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On Topological Stability of Iterated Function Systems

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Dissertação de mestrado apresentada ao Programa de Pós-graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Mestre em Matemática

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Este trabalho é dedicado às crianças adultas que,
quando pequenas, sonharam em se tornar cientistas.

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Sumário

1 Introduction

The theory of dynamical systems deals with the long time behaviour of a particle submitted to a law of iteration. The cornerstone of the theory is to understand this behaviour. Obviously, this is an unrealistic quest, since it is easy to produce many dynamics as the set of closed sets in the euclidean space with dimension at least two.

Even so, many research was done to understand this type of question for the majority of orbits and for the majority of systems. The notion of majority here leads to many viewpoints, like generic systems, generic points, Lebesgue almost every points and so on. This is a fruitful field of research. It can be seen in [12, 17].

Inspired by the theory of group actions a natural question arises, what is the actual dynamics if we randomly choose the law? This is still a difficult question, since it deals with a notion of randomness and also with the regularity of the dynamical systems.

In this spirit, people like Arbieto, Junqueira, Santiago, Thakkar, Das, Glavan, Gutu decided to study the whole behaviour. This is called the Iterated Function Systems (IFS) theory, where we deal with all possible orbits generated by all possible iterations combining the maps. It turns out that this could upgrade trivial dynamics to non-trivial ones, like, it is possible to use two Morse-Smale systems and show that their combined iteration leads to minimal dynamics (even in a robust way!).

In this viewpoint, a natural question is to understand the omega-limit set of the iterations (that now we call an orbit for the IFS). If we think of a contraction map, the Banach fixed point tell us that the omega-limit set is only a point (fixed). However, we can take two contractions on the interval as in [10], it can be showed that the omega-limit set is the Cantor ternary set.

This remark was strong enough to link the IFS theory with the theory of Fractal sets. This theory was very popular in other fields of science, like it is expressed in the book of Barnsley [5].

One of the first and seminal work was developed by Hutchinson in [10]. He address the general question of the dynamics of finitely many contractions. He realize that the study of the omega-limit set of a collection of maps is connected with the iteration of compact sets. It is like a collective dynamics, and he found that the base space was already been study by topologists, the so called hyperspace of a compact metric space, using the Hausdorff metric. This is very important in continuum theory and so on, see [4, 13]

He introduces an operator, nowadays called the Hutchinson-Barnsley operator, that evolves any point in the hyperspace using the whole IFS. He realizes that uniform contractions of the maps induces a uniform contraction in the hyperspace. Thus, using Banach's fixed point again an attractor appears. It turns out that this attractor is the omega-limit set of all possible the iterations, and in applications is the Fractal that were searched.

In some sense, this leads to investigations if weak sources of contractions could be enough to obtain such attractors. Moreover, he also produces other type of attractors, statistical ones, using other operator: the Transfer Operator acting on the set of measures. However, we will not pursuit this here.

Eventually, people realize that the theory of IFS can be seen as the action of a atomic measure on the space of dynamics over the phase space, generating a random dynamical system. So, a natural question arises: Instead of an atomic measure (related with finitely maps) could be used a Radon measure with compact support? In other words, could be the parameter space compact instead of finite?

This was pursuit by many authors, and eventually Arbieto, Junqueira e Santiago [2] obtained several results in this setting asking very weak sources of contractions. After all, Melo [11] generalized this in his thesis.

One of the purposes of this work is to give the dynamical and topological basis to this theory, also with examples, and to describe Hutchinson theory in the case of a compact parameter space.

The theory of dynamical systems had a boost in mathematics with the advent of hyperbolic theory due to Smale [15]. The horseshoe became the paradigmatic example and had two important topological dynamical features: expansiveness and the shadowing property. This two notions were extensively studied by many mathematicians like Sakai, Kato, Thakkar, Das and many others. One of theirs best feature was that they were heavily used in the stability of hyperbolic differentiable dynamics, see [18].

It turns out, that in topological dynamics, this also leads to some type of stability (nowadays called topological stability), see [1]. Moreover, it was shown that the shadowing property is a necessary condition to topological stability. This was a seminal result that gives rise to the study of shadowing-type properties and even stronger forms of stability, like the Gromov-Hausdorff Stability by Arbieto and Morales, see [3].

Naturally, this notion was exploited for IFS by some authors, see [14]. However, in our opinion it is not so well precise. Moreover, the study is done in the finite case. So, the second purpose of this work is to clarify this issues and to prove in the case of a compact parameter space.

For this, we propose a new type of shadowing for IFS and a another definition of topological stability. These new definitions permit us to obtain our main results showing that this new shadowing is a necessary condition for topological stability, extending the original result of Pilyugin to the context of IFS and we also show that this shadowing added to the expansiveness are sufficient condition for topological stability.

One interesting feature in the theory of dynamical systems is the concept of entropy. It measures the complexity generated by the orbits in a system. This is a fruitful area of research in many aspects, can be seen in [6]. It is natural to try to study this in the theory of IFS. However, to formalize it is not too simple, and several definitions appear. So, we will not pursuit it in this work, but we have some partial results and many questions involving that. We hope that this could be analyzed in future works and inspire more delicate questions.

2 Iterated Function Systems

In 1981, Hutchinson introduced the IFS in the way of studying fractals in [10]. In that case he studied only hyperbolic IFS with finite parameter space, a finite collection of contractions. His theory and fractal theory was disseminated by different books, like [5, 7].

One way of extend the results of Hutchinson is to enlarge the parameter space, Thakkar and Das admitted the parameter space to be an infinite countable set in [16] while others authors considered the parameter space as a compact metric space. Our study was completely based on the last one, which permit us to work in a more general case, without losing the vastness of examples that already exist.

A family $\{\omega_\lambda : \lambda \in \Lambda\} \subset C^0(X)$ such that Λ is a compact metric space and $\omega : \Lambda \times X \to X$, given by $\omega(\lambda, x) := \omega_\lambda(x)$ is a continuous map, is said to be an IFS with compact parameter space. We call ω its general map and we call each ω_λ a partial map. The space Λ is called the *parameter space* and X is called the *phase space* of this IFS. We will often refer to an IFS by its general map but it is important observe that different general maps can represent the same IFS.

The space $\Lambda^{\mathbb{N}}$ endowed with the product topology will be denoted by $\Omega := \Lambda^{\mathbb{N}}$. For $\sigma = (\lambda_1, \lambda_2, ...) \in \Omega$ we will denote the map $\omega_{\sigma_k} := \omega_{\lambda_k} \circ ... \circ \omega_{\lambda_1}$ and $\omega_{\sigma_0} := id$. A sequence $\{x_n\}$ is called a *chain* for the IFS ω if for any n there exists $\lambda_n \in \Lambda$ such that $x_{n+1} = \omega_{\lambda_n}(x_n)$.

A first observation is that any finite set with the discrete metric is a compact metric space. So any IFS with finitely many partial maps is automatically included in our definition.

A second observation is that the family of partial maps is uniformly equicontinuous, and it comes directly from the fact that the general maps is continuous, as we can see bellow.

Proposition 1. *The family of partial maps of an IFS is uniformly equicontinuous.*

Proof. Let $\lambda \in \Lambda$. Since ω is continuous and $\Lambda \times X$ is compact, then ω is uniformly continuous which means that for any given $\varepsilon > 0$ there exists $\delta > 0$ such that for any $d(x, y) = d((\lambda, x), (\lambda, y)) < \delta$ implies that $d(\omega(\lambda, x), \omega(\lambda, y)) = d(\omega_\lambda(x), \omega_\lambda(y)) < \varepsilon$. □

A first and simple example with only two partial maps will be exhibited bellow. This example will reappear in futures chapters when it will be clear its properties.

Example 1. *Let* $\omega_1, \omega_2 : [0,1] \to [0,1]$ *given by:*

$$\omega_1(x) = \tfrac{1}{3}x,\ \omega_2(x) = \tfrac{1}{3}x + \tfrac{2}{3}$$

The collection $\{\omega_1, \omega_2\}$ *is an IFS with parameter space* $\{1,2\}$ *and phase space* $[0,1]$.

To simplify the notation let us denote $I := [0,1]$. *If we apply both partial maps in* I *repeatedly, we obtain the sets exhibited in the figure 1.*

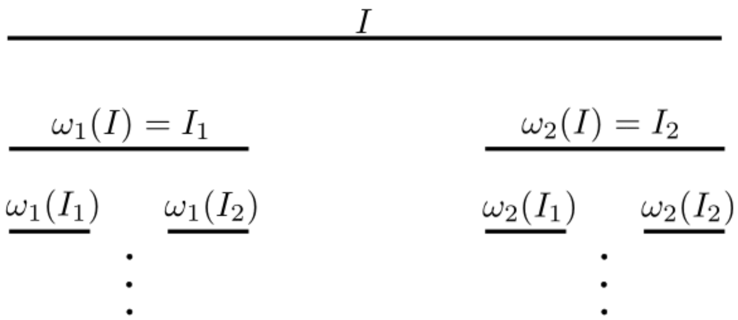

Figura 1 – Image of I by the partial maps.

Observe that the successive applications of the partial maps in I converges to the Cantor ternary set.

The convergence of this example is not an isolated case, a large class of IFS has a similar behavior and conditions for it happens will be studied in future chapters. By now, we will focus on present another example, but differently than the before, now with infinitely many partial maps

Example 2. *Let us consider* $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ *the unit circle and the IFS given for the following general map:*

$$\begin{aligned} \omega : &[0,1] \times \mathbb{T}^1 \to \mathbb{T}^1 \\ &(\lambda, x) \mapsto x + \lambda \mod 1 \end{aligned}$$

An interesting property of this IFS is that for any fixed $x \in \mathbb{T}^1$ *we have* $\omega([0,1]\times\{x\}) = \mathbb{T}^1$ *which implies that any sequence* $\{x_n\}$ *in* $\mathbb{T}^1$ *is a chain for the IFS.*

3 Examples

Example 3. *Let us consider* $\{1, ..., n\}^{\mathbb{N}} = \Sigma$ *with the metric* $d(x, y) = 2^{-k}$ *where* k *is the smallest natural number where* $x_k \neq y_k$ *and if* $x = y$, $d(x, y) = 0$ *and the IFS whose general map is* $\omega : \Sigma \times \Sigma \to \Sigma$ *where* $\omega(\lambda, x)_{2i-1} = \lambda_i$ *and* $\omega(\lambda, x)_{2i} = x_i$ *for all* i.

For a fixed $\lambda \in \Sigma$ *we have that if* $d(x, y) = 2^{-k}$ *then* $d(\omega(\lambda, x), \omega(\lambda, y)) = 2^{-2k} \leq 2^{-2-k} = 2^{-2}d(x, y)$ *for* $k \geq 2$. *If* $k = 1$ *then* $d(\omega(\lambda, x), \omega(\lambda, y)) = 2^{-2} = 2^{-1}d(x, y)$, *so* ω *is uniformly contracting with contraction ratio* 2^{-1}. *Consequently, by theorem 9 it has strong shadowing property.*

Example 4. *Consider the compact metric space* $\Lambda = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ *and the IFS* $\omega : \Lambda \times [0, 1] \to [0, 1]$, *where* $\omega_0(x) = x$ *and* $\omega_{1/n}(x) = (1 - 1/2n)x$ *for* $n \geq 1$. *We can show that this IFS does not have shadowing property. If we take* $\varepsilon \leq 1/10$ *for any* $\delta > 0$ *there exists* n_0 *such that* $1/n_0 < \delta$. *The sequence* $\{y_k\}$ *where* $y_k = k/n_0$ *for* $0 \leq k \leq n_0$ *and* $y_k = 1$ *for* $k > n_0$ *is a* δ*-chain for the sequence* $(0, 0, 0...)$ *that cannot be shadowable, because for any* $x \in B(0, \varepsilon)$ *and for any sequence* $\sigma \in \Omega$ *the sequence* $\{\omega_{\sigma_k}(x)\}$ *is decreasing, thus* $\omega_{\sigma_k}(x) < \varepsilon$ *and it cannot shadow* $\{y_k\}$.

Example 5. *By now, we will consider* $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ *and the IFS defined by the general map bellow.*

$$\begin{aligned} \omega : [0, 1] \times \mathbb{T}^2 &\to \mathbb{T}^2 \\ (\lambda, x_1, x_2) &\mapsto (2x_1 + x_2, x_1 + x_2) - (\lambda, 0) \mod 1 \end{aligned}$$

Firstly, we observe that this IFS is expansive. This is a consequence of the linearity of $f(x_1, x_2) = (2x_1 + x_2, x_1 + x_2)$. *Let us consider* $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{T}^2$ *and* $\sigma = (\lambda_1, \lambda_2, ...) \in \Omega$. *So, we have:*

$$\omega_{\lambda_1}(x) = f(x) - (\lambda_1, 0) \mod 1$$

The second iterate for this sequence is:

$$\begin{aligned} \omega_{\sigma_2}(x) &= \omega_{\lambda_2}(\omega_{\lambda_1}(x)) \\ &= f(f(x) - (\lambda_1, 0)) - (\lambda_2, 0) \mod 1 \\ &= f^2(x) - f(\lambda_1, 0) - (\lambda_2, 0) \mod 1 \end{aligned}$$

By induction, we obtain for each k:

$$\omega_{\sigma_k}(x) = f^k(x) - f^{k-1}(\lambda_1, 0) - ... - f(\lambda_{k-1}, 0) - (\lambda_k, 0) \mod 1$$

For $\omega_{\sigma_k}(y)$ *we have the same expression. Thus, we have the following:*

$$d(\omega_{\sigma_k}(x), \omega_{\sigma_k}(y)) = d(f^k(x), f^k(y))$$

Since f *is an expansive map, the IFS* ω *must be too. More than this, the time of expansion is independent of the sequence of parameters.*

4 The Hyperspace

In this chapter we will present the Hyperspace, that will be an important tool for studying IFS. It will permit us to prove some important results for hyperbolic IFS and it will also be important to understand the space of IFS of a compact metric space X.

Let (X, d) be a metric space, we will denote $\mathcal{K}(X)$ as the collection of all nonempty compact subsets of X, and call it the *Hyperspace of* X. We shall define a metric for this space, known as Hausdorff Metric, and prove some interesting properties relating X and its hyperspace.

For $A, B \in \mathcal{K}(X)$ we can define the following operation:

$$\rho(A, B) = \sup_{a \in A} d(a, B)$$

This operation and its properties will be important to define the Hausdorff distance on the Hyperspace.

Proposition 2. *Let $x \in X$ and $A, B, C \in \mathcal{K}(\mathrm{X})$ then:*

1. *There exists $a_0 \in A$ and $b_0 \in B$ such that $\rho(A, B) = d(a_0, b_0)$.*

2. *$\rho(A, B) = 0$ if and only if $A \subset B$.*

3. *$\rho(A, C) \leq \rho(A, B) + \rho(B, C)$.*

Proof. To prove the first property, since $\rho(A, B) = \sup\{d(a, B) : a \in B\}$ there exists a sequence $\{a_n\}$ such that $\lim_{n\to\infty} d(a_n, B) = \rho(A, B)$. As A is compact, we can consider the sequence convergent and a_0 its limit. Since $B \in \mathcal{K}(X)$ there exists b_0 such that $d(a_0, B) = d(a_0, b_0)$.

So, $\rho(A, B) = \lim_{n\to\infty} d(a_n, B) = d(a_0, B) = d(a_0, b_0)$.

For the third property, suppose $A \subset B$, then for each $a \in A$ and consequently $a \in B$. So $d(a, B) = 0$ for all $a \in A$. Hence, $\rho(A, B) = 0$.

Conversely, as $\rho(A, B) = 0$, if $a \in A$, then $d(a, B) = 0$. By property (1) there exists $b \in B$ such that $d(a, b) = d(a, B) = 0$. So, $a = b$ and $A \subset B$.

Finally, we prove the last property using that, for each $a \in A$ there exists b_0 such that $d(a, b_0) = d(a, B)$. Then, we have:

$$\begin{aligned} d(a, C) &= \inf\{d(a, c) : c \in C\} \\ &\leq \inf\{d(a, b_0) + d(b_0, c) : c \in C\} \\ &= d(a, b_0) + \inf\{d(b_0, C) : c \in C\} \\ &= d(a, B) + d(b_0, C) \\ &\leq \rho(A, B) + \rho(B, C) \end{aligned}$$

As, a is arbitrary we obtain $\rho(A, C) \leq \rho(A, B) + \rho(B, C)$. □

Clearly ρ is not symmetrical. Because of this it cannot be a metric for $\mathcal{K}(X)$, but this problem can be easily solved defining the Hausdorff Distance as bellow:

$$d_H(A, B) := \max\{\rho(A, B), \rho(B, A)\}.$$

Theorem 1. *The Hausdorff Distance is a metric for the hyperspace.*

Proof. Firstly, the symmetry is an immediate consequence of the definition of d_H. By definition we have $d \geq 0$, which implies $\rho \geq 0$ and consequently $d_H \geq 0$.

Let $A, B \in \mathcal{K}(X)$. By property (2), if $A = B$, then $\rho(A, B) = \rho(B, A) = 0$ and then $d_H(A, B) = 0$. Conversely, if $d_H(A, B) = 0$, then $\rho(A, B) = \rho(B, A) = 0$ and also by property (2), we have $A = B$. So, $d_H(A, B) = 0$ if and only if $A = B$.

Lastly, the triangle inequality is obtained by the property (3), added to the definition of d_H. Let $C \in \mathcal{K}(x)$, so we have:

$$\rho(A, C) \leq \rho(A, B) + \rho(B, C) \leq d_H(A, B) + d_H(B, C)$$

Similarly, we have:

$$\rho(C, A) \leq d_H(A, B) + d_H(B, C)$$

So, by definition of d_H:

$$d_H(A, C) \leq d_H(A, B) + d_H(B, C)$$

□

The Hausdorff metric is important in the sense that it makes the hyperspace inherits topological properties from the original space, as examples we have completeness and compactness that will be proved during this chapter. The first one of those has a fundamental role in the main theorem of the next chapter.

In the way of prove the completeness, we will use the notion of closed ε-neighborhood. For $A \subset X$ and $\varepsilon > 0$ we can define $A + \varepsilon := \{x \in X : d(x, A) \leq \varepsilon\}$.

Theorem 2. *If $A \in \mathcal{K}(X)$ and $\varepsilon > 0$, then $A + \varepsilon$ is closed.*

Proof. Let x be a limit point of $A + \varepsilon$. Then there exists a sequence $\{x_n\}$ of points in $A + \varepsilon \backslash \{x\}$ converging to x. As $x_n \in A + \varepsilon$, we obtain that $d(x_n, A) \leq \varepsilon$ for all $n \in \mathbb{N}$. For each n there exists $a_n \in A$ such that $d(x_n, A) = d(x_n, a_n)$. By compactness of A we can consider a_n convergent and $a \in A$ its limit. Thus, $d(x, A) \leq d(x, a) \leq \varepsilon$ which implies $x \in A + \varepsilon$ and then $A + \varepsilon$ is closed. □

Remark. Although being closed $A + \varepsilon$ is not necessarily compact. If we consider $\mathbb{R}$ with the discrete metric, $A \subset \mathbb{R}$ a nonempty set and $\varepsilon > 1$ then $A + \varepsilon = \mathbb{R}$ that is not compact.

This concept will be useful for the study of convergence of sequences in the hyperspace in the sense that it makes easier verify if two compact sets are ε-close in the Hausdorff metric. A simplification is given by the following theorem.

Proposition 3. *Suppose $A, B \in \mathcal{K}(X)$ and $\varepsilon > 0$. Then $d_H(A, B) \leq \varepsilon$ if and only if $A \subset B + \varepsilon$ and $B \subset A + \varepsilon$*

Proof. By symmetry it is sufficient to prove $\rho(B, A) \leq \varepsilon$ if and only if $B \subset A + \varepsilon$.

If $B \subset A + \varepsilon$, then by definition $d(b, A) \leq \varepsilon$ for all $b \in B$. It follows that $\rho(B, A) \leq \varepsilon$. Conversely, if $\rho(B, A) \leq \varepsilon$, then $d(b, A) \leq \varepsilon$ for all $b \in B$ which implies $B \subset A + \varepsilon$. □

As said before, we want to prove that the completeness of the hyperspace is inherited from the completeness of the original space. For this we will take a Cauchy sequence in the hyperspace and we will exhibit a limit for it.

The first step for the construction of this limit is an extension lemma.

Lemma 1. *If $\{A_n\}$ be a Cauchy sequence in $\mathcal{K}(X)$, $\{n_k\}$ an increasing sequence of natural number and $\{x_{n_k}\}$ a Cauchy sequence in X satisfying $x_{n_k} \in A_{n_k}$ for all k, then there exists $\{y_n\}$ a Cauchy sequence in X such that $y_{n_k} = x_{n_k}$ for all k and $y_n \in A_n$ for all n.*

Proof. We can consider $n_1 = 1$. For each $n_{k-1} < n \leq n_k$ we take y_n such that $d(x_{n_k}, A_n) = d(x_{n_k}, y_n)$. Since $x_{n_k} \in A_{n_k}$ we obtain $d(x_{n_k}, A_{n_k}) = 0$ and consequently $y_{n_k} = x_{n_k}$ as desired. From the construction we also obtain:

$$d(x_{n_k}, y_n) = d(x_{n_k}, A_n) \leq \rho(A_{n_k}, A_n) \leq d_H(A_{n_k}, A_n)$$

Given $\varepsilon > 0$, since $\{x_{n_k}\}$ is a Cauchy sequence, there exists k_0 such that $d(x_{n_k}, x_{n_j}) < \frac{\varepsilon}{3}$ for all $k, j \geq k_0$. Since $\{A_n\}$ is also a Cauchy sequence, there exists $n_0 \geq n_{k_0}$ such that $d(A_n, A_m)$ for all $n, m \geq n_0$.

For each $n, m \geq n_0$ there exists $k, j > n_{k_0}$ such that $n_{k-1} < n \leq n_k$ and $n_{j-1} < m \leq n_j$. Then we have:

$$\begin{aligned} d\left(y_n, y_m\right) &\leq d\left(y_n, x_{n_k}\right) + d\left(x_{n_k}, x_{n_j}\right) + d\left(x_{n_j}, y_m\right) \\ &= d\left(x_{n_k}, A_n\right) + d\left(x_{n_k}, x_{n_j}\right) + \left(x_{n_j}, A_m\right) \\ &\leq d_H\left(A_{n_k}, A_n\right) + d\left(x_{n_k}, x_{n_j}\right) + d_H\left(A_{n_j}, A_m\right) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

Therefore, the sequence $\{y_n\}$ is also a Cauchy sequence with all the properties desired. □

For a sequence $\{A_n\}$ we can define the set A as the set of all $x \in X$ such that there exists a sequence $\{x_n\}$ converging to x satisfying $x_n \in A_n$ for all n. If $\{A_n\}$ is a Cauchy sequence in, then A will be a candidate for its limit. If we want completeness of the hyperspace, then A must be compact and its compactness will be given by the next lemmas.

Lemma 2. *If $\{A_n\}$ is a Cauchy sequence in $\mathcal{K}(X)$ and X is complete, then A is a nonempty closed set.*

Proof. We start by proving that A is nonempty. As $\{A_n\}$ is a Cauchy sequence, there exists n_1 such that $d_H(A_n, A_m) < \frac{1}{2}$ for all $n, m \geq n_1$. There is also $n_2 > n_1$ such that $d_H(A_n, A_m) < \frac{1}{2^2}$. By continuing the process we obtain an increasing sequence $\{n_k\}$ such that $d_H(A_n, A_m) < \frac{1}{2^k}$ for all $n, m \geq n_k$.

Let x_{n_1} be a fixed point in A_{n_1}. There exists $x_{n_2} \in A_{n_2}$ such that $d(x_{n_1}, A_{n_2}) = d(x_{n_1}, x_{n_2})$. So we can construct a sequence $\{x_{n_k}\}$ by using this method and we obtain $x_{n_{k+1}}$ satisfying $d(x_{n_k}, A_{n_{k+1}}) = d(x_{n_k}, x_{n_{k+1}})$ for all k.

We claim that $\{x_{n_k}\}$ is a Cauchy sequence. It follows directly from the construction of $\{n_k\}$.

$$d(x_{n_1}, x_{n_2}) = d(x_{n_1}, A_{n_2}) \leq \rho(A_{n_1}, A_{n_2} \leq d_H(A_{n_1}, A_{n_2}) < \frac{1}{2}$$

Similarly, we obtain:

$$d(x_{n_k}, x_{n_{k+1}}) \leq d_H(A_{n_k}, A_{n_{k+1}}) < \frac{1}{2^k}$$

So $\{x_{n_k}\}$ is a Cauchy sequence. By the extension lemma there exists $\{y_n\}$ Cauchy sequence satisfying $y_n \in A_n$. Since X is complete $\{y_n\}$ is convergent and by definition its limit is in A. Hence, A is nonempty.

Now, let a be a limit point of A. Then, there exists a sequence $\{a_k\}$ in $A \backslash \{a\}$ converging to a. By definition of A, each a_k is limit of a sequence $\{z_n\}$ converging to a_k

such that $z_n \in A_n$ for all n. So, there exists $z_{n_1} \in A_{n_1}$ such that $d(a_1, z_{n_1}) < 1$. Similarly, we obtain $z_{n_2} \in A_{n_2}$ such that $d(a_2, z_{n_2}) < \frac{1}{2}$ with $n_2 > n_1$. By repeating the process we obtain a sequence $\{z_{n_k}\}$ such that $d(a_k, z_{n_k}) < \frac{1}{k}$ for all k.

Given $\varepsilon > 0$, there exists k_1 such that $\frac{1}{k_1} < \frac{\varepsilon}{2}$. Since $\{a_k\}$ converges to a, there exists $k_0 > k_1$ such that $d(a_k, a) < \frac{\varepsilon}{2}$ for all $k \geq k_0$. Then, for all $k \geq k_0$ we have:

$$d(z_{n_k}, a) \leq d(z_{n_k}, a_k) + d(a_k, a) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So $\{z_{n_k}\}$ converges to a and since it is a convergent sequence, it is also a Cauchy sequence. By the extension lemma there exists $\{y_n\}$ a Cauchy sequence satisfying $y_n \in A_n$ for all n and $y_{n_k} = z_{n_k}$. Since X is complete, the sequence $\{y_n\}$ is convergent and by the uniqueness of the limit, it must converge to a and consequently $a \in A$. Therefore, A is closed. $\square$

As being closed is not enough to be a compact set, it is missing to check if the set A as defined above is also totally bounded. The following lemma guarantee that in the context we have for completeness, the set A will be in fact totally bounded.

Lemma 3. *If A is a subset of X and $\{D_n\}$ is a sequence in $\mathcal{K}(X)$ such that for any $\varepsilon > 0$ there exists N such that $A \subset D_N + \varepsilon$, then A is totally bounded.*

Proof. Let $\varepsilon > 0$. By hypothesis there exists N such that $A \subset D_N + \frac{\varepsilon}{4}$. Since D_N are compact, they are totally bounded and there exists $\{x_1, ...x_q\}$ such that $D_N \subset \bigcup_{i=1}^{q} B_d\left(x_i, \frac{\varepsilon}{4}\right)$. We can consider that $B_d\left(x_i, \frac{\varepsilon}{2}\right) \cap A \neq \emptyset$ for $1 \leq i \leq p$ and $B_d\left(x_i, \frac{\varepsilon}{2}\right) \cap A = \emptyset$ for $p < i$. For each $1 \leq i \leq p$, let $y_i \in B_d\left(x_i, \frac{\varepsilon}{2}\right) \cap A$. We claim that $A \subseteq \bigcup_{i=1}^{p} B_d\left(y_i, \varepsilon\right)$. Let $a \in A$, then $a \in D_N + \frac{\varepsilon}{4}$ which implies $d(a, D_N) \leq \frac{\varepsilon}{4}$. Since D_N is compact, there exists $x \in D_N$ such that $d(a, x) = d(a, D_N)$. Since $x \in D_N$, we have that $x \in B_d(x_i, \frac{\varepsilon}{4})$ for some $1 \leq i \leq q$. On the other hand, we have that:

$$d(a, x_i) \leq d(a, x) + d(x, x_i) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

Thus $1 \leq i \leq p$ and for the respective y_i we have:

$$d(a, y_i) \leq d(a, x_i) + d(x_i, y_i) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore, $a \in B_d(y_i, \varepsilon)$, consequently $A \subset \displaystyle\bigcup_{i=1}^{p} B_d\left(y_i, \varepsilon\right)$ and we obtain that A is totally bounded. $\square$

We are now ready to prove both theorems mentioned before about the properties of the hyperspace inherited from the original space.

Theorem 3. *If (X, d) is complete, then $(\mathcal{K}(X), d_H)$ is complete.*

Proof. Let $\{A_n\}$ be a Cauchy sequence in $\mathcal{K}(X)$ and A be the set of all points $x \in X$ such that there exists a sequence $\{x_n\}$ converging to x satisfying $x_n \in A_n$ for all n. By 2 we have that A is a nonempty closed set.

Let $\varepsilon > 0$. Since $\{A_n\}$, there exists $n_0 > 0$ such that $d_H(A_n, A_m) < \varepsilon$ for all $n, m \geq n_0$ which implies $A_m \subset A_n + \varepsilon$, by lemma 3. Fix $N \geq n_0$. For any sequence $\{x_n\}$ satisfying $x_n \in A_n$ have that $x_n \in A_N + \varepsilon$ for all $n > N$. By proposition 2 $A_N + \varepsilon$ is closed, so if $\{x_n\}$ converges to x, then $x \in A_N + \varepsilon$. Thus, $A \subset A_N + \varepsilon$. By lemma 3 A is totally bounded and consequently A is compact, so $A \in \mathcal{K}(X)$.

Since $N > n_0$ is arbitrary, for all $n \geq n_0$ we have that $A \subset A_n + \varepsilon$ for all $n \geq n_0$. As $\{A_n\}$ is Cauchy, there exists $n_1 > n_0$ such that $d_H(A_n, A_m) < \frac{\varepsilon}{2^2}$ for all $n, m \geq n_1$. We can also obtain $n_2 > n_1$ such that $d_H(A_n, A_m) < \frac{\varepsilon}{2^3}$ for all $n, m \geq n_2$. By continuing the process we obtain an increasing sequence $\{n_k\}$ such that $d_H(A_n, A_m) < \frac{\varepsilon}{2^{k+1}}$ for all $n, m \geq n_k$.

Let $n > n_1$ and $y \in A_n$. We have that $A_n \subset A_{n_1} + \frac{\varepsilon}{2^2}$, so there exists $x_{n_1} \in A_{n_1}$ such that $d(y, A_{n_1}) = d(y, x_{n_1}) \leq \frac{\varepsilon}{2^2}$. We also have that $A_{n_1} \subset A_{n_2} + \frac{\varepsilon}{2^2}$, so there exists x_{n_2} such that $d(x_{n_1}, A_{n_2}) = d(x_{n_1}, x_{n_2}) < \frac{\varepsilon}{2^2}$. repeating the argument, we obtain $x_{n_3} \in A_{n_3}$ such that $d(x_{n_2}, x_{n_3} < \frac{\varepsilon}{2^3}$ By continuing the process we obtain a sequence $\{x_{n_k}\}$ such that $d(x_{n_k}, x_{n_{k+1}} < \frac{\varepsilon}{2^{k+2}}$, so $\{x_{n_k}\}$ is a Cauchy sequence and by construction $x_{n_k} \in A_{n_k}$. Since X is complete we can admit that the sequence converges to $a \in X$. By the extension lemma $a \in A$. On the other hand, for any k we find that

$$
\begin{aligned}
d(y, x_{n_k}) &\leq d(y, x_{n_1}) + d(x_{n_1}, x_{n_2}) + d(x_{n_2}, x_{n_3}) + \ldots + d(x_{n_{k-1}}, x_{n_k}) \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \ldots + \frac{\varepsilon}{2^k} < \varepsilon
\end{aligned}
$$

It follows that $y \leq \varepsilon$, then $y \in A + \varepsilon$ and consequently $A_n \subset A + \varepsilon$ for all $n \geq n_1$.

Therefore, $d_H(A_n, A) \leq \varepsilon$ for all $n \geq n_1$ and $\mathcal{K}(X)$ is complete. □

The second property mentioned above relating X and $\mathcal{K}(X)$ by the Hausdorff metric says that the hyperspace inherits compactness from the original space and it is proved in the next theorem.

Theorem 4. *If (X, d) is compact, then $(\mathcal{K}(X), d_H)$ is compact.*

Proof. From the previous theorem, we obtain the completeness of $\mathcal{K}(X)$. So it is missing to prove that $\mathcal{K}(X)$ is totally bounded.

Let $\varepsilon > 0$. Since X is totally bounded there exists $\{x_1, \ldots, x_n\}$ such that $X \subset \displaystyle\bigcup_{i=1}^{n} B_d\left(x_i, \frac{\varepsilon}{3}\right)$. Let $\{C_k : 1 \leq k \leq 2^n - 1\}$ be the collection of all possibles nonempty union of the closures of these balls. Since X is compact the closure of each ball is also compact. As each C_k is a finite union of compact sets, it is also compact. So, $C_k \in \mathcal{K}(X)$ for $1 \leq k \leq 2^n - 1$. We claim that $\mathcal{K}(X) \subset \bigcup_{k=1}^{n} B_{d_H}(C_K, \varepsilon)$.

Let $K \in \mathcal{K}(X)$ and $S = \{i : K \cap \overline{B_d\left(x_i, \frac{\varepsilon}{3}\right)} \neq \emptyset\}$. Let $C_j = \bigcup_S \overline{B_d\left(x_i, \frac{\varepsilon}{3}\right)}$, so $K \subset C_j$ which implies $\rho(K, C_j) = 0$.Let $x \in C_j$. Then $x \in B_d(x_i, \frac{\varepsilon}{3})$ for some $i \in S$. By definition of S there exists $z \in K \cap B_d(x_i, \frac{\varepsilon}{3})$. By triangle inequality we obtain that $d(x, K) \leq d(x, z) \leq \frac{2\varepsilon}{3}$. Since x is arbitrary, we find that $\rho(C_j, K) \leq \frac{2\varepsilon}{3} < \varepsilon$. Therefore, $d_H(K, C_j) < \varepsilon$, so $K \in B_{d_H}(C_j, \varepsilon)$. Since K is arbitrary, we find that $\mathcal{K}(X)$ is totally bounded and consequently is compact. □

The notion of hyperspace and IFS are close related. In the next chapter we will see that this new tool will permit us to find attractors for hyperbolic IFS. More than that, after some studies we observed that the hyperspace were even more related to the IFS. One of our results is to explicit the relation between them which is exposed in the next theorem.

Theorem 5. *The space of IFS with phase space X is the hyperspace of $C^0(X)$.*

Proof. To see that $K \in \mathcal{K}(X)$ is an IFS we just need to consider the parameter space as K with the C^0-metric. The other continence requires a little bit more.

Let $\{\omega_\lambda : \lambda \in \Lambda\}$ be an IFS, we want to show that it is a compact subset of $C^0(X)$. For this, we define $\varphi : \Lambda \to C^0(X)$ given by $\varphi(\lambda) = \omega_\lambda$ and we claim that φ is continuous.

Let $\{\lambda_n\}$ be a sequence in Λ converging to λ. By the continuity of the first variable of $\omega : \Lambda \times X \to X$, we obtain that $\{\omega_{\lambda_n}\}$ converges pointwise to ω_λ. As $\{\omega_\lambda : \lambda \in \Lambda\}$ is equicontinuous, so is $\{\omega_{\lambda_n}\}$. Since X is compact, we obtain that the convergence is uniform, which implies convergence in the C^0-topology and consequently the continuity of φ. Therefore, $\varphi(\Lambda) = \{\omega_\lambda : \lambda \in \Lambda\}$ is compact and that completes the proof. □

The theorem 3 added to the last theorem permit us to conclude that the space of IFS with phase space X is a complete metric space with the Hausdorff metric, and it comes from the fact that if X is a compact metric space, then $C^0(X)$ is a complete space with C^0-topology. So, to fix the notation, by now for $\omega : \Lambda \times X \to X$ and $\tilde{\omega} : \tilde{\Lambda} \times X \to X$ both IFS, we will denote $d_H(\omega, \tilde{\omega}) := d_H(\{\omega_\lambda : \lambda \in \Lambda\}, \{\tilde{\omega}_\lambda : \lambda \in \tilde{\Lambda}\})$.

5 The Hyperbolic Case

Hutchinson introduced the notion of IFS in [10] and also proved an important theorem guaranteeing the existence of a compact invariant attractor set for the IFS if all the maps are contractions. He did it all for finitely many maps, but we can also obtain the same result for IFS with compact parameter space by making simple changes.

In the way of proving the existence of this attractor for an IFS ω, we will use the *Hutchinson-Barnsley Operator* $\mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X)$ which is given by:

$$\mathcal{F}(K) := \bigcup_{\lambda \in \Lambda} \omega_\lambda(K) = \omega(\Lambda \times K)$$

Since ω is continuous and $\Lambda \times K$ is compact for all $K \in \mathcal{K}(X)$, we obtain that the operator is well defined.

We say that a $K \in \mathcal{K}(X)$ is *invariant* for the IFS ω if it is a fixed point for the Hutchinson-Barnsley Operator, that is, $\mathcal{F}(K) = K$. If for any $A \in \mathcal{K}(X)$ we have that $\mathcal{F}^n(A) \to K$ in the Hausdorff topology, we say that K is an *attractor* for the IFS.

The proof of the existence of an invariant attractor for the IFS of contractions of Hutchinson is based on the Banach Fixed-Point Theorem, applying it on the Hutchinson-Barnsley operator. We will do the same, but to use the theorem, the operator must be a contraction. This is easy to prove if we consider the IFS uniformly contracting.

We say that the an IFS $\{\omega_\lambda : \lambda \in \Lambda\}$ is *uniformly contracting* if

$$\beta := \sup_{\lambda \in \Lambda} \sup_{x \neq y} \frac{d(\omega_\lambda(x), \omega_\lambda(y))}{d(x, y)} < 1$$

And this number β is called *contraction ratio*.

Lemma 4. *If an IFS ω is uniformly contracting with contraction ratio β, then the Hutchinson-Barnsley generated by this IFS is a contraction with contractivity factor β.*

Proof. We want to prove that $d_H(\mathcal{F}(A), \mathcal{F}(B)) \leq \beta d_H(A, B)$.

Let $z \in \mathcal{F}(A)$. By definition of the $\mathcal{F}$ there exists $x \in A$ and $\lambda \in \Lambda$ such that $z = \omega_\lambda(x)$. From the previous chapter we know that there exists $y \in B$ such that

$d(x,y) = d(x,B)$. By construction and the contractivity of ω_λ we obtain:

$$\begin{aligned} d(z,\mathcal{F}(B)) &\leq d(\omega_\lambda(x),\omega_\lambda(y)) \\ &\leq \beta d(x,y) \\ &= \beta d(x,B) \\ &\leq \beta \rho(A,B) \\ &\leq \beta d_H(A,B) \end{aligned}$$

Since z is arbitrary if we take the supremum of $z \in \mathcal{F}(A)$ we obtain that:

$$\rho(\mathcal{F}(A),\mathcal{F}(B)) \leq \beta d_H(A,B)$$

Analogously, we obtain that:

$$\rho(\mathcal{F}(B),\mathcal{F}(A)) \leq \beta d_H(A,B)$$

Thus, by the definition of the Hausdorff metric, we conclude that:

$$d_H(\mathcal{F}(A),\mathcal{F}(B)) \leq \beta d_H(A,B)$$

□

So we have an extension for Hutchinson's theorem about the existence of an unique invariant attractor for IFS with compact parameter space using the same strategy of applying Banach Fixed-Point Theorem.

Theorem 6. *If an IFS is uniformly contracting, then there exists an unique invariant attractor for the IFS.*

Proof. In the previous chapter, we proved that the Hyperspace of X inherited the completeness from the completeness of X. The previous lemma guarantees that with our hypothesis of the IFS be uniformly contracting, the Hutchinson-Barnsley operator is a contraction on $\mathcal{K}(X)$. Thus, by Banach Fixed-Point Theorem there exists $K \in \mathcal{K}(X)$ an unique fixed point for Hutchinson-Barnsley operator which means that K is the unique invariant compact subset for the IFS. Moreover, if $A \in \mathcal{K}(X)$ then $\mathcal{F}^n(A)$ converges to K in the Hausdorff topology and we conclude that K is an invariant attractor for the IFS. □

One example of attractor of an IFS is the Cantor Ternary set obtained in example 1. Many others examples are exhibited in [5].

6 Expansiveness

For maps, expansiveness means essentially that for any two different points, their orbits move away from each other at least a constant. This notion can be translated for IFS trading orbits for chains for any sequence.

Formally, we say that an IFS ω is *expansive* if there exists a constant $\eta > 0$, called expansivity constant, such that for any $\sigma \in \Omega$ if $x, y \in M$ satisfy $d(\omega_{\sigma_n}(x), \omega_{\sigma_n}(y)) \leq \eta$ for all $n \in \mathbb{N}$, then $x = y$. We usually say that ω is $\eta -$ *expansive*.

The following theorem helps us to understand a little bit more about expansiveness and will be very important in the chapter of topological stability. Essentially, it says that for an expansive IFS if two points have their chains with the same sequence remaining close at most the expansivity constant for an enough long time, then these points must be close too.

Theorem 7. *If ω is an IFS η-expansive and $\sigma = (\lambda_1, \lambda_2, ...)$ a sequence, then for any given $\mu > 0$ there exists $N > 0$ such that if $x, y \in M$ and $d(\omega_{\sigma_n}(x), \omega_{\sigma_n}(y)) \leq \eta$ for all $n \leq N$, then $d(x, y) < \mu$.*

Proof. Suppose that exists μ that fails the lemma. Then for each $N \in \mathbb{N}$ there exists x_N and y_N such that $d(\omega_{\sigma_k}(x_N), \omega_{\sigma_k}(y_N)) \leq \eta$ for all $k \leq N$ but $d(x_N, y_N) \geq \mu$. So, we obtain $\{x_N\}_{N\in\mathbb{N}}$ and $\{y_N\}_{N\in\mathbb{N}}$ and by compactness we can assume they are convergent, respectively to x and y. Now fixed $n \in \mathbb{N}$, by continuity of the IFS we have that $\omega_{\sigma_n}(x_N)$ converges to $\omega_{\sigma_n}(x)$ and $\omega_{\sigma_n}(y_N)$ converges to $\omega_{\sigma_n}(y)$. As $d(\omega_{\sigma_n}(x_N), \omega_{\sigma_n}(y_N)) \leq \eta$ for all $n \leq N$, we obtain that $d(\omega_{\sigma_n}(x), \omega_{\sigma_n}(y)) \leq \eta$. On the other hand, $d(x_N, y_N) \geq \mu$ for all $N \in \mathbb{N}$ wich implies $d(x, y) \geq \mu$. This contradicts the hypothesis of ω be η-expansive. □

In the last chapter we talked about uniformly contracting IFS, similarly, we say that an IFS ω is *uniformly expanding* if

$$\alpha := \inf_{\lambda\in\Lambda} \inf_{x\neq y} \frac{d(\omega_\lambda(x), \omega_\lambda(y)}{d(x, y)} > 1$$

.

If an IFS ω is uniformly expanding with expanding ratio α, $x \neq y \in X$ and $\sigma \in \Omega$, we have:

$$d(\omega_{\lambda_1}(x), \omega_{\lambda_1}(y)) \geq \alpha d(x, y).$$

By induction, we have that for any $i \geq 1$:

$$d(\omega_{\sigma_i}(x), \omega_{\sigma_i}(y)) \geq \alpha^i d(x, y).$$

Since $\alpha > 1$ and $d(x, y) > 0$, for i large enough we have that

$$d(\omega_{\sigma_i}(x), \omega_{\sigma_i}(y)) > 1.$$

Thus, any uniformly expanding IFS is expansive. A natural question is if all expansive IFS is also uniformly expansive. In 2018, Nia answered the question in [8] exposing an example similar to the one bellow that is expansive but it is not uniformly expanding.

Example 6. *If we consider Σ as the space of binary sequences and $d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$ the space (Σ, d) is a compact metric space. It is well known that the shift map $\omega_1 : \Sigma \to \Sigma$ defined by $(\omega_1(x))_i = x_{i+1}$ is continuous. Let us consider $\omega_2 : \Sigma \to \Sigma$ defined by $(\omega_2(x))_i = x_{i+2}$ which is continuous, since it is the composition of ω_1 twice. So, $\omega : \{1, 2\} \times \Sigma \to \Sigma$ is an IFS. If $x \neq y$ then, for any sequence $\sigma \in \{1, 2\}^{\mathbb{N}}$ there exists k such that $d(\omega_{\sigma_k}(x), \omega_{\sigma_k}(y) \geq \frac{1}{4}$, so it is expansive. On the other hand if we consider $x = (1, 0, 0, ...)$ and $y = (0, 0, 0, ...)$ we have that $\frac{d(\omega_1(x), \omega_1(y))}{d(x,y)} = 0$ so, by definition, it cannot be uniformly expanding.*

7 Shadowing Property

In Dynamical Systems, a well known property studied for maps is the shadowing property, which consists in guaranteeing the existence of a real orbit close to a pseudo-orbit, which is a sequence similar to an orbit, but where small errors is permitted for each iterate. We studied a similar idea for IFS.

Given a sequence $\{x_k\}$ in X and $\delta > 0$, this sequence is said to be a *δ-chain* if for each k there exists $\lambda_k \in \Lambda$ such that $d(x_k, \omega_{\lambda_k}(x_{k-1})) \leq \delta$. If the sequence is finite, we call it *finte δ-chain*.

In the context of IFS, we say that an IFS ω has the *Shadowing Property* if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that for any δ-chain $\{x_k\}$ there exists a chain $\{y_k\}$ such that $d(x_k, y_k) < \varepsilon$ for all $k \geq 0$. In this case, we say that $\{y_k\}$ *(ε)-shadows* $\{x_k\}$. Similarly, we say that an IFS ω has the *Finite Shadowing Property* if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that for any finite δ-chain $\{x_1, ..., x_n\}$ there exists a chain $\{y_k\}$ such that $d(x_k, y_k) < \varepsilon$, for $k = 0, ..., n$.

*Remark.*We remark that these definitions of shadowing property and finite shadowing property do not guarantee any relations between the sequence of the shadow and the sequence of the δ-chain.

As we are considering Λ and X compact, we have an equivalence between these two definitions. Clearly shadowing property implies finite shadowing property and the converse is given in the following lemma.

Theorem 8. *If an IFS has finite shadowing property, then it has shadowing property.*

Proof. Let $\varepsilon > 0$, by hypothesis there exists $\delta > 0$ such that every finite δ-chain can be ε-shadowable. Let $\{x_k\}$ be a δ-chain, then for each natural number i there exists $y_i \in M$ and $\sigma^i = (\lambda_1^i, \lambda_2^i, ...) \in \Omega$ such that:

$$d(\omega_{\sigma_j^i}(y_i), x_j) < \varepsilon, j = 0, ..., i. \tag{7.1}$$

By compactness of Λ and X, we can assume that $\{y_i\}$ converges to y and $\{\lambda_k^i\}$ converges to λ_k for all k. From this we can construct $\sigma = (\lambda_1, \lambda_2, ...)$. Fixed n, we have that $(\lambda_1^i, ..., \lambda_n^i)$ converges to $(\lambda_1, ..., \lambda_n)$. As ω is continuous, $\omega_{\sigma_n^i}$ converges to ω_{σ_n} and by (7.1) $d(\omega_{\sigma_n}(y), x_n) < \varepsilon$. Thus, as n is arbitrary, we have that $\{\omega_{\sigma_k}(y)\}$ ε-shadows $\{x_k\}$ and the IFS has the shadowing property. □

For some technical reasons sometimes we would like to be able to shadow the δ-chain with a chain having the same sequence. We say that an IFS has *Sequential*

Shadowing Property if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-chain can be ε-shadowable by a chain with the same sequence.

A natural question is: Is there any IFS with shadowing property by without sequential shadowing property?

Fortunately, the answer is positive! The example 2 answers the question. By the observation made in the example that any sequence of points in $\mathbb{T}^1$ is a chain we conclude that the IFS has shadowing property, because any δ-chain is, in fact, a chain and shadows itself. On the other hand, if we fix $\lambda \in [0, 1]$ the partial map generated by λ is a rotation, so it does not have the shadowing property. If we consider $\{x_n\}$ a δ-pseudo orbit for the map ω_λ that cannot be shadowable, it is a δ-chain for the IFS with sequence $\sigma = (\lambda, \lambda, ...) \in \Omega$. By the same observation made when the example was exposed this δ-chain is also a chain for the IFS and also shadows itself, as the map ω_λ does not have shadowing property it is impossible to shadow $\{x_n\}$ by a chain with sequence σ. Another curiosity about this example is that, since none rotation has shadowing property, we conclude that it is not required the partial maps to have shadowing property to the entire IFS to have it.

In 2001, Glavan and Gutu proved that sequential shadowing property comes immediately from uniformly contraction in [9]. The results exhibited by them are really general, in the sense that they work for any IFS regardless of the parameter space.

Theorem 9. *Every IFS uniformly contracting has sequential shadowing property.*

Proof. Let ω be an IFS uniformly contracting with contraction ratio β. For any given $\varepsilon > 0$ take $\delta = (1-\beta)\frac{\varepsilon}{2}$. Let $\{x_n\}$ be a δ-chain with sequence σ and y_0 satisfying $d(x_0, y_0) < \frac{\varepsilon}{2}$. Let us consider the chain $\{y_n\}$ given by $y_n = \omega_{\sigma_n}(y_0)$ Firstly, we observe that:

$$d(x_1, y_1) \leq d(x_1, \omega_{\lambda_1}(x_0)) + d(\omega_{\lambda_1}(x_0), \omega_{\lambda_1}(y_0)) \leq \delta + \beta d(x_0, y_0)$$

By induction we obtain that for any $n \geq 1$

$$d(x_n, y_n) \leq \delta(1 + \beta + ... + \beta^{n-1}) + \beta^n d(x_0, y_0) \leq \delta\frac{1}{1-\beta} + \beta^n d(x_0, y_0)$$

By definition of δ and the choice of y_0 we finally conclude that:

$$d(x_n, y_n) \leq \delta\frac{1}{1-\beta} + \beta^n d(x_0, y_0) \leq \frac{\varepsilon}{2} + \beta^n\frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, $\{y_n\}$ ε-shadows $\{x_n\}$. Since both of them have the same sequence we obtain that ω has the sequential shadowing property. □

Also in [9], they proved that this statement is also true for IFS uniformly expanding by adding the hypothesis of all partial maps being surjective.

Theorem 10. *Every IFS uniformly expanding with all partial maps being surjective has strong shadowing property.*

These theorems show us that we several examples with this new property.

Sometimes more than be able to shadow with the same sequence we want uniqueness of this shadow and some IFS have this property. We say that an IFS has *Uniqueness Shadowing Property* if it has sequential shadowing property and there exists $\varepsilon > 0$ such that for its respective δ from the sequential shadowing property we have that for any $\{x_k\}$ δ-chain with sequence σ there exists an unique y such that $\{\omega_{\sigma_k}(y)\}$ ε-shadows $\{x_k\}$.

This uniqueness will have a fundamental role in the next chapter, when we show sufficient conditions for an IFS be topologically stable. There we will use the following proposition that is a version for IFS of a well known property for maps.

Proposition 4. *If ω is an η-expansive IFS and it has sequential shadowing property, then ω has the shadowing uniqueness property.*

Proof. Let $\varepsilon < \frac{\eta}{2}$. As ω has strong shadowing property, we obtain $\delta > 0$ such that ever δ-chain is ε-shadowable by a chain with the same sequence . Let $\{x_k\}$ be a δ-chain with sequence σ, so there exists y such that $\{\omega_{\sigma_k}(y)\}$ ε-shadows $\{x_k\}$.

Now suppose there exists z such that $\{\omega_{\sigma_k}(z)\}$ ε-shadows $\{x_k\}$, then we have:

$$d(y_k, z_k) \leq d(y_k, x_k) + d(y_k, z_k) < 2\varepsilon < \eta.$$

Thus, as ω is η-expansive $z = y$ and ω has shadowing uniqueness property. $\square$

8 Topological Stability

In this chapter we introduce a property that permit us to understand, in a topological point of view, the behaviour of another IFS sufficiently close to an IFS with this particular property. We will also see consequences and relations between this property and the others saw in the previous chapters.

For some technical reasons sometimes we will need the phase space to be a smooth compact manifold, in those cases we will replace X by M and d will be a Riemannian metric on M.

Differently than an orbit of a point for a map, for an IFS a point can have infinitely many chains, which together consist the entire orbit of the point. It is impossible compare any two chains between any different IFS, our goal is to analyze similar chains in similar IFS, and because of this we introduce the notion of *δ-compatibility*, that will permit us to define the topological stability.

For ω, $\tilde{\omega}$ two IFS and σ, $\tilde{\sigma}$ sequences (both finite or infinite) in each parameter space, we say that $(\sigma, \tilde{\sigma})$ is *δ-compatible* if for all k we have:

$$d_{C^0}(\omega_{\lambda_k}, \tilde{\omega}_{\tilde{\lambda}_k}) < \delta.$$

Remark. A simple remark is that if $d_H(\omega, \tilde{\omega}) \geq \delta$, then there is no δ-compatible pair of sequences.

We say that an IFS ω is *topologically stable* if given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\tilde{\omega}$ is an IFS and $d_H(\omega, \tilde{\omega}) \leq \delta$, then for each $(\sigma, \tilde{\sigma})$ δ-compatible there exists a continuous map $h : X \to X$(respectively $h : M \to M$) with the following properties:

(i) $d_{C^0}(\omega_{\sigma_k} \circ h, \tilde{\omega}_{\tilde{\sigma}_k}) < \varepsilon$ for all $k \in \mathbb{N}$.

(ii) $d_{C^0}(h, id) < \varepsilon$.

One of our main results, extending and clarifying the work of Rezaei and Nia in [14] is to show that a consequence of topological stability is the shadowing property, actually we go further and we prove that topological stability implies sequential shadowing property for IFS having a manifold with dimension at least 2 as phase space.

The hypothesis of having phase space being a manifold with dimension at least 2 is required because this structure permit us to obtain the following lemma which is fundamental to the entire construction in the theorem. The proof is omitted in this work but is completely exhibited in the master thesis of Bernardo de Carvalho [**?**].

Lemma 5. *Let M be a manifold with $dimM \geq 2$. For any given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\{x_1, ..., x_k, y_1, .., y_k\}$ is a list of distinct points in M satisfying $d(x_j, y_j) < \delta$, then there exists a diffeomorphism f such that:*

(i) $d_{C^0}(f, id) < \varepsilon$.

(ii) $f(x_j) = y_j$, $j = 1, .., k$.

The following is the key lemma for the proof of the first theorem of this chapter. Having an IFS and a δ-chain we shall construct another IFS as close as wanted to the one we have. We also construct a chain for this new IFS close to the initial δ-chain and such that their sequences are δ-compatible, with this, the theorem becomes easy.

Lemma 6. *Let ω be an IFS with $dimM \geq 2$. Given $\Delta > 0$, there exists $\delta > 0$ such that if $\{x_0, ...x_n\}$ is δ-chain with sequence σ, then there exists an IFS $\tilde{\omega}$ satisfying $d_H(\omega, \tilde{\omega}) < \Delta$, a sequence $\tilde{\sigma}$ such that $(\sigma, \tilde{\sigma})$ is Δ-compatible and $y_0 \in M$ satisfying $d(\tilde{\omega}_{\tilde{\sigma}_k}(y_0), x_k) < \Delta$ for $k = 0, ..., n$.*

Proof. As $dimM \geq 2$, for $\Delta > 0$, from lemma 5 we obtain $\delta > 0$ such that if $\{x_0, ..., x_k, y_0, .., y_k\}$ is a list of distinct points in M satisfying $d(x_j, y_j) < 3\delta$, then there exists a diffeomorphism f such that:

(i) $d_{C^0}(f, id) < \Delta$.

(ii) $f(x_j) = y_j$, $j = 0, .., k$.

Let $\{x_1, ..., x_n\}$ be a δ-chain. Since it is finite, using triangle inequality we obtain $\{y_1, ..., y_n\}$ a 3δ-chain such that:

(i) $d(x_k, y_k) < \Delta$, $k = 0, .., n$.

(ii) $\omega_{\lambda_{k+1}}(y_k) \neq y_{k+1}$, $k = 0, ..., n-1$.

For $k = 0, ..., n-1$ there exists h_k such that $d_{C^0}(h_k, id) < \Delta$ and $h_k(\omega_{\lambda_{k+1}}(y_k)) = y_{k+1}$.

We define $\tilde{\Lambda} = \{0, ..., n-1\} \times \Lambda$ and $\tilde{\omega} : \tilde{\Lambda} \times M \to M$ where $\tilde{\omega}(k, \lambda, x) := h_k \circ \omega_\lambda(x)$. As ω is continuous, so is $\tilde{\omega}$ and then it is a general map of an IFS. We claim that for each $k \in \{0, ..., n-1\}$ and $\lambda \in \Lambda$ we have $d_{C^0}(\tilde{\omega}_{(k,\lambda)}, \omega_\lambda) < \Delta$. Let $x \in M$, then we have:

$$d(\tilde{\omega}_{(k,\lambda)}(x), \omega_\lambda(x)) = d(h_k(\omega_\lambda(x)), \omega_\lambda(x)) \leq d_{C^0}(h_k, id) < \Delta \tag{8.1}$$

As x was arbitrary, we have $d_{C^0}(\tilde{\omega}_{(k,\lambda)}, \omega_\lambda) < \Delta$ and then $d_H(\omega, \tilde{\omega}) < \Delta$.

We define $\tilde{\sigma} = \{\tilde{\lambda}_1, ..., \tilde{\lambda}_n\}$, where $\tilde{\lambda}_k := (k, \lambda_k)$. For y_0 obtained above we have $\tilde{\omega}_{\tilde{\sigma}_k}(y_0) = y_k$ and from (8.1) $(\sigma, \tilde{\sigma})$ is Δ-compatible.

□

Using the previous lemmas we show our first theorem of this chapter, which says that under the conditions mentioned before about the phase space, topological stability implies finite shadowing property, and using results of the last chapter, this is equivalent to prove that topological stability implies shadowing property.

Theorem 11. *Every IFS topologically stable having a manifold with dimension at least* 2 *as phase space has finite shadowing property.*

Proof. Let $\omega : \Lambda \times M \to M$ be an IFS topologically stable with $dimM \geq 2$. For a given $\varepsilon > 0$, from the definition of topological stability we obtain a $\Delta > 0$ such that if $\tilde{\omega}$ is an IFS with $d_H(\omega, \tilde{\omega}) < \Delta$, for each $(\sigma, \tilde{\sigma})$ Δ-compatible there exists a continuous map $h : M \to M$ with the following properties:

(i) $d_{C^0}(\omega_{\sigma_k} \circ h, \tilde{\omega}_{\tilde{\sigma}_k}) < \frac{\varepsilon}{2}$ for all $k \in \mathbb{N}$.

(ii) $d_{C^0}(h, id) < \frac{\varepsilon}{2}$.

We assume $\Delta < \frac{\varepsilon}{2}$.

Let $\delta < \frac{\Delta}{6}$ and $\{x_0, ..., x_n\}$ a δ-chain with sequence σ, then from the previous lemma there exists $\tilde{\omega}$ with $d_H(\omega, \tilde{\omega}) < \Delta$, $\tilde{\sigma}$ such that $(\sigma, \tilde{\sigma})$ is Δ-compatible and y_0 such that $d(\tilde{\omega}_{\tilde{\sigma}_k}(y_0), x_k) < \Delta$ for $k = 0, ..., n$.

We will consider, by now, σ and $\tilde{\sigma}$ infinite by complete with $\lambda_k = \lambda_1$ and $\tilde{\lambda}_k = (1, \lambda_1)$ for $k \geq n$. We remark that $(\sigma, \tilde{\sigma})$ is still δ-compatible.

So, we obtain $h : M \to M$ a continuous map with the properties mentioned above.

We consider $z_0 = h(y_0)$, then $\{\omega_{\sigma_k}(z_0)\}_{k=0}^{\infty}$ is clearly a chain for ω and we observe that for $k = 1, ..., n$:

$$\begin{aligned}
d(x_k, \omega_{\sigma_k}(z_0)) &= d(x_k, \omega_{\sigma_k}(h(y_0)) \\
&\leq d(x_k, \tilde{\omega}_{\tilde{\sigma}_k}(y_0)) + d(\tilde{\omega}_{\tilde{\sigma}_k}(y_0), \omega_{\sigma_k}(h(y_0)) \\
&\leq \Delta + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

□

Corollary 1. *Every topologically stable IFS having a manifold with dimension at least* 2 *as phase space has shadowing property.*

The proof of the theorem gives us more than this, each finite δ-chain is shadowed by chain whose sequence has the same first n elements of the sequence of the δ-chain, since the phase space is a compact manifold, by this observation we obtain another stronger theorem.

Theorem 12. *Every topologically stable IFS with dimension of phase space at least 2 has sequential shadowing property.*

Proof. Let $\varepsilon > 0$. Let us consider $\delta > 0$ as obtained in the proof theorem 11 and let $\{x_k\}_{k=0}^{\infty}$ be a δ-chain with with sequence $\sigma = (\lambda_1, \lambda_2, ...)$. Repeating the proof of that theorem for each $n \in \mathbb{N}$ if we consider $\{x_k\}_{k=0}^{n}$ we obtain z_n and $\sigma^n = (\lambda_1^n, \lambda_2^n, ...)$ such that for all $0 \leq j \leq n$ we have $\lambda_j^n = \lambda_j$ and consequently

$$d(x_k, \omega_{\sigma_k}(z_n)) = d(x_k, \omega_{\sigma_k^n}(z_n)) < \varepsilon. \tag{8.2}$$

By compactness of M we can consider $\{z_n\}_{n\in\mathbb{N}}$ convergent and z_0 its limit. Fixed $k \in \mathbb{N}$, from (8.2) we have that $d(x_k, \omega_{\sigma_k}(z_0)) < \varepsilon$. So, $\{\omega_{\sigma_k}(z_0)\}_{k=0}^{\infty}$ is a chain that ε-shadows $\{x_k\}_{k=0}^{\infty}$ with the same sequence. □

For maps we have a converse for this theorem can be obtained by adding the hypothesis of expansiveness, we expected that this should be true for IFS too. Using the fact that expansiveness added to sequential shadowing property implies uniqueness shadowing property we construct for each pair of sequences δ-compatible a continuous map with the properties desired for topological stability. After give this proof, we observed that it is essentially the same of the proof given by Thakkar and Das in [16] the observation is that a chains with pair of sequences δ-compatible is equivalent to the time varying maps δ-closed for them.

Theorem 13. *Every expansive IFS with sequential shadowing property is topologically stable.*

Proof. Let $\varepsilon > 0$, ω be an IFS expansive with sequential shadowing property and $\eta > 0$ be the expansivity constant of ω. From the theorem 4 in the last chapter, we obtain that ω has shadowing uniqueness property, moreover from the proof we know that any $\varepsilon < \frac{\eta}{2}$ satisfies the shadowing uniqueness property, so let us consider $\varepsilon < \frac{\eta}{2}$, from the sequential shadowing property we have $\delta > 0$ such that any δ-chain is uniquely ε-shadowable by a chain with the same sequence.

Let $\tilde{\omega}$ be an IFS with $d_H(\omega, \tilde{\omega}) < \delta$. Fix $\sigma \in \Omega$ and $x \in X$. Let $\tilde{\sigma}$ be a sequence such that $(\sigma, \tilde{\sigma})$ is δ-compatible.

Since $(\sigma, \tilde{\sigma})$ is δ-compatible, we observe that $\{\tilde{\omega}_{\tilde{\sigma}_k}(x)\}_{k=0}^{\infty}$ is a δ-chain for ω with sequence σ, then there exists a unique point y_x such that the chain $\{\omega_{\sigma_k}(y_x)\}_{k=0}^{\infty}$ ε-shadows $\{\tilde{\omega}_{\tilde{\sigma}_k}(x)\}_{k=0}^{\infty}$.

We define $h : X \to X$, by $h(x) := y_x$ and we observe that from the shadowing uniqueness property h is well defined and by construction $d(x, h(x)) < \varepsilon$ for all $x \in X$.

Thus, if h is continuous, then $d_{C^0}(h, id) < \varepsilon$. We also observe that by construction $d(\omega_{\sigma_k}(h(x)), \tilde{\omega}_{\tilde{\sigma}_k}(x)) < \varepsilon$ for all $k \in \mathbb{N}$ and $x \in M$. So, if h is continuous, we also have $d_{C^0}(\omega_{\sigma_k} \circ h, \tilde{\omega}_{\tilde{\sigma}_k}) < \varepsilon$ for all $k \in \mathbb{N}$.

We claim that h is continuous. Let $\mu > 0$ be given. By theorem 7 there exists $N \in \mathbb{N}$ such that if $x, y \in X$ and $d(\omega_{\sigma_k}(x), \omega_{\sigma_k}(y)) \leq \eta$, for all $k \leq N$ then $d(x, y) < \mu$. For each $k \in \{0, ..., N\}$, ω_{σ_k} and $\tilde{\omega}_{\tilde{\sigma}_k}$ are continuous, as M is compact, they are uniformly continuous and then for each k, there exists $\beta_k > 0$ and $\tilde{\beta}_k > 0$ such that if $d(x, y) < \beta_k$, then $d(\omega_{\sigma_k}(x), \omega_{\sigma_k}(y)) < \epsilon$, and if $d(x, y) < \tilde{\beta}_k$, then $d(\tilde{\omega}_{\tilde{\sigma}_k}(x), \tilde{\omega}_{\tilde{\sigma}_k}(y)) < \varepsilon$. Take $\beta = \min\{\beta_k, \tilde{\beta}_k : k = 0, ..., N\}$. We observe that if $d(x, y) < \beta$ then for $k = 0, ..., N$ we have:

$$\begin{aligned} d(\omega_{\sigma_k}(h(x)), \tilde{\omega}_{\tilde{\sigma}_k}(h(y))) &\leq d(\omega_{\sigma_k}(h(x)), \tilde{\omega}_{\tilde{\sigma}_k}(x)) + d(\tilde{\omega}_{\tilde{\sigma}_k}(x), \tilde{\omega}_{\tilde{\sigma}_k}(y)) \\ &+ d(\tilde{\omega}_{\tilde{\sigma}_k}(y), \omega_{\sigma_k}(h(y))) \\ &< \varepsilon + \varepsilon + \varepsilon < \eta \end{aligned}$$

Thus, $d(x, y) < \beta$ implies $d(\omega_{\sigma_k}(h(x)), \tilde{\omega}_{\tilde{\sigma}_k}(h(y))) < \eta$ for $0, ..., N$, which implies $d(h(x), h(y)) < \mu$ and consequently h continuous and ω is topologically stable. □

Remark. We remark that for this last theorem it is not required the phase space to be a manifold, it works for any compact metric space.

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