,0) \ ,$$

and it follows that

$$\varphi_a(h,j) \;=\; (h-j)\varphi_a(1,0) \;=\; (h-j)c^a_{1,0}[u_a] \;=\; c_a(h-j)[u_a] \ ,$$

as we wanted to prove.

$\square$

Proposition 3.26. *We have* $p_{a/4}(E^a_{h,j}) = c_a(h-j)[\omega^a_{h,j}]$, *where* $[\omega^a_{h,j}] := (p^a_{h,j})^*([u_a])$.

Proof. Roughly speaking, the tangent space to each point $v \in E^a_{h,j}$ is given by the combination of two directions:

1. a "*horizontal direction*", given by the tangent space $T\mathbb{S}^a_{p^a_{h,j}(v)}$ at $p^a_{h,j}(v)$ and

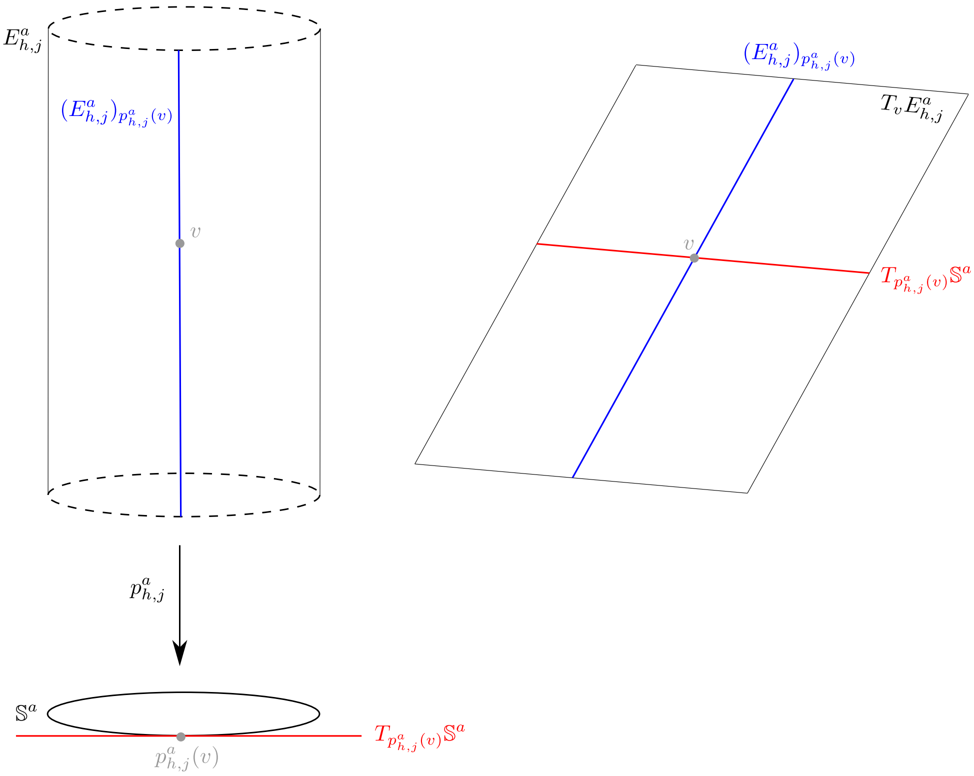

Figure 8 – Two directions spanning $T_v E^a_{h,j}$

2. a "*vertical direction*", given by the fibre $\left(E^a_{h,j}\right)_{p^a_{h,j}(v)}$ over $p^a_{h,j}(v)$ itself.

To be more precise, the tangent bundle $\tau_{E^a_{h,j}}$ is $E^a_{h,j}$-isomorphic (not canonically, though) to the Whitney sum $(p^a_{h,j})^*(\tau_{\mathbb{S}^a}) \oplus (p^a_{h,j})^*(\xi^a_{h,j})$. Thus,

$$p_{a/4}(\tau_{E^a_{h,j}}) \quad = \quad p_{a/4}\left((p^a_{h,j})^*(\xi^a_{h,j}) \ \oplus \ (p^a_{h,j})^*(\tau_{\mathbb{S}^a})\right) \ ,$$

and by Whitney sum formula, we have

$$p_{a/4}(E^a_{h,j}) \ = \ p_{a/4}(\tau_{E^a_{h,j}}) \ = \ \sum_{k=0}^{a/4} p_k\left((p^a_{h,j})^*(\xi^a_{h,j})\right) p_{(a/4)-k}\left((p^a_{h,j})^*(\tau_{\mathbb{S}^a})\right) \ .$$

Now, by naturality of Pontryagin classes, we obtain

$$p_{a/4}(E^a_{h,j}) \ = \ \sum_{k=0}^{a/4}(p^a_{h,j})^*\left(p_k(\xi^a_{h,j})\right)(p^a_{h,j})^*\left(p_{(a/4)-k}(\tau_{\mathbb{S}^a})\right) \ = \ (p^a_{h,j})^*\left(p_{a/4}(\xi^a_{h,j})\right) \ ,$$

where the last equation follows from the fact that the only non-vanishing Pontryagin class of $\tau_{\mathbb{S}^a}$ is the 0-th one, which is precisely 1. See Example 3.16. It then follows from Proposition 3.24 that

$$p_{a/4}(E^a_{h,j}) \ = \ (p^a_{h,j})^*\left(c_a(h-j)[u_a]\right) \ = \ c_a(h-j)(p^a_{h,j})^*([u_a]) \ = \ c_a(h-j)[\omega^a_{h,j}] \ ,$$

as we wanted to prove.

□

Let $\iota^a_{h,j} : D^a_{h,j} \hookrightarrow E^a_{h,j}$ be the inclusion and set $[\theta^a_{h,j}] := (\iota^a_{h,j})^*([\omega^a_{h,j}]) = (p^a_{h,j} \circ \iota^a_{h,j})^*([u_a])$.

Corollary 3.27. *We have* $p_{a/4}(D^a_{h,j}) = c_a(h-j)[\theta^a_{h,j}]$.

Proof. The tangent bundle $\tau_{D^a_{h,j}}$ of $D^a_{h,j}$ is isomorphic to the pullback bundle $(\iota^a_{h,j})^*(\tau_{E^a_{h,j}})$. Thus,

$$p_{a/4}(D^a_{h,j}) \;=\; p_{a/4}(\tau_{D^a_{h,j}}) \;=\; p_{a/4}\left((\iota^a_{h,j})^*\tau_{E^a_{h,j}}\right) \;=\; (\iota^a_{h,j})^*(p_{a/4}(\tau_{E^a_{h,j}})) \;=\; (\iota^a_{h,j})^*\left(p_{a/4}(E^a_{h,j})\right) \;,$$

where the second equality follows from the fact that isomorphic vector bundles have the same Pontryagin classes and the third, by naturality. The thesis follows from the preceding proposition, for

$$(\iota^a_{h,j})^*\left(p_{a/4}(E^a_{h,j})\right) \;=\; (\iota^a_{h,j})^*\left(c_a(h-j)[\omega^a_{h,j}]\right) \;=\; c_a(h-j)[\theta^a_{h,j}] \;.$$

□

Let I_a be the weighted partition of $2a$ given by 2 in the $(a/4)$-entry and 0, otherwise. For $h+j=1$, the relative long exact sequence and the fact that $S^a_{h,j}$ is homeomorphic to $\mathbb{S}^{2a-1}$ imply that I_a is compatible with $D^a_{h,j}$. Let

$$[\alpha^a_{h,j}]_\partial \;:=\; (j^*)^{-1}([\theta^a_{h,j}]) \;\in\; H^a_{dR}(D^a_{h,j}, S^a_{h,j}) \;.$$

Proposition 3.28. $\displaystyle\int_{D^a_{h,j}} \alpha^a_{h,j} \wedge \alpha^a_{h,j} \;=\; \pm 1 \;.$

Proof. Observe that the image of the zero section $M^a_{h,j} := Z^a_{h,j}(\mathbb{S}^a)$ is an oriented submanifold of $D^a_{h,j}$. Moreover, the class

$$[\theta^a_{h,j}] \;\in\; H^a_{dR}(D^a_{h,j}) \;\simeq\; Hom\left(H_a(D^a_{h,j}), \mathbb{R}\right)$$

is characterized by

$$\int_{M^a_{h,j}} \theta^a_{h,j} \;=\; 1 \;.$$

In order to compare $[\theta^a_{h,j}]$ with $j_*(PD(M^a_{h,j}))$, for $PD(M^a_{h,j})$ *Poincaré dual* (see [Lef26]) of $M^a_{h,j}$, we recall that

$$\int_{M^a_{h,j}} PD(M^a_{h,j}) \;=\; \int_{M^a_{h,j}} e \;,$$

where $e \in H^a(\mathbb{S}^a;\mathbb{Z})$ is the *Euler class* (see [BT82] and [MS74]) of the vector bundle $\xi^a_{h,j}$. Then we look at the following piece of the *Gysin sequence* (see [Gys42]) for the sphere bundle $S^a_{h,j} \to \mathbb{S}^a$:

$$H^{a-1}(S^a_{h,j};\mathbb{Z}) \to H^0(\mathbb{S}^a;\mathbb{Z}) \xrightarrow{\cup e} H^a(\mathbb{S}^a;\mathbb{Z}) \to H^a(S^a_{h,j}) \ .$$

Since $S^a_{h,j}$ is homeomorphic to $\mathbb{S}^{2a-1}$, it follows that $\cup e$ is an isomorphism, and hence

$$\int_{M^a_{h,j}} e \ = \ \pm 1 \, .$$

Therefore, $j_*(PD(M^a_{h,j})) = \pm[\theta^a_{h,j}]$ or, $PD(M^a_{h,j}) = \pm[\alpha^a_{h,j}]$. Hence,

$$\int_{D^a_{h,j}} \alpha^a_{h,j} \wedge \alpha^a_{h,j} \ = \ \pm \int_{D^a_{h,j}} \alpha^a_{h,j} \wedge PD(M^a_{h,j}) \ = \ \pm \int_{M^a_{h,j}} \alpha^a_{h,j} \ = \ \pm 1 \, .$$

□

Corollary 3.29. *If k is an odd integer, then $p_{I_a}(D^{2a}_k) = c_a^2 k^2$.*

Proof. Observe that

$$(j^*)^{-1}(p_{a/4})(D^a_{h_k,j_k}) \ = \ (j^*)^{-1}(c_a(h_k - j_k)[\theta^a_{h_k,j_k}]) \ = \ c_a k[\alpha^a_{h_k,j_k}] \, ,$$

so the corresponding relative Pontryagin number of $D^a_{h_k,j_k}$ is

$$p_{I_a}(D^a_{h_k,j_k}) \ = \ \int_{D^a_{h_k,j_k}} c_a k \alpha^a_{h_k,j_k} \wedge c_a k \alpha^a_{h_k,j_k} \ = \ c_a^2 k^2 \int_{D^a_{h_k,j_k}} \alpha^a_{h_k,j_k} \wedge \alpha^a_{h_k,j_k} \ .$$

By the proposition above, the integral on the right is equal to ± 1. If it is equal to 1, then $D^{2a}_k = D^a_{h_k,j_k}$ and $p_{I_a}(D^{2a}_k) = p_{I_a}(D^a_{h_k,j_k}) = c_a^2 k^2$. Otherwise, such integral equals -1 and $D^{2a}_k = -D^a_{h_k,j_k}$, from what follows

$$\int_{D^{2a}_k} \alpha^a_{h_k,j_k} \wedge \alpha^a_{h_k,j_k} \ = \ \int_{-D^a_{h_k,j_k}} \alpha^a_{h_k,j_k} \wedge \alpha^a_{h_k,j_k} \ = \ - \int_{D^a_{h_k,j_k}} \alpha^a_{h_k,j_k} \wedge \alpha^a_{h_k,j_k} \ = \ 1 \ .$$

In any case, $p_{I_a}(D^{2a}_k) = c_a^2 k^2$.

□

Proposition 3.30. $c_4^2 = 4$ *and* $c_8^2 = 36$.

Proof. Recall that $\mathbb{A} = \mathbb{H}$ or $\mathbb{A} = \mathbb{O}$, accordingly to $a = 4$ or $a = 8$. In order to determine c_a^2, we remark that $D^a_{1,0}$ is diffeomorphic to $M_a := \mathbb{A}P^2 - \mathring{B}^{2a}$, i.e., the projective plane $\mathbb{A}P^2$ with an open $2a$-cell removed. The proof of this fact is technically involved. See the proof of Proposition 2.3.1 in [Gre12] for the case $\mathbb{A} = \mathbb{O}$. One can mimic the ideas involved there for the case $\mathbb{A} = \mathbb{H}$ because $\mathbb{H}P^2$ and $\mathbb{O}P^2$ are both obtained by similar constructions involving Hopf maps. Notice that ∂M_a is diffeomorphic to $\mathbb{S}^{2a-1}$. Therefore, orienting M_a as a submanifold of $\mathbb{A}P^2$ and orienting $\mathbb{D}^{2a}$ in such a way that the induced orientation on the boundary $\mathbb{S}^{2a-1}$ is the same one of ∂M_a, we see that $\mathbb{A}P^2$ is diffeomorphic to $M_a \diamond \mathbb{D}^{2a}$. In particular, Proposition 3.19 and Example 3.21 imply

$$p_I(\mathbb{A}P^2) \;=\; p_I(M_a) \;-\; p_I(\mathbb{D}^{2a}) \;=\; p_I(M_a) \;,$$

for all weighted partition I of $2a$ compatible with M_a and $\mathbb{D}^{2a}$. Since $D^a_{1,0}$ and M_a are diffeomorphic, their relative Pontryagin number are the same up to sign, accordingly to the diffeomorphism between them being orientation preserving or reversing . Moreover, I_a is a partition of $2a$ compatible with booth M_a and $D^a_{1,0}$, and so we have

$$p_{I_a}(D^a_{1,0}) \;=\; \pm\, p_{I_a}(M_a) \;=\; \pm\, p_{I_a}(\mathbb{A}P^2) \;.$$

We will complete our analysis by considering the cases $a = 4$ and $a = 8$ separately.

For $a = 4$, we have $p_{I_4}(\mathbb{H}P^2) = p_{2,0}(\mathbb{H}P^2) = 4$. See [BH58, Section 18.1]. Thus,

$$c_4^2 \;=\; \pm\, p_{2,0}(\mathbb{H}P^2) \;=\; \pm\, 4 \;.$$

But since c_4 is a real number, we must have $c_4^2 = 4$.

Now let $a = 8$. It follows from [BH58, Theorem 19.4] that $p_{0,2,0,0}(\mathbb{O}P^2) = 36$. Thus,

$$c_8^2 \;=\; \pm\, p_{0,2,0,0}(\mathbb{O}P^2) \;=\; \pm\, 36 \;.$$

Being c_8 a real number, we must have $c_8^2 = 36$.

□

Proof. (Proposition 3.23) As we have already pointed out, the definition of D^{2a}_k implies that $\sigma_\partial(D^{2a}_k) = 1$. Moreover, Corollary 3.29 and Proposition 3.30 imply that

$$p_{2,0}(D^8_k) \;=\; 4k^2 \qquad\qquad \text{and} \qquad\qquad p_{0,2,0,0}(D^{16}_k) \;=\; 36k^2$$

It only remains to show that

$$p_{1,0,1,0}(D^{16}_k) \;=\; p_{2,1,0,0}(D^{16}_k) \;=\; p_{4,0,0,0}(D^{16}_k) \;=\; 0 \;.$$

Since D^{16}_k has the same cohomology of $\mathbb{S}^8$, the de Rham relative long exact sequence implies that $j^* : H^l_{dR}(D^{16}_k, \partial D^{16}_k) \to H^l_{dR}(D^{16}_k)$ is an isomorphism for $l = 4, 16$. Furthermore, for these values of l, the cohomology group $H^l_{dR}(D^{16}_k)$ is trivial and, therefore, so is $H^l_{dR}(D^{16}_k, \partial D^{16}_k)$. This implies that the forms figuring in the integrals that give the above three relative Pontryagin numbers are exact and, since they all vanish on the boundary ∂D^{16}_k, Stokes theorem implies that they integrate to zero.

□

3.3 Hirzebruch Signature Theorem

The content of Hirzebruch signature theorem is that the signature of any closed, oriented $4n$-manifold is expressible as a $\mathbb{Q}$-linear combination of its Pontryagin numbers. Therefore, this theorem relates a *homotopy type invariant* (the signature) with a *diffeomorphims invariant* (Pontryagin numbers).

Theorem 3.31 (Hirzebruch)**.** *For a closed, oriented $4n$-manifold M, we have*

$$\sigma(M) \quad = \quad \int_M L_n(p_1(M), \cdots, p_n(M)) \ ,$$

where $L_n(x_1, \cdots, x_n) \in \mathbb{Q}[x_1, \cdots, x_n]$ is the n-th L-polynomial.

For a proof, we refer the reader to [MS74]. In an appendix of [Tu17], the reader can consult the definition of the polynomials L_n. It is possible to compute the first few of them by hand, but we will not follow this approach. Instead, we register that

$$L_2(x_1, x_2) \ = \ \frac{1}{45}(7x_2 - x_1^2)$$

and

$$L_4(x_1, x_2, x_3, x_4) \ = \ \frac{1}{14175}(381x_4 - 71x_1x_3 - 19x_2^2 + 22x_1^2x_2 - 3x_1^4) \ ,$$

together with Theorem 3.31, immediately yield the following corollaries.

Corollary 3.32. *For a closed, oriented 7-manifold M, we have*

$$\sigma(M) \ = \ \frac{1}{45}(7p_{0,1}(M) \ - \ p_{2,0}(M)) \ .$$

Corollary 3.33. *For a closed, oriented 15-manifold M, we have*

$$\begin{aligned} \sigma(M) \ = \ \frac{1}{14175}\{381\, p_{0,0,0,1}(M) \ - \ 71\, p_{1,0,1,0,}(M) \ - \ 19\, p_{0,2,0,0}(M) \\ + \ 22\, p_{2,1,0,0}(M) \ - \ 3\, p_{4,0,0,0}(M)\} \ . \end{aligned}$$

The reader can find a *Mathematica* code, due to Carl McTague, to compute L-polynomials in the page `https://oeis.org/A237111`. For a first hand account on Hirzebruch siganture theorem, we refer the reader to [Hir71].

4 Exotic Spheres

4.1 Milnor's invariant

Let M be a closed, oriented 7-manifold such that $H^3_{dR}(M) = H^4_{dR}(M) = \{0\}$, e.g., $M = \mathbb{S}^7$. Assume further that M bounds a compact, oriented 8-manifold B. Then, by the relative long exact sequence, $j^* : H^4_{dR}(B, M) \to H^4_{dR}(B)$ is an isomorphism, i.e., the weighted partition $(2, 0)$ of 8 is compatible with B and we can consider the relative Pontryagin class $p_{2,0}(B)$. We define Milnor's invariant as

$$\lambda(M) \quad \stackrel{\text{def}}{\equiv} \quad 2\,p_{2,0}(B) \;-\; \sigma_\partial(B) \quad \pmod 7 \;.$$

We claim that λ is well-defined. Indeed, let B_i $(i = 1, 2)$ be compact, oriented 8-manifolds whose boundaries are M. By *Hirzebruch's Signature Theorem*,

$$\sigma(B_1 \diamond B_2) \;=\; \frac{1}{45}\,(7\,p_{0,1}(B_1 \diamond B_2) \;-\; p_{2,0}(B_1 \diamond B_2)) \;\;,$$

from what follows that

$$45\,\sigma(B_1 \diamond B_2) \;+\; p_{2,0}(B_1 \diamond B_2) \;\equiv\; 7\,p_{0,1}(B_1 \diamond B_2) \;\equiv\; 0 \quad \pmod 7 \;.$$

By multiplying both sides by 2, we get

$$2\,p_{2,0}(B_1 \diamond B_2) \;-\; \sigma(B_1 \diamond B_2) \;\equiv\; 0 \quad \pmod 7 \;. \tag{4.1}$$

From Corollary 2.48 and Corollary 3.20, we see that

$$\sigma(B_1 \diamond B_2) \;=\; \sigma_\partial(B_1) \;-\; \sigma_\partial(B_2)$$

and

$$p_{2,0}(B_1 \diamond B_2) \;=\; p_{2,0}(B_1) \;-\; p_{2,0}(B_2) \;,$$

respectively. Therefore, substituting these values in equation (4.1) and rearranging the terms, we conclude that

$$2\,p_{2,0}(B_1) \;-\; \sigma_\partial(B_1) \;\equiv\; 2\,p_{2,0}(B_2) \;-\; \sigma_\partial(B_2) \quad \pmod 7 \;,$$

proving that λ is well-defined.

Remark 4.1. The french mathematician R. Thom showed that the 7-th oriented cobordism group is trivial [Tho54, Théorème IV.13]. In other words, every closed, oriented 7-manifold is the boundary of a compact, oriented 8-manifold. Thus, Milnor's invariant is actually defined for every closed, oriented manifold M for which the third and the fourth Betti numbers are zero.

Proposition 4.2. *Let M and N be closed, oriented 7-manifolds for which Milnor's invariant is defined. If they are diffeomorphic, then*

$$\lambda(M) \equiv \pm\,\lambda(N) \pmod 7 ,$$

according as the diffeomorphism preserves or reverses orientation. In particular, $\lambda(-M) \equiv -\lambda(M) \pmod 7$.

Proof. Let B be a compact, oriented 8-manifold with $\partial B = M$. If M and N are orientation-preserving (resp.orientation-reversing) diffeomorphic, then we may regard N as the boundary of B (resp. $-B$). In particular, $\lambda(M) \equiv \lambda(N) \pmod 7$ (resp. $\lambda(M) \equiv -\lambda(N) \pmod 7$).

□

Corollary 4.3. *Suppose M is a closed, oriented 7-manifold with $\lambda(M) \not\equiv 0 \pmod 7$. Then, there is no orientation reversing diffeomorphism of M onto itself.*

Proof. Assume there exists $\varphi : M \to M$ an orientation-reversing diffeomorphism. Composing it with $\psi := id : M \to -M$, we get an orientation preserving diffeomorphism $\psi \circ \varphi : M \to -M$. Thus,

$$\lambda(M) \equiv \lambda(-M) \equiv -\lambda(M) \pmod 7 ,$$

from what follows $\lambda(M) \equiv 0 \pmod 7$. This is a contradiction.

□

Example 4.4. The standard 7-sphere $\mathbb{S}^7$ is a closed, oriented 7-manifold for which the cohomology groups $H^3_{dR}(\mathbb{S}^7)$ and $H^4_{dR}(\mathbb{S}^4)$ are trivial. Moreover, it bounds the unit disk $\mathbb{D}^8$, which is a compact, oriented 8-manifold. Thus, we can consider $\lambda(\mathbb{S}^7)$. By Examples 2.44 and 3.21, it follows that

$$\lambda(\mathbb{S}^7) \equiv 2p_{2,0}(\mathbb{D}^8) - \sigma_\partial(\mathbb{D}^8) \equiv 2 \times 0 - 0 \equiv 0 \pmod 7 .$$

We can generalize the computations in the preceding example in the following way. Let B be a compact, oriented 8-manifold with non-empty boundary. Assume further that $H^3_{dR}(\partial B)$ and $H^4_{dR}(\partial B)$ are trivial. Thus, we can consider $\lambda(\partial B)$. If $H^4_{dR}(B) = \{0\}$, the relative long exact sequence implies that we also have $H^4_{dR}(B, \partial B) = \{0\}$. In particular,

$$\sigma_\partial(B) = 0 \qquad \text{and} \qquad p_{2,0}(B) = 0 ,$$

Therefore,

$$\lambda(\partial B) \equiv 2p_{2,0}(B) - \sigma_\partial(B) \equiv 0 \pmod 7 .$$

This proves part of the following result.

Proposition 4.5. *Let S be a compact, oriented 7-manifold such that $H^3_{dR}(S) = H^4_{dR}(S) = \{0\}$ and suppose it bounds a compact, oriented 8-manifold B. If $\lambda(S) \not\equiv 0 \pmod 7$, then the fourth Betti number of B is non-zero. In particular, S can not be diffeomorphic to $\mathbb{S}^7$.*

Proof. Suppose the fourth Betti number is zero, i.e., $H^4_{dR}(B) = \{0\}$. Then,

$$\lambda(S) \quad \equiv \quad \lambda(\partial B) \quad \equiv \quad 0 \qquad (\text{mod } 7) \ ,$$

by the preceding discussion. This is a contradiction, since we have assumed $\lambda(S) \neq 0 \pmod 7$.

Finally, if the oriented manifold S were diffeomorphic to $\mathbb{S}^7$, then we would have

$$\lambda(S) \quad \equiv \quad \pm\,\lambda(\mathbb{S}^7) \quad \equiv \quad 0 \qquad (\text{mod } 7) \ .$$

This is a contradiction, for $\lambda(S) \not\equiv 0 \pmod 7$, by hypothesis.

$\square$

Example 4.6. For each odd integer k, let S^7_k be the boundary of D^8_k with the induced orientation. Then, Proposition 3.23 gives

$$p_{2,0}(D^8_k) \quad = \quad 4k^2 \qquad\qquad \text{and} \qquad\qquad \sigma_\partial(D^8_k) \quad = \quad 1 \ ,$$

from what we get

$$\lambda(S^7_k) \quad \equiv \quad 8k^2 \ - \ 1 \quad \equiv \quad k^2 \ - \ 1 \qquad (\text{mod } 7) \ .$$

Remark 4.7. In particular, for k an odd integer with $k^2 \not\equiv 1 \pmod 7$, the manifold S^7_k does not admit an orientation reversing diffeomorphism onto itself.

The oriented 7-manifold S^7_k is homeomorphic to $\mathbb{S}^7$. When $\lambda(S^7_k) \not\equiv 0 \pmod 7$, they can not be diffeomorphic and, in this case, S^7_k is an exotic 7-sphere. Besides, if $\lambda(S^7_k) \not\equiv \lambda(S^7_l) \pmod 7$, there is no orientation preserving diffeomorphism between S^7_k and S^7_l. In particular, if both values are non-zero, we have non-equivalent oriented exotic spheres. Indeed, recall that we consider oriented exotic spheres equivalent if, and only if, there is an orientation preserving diffeomorphism between them. Therefore, since

$$\#\,\{\,\lambda(S^7_k) \ | \ k \text{ is an odd integer}\,\} \quad = \quad \#\,\{6, 0, 3, 1\}\ ,$$

we get the following result.

Theorem 4.8. *There are at least three distinct equivalence classes of oriented exotic 7-spheres. In particular, exotic 7-spheres exist.*

Remark 4.9. From the 27 distinct equivalence classes of oriented exotic 7-spheres, precisely 15[1] are represented by total spaces of $\mathbb{S}^3$-bundles over $\mathbb{S}^4$ (see [EK62]). With Milnor's invariant, we were able distinguish between 3 classes in the last collection.

4.2 Shimada's invariant

To simplify notation, for any closed, oriented 16-manifold N, we write

$$Q(N) \;:=\; 71\, p_{1,0,1,0,}(N) \;+\; 19\, p_{0,2,0,0}(N) \;-\; 22\, p_{2,1,0,0}(N) \;+\; 3\, p_{4,0,0,0}(N) \;.$$

Now, if N is a compact, oriented 16-manifold with non-empty boundary, we write

$$q(N) \;:=\; 71\, p_{1,0,1,0,}(N) \;+\; 19\, p_{0,2,0,0}(N) \;-\; 22\, p_{2,1,0,0}(N) \;+\; 3\, p_{4,0,0,0}(N) \;,$$

whenever all the relative Pontryagin numbers figuring in the sum above are defined for N.

Let M be a closed, oriented 15-manifold such that

$$H^k_{dR}(M) \;=\; \{0\} \qquad (k \;=\; 3,4,7,8,11,12) \;.$$

Assume further that M bounds a compact, oriented 16-manifold B. Then, by the relative long exact sequence, the weighted partitions

$$(1,0,1,0)\,,\; (0,2,0,0)\,,\; (2,1,0,0) \qquad \text{and} \qquad (4,0,0,0)$$

of 16 are compatible with B and we can consider $q(B)$. We define Shimada's invariant by

$$\theta(M) \quad \overset{\text{def}}{\equiv} \quad q(B) \;+\; 14175\,\sigma_\partial(B) \;\equiv\; q(B) \;+\; 78\,\sigma_\partial(B) \qquad (\text{mod } 381) \;.$$

We claim that θ is well-defined. Indeed, let B_i $(i = 1,2)$ be compact, oriented 8-manifolds whose boundaries are M. By *Hirzebruch's Signature Theorem*,

$$\sigma(B_1 \diamond B_2) \;=\; \frac{1}{14175}\,(381\; p_{0,0,0,1}(B_1 \diamond B_2) \;-\; Q(B_1 \diamond B_2))\;,$$

from what follows that

$$14175\,\sigma(B_1 \diamond B_2) \;+\; Q(B_1 \diamond B_2) \;\equiv\; 381\; p_{0,0,0,1}(B_1 \diamond B_2) \;\equiv\; 0 \qquad (\text{mod } 381)\;. \tag{4.2}$$

From Corollary 2.48 and Corollary 3.20, we see that

$$\sigma(B_1 \diamond B_2) \;=\; \sigma_\partial(B_1) \;-\; \sigma_\partial(B_2)$$

[1] In [EK62], the reader will actually find out that 16 equivalence classes in Θ_7 can be represented by total spaces of $\mathbb{S}^3$-bundles over $\mathbb{S}^4$. However, one of these classes is the class of the oriented standard 7-sphere. Indeed, $\mathbb{S}^7$ is diffeomorphic to S^7_1, as pointed out in [Mil56].

and

$$Q(B_1 \diamond B_2) \;=\; q(B_1) \;-\; q(B_2) \;,$$

respectively. Therefore, substituting these values in equation (4.2) and rearranging the terms, we conclude that

$$q(B_1) \;+\; 14175\,\sigma_\partial(B_1) \;\equiv\; q(B_2) \;+\; 14175\,\sigma_\partial(B_2) \pmod{381} \;,$$

proving that θ is well-defined.

Example 4.10. The standard 15-sphere $\mathbb{S}^{15}$ is a closed, oriented 7-manifold for which

$$H^k_{dR}(\mathbb{S}^{15}) \;=\; \{0\} \qquad (k = 3, 4, 7, 8, 11, 12) \;.$$

Moreover, it bounds the unit disk $\mathbb{D}^{16}$, which is a compact, oriented 8-manifold. Thus, we can consider $\theta(\mathbb{S}^{15})$. By Examples 2.44 and 3.21, it follows that

$$\theta(\mathbb{S}^{15}) \;=\; q(\mathbb{D}^{16}) \;+\; 78\,\sigma_\partial(\mathbb{D}^{16}) \;=\; 0 \;+\; 78 \times 0 \;\equiv\; 0 \pmod{381} \;.$$

The following proposition collects the algebraic rules involving Shimada's invariant. Note that they are similar to the ones satisfied by Milnor's invariant. Since the proofs of such properties are, mutatis mutandis, the same as for Milnor's invariant, we omit the demonstration.

Proposition 4.11. *The following hold.*

1. *Let M and N be closed, oriented 15-manifolds for which Shimada's invariant is defined. If they are diffeomorphic, then*

   $$\theta(M) \;\equiv\; \pm\,\theta(N) \pmod{381} \;,$$

   according as the diffeomorphism preserves or reverses orientation. In particular, $\theta(-M) \equiv -\theta(M) \pmod{381}$.

2. *Let M be a closed, oriented 15-manifold for which*

   $$H^k_{dR}(M) \;=\; \{0\} \qquad (k = 3, 4, 7, 8, 11, 12) \;.$$

   Assume further that M bounds a compact, oriented 16-manifold B. If $\theta(M) \not\equiv 0 \pmod{381}$, then the Betti numbers β_4, β_8 and β_{12} of B are non-zero. In particular, M can not be diffeomorphic to $\mathbb{S}^{15}$.

3. *Let M be a closed, oriented 15-manifold for which $\theta(M)$ is defined. If $\theta(M) \not\equiv 0 \pmod{381}$, then M has no orientation reversing diffeomorphism onto itself.*

Example 4.12. For each odd integer k, let S_k^{15} be the boundary of D_k^{16} with the induced orientation. Then, Proposition 3.23 gives

$$q(D_k^8) \;=\; 19 \times 36k^2 \qquad\qquad \text{and} \qquad\qquad \sigma_\partial(D_k^8) \;=\; 1 \;,$$

from what we get

$$\theta(S_k^{15}) \;\equiv\; 303k^2 \,+\, 78 \;\equiv\; 78(1 \,-\, k^2) \qquad (\mathrm{mod}\ 381) \;.$$

Remark 4.13. In particular, for k an odd integer with $\theta(S_k^{15}) \not\equiv 0 \pmod{381}$, the manifold S_k^{15} has no orientation reversing diffeomorphism onto itself.

The oriented 15-manifold S_k^{15} is homeomorphic to $\mathbb{S}^{15}$. When $\theta(S_k^{15}) \not\equiv 0 \pmod{381}$, they can not be diffeomorphic, so S_k^{15} is an oriented exotic 15-sphere. Moreover, if $\theta(S_k^7) \not\equiv \theta(S_l^7) \pmod{381}$, there is no orientation preserving diffeomorphism between S_k^{15} and S_l^{15}. In particular, if both values are non-zero, we have non-equivalent oriented exotic spheres. Therefore, since

$$\#\,\{\,\theta(S_k^{15}) \;|\; k \text{ is an odd integer}\,\} \;=\; 64\;,$$

we get the following result.

Theorem 4.14. *There are at least* 63 *distinct equivalence classes of oriented exotic* 15*-spheres. In particular, exotic* 15*-spheres exist.*

Remark 4.15. From the 16255 distinct equivalence classes of oriented exotic 15-spheres, precisely 4095[2] are represented by total spaces of $\mathbb{S}^7$-bundles over $\mathbb{S}^8$ (see [EK62]). With Shimadas's invariant, we were able to distinguish between 63 classes of the last kind.

[2] In [EK62], the reader will actually find out that 4096 equivalence classes in Θ_{15} can be represented by total spaces of $\mathbb{S}^7$-bundles over $\mathbb{S}^8$. However, one of these classes is the class of the oriented standard 15-sphere. Indeed, $\mathbb{S}^{15}$ is diffeomorphic to S_1^{15}, as pointed out in [Shi57]

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